Towards An Approach to Hilbert's Sixth Problem: A Brief Review

Yuri Ximenes Martins, Rodney Josué Biezuner

Towards An Approach to Hilbert's Sixth Problem: A Brief Review

Yuri Ximenes Martins*, Rodney Josué Biezuner

Departamento de Matemática, ICEx, Universidade Federal de Minas Gerais, Av. Antônio Carlos 6627, Pampulha, CP 702, CEP 31270-901, Belo Horizonte, MG, Brazil

Abstract

In 1900 David Hilbert published his famous list of 23 problems. The sixth of them - the axiomatization of Physics - remains partially unsolved. In this work we will give a gentle introduction and a brief review to one of the most recent and formal approaches to this problem, based on synthetic higher categorical languages. This approach, developed by Baez, Schreiber, Sati, Fiorenza, Freed, Lurie and many others, provides a formalization to the notion of classical field theory in terms of twisted differential cohomologies in cohesive $(\infty, 1)$-topos. Furthermore, following the Atiyah-Witten functorial style of topological quantum field theories, it provides a nonperturbative quantization for classical field theories whose underlying cohomology satisfies orientability and duality conditions. We follow a pedagogical and almost non-technical style trying to minimize the mathematical prerequisites. No categorical background is required.

1. Introduction

In the year of 1900, David Hilbert (a mathematician which is mostly known in Physics for his contribution to the Einstein-Hilbert action functional) published a list containing 23 problems (usually known as the *Hilbert's problems*) which in his opinion would strongly influence the development of 20th century mathematics [107]. The sixth problem is about the axiomatization of the whole physics and, presently, it remains partially unsolved. Our aim is to introduce basic ideas of an approach to this problem following works of Urs Schreiber, Domenico Fiorenza, Hisham Sati, Daniel Freed, John Baez and many others, mostly inspired by the seminal topos-theoretic works of William Lawvere [134, 137, 135, 138, 136] and formalized through the development of higher topos theory by André Joyal [118, 120, 119], Michael Batanin [27], Ross Street [223], Charles Rezk [183, 184], Carlos Simpson [213], Clark Barwick [24, 25, 26], Tom Leinster [140, 48], Dominic Verity [236, 235], Jacob Lurie [147, 148] and many others. For reviews on these different approaches to higher category theory, see [47, 141, 140].

The idea goes as follows. Some points are also discussed in [179, 191, 58, 58, 44]. See [202] for a technical introduction and [160] for a gentle exposition. See [128] for a discussion on other mathematical problems deeply influenced by Hilbert's works, [215, 178] for a more philosophical discussion and [128] for the history of the sixth problem.

There are two kinds of math: *naive* (or *intuitive*) *math* and *axiomatic* (or *rigorous*) *math*. In naive math the fundamental objects are *primitives*, while in axiomatic math they are defined in terms of more elementary structures. For instance, we have *naive set theory* and *axiomatic set theory*. In both cases (naive or axiomatic) we need a *background language* (also called *logic*) in order to develop the theory, as in the diagram below. In the context of set theory this background language is just *classical logic*.

$$
\begin{tikzcd}
 & \textbf{naive math} \\
\textbf{background language} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & \\
 & \textbf{axiomatic math}
\end{tikzcd}
$$

Notice that, when a mathematician is working (for instance, when he is trying to prove some new result in his area of research), at first he does not make use of completely rigorous arguments. In fact, he first uses his intuition, making some scribbles in pieces of paper or in a blackboard and usually considers many wrong strategies before finally discovering a good sequence of arguments which can be used to prove (or disprove) the desired result. It is only at this later moment that he tries to introduce rigor in his ideas, in order for his result to be communicated and accepted by the other members of the mathematical community.

Thus we can say that in the process of producing new mathematics, *naive arguments come before rigorous ones*. More precisely, we can say that naive mathematics produces conjectures and rigorous mathematics turns these conjectures into theorems. It happens that the same conjecture generally can be proved (or disproved) in different ways, say by using different tools or by considering different models. For instance,

*Corresponding author.

Email addresses: `yurixm@ufmg.br` (Yuri Ximenes Martins), `rodneyjb@ufmg.br` (Rodney Josué Biezuner)

many concepts have algebraic and geometric incarnations (e.g, Serre-Swan theorem [224, 210, 175], Stone diality [116], Gelfand duality [174, 180] and Tannaka duality [59, 145]), suggesting the existence of a duality between algebra and geometry, as in diagram (1) below.

As will be discussed, this duality between algebra and geometry, which in some generality is formalized by Isbell-type dualities [112, 113, 220, 22], extends to the context of physics, leading to dual approaches to Hilbert's sixth problem. For another incarnation of this duality, notice that when a geometric entity x has a *commutative* algebraic incarnation $A(X)$, then we can define the *noncommutative* version of X simply by dropping the commutativity requirement on $A(X)$. This is precisely the approach used in noncommutative geometry [97, 52].

$$\begin{tikzcd}
 & \textbf{naive math} \arrow[r, Rightarrow, "\text{produces}"] \arrow[dd, Rightarrow, "\text{model}"] & \textbf{conjectures} \arrow[dd, Rightarrow, shift right, "\text{algebra}"'] \arrow[dd, Rightarrow, shift left, "\text{geometry}"] \\
\textbf{background language} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & \\
 & \textbf{axiomatic math} \arrow[r, Rightarrow, "\text{produces}"'] & \textbf{theorems}
\end{tikzcd} \tag{1}$$

When compared with mathematics, physics is a totally different discipline. Indeed, in physics we have a restriction on the *existence* of physical objects, meaning that we have a connection with *ontology*, which is given by *empiricism* [215]. More precisely, while math is a strictly logical discipline, physics is logical and ontological. This restriction on existence produces many difficulties. For instance, logical consistence is no longer sufficient in order to establish a given sentence as physically true: ontological consistence is also needed (see [178] for a comparison between philosophical aspects of math and physics). Thus, *even though a sentence is logically consistent, in order to be considered physically true it must be consistent with all possible experiments!* This fact can be expressed in terms of a commutative diagram:

$$\begin{tikzcd}
\textbf{philosophy} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \textbf{logic} \arrow[d, Rightarrow, "\text{mathematical modelling}"] \\
\textbf{ontology} \arrow[r, Rightarrow, "\text{empiricism}"'] & \textbf{physics}
\end{tikzcd}$$

But now, recall that logic determines the validity of mathematical arguments, which was also translated in terms of the commutative diagram (1). So, gluing the diagram of physics with the diagram of math, we have a new commutative diagram:

$$\begin{tikzcd}
 & \textbf{conjectures} \arrow[r, Rightarrow] & \textbf{theorems} \\
 & \textbf{naive math} \arrow[u, Rightarrow] & \\
\textbf{philosophy} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \textbf{logic} \arrow[u, Rightarrow] \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \textbf{axiomatic math} \arrow[uu, Rightarrow] \\
\textbf{ontology} \arrow[r, Rightarrow, "\text{empiricism}"'] & \textbf{physics} &
\end{tikzcd}$$

One of the most important facts concerning the relation between physics and mathematics is that, in the above diagram, the arrow

$$\textbf{logic} \Longrightarrow \textbf{physics}\,.$$

has an inverse

$$\textbf{logic} \Longleftarrow \textbf{physics}\,.$$

This last arrow is called *physical insight*. The composition of physical insight with axiomatic math produces a new arrow, called *mathematical physics*.

$$\begin{tikzcd}
\textbf{logic} \arrow[r, Rightarrow] & \textbf{axiomatic math} \arrow[dl, Rightarrow, "\text{math. phys.}"] \\
\textbf{physics} \arrow[u, Rightarrow, "\text{insight}"] &
\end{tikzcd}$$

Therefore, the full relation between physics and mathematics is given by the following diagram

$$\begin{tikzcd}
 & \textbf{conjectures} \arrow[r, Rightarrow] & \textbf{theorems} \\
 & \textbf{naive math} \arrow[u, Rightarrow] & \\
\textbf{philosophy} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \textbf{logic} \arrow[u, Rightarrow] \arrow[r, Rightarrow] \arrow[d, Leftrightarrow] & \textbf{axiomatic math} \arrow[uu, Rightarrow] \arrow[dl, Rightarrow] \\
\textbf{ontology} \arrow[r, Rightarrow, "\text{empiricism}"'] & \textbf{physics} &
\end{tikzcd}$$

leading us to the following conclusion:

Conclusion: *we can use physical insight in order to do naive mathematics and, therefore, in order to produce conjectures. These conjectures can eventually be proven, producing theorems, which in turn can be used to create mathematical models for physics* (*mathematical physics*).

The axiomatization problem of Physics[1] concerns precisely in building and studying concrete realizations of the following sequence.

$$
\begin{tikzcd}
\textbf{physics} \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & \textbf{logic} \arrow[r, Rightarrow] & \textbf{naive math} \arrow[dl, Rightarrow] \\
\textbf{theorems} & \textbf{conjectures} \arrow[l, Rightarrow] &
\end{tikzcd}
$$

Since this sequence depends explicitly on a background language and since there is a duality between algebra and geometry, it is natural to expect that this duality induces a duality in the level of mathematical physics, as shown in the following diagram.

$$
\begin{tikzcd}
\textbf{physics} \arrow[r, Leftrightarrow] \arrow[d, Rightarrow, shift right, "\text{geometry}"'] \arrow[d, Rightarrow, shift left, "\text{algebra}"] & \textbf{logic} \arrow[r, Rightarrow] & \textbf{naive math} \arrow[dl, Rightarrow] \\
\textbf{theorems} & \textbf{conjectures} \arrow[l, Rightarrow, shift right, "\text{algebra}"'] \arrow[l, Rightarrow, shift left, "\text{geometry}"] &
\end{tikzcd}
$$

This is really the case. At the quantum level it dates from the dual Heisenberg and Schrödinger approaches to quantum mechanics (QM) and it extends to the context of Quantum Field Theories (QFT) with the so-called *Functorial* (or *Topological*) QFT (FQFT or TQFT) and *Algebraic QFT* (AQFT). Both of them are based on a functorial approach capturing a locality principle. More precisely, a TQFT is a functor F assigning to each instant of time t a Hilbert space $F(t) = \mathcal{H}_t$ and to each trajectory $\Sigma : t \to t'$ an operator $F(\Sigma) = U(t';t)$ representing time evolution [7, 238]. On the other hand, an AQFT can be defined as a functor assigning to each small region $U \subseteq M$ of spacetime the corresponding local algebra $\mathcal{A}(U)$ of observables, such that causality conditions are satisfied [18, 74].

At the classical level, it begins with the Gelfand duality between the configuration space of Classical Mechanics M and its algebra of observables $C^\infty(M)$ and generalizes to a duality between an arbitrary symplectic manifold (M,ω) and its corresponding Poisson algebra of observables.

As we will see, in order to take the axiomatization problem seriously one needs to consider arbitrarily abstract languages, which are "higher" versions of the classical language, so that one has to talk about "higher algebra" and "higher geometry", which should play a dual role. In this paper we will follow only the geometric side, so that we will review an approach to Hilbert's sixth problem by means of higher geometrical language. The "higher TQFT" are called *extended TQFT*. For the higher algebraic approach see [30, 34, 32, 35, 33, 31] and [54, 170, 171]. To a comparison between both approaches, see [200, 198]. In sum, we have the following table[2].

For us, "higher" means *homotopical*. Thus, "higher geometry" is about the coupling between homotopy theory

geometry	*algebra*
Schrödinger	Heisenberg
TQFT	AQFT
phase space	alg. observables
higher Poisson geometry	higher Poisson algebra

Table 1: geometry *vs* algebra

and geometry [144, 229, 146, 230, 201]. But since in modern algebraic topology homotopy theory is described by ∞-category theory [163, 147, 49], "higher geometry" is also about the ∞-categorical version of geometry.

$$
\boxed{\begin{array}{c}\textbf{higher}\\ \textbf{geometry}\end{array} = \begin{array}{c}\textbf{homotopy}\\ \textbf{+}\\ \textbf{geometry}\end{array} = \begin{array}{c}\boldsymbol{\infty}\textbf{-categorical}\\ \textbf{geometry}\end{array}} \tag{2}
$$

Because a TQFT is a functor (a categorical object) assigning time evolution operators to trajectories, it is natural do regard extended TQFT's as ∞-functors (which are ∞-categorical objects) assigning trajetories and higher trajectories (which are equivalent to trajectories of arbitrarily dimensional objects) to higher operators acting on ∞-Hilbert spaces. This is the approach of Baez-Dolan-Freed [7, 238, 81], formalized and classified by Lurie [149].

On the other hand, recalling that functors are sources of mathematical invariants, from (2) we conclude that homotopical invariants should play an important role. There are basically two kinds of such invariants: *homotopy groups* and *cohomology groups*. Gauge theories are about G-bundles with connections, which are classified by a flavor of cohomology: the *nonabelian differential cohomology* [42, 211, 109, 201]. On the other hand, the D-brane charges are supposed to be represented in twisted cohomology [239, 39], so that *twisted nonabelian differential cohomology* should be the natural context to describe gauge theories and string theory. What about M-theory? In recent years, Schreiber, Fiorenza and Sati showed that if the C-field of supergravity is quantized in twisted cohomotopy cohomology (assumption known as the *Hypothesis H*), then several folks involving M-theory, including anomaly cancellations, can be proved in all needed mathematical rigor [76, 75, 193, 43, 192, 194, 77]. Thus:

Conclusion: *the ∞-categorical language, when regarded as background language, seems to produce a promising approach for working in Hilbert's sixth problem.*

The remaining of this paper is organized as follows. In Section 2 we emphasize the difference between *axiomatization* and *unification*, where the latter is introduced as a particular approach to the former. In Section 3 we discuss why the classical set-theoretic language is not enough to attack Hilbert's sixth problem, and in Section 4 we consider the categorical language as the next natural candidate. We discuss that it is a nice choice when trying to axiomatize *particle* physics, but in Section 5 we argue that it is not sufficient to the entire Hilbert's sixth problem, so that one has to consider more

[1] As will be discussed in next section, there exists another different (but strictly related) problem: the *unification problem of physics*.

[2] For a complete version of this table, see [1].

abstract languages. Section 6 is devoted to the discussion of an "abstractification process", which assigns to each language a more abstract language by means of *categorification* [14, 160]. Iterating the process and taking the limit one gets an infinitely abstract language: the ∞-categorical language. It is argued that this language solves the problems of Section 5 faced by categorical language. In Section 7 the role of abstract cohomology theory in the description of classical theories is introduced and in Section 8 the quantization problem is briefly considered and reviewed. Finally, some concluding remarks are presented in Section 9.

2. Axiomatization *vs* Unification

Recall that the existence of physical insight gives the "mathematical physics" arrow, as shown below. An approach to Hilbert's sixth problem can then be viewed as a way to present mathematical physics as a *surjective* arrow.

$$
\begin{tikzcd}
\textbf{axiomatic math} \arrow[r, Rightarrow, "\text{mathematical}", "\text{physics}"'] & \textbf{physics}
\end{tikzcd}
$$

Because axiomatic math is described by some kind of logic, the starting point is to select a proper background language. The selected background language determines directly the *naive* math, so that the next step is to analyze the following loops:

$$
\begin{tikzcd}
& \textbf{naive math} \arrow[dr, Rightarrow] & \\
\textbf{background language} \arrow[ur, Rightarrow] & & \textbf{physics} \arrow[ll, Rightarrow]
\end{tikzcd}
\tag{3}
$$

Then, once some model has been selected, we lift it to axiomatic math, as in the first diagram below. The final step is to verify if the corresponding "mathematical physics" arrow is surjective or not. In other words, we have to verify if the axiomatic concepts produced by the selected logic are general enough to model all physical phenomena.

$$
\begin{tikzcd}
& \textbf{axiomatic math} \arrow[ddr, dashed, bend left] & \\
& \textbf{naive math} \arrow[u, Rightarrow, "\text{model}"'] \arrow[dr, Rightarrow] & \\
\textbf{background language} \arrow[ur, Rightarrow] \arrow[uur, Rightarrow, bend left] & & \textbf{physics} \arrow[ll, Rightarrow]
\end{tikzcd}
$$

Presently, physical theories can be divided into two classes depending on the scale of energy involved: the *classical theories* and the *quantum theories*, as shown below.

$$
\begin{tikzcd}
\textbf{classical physics} \arrow[d, Rightarrow] & \\
\textbf{physics} & \textbf{quantum physics} \arrow[l, Rightarrow]
\end{tikzcd}
$$

Thus, there are essentially two ways to build a surjective "mathematical physics" arrow. Either we build the arrow directly (as in the diagram above), or we first build surjective arrows (a) and (b), which respectively axiomatize classical and quantum physics, and then another arrow (c) linking these two axiomatizations, as in the diagram below.

$$
\begin{tikzcd}
& \textbf{axiomatic math} \arrow[dr, dashed, "(a)"] \arrow[ddrr, dashed, bend left, "(b)"] & & \\
& \textbf{naive math} \arrow[u, Rightarrow, "\text{model}"'] \arrow[dr, Rightarrow] & \textbf{classical physics} \arrow[d, Rightarrow] \arrow[dr, dotted, "(c)"] & \\
\textbf{brackground language} \arrow[ur, Rightarrow] \arrow[uur, Rightarrow, bend left] & & \textbf{physics} \arrow[ll, Rightarrow] & \textbf{quantum physics} \arrow[l, Rightarrow]
\end{tikzcd}
$$

We emphasize the difference between the two approaches: in the first one, all physical theories are described by the *same* set of axioms, while in the second classical and quantum theories are described by *different* axioms, but that are related by some process. This means that if we choose the first approach we need to *unify all physical theories*.

There are some models to this unification arrow, but they are not our focus. Just to mention one, string theory is maybe the most well-known. It is based on the assumption that the "building blocks" of nature are not particles, but rather one dimensional entities called *strings*, i.e, connected one-dimensional manifolds, which are diffeomorphic to some interval (when it has boundary or when it is not compact) or to the circle (if it is boundaryless and compact). In the first case we say that we have *open strings*, while in the second we say that we have *closed strings*.

From the mathematical viewpoint, string theory is very fruitful - see [58, 191, 241, 67] for general discussions and [62] for a brief review. This mathematical interest arises in part from the following facts. Since strings are 1-dimensional objects, their worldsheets are 2-dimensional manifolds (typically oriented). In the case of closed strings, these are Riemann surfaces, whose moduli space is finite-dimensional [60, 172]. Since the path integrals are defined on the moduli space of trajectories, it follows that for closed strings they are well-defined [61]. Not only this, they compute many geometric-topological invariants [154, 164, 69, 46], playing a fundamental role in many areas of math, such as complex algebraic geometry [154, 46, 55], complex dynamics [70, 71, 108] and symplectic topology [87, 226, 164]. See [233] for a nice pedagogical overview. In the case of open strings, they start and end in high dimensional objects, called *D-branes*, which fits into a (higher) category called *Fukaya category*, also playing a fundamental role in symplectic topology. Besides that, they appear in Mirror Symmetry, which became a highly active branch of algebraic geometry since the seminal contribution [127] of Kontsevich. See also [110, 55, 122].

On the other hand, one needs to recall that physics is not completely determined by the arrow **logic** $\Rightarrow$ **physics**, but there is also the ontological (i.e, empirical) branch, and presently there is no concrete empirical proof that strings (instead of particles) really are the most fundamental objects of nature [231, 8].

In this text, we will describe an attempt to realize the second approach. As we will see, if one starts with a sufficiently abstract (or powerful) background language, then one can effectively axiomatize *separately* a huge amount of classical and quantum theories, leaving the problem of finding a suitable quantization process, which is the really difficult part[3]. At least in the approach that will be discussed here, this difficulty comes in part from the fact that the underlying background language is itself under construction. Even so, there is a very promising process, intended to generalize geometric quantization from geometry to higher geometry, known as *motivic* or *cohomological* or *pull-push quantization* [201, 176].

3. Towards The Correct Language

As commented in the last section, independently of the approach used in order to attack Hilbert's sixth problem, the starting point is to select a proper background language. The most obvious choice is the *classical logic* used to describe set theory. This logic produces, via the arrow **logic** $\Rightarrow$ **axiomatic math**, not only set theory, but indeed all classical areas of math, such as group theory, topology and differential geometry, since each of them can be described in terms of set theory. Therefore, for this choice of background language, the different known areas of mathematics could be used in order to describe classical and quantum physical phenomena.

Since 1900, when Hilbert published his list of problems, many classical and quantum theories were formalized by means of these by now well known areas of mathematics. Indeed, quantum theories were observed to have a more algebraic and probabilistic nature, while classical theories were presumably more geometric in character.

For instance, a system in quantum mchanics (which is about quantum *particles*) can be formalized in the Schrödinger approach as a pair $(\mathcal{H}, \hat{H})$, where $\mathcal{H}$ is a complex (separable) Hilbert space and $\hat{H} : D(\hat{H}) \to \mathcal{H}$ is a self-adjoint operator defined in a (dense) subspace $D(\hat{H}) \subseteq \mathcal{H}$ [181, 103]. We say that $\hat{H}$ is the *Hamiltonian* of the system and the fundamental problem is to determine its *spectrum*, which on one hand is the set of all information about $\hat{H}$ that can be accessed experimentally and on the other hand is the spectrum (in the mathematical sense, i.e, the generalized eigenvalues) of $\hat{H}$ [227, 57]. The dynamics of the system from a instant t_0 to a instant t_1 is guided by the unitary operator $U(t_1; t_0) = e^{\frac{i}{\hbar}(t_1 - t_0)\hat{H}}$ associated to $\hat{H}$ or, equivalently, by the (time independent) Schrödinger equation $i\hbar \frac{d\psi}{dt} = \hat{H}\psi$.

On the other hand, classical theories for *particles* are given by some *action functional* $S : \mathrm{Fields}(M) \to \mathbb{R}$, defined in some "space of configurations" (or "space of fields") over a spacetime M. These configurations (or fields) generally involve paths $\gamma : I \to M$, interpreted as the trajectories of particles moving in some spacetime. If the particles are not free, i.e, if they are subjected to some interaction, we have also to consider its source as part of the configurations. The presence of an interaction can be measured by means of a force. Furthermore, it may (or not) be intrinsic to the spacetime M. For the geometrical aspects of classical field theories, see [91, 153, 104, 16] and the standard references [173, 79].

For instance, since the development of general relativity in 1916, gravity is supposed to be an intrinsic force[4], meaning that it will act on *any* particle. The presence of an intrinsic force is formalized by the assumption of a certain additional geometric structure on the spacetime M - in the case of gravity, a Lorentzian metric g on M. Other intrinsic interactions are modeled by other types of geometric structure. But not all manifolds may carry a given geometric structure, specially if integrability conditions are required [102, 221, 56]. This means that not all manifolds can be used to model the spacetime. Indeed, each geometric structure exists on a given manifold only if certain quantities, called *obstruction characteristic classes* vanish. For example, a compact manifold admits a Lorentzian metric iff its Euler characteristic $\chi(M)$ vanishes, which implies that $\mathbb{S}^4$ cannot be used to model the universe [29, 56]. On the other hand, M has a spin structure (needed to consider spinorial fields on M) iff its first two Stiefel-Whitney classes are null [133]. Furthermore, if one tries to model gravity using geometries other than the Lorentzian one the obstructions are even stronger [157, 156].

An important class of non-intrinsic interactions are given by Yang-Mills theories (or more general gauge theories). These depend on a Lie group G, called the *gauge group*, and on a G-principal bundle $P \to M$ over the spacetime M. The interaction is modelled by a connection on P, which is just a vertical equivariant $\mathfrak{g}$-valued 1-form $A : TP \to \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G. We can think of A as the *potential* of the interaction, so that the *force* is just the (exterior covariant) derivative $d_A A$ of A. Here, the standard example is electromagnetism, for which G is the abelian group $U(1)$ and A is the eletromagnetic vector potential. We can think of an arbitrary Yang-Mills interaction as some kind of "nonabelian" version of electromagnetism, leading, e.g, to the development of nonabelian Hodge theory [214, 212, 89, 111] and to the formulation of powerful topological invariants, such as Donaldson invariants [65, 64, 85] and Seiberg-Witten invariants [208, 63, 226]. Note that if one tries to extend or modify

[3]Notice that one of the Millennium Problems is part of this quest [115].

[4]There is an old but active discussion whether gravity is actually an external force (gauge theory). It is not the focus of this review. See [130, 66, 232, 114, 190, 189].

the Yang-Mills-type interactions one also finds obstructions [158, 159].

Now, let us return to focus on Hilbert's sixth problem. It requires answering for questions like these [201, 202]:

1. What is a classical theory?
2. What is a quantum theory?
3. What is quantization?

Notice that the previous discussion *does not* answer these questions. Indeed, it only reveals some examples and aspects of what classical and quantum theories should be; it does not say, axiomatically, what they really are. Furthermore, the presence of obstructions for freely extending the Lorentzian geometry of gravity and Yang-Mills theories suggests the existence of another background language in which these obstructions disappear. This leads us to the following conclusion:

Conclusion: *classical logic, viewed as a background language, is very nice in order to formulate and study properties of particular classical and quantum theory. On the other hand, a priori it does not give tools to study more deeper questions as those required by Hilbert's sixth problem.*

4. The Role of Categorical Language

The above conclusion is that, in order to attack Hilbert's sixth problem, the natural strategy is to replace classical logic by a more abstract background language. *What kind of properties should this new language have?*

Recall that by making use of classical logic we learn that classical theories are generally described by *geometric notions*, while quantum theories are described by *algebraic and probabilistic tools*. Since the quantization process is some kind of process linking classical theories to quantum theories, it should therefore connect geometric areas to algebraic/probabilistic areas. So, the idea is to search for a language which formalizes the notion of "area of mathematics" and the notion of "map between two areas".

This language actually exists: it is *categorical language*. In categorical language, an area of mathematics is determined by specifying which are the objects of interest, which are the morphisms between these objects and which are the possible ways to compose two given morphisms. This data defines a *category*. The link between two areas of mathematics described by categories $\mathbf{C}$ and $\mathbf{D}$ is formalized by the notion of *functor*. This is given by a rule $F : \mathbf{C} \to \mathbf{D}$ assigning objects into objects and mappings into mappings in such a way that compositions are preserved. See [37, 168, 150] for classical books on category theory and [186, 142, 185, 155] for more recent, introductory and pedagogical expositions. See [139] for a very elementary text. See [50, 51] for introductions devoted to physicists.

Notice that we have a category **Set**, describing set theory, whose fundamental objects are sets, whose morphisms are just maps between sets and whose composition laws are the usual compositions between functions. That all classical areas of mathematics can be described by categorical language comes from the the fact that in each of them the fundamental objects are just sets endowed with some further structure, while the morphisms are precisely the maps between the underlying sets which preserve this additional structure. For instance, linear algebra is the area of mathematics which study vector spaces and linear maps. But a vector space is just a set endowed with a linear structure, while a linear map is just a map preserving the linear structure.

Therefore, each classical area of math defines a category $\mathbf{C}$ equipped with an inclusion functor $\imath : \mathbf{C} \hookrightarrow \mathbf{Set}$ which only forgets all additional structures (in the context of linear algebra, this functor forgets the linear structure). The categories which can be included into **Set** are called *concrete*. Thus, *in order to recover classical logic from categorical language it is enough to restrict the latter to the class of concrete categories.*

The fact that categorical language is really more abstract than classical logic comes from the existence of *non-concrete* (also called *abstract*) categories. There are many examples of them. For instance, given a natural number p, we can build a category $\mathbf{Cob}_{p+1}$ whose objects are p-dimensional smooth manifolds and whose mappings $\Sigma : M \to N$ are *cobordisms* between them, i.e, $(p+1)$-manifolds Σ such that $\partial\Sigma = M \sqcup N$. The abstractness of this category comes from the fact that the morphisms are not mappings satisfying some condition, but actually higher dimensional manifolds. For a formal definition of $\mathbf{Cob}_{p+1}$ with applications to homotopical aspects of cobordism theory, see [83]. See also [222, 188, 88]. We recall that the study of cobordisms dates from the seminal work of Thom [6, 228].

Observe that for $p = 0$ the objects of $\mathbf{Cob}_1$ are just 0-manifolds: finite collection of points. The cobordisms between them are 1-manifolds having these 0-manifolds as boundaries. In other words, the morphisms are just disjoint unions of intervals, while the composition between intervals $[t_0; t_1]$ and $[t_1; t_2]$ is the interval $[t_0; t_1]$.

With categorical language on hand, let us try to attack Hilbert's sixth problem. We start by recalling that the dynamics of a system in quantum mechanics is guided by the time evolution operators $U(t_1; t_0) = e^{\frac{i}{\hbar}(t_1 - t_0)\hat{H}}$. Notice that when varying t_0 and t_1, all corresponding operators can be regarded as a unique functor $U : \mathbf{Cob}_1 \to \mathbf{Vec}_{\mathbb{C}}$, where $\mathbf{Vec}_{\mathbb{C}}$ denotes the category delimiting complex linear algebra (i.e, it is the category of complex vector spaces and linear transformations). Such a functor assigns to any instant t_0 a complex vector space $U(t) = \mathcal{H}_t$ and to any interval $[t_0; t_1]$ an operator $U(t_1; t_0) : \mathcal{H}_0 \to \mathcal{H}_1$.

At this point, the careful reader could make some remarks:

1. As commented previously, a system in quantum mechanics is defined by a pair $(\mathcal{H}, \hat{H})$, where we have a single space $\mathcal{H}$ which does not depend on time, so that for any interval $[t_0; t_1]$ the time evolution operator $U(t_1; t_0)$ is defined in the same space. On the other hand, for a functor $U : \mathbf{Cob}_1 \to \mathbf{Vec}_{\mathbb{C}}$ we have a space $\mathcal{H}_t$ for each instant of time and, therefore, for any interval the corresponding operators are defined on different spaces.
2. In quantum mechanics, the evolution is guided not

by an arbitrary operator, but by a unitary operator. Furthermore, in Quantum Mechanics the space $\mathcal{H}$ for systems describing more than one (say k) particles decomposes as a tensor product $\mathcal{H}^1 \otimes ... \otimes \mathcal{H}^k$. However both conditions are not contained in the data defining a functor $U : \mathbf{Cob}_1 \to \mathbf{Vec}_{\mathbb{C}}$.

About the first remark, notice that if there is an interval $[t_0; t_1]$ connecting two different time instants t_0 and t_1, then there exists a inverse interval $[t_0; t_1]^{-1}$ such that, when composed (in $\mathbf{Cob}_1$) with the original interval, gives precisely the trivial interval. In fact, the inverse is obtained simply by flowing time in the inverse direction. In more concise terms, any morphism in the category $\mathbf{Cob}_1$ is indeed an isomorphism. But functors preserve isomorphisms, so that for any interval $[t_0; t_1]$ the corresponding spaces $\mathcal{H}_{t_0} = U(t_0)$ and $\mathcal{H}_{t_1} = U(t_1)$ are isomorphic. This means that the spaces $\mathcal{H}_t$ actually do not dependent on the time t: up to isomorphisms they are all the same.

Concerning the second remark, let us say that there are some reasons to believe that the unitarity of time evolution should not be *required* as a fundamental axiom of the true fundamental physics, but instead it should *emerge* as a consequence (or as an additional assumption) of the correct axioms. For instance, in quantum mechanics itself the unitarity of $U(t_0; t_1)$ can be viewed as a consequence of the Schrödinger equation, because its solutions involve the exponential of a Hermitian operator which by Stone's theorem is automatically unitary [227]. There are even more fundamental reasons involving the possibility of topology change in quantum gravity [9].

On the other hand, it is really true that an arbitrary functor $U : \mathbf{Cob}_1 \to \mathbf{Vec}_{\mathbb{C}}$ does not take into account the fact that in a system with more than one particle the total space of states decomposes as a tensor product of the state of spaces of each particle. In order to incorporate this condition, notice that when we say that we have a system of two fully independent particles, we are saying that time intervals corresponding to their time evolution are *disjoint*. Thus, we can interpret the time evolution of a system with many particles as a disconnected one dimension manifold, i.e, as a disconnected morphism of $\mathbf{Cob}_1$. Consequently, the required condition on the space of states can be obtained by imposing the properties

$$U(t \sqcup t') \simeq U(t) \otimes U(t') \quad \text{and} \quad U(\varnothing) \simeq \mathbb{C},$$

where the second condition only means that a system with zero particles must have a trivial space of states.

Now, notice that both categories $\mathbf{Cob}_1$ and $\mathbf{Vec}_{\mathbb{C}}$ are equipped with binary operations (respectively given by $\sqcup$ and $\otimes$), which are associative and commutative up do isomorphisms, together with distinguished objects (given by $\varnothing$ and $\mathbb{C}$), which behave as neutral elements for these operations. A category with this kind of structure is called a *symmetric monoidal category*. A functor between monoidal categories which preserves the operations and the distinguished object is called a *monoidal functor* (see [5, 150, 73] for a formal definition of monoidal categories and monoidal functors, and [209] for an interesting survey on the different flavors of monoidal structures). Therefore, these observations lead us to the following conclusion:

Conclusion. *A system in quantum mechanics is a special flavor of monoidal functor from the category* $\mathbf{Cob}_1$ *of 1-dimensional cobordisms to the category* $\mathbf{Vec}_{\mathbb{C}}$ *of complex vector spaces.*

Consequently, eyeing Hilbert's sixth problem we can use the above characterization in order to *axiomatize* a quantum theory of *particles* as being an arbitrary monoidal functor

$$U : (\mathbf{Cob}_{p+1}, \sqcup, \varnothing) \to (\mathbf{C}, \otimes, 1) \tag{4}$$

for $p = 0$ and taking values in some symmetric monoidal category. This is precisely the 1-dimensional version of the functorial topological quantum field theories (TQFT) of Atiyah-Witten [7, 238], inspired in the functorial formulation of Conformal Field Theories by Segal [207, 206, 36]. See also [9] for a comprehensive introduction. For the classification of the functors (4) in the case $p = 1$, see [2, 126, 132] and compare with the 2-dimensional conformal case [206]. For discussions on the beautiful math arising in the $p = 2$ case, see [19, 41, 23, 125, 20].

A natural question is on the viability of using the same kind of argument (and therefore the same background language) in order to get an axiomatization of the classical theories of *particles*. This really can be done, as we will outline following ideas of Chapter 1 of [201]. We start by recalling that a classical theory of particles is given by an action functional $S : \mathrm{Fields}(M) \to \mathbb{R}$, defined in some space of fields. Therefore, the first step is to axiomatize the notion of *space of fields*. In order to do this, recall that it is generally composed by smooth paths $\gamma : I \to M$, representing the trajectories of the particle into some spacetime M, and by the interacting fields, corresponding to configurations of some kind of geometric structure put in M. The canonical examples of interacting fields are metrics (describing gravitational interaction) and connections over bundles (describing gauge interactions), as discussed in Section 3.

Now we ask: *which property smooth functions, metrics and connections have in common?* The answer is: *locality*. In fact, in order to conclude that a map $\gamma : I \to M$ is smooth, it is enough to analyze it relative to an open cover $U_i \hookrightarrow M$ given by coordinate charts $\varphi_i : U_i \to \mathbb{R}^n$. Similarly, a metric g on M is totally determined by its local components g_{ij}. Finally, it can also be shown that to give a connection $A : TP \to \mathfrak{g}$ is the same as giving a family of 1-forms $A_i : U_i \to \mathfrak{g}$ fulfilling compatibility conditions at the intersections $U_i \cap U_j$. Therefore, one can say that a space of fields over a fixed spacetime M is some kind of set $\mathrm{Fields}(M)$ of structures which are local in the sense that, for any cover $U_i \hookrightarrow M$ by coordinate systems, we can reconstruct the total space $\mathrm{Fields}(M)$ from the subset of all $s_i \in \mathrm{Fields}(U_i)$ that are compatible in the intersections $U_i \cap U_j$.

Notice that we are searching for a notion of *space of fields* (without mention of any spacetime), but up to this point we got a notion of space of fields over a *fixed* spacetime. The main idea is to consider the rule $M \mapsto \mathrm{Fields}(M)$ from which follows the immediate hypothesis that it is functorial. Therefore, we could axiomatize a space of fields as a functor

$$\text{Fields} : \mathbf{Diff}^{op} \to \mathbf{Set}$$

assigning to any manifold a set of local structures. This kind of functor is called a *sheaf on the site of manifolds and coordinate coverings* (or a *smooth set*, a *smooth sheaf* or even a *generalized smooth space*). See [152, 117, 123] for general discussions on sheaves on sites, and [11, 131, 201] for smooth sets and generalized smooth spaces. See also [160].

This is still not the correct notion of space of fields, however. Indeed, recall that in typical situations the set Fields(M) contains geometric structures. But when doing geometry we only consider the entities up to their natural equivalences (their "congruences"). This means that for a fixed space M we should take the quotient space

$$\text{Fields}(M)/\text{congruences}. \tag{5}$$

In order to do calculations with a quotient space we have to select an element within each equivalence class (a representative) and then prove that the results of the computations do not depend on this choice. The problem is that when we select an element we are automatically privileging it, but *there is no physically privileged element*. So, in order to be physically correct we have to work with all elements of the equivalence class *simultaneously*. This can be done by replacing the set (5) with the category whose objects are elements of Fields(M) and such that there is a mapping between s, s' iff they are equivalent. In this category, all mappings are obviously isomorphisms, so that it is indeed a *groupoid*. The sheaves on the site of manifolds which take values in **Gpd** are called *smooth stacks* or *smooth groupoids*, and they finally give the correct way of thinking about the space of fields for the case of *particles* [165]. See Chapter 1 of [201] for a very nice discussion and [160] for a pedagogical approach.

Therefore, in order to finish the axiomatization of the "classical theory" notion for particles we need to define what is the action functional. Given a spacetime M this should be some kind of map between the space of fields Fields(M) to $\mathbb{R}$. Notice that $\mathbb{R}$ is a *set*, but we promoted the space of fields to a *groupoid*. So, in order to define a map between these two entities we also need to promote $\mathbb{R}$ to a groupoid. This is done defining a groupoid whose objects are real numbers and having only trivial morphisms.

We can then finally axiomatize a classical theory of *particles* as being given by a smooth stack Fields of fields and a rule S assigning to every manifold M the action functional

$$S_M : \text{Fields}(M) \to \mathbb{R}.$$

On the other hand, from the higher Yoneda embedding, every smooth groupoid X can be regarded as a smooth ∞-stack $y(X)$ [147]. Thus, $\mathbb{R}$ itself can be regarded as smooth ∞-stack $y(\mathbb{R})$ and the action functional is just a morphism $S : \text{Fields} \Rightarrow y(\mathbb{R})$ between smooth ∞-stacks [201].

5. The Need of Highering

Up to this point we have seen that starting with categorical logic as the background language we can effectively axiomatize the notions of classical and quantum theories for *particles*. Indeed, a quantum theory is a symmetric monoidal functor $U : \mathbf{Cob}_1 \to \mathbf{Vec}_{\mathbb{C}}$, while a classical theory is given by a smooth stack Fields and by an action functional

$$S : \text{Fields}(-) \to \mathbb{R}.$$

On the other hand, the nonexistence of *clues* that strings and higher dimensional objects are the correct building blocks of nature cannot be used to completely exclude them. Thus, any approach to Hilbert's sixth problems which intends to be stable under new physics should be based on a background language which allows the axiomatization of classical and quantum theories not only for *particles*, but for objects of arbitrary dimension. In this regard, categorical logic fails as background language. Indeed, we have at least the following problems (see also [201, 191, 15, 219, 160]):

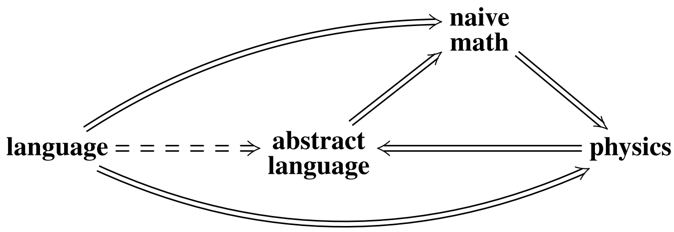
- *in quantum theories*. Recall that for any p we have the category $\mathbf{Cob}_{p+1}$, so that we could immediately extend the notion of quantum theory for particles by defining a quantum theory for *p-branes* as a symmetric monoidal functor $U : \mathbf{Cob}_{p+1} \to \mathbf{Vec}_{\mathbb{C}}$, which coincides with the Atiyah-Witten p-dimensional TQFT [7, 238]. The problem is that in this case we are only replacing the assumption that particles are the correct building blocks of nature with the assumption that p-branes are the correct building blocks of nature. Indeed, in both cases we can only talk about quantum theories for a specific kind of object, while Hilbert's sixth problem requires an *absolute notion of quantum theory* (this is related with the locality principle and with the principle of general covariance [9, 200, 197, 203]);

- *in classical theories*. The trajectories of particles (which are 0-dimensional objects) on a spacetime M is described by smooth paths $\varphi : I \to M$, which are smooth maps defined on a 1-dimensional manifold I. Therefore we can easily define the trajectory of p-branes (which are p-dimensional objects) on M as smooth maps $\varphi : \Sigma \to M$ defined on a $(p+1)$-dimensional manifold Σ, leading us to sigma-model theories [58]. The problem is that in the space of fields of a classical theory we have to consider not only the trajectories but also the interactions which act on the object. For particles we could consider these interactions as modeled by connections because the *notion of connection is equivalent to the notion of parallel transport along paths*. But, to the best of our knowledge, there is no obvious global notion of parallel transport along arbitrary higher dimensional manifolds[5].

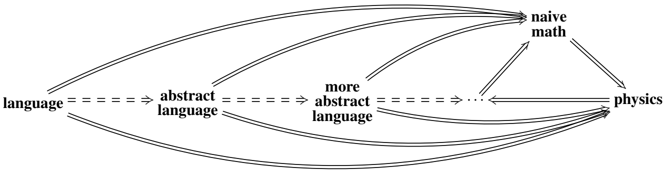
[5] Actually, all attempts to higher dimensional parallel transport that we know make use, at least implicitly, of higher categorical structures [151, 205, 204, 12, 92, 217, 124].

Conclusion. *In order to axiomatize the notions of classical and quantum theories for higher dimensional objects we need to start with a language which is more abstract than categorical language.*

The immediate idea that comes in mind is to try building some kind of process which takes a language and returns a more abstract language in such a way that set-theoretical things are replaced by their categorical analogue. Before discussing how this can be done, recall that for a selected background language, the first step in solving Hilbert's sixth problem is to consider the loop (3) involving naive math, the language itself and physics. So, having constructed a more abstract language from a given one, we would like to consider loops for the new language as arising by extensions of the loop for the starting language, as in the diagram below.



This condition is very important, because it ensures that the physics axiomatized by the initial language is contained in the physics axiomatized by the new language. So, this means that when applying the process of "abstractification" we are getting languages that axiomatize more and more physics. Consequently, by iterating the process and taking the limit one hopes to get a background language which is abstract enough to axiomatize the whole physics. See the discussion at the introduction and section 2.3 of [160].



6. Highering

In the last sections we saw that category theory is useful to axiomatize particle physics, but not string physics and physics of higher-dimensional objects, so that we need to build some "abstractification process" which will be used to replace category theory by other more abstract theory. Notice that categorical language is more abstract than classical language, so that learning how to characterize the passage from classical logic to categorical logic should help knowing how to iterate the construction, getting languages more abstract than categorical language. In other words, the main approach to the "abstractification process" should be some kind of *categorification process*.

In order to get some feeling for this categorification process, notice that a set contains less information than a category. Indeed, sets consist of only one type of information: their elements. On the other hand, categories have three layers of information: objects, morphisms and compositions. Thus, we can understood the passage from set theory to category theory (and, therefore, from classical logic to categorical logic) as the *addition of information layers* (see [14] for an interesting discussion; see also Part 2 of [160]).

When iterating this process we expect to get a language describing entities containing more information than usual categories. Indeed, we expect to have not only objects, morphisms between objects and compositions between morphisms, but also morphisms between morphisms (called *2-morphisms*) and compositions of 2-morphisms. Thus, if we call such entities 2-*categories*, adding another information layer we get 3-*categories*, and so on. Taking the limit we then get ∞-*category*. The basic example, which reveals the connection between higher category theory and homotopy theory, it the ∞-categories of topological spaces, continuous maps, homotopies, homotopies between homotopies, and so on.

Similarly, if a functor between categories maps objects into objects and morphisms into morphisms, then a k-*functor* between k-categories maps objects into objects and i-morphisms into i-morphisms for every $1 \leq i \leq k$. Taking the limit $k \to \infty$, we can define ∞-*functors*. Since the categorical language is about categories and functors (and natural transformations), then we obtain a ∞-*categorical language*, consisting of ∞-categories and ∞-functors (and higher natural transformations), which by constructions is arbitrarily abstract. See Part 3 of [160] for a more detailed intuitive discussion on this new language. We then have the following conclusion:

Conclusion. *A natural candidate for a background language sufficient to solve Hilbert's sixth problem is ∞-categorical language.*

With this conclusion in mind we need to build a math-language-physics loop (3) for the ∞-categorical language by means of extending the loop obtained using categorical language. This can be done as follows:

- *in quantum theories*: Recall that the problem with the definition of quantum theories as symmetric monoidal functors $U : \mathbf{Cob}_{p+1} \to \mathbf{Vec}_{\mathbb{C}}$ involves the fact that such functors take into account only p-branes for a *fixed* p, implying that we need to know previously what are the correct building blocks of nature. This can be avoided in the ∞-categorical context. Indeed, we can define a ∞-category $\mathbf{Cob}(\infty)$ having 0-manifolds as objects, 1-cobordisms (i.e., cobordisms between 0-manifolds) as morphisms, 2-cobordisms (i.e., cobordisms between 1-cobordisms) as 2-morphisms and so on. Notice that differently from $\mathbf{Cob}_{p+1}$ (which contains only p-manifolds and cobordisms between them), the defined ∞-category $\mathbf{Cob}(\infty)$ contains cobordisms of all orders and, therefore, describe p-branes for every p simultaneously. Thus, one can define an *absolute (or extended) quantum theory* (as required by Hilbert's sixth problem) as some kind of ∞-

functor $U : \mathbf{Cob}(\infty) \to \infty\mathbf{Vec}_{\mathbb{C}}$, where $\infty\mathbf{Vec}_{\mathbb{C}}$ is some ∞-categorical version of $\mathbf{Vec}_{\mathbb{C}}$ (i.e, is some ∞-category of ∞-vector spaces). See [81, 13, 149].

- *in classical theories*: The problem with the axiomatization of classical theories via categorical language was that in the space of fields we have to consider interacting fields. For particles, these fields are modeled by connections on bundles, which are equivalent to parallel transport along paths. But, as commented, there is no canonical notion of transportation along arbitrary higher dimensional manifolds [151, 205, 204, 12, 92, 217, 124]. Another way to motivate the problem is the following: in order to define a connection A locally (i.e, in terms of data over an open covering $U_i \hookrightarrow M$) we need a family of 1-forms A_i on U_i subject to compatibility conditions on $U_i \cap U_j$. The information U_i and $U_i \cap U_j$ belong to an usual category $\check{C}(U_i)$, whose objects are elements $x_i \in U_i$ and there is a morphism $x_i \to x_j$ iff $x_i, x_j \in U_i \cap U_j$. On the other hand, if we try to define higher transportation locally we need to take into account data on U_i which is compatible not only on $U_i \cap U_j$, but also on $U_i \cap U_j \cap U_k$, on $U_i \cap U_j \cap U_k \cap U_l$, and so on. Thus, we need to consider much more information layers than can be put inside an usual category, justifying the nonexistence of connections along arbitrary higher dimensional manifolds. But they can be put inside a ∞-category $\check{C}_\infty(U_i)$, meaning that we actually have a notion of ∞-connection when we consider ∞-categorical language (see [12, 204, 195, 205]). More precisely, the initial problem is avoided if we define the space of fields not as a smooth stack, but as a *smooth ∞-stack* (or smooth *∞-groupoid*: a ∞-functor $\mathrm{Fields} : \mathbf{Diff}^{op} \to \infty\mathbf{Gpd}$ such that for any M the quantity $\mathrm{Fields}(M)$ is determined not only by $\mathrm{Fields}(U_i)$ and $\mathrm{Fields}(U_i \cap U_j)$, but also by (see [197, 203, 201, 202, 160]):

$$\mathrm{Fields}(U_i \cap U_j \cap U_k) \tag{6}$$

$$\mathrm{Fields}(U_i \cap U_j \cap U_k \cap U_l) \tag{7}$$

$$\text{and so on.} \tag{8}$$

Remark. In parts, the difficulties with this approach to Hilbert's sixth problem arise from the fact that the ∞-categorical language is still in development. Recall that a groupoid is a category such that all morphisms are invertible. When we are in a higher category we have higher morphisms, so that given a k-morphism, it can be invertible (in the classical sense) or invertible up to $k+1$-morphisms. A ∞-groupoid is a ∞-category such that each k-morphism is invertible up to $k+1$-morphism, for every $k > 0$. The theory for this kind of ∞-category is well-studied and in some cases it is equivalent to the homotopy theory of topological CW-complexes, which since [10] is known as the *Homotopy Hypothesis* [182, 101] (see Section 8.4 of [160] for a discussion on its role in physics). More generally, we can define a (∞, n)-category as a ∞-category such that each k-morphism is invertible up to $k+1$-morphisms, for every $k > n$. Thus, ∞-groupoids are the same thing as $(\infty, 0)$-categories. The theory of $(\infty, 1)$-categories is also well-studied [118, 120, 119, 27, 140, 48, 147, 148]. But, in order to describe extended TQFT and higher prequantization we need to work on (∞, n)-categories for $n > 1$ [149, 202, 197, 203, 201], whose theory is not as well developed as the theory for $n = 0, 1$ [24, 25, 26].

7. The Cohomological Description

In theoretical physics, the concept of "cohomology" is more well known as the cohomology groups $H^n(X^\bullet; Q)$ induced by a nilpotent operator $Q : X^\bullet \to X^\bullet$ (i.e., such that $Q^2 = 0$), defined in a graded vector space $X^\bullet = \oplus X^n$ (or in a graded algebra) which is typically related to the conservation of some gauge charge [218, 219, 21]. In mathematics, on the other hand, it has not only this fully algebraic meaning (which arises from homological algebra [237, 90]), but also a more topological one: the *generalized cohomology theories*. These consist of sequences of functors H^n, assigning abelian groups to topological spaces, subjected to the so-called *Eilenberg-Steenrod axioms*. Examples include the ordinary cohomology theory (which in real coefficients is equivalent to de Rham cohomology), K-theory, cobordism theory and topological modular forms [143, 4, 72]. The facts that particles are charged in ordinary cohomology and that D-brane charges can be described by K-theory show that generalized cohomology theoy is also important in physics. Thus, in view of Hilbert's sixth problem, it is natural to search for an axiomatization to cohomology which contains both flavors of cohomology as particular examples.

By Brown's representability theorem, generalized cohomologies are fully determined by sequences $\mathbb{E} = (E_n)_n$ of topological spaces known as *spectra* [143, 4, 72]. More precisely, for every $n \geq 0$ there is a topological space E_n such that for every topological space X we have $H^n(X) \simeq [X; E_n]$, where the right-hand side denotes the *set* of homotopy classes. Via Dold-Kan correspondence N, the algebraic cohomology groups $H^n(X^\bullet; Q)$ are also determined by a sequence of entities E_n such that $H^n(X^\bullet; Q) \simeq [N(X_\bullet); E_n]$, where $X_\bullet$ is the dual chain sequence of $X^\bullet$ (see Chapter 1 of [201] and Chapter 9 of [160]). The only difference is that now the representing objects are higher-categorical entities and therefore belong to a $(\infty, 1)$-category instead of to a classical category, but recall that topological spaces, continuous maps, homotopies and higher homotopies also define a $(\infty, 1)$-category, so that in the first case we also are in the context of ∞-categories.

Another similarity between both flavors of cohomology is the following. We say that a topological space Y is *deloopable* if there is another topological space BY whose loop space is (weak) homotopic to Y, i.e., such that $\Omega BY \simeq Y$. In a generalized cohomology theory each E_n is deloopable, with $BE_n = E_{n+1}$. In particular, $E_n = B^n E_0 = B(B...(BE_0)...)$ [162, 3]. We note that in every suitable[6] $(\infty, 1)$-category we can also consider loop spaces $\mathbf{\Omega} Y$, so that we can also

[6] With $(\infty, 1)$-pullbacks or homotopy pullbacks.

talk about deloopable objects [93, 147, 201]. But via Dold-Kan correspondence the spaces E_n characterizing algebraic cohomology are also deloopable, with $\mathbf{B}E_n \simeq E_{n+1}$, so that again we have $E_n = \mathbf{B}^n E_0$. This motivates us to axiomatize the notion of cohomology in any $(\infty, 1)$-category as follows [40, 68, 147, 201, 160]. Given an object E_0 in an $(\infty, 1)$-category, for every other object X, the *0th cohomology of X with coefficients in E_0* is given by $H^0(X; E_0) := [X; E_0]$. Analogously, if E_0 has deloopings $B^n E_0$ we define the nth cohomology as $H^n(X; E_0) := [X; E_n]$.

The amazing fact of this abstract axiomatization is that it does not only unify different flavors of cohomology, but also sets classical theories in a new powerful language [201]. More precisely, recall that, as discussed in Section 6 the space of fields is supposed to be a smooth ∞-stack $\mathrm{Fields}(-) : \mathbf{Diff}^{op} \to \infty\mathbf{Grpd}$. Via higher Yoneda embedding [147], each smooth manifold M can itself be regarded as a smooth ∞-stack $y(M)$ and $\mathrm{Fields}(M) \simeq [y(M); \mathrm{Fields}]$, so that the space of fields are actually the coefficients of a cohomology in the $(\infty, 1)$-category of all smooth ∞-stacks [201]. On the other hand, $y(\mathbb{R})(M) = [y(M); y(\mathbb{R})] = H^0(M; \mathbb{R})$. Therefore, the action functional $S : \mathrm{Fields} \Rightarrow y(\mathbb{R})$ is secretly a morphism between cohomology theories. In homotopy theory, this is known as a *cohomology operation*.

Let us look at a particular example. Every Lie group G is deloopable and by the classification theorem of principal bundles, the cohomology $H^1(M; G) := [M; BG]$ classifies precisely G-bundles over M [161, 56]. The interacting fields acting on particles are not G-bundles, but G-bundles with connections, so that it is natural to ask if this classification theorem cannot be lifted to the context of bundles with connections. This is really the case if we look at the language of smooth ∞-stacks. Indeed, by the higher Yoneda embedding, we have $H^1(M; G) \simeq [y(M); \mathbf{B}G]$. On the other hand, there is another smooth ∞-stack $\mathbf{B}G_{\mathrm{diff}}$ such that fixing a connection A in the G-bundle classified by a map $f : y(M) \to \mathbf{B}(G)$ is equivalent to giving a lifing $\hat{f} : y(M) \to \mathbf{B}\mathrm{diff}(G)$ of f. Consequently, the abstract cohomology $H^1_{\mathrm{diff}}(M; G) := H^0(M; \mathbf{B}_{\mathrm{diff}}(G) = [y(M); \mathbf{B}_{\mathrm{diff}}(G)]$, called *nonabelian differential G-cohomology* describes the space of all gauge fields [199, 42, 211, 109]. Thus, gauge theories in the classical sense are the same thing as cohomology operations on nonabelian differential cohomology.

If G is only a group, then the smooth ∞-stack $\mathbf{B}(G)$ needs not be deloopable. On the other hand, if G is a more general higher group, then $\mathbf{B}^2(G)$ exists and, in similar way, there also exists the differential refinement $\mathbf{B}^2_{\mathrm{diff}}(G)$, leading us to define the 2th differential cohomology $H^2(M; G) := [y(M); \mathbf{B}^2_{\mathrm{diff}}(G)]$ [199, 201]. Similarly, if G admits higher deloopings we can define higher-degree differential cohomology. Notice that if cohomology operations in 1th differential cohomology describes gauge theoris, cohomology operations in higher differential cohomology should describe higher gauge theory, i.e, classical theories whose interacting fields are not vectorial, but tensorial, as typically string theory (with the B-field) and supergravity/M-theory (with C-field). This is really the case [82, 199, 201, 225, 169, 17].

Finally, recall that cohomology theories are the natural environments in which obstruction theory can be developed [105, 161, 28]. On the other hand, there are typical anomaly cancellation conditions in (higher) gauge theories, which typically can be interpreted as obstructions to the preservation of some symmetry at the quantum level or as a charge quantization condition. Examples such as the quantization of D-brane charges in (twisted) K-theories [239, 39, 167, 84, 98, 99], the Freed-Witten anomaly as pull-push operations in (twisted) K-theory [86, 121, 45, 176], the fermionic anomaly as the trivialization of a line bundle [78, 80], the Green-Schwarz mechanism and its description in terms of liftings to twisted differential cohomology [82, 53, 196] and, more recently, the Hypothesis H, about the quantization of the C-field on (twisted) cohomotoy cohomology theory and its many consequences [76, 75, 193, 43, 192, 194, 77] seem to reveal that this cohomological approach to physics, where the meaning of obstruction is very clear, is not only formal, but very fruitful.

8. Quantization

In the previous sections we discussed that ∞-categorical language is abstract enough to provides suitable axiomatization of classical and quantum physics, making it a natural candidate to attack Hilbert's sixth problem. Thus, using this class of languages, we need to build a quantization process linking *any* classical theory to a corresponding quantum theory. Besides the problematics involving the current development of (∞, n)-categorical languages for higher n, building this full quantization process remains an open problem. Discussing complete details is beyond the scope of this paper. See [202] for a concise discussion, [160] for a guide and [201] for a complete reference. Even so, let us say a few words.

With the axiomatization of classical and quantum theories on hand we can a priori build ∞-categories Class and Quant, so that it is expected that a quantization process should be some kind of ∞-functor

$$Q : \mathrm{Class} \to \mathrm{Quant}$$

such that it assigns to each system of Classical Mechanics a corresponding system of quantum mechanics. Since as discussed in Section 3 these disciplines can be axiomatized using 1-categorical language, one should expect that, when restricted to them, the quantization ∞-functor Q becomes a functor. But, by counterexamples of van Hove and others [234, 100, 95, 94, 96], this quantization 1-functor does not exist, forcing us to take into account higher layers even in the simple case.

On the other hand, there is a canonical way to map classical mechanics into quantum mechanics given by the so-called *geometric quantization* [240, 216], where the main idea was to incorporate the higher layer on it. The first step was to observe that part of the data needed to apply that quantization corresponds to an orientation in complex K-theory [166, 129]. The second one was the extension of geometric quantization from symplectic manifolds to symplectic groupoids (and therefore from set-theoretic objects to categorical-theoretic entities) [38, 106, 187]. The *higher geometric quantization* is then a program aiming to join both observations and build a quantization for higher groupoids by means of cohomological methods, which is a natural strategy in virtue of the discussion in Section 7. A very promising approach which has been more extensively developed in the last decade is the so-called *cohomological quantization*, also known as *pull-push quantization* or *motivic quantization*. See [197, 203, 177, 187, 202, 201] and the references therein.

9. Conclusion

In this paper we described and surveyed in a pedagogical way a formal approach to Hilbert's sixth problem based on works of Urs Schreiber, John Baez, Daniel Freed, Jacob Lurie and many other researchers. We discussed that the problem requires languages which are arbitrarily abstract and synthetic, so that higher categorical languages were presented as natural candidates. We saw that these higher languages seem to be abstract enough to axiomatize classical and quantum physics, leaving open the problem of building a global quantization ∞-functor. Some comments on recent cohomological quantization were made.

Acknowledgements

Both authors would like to thank Fernando Pereira Paulucio Reis and Maico Felipe Silva Ribeiro for reading a version of the text. This text was written based on many invited minicourses and talks given by Y. X. Martins at UFLA and UFMG. He would like to thank the corresponding organizers for the oportunity. He would also like Maurício Corrêa Barros Júnior for fruitful discussions. Y. X. Martins was supported by CAPES (grant number 88887.187703/2018-00).

References

[1] Algebra-Geometry duality in physics. `https://ncatlab.org/nlab/show/Isbell+duality+-+table`. Accessed: 2020-07-21.

[2] L. Abrams. Two-dimensional topological quantum field theories and frobenius algebras. *Journal of Knot theory and its ramifications*, 5(05):569–587, 1996.

[3] J. F. Adams. *Infinite loop spaces*. Number 90. Princeton University Press, 1978.

[4] J. F. Adams and J. F. Adams. *Stable homotopy and generalised homology*. University of Chicago press, 1974.

[5] M. Aguiar and S. A. Mahajan. *Monoidal functors, species and Hopf algebras*, volume 29. American Mathematical Society Providence, RI, 2010.

[6] M. Atiyah. The impact of thomâs cobordism theory. *Bulletin of the American Mathematical Society*, 41(3):337–340, 2004.

[7] M. F. Atiyah. Topological quantum field theory. *Publications Mathématiques de l'IHÉS*, 68:175–186, 1988.

[8] C. Bachas, L. Baulieu, M. Douglas, E. Kiritsis, E. Rabinovici, P. Vanhove, P. Windey, and L. Cugliandolo. *String Theory and the Real World: From particle physics to astrophysics: Lecture Notes of the Les Houches Summer School 2007*. Elsevier, 2008.

[9] J. Baez. Quantum quandaries: a category-theoretic perspective. *The structural foundations of quantum gravity*, pages 240–265, 2006.

[10] J. Baez. The homotopy hypothesis. *Fields Institute. Available online at http://math. ecr. edu/home/baez/homotopy*, 2007.

[11] J. Baez and A. Hoffnung. Convenient categories of smooth spaces. *Transactions of the American Mathematical Society*, 363(11):5789–5825, 2011.

[12] J. Baez and U. Schreiber. Higher gauge theory: 2-connections on 2-bundles. *arXiv preprint hep-th/0412325*, 2004.

[13] J. C. Baez and J. Dolan. Higher-dimensional algebra and topological quantum field theory. *Journal of Mathematical Physics*, 36(11):6073–6105, 1995.

[14] J. C. Baez and J. Dolan. Categorification. *arXiv preprint math/9802029*, 1998.

[15] J. C. Baez and A. Lauda. A prehistory of n-categorical physics. *Deep Beauty: Understanding the Quantum World Through Mathematical Innovation*, pages 13–128, 2011.

[16] J. C. Baez and J. P. Muniain. *Gauge fields, knots and gravity*, volume 4. World Scientific Publishing Company, 1994.

[17] J. C. Baez and U. Schreiber. Higher gauge theory. *arXiv preprint math/0511710*, 2005.

[18] J. C. Baez, I. E. Segal, and Z. Zhou. *Introduction to algebraic and constructive quantum field theory*, volume 47. Princeton University Press, 2014.

[19] B. Bakalov and A. A. Kirillov. *Lectures on tensor categories and modular functors*, volume 21. American Mathematical Soc., 2001.

[20] B. Balsam. Turaev-viro invariants as an extended tqft iii. *arXiv preprint arXiv:1012.0560*, 2010.

[21] G. Barnich, F. Brandt, and M. Henneaux. Local brst cohomology in gauge theories. *Physics Reports*, 338(5):439–569, 2000.

[22] M. Barr, J. F. Kennison, and R. Raphael. Isbell duality. *Theory Appl. Categories*, 20:504–542, 2008.

[23] J. W. Barrett and B. W. Westbury. Spherical categories. *arXiv preprint hep-th/9310164*, 1993.

[24] C. Barwick. (∞, n) -cat as a closed model category. 2005.

[25] C. Barwick and D. M. Kan. Relative categories: another model for the homotopy theory of homotopy theories. *Indagationes Mathematicae*, 23(1-2):42–68, 2012.

[26] C. Barwick and C. Schommer-Pries. On the unicity of the homotopy theory of higher categories. *arXiv preprint arXiv:1112.0040*, 2011.

[27] M. A. Batanin. Monoidal globular categories as a natural environment for the theory of weak n-categories. *Advances in Mathematics*, 136(1):39–103, 1998.

[28] H. J. Baues. *Obstruction theory: On homotopy classification of maps*, volume 628. Springer, 2006.

[29] J. K. Beem. *Global lorentzian geometry*. Routledge, 2017.

[30] M. Benini, C. Dappiaggi, and A. Schenkel. Quantized abelian principal connections on lorentzian manifolds. *Communications in Mathematical Physics*, 330(1):123–152, 2014.

[31] M. Benini, M. Perin, A. Schenkel, and L. Woike. Categorification of algebraic quantum field theories. *arXiv preprint arXiv:2003.13713*, 2020.

[32] M. Benini and A. Schenkel. Quantum field theories on categories fibered in groupoids. *Communications in Mathematical Physics*, 356(1):19–64, 2017.

[33] M. Benini and A. Schenkel. Higher structures in algebraic quantum field theory: Lms/epsrc durham symposium on higher structures in m-theory. *Fortschritte der Physik*, 67(8-9):1910015, 2019.

[34] M. Benini, A. Schenkel, and R. J. Szabo. Homotopy colimits and global observables in abelian gauge theory. *Letters in Mathematical Physics*, 105(9):1193–1222, 2015.

[35] M. Benini, A. Schenkel, and L. Woike. Homotopy theory of algebraic quantum field theories. *Letters in Mathematical Physics*, 109(7):1487–1532, 2019.

[36] R. Blute, P. Panangaden, and D. Pronk. Conformal field theory as a nuclear functor. *Electronic Notes in Theoretical Computer Science*, 172:101–132, 2007.

[37] F. Borceux. Handbook of categorical algebra, volumes 1, 2, 3, 1994.

[38] R. D. Bos. *Groupoids in geometric quantization*. PhD thesis, Nijmegen:[Sn], 2007.

[39] P. Bouwknegt and V. Mathai. D-branes, b-fields and twisted k-theory. *Journal of High Energy Physics*, 2000(03):007, 2000.

[40] K. S. Brown. Abstract homotopy theory and generalized sheaf

cohomology. *Transactions of the American Mathematical Society*, 186:419–458, 1973.

[41] A. Bruguières and A. Virelizier. Categorical centers and reshetikhin-turaev invariants. *arXiv preprint arXiv:0812.2426*, 2008.

[42] U. Bunke, T. Nikolaus, and M. Völkl. Differential cohomology theories as sheaves of spectra. *Journal of Homotopy and Related Structures*, 11(1):1–66, 2016.

[43] S. Burton, H. Sati, and U. Schreiber. Lift of fractional d-brane charge to equivariant cohomotopy theory. *arXiv preprint arXiv:1812.09679*, 2018.

[44] D. Calaque and T. Strobl. *Mathematical Aspects of Quantum Field Theories*. Springer, 2015.

[45] A. L. Carey and B.-L. Wang. Thom isomorphism and push-forward map in twisted k-theory. *arXiv preprint math/0507414*, 2005.

[46] R. Catenacci, M. Cornalba, M. Martellini, and C. Reina. Algebraic geometry and path integrals for closed strings. *Physics Letters B*, 172(3-4):328–332, 1986.

[47] E. Cheng and A. Lauda. Higher-dimensional categories: an illustrated guide book. *Preprint*, 2004.

[48] E. Cheng and T. Leinster. Weak ω-categories via terminal coalgebras. *ArXiv e-prints*, 2012.

[49] D.-C. Cisinski. Higher category theory and homotopical algebra. *Camb. Stud. Adv. Math*, 180, 2019.

[50] B. Coecke. Introducing categories to the practicing physicist. In *What is category theory*, pages 45–74. Polimetrica Publishing Milan, Italy, 2006.

[51] B. Coecke and E. O. Paquette. Categories for the practising physicist. In *New structures for physics*, pages 173–286. Springer, 2010.

[52] A. Connes. Noncommutative geometry. *San Diego*, 1994.

[53] C. Córdova, T. T. Dumitrescu, and K. Intriligator. Exploring 2-group global symmetries. *Journal of High Energy Physics*, 2019(2):184, 2019.

[54] K. Costello and O. Gwilliam. *Factorization algebras in quantum field theory*, volume 1. Cambridge University Press, 2016.

[55] D. A. Cox and S. Katz. *Mirror symmetry and algebraic geometry*. Number 68. American Mathematical Soc., 1999.

[56] F. Dadam and Y. X. Martins. *Topologia, Geometria e Buracos Negros*. NEA, 2016.

[57] C. R. De Oliveira. *Intermediate spectral theory and quantum dynamics*, volume 54. Springer Science & Business Media, 2008.

[58] E. P. Deligne, P. Etingof, D. S. Freed, L. C. Jeffrey, D. Kazhdan, J. W. Morgan, D. R. Morrison, and E. Witten. Quantum fields and strings. a course for mathematicians. In *Material from the Special Year on Quantum Field Theory held at the Institute for Advanced Study*. Citeseer, 1999.

[59] P. Deligne. Catégories tannakiennes. In *The Grothendieck Festschrift*, pages 111–195. Springer, 2007.

[60] P. Deligne and D. Mumford. The irreducibility of the space of curves of given genus. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, 36(1):75–109, 1969.

[61] E. D'Hoker and D. Phong. The geometry of string perturbation theory. *Reviews of Modern Physics*, 60(4):917, 1988.

[62] R. Dijkgraaf. The mathematics of string theory. *Gazette des Mathématiciens*, 106(supplement):45–64, 2005.

[63] S. Donaldson. The seiberg-witten equations and 4-manifold topology. *Bulletin of the American Mathematical Society*, 33(1):45–70, 1996.

[64] S. K. Donaldson, S. K. Donaldson, and P. Kronheimer. *The geometry of four-manifolds*. Oxford University Press, 1990.

[65] S. K. Donaldson et al. An application of gauge theory to four-dimensional topology. *Journal of Differential Geometry*, 18(2):279–315, 1983.

[66] C. Doran, A. Lasenby, and S. Gull. Gravity as a gauge theory in the spacetime algebra. In *Clifford algebras and their applications in mathematical physics*, pages 375–385. Springer, 1993.

[67] M. Douglas et al. Strings and geometry. *American Mathematical Society*, 2002.

[68] D. Dugger. Sheaves and homotopy theory. *preprint*, 1999.

[69] T. Eguchi, K. Hori, and C.-S. Xiong. Gravitational quantum cohomology. *International Journal of Modern Physics A*, 12(09):1743–1782, 1997.

[70] I. Ekeland and H. Hofer. Symplectic topology and hamiltonian dynamics. *Mathematische Zeitschrift*, 200(3):355–378, 1989.

[71] I. Ekeland and H. Hofer. Syplectic topology and hamiltonian dynamics ii. *Mathematische Zeitschrift*, 203(1):553–567, 1990.

[72] A. Elmendorf, I. Kriz, M. Mandell, and J. May. Modern foundations for stable homotopy theory. *Handbook of algebraic topology (ed. IM James, North-Holland, Amsterdam, 1995)*, pages 213–253, 1995.

[73] P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik. *Tensor categories*, volume 205. American Mathematical Soc., 2016.

[74] C. J. Fewster and K. Rejzner. Algebraic quantum field theory. In *Progress and Visions in Quantum Theory in View of Gravity*, pages 1–61. Springer, 2020.

[75] D. Fiorenza, H. Sati, and U. Schreiber. Twisted cohomotopy implies m5 wz term level quantization. *arXiv preprint arXiv:1906.07417*.

[76] D. Fiorenza, H. Sati, and U. Schreiber. Twisted cohomotopy implies m-theory anomaly cancellation on 8-manifolds. *arXiv preprint arXiv:1904.10207*, 2020.

[77] D. Fiorenza, H. Sati, and U. Schreiber. Twisted cohomotopy implies twisted string structure on m5-branes. *arXiv preprint arXiv:2002.11093*, 2020.

[78] R. Flauger. Anomalies and the atiyah-singer index theorem.

[79] T. Frankel. *The geometry of physics: an introduction*. Cambridge university press, 2011.

[80] D. S. Freed. Determinants, torsion, and strings. *Communications in mathematical physics*, 107(3):483–513, 1986.

[81] D. S. Freed. Higher algebraic structures and quantization. *Communications in mathematical physics*, 159(2):343–398, 1994.

[82] D. S. Freed. Dirac charge quantization and generalized differential cohomology. *arXiv preprint hep-th/0011220*, 2000.

[83] D. S. Freed. Bordism: Old and new. 2013.

[84] D. S. Freed and M. Hopkins. On ramond-ramond fields and k-theory. *Journal of High Energy Physics*, 2000(05):044, 2000.

[85] D. S. Freed and K. K. Uhlenbeck. *Instantons and four-manifolds*, volume 1. Springer Science & Business Media, 2012.

[86] D. S. Freed and E. Witten. Anomalies in string theory with d-branes. *arXiv preprint hep-th/9907189*, 1999.

[87] K. Fukaya and K. Ono. Arnold conjecture and gromov-witten invariant for general symplectic manifolds. *The Arnoldfest (Toronto, ON, 1997)*, 24:173–190, 1999.

[88] S. Galatius, I. Madsen, U. Tillmann, M. Weiss, et al. The homotopy type of the cobordism category. *Acta mathematica*, 202(2):195–239, 2009.

[89] A. García-Raboso and S. Rayan. Introduction to nonabelian hodge theory. In *Calabi-Yau varieties: arithmetic, geometry and physics*, pages 131–171. Springer, 2015.

[90] S. I. Gelfand and Y. I. Manin. *Methods of homological algebra*. Springer Science & Business Media, 2013.

[91] G. Giachetta, L. Mangiarotti, et al. *Advanced classical field theory*. World Scientific, 2009.

[92] F. Girelli and H. Pfeiffer. Higher gauge theoryâdifferential versus integral formulation. *Journal of mathematical physics*, 45(10):3949–3971, 2004.

[93] P. G. Goerss and J. F. Jardine. *Simplicial homotopy theory*. Springer Science & Business Media, 2009.

[94] M. Gotay, J. Grabowski, and H. Grundling. An obstruction to quantizing compact symplectic manifolds. *Proceedings of the American Mathematical Society*, 128(1):237–243, 2000.

[95] M. J. Gotay. Obstructions to quantization. In *Mechanics: from theory to computation*, pages 171–216. Springer, 2000.

[96] M. J. Gotay and H. B. Grundling. Nonexistence of finite-dimensional quantizations of a noncompact symplectic manifold. *arXiv preprint dg-ga/9710024*, 1997.

[97] J. M. Gracia-Bondía, J. C. Várilly, and H. Figueroa. *Elements of noncommutative geometry*. Springer Science & Business Media, 2013.

[98] D. Grady and H. Sati. Ramond-ramond fields and twisted differential k-theory. *arXiv preprint arXiv:1903.08843*, 2019.

[99] D. Grady and H. Sati. Twisted differential ko-theory. *arXiv preprint arXiv:1905.09085*, 2019.

[100] H. J. Groenewold. On the principles of elementary quantum mechanics. In *On the Principles of Elementary Quantum Mechanics*, pages 1–56. Springer, 1946.

[101] A. Grothendieck. *Pursuing stacks*. unpublished manuscript, 1983.

[102] V. Guillemin. The integrability problem for g-structures. *Transactions of the American Mathematical Society*, 116:544–560, 1965.

[103] B. C. Hall. *Quantum theory for mathematicians*. Springer, 2013.

[104] M. J. Hamilton. *Mathematical gauge theory*. Springer, 2017.

[105] A. Hatcher. *Algebraic topology*. 2005.
[106] E. Hawkins et al. A groupoid approach to quantization. *Journal of Symplectic Geometry*, 6(1):61–125, 2008.
[107] D. Hilbert. Mathematical problems. *Bulletin of the American Mathematical Society*, 8(10):437–479, 1902.
[108] H. Hofer and E. Zehnder. *Symplectic invariants and Hamiltonian dynamics*. Birkhäuser, 2012.
[109] M. J. Hopkins and I. M. Singer. Quadratic functions in geometry, topology, and m-theory. *arXiv preprint math/0211216*, 2002.
[110] K. Hori, R. Thomas, S. Katz, C. Vafa, R. Pandharipande, A. Klemm, R. Vakil, and E. Zaslow. *Mirror symmetry*, volume 1. American Mathematical Soc., 2003.
[111] P. Huang et al. Non-abelian hodge theory and related topics. *SIGMA. Symmetry, Integrability and Geometry: Methods and Applications*, 16:029, 2020.
[112] J. R. Isbell. Structure of categories. *Bulletin of the American Mathematical Society*, 72(4):619–655, 1966.
[113] J. R. Isbell. Normal completions of categories. In *Reports of the Midwest Category Seminar*, pages 110–155. Springer, 1967.
[114] D. Ivanenko and G. Sardanashvily. The gauge treatment of gravity. *Physics Reports*, 94(1):1–45, 1983.
[115] A. Jaffe and E. Witten. Quantum yang-mills theory. *The millennium prize problems*, pages 129–152, 2006.
[116] P. T. Johnstone. *Stone spaces*, volume 3. Cambridge university press, 1982.
[117] P. T. Johnstone. Sketches of an elephant (2 vols.), volume 43 of oxford logic guides, 2002.
[118] A. Joyal. Quasi-categories and kan complexes. *Journal of Pure and Applied Algebra*, 175(1-3):207–222, 2002.
[119] A. Joyal. Notes on quasi-categories. *preprint*, 2008.
[120] A. Joyal. The theory of quasi-categories and its applications. 2008.
[121] A. Kapustin. D-branes in a topologically nontrivial b-field. *arXiv preprint hep-th/9909089*, 1999.
[122] A. Kapustin, M. Kreuzer, and K.-G. Schlesinger. *Homological mirror symmetry: new developments and perspectives*, volume 757. Springer, 2008.
[123] H. Kashiwara and P. Schapira. Sheaves and categories, 2006.
[124] H. Kim and C. Saemann. Adjusted parallel transport for higher gauge theories. *Journal of Physics A: Mathematical and Theoretical*, 2020.
[125] A. Kirillov Jr and B. Balsam. Turaev-viro invariants as an extended tqft. *arXiv preprint arXiv:1004.1533*, 2010.
[126] J. Kock. *Frobenius algebras and 2-d topological quantum field theories*, volume 59. Cambridge University Press, 2004.
[127] M. Kontsevich. Homological algebra of mirror symmetry. In *Proceedings of the international congress of mathematicians*, pages 120–139. Springer, 1995.
[128] J. Kouneiher. *Foundations of Mathematics and Physics One Century After Hilbert: New Perspectives*. Springer, 2018.
[129] N. Landsman. Functorial quantization and the guillemin–sternberg conjecture. In *Twenty Years of Bialowieza: A Mathematical Anthology: Aspects of Differential Geometric Methods in Physics*, pages 23–45. World Scientific, 2005.
[130] A. Lasenby, C. Doran, and S. Gull. Gravity, gauge theories and geometric algebra. *Philosophical Transactions of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences*, 356(1737):487–582, 1998.
[131] M. Laubinger. Differential geometry in cartesian closed categories of smooth spaces. 2008.
[132] A. D. Lauda and H. Pfeiffer. Open–closed strings: Two-dimensional extended tqfts and frobenius algebras. *Topology and its Applications*, 155(7):623–666, 2008.
[133] H. B. Lawson and M.-L. Michelsohn. *Spin geometry*, volume 38. Princeton university press, 2016.
[134] F. W. Lawvere. Categorical dynamics. *Topos theoretic methods in geometry*, 30:1–28, 1979.
[135] F. W. Lawvere. Taking categories seriously. *Revista colombiana de matematicas*, 20(3-4):147–178, 1986.
[136] F. W. Lawvere. Categories of space and of quantity. *The Space of mathematics*, pages 14–30, 1992.
[137] F. W. Lawvere. Toposes of laws of motion. *American Mathematical Society, Transcript from video, Montreal-September*, 27:1997, 1997.
[138] F. W. Lawvere. Axiomatic cohesion. *Theory and Applications of Categories [electronic only]*, 19:41–49, 2007.
[139] F. W. Lawvere and S. H. Schanuel. *Conceptual mathematics: a first introduction to categories*. Cambridge University Press, 2009.
[140] T. Leinster. A survey of definitions of n-category. *arXiv preprint math/0107188*, 2001.
[141] T. Leinster. *Higher operads, higher categories*, volume 298. Cambridge University Press, 2004.
[142] T. Leinster. *Basic category theory*, volume 143. Cambridge University Press, 2014.
[143] E. L. Lima. *Duality and Postnivkov Invariants*. University of Chicago, 1958.
[144] J. Lurie. *Derived algebraic geometry*. PhD thesis, Massachusetts Institute of Technology, 2004.
[145] J. Lurie. Tannaka duality for geometric stacks. *arXiv preprint math/0412266*, 2004.
[146] J. Lurie. Derived algebraic geometry v: Structured spaces. *arXiv preprint arXiv:0905.0459*, 2009.
[147] J. Lurie. *Higher topos theory*. Princeton University Press, 2009.
[148] J. Lurie. Higher algebra, 2012.
[149] J. Lurie et al. On the classification of topological field theories. *Current developments in mathematics*, 2008:129–280, 2009.
[150] S. Mac Lane. *Categories for the working mathematician*, volume 5. Springer Science & Business Media, 2013.
[151] M. Mackaay and R. Picken. Holonomy and parallel transport for abelian gerbes. *arXiv preprint math/0007053*, 2000.
[152] S. MacLane and I. Moerdijk. *Sheaves in geometry and logic: A first introduction to topos theory*. Springer Science & Business Media, 2012.
[153] L. Mangiarotti and G. A. Sardanashvily. *Connections in classical and quantum field theory*. World Scientific, 2000.
[154] Y. I. Manin. *Frobenius manifolds, quantum cohomology, and moduli spaces*, volume 47. American Mathematical Soc., 1999.
[155] Y. X. Martins. *Introdução à Teoria da Homotopia Abstrata*. NEA, 2018.
[156] Y. X. Martins and R. J. Biezuner. Geometric obstructions on gravity. *arXiv preprint arXiv:1912.11198*, 2019.
[157] Y. X. Martins and R. J. Biezuner. Topological and geometric obstructions on einstein–hilbert–palatini theories. *Journal of Geometry and Physics*, 142:229–239, 2019.
[158] Y. X. Martins, L. F. A. Campos, and R. J. Biezuner. On extensions of yang-mills-type theories, their spaces and their categories. *arXiv preprint arXiv:2007.01660*, 2020.
[159] Y. X. Martins, L. F. A. Campos, and R. J. Biezuner. On maximal, universal and complete extensions of yang-mills-type theories. *arXiv preprint arXiv:2007.08651*, 2020.
[160] Y. X. Martins et al. Categorical and geometrical methods in physics. 2018.
[161] J. P. May. *A concise course in algebraic topology*. University of Chicago press, 1999.
[162] J. P. May. *The geometry of iterated loop spaces*, volume 271. Springer, 2006.
[163] J. P. May and J. C. Baez. *Towards higher categories*. Springer, 2010.
[164] D. W. McDuff, D. Salamon, et al. *J-Holomorphic Curves and Quantum Cohomology*. Number 6. American Mathematical Soc., 1994.
[165] D. Metzler. Topological and smooth stacks. *arXiv preprint math/0306176*, 2003.
[166] D. S. Metzler. A k-theoretic note on geometric quantization. *manuscripta mathematica*, 100(3):277–289, 1999.
[167] R. Minasian and G. Moore. K-theory and ramond-ramond charge. *Journal of High Energy Physics*, 1997(11):002, 1997.
[168] B. Mitchell. *Theory of categories*. Academic Press, 1965.
[169] G. Moore. A minicourse on generalized abelian gauge theory, self-dual theories, and differential cohomology. *Lectures at the Simons Center for Geometry and Physics*, 2011.
[170] S. Morrison and K. Walker. Higher categories, colimits, and the blob complex. *Proceedings of the National Academy of Sciences*, 108(20):8139–8145, 2011.
[171] S. Morrison and K. Walker. Blob homology. *Geometry & Topology*, 16(3):1481–1607, 2012.
[172] D. Mumford. Towards an enumerative geometry of the moduli space of curves. In *Arithmetic and geometry*, pages 271–328. Springer, 1983.
[173] M. Nakahara. *Geometry, topology and physics*. CRC Press, 2003.

[174] J. W. Negrepontis. Duality in analysis from the point of view of triples. *Journal of Algebra*, 19(2):228–253, 1971.
[175] J. Nestruev. *Smooth manifolds and observables*, volume 220. Springer Science & Business Media, 2006.
[176] J. Nuiten. Cohomological quantization of local prequantum boundary field theory. Master's thesis, 2013.
[177] J. Nuiten. Cohomological quantization of local prequantum boundary field theory. Master's thesis, 2013.
[178] R. Omnès. *Converging realities: toward a common philosophy of physics and mathematics*. Princeton University Press, 2005.
[179] F. Paugam. *Towards the mathematics of quantum field theory*, volume 59. Springer Science & Business Media, 2014.
[180] H.-E. Porst and M. B. Wischnewsky. Every topological category is convenient for gelfand duality. *manuscripta mathematica*, 25(2):169–204, 1978.
[181] E. Prugovecki. *Quantum mechanics in Hilbert space*. Academic Press, 1982.
[182] D. G. Quillen. *Homotopical algebra*, volume 43. Springer, 2006.
[183] C. Rezk. A model for the homotopy theory of homotopy theory. *Transactions of the American Mathematical Society*, 353(3):973–1007, 2001.
[184] C. Rezk. Toposes and homotopy toposes. *Lectures at University of Illinois at Urbana-Champaign*, 2005.
[185] M. F. S. Ribeiro. *Teoria das Categorias para Matemáticos: uma breve introdução*. SBM, 2020.
[186] E. Riehl. *Category theory in context*. Courier Dover Publications, 2017.
[187] C. L. Rogers. Higher symplectic geometry. *arXiv preprint arXiv:1106.4068*, 2011.
[188] Y. B. Rudyak and J. Rudjak. *On Thom spectra, orientability, and cobordism*. Springer Science & Business Media, 1998.
[189] G. Sardanashvily. Classical gauge gravitation theory. *International Journal of Geometric Methods in Modern Physics*, 8(08):1869–1895, 2011.
[190] G. Sardanashvily and O. Zakharov. *Gauge gravitation theory*. World Scientific, 1992.
[191] H. Sati and U. Schreiber. *Mathematical foundations of quantum field theory and perturbative string theory*, volume 83. American Mathematical Soc., 2011.
[192] H. Sati and U. Schreiber. Differential cohomotopy implies intersecting brane observables via configuration spaces and chord diagrams. *arXiv preprint arXiv:1912.10425*, 2019.
[193] H. Sati and U. Schreiber. Equivariant cohomotopy implies orientifold tadpole cancellation. *Journal of Geometry and Physics*, page 103775, 2020.
[194] H. Sati and U. Schreiber. Twisted cohomotopy implies m5-brane anomaly cancellation. *arXiv preprint arXiv:2002.07737*, 2020.
[195] H. Sati, U. Schreiber, and J. Stasheff. Lâ-algebra connections and applications to string-and chern-simons n-transport. In *Quantum field theory*, pages 303–424. Springer, 2009.
[196] H. Sati, U. Schreiber, and J. Stasheff. Twisted differential string and fivebrane structures. *Communications in Mathematical Physics*, 315(1):169–213, 2012.
[197] U. Schreiber. Higher geometric pre-quantization on moduliâ-stacks.
[198] U. Schreiber. Aqft from n-functorial fqft and applications. 2008.
[199] U. Schreiber. On nonabelian differential cohomology, 2008.
[200] U. Schreiber. Aqft from n-functorial qft. *Communications in Mathematical Physics*, 291(2):357–401, 2009.
[201] U. Schreiber. Differential cohomology in a cohesive infinity-topos. *arXiv preprint arXiv:1310.7930*, 2013.
[202] U. Schreiber. What, and for what is higher geometric quantization. 2014.
[203] U. Schreiber. Higher prequantum geometry. *arXiv preprint arXiv:1601.05956*, 2016.
[204] U. Schreiber and K. Waldorf. Connections on non-abelian gerbes and their holonomy. *arXiv preprint arXiv:0808.1923*, 2008.
[205] U. Schreiber, K. Waldorf, et al. Smooth functors vs. differential forms. *Homology, Homotopy and Applications*, 13(1):143–203, 2011.
[206] G. Segal. Two-dimensional conformal field theories and modular functors. In *IXth International Congress on Mathematical Physics, Swansea, 1988*, pages 22–37. Hilger, 1989.
[207] G. B. Segal. The definition of conformal field theory. In *Differential geometrical methods in theoretical physics*, pages 165–171. Springer, 1988.
[208] N. Seiberg and E. Witten. Electric-magnetic duality, monopole condensation, and confinement in n= 2 supersymmetric yang-mills theory. *Nuclear Physics B*, 426(1):19–52, 1994.
[209] P. Selinger. A survey of graphical languages for monoidal categories. In *New structures for physics*, pages 289–355. Springer, 2010.
[210] J.-P. Serre. Faisceaux algébriques cohérents. *Annals of Mathematics*, pages 197–278, 1955.
[211] J. Simons and D. Sullivan. Axiomatic characterization of ordinary differential cohomology. *Journal of Topology*, 1(1):45–56, 2008.
[212] C. Simpson. Algebraic aspects of higher nonabelian hodge theory. *arXiv preprint math/9902067*, 1999.
[213] C. Simpson. *Homotopy Theory of Higher Categories: From Segal Categories to n-Categories and Beyond*, volume 19. Cambridge University Press, 2011.
[214] C. T. Simpson. Nonabelian hodge theory. In *Proceedings of the International Congress of Mathematicians*, volume 1, pages 747–756, 1990.
[215] L. Sklar. *Philosophy of physics*. OUP Oxford, 1992.
[216] J. Sniatycki. *Geometric quantization and quantum mechanics*, volume 30. Springer Science & Business Media, 2012.
[217] E. Soncini and R. Zucchini. A new formulation of higher parallel transport in higher gauge theory. *Journal of Geometry and Physics*, 95:28–73, 2015.
[218] J. Stasheff. Cohomological physics. In *Algebraic Topology Rational Homotopy*, pages 228–237. Springer, 1988.
[219] J. Stasheff. A survey of cohomological physics, 2000.
[220] H. Stel. Stacks and their function algebras. Master's thesis, 2011.
[221] S. Sternberg. *Lectures on differential geometry*, volume 316. American Mathematical Soc., 1999.
[222] R. E. Stong. *Notes on cobordism theory*. Princeton University Press, 2015.
[223] R. Street. The role of michael batanin's monoidal globular categories. *Contemporary Mathematics*, 230:99–116, 1998.
[224] R. G. Swan. Vector bundles and projective modules. *Transactions of the American Mathematical Society*, 105(2):264–277, 1962.
[225] R. J. Szabo. Quantization of higher abelian gauge theory in generalized differential cohomology. *arXiv preprint arXiv:1209.2530*, 2012.
[226] C. Taubes and R. Wentworth. *Seiberg Witten and Gromov invariants for symplectic 4-manifolds*, volume 2. International Press Somerville, MA, 2000.
[227] G. Teschl. Mathematical methods in quantum mechanics. *Graduate Studies in Mathematics*, 99:106, 2009.
[228] R. Thom. Quelques propriétés globales des variétés différentiables. *Commentarii Mathematici Helvetici*, 28(1):17–86, 1954.
[229] B. Toën. Derived algebraic geometry. *arXiv preprint arXiv:1401.1044*, 2014.
[230] B. Toën, B. Toën, and G. Vezzosi. *Homotopical Algebraic Geometry II: Geometric Stacks and Applications: Geometric Stacks and Applications*, volume 2. American Mathematical Soc., 2008.
[231] A. Uranga. String theory phenomenology. In *Journal of Physics: Conference Series*, volume 259, page 012014. IOP Publishing, 2010.
[232] R. Utiyama. Invariant theoretical interpretation of interaction. *Physical Review*, 101(5):1597, 1956.
[233] R. Vakil. The moduli space of curves and gromov–witten theory. In *Enumerative invariants in algebraic geometry and string theory*, pages 143–198. Springer, 2008.
[234] L. C. P. Van Hove. *Sur certaines représentations unitaires d'un groupe infini de transformations*. PhD thesis, Bruxelles U., 1951.
[235] D. Verity. Nerves of complicial gray-categories. In *Categories in Algebra, Geometry and Mathematical Physics: Conference and Workshop in Honor of Ross Street's 60th Birthday, July 11-16/July 18-21, 2005, Macquarie University, Sydney, Australia, Australian National University, Canberra, Australia*, volume 431, page 441. American Mathematical Soc., 2007.
[236] D. R. Verity. Weak complicial sets i. basic homotopy theory. *Advances in Mathematics*, 219(4):1081–1149, 2008.
[237] C. A. Weibel. *An introduction to homological algebra*. Number 38. Cambridge university press, 1995.
[238] E. Witten. Topological quantum field theory. *Communications in Mathematical Physics*, 117(3):353–386, 1988.

[239] E. Witten. D-branes and k-theory. *Journal of High Energy Physics*, 1998(12):019, 1999.

[240] N. M. J. Woodhouse. *Geometric quantization*. Oxford University Press, 1997.

[241] S.-T. Yau. *Mathematical aspects of string theory: proceedings of the conference held at University of California, San Diego, July 21-August 1, 1986*. World Scientific, 1987.