Refundable deductible insurance

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Abstract

Most insurance policies include a deductible, so that a part of the claim is assumed by the insured. In order to get a full coverage of the claim, the insured has two options: hire a Zero Deductible Insurance or take out an insurance policy with deductible and, simultaneously, a Refundable Deductible Insurance. The objective of this paper is to analyze these two options, comparing the premium paid. We define $dif(F)$ as the difference between the premiums paid. This function depends on the parameters of the deductible applied, and we focus our attention on the sign of this difference and the calculation of the optimal deductible, that is, the values of the parameters of the deductible that allows us to obtain the greatest reduction in the global premium.

Keywords: premium calculation, variance criterion, optimization

1 Introduction

Nowadays, the car rental sector is booming and it is supposed to experience an important future growth, especially in large urban centers. Specifically, the global car rental market is expected to register a CAGR (Compound Annual Growth Rate) of 7.5, during the forecast period 2019-2024 ([2]). We could consider several reason for this trend: a growing environmental awareness, growing traffic problems, traffic restrictions due to pollution or the explosion of car sharing.

Due to the positive evolution of this sector, companies offering insurance policies to rental companies and their customers/users have developed new strategies to complete the coverage offered and decrease the price of car hire insurance, which can represent, in some cases, half of the rental cost. For more information about the car rental insurance sector in Europe, see [12].

In Europe, most car rental companies offer rental insurance coverage that guarantees protection against damage, theft, and liability insurance against third parties. It is what is known as CDW (Collision Damage Waiver), that provides cover in the event of collision or damage, and LDW (Loss Damage Waiver), that provides cover in the event of theft or loss of use of the vehicle. These policies offer varying levels of protection against damage. In other countries, such as USA or Canada, CDW and LDW insurance must be contracted separately. Most of these insurance

policies include a deductible, which in case of accident must be paid by the user to the rental company.

In order to cover the deductible, the insured has two options: The first one is to contract an extension to a Super CDW offered by the car rental company, that is in fact a Zero Deductible Insurance. The second one is to contract a RDI (Refundable Deductible Insurance) that covers the cost of the deductible. In recent years there have been online companies that, at a lower cost, cover the deductible. Then, if a claim occurs, the amount of the deductible that the car rental company has charged to the user is refunded. Thereby, if the insured does not want to pay anything if a claim occurs, he/she has two possibilities: take out a Zero Deductible Insurance, or, take out a Deductible Insurance and simultaneously a Refundable Deductible Insurance.

In the actuarial literature the studies about deductibles have concentrated in several aspects. One of the main topics is the analysis of the problem of optimal coverage and deductible through expected-utility (see [1], [19] or [24]) and stochastic dominance ([13] or [23]). The interaction between deductibles and bonus-malus systems and its repercussion on the efficiency of the bonus-malus system have been studied in [21], [18], [22] or [6]. Another topic is the optimal allocation of policy limits and deductibles from the viewpoint of a risk-averse policyholder ([4] and [15]) or from the the viewpoint of the insurer ([16]). Although the introduction of deductibles on insurance contracts has been widely analyzed in the actuarial literature, as far as we know, there is a lack concerning the Refundable Deductible Insurance.

The objective of our research is to analyze the advantage the insured can have by contracting a Deductible Insurance and a Refundable Deductible Insurance, option covering the whole cost of the claim, as opposed to the alternative of directly contracting a Super CDW, that is to say a Zero Deductible Insurance. We measure this advantage by comparing the premium paid in these two alternatives considering different rules to share the cost of the claim between the insurer and the insured. Precisely we work with the absolute deductible, proportional deductible, mixture of absolute deductible and proportional deductible and all-nothing deductible. For a definition of these deductibles see [5] or [3] among others.

As preliminary we recall some concepts and facts that are useful in what follows. Notation and conventions used throughout the paper are also established. Let us consider X_i r.v. the cost of the i-th individual claim, $A(X_i)$ the part of the cost paid by the insured in an insurance with deductible, and $C(X_i)$ the part of the claim paid by the insurer. The r.v. number of claims in a period, denoted by N, allows us to calculate the r.v. total cost in a period, S,

$$S = \sum_{i=0}^{N} X_i. \tag{1}$$

Assuming the usual hypothesis of risk theory (see [10] or [20]), we consider that the $X_i, i \geq 1$ are identical and independently distributed (i.i.d.). From now on, for reasons of simplicity, $A(X)$ and $C(X)$ are denoted as A and C. For a random variable Y, its expected value and variance are denoted by $E(Y)$ and $V(Y)$ respectively. Let's also denote by $F_Y(y)$ the cumulative distribution function and by $\bar{F}_Y(y) = 1 - F_Y(y)$ the survival function of Y. The moments of the total cost are easily calculated from the moments of X and N, being the expected value of the total cost S

$$E(S) = E(N)E(X), \tag{2}$$

and, its variance,

$$V(S) = E(N)V(X) + E(X)^2V(N). \tag{3}$$

The insurer calculates the premium considering only the part of the claim that he/she pays. Then, if the contract includes a deductible, the total cost for the insured and its moments are obtained from 1 to 3, substituting X by C.

Let Π the premium paid for a Zero Deductible Insurance, Π^F the premium paid for a Deductible Insurance, and Π^* the premium paid for a Refundable Deductible Insurance, that is to say an insurance policy that covers A. To measure the effect of the two alternatives that the insured has to get a full coverage, we compare Π and $\Pi^F + \Pi^*$. The function $dif(F)$ measures the difference for the insured between the two options,

$$dif(F) = \Pi - \Pi^F - \Pi^*,$$

where we explicit the dependence on F, the type of deductible applied defined by a set of parameters $\{x_1, ..., x_n\}$. Then, if $\Pi = \Pi^F + \Pi^*$, $dif(F)$ equals to zero wich implies that the two alternatives are indifferent to the insured. If $\Pi < \Pi^F + \Pi^*$, $dif(F)$ is negative and the insured prefers to contract just a Zero Deductible Insurance. And lastly, if $\Pi > \Pi^F + \Pi^*$, $dif(F)$ is positive so the insured prefers to contract a Deductible Insurance and a Refundable Deductible Insurance. This analysis depends on the type of deductible applied and the premium principle used, in other words, the mathematical method used to fix the insurance premium. In this paper all premiums are calculated with the same premium criterion, in particular we consider the mean principle and the variance principle.

In the mean principle, the premium is calculated as the expected value of the risk plus a proportional loading to this expected value,

$$\Pi = E(S)(1 + \delta), \qquad \delta > 0,$$

whereas in the variance criterion, the loading is proportional to the variance,

$$\Pi = E(S) + \delta V(S), \qquad \delta > 0.$$

For more information about premium principles and their properties see [14] and [17].

If the mean criterion is applied, $\Pi = \Pi^F + \Pi^*$, then $dif(F) = 0$ for any X and N and whatever deductible is used. In this case there is no advantage from choosing one alternative or the other one. Whereas, if we focus on the variance criterion, it is easy to see that $dif(F)$ is not always equal to zero, and then a deep analysis is needed in order to optimize the advantage that the insured can get.

In this framework, the contribution of the paper refers to three aspects. The first one, consists on proposing a theoretical framework to the market practice of contracting a Refundable Deductible Insurance. As a second contribution, we find a sufficient condition, fulfilled by almost all of the deductible types, that guarantees the advantage for the insured of the Refundable Deductible Insurance under the variance criterion. And lastly, we present the optimal deductible parameters allowing the insured to obtain the maximum gain, for different kinds of deductibles.

After this introduction, the paper is structured as follows. In Section 2, we present general results for the variance criterion and in Section 3, we develop the specific results for the different deductibles considered in this paper: absolute deductible, proportional deductible, mixture of absolute deductible and proportional deductible and all-nothing deductible. The paper ends with some conclusions.

2 Variance criterion

In this Section the variance criterion is used to calculate the premiums. A general expression for $dif(F)$ is obtained and the conditions that allow to maximize this function are presented.

Using the variance criterion, the premium of the Deductible Insurance is

$$\Pi^F = E(N)E(C) + \delta[E(N)V(C) + E(C)^2V(N)],$$

whereas the premium for the Refundable Deductible Insurance is given by

$$\Pi^* = E(N)E(A) + \delta[E(N)V(A) + E(A)^2V(N)],$$

and the premium that would be paid for a complete insurance covering all X, a Zero Deductible Insurance, is given by

$$\Pi = E(N)E(X) + \delta[E(N)V(X) + E(X)^2V(N)].$$

The random variables A and C depend not only on X but also on the parameters of the deductible. In order to simplify the expressions, we will not make explicit these dependencies. Then with the variance criterion,

$$\begin{aligned} dif(F) &= \delta E(N)V(X) + \delta V(N)E(X)^2 - \delta E(N)[V(C) + V(A)] - \delta V(N)[E(C)^2 + E(A)^2] \\ &= 2\delta E(N)Cov(A,C) + 2\delta V(N)E(A)E(C). \end{aligned} \tag{4}$$

Taking into account that $Cov(A,C) = E(AC) - E(A)E(C)$ and substituting C by $X - A$, an alternative expression for $dif(F)$ is obtained,

$$dif(F) = 2\delta\left([E(A)E(X) - E(A)^2][V(N) - E(N)] + E(N)[E(AX) - E(A^2)]\right). \tag{5}$$

Let us analyze this function $dif(F)$. The first aspect is the sign of this difference, Proposition 2.1 establish its positiveness under an usual condition. The second aspect studied is the calculus, if it exists, of the optimal deductible, that is, the values of the parameters of the deductible such that the insured obtains the greatest reduction in the global premium paid with the same coverage. In the present section we include the general condition (first order condition) for this optimization problem. Next sections are dedicated to these questions regarding the absolute deductible, the proportional deductible with maximum loss, a mixture of an absolute deductible and a proportional one and the all-nothing deductible

Proposition 2.1. *For a deductible such that A and C are comonotonic risks, $dif(F) > 0$.*

Proof. Two risks X and Y are comonotone (see [25]) if their bivariate cumulative distribution function, $F_{XY}(x,y)$, satisfies $F_{XY}(x,y) = min[F_X(x), F_Y(y)]$ for all $x, y \geq 0$. From an intuitive point of view, the comonotonicity of two risks means that these risks are not able to compensate each other. If two risks are comonotone, its covariance is positive (see [11]). Then, from (4), if A and C are comonotone, $dif(F) > 0$. □

Then, the comonotonicity is a sufficient condition for the positiveness of $dif(F)$. For more information on comonotonicity, we refer to [8], [9] and [7]. The r.v.'s A and C are comonomotic in almost all of the deductible, see [25]. This is the case for the absolute deductible (Section 3.1), the proportional deductible with maximum loss (Section 3.2) and a mixture of an absolute deductible and a proportional one (Section 3.3). In Section 3.4. we analyze a kind of deductible such that A and C are not comonotone, the all-nothing deductible.

The optimal deductible problem is

$$\max dif(F),$$

w.r.t. the parameters $\{x_1, ..., x_n\}$ that define the deductible. The first order condition is

$$\nabla dif(F) = \left(\frac{\partial dif(F)}{\partial x_1}, ..., \frac{\partial dif(F)}{\partial x_n} \right) = 0,$$

that applied to (5) and defining $g(N) = \frac{E(N)-V(N)}{E(N)}$, allows us to obtain the following system of equations

$$\frac{\partial E(AX)}{\partial x_i} - \frac{\partial E(A^2)}{\partial x_i} = g(N) \left(\frac{\partial E(A)E(X)}{\partial x_i} - \frac{\partial E(A)^2}{\partial x_i} \right), \quad i = 1, ..., n. \tag{6}$$

From (6) we see that, regarding the number of claims, only its expected value and variance affect the optimization problem and in fact, when the expected value and the variance of N are equal, the first order conditions simplify. In the Poisson case, $N \sim Po(\lambda)$, $E(N) = V(N)$, so $g(N) = 0$, and (5) becomes

$$dif(F) = 2\delta E(N)[E(AX) - E(A^2)],$$

with the first order conditions from (6)

$$\frac{\partial E(AX)}{\partial x_i} - \frac{\partial E(A^2)}{\partial x_i} = 0, \quad i = 1, ..., n.$$

In Section 3 we apply this general result to obtain the parameters of the different deductibles that minimize the premium paid by the insured.

3 Specific results assuming different types of deductibles

In this Section we present the results obtained in the optimization of $dif(F)$ for different types of deductibles under the variance criterion. Specifically, we consider absolute deductible, proportional deductible, mixture of absolute deductible and proportional deductible and all-nothing deductible. In the first three types of deductibles, Proposition 2.1. is fulfilled, that's to say, A and C are comonotonic risks and therefore $dif(F)$ is always positive. In the all-nothing deductible, A and C are not comonotonic risks, thence $dif(F)$ can be positive or negative so, for the insured, the option of taking out a Deductible Insurance and a Refundable Deductible Insurance is not always better than contracting a Zero Insurance.

3.1 Absolute Deductible

In an absolute deductible, the insured pays the first a monetary units of each claim X, and the insurer pays the excess over a, $X - a$. Then, in an absolute deductible with parameter $a \geq 0$, A and C are defined in Table 1.

X	A	C
$X < a$	X	0
$X > a$	a	$X - a$

Table 1: Absolute deductible with parameter $a \geq 0$.

Following [26], we define the n-th partial moment of X about the origin over $(0, x_0)$ as the partial expectation of X^n, $H_{X^n}(x_0) = \int_0^{x_0} x^n f(x)dx$. Hence, the expectations of A, AX and A^2 are

$$E(AX) = H_{X^2}(a) + aE(X) - H_X(a), \tag{7}$$
$$E(A^2) = H_{X^2}(a) + a^2\bar{F}_X(a), \tag{8}$$
$$E(A) = H_X(a) + a\bar{F}_X(a). \tag{9}$$

From (5) and using (7), (8) and (9), $dif(F)$ is given by

$$\begin{aligned} dif(F) \quad &= 2\delta[H_X(a) + a\bar{F}_X(a)][E(X) - H_X(a) - a\bar{F}_X(a)][V(N) - E(N)] \\ &\quad +2\delta E(N)a[E(x) - H_X(a) - a\bar{F}_X(a)]. \end{aligned} \tag{10}$$

As $dif(F)$ is function of only just one variable, in order to find the value of a that maximizes the difference, we substitute the gradient by the derivative with respect to a. The system (6) becomes

$$E'(AX) - E'(A^2) = g(N)\{E'(A)E(X) - [E(A)^2]'\}. \tag{11}$$

From (7), (8) and (9), derivating with respect to a,

$$E'(AX) = E(X) - H_X(a), \tag{12}$$
$$E'(A^2) = 2a\bar{F}_X(a), \tag{13}$$
$$E'(A) = \bar{F}_X(a), \tag{14}$$
$$(E(A)^2)' = 2\bar{F}_X(a)[H_X(a) + a\bar{F}_X(a)]. \tag{15}$$

Then, using these last derivatives, (11) is

$$E(X) - H_X(a) - 2a\bar{F}_X(a) = g(N)\bar{F}_X(a)[E(X) - 2H_X(a) - 2a\bar{F}_X(a)]. \tag{16}$$

From (16) it is clear that the existence of a maximum value depends on the distribution both the number of claims and the claim amount.

Let's consider two claim amount distributions: exponential and Pareto-Lomax.

Exponential case: $X \sim \exp(\gamma)$, being $f(x) = \gamma e^{-\gamma x}$, then

$$H_X(a) = \frac{1}{\gamma} - (\frac{1}{\gamma} + a)e^{-a\gamma}, \tag{17}$$
$$\bar{F}_X(a) = e^{-a\gamma}. \tag{18}$$

Substituting the previous expressions in (10)

$$dif(F) = \frac{2\delta}{\gamma} e^{-a\gamma} \left\{ \left(1 - e^{-a\gamma}\right) \left(\frac{1}{\gamma} + 2a\right) [V(N) - E(N)] + E(N)a \right\}.$$

Considering (17) and (18), from (16), the first order condition is

$$1 - a\gamma = g(N)(2e^{-a\gamma} - 1).$$

The values of a that fulfill the previous expression, a^*, depend on the relationship between $E(N)$ and $V(N)$. So, if $E(N) > V(N)$, $a^* \in (0, \frac{\ln 2}{\gamma}) \cup (\frac{1}{\gamma}, \infty)$; if $E(N) < V[N]$, $a^* \in (\frac{\ln 2}{\gamma}, \frac{1}{\gamma})$, and lastly when $E(N) = V(N)$, $a^* = \frac{1}{\gamma}$. If N is Poisson distributed with parameter λ, the optimal value for a coincides with the mean claim amount, the maximum difference is $\frac{2\delta E(N)}{\gamma^2 e}$ and hence, for the insured, the best option is to contract a deductible insurance with $a = E(X)$ and a refundable insurance that covers a.

In Figure 1 we plot $dif(F)$, in the Poisson-exponential case, as a function of a for different values of γ.

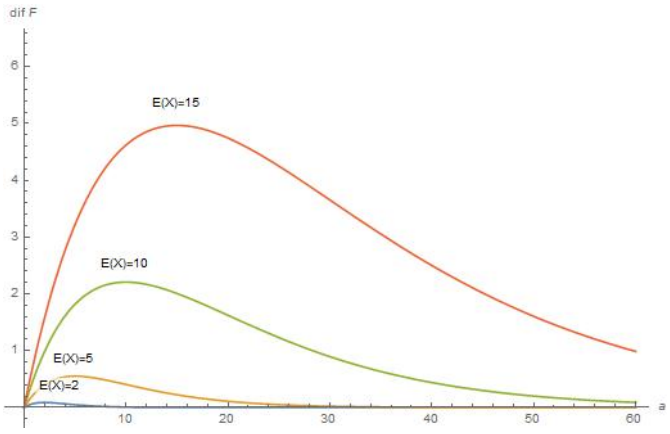


Figure 1: $dif(F)$ as a function of a in the Poisson-exponential case for $\delta = 0.03$ and $\lambda = 1$ for $E(X) = 2, 5, 10, 15$.

Pareto-Lomax case: $X \sim Pareto(\theta, \nu)$, being $f(x) = \theta\nu^\theta(\nu + x)^{-\theta-1}$ and $E(X) = \frac{\nu}{\theta-1}$, $\nu > 0$, $\theta > 2$ (so that the variance is finite) then

$$H_X(a) = \frac{1}{1-\theta} \left[\nu^\theta(\nu + a)^{1-\theta} - \nu\right] - \frac{a\nu^\theta}{(\nu + a)^\theta},$$
$$\bar{F}_X(a) = \left(\frac{\nu}{\nu + a}\right)^\theta.$$

Substituting the previous expressions in (16)

$$\nu + a(2 - \theta) = g(N)\nu \left[2\nu^{\theta-1}(\nu + a)^{1-\theta} - 1\right]. \tag{19}$$

If $E(N) > V(N)$, $a^* > \frac{\nu}{\theta-2}$, if $E(N) = V(N)$ (Poisson case) $a^* = \frac{\nu}{\theta-2}$, and if $E(N) < V(N)$, $a^* < \frac{\nu}{\theta-2}$.

	$E(N) < V(N)$	$E(N) = V(N)$	$E(N) > V(N)$
Exp(γ)	$a^* \in (\frac{\ln 2}{\gamma}, \frac{1}{\gamma})$	$a^* = \frac{1}{\gamma}$	$a^* \in (0, \frac{\ln 2}{\gamma}) \cup (\frac{1}{\gamma}, \infty)$
Pareto(θ,ν)	$a^* < \frac{\nu}{\theta-2}$	$a^* = \frac{\nu}{\theta-2}$	$a^* > \frac{\nu}{\theta-2}$

Table 2: Optimal a in absolute deducible in the Poisson-exponential and Poisson-Pareto-Lomax cases.

Summarizing the results for the exponential and the Pareto-Lomax cases, in Table 2 , the optimal values of a that maximizes $dif(F)$ are included,

Let us now consider that N is Poisson distributed with parameter λ and X is a general distribution with p.d.f. $f(x)$. Now, $dif(F)$ is linear on δ and depends on a, the only parameter of the deductible,

$$dif(F) = \delta E(N) 2a[E(X) - H_X(a) - a\bar{F}_X(a)].$$

The value of a that maximizes $dif(F)$ is the solution of

$$E(X) - H_X(a) - 2a\bar{F}_X(a) = 0, \tag{20}$$

that can be rewritten as

$$\int_a^\infty (x - 2a) f(x) dx = 0. \tag{21}$$

Via a complete numerical analysis it can be deduced that the above optimization problem can always be solved. The solution of (23) depends only on the distribution of X. Explicit expressions have been obtained in the previous two subsections for the exponential and the Pareto-Lomax distributions. For other claim amount distributions only numerical solutions can be found. In order to realize the sensibility of the solution on the claim amount distribution, Table 3 includes the optimal value of a and the maximum difference reached for two distributions of the claim amount: exponential and lognormal. The parameters of the distributions are such that the expected value and the variance of the claim amount is the same, so the premium with a zero deductible insurance, Π would be the same. The Pareto-Lomax distribution is not included in the comparison because there exist no a combination of θ and ν that fulfills $E(X)^2 = V(X)$, as in the exponential case. For the lognormal distribution, with parameters μ and σ, the equation to be solved (23) is

$$E(X)\left[1 - \Phi\left(\frac{\ln a - (\mu + \sigma^2)}{\sigma}\right)\right] - 2a\left[1 - \Phi\left(\frac{\ln a - \mu}{\sigma}\right)\right] = 0, \tag{22}$$

that, for the specific choice of the parameters that fulfils $E(X)^2 = V(X)$, in order to compare with the exponential distribution, we obtain

$$E(X)\left\{1 - \Phi\left[\frac{1}{\sqrt{\ln 2}} \ln\left(\frac{a}{E(X)}\right) - 0.5\sqrt{\ln 2}\right]\right\} - 2a\left\{1 - \Phi\left[\frac{1}{\sqrt{\ln 2}} \ln\left(\frac{a}{E(X)}\right) + 0.5\sqrt{\ln 2}\right]\right\} = 0.$$

If the individual claim amount follows an Inverse-Gaussian distribution, $X \sim IG(\mu, \lambda)$, $\mu > 0$, $\lambda > 0$, being $f(x) = \sqrt{\frac{\lambda}{2\pi x^3}} e^{\frac{-\lambda(x-\mu)^2}{2\mu^2 x}}$, $E(X) = \mu$ and $V(X) = \frac{\mu^3}{\lambda}$, the combination of μ and λ that

fulfills $E(X)^2 = V(X)$ is $\mu = \lambda$. Then, the optimal value of a is obtained from,

$$\int_a^{\infty} (x - 2a)\sqrt{\frac{\lambda}{2\pi x^3}} e^{\frac{-\lambda(x-\mu)^2}{2\mu^2 x}} dx = 0, \tag{23}$$

expression that can be solved numerically.

In Table 3 the results of the optimal value a^* and the maximum value of $dif(F)$ depending on $E(N)$ and δ are presented for an Exponential, a Lognormal and an Inverse-Gaussian distribution,

$E(X)$	$V(X)$	$Exp(\gamma)$	$Log(\mu, \sigma)$	$IG(\mu, \lambda)$
0.1	0.01	$(0.1, 0.0074E(N)\delta)$	$(0.0934, 0.0012E(N)\delta)$	$(0.1016, 0.0067E(N)\delta)$
0.5	0.25	$(0.5, 0.1839E(N)\delta)$	$(0.4679, 0.0294E(N)\delta)$	$(0.5082, 0.1681E(N)\delta)$
1	1	$(1, 0.7358E(N)\delta)$	$(0.9383, 0.3322E(N)\delta)$	$(1.0165, 0.6724E(N)\delta)$
2	4	$(2, 2.9430E(N)\delta)$	$(1.8768, 1.6212E(N)\delta)$	$(2.0331, 2.6899E(N)\delta)$
5	25	$(5, 18.3940E(N)\delta)$	$(4.6919, 9.9024E(N)\delta)$	$(5.082, 16.812E(N)\delta)$
10	100	$(10, 73.5759E(N)\delta)$	$(9.3878, 39.4688E(N)\delta)$	$(10.1656, 67.2479E(N)\delta)$

Table 3: Optimal value of a and difference $(a^*, dif(a^*))$ for $Exp(\gamma)$, $Log(\mu, \sigma)$ and $IG(\mu, \lambda)$ claim amount distributions.

We observe in Table 3 that the maximum difference is proportional to the expected value of N and to the proportional loading and this value is the greater in the exponential case and the smaller in the Lognormal case. The value a^* that allows reaching this maximum is the smaller in the Lognormal case and the greater in the Inverse-Gaussian case.

3.2 Proportional deductible

In this section we focus our attention on the proportional deductible. We work with two types of proportional deductible: a first type in which the insured pays a percentage α of each claim, and, a second type, in which we include a maximum loss for the insured, B.

In a deductible with participation $\alpha \in (0, 1)$, A and C are defined in Table 4.

X	A	C
$\forall X$	αX	$(1 - \alpha)X$

Table 4: Deductible with participation $\alpha \in (0, 1)$.

In this case, $E(AX) = \alpha E(X^2)$, $E(A^2) = \alpha^2 E(X^2)$ and $E(A) = \alpha E(X)$, then

$$dif(F) = 2\delta\alpha(1 - \alpha)\{E(X)^2[V(N) - E(N)] + E(N)E(X^2)\}, \tag{24}$$

being the value α that maximizes the difference, from (6),

$$(1 - 2\alpha)\left[E(X^2) - g(N)E(X)^2\right] = 0.$$

Independently of the distribution of N and X, $\alpha = \frac{1}{2}$ is the value that maximizes the difference, being the maximum difference $\frac{\delta}{2}V(S)$. If $N \sim Po(\lambda)$, the maximum difference is $\frac{\delta\lambda E(X^2)}{2}$.

Now, let's generalize the proportional deductible including a maximum loss for the insured. In Table 5 a deductible with participation $\alpha \in (0,1)$ and a maximum loss $B > 0$ is defined,

X	A	C
$X < \frac{B}{\alpha}$	αX	$(1-\alpha)X$
$X > \frac{B}{\alpha}$	B	$X - B$

Table 5: Deductible with participation $\alpha \in (0,1)$ with limit $B > 0$.

If B tends to infinity, the first kind of proportional deductible is obtained as a particular case. In this deductible,

$$E(AX) = \alpha H_{X^2}\left(\frac{B}{\alpha}\right) + B\left[E(X) - H_X\left(\frac{B}{\alpha}\right)\right], \tag{25}$$

$$E(A^2) = \alpha^2 H_{X^2}\left(\frac{B}{\alpha}\right) + B^2 \bar{F}_X\left(\frac{B}{\alpha}\right), \tag{26}$$

$$E(A) = \alpha H_X\left(\frac{B}{\alpha}\right) + B\bar{F}_X\left(\frac{B}{\alpha}\right). \tag{27}$$

From (5), $dif(F)$ is

$$\begin{aligned} dif(F) \quad &= 2\delta\left[\alpha H_X\left(\tfrac{B}{\alpha}\right) + B\bar{F}_X\left(\tfrac{B}{\alpha}\right)\right]\left[E(X) - \alpha H_X\left(\tfrac{B}{\alpha}\right) - B\bar{F}_X\left(\tfrac{B}{\alpha}\right)\right][V(N) - E(N)] \\ &+2\delta E(N)\left\{(\alpha - \alpha^2)H_{X^2}\left(\tfrac{B}{\alpha}\right) + B\left[E(X) - H_X\left(\tfrac{B}{\alpha}\right)\right] - B^2\bar{F}_X\left(\tfrac{B}{\alpha}\right)\right\} \end{aligned} \tag{28}$$

The deductible depends on two parameters, α and B, being the partial expectations, from (25), (26) and (27)

$$\frac{\partial E(AX)}{\partial B} = E(X) - H_X\left(\frac{B}{\alpha}\right), \quad \frac{\partial E(AX)}{\partial \alpha} = H_{X^2}\left(\frac{B}{\alpha}\right), \tag{29}$$

$$\frac{\partial E(A^2)}{\partial B} = 2B\bar{F}_X\left(\frac{B}{\alpha}\right), \quad \frac{\partial E(A^2)}{\partial \alpha} = 2\alpha H_{X^2}\left(\frac{B}{\alpha}\right), \tag{30}$$

$$\frac{\partial E(A)}{\partial B} = \bar{F}_X\left(\frac{B}{\alpha}\right), \quad \frac{\partial E(A)}{\partial \alpha} = H_X\left(\frac{B}{\alpha}\right). \tag{31}$$

Then, the system of equations that allows us to maximize $dif(F)$ is

$$\left.\begin{aligned} E(X) - H_X\left(\tfrac{B}{\alpha}\right) - 2B\bar{F}_X\left(\tfrac{B}{\alpha}\right) &= g(N)\bar{F}_X\left(\tfrac{B}{\alpha}\right)\left\{E(X) - 2\left[\alpha H_X\left(\tfrac{B}{\alpha}\right) + B\bar{F}_X\left(\tfrac{B}{\alpha}\right)\right]\right\} \\ (1-2\alpha)H_{X^2}\left(\tfrac{B}{\alpha}\right) &= g(N)H_X\left(\tfrac{B}{\alpha}\right)\left\{E(X) - 2\left[\alpha H_X\left(\tfrac{B}{\alpha}\right) + B\bar{F}_X\left(\tfrac{B}{\alpha}\right)\right]\right\} \end{aligned}\right\} \tag{32}$$

From now on, in order to obtain analytical solutions for different individual claim amounts, we assume the Poisson case. The analysis of (32) for $g(N) \neq 0$ only can be done using numerical methods. If $N \sim Po(\lambda)$, $dif(F)$ is

$$dif(F) = 2\delta E(N)\left\{(\alpha - \alpha^2)H_{X^2}\left(\frac{B}{\alpha}\right) + B\left[E(X) - H_X\left(\frac{B}{\alpha}\right)\right] - B^2\bar{F}_X\left(\frac{B}{\alpha}\right)\right\}. \tag{33}$$

And, from (32) the system of equations that allows us to maximize $dif(F)$ is

$$\left.\begin{array}{c} E(X) - H_X\left(\frac{B}{\alpha}\right) - 2B\bar{F}_X\left(\frac{B}{\alpha}\right) = 0 \\ (1-2\alpha)H_{X^2}\left(\frac{B}{\alpha}\right) = 0 \end{array}\right\} \tag{34}$$

Proposition 3.1. *If $N \sim Po(\lambda)$, there is no value of (B, α) that maximizes* (34)*, but marginal optimums exist: for a fixed B, $\alpha^* = \frac{1}{2}$ maximizes the difference, and for a fixed $\alpha > \frac{1}{2}$ the optimal value of B is the one that fulfils $\int_{\frac{B}{\alpha}}^{\infty}(x-2B)f(x)dx = 0$. If $\alpha \leq \frac{1}{2}$, the optimal value of B does not exist.*

Proof. From the second equation, knowing that $\int_0^{\frac{B}{\alpha}} x^2 f(x)dx \neq 0$, we obtain $\alpha = \frac{1}{2}$. Substituting in the first one, it is reduced to $\int_{2B}^{\infty}(x-2B)f(x)dx = 0$ that is impossible because the integral is always different from 0. Then, there is no a value of (B, α) that maximizes (34). Analyzing $\frac{\partial \cdot}{\partial \alpha}$, $\alpha = \frac{1}{2}$ is the value that maximizes $dif(F)$ for any value of B. And analyzing $\frac{\partial \cdot}{\partial B}$, if $\alpha < \frac{1}{2}$, the value of B that maximizes does not exist because $\int_{\frac{B}{\alpha}}^{\infty}(x-2B)f(x)dx$ is always different from 0 as $\frac{B}{\alpha} > 2B$, and if $\alpha > \frac{1}{2}$ the value of B that optimizes is the one obtained from $\int_{\frac{B}{\alpha}}^{\infty}(x-2B)f(x)dx = 0$. □

Poisson-Exponential case: If $N \sim Po(\lambda)$ and $X \sim \exp(\gamma)$, following Proposition 3.1., we know that there is no value of (α, B) that maximizes (33), but marginal optimums exist. For a fixed B, $\alpha^* = \frac{1}{2}$ maximizes the difference and for a fixed $\alpha > 0.5$, the value $B^* = \dfrac{\alpha}{\gamma(2\alpha-1)}$ maximizes $dif(F)$. For example, assuming $\gamma = 0.6$, $\delta = 0.03$ and $\lambda = 1$, $dif(F)$ as a function of α and B is plotted in Figure 2.

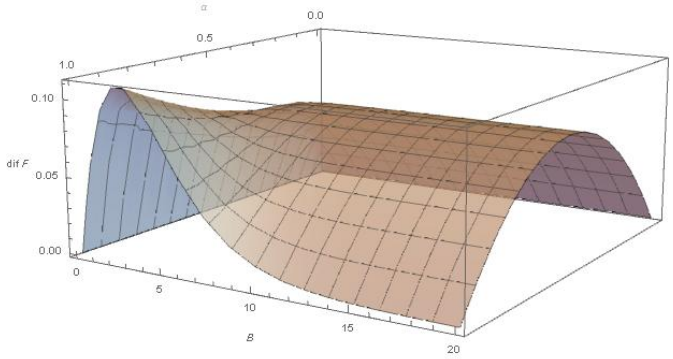


Figure 2: $dif(F)$ in the Poisson-exponential case with $\delta = 0.03, \lambda = 1, \gamma = 0.6$.

Poisson-Pareto case: If $N \sim Po(\lambda)$ and c for a fixed B, $\alpha^* = \frac{1}{2}$ maximizes $dif(F)$ and for a fixed $\alpha > \frac{1}{2}$, $B^* = \dfrac{\alpha}{2\alpha(\theta-1)-\theta}$ also maximizes the difference. For example, if $\theta = 3$, $\delta = 0.03$ and $\lambda = 1$, $dif(F)$ is plotted in Figure 3

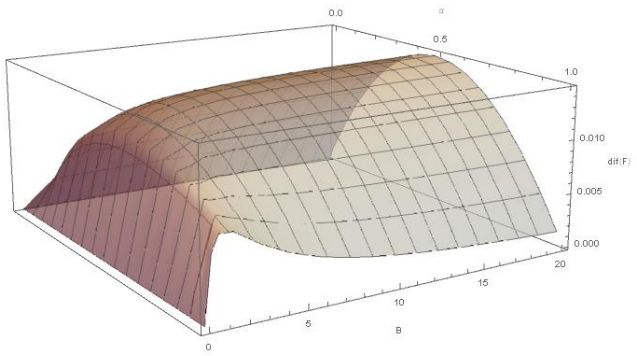


Figure 3: $dif(F)$ in the Poisson-Pareto case with $\delta = 0.03, \lambda = 1, \theta = 3$.

Poisson-Lognormal case: If $X \sim LN(\mu, \sigma)$, for a fixed $\alpha > \frac{1}{2}$, the optimal value of B is the one that fulfils $\int_{\frac{B}{\alpha}}^{\infty}(x - 2B)f(x)dx = 0$, that can can be written as,

$$E(X)\left[1 - \Phi\left(\frac{\ln\frac{B}{\alpha} - (\mu + \sigma^2)}{\sigma}\right)\right] - 2B\left[1 - \Phi\left(\frac{\ln\frac{B}{\alpha} - \mu}{\sigma}\right)\right] = 0. \tag{35}$$

In this case, it's not possible to obtain analytical results of the value B that optimizes the difference. Some numerical results are presented in Table 6 for differents values of $\alpha > \frac{1}{2}$.

α	0.55	0.6	0.65	0.7	0.75	0.8	0.85	0.9	0.95	1
B^*	11.79	3.1743	2.0103	1.5907	1.38081	1.2573	1.1773	1.1222	1.0824	1.0529

Table 6: Optimal B in proportional deducible in the Poisson-Lognormal case for $\alpha > \frac{1}{2}$.

From Table 6, we see that B^* decreases with α and if $\alpha = 1$, we have in fact an absolute deductible and thus (35) reduces to (22).

3.3 Mixture of absolute deductible and proportional deductible

In a mixture of absolute deductible $a > 0$ and proportional deductible with $\alpha \in (0, 1)$, the insured doesn't pay anything if the claim amount is less than a, pays a if the claim amount is greater than a and less than $\frac{a}{\alpha}$, and if the claim amount is greater than $\frac{a}{\alpha}$ the proportional deductible is applied and the insured pays a percentage α of the claim amount. A and C are defined in Table 7.

X	A	C
$X < a$	0	X
$a < X < \frac{a}{\alpha}$	a	$X - a$
$X > \frac{a}{\alpha}$	$a\alpha$	$(1 - \alpha)X$

Table 7: Mixture of absolute deductible $a > 0$ and proportional deductible $\alpha \in (0, 1)$.

Let's define the different elements we apply in the optimization process:

$$E(AX) = H_{X^2}(a) + a\left[H_X\left(\frac{a}{\alpha}\right) - H_X(a)\right] + \alpha\left[E(X^2) - H_{X^2}\left(\frac{a}{\alpha}\right)\right], \tag{36}$$

$$E(A^2) = H_{X^2}(a) + a^2\left[\bar{F}(a) - \bar{F}\left(\frac{a}{\alpha}\right)\right] + \alpha^2\left[E(X^2) - H_{X^2}\left(\frac{a}{\alpha}\right)\right], \tag{37}$$

$$E(A) = H_X(a) + a\left[\bar{F}(a) - \bar{F}\left(\frac{a}{\alpha}\right)\right] + \alpha\left[E(X) - H_X\left(\frac{a}{\alpha}\right)\right]. \tag{38}$$

This deductible depends on two parameters. Let's calculate the partial derivatives with respect to α and a,

$$\frac{\partial E(AX)}{\partial a} = H_X\left(\frac{a}{\alpha}\right) - H_X(a), \tag{39}$$

$$\frac{\partial E(A^2)}{\partial a} = 2a\left[\bar{F}(a) - \bar{F}\left(\frac{a}{\alpha}\right)\right], \tag{40}$$

$$\frac{\partial E(A)}{\partial a} = \bar{F}(a) - \bar{F}\left(\frac{a}{\alpha}\right), \tag{41}$$

$$\frac{\partial E(AX)}{\partial \alpha} = E(X^2) - H_{X^2}\left(\frac{a}{\alpha}\right), \tag{42}$$

$$\frac{\partial E(A^2)}{\partial \alpha} = 2\alpha\left[E(X^2) - H_{X^2}\left(\frac{a}{\alpha}\right)\right], \tag{43}$$

$$\frac{\partial E(A)}{\partial \alpha} = E(X) - H_X\left(\frac{a}{\alpha}\right). \tag{44}$$

And the equation system that allows us to obtain the optimal value of the parameters of the deductible to maximize $dif(F)$,

$$\left.\begin{array}{c} H_X\left(\frac{a}{\alpha}\right) - H_X(a) - 2a\left[\bar{F}(a) - \bar{F}(\frac{a}{\alpha})\right] = g(N)\left[\bar{F}(a) - \bar{F}(\frac{a}{\alpha})\right]\left[E(X) - 2E(A)\right] \\ (1-2\alpha)\left[E(X^2) - H_{X^2}(\frac{a}{\alpha})\right] = g(N)\left[E(X) - H_X(\frac{a}{\alpha})\right]\left[E(X) - 2E(A)\right] \end{array}\right\} \tag{45}$$

In the Poisson case, $N \sim Po(\lambda)$, $dif(F)$ is

$$dif(F) = 2\delta E(N)\left\{(\alpha - \alpha^2)\left[E(X^2) - H_{X^2}\left(\frac{a}{\alpha}\right)\right] + a\left[H_X\left(\frac{a}{\alpha}\right) - H_X(a)\right] - a^2\left[\bar{F}(a) - \bar{F}\left(\frac{a}{\alpha}\right)\right]\right\}, \tag{46}$$

and (45) is

$$\left.\begin{array}{c} H_X(\frac{a}{\alpha}) - H_X(a) - 2a\left[\bar{F}(a) - \bar{F}(\frac{a}{\alpha})\right] = 0 \\ (1-2\alpha)\left[E(X^2) - H_{X^2}(\frac{a}{\alpha})\right] = 0 \end{array}\right\}, \tag{47}$$

or alternatively

$$\left.\begin{array}{c} \int_a^{\frac{a}{\alpha}} (x - 2a) f(x) dx = 0 \\ (1-2\alpha)\int_{\frac{a}{\alpha}}^{\infty} x^2 f(x) dx = 0 \end{array}\right\}. \tag{48}$$

In order to obtain analytical results, henceforth we work with the Poisson model, $N \sim Po(\lambda)$.

Proposition 3.2. *If $N \sim Po(\lambda)$, there is no value of (a, α) that maximizes (48), but marginal optimums exist: for a fixed a, $\alpha^* = \frac{1}{2}$ maximizes the difference, and for a fixed $\alpha < \frac{1}{2}$ the optimal value of a that maximizes $dif(F)$ is the one that fulfils $\int_a^{\frac{a}{\alpha}} (x - 2a) f(x) dx = 0$. If $\alpha > \frac{1}{2}$, the optimal value of a that maximizes $dif(F)$ does not exist.*

Proof. From the second equation of (48), knowing that $\int_{\frac{a}{\alpha}}^{\infty} x^2 f(x)dx \neq 0$, we obtain $\alpha^* = \frac{1}{2}$. Substituting in the first one, we obtain $\int_a^{2a}(x-2a)f(x)dx = 0$ that is impossible because the integral is always different from 0. Then, there is no a value of (a, α) that maximizes (48). Analyzing $\frac{\partial \cdot}{\partial \alpha}$, $\alpha = \frac{1}{2}$ is the value that maximizes $dif(F)$ for any value of a. And analyzing $\frac{\partial \cdot}{\partial a}$, if $\alpha > \frac{1}{2}$, the value of a that maximizes does not exist because $\int_a^{\frac{a}{\alpha}}(x-2a)f(x)dx$ is always different from 0, and if $\alpha < \frac{1}{2}$ the value of a that optimizes is the one obtained from $\int_a^{\frac{a}{\alpha}}(x-2a)f(x)dx = 0$. □

Poisson-Exponential case: For $X \sim \exp(\gamma)$, if $\alpha > \frac{1}{2}$, the value of a that optimizes the difference is obtained from the equation $\frac{2a-\frac{a}{\alpha}-\frac{1}{\gamma}}{(a-\frac{1}{\gamma})} = e^{(\frac{1}{\alpha}-1)\gamma a}$ that has no explicit solution.

Let's visualize graphically the behaviour of $dif(F)$ and its optimal values. For example, if $\gamma = 0.6$, $\delta = 0.03$ and $\lambda = 1$, $dif(F)$ is plotted in Figure 4.

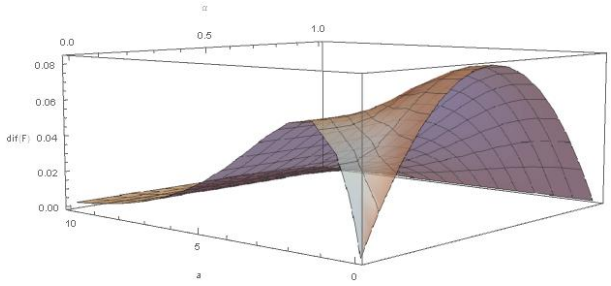


Figure 4: $dif(F)$ in the Poisson-exponential case with $\delta = 0.03, \lambda = 1, \gamma = 0.6$.

Partial analysis of $dif(F)$ with respect to a and α are plotted in Figure 5 and in Figure 6 respectively.

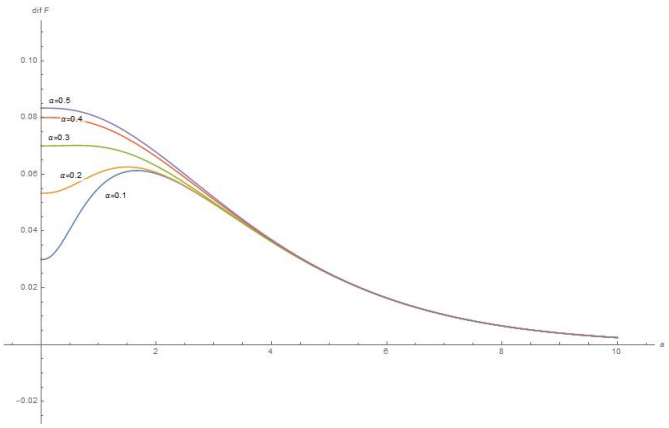


Figure 5: $dif(F)$ for $\alpha = 0.1, 0.2, 0.3, 0.4, 0.5$ in the Poisson-exponential case with $\delta = 0.03, \lambda = 1, \gamma = 0.6$.

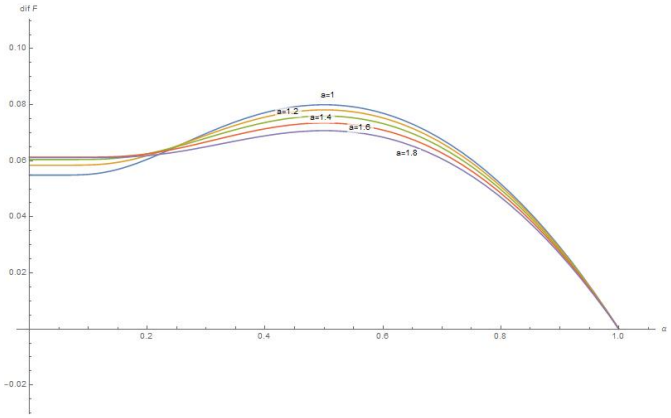


Figure 6: $dif(F)$ for $a = 1, 1.2, 1.4, 1.6, 1.8$ in the Poisson-exponential case with $\delta = 0.03, \lambda = 1, \gamma = 0.6$.

Poisson-Pareto case: If $X \sim Pareto(\theta, 1), \theta > 2$, the value of a that optimizes the difference is obtained from the equation $\frac{2a(\gamma-1)-\gamma a-1}{2a(\gamma-1)-\frac{\gamma}{\alpha}a-1} = \left(\frac{\frac{a}{\alpha}+1}{a+1}\right)^{-\theta}$.

In Figure 7, a graphical illustration of $dif(F)$ is included for $\theta = 3$, $\delta = 0.03$ and $\lambda = 1$,

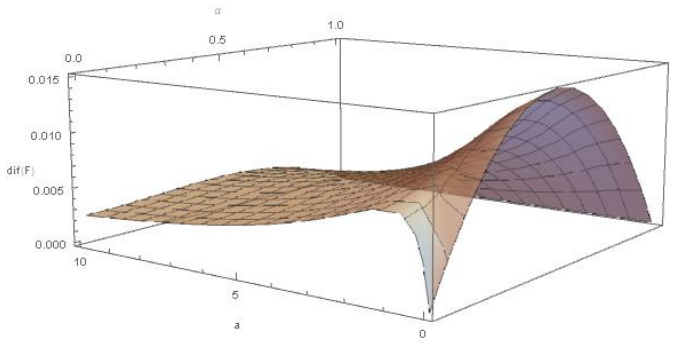


Figure 7: $dif(F)$ in the Poisson-Pareto case with $\delta = 0.03, \lambda = 1, \theta = 0.6$

Partial analysis of $dif(F)$ with respect to a and α are plotted in Figure 8 and in Figure 9 respectively.

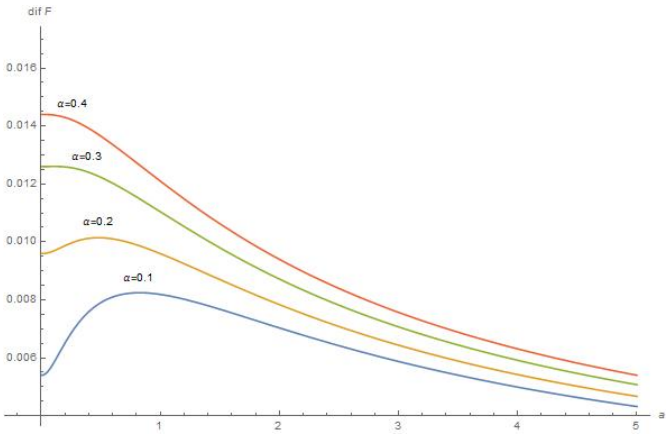


Figure 8: $dif(F)$ for $\alpha = 0.1, 0.2, 0.3, 0.4$ in the Poisson-Pareto case with $\delta = 0.03, \lambda = 1, \theta = 3$

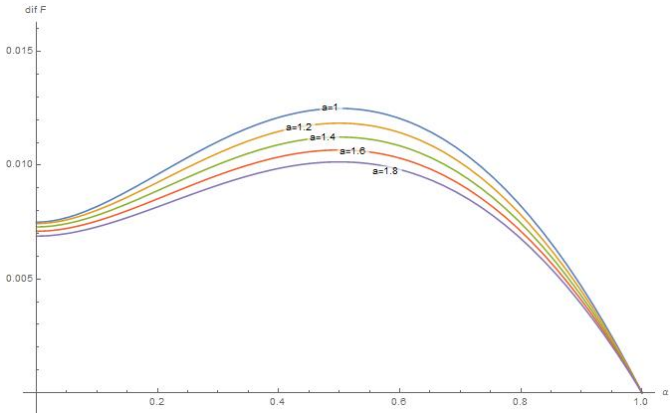


Figure 9: $dif(F)$ for $a = 1, 1.2, 1.4, 1.6, 1.8$ in the Poisson-Pareto case with $\delta = 0.03, \lambda = 1, \theta = 3$

We can observe in the figures obtained for the exponential case (Figures 5 and 6) and the ones obtained for the Pareto case (Figures 8 and 9) a similar behavior in the optimization problem with respect to a and α.

Poisson-Lognormal case: If $X \sim LN(\mu, \sigma)$, in order to obtain the optimal value of B that maximizes $dif(F)$ for values of $\alpha \geq \frac{1}{2}$, the integral $\int_a^{\frac{a}{\alpha}} (x - 2a) f(x) dx = 0$ can be written as,

$$E(X)\left[\Phi\left(\frac{\ln\frac{a}{\alpha} - (\mu + \sigma^2)}{\sigma}\right) - \Phi\left(\frac{\ln a - (\mu + \sigma^2)}{\sigma}\right)\right] - 2a\left[\Phi\left(\frac{\ln\frac{a}{\alpha} - \mu}{\sigma}\right) - \Phi\left(\frac{\ln a - \mu}{\sigma}\right)\right] = 0.$$

Due to the impossibility of solving analytically the previous equation, in Table 8 we present some numerical results from which we see that B^* decreases with α.

α	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
B^*	1.05293	1.05199	1.04197	1.00751	0.934123	0.80981	0.62501	0.376775	0.09619	0.00087

Table 8: Optimal value of B in the Poisson-Lognormal case for $\alpha < \frac{1}{2}$.

3.4 All-nothing deductible

In this section we analyze the all-nothing deductible with participation $M > 0$. The idea is that if the individual claim amount, X, is less than M, the insurer pays the whole claim, but if X is greater than M the insurer doesn't pay anything (see [5]).

In Table 9, A and C are defined,

X	A	C
$X < M$	0	X
$X > M$	X	0

Table 9: All-nothing deductible M.

This deductible doesn't fulfill Proposition 2.1. since A and C are not commonotic risks. Then, the following results are not focused on obtain the optimal deductible for the insured because its not possible to maintain that the function $dif(F)$ is always positive.

In the all-nothing deductible, $E(AX) = E(A^2) = E(X^2) - H_{X^2}(M)$, and $E(A) = E(X) - H_X(M)$. Hence the expression for $dif(F)$ is

$$dif(F) = 2\delta[E(X) - H_X(M)]H_X(M)\,[V(N) - E(N)] \tag{49}$$

If $N \sim Po(\lambda)$, then $dif(F) = 0$ regardless of the claim amount distribution. Then, for the Poisson case, for the insured is the same to cover the risk with a Zero Deductible Insurance or with a Deductible Insurance jointly with a Refundable deductible Insurance.

Knowing that $E(X) - H_X(M) > 0$, we can observe in (49) that the sign of $dif(F)$ depends on the sign of $V(N) - E(N)$. Then, $dif(F)$ can be positive or negative. The explanation is that, in this deductible, A and C are not commonotic risks (see [25]), and therefore the initial hypothesis of this paper is not fulfilled. That is to say, in this deductible, $Cov(A, C)$ is not a positive value. In fact, knowing that $E(AC) = E(AX) - E(A^2) = 0$, $Cov(A, C) = -E(A)E(C) < 0$. From (4), the positiveness of $Cov(A, C)$ is a sufficient, but not necessary, condition for the positiveness of $dif(F)$. We are only interested in the situations in which $dif(F)$ is positive, therefore we impose that $V(N) > E(N)$.

In order to obtain the value that optimizes $dif(F)$ we need the following two previous derivatives,

$$E'(AX) = E'(A^2) = -M^2 f(M), \tag{50}$$

$$E'(A) = -Mf(M), \tag{51}$$

and, from (6), the value that optimizes $dif(F)$ is,

$$E(X) = 2H_X(M). \tag{52}$$

In the exponential case, it is easy to deduce from (52) that the optimal M fulfills $(2\gamma M + 2)e^{-M\gamma} = 1$, and for the $Pareto(\theta, \nu)$, it can be obtained from

$$\left(\frac{\nu}{\nu + M}\right)^{\theta} = \frac{\nu}{2(\nu + M\theta)}. \tag{53}$$

In Table 10 some numerical results for the exponential distribution are obtained

$E(X)$	1	2	3	4	5	6	10	20
M	1.6783	3.3567	5.035	6.7134	8.3917	10.07	16.783	33.567

Table 10: Optimal value of M assuming exponential distribution

and if we assume $Pareto(\theta, \nu)$ the results are shown in Table 11.

ν θ	2.1	2.5	3	3.5	4	5
1	2.1189	1.4176	1	0.771650	0.627942	0.457323
1.5	3.17833	2.12645	1.5	1.15747	0.941913	0.685984
2	4.23778	2.83527	2	1.5433	1.2559	0.914645
2.5	5.29722	3.54409	2.5	1.92912	1.5698	1.14331

Table 11: Optimal value of M assuming Pareto distribution

If $X \sim LN(\mu, \sigma)$, then $E(X) = 2H_X(M)$ is,

$$\Phi\left(\frac{\ln M - (\mu + \sigma^2)}{\sigma}\right) = \frac{1}{2},$$

that implies $\ln M = (\mu + \sigma^2)$, hence the value of M that maximizes $dif(F)$ is $M = e^{(\mu+\sigma^2)}$.

4 Conclusions

In this paper, we present a theoretical framework to analyse the advantage that the insured can obtain by contracting a Refundable Deductible Insurance. Several criterion can be applied by the insurer to calculate premiums: if the mean criterion is applied this advantage is null, whereas with the variance criterion, the commonotonicity of the parts of the claim covered by the insurer and the insured guarantees the advantage. The deductible parameters that maximize the difference obtained by the insured depend on the expected value and variance of the number of claims and the distribution of the individual claim amount. Explicit expressions of these optimal parameters are obtained in several cases, in the others, a numerical analysis is developed.

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