Pressure at infinity and strong positive recurrence in negative curvature

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With an appendix by Felipe Riquelme

Abstract

In the context of geodesic flows of noncompact negatively curved manifolds, we propose three different definitions of entropy and pressure at infinity, through growth of periodic orbits, critical exponents of Poincaré series, and entropy (pressure) of invariant measures. We show that these notions coincide.

Thanks to these entropy and pressure at infinity, we investigate thoroughly the notion of strong positive recurrence in this geometric context. A potential is said strongly positively recurrent when its pressure at infinity is strictly smaller than the full topological pressure. We show in particular that if a potential is strongly positively recurrent, then it admits a finite Gibbs measure. We also provide easy criteria allowing to build such strong positively recurrent potentials and many examples.

([1]) ([2])

Contents

1. Keywords : Entropy at infinity, pressure at infinity, strong positive recurrence, noncompact negatively curved manifolds, thermodynamic formalism
2. MSC 2020 classification : 37A35, 37B40, 37C35, 37D20, 37D35, 37D40, 37F32

1 Introduction

The geodesic flow on a compact negatively curved manifold M is the typical geometrical example of an *Anosov flow*. Its chaotic behavior reveals itself in particular through the existence of infinitely many possible different behaviours of orbits, and even of all imaginable behaviours.

A *Gibbs measure* is an ergodic invariant (probability) measure associated to a given continuous map $F : T^1M \to \mathbb{R}$, with respect to which almost all orbits will spend most of their time in the subsets of T^1M where the potential F is large (see Section 3.4 for the precise definition). In particular, the existence of a Gibbs measure for all (Hölder) continuous maps is a quantified way to express the above idea that all possible behaviours of orbits are indeed realized as typical trajectories w.r.t. the Gibbs measures of all Hölder potentials.

When the manifold M is not compact anymore, a geometric construction developed in [PPS15] allows to build good candidates for Gibbs measures. However, due to noncompactness of M and T^1M, these measures are not necessarily finite, and therefore not always extremely useful.

In [PS18], Pit and Schapira characterized the finiteness of these measures in terms of the convergence of some geometric series. In [ST19], in the case of the zero potential $F = 0$, building on [PS18], Schapira and Tapie proposed a criterion, called *strong positive recurrence*, which implies the finiteness of the associated measure, known as the *Bowen-Margulis-Sullivan measure.* This criterion is the following. If $\Gamma = \pi_1(M)$, recall that the *critical exponent of* Γ is the exponential growth rate of any orbit of Γ acting on the universal cover $\widetilde{M}$ of M. By a result of Otal and Peigné [OP04], it also coincides with the topological entropy of the geodesic flow on T^1M. In [ST19], a *critical exponent at infinity* δ_Γ^∞ is defined, and the authors prove that a critical gap $\delta_\Gamma^\infty < \delta_\Gamma$ implies that the Bowen-Margulis-Sullivan measure is finite. This had been previously shown by Dal'bo, Otal and Peigné in [DOP00] for *geometrically finite manifolds*, for which the critical exponent at infinity is the maximum of the critical exponents among parabolic subgroups. In general, this critical exponent at infinity should be seen as a kind of *entropy at infinity*. Other striking applications of this critical gap have been proved in [CDST19].

The main goal of this paper is to produce a complete study of strong positive recurrence in negative curvature. First, in sections 4, 5, 6, we compare this critical exponent at infinity with other, new and old, possible definitions of entropy at infinity and show that they all coincide. At the same time, considering pressures and pressures at infinity instead of entropies, we generalize this study to all Gibbs measures studied in [PPS15, PS18]. In a second part (section 7), we give a detailed study of strong positive recurrence in negative curvature. The appendix by F. Riquelme proves important properties of entropy, that are classical in the compact case, but need a careful proof in the noncompact case.

Analogous results were known since years in the context of symbolic dynamics over a countable alphabet, see [Gur69, Gur70, GS98, Sar99, Sar01, Rue03, BBG06, BBG14] .

Let us present our results with more details.

The *topological pressure* of a (Hölder) potential $F : T^1M \to \mathbb{R}$ is a weighted version of entropy. For a dynamical system on a compact space, there are a lot of different definitions, which all coincide, see for example [Wal82, ch 9] or [Bow75]. In the noncompact setting, some of these definitions are meaningless. In [PPS15], following the works of [Rob03, OP04] on entropy, three definitions were compared. The *Gurevič Pressure* $P_{\mathrm{Gur}}(F)$ is the (weighted) exponential growth rate of the periodic orbits of the geodesic flow. The *variational pressure* $P_{\mathrm{var}}(F)$([3]) is the supremum over all invariant probability measures of their measure-theoretic pressures, that is a weighted version of their Kolmogorov-Sinai entropies. The *critical pressure* $\delta_\Gamma(F)$, a geometric notion specific to geodesic flows, is the (weighted) exponential growth rate of the orbits of the fundamental group Γ of M acting on its universal cover $\widetilde{M}$.

It has been shown in [Rob03, OP04] when $F \equiv 0$ and [PPS15, thm 1.1] for general potentials that all these pressures coincide.

Theorem 1.1 (Roblin, Otal-Peigné, Paulin-Pollicott-Schapira)**.** *Let M be a nonelementary complete connected negatively curved manifold with pinched negative curvature, and bounded first derivative of the curvature. Let $F : T^1M \to \mathbb{R}$ be a Hölder continuous map. Then we have*

$$\delta_\Gamma(F) = P_{\mathrm{var}}(F) = P_{\mathrm{Gur}}(F). \tag{1}$$

We denote this common value by $P_{\mathrm{top}}(F)$.

We propose here three notions of pressure at infinity, whose precise definitions will be given in Section 4. The *Gurevič pressure at infinity* $P^\infty_{\mathrm{Gur}}(F)$ measures the exponential growth rate of periodic orbits staying most of the time outside any given compact set. The *variational pressure at infinity* $P^\infty_{\mathrm{var}}(F)$ measures the supremum of measure-theoretic pressures of invariant probability measures supported mostly outside any given compact set. The *critical exponent at infinity* $\delta^\infty_\Gamma(F)$ measures the (weighted) exponential growth rate of those orbits of the fundamental group Γ corresponding to excursions outside any given compact set.

The first main result of this article is the following.

Theorem 1.2. *Let M be a nonelementary complete connected negatively curved manifold with pinched negative curvature, and bounded first derivative of the curvature. Let $F : T^1M \to \mathbb{R}$ be a Hölder continuous map. Then, we have*

$$\delta^\infty_\Gamma(F) = P^\infty_{\mathrm{var}}(F) = P^\infty_{\mathrm{Gur}}(F).$$

We denote this common value by $P^\infty_{\mathrm{top}}(F)$.

In the special case where F is constant at infinity, the equality $\delta^\infty_\Gamma(F) = P^\infty_{\mathrm{var}}(F)$ has also been obtained by completely distinct methods in [Vel19].

As already implicitely or explicitely noticed for example in [EK12, EKP15, IRV18, RV19], this pressure at infinity is deeply related to the phenomenon of loss of mass. In the vague topology, on a noncompact space, a sequence of probability measures (with mass 1) may converge to a finite measure with smaller total mass. As proven by the above authors, if these probability measures have a larger entropy than the entropy at infinity, then they cannot loose the whole mass and converge to the zero measure. In this spirit, as a corollary of Theorem 6.10, we obtain in Corollary 6.11 the following result.

3. It was denoted by $P_{\mathrm{top}}(F)$ in [PPS15], but it seems to us now better to say that the *topological pressure* is the common value of all these definitions of pressure, once Theorem 1.1 is known.

Theorem 1.3. *Let M be a nonelementary complete connected negatively curved manifold with pinched negative curvature, and bounded first derivative of the curvature. Let $F : T^1M \to \mathbb{R}$ be a Hölder continuous map with finite pressure. Let (μ_n) be a sequence of probability measures converging in the vague topology to a finite measure μ, with mass $0 \leq \|\mu\| \leq 1$. Then*

$$\limsup_{n\to\infty} h_{KS}(\mu_n) + \int F \,\mathrm{d}\mu_n \leq (1 - \|\mu\|) \times P^{\infty}_{\mathrm{top}}(F) + \|\mu\| P_{\mathrm{top}}(F)\,.$$

In particular, if $\mu_n \to 0$, then

$$\limsup h_{KS}(\mu_n) + \int F \,\mathrm{d}\mu_n \leq P^{\infty}_{\mathrm{top}}(F)\,.$$

In [IRV18, RV19], in the geometrically finite case, and in Velozo's phd ([Vel18], cf also [Vel19, Thm 1.1]) for general manifolds, they obtained an improvement of the conclusion of the Theorem, with $P_\mu(F)$ instead of $P_{\mathrm{top}}(F)$ on the right, but only for the particular class of potentials F which converge to 0 at infinity for which $P^{\infty}_{\mathrm{top}}(F) = P^{\infty}_{\mathrm{top}}(0)$. The approach used in these papers is completely different to ours, and does not work (at the moment) for potential which are non-constant at infinity. It would be interesting to obtain their sharper inequality under our weaker assumptions (cf [Vel19, Conjecture 5.5]).

Once Theorem 1.2 is proven, we can say that a potential F is *strongly positively recurrent* (SPR) when the following *pressure gap* holds:

$$P^{\infty}_{\mathrm{top}}(F) < P_{\mathrm{top}}(F)\,. \tag{2}$$

We refer the reader to Section 7 for the notions of recurrence, positive recurrence, strong positive recurrence.

An analogous notion of *pressure gap* for potentials on nonpositively curved manifolds, w.r.t. the set of singular vectors instead of infinity, has been introduced in [BCFT18].

As in [ST19, Thm 7.1] when $F = 0$, we prove the following extremely useful property of SPR potentials.

Theorem 1.4. *Let M be a nonelementary complete connected negatively curved manifold with pinched negative curvature, and bounded first derivative of the curvature. Let $F : T^1M \to \mathbb{R}$ be a Hölder continuous map. If F is strongly positively recurrent, then it admits a finite Gibbs measure.*

For potentials which vanish at infinity, this has also been obtained in [Vel19, Theorem 1.3] using a different strategy. We will show that, on any negatively curved manifold, there exist strongly positively recurrent potentials, see Corollary 4.12. This implies the following new result.

Corollary 1.5. *Let M be a nonelementary complete connected negatively curved manifold with pinched negative curvature, and bounded first derivative of the curvature. There exists a Hölder continuous potential $F : T^1M \to \mathbb{R}$ which admits a finite Gibbs measure.*

It may worth pointing that in their current proof, all results of [Vel19] which we previously quoted actually rely on the existence of such potential with finite Gibbs measure. Nevertheless to our knowledge, this fact had not been established beyond geometrically finite manifolds.

We also establish other useful properties. Let $F : T^1M \to \mathbb{R}$ be a Hölder continuous map which admits a finite Gibbs measure m_F. This measure is automatically ergodic and therefore conservative, so that almost all orbits come back infinitely often to a set of finite measure. For a given compact set $K \subset T^1M$, consider the set $V_{T_0,T}(K)$ of vectors v, such that $(g^t v)_{t\geq 0}$ leaves K and does not return in K during the interval of time $[T_0, T]$. These sets $(V_{T_0,T}(K))_{T>0}$ decrease when $T \to +\infty$. We say that the measure m_F is *exponentially recurrent* if there exist $K, C, \alpha, T_0 > 0$ such that for all $T > 0$,

$$m_F(V_{T_0,T}(K)) \leq C e^{-\alpha T}\,. \tag{3}$$

In Section 7.4, we establish the following theorem.

Theorem 1.6. *Let M be a nonelementary complete connected negatively curved manifold with pinched negative curvature, and bounded first derivative of the curvature. Let $F : T^1M \to \mathbb{R}$ be a Hölder continuous map with finite pressure and finite Gibbs measure. Then F is strongly positively recurrent iff it is exponentially recurrent.*

We finish this work with Theorem 7.8, showing that strong positive recurrence does not really depend on the chosen compact set K. More precisely, the critical pressure at infinity is defined as the infimum over all compact sets K of the weighted exponential growth rate of the excursions outside K. We show in Theorem 7.8 that if the potential F is strongly positively recurrent, then for any compact set K, as soon as the interior of K meets a closed geodesic, this exponential growth rate of excursions outside K is strictly smaller than the full pressure.

The first two sections 2 and 3 contain preliminaries, on the one hand on negatively curved geometry and dynamics, and on the other hand on thermodynamical formalism, in particular all different notions of pressures, and the construction of the measure m^F.

Sections 4, 5, 6 on the one hand, and Section 7 on the other hand can be read independently.

Section 4 contains three different definitions of pressures at infinity. In section 5, we give upper bounds on the growth of certain sets of periodic orbits in terms of entropy and entropy at infinity. We deduce equality of the geometric and Gurevič Pressures at infinity $\delta_\Gamma^\infty(F)$ and $P_{\mathrm{Gur}}^\infty(F)$. In section 6, we show that geometric and variational pressures at infinity $\delta_\Gamma^\infty(F)$ and $P_{\mathrm{var}}^\infty(F)$ coincide. These sections are the technical heart of the paper.

Section 7 is more conceptual. We investigate the notion of strongly positively recurrent potentials in our geometric context, and prove Theorems 1.4 and 1.6.

The appendix by Felipe Riquelme (Theorem A.1) shows that different possible definitions of measure-theoretic entropy, the Kolmogorov-Sinai entropy, the Brin-Katok entropy, and the Katok entropy coincide. This result is well known in the compact case, but not obvious at all without compactness.

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2 Negative curvature, geodesic flow

2.1 Geometric preliminaries

Our assumptions and notations are close to those of [PPS15, PS18, ST19].

Let (M, g) be a smooth complete connected noncompact Riemannian manifold with pinched negative sectional curvatures $-b^2 \le K_g \le -a^2$, for some $a, b > 0$, and bounded first derivative of the curvature. Let $\widetilde{M}$ be its universal cover, $\Gamma = \pi_1(M)$ its fundamental group, and $p_\Gamma : \widetilde{M} \to M = \widetilde{M}/\Gamma$ the quotient map. We assume that M admits at least two distinct closed geodesics, or in other words, that the group Γ is nonelementary. In particular it contains at least a free group (see for instance [Bow95]). We denote by T^1M and $T^1\widetilde{M}$ the unit tangent bundles of M and $\widetilde{M}$, and by $\pi : T^1M \to M$ or $\pi : T^1\widetilde{M} \to \widetilde{M}$ the canonical bundle projection. By abuse of notation, we also write $p_\Gamma : T^1\widetilde{M} \to T^1M$ for the differential of p_Γ.

Given any two points $x, y \in \widetilde{M}$, the set $[x, y] \subset \widetilde{M}$ will denote the (unique) geodesic segment between x and y.

We fix arbitrarily a point $o \in \widetilde{M}$ which we call *origin*. The boundary at infinity $\partial\widetilde{M}$ is the set of equivalence classes of geodesic rays staying at bounded distance one from another. The *limit set* $\Lambda_\Gamma \subset \partial\widetilde{M}$ is the set of accumulation points $\Lambda_\Gamma = \overline{\Gamma o} \backslash \Gamma o$ of the orbit of o. As shown by Eberlein [Ebe72], the nonwandering set $\Omega \subset T^1M$ of the geodesic flow is the set of geodesic orbits which admit a lift whose negative and positive endpoints belong to Λ_Γ. The *radial limit set* $\Lambda_\Gamma^{\mathrm{rad}} \subset \Lambda_\Gamma$ is the set of endpoints of geodesics whose images through p_Γ return infinitely often in some compact set:

$$\Lambda_\Gamma^{\mathrm{rad}} := \{\xi \in \Lambda_\Gamma, \exists C > 0, \exists (\gamma_n) \in \Gamma^{\mathbb{N}}, \gamma_n o \to \xi, d(\gamma_n o, [o\xi)) \le C\}\,.$$

We denote by $(g^t)_{t\in\mathbb{R}}$ the geodesic flow acting on T^1M or $T^1\widetilde{M}$. The metric g induces a distance on M and $\widetilde{M}$ that we will simply denote by d. We will also denote by d the distance on T^1M (resp. on $T^1\widetilde{M}$) defined as follows: for all $v,w\in T^1M$ (resp. in $T^1\widetilde{M}$), let

$$d(v,w) := \sup_{t\in[-1,1]} d(\pi g^t v, \pi g^t w).$$

This distance is not Riemannian but it is equivalent to the standard Sasaki metric on T^1M (resp. on $T^1\widetilde{M}$), see [PPS15, Chap. 2] for a discussion on the subject. We will often make use of the following standard lemmas.

The Busemann cocycle is defined by

$$\beta_\xi(x,y) = \lim_{z\to\xi} d(x,z) - d(y,z) \tag{4}$$

We will sometimes also write, for all $x,y,z\in\widetilde{M}$,

$$\beta_z(x,y) = d(x,z) - d(y,z).$$

The set of oriented geodesics of $\widetilde{M}$ can be identified with

$$\partial^2\widetilde{M} = (\partial\widetilde{M}\times\partial\widetilde{M})\backslash\mathrm{Diag}\,.$$

For all $v\in T^1\widetilde{M}$, denote by $v^\pm$ the negative and positive endpoints in $\partial\widetilde{M}$ of the geodesic tangent to v. The unit tangent bundle $T^1\widetilde{M}$ is homeomorphic to $\partial^2\widetilde{M}\times\mathbb{R}$ via the *Hopf parametrization*

$$\mathcal{H} : \left\{ \begin{array}{ccc} T^1\widetilde{M} & \to & \partial^2\widetilde{M}\times\mathbb{R} \\ v & \mapsto & (v^-, v^+, \beta_{v^+}(o,\pi v)) \end{array} \right. . \tag{5}$$

The geodesic flow acts by translation in these coordinates: for all $v=(v^-,v^+,s)$ and $t\in\mathbb{R}$,

$$g^t(v^-,v^+,s) = (v^-,v^+,t+s)\,.$$

The group Γ acts in these coordinates by

$$\gamma(v^-,v^+,s) = \left(\gamma v^-, \gamma v^+, s+\beta_{v^+}(\gamma^{-1}o,o)\right)\,.$$

In terms of these Hopf coordinates, the nonwandering set Ω is identified with $(\Lambda^2_\Gamma\times\mathbb{R})/\Gamma$.

Recall that an isometry $\gamma\in\Gamma$ is *hyperbolic* when it admits two fixed points in $\partial\widetilde{M}$. In this case, it acts by translation on the geodesic joining them. The set $\mathcal{P}$ of periodic orbits of the geodesic flow on T^1M is in $1-1$ correspondence with the set of conjugacy classes of hyperbolic elements of Γ. Indeed, a periodic orbit p can be lifted to a collection $p_\Gamma^{-1}(p)$ of geodesics of $T^1\widetilde{M}$, and each of them, once projected on $\widetilde{M}$, is the oriented axis of a unique hyperbolic element γ_p, which acts by translation in the positive direction on the axis, with translation length equal to $\ell(p)$. By construction, all these elements are conjugated one to another.

Not all elements of Γ are hyperbolic. However, the following lemma from [PS18, lemma 2.6], variant of the well known point of view, due to Margulis, of counting elements of Γ inside cones, will allow us to consider only hyperbolic elements.

Lemma 2.1. *Let $\widetilde{M}$ be a Hadamard manifold with sectional curvature bounded from above by a negative constant. Let $\widetilde{\mathcal{K}}\subset T^1\widetilde{M}$ be a compact set whose interior intersects $\widetilde{\Omega}$. There exist finitely many elements $g_1,\dots,g_k$ depending on $\widetilde{\mathcal{K}}$ such that for every $\gamma\in\Gamma$, there exist g_i,g_j such that $g_j^{-1}\gamma g_i$ is hyperbolic, and its axis intersects $\widetilde{\mathcal{K}}$.*

Proof. By Lemma [PS18, lemma 2.6], there exists a finite set $F=\{g_1,\dots,g_k\}$ such that every $\gamma\in\Gamma\setminus S$ satisfies the conclusion of the lemma with respect to F, where $S=\{s_1,\dots,s_j\}$ is a finite set of exceptions. Consider a hyperbolic element h whose axis intersects $\widetilde{\mathcal{K}}$. Then the set $F'=\{g_1,\dots,g_k,s_1,\dots,s_j,h\}$ works for every $\gamma\in\Gamma$. Indeed, it works for $\gamma\notin S$ by assumption, and for $\gamma=s_i\in S$ then $s_i^{-1}\gamma h = h$ has an axis intersecting $\widetilde{\mathcal{K}}$, with $s_i,h\in F'$. $\square$

Let us point the following elementary lemma, that we will use many times.

Triangle **Lemma 2.2.** *Let $\widetilde{M}$ be a geodesic metric space. For all $x, y, z \in \widetilde{M}$, we have*

$$d(y,x) + d(x,z) - 2d(x,[y,z]) \leq d(y,z) \leq d(y,x) + d(x,z)\,.$$

We will often need more precise distance estimates, which rely on a negative upperbound of the curvature. The next lemma follows from [PPS15, Lemma 2.5].

v4Points **Lemma 2.3.** *Let $\widetilde{M}$ be a Hadamard manifold with sectional curvature pinched between two negative constants. For all $D > 0$ and all $\varepsilon > 0$, there exists $T_0 = T_0(D,\varepsilon) > 0$ such that if $x, x', y, y' \in T^1\widetilde{M}$ satisfy $d(x,x') \leq D$, $d(y,y') \leq D$ and $d(x,y) \geq 2T_0$, then there exists $s_0 \in [0,T_0]$ such that, if v_{xy} (resp. $v_{x'y'}$) denotes the unit tangent vector based at x (resp. x') tangent to the segment $[x,y]$ (resp. $[x',y']$), then for all $t \in [T_0, d(x,y) - T_0]$*

$$d(g^t v_{xy}, g^{t+s_0} v_{x'y'}) \leq \varepsilon.$$

We will also need the following lemma which allows to approximate broken geodesics by axes of hyperbolic elements. If $x, y \in \widetilde{M}$, let v_{xy} denote the (oriented and unitary) tangent vector of the geodesic segment $[x,y]$ at x. If $v, w \in T^1_x\widetilde{M}$, set $\measuredangle(v,w) \in (0,\pi)$ for their geometric angle. If $v \in T^1_x\widetilde{M}$ and $w \in T^1_y\widetilde{M}$, denote by $\measuredangle(v,w) \in (0,\pi)$ the geometric angle between v and the image of w through the parallel transport from y to x along $[x,y]$.

odBrisee **Lemma 2.4.** *For all $\theta \in (0,\pi)$, and all $\varepsilon > 0$, there exists $C = C(\theta,\varepsilon) > 0$ such that the following holds. Let $x, y, z, b \in \widetilde{M}$ and $\gamma \in \Gamma$ be such that $d(x,y), d(y,z)$ and $d(z,b)$ are at least $2C$, and $d(b,\gamma x) \leq 1$. Assume moreover that the angles $\measuredangle(v_{yx}, v_{yz})$, $\measuredangle(v_{zy}, v_{zb})$, and $\measuredangle(\gamma v_{xy}, v_{bz})$ are at least θ. Then γ is hyperbolic, the piecewise geodesics $[x,y] \cup [y,z] \cup [z,b]$ is in the ε-neighbourhood of its axis except in the C-neighbourhood of the points x, y, z and b. Moreover, the period T_γ of γ satisfies*

$$T_\gamma - (6C+1) \leq d(x,y) + d(y,z) + d(z,b) \leq T_\gamma + 6C + 1.$$

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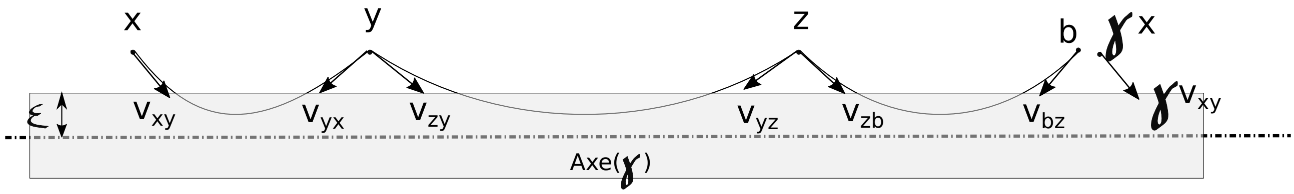

Figure 1 – Broken geodesic close to a hyperbolic axis

Sketch of proof. Since the sectional curvature are bounded from above by some $-a^2 < 0$, for all $\varepsilon > 0$ there exists $C > 0$ such that if x and b are on the same horosphere $\mathcal{H}_\xi(x)$ centered at some $\xi \in \partial\widetilde{M}$ with $d(x,b) \geq C$, then v_{xb} and v_{bx} are ε-close to the inward normal to $\mathcal{H}_\xi(x)$ at their base point. Therefore, since $\measuredangle(\gamma v_{xy}, v_{bz}) \geq \theta$ and $d(b,\gamma x) \leq 1$, the element γ cannot be parabolic as soon as $C > 0$ is large enough (depending on θ). Therefore it is hyperbolic.

The rest of the proof is an immediate adaptation of the arguments presented in [PPS15, p. 98]. $\square$

2.2 Dynamical properties of the geodesic flow

Given any vector $v \in T^1M$, its strong stable manifold is defined by

$$W^{ss}(v) = \{w \in T^1M,\ d(g^t v, g^t w) \to 0 \quad \text{when} \quad t \to +\infty\}$$

The local strong stable manifold $W^{ss}_\varepsilon(v)$ is the ε-neighbourhood of v for the induced metric on $W^{ss}(v)$ by the Riemannian metric. The strong unstable manifold $W^{su}(v)$ (resp. the local strong unstable manifold $W^{su}_\varepsilon(v)$) is defined similarly but with $t \to -\infty$.

The following result is well known. In the non-compact setting, it has been shown by Eberlein in [Ebe96, Prop. 4.5.15], see also [Cou04, CS10] for details.

Proposition 2.5. *Let M be a nonelementary complete connected negatively curved manifold with pinched negative curvature, and bounded first derivative of the curvature.*

- *The geodesic flow is* transitive *on the non-wandering set Ω: for all open sets $U, V \subset \Omega$, there exists $T > 0$ such that $g^T U \cap V \neq \emptyset$;*
- *The geodesic flow admits a* local product structure *on Ω: for all $\varepsilon > 0$, there exists $\eta > 0$ such that for all $u, v \in \Omega$ with $d(u, v) \leq \eta$, there exists $w \in \Omega$ and a real number t with $|t| \leq \varepsilon$ such that $w \in W^{ss}_{\varepsilon}(u) \cap W^{su}_{\varepsilon}(g^t v)$;*
- *The geodesic flow satisfies the* closing lemma*: for all $\varepsilon > 0$, there exists $\eta > 0$, such that for all $v \in \Omega$, and $t > 0$ such that $d(g^t v, v) \leq \eta$, there exists a periodic vector p whose period satisfies $|\ell(p) - t| \leq \varepsilon$, and for all $0 \leq s \leq t$, $d(g^t p, g^t v) \leq \varepsilon$.*

We will use several times the following proposition.

Proposition 2.6 (Connecting lemma)**.** *Let K and K' be compact sets of M whose interior intersects $\pi(\Omega)$, and $\widetilde{K} \subset \widetilde{M}$ a compact set such that $p_\Gamma(\widetilde{K}) = K$. For all $\varepsilon > 0$, there exists $T_0 = T_0(K, K', \varepsilon) > 0$ and $C_0 = C_0(\widetilde{K}, \varepsilon, T_0) > 0$ such that the following holds.*

1. *(Shadowing) For all $T \geq 2T_0$ and all $v \in T^1K$ such that $g^T v \in T^1 K$, there exists a periodic orbit $\wp = (g^t u)_{t \in \mathbb{R}}$ whose period is in $[T, T + T_0]$, that intersects the interior of T^1K', such that for all $t \in [T_0, T - T_0]$, $d(g^t v, g^t u) \leq \varepsilon$.*
2. *(Bounded multiplicity) For every periodic orbit $\wp \subset T^1M$ obtained in this way, the number of elements $\gamma \in \Gamma$ such that, for some $x, y \in \widetilde{K}$, the periodic orbit associated to the unit vector $v_{x,\gamma y}$ tangent to the loop $p_\Gamma([x, \gamma y])$, with return time $T = d(x, \gamma y)$ is bounded from above by $C_0 T = C_0 \times d(x, \gamma y)$.*

It would be a standard consequence of Proposition 2.5 in the case $v \in \Omega$. However, we wish to apply this proposition to vectors which may be wandering. Therefore we provide a more detailed proof, using Proposition 2.5 together with Lemma 2.4.

Proof. **Item 1**. The reader may follow the proof on Figure 2.2

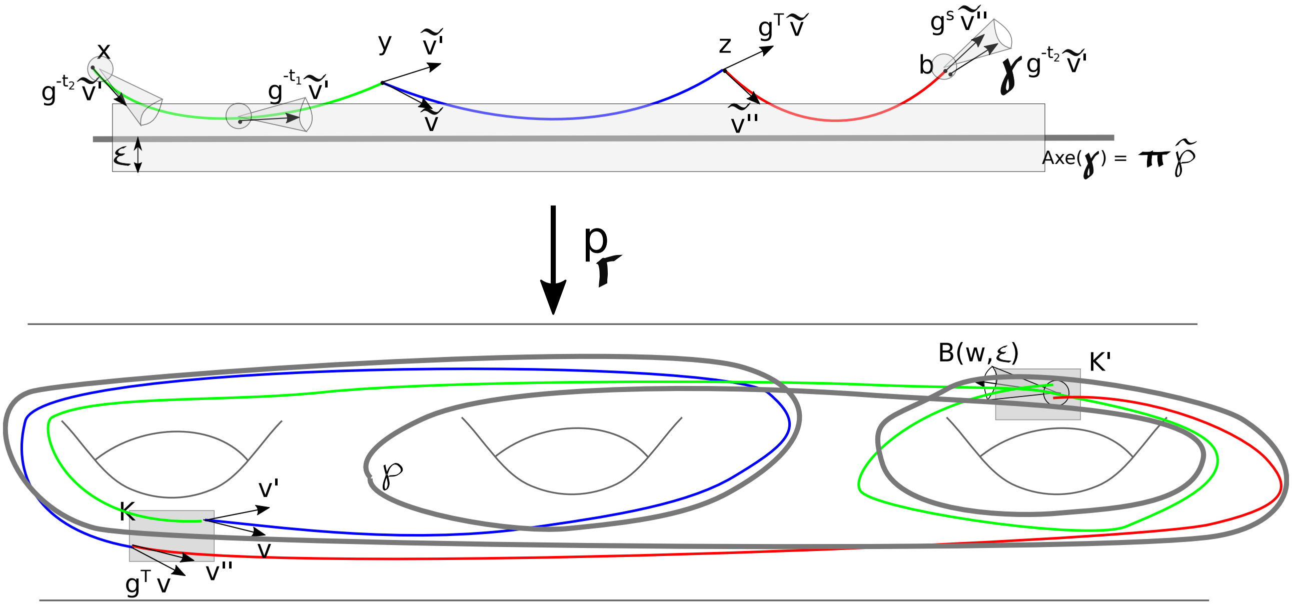

Figure 2 – Connecting lemma

We can assume that 2ε is smaller than 1 and than the injectivity radius at any point of K'. We fix once for all a vector $w \in T^1K' \cap \Omega$ such that $B(\pi w, 2\varepsilon) \subset K'$.

By compactness of $\widetilde{K}$ and Λ_Γ, there exists $\theta = \theta(K) > 0$ such that for all $y \in \widetilde{K}$, and $v \in T^1_y \widetilde{K}$, there exists $\xi \in \Lambda_\Gamma$, such that $\measuredangle(v_{y\xi}, v) \geq \theta$. As the geodesic flow is topologically transitive, and the action of Γ on Λ_Γ is minimal, we can assume moreover that the geodesic orbit on T^1M associated to $(g^t v_{y\xi})_{t\geq 0}$ is "dense in Ω" (in the sense that it contains Ω in its closure). Let $C = C(\theta, \varepsilon)$ be the constant provided by Lemma 2.4. By compactness of $T^1K \cap \Omega$ and of $T^1B(w,\varepsilon)$, a uniform property of transitivity holds, in the following sense. There exists $T_1 > 0$ such that the vector $v = v_{y\xi}$ can be chosen in such a way that $g_{[2C,T_1]}(v_{y\xi})$ intersects $B(w,\varepsilon)$. Similarly, there exists $T_2 > 0$ such that, if $v_{y\xi}$ is conveniently chosen, $g_{[T_1+2C,T_2]}(v_{y\xi})$ intersects once again $B(w,\varepsilon)$.

Let $v \in T^1K$. Set $y_0 = \pi v \in M$ and $y \in T^1\widetilde{K}$ such that $p_\Gamma(y) = y_0$. Let $\widetilde{v} \in T^1_y\widetilde{M}$ be such that $p_\Gamma(\widetilde{v}) = v$. By the above applied to $-v$, there exists $\widetilde{v}' \in T^1_y\widetilde{M}$ with $\measuredangle(\widetilde{v},\widetilde{v}') \leq \pi - \theta$ such that the half orbit $(\{g^{-t}v', t \geq 0\})$ is dense in Ω, and at two distincts times $t_1 \in [2C, T_1]$ and $t_2 \in [T_1 + 2C, T_2]$, we have $g^{-t_1}v' \in B(w,\varepsilon)$, and $g^{-t_2}v' \in B(w,\varepsilon)$. We will see below how it will be important. Set $x = \pi g^{t_2}\widetilde{v}'$.

By assumption, $g^T v \in T^1K$ for some $T \geq 0$. Set $z = \pi g^T \widetilde{v}$. By the same arguments, there exists $\widetilde{v}'' \in T^1_z\widetilde{M}$ with $\measuredangle\left(g^T\widetilde{v}, \widetilde{v}''\right) \leq \pi - \theta$ such that, if $v'' = p_\Gamma(\widetilde{v}'')$, the half orbit $(g^t v'')_{t\geq 0}$ is dense in Ω, and for some $s \in (2C, T_1)$, $g^s v'' \in B(w,\varepsilon)$. Let $b = \pi g^s \widetilde{v}''$ be the base point of $\widetilde{v}''$.

Consider now the broken geodesic $(g^t g^{-t_2}\widetilde{v}')_{0\leq t\leq t_2} \cup (g^t\widetilde{v})_{0\leq t\leq T} \cup (g^t\widetilde{v}'')_{0\leq t\leq s}$. It starts from $x = \pi(g^{-t_2}\widetilde{v}')$, has an angle at least θ at $y = \pi(\widetilde{v})$, a second angle at least θ at $z = \pi(g^T\widetilde{v})$, and finishes at $b = \pi(g^s\widetilde{v}'')$. Since $p_\Gamma(x)$ and $p_\Gamma(b)$ are both in $\pi B(w,\varepsilon)$, with ε less than the injectivity radius at πw, there exists $\gamma \in \Gamma$ such that $d(\gamma x, b) \leq \varepsilon$. Moreover, if ε is small enough, since $g^{-t_2}v' \in B(w,\varepsilon)$ and $g^s v'' \in B(w,\varepsilon)$, the angle $\measuredangle\left(\gamma g^{-t_2}\widetilde{v}', g^s\widetilde{v}''\right)$ is at most $\pi - \theta$.

Assume that $T \geq 2T_1 + T_2$. By Lemma 2.4, the broken geodesic $[x,y] \cup [y,z] \cup [z,b]$ is in the ε neighbourhood of the axis $\pi\widetilde{\wp}$ of γ, except maybe in the C-neighbourhood of x, y, z, b. But as we chose $\widetilde{v}'$ so that $g^{-t_1}\widetilde{v}' \in B(w,\varepsilon)$, the geodesic segment $p_\Gamma([x,y])$ intersects K' far from $p_\Gamma(x)$ and $p_\Gamma(y)$. In particular, since $t_1 \in (2C, d(y,x) - 2C)$ and the periodic orbit $\wp = p_\Gamma(\widetilde{\wp})$ intersects $B(w, 2\varepsilon) \subset T^1K'$. Moreover, it follows from the previous construction that the period of γ satisfies

$$T - 6C + 1 \leq \ell(\gamma) \leq T + 2T_1 + T_2 + 6C + 1\,.$$

To conclude, choose some point q on the axis of γ which projects to a point q' on $[y,z]$ with $d(q,q') \leq \varepsilon$. Let $\sigma > 0$ be such that $q' = \pi g^\sigma \widetilde{v}$. The vector u in the statement of item 1 is defined as $u = p_\Gamma(\widetilde{u})$, where $\widetilde{u}$ is a tangent vector to the axis of γ pointing in the same direction as $g^\sigma\widetilde{v}$ and defined by $\pi(g^\sigma u) = q$ and $g^\sigma u$.

Item 2. Let $\wp =\subset T^1M$ be a closed orbit obtained by the previous construction, with $\ell(p) \in [T, T+T_0]$. Assume that $T \geq T_0$. Let us bound the number of possible $\gamma \in \Gamma$ such that there exists $x, y \in \widetilde{K}$ with $d(x,\gamma y) = T$ and $\wp = \wp(v_{x,\gamma y})$.

Let $\widetilde{K}_0 \subset \widetilde{M}$ be the $(T_0+\varepsilon)$-neighbourhood of $\widetilde{K}$, and $K_0 = p_\Gamma(K_0)$. By construction, for any such $\gamma \in \Gamma$, the orbit $\wp$ has a lift $\widetilde{\wp} \subset T^1\widetilde{M}$ such that $[x, \gamma y]$ belongs to the ε-neighbourhood of $\pi\widetilde{\wp}$, except maybe in the T_0-neighbourhood of x and γy. In particular, $\pi\widetilde{\wp} \cap \widetilde{K}_0 \neq \emptyset$.

Choose such a lift $\widetilde{\wp}$, which is the axis of some hyperbolic element $g \in \Gamma$, Let $\gamma \in \Gamma$ be such that there exist $x, y \in \widetilde{K}$ with $[x,\gamma y]$ in the ε-neighbourhood of $\widetilde{\wp}$ except in $B(x,T_0) \cup B(\gamma y, T_0)$. Note that moving x and y of less than the injectivity radius ρ_K of K will not change γ. Therefore, if $C_2 = C_2(\widetilde{K}, T_0)$ is the number of balls of radius ρ_K needed to cover $\widetilde{K}_0$ (or $g\widetilde{K}_0$), the number of possible γ associated to this axis $\widetilde{\wp}$ is at most $(C_2)^2$.

It remains to bound the number of such lifts $\widetilde{\wp}$ of $\wp$. This is done in Lemma 2.7 below, and concludes the proof of item 2. $\square$

Let us denote as in [PS18], for every compact set $\widetilde{W} \subset \widetilde{M}$ and any periodic orbit $p \subset T^1M$, the number of axes of hyperbolic elements associated to the closed orbit p that intersect $\widetilde{W}$, by

$$n_{\widetilde{W}}(p) = \#\{g \in \Gamma; \exists x \in \widetilde{W},\ d(x,gx) = \ell(p) \text{ and } p_\Gamma([x,gx]) = \pi(p)\}\,.$$

It is a geometric way of estimating the number of returns of p in W.

Lemma 2.7. *For every compact set $\widetilde{W} \subset \widetilde{M}$, there exists $C_{\widetilde{W}} > 0$ such that for every periodic orbit $p \subset T^1M$,*

$$n_{\widetilde{W}}(p) \leq C_{\widetilde{W}}\ell(p).$$

Proof. First assume that $\widetilde{W} = B(x, \rho)$, where $\rho \leq \frac{\mathrm{inj}(p_\Gamma(x))}{2}$. Then

$$n_{\widetilde{W}}(p) \leq \frac{\ell(p)}{2\rho}. \tag{6}$$

Indeed, if $y, z \in B(x, \rho)$ belong to two distinct axes, say of g_y and g_z both projecting to p, there exists $\gamma \in \Gamma$ such that γz is on the axis of g_y. Since $p_\Gamma([z, y] \cup [y, \gamma z])$ is a geodesic bigone based at $p_\Gamma(y)$, each of its sides has length at least $\mathrm{inj}(y) \geq 2\rho$. Therefore $d([y, \gamma z]) \geq 2\rho$ which implies (6).

Now, let $\widetilde{W} \subset \widetilde{M}$ be an arbitrary compact set, and $\rho_{\widetilde{W}} > 0$ be half of the minimal injectivity radius in $W = p_\Gamma(\widetilde{W})$. Cover $\widetilde{W}$ by a finite number of balls of the form $B(x_i, \rho_{\widetilde{W}})$ with $x_i \in \widetilde{W}$. The result follows from [PS18, Lemma 3.2]. □

3 Thermodynamical formalism

Entropy is a well-known measure of the exponential rate of complexity of a dynamical system, and the measure of maximal entropy is an important tool in the ergodic study of hyperbolic dynamical systems.

Pressure is a weighted version of entropy, which is particularly useful for the study of perturbations of hyperbolic systems. The notion of *equilibrium state* is the weighted analogue of the measure of maximal entropy.

In this section, for the geodesic flow of noncompact negatively curved manifolds, we recall some well known notions and facts from [PPS15] and [PS18] on pressure and the construction of the *equilibrium state* or *Gibbs measure* associated with a Hölder-continuous map $F : T^1M \to \mathbb{R}$. This construction has a long story, initiated by the works of Patterson [Pat76] and Sullivan [Sul79] when $F = 0$, by Hamenstädt [Ham89] and Ledrappier [Led95]. We refer to [PPS15] for detailed historical background and proofs of the assertions in this paragraph. We follow here mainly [PPS15, Chap 3.] and [Sch04].

3.1 Hölder potentials

We follow the notations of Section 2 and [PPS15] and [PS18].

Let $F : T^1M \to \mathbb{R}$ be a Hölder-continuous map in the following sense: there exist $0 < \beta \leq 1$ and $C > 0$ such that for all $v, w \in T^1M$ with $d(v, w) \leq 1$, we have

$$|F(v) - F(w)| \leq C d(v, w)^\beta.$$

Such a map F will be said (β, C)-Hölder. Let $\widetilde{F} = F \circ p$ be the Γ-invariant lift of F to $T^1\widetilde{M}$.

Lemma 3.2 of [PPS15] and the remark (ii) page 34 which follows this lemma give the following statement.

Lemma 3.1. *Let $F : T^1M \to \mathbb{R}$ be a (β, C_F)-Hölder map on T^1M, $\widetilde{F}$ its Γ-invariant lift. Let K be a compact set of $\widetilde{M}$, with diameter D. There exist constants $c_1 > 0$ and $c_2 > 0$ depending only on the upper bound of the curvature, the Hölder constants (β, C_F) and the diameter D, such that for all $x, y \in K$, all $\gamma \in \Gamma$ and all $x', y' \in \gamma K$, we have*

$$\left| \int_x^{x'} \widetilde{F} - \int_y^{y'} \widetilde{F} \right| \leq c_1 D^{c_2} + 2D \max_{T^1 K_D} \left| \widetilde{F} \right|,$$

where K_D is the D-neighborhood of K.

3.2 Pressures of Hölder potentials

There are several natural definitions of pressure, that all coincide, as proven in [PPS15, Theorems 4.7 and 6.1], see Theorem 1.1. We recall here these three definitions.

3.2.1 Geometric pressure as a critical exponent

Recall that some point $o \in \widetilde{M}$ has been chosen once and for all. The Poincaré series associated to (Γ, F) is defined by

$$P_{\Gamma,o,F}(s) = \sum_{\gamma \in \Gamma} e^{-sd(o,\gamma o) + \int_o^{\gamma o} \widetilde{F}} .$$

The following lemma is elementary, see for instance [PPS15, p. 34-35].

Lemma 3.2 (Geometric pressure)**.** *The above series admits a* critical exponent $\delta_\Gamma(F) \in \mathbb{R} \cup \{+\infty\}$ *defined by the fact that for all $s > \delta_\Gamma(F)$ (resp. $s < \delta_\Gamma(F)$), the series $P_{\Gamma,o,F}(s)$ converges (resp. diverges). Moreover, $\delta_\Gamma(F)$ does not depend on the choice of o and satisfies for any $c > 0$,*

$$\delta_\Gamma(F) = \limsup_{T \to +\infty} \frac{1}{T} \log \sum_{\gamma \in \Gamma, T-c \leq d(o,\gamma o) \leq T} e^{\int_o^{\gamma o} \widetilde{F}} .$$

We call $\delta_\Gamma(F)$ the critical exponent *of (Γ, F) or the* geometric pressure *of F.*

As Γ is nonelementary, one can show that $\delta_\Gamma(F) > -\infty$. Moreover, observe that $\delta_\Gamma(F)$ is finite as soon as F is bounded from above. In [PPS15, thm 4.7], it has been shown that the above limsup is in fact a true limit. In what follows, we will never require F to be bounded above, but we will sometimes assume that $\delta_\Gamma(F)$ is finite.

3.2.2 Variational pressure

Let $\mathcal{M}_1$ be the set of probability measures invariant by the geodesic flow, and $\mathcal{M}_{1,\mathrm{erg}}$ the subset of ergodic probability measures. For a given Hölder potential $F : T^1M \to \mathbb{R}$, consider the subsets $\mathcal{M}_1^F$ and $\mathcal{M}_{1,\mathrm{erg}}^F$ of (ergodic) probability measures with $\int F^- \,\mathrm{d}\mu < \infty$, where $F^- = -\inf(F, 0)$ is the negative part of F. Given a probability measure μ on T^1M, invariant under the geodesic flow $(g^t)_{t\in\mathbb{R}}$, we denote by $h_{KS}(\mu) = h_{KS}(g^1, \mu)$ its *Kolmogorov-Sinai, or measure-theoretic entropy* with respect to g^1 (see the appendix for the definition).

Definition 3.3. *The* variational pressure *of F is defined by*

$$P_{\mathrm{var}}(F) = \sup_{\mu \in \mathcal{M}_1^F} h_{KS}(\mu) + \int F \,\mathrm{d}\mu = \sup_{\mu \in \mathcal{M}_{1,\mathrm{erg}}^F} h_{KS}(\mu) + \int F \,\mathrm{d}\mu .$$

3.2.3 Growth of periodic geodesics and Gurevič pressure

We denote by $\mathcal{P}$ (resp. $\mathcal{P}'$) the set of periodic (resp. primitive periodic) orbits of the geodesic flow. Let now $K \subset M$ be a compact set whose interior intersects at least a closed geodesic, and $c > 0$ be fixed. Let us denote by $\mathcal{P}_K(t)$ (resp. $\mathcal{P}_K(t-c, t)$) the set of periodic orbits $p \subset T^1M$ of the geodesic flow whose projection $\pi(p)$ on M intersects K and such that $\ell(p) \leq t$ (resp. $\ell(p) \in [t-c, t]$). The subsets $\mathcal{P}'_K$, $\mathcal{P}'_K(t)$, $\mathcal{P}'_K(t-c, t)$ of $\mathcal{P}'$ are defined similarly.

By [PPS15, thm 4.7], the definition below makes sense.

Definition 3.4 (Gurevič pressure)**.** *For any compact set $K \subset M$ whose interior intersects a closed geodesic and any $c > 0$, the* Gurevič pressure *of F is defined by*

$$P_{\mathrm{Gur}}(F) = \lim_{T \to +\infty} \frac{1}{T} \log \sum_{p \in \mathcal{P}_K(T-c,T)} e^{\int_p F} .$$

It does not depend on K nor c. Moreover, when $P_{\mathrm{Gur}}(F) > 0$, then

$$P_{\mathrm{Gur}}(F) = \lim_{T \to +\infty} \frac{1}{T} \log \sum_{p \in \mathcal{P}_K(T)} e^{\int_p F} .$$

Gurevič was the first to introduce this definition (for the potential $F = 0$) in the context of symbolic dynamics, see [Gur69]. The equality $P_{Gur}(F) = P_{var}(F)$ has been proven in [Bow72] for compact manifolds and $F = 0$, in [BR75] for compact manifolds and Hölder potentials. The equality $\delta_\Gamma(F) = P_{Gur}(F)$ is due to Ledrappier [Led95] in the compact case.

In the noncompact case, when $F \equiv 0$, Sullivan [Sul84] and Otal-Peigné [OP04] proved that $\delta_\Gamma = P_{var}$, and Roblin [Rob03] proved that $P_{Gur} = \delta_\Gamma$ The equality between the three notions of pressures for general Hölder potentials on noncompact manifolds is done in [PPS15, Thm. 4.2].

3.3 Patterson-Sullivan-Gibbs construction

Let $F : T^1M \to \mathbb{R}$ be a Hölder continuous potential with finite pressure, i.e. $\delta_\Gamma(F) < +\infty$). As will be seen in Paragraph 3.4, the construction of a good invariant measure associated to F will use the product structure $\Omega \simeq (\Lambda_\Gamma^2 \times \mathbb{R})/\Gamma$. The main step is the definition of a good measure on Λ_Γ, the so-called *Patterson-Sullivan-Gibbs measure* ν_F. We recall it below with more care than usually done, because we will need in Section 7.3 to deal with technical points of the construction.

As stated in Lemma 3.1, the Poincaré series

$$P_{\Gamma,o,F}(s) = \sum_{\gamma\in\Gamma} e^{-sd(o,\gamma o)+\int_o^{\gamma o}\widetilde{F}}$$

admits a critical exponent $\delta_\Gamma(F) \in \mathbb{R} \cup \{+\infty\}$. We say that (Γ, F) is *divergent* if this series diverges at $s = \delta_\Gamma(F)$, and *convergent* if the series converges.

Following the famous Patterson trick, see [Pat76], when (Γ, F) is convergent, we choose a positive increasing map $h : \mathbb{R}^+ \to \mathbb{R}^+$ with subexponential growth such that for all $\eta > 0$, there exist $C_\eta > 0$ and $r_\eta > 0$ such that

$$\forall r \geq r_\eta, \quad \forall t \geq 0, \quad h(t + r) \leq C_\eta e^{\eta t} h(r)\,, \tag{7}$$

and the series $\widetilde{P}_{\Gamma,F}(o, s) = \sum_{\gamma\in\Gamma} h(d(o, \gamma o))\, e^{-sd(o,\gamma o)+\int_o^{\gamma o}\widetilde{F}}$ has the same critical exponent $\delta_\Gamma(F)$, but diverges at the critical exponent $\delta_\Gamma(F)$.

Define now for all $s > \delta_\Gamma(F)$ a measure on $\widetilde{M} \cup \partial\widetilde{M}$ by

$$\nu^{F,s} = \frac{1}{\widetilde{P}_{\Gamma,F}(o, s)} \sum_{\gamma\in\Gamma} h(d(o, \gamma o)) e^{-sd(o,\gamma o)+\int_o^{\gamma o}\widetilde{F}} \mathscr{D}_{\gamma o}\,, \tag{8}$$

where $\mathscr{D}_x$ stands for the Dirac mass at x.

By compactness of $\widetilde{M} \cup \partial\widetilde{M}$, we can choose a decreasing sequence $s_k \to \delta_\Gamma(F)$ such that ν^{F,s_k} converges to a probability measure ν^F. As $\widetilde{P}_{\Gamma,o,F}$ diverges at $s = \delta_{\Gamma,F}$, we deduce that ν^F is supported on $\Lambda_\Gamma \subset \partial\widetilde{M}$.

For all $x, y \in \widetilde{M}$ and $\xi \in \partial\widetilde{M}$, recall the following notation from [Sch04, sec 2.2.1]:

$$\rho_\xi^F(x, y) = \lim_{z\in[x,\xi),\, z\to\xi} \int_x^z \widetilde{F} - \int_y^z \widetilde{F} = \text{"} \int_x^\xi \widetilde{F} - \int_y^\xi \widetilde{F}\,.$$

Observe that $\rho_\xi^0 = 0$ and more generally, when $F \equiv c$ is constant, $\rho^c = c \times \beta$, where β is the usual Busemann cocycle defined in Equation (4).

The measure ν^F satisfies the following crucial properties. For all $\gamma \in \Gamma$, and ν^F-almost all $\xi \in \partial\widetilde{M}$,

$$\frac{d\gamma_*\nu^F}{\mathrm{d}\nu^F}(\xi) = e^{-\delta_\Gamma(F)\beta_\xi(o,\gamma o)+\rho_\xi^F(o,\gamma o)}\,. \tag{9}$$

A version of this quasi-invariance property holds for the family of measures $\nu^{F,s}$. More precisely, for all $\gamma \in \Gamma$, $\delta_\Gamma(F) < s < 2\delta_\Gamma(F)$ there exists $C > 0$ and $T > 0$ such that for all $y \in \Gamma o$ with $d(o, y) \geq T$,

$$\frac{1}{C} e^{-s\beta_y(o,\gamma o)+\rho_y^F(o,\gamma o)} \leq \frac{d\gamma_*\nu^{F,s}}{\mathrm{d}\nu^{F,s}}(y) \leq C e^{-s\beta_y(o,\gamma o)+\rho_y^F(o,\gamma o)}\,. \tag{10}$$

As a consequence of (9), one gets the following key property, proved in [Moh07]. Recall that for a given set $A \subset \widetilde{M}$, the *Shadow* $\mathcal{O}_x(A)$ of A viewed from x is by definition the set of points $y \in \widetilde{M} \cup \partial\widetilde{M}$ such that the geodesic (x, y) intersects the set A.

Proposition 3.5 (Shadow Lemma)**.** *There exists $R_0 > 0$ such that for every given $R \geq R_0$, there exists a constant $C > 0$ such that for all $\gamma \in \Gamma$,*

$$\frac{1}{C}\, e^{-\delta_\Gamma(F) d(o,\gamma o) + \int_o^{\gamma o} \widetilde{F}} \leq \nu_o^F(\mathcal{O}_o(B(\gamma o, R)) \leq C e^{-\delta_\Gamma(F) d(o,\gamma o) + \int_o^{\gamma o} \widetilde{F}}\,.$$

Observe that the measure ν^F constructed above is not unique a priori, but it will be unique in all interesting cases, see Section 7.1 for details.

In fact, we will need a shadow lemma for the family of measures $\nu^{F,s}$, for $s > \delta_\Gamma(F)$. As the uniformity of the constants in the statements w.r.t. $s > \delta_\Gamma(F)$ will be crucial, we provide a detailed proof.

For $A, B \subset \widetilde{M}$ two sets, introduce the enlarged shadow $\mathcal{O}_B(A) = \cup_{x \in B} \mathcal{O}_x(A)$ as the set of points $y \in \widetilde{M} \cup \partial \widetilde{M}$ such that there exists some $x \in B$ such that the geodesic (x, y) intersects A.

Lemma 3.6 (Orbital Shadow Lemma)**.** *There exist $R_1 > 0$ and $\tau > 0$ such that for every $R \geq R_1$, every compact set $\widetilde{K} \subset \widetilde{M}$ which contains the ball $B(o, R)$, and every $\eta > 0$, there exist $r_\eta > 0$, $C > 0$, such that for all $\delta_\Gamma(F) < s \leq \delta_\Gamma(F) + \tau$ and for all $\gamma \in \Gamma$ with $d(o, \gamma o) \geq r_\eta + 2D$, we have*

$$\frac{1}{C} e^{-s d(o,\gamma o) + \int_{\gamma^{-1} o}^{o} \widetilde{F}} \leq \nu^{F,s}(\mathcal{O}_o(\gamma \widetilde{K})) \leq C e^{-(s-\eta) d(o,\gamma o) + \int_{\gamma^{-1} o}^{o} \widetilde{F}}$$

Proof. Observe first that by Lemma 2.3, for all $D > 0$ there exists $\varepsilon > 0$ such that for every compact set $\widetilde{K}$ with diameter at most D, we have the following inclusion:

$$\mathcal{O}_o(\gamma \widetilde{K}) \subset \mathcal{O}_{\widetilde{K}}(\gamma \widetilde{K}) \subset \mathcal{O}_o(\gamma \widetilde{K}_\varepsilon)\,.$$

We follow the classical proof of the Shadow lemma, with $\nu^{F,s}$ on $\widetilde{M}$ instead of ν^F on $\partial \widetilde{M}$. By definition, for all $y \in \widetilde{M}$ we have

$$\frac{d(g_* \nu^{F,s})}{\mathrm{d}\nu^{F,s}}(y) = \frac{h(d(go, y))}{h(d(o, y))} e^{-s(d(go,y) - d(o,y)) + \int_{go}^{y} \widetilde{F} - \int_o^y \widetilde{F}}\,.$$

We deduce that

$$\nu^{F,s}\left(\mathcal{O}_o(\gamma \widetilde{K})\right) = \gamma_*^{-1} \nu^{F,s}(\mathcal{O}_{\gamma^{-1} o}(\widetilde{K})) = \int_{\mathcal{O}_{\gamma^{-1} o}(\widetilde{K})} \frac{h(d(\gamma^{-1} o, y))}{h(d(o, y))} e^{-s(d(\gamma^{-1} o,y) - d(o,y)) + \int_{\gamma^{-1} o}^{y} \widetilde{F} - \int_o^y \widetilde{F}} \mathrm{d}\nu^{F,s}\,.$$

The triangular inequality gives $d(\gamma^{-1} o, y) \leq d(\gamma^{-1} o, o) + d(o, y)$. Moreover, as $o \in \widetilde{K}$ and $y \in \mathcal{O}_{\gamma^{-1} o}(\widetilde{K})$, by Lemma 2.2, we have $d(\gamma^{-1} o, y) \geq d(\gamma^{-1} o, o) + d(o, y) - 2D$. By construction, the map h is increasing and for all $\eta > 0$, there exists $r_\eta > 0$ such that for $r \geq r_\eta$, $t \geq 0$, $h(t + r) \leq C_\eta e^{\eta t} h(r)$. Thus, if $d(\gamma^{-1} o, o) \geq r_\eta + 2D$, then $d(\gamma^{-1} o, y) \geq r_\eta$, so that independently of $s > \delta_\Gamma(F)$, we have

$$1 \leq \frac{h(d(\gamma^{-1} o, y))}{h(d(o, y))} \leq \frac{h(d(\gamma^{-1} o, o) + d(o, y))}{h(d(o, y))} \leq C_\eta e^{\eta d(\gamma^{-1} o, o)}\,.$$

As the curvature of $\widetilde{M}$ is bounded from above by a negative constant, triangles are thin, see Lemma 2.2. Thus, by Lemmas 2.3 and 3.1, there exists a positive constant $C(F, \widetilde{K}, \varepsilon)$, such that uniformly in $y \in \mathcal{O}_o(\gamma \widetilde{K}_\varepsilon)$ and $s > \delta_\Gamma(F)$, we have

$$|d(\gamma^{-1} o, y) - d(o, y) - d(\gamma^{-1} o, o)| \leq 2D + 2\varepsilon \quad \text{and} \quad \left| \int_{\gamma^{-1} o}^{y} \widetilde{F} - \int_o^y \widetilde{F} - \int_{\gamma^{-1} o}^{o} \widetilde{F} \right| \leq C(F, \widetilde{K}, \varepsilon)\,.$$

We deduce that for some positive constant $C > 0$,

$$\nu^{F,s}\left(\mathcal{O}_o(\gamma \widetilde{K}_\varepsilon)\right) \leq C_\eta e^{sC} e^{-s d(\gamma^{-1} o, o) + \int_{\gamma^{-1} o}^{o} \widetilde{F}} \times \nu^{F,s}(\mathcal{O}_{\gamma^{-1} o}(\widetilde{K}_\varepsilon)) \leq C_\eta e^{2\delta_\Gamma(F) C} e^{-s d(\gamma^{-1} o, o) + \int_{\gamma^{-1} o}^{o} \widetilde{F}}\,.$$

For the lower bound, we have

$$\nu^{F,s}(\mathcal{O}_o\left(\gamma \widetilde{K}\right)) \geq e^{-sC} e^{-s d(\gamma^{-1} o, o) + \int_{\gamma^{-1} o}^{o} \widetilde{F}} \times \nu^{F,s}(\mathcal{O}_{\gamma^{-1} o}(\widetilde{K}))\,.$$

The crucial point is to get a lower bound of this measure. We write

$$\nu^{F,s}\left((\mathcal{O}_{\gamma^{-1}o}(\widetilde{K})\right) \geq \liminf_{s\to\delta_\Gamma(F)} \inf_{y\in\widetilde{M}\cup\partial\widetilde{M}} \nu^{F,s}(\mathcal{O}_y(\widetilde{K}) \geq \liminf_{s\to\delta_\Gamma(F)} \inf_{y\in\widetilde{M}\cup\partial\widetilde{M}} \nu^{F,s}(\mathcal{O}_y(\widetilde{K})\,.$$

Let us show that if $\widetilde{K}$ is large enough, this infimum is positive. The usual argument which concludes the proof of the classical Shadow Lemma is as follows. Imagine that it is not the case. Assume that $\widetilde{K}$ is a ball $B(o,R)$, for R arbitrarily large, so that

$$\nu^{F,s}(\mathcal{O}_{\gamma^{-1}o}(B(o,R)) \geq \liminf_{s\to\delta_\Gamma(F)} \inf_{y\in\widetilde{M}\cup\partial\widetilde{M}} \nu^{F,s}(\mathcal{O}_y(B(o,R))\,.$$

As ν^F has support the full limit set Λ_Γ, it is not a purely atomic measure. In particular, for all $y\in\partial\widetilde{M}$, $\nu^F(\mathcal{O}_y(B(o,R))\to\nu^F(\partial\widetilde{M}\setminus\{y\})\geq 1-\alpha$, where α is the largest mass of an atom of ν^F. Therefore, there exists $R_0>0$ such that for $R\geq R_0$ large enough, uniformly in y, we have $\nu^F(\mathcal{O}_y(B(o,R))\geq 1-\frac{\alpha}{2}$.

Suppose by contradiction that for all $R>0$, the above $\liminf$ is zero. It would mean that there exists $s_n\to\delta_\Gamma(F)$, $R_n\to\infty$ and $y_n\to y_\infty\in\partial\widetilde{M}$ such that $\nu^{F,s_n}(\mathcal{O}_{y_n}(\widetilde{B}(o,R_n))\to 0$. The sequence s_n is not necessarily the sequence along which $\nu^{F,s}$ converges to ν^F but we don't care. There exists a subsequence s_{n_k} such that $\nu^{F,s_{n_k}}$ converges to some measure ν' on the limit set which is also supported on the full limit set. The above classical argument gives a contradiction. □

ec:Gibbs

3.4 Gibbs measures

Let $F: T^1M\to\mathbb{R}$ be a Hölder potential with finite pressure, and let ν^F be a Patterson-Sullivan measure associated to F, as constructed in the previous paragraph.

Denote by $\iota: T^1M\to T^1M$ the involution $v\to -v$, and let $\nu^{F\circ\iota}$ be a Patterson-Sullivan measure associated to $F\circ\iota$. Hopf coordinates allow us to define a Radon measure on $T^1\widetilde{M}$ by the formula

$$d\widetilde{m}^F(v) = e^{\delta_\Gamma(F)\beta_{v^-}(o,\pi(v)) - \rho^{F\circ\iota}_{v^-}(o,\pi(v)) + \delta_\Gamma(F)\beta_{v^+}(o,\pi(v)) - \rho^F_{v^+}(o,\pi(v))}\, d\nu_o^{F\circ\iota}(v_-)\, d\nu_o^F(v_+)\, dt\,. \qquad (11)$$ Gibbs-pr

By construction, $\widetilde{m}^F$ is invariant under the geodesic flow and it follows from (9) that it is invariant under the action of Γ on $T^1\widetilde{M}$, so that it induces a Radon measure m_F on T^1M.

The following crucial result was shown in [OP04] for $F=0$ and in [PPS15, Chap. 6] in general.

eo:Gibbs **Theorem 3.7** ([OP04]–[PPS15])**.** *Let M be a nonelementary complete connected negatively curved manifold with sectional curvatures pinched between two negative constants and bounded first derivative of the curvature. Let $F: T^1M\to\mathbb{R}$ be a Hölder-continuous potential with finite pressure. Then the following alternative holds. If a measure m^F on T^1M given by the Patterson-Sullivan-Gibbs construction is finite, then (once normalized into a probability measure) it is the unique probability measure realizing the supremum in the variational principle:*

$$P(F) = \sup_{\mu\in\mathcal{M}_1^F} h_{KS}(\mu) + \int_{T^1M} F\, d\mu = h_{KS}(m^F) + \int_{T^1M} F\, dm^F\,.$$

If such a measure m^F is infinite, then there is no probability measure realizing this supremum.

We will also need the following result, called *Hopf-Tsuji-Sullivan-Roblin* Theorem, see [PPS15, Theorem 5.3] for a more complete statement and a proof.

theo:HTS **Theorem 3.8** (Hopf-Tsuji-Sullivan theorem, [PPS15])**.** *Let M be a nonelementary complete connected negatively curved manifold with sectional curvatures pinched between two negative constants and bounded first derivative of the curvature. The following assertions are equivalent.*

1. *The pair (Γ,F) is divergent, i.e., the Poincaré series $P_{\Gamma,o,F}(s)$ diverges at the critical exponent $\delta_\Gamma(F)$;*
2. *the measure ν^F gives positive measure to the radial limit set $\nu^F(\Lambda_\Gamma^{\mathrm{rad}})>0$;*

3. *the measure ν^F gives full measure to the radial limit set $\nu^F(\Lambda_\Gamma^{\mathrm{rad}}) = 1$;*
4. *the measure m^F is conservative for the action of the geodesic flow on T^1M;*
5. *the measure m^F is ergodic and conservative for the action of the geodesic flow on T^1M.*

Together with the above Hopf-Tsuji-Sullivan Theorem, the Poincaré recurrence Theorem implies the following crucial observation:

When the measure m^F is finite, it is ergodic and conservative.

4 Pressures at infinity

In this section, we recall first the notion of *fundamental group outside a compact set* introduced in [PS18]. Then, to each of the three notions of pressures recalled in section 3.2, we associate now a natural notion of *pressure at infinity.*

4.1 Fundamental group outside a given compact set

For any compact set $\widetilde{K} \subset \widetilde{M}$, as in [PS18, ST19, CDST19] we define the *fundamental group outside* $\widetilde{K}$, denoted by $\Gamma_{\widetilde{K}}$ as

$$\Gamma_{\widetilde{K}} = \left\{ \gamma \in \Gamma, \exists x, y \in \widetilde{K},\ [x, \gamma y] \cap \Gamma\widetilde{K} \subset \widetilde{K} \cup \gamma\widetilde{K} \right\} .$$

Considering the last point on such a geodesic segment in $\widetilde{K}$, and the first point in $\gamma\widetilde{K}$, it follows that this set can equivalently be written as

$$\Gamma_{\widetilde{K}} = \left\{ \gamma \in \Gamma, \exists x, y \in \widetilde{K},\ [x, \gamma y] \cap \Gamma\widetilde{K} = \{x, \gamma y\} \right\} .$$

This subset of Γ corresponds to long excursions of geodesics outside of K. We stress that this is not a subgroup in general, see examples in [ST19, Section 7].

Recall from [ST19, Prop. 7.9] and [ST19, prop 7.7] the following results.

Proposition 4.1. *Let M be a nonelementary complete connected Riemannian manifold with pinched negative curvature.*

1. *Let $\widetilde{K} \subset \widetilde{M}$ be a compact set, and $\alpha \in \Gamma$. Then $\Gamma_{\alpha\widetilde{K}} = \alpha\Gamma_{\widetilde{K}}\alpha^{-1}$.*
2. *If $\widetilde{K}_1$ and $\widetilde{K}_2$ are compact sets of $\widetilde{M}$ such that $\widetilde{K}_1$ is included in the interior of $\widetilde{K}_2$, then there exist finitely many $\alpha_1, \ldots, \alpha_k \in \Gamma$ such that $\Gamma_{\widetilde{K}_2} \subset \bigcup_{i,j=1}^{k} \alpha_i \Gamma_{\widetilde{K}_1} \alpha_j^{-1}$.*

In some circumstances, it may be useful to consider different Riemannian structures (M, g_0) and (M, g) on the same manifold, and compare their fundamental groups outside a given compact set, denoted $\Gamma^{g_0}_{\widetilde{K}}$ and $\Gamma^{g}_{\widetilde{K}}$ to avoid confusions.

Proposition 4.2. *Let (M, g_0) be a nonelementary complete Riemannian manifold with pinched negative curvature. Let $\widetilde{K} \subset \widetilde{M}$ be a compact set. Let g be another complete Riemannian metric which coincides with g_0 outside $p_\Gamma(\widetilde{K})$. Then*

$$\Gamma^{g}_{\widetilde{K}} = \Gamma^{g_0}_{\widetilde{K}} .$$

4.2 Critical exponent at infinity

Consider the associated restricted Poincaré series

$$P_{\Gamma_{\widetilde{K}}}(s, F) = \sum_{\gamma \in \Gamma_{\widetilde{K}}} e^{-sd(o,\gamma o) + \int_o^{\gamma o} \widetilde{F}} .$$

Its critical exponent, denoted by $\delta_{\Gamma_{\widetilde{K}}}(F)$, satisfies for all $c > 0$

$$\delta_{\Gamma_{\widetilde{K}}}(F) = \limsup_{t \to +\infty} \frac{1}{t} \log \sum_{\gamma \in \Gamma_{\widetilde{K}}, t-c \leq d(o,\gamma o) \leq t} e^{\int_o^{\gamma o} \widetilde{F}}.$$

We call it the *critical exponent* or *geometric pressure of* F *outside* $\widetilde{K}$. By construction,

$$\delta_{\Gamma_{\widetilde{K}}}(F) \leq \delta_\Gamma(F).$$

Definition 4.3. *The* critical exponent at infinity *or* geometric pressure at infinity *of F is defined as*

$$\delta_\Gamma^\infty(F) = \inf_{\widetilde{K}} \delta_{\Gamma_{\widetilde{K}}}(F),$$

where the infimum is taken over all compact sets $\widetilde{K} \subset \widetilde{M}$.

An immediate corollary of Proposition 4.1 is the following.

Corollary 4.4. *Let M be a nonelementary complete connected Riemannian manifold with pinched negative curvature.*

1. *Let $\widetilde{K} \subset \widetilde{M}$ be a compact set, and $\alpha \in \Gamma$. Then $\delta_{\Gamma_{\alpha\widetilde{K}}}(F) = \delta_{\Gamma_{\widetilde{K}}}(F)$.*
2. *If $\widetilde{K}_1$ and $\widetilde{K}_2$ are compact sets of $\widetilde{M}$ such that $\widetilde{K}_1$ is included in the interior of $\widetilde{K}_2$, then*

$$\delta_{\Gamma_{\widetilde{K}_2}}(F) \leq \delta_{\Gamma_{\widetilde{K}_1}}(F).$$

Corollary 4.4 implies for any Hölder potential F the very convenient following fact:

$$\delta_\Gamma^\infty(F) = \lim_{R \to +\infty} \delta_{\Gamma_{B(o,R)}}(F). \tag{12}$$

It is worth noting that this critical exponent at infinity can be equal to $-\infty$, in particular in the trivial situations described in the following lemma, where *all* potentials have critical exponent at infinity equal to $-\infty$.

Lemma 4.5. *Let M be a compact or convex-cocompact Riemannian manifold with pinched negative curvature. Then, for every Hölder potential $F : T^1M \to \mathbb{R}$,*

$$\delta_\Gamma^\infty(F) = -\infty.$$

Proof. By [ST19, Prop. 7.17], for $\widetilde{K} \subset \widetilde{M}$ large enough, the set $\Gamma_{\widetilde{K}}$ is finite. It immediately implies

$$\delta_\Gamma^\infty(F) \leq \delta_{\Gamma_{\widetilde{K}}}(F) = -\infty.$$

□

We refer to Corollary 7.6 for more interesting situations where $\delta_\Gamma^\infty(0) \geq 0$ and there exists a Hölder map $F : T^1M \to \mathbb{R}$ with $\delta_\Gamma^\infty(F) = -\infty$.

4.3 Variational pressure at infinity

Recall that the *vague topology* on the space of Radon measures on T^1M is the weak-* topology on the space of Radon measures viewed as the dual of the space $C_c(T^1M)$ of continuous maps with compact support on T^1M. A sequence of probability measures $(\mu_n)_{n\in\mathbb{N}}$ converges to 0 for the vague topology if and only if for every map $\varphi \in C_c(T^1M)$, it satisfies $\lim_{n\to+\infty} \int \varphi \, \mathrm{d}\mu_n = 0$. We write this $\mu_n \rightharpoonup 0$. This provides the following other natural notion of pressure at infinity.

Definition 4.6. *Let F be a Hölder potential with finite pressure on T^1M. The* variational pressure at infinity *of F is*

$$\begin{aligned}
P^\infty_{\mathrm{var}}(F) &= \sup\left\{\lim_{n\to+\infty} h_{KS}(\mu_n) + \int_{T^1M} F\,\mathrm{d}\mu_n \ ;\ (\mu_n)_{n\in\mathbb{N}} \in \left(\mathcal{M}_1^F\right)^{\mathbb{N}} \ s.t.\ \mu_n \rightharpoonup 0\right\} \\
&= \lim_{\varepsilon\to 0} \inf_{K\subset M, K \text{ compact}} \sup\left\{h_{KS}(\mu) + \int_{T^1M} F\,\mathrm{d}\mu \ ;\ \mu\in\mathcal{M}_1^F \ s.t.\ \mu(T^1K)\le\varepsilon\right\} \\
&= \inf_{K\subset M, K \text{ compact}} \lim_{\varepsilon\to 0} \sup\left\{h_{KS}(\mu) + \int_{T^1M} F\,\mathrm{d}\mu \ ;\ \mu\in\mathcal{M}_1^F \ s.t.\ \mu(T^1K)\le\varepsilon\right\}.
\end{aligned}$$

It is a standard exercise to check that these three definitions coincide:

Proof. The limit in ε in the last two lines is a decreasing limit, i.e., an infimum, so it commutes with the infimum over K. Hence, it suffices to show that the quantity on the first line, say A, coincides with the quantity on the second line, say B. If a sequence μ_n realizes the supremum in A, then for any $\varepsilon > 0$ and for any compact set K, one has eventually $\mu_n(T^1K) \le \varepsilon$ by definition of the vague convergence to 0. Therefore, $A \le B$. Conversely, consider sequences ε_n and K_n realizing the infimum in B. Since decreasing ε_n and increasing K_n can only make the infimum smaller, it follows that $\varepsilon'_n = \min(\varepsilon_n, 1/n)$ and $K'_n = K_n \cup B(o,n)$ also realize the infimum in B. We get a sequence of measures $\mu_n \in \mathcal{M}_1^F$ with $\mu_n(T^1K'_n) \le \varepsilon'_n$ and $h_{KS}(\mu_n) + \int_{T^1M} F\,\mathrm{d}\mu_n \to B$. Since $T^1K'_n$ increases to cover the whole space and ε'_n tends to 0, we have $\mu_n \rightharpoonup 0$. Therefore, $B \le A$. □

From a dynamical point of view, it would be more natural, and apparently more general to consider all compact sets $\mathcal{K}$ of T^1M, instead of restricting to unit tangent bundles $\mathcal{K} = T^1K$ of compact sets of M. However, the equality between the three above quantities shows that it would not bring anything to the definition.

In the case $F \equiv 0$, in the context of symbolic dynamics, this definition already appeared in different works, see for example [GS98, Rue03, BBG06, BBG14].

One can consider a variation around the above definition, requiring additionally that all the measures μ_n are ergodic. We will denote this pressure by $P^\infty_{\mathrm{var,erg}}(F)$. We will see in Corollary 6.12 that it coincides with $P^\infty_{\mathrm{var}}(F)$, as a byproduct of the proof of Theorem 1.2.

4.4 Gurevič pressure at infinity

To the Gurevič pressure is naturally associated a notion of *Gurevič pressure at infinity*, when considering only periodic orbits that spend an arbitrarily small proportion of their period in a given compact set. This only makes sense for compact sets on T^1M whose interior intersects the nonwandering set Ω. As in the preceding sections, we consider only compact sets K on M or $\widetilde{M}$, so that we require that the interior of K, denoted by $\overset{\circ}{K}$, intersects the projection $\pi(\Omega)$ of the nonwandering set on M.

Definition 4.7. *Let F be a Hölder potential on T^1M. For any $c > 0$, the* Gurevič pressure at infinity *of F is*

$$\begin{aligned}
P^\infty_{\mathrm{Gur}}(F) &= \inf_{\substack{K\subset M, K \text{ compact} \\ \overset{\circ}{K}\cap\pi(\Omega)\neq\emptyset}} \lim_{\alpha\to 0} \limsup_{T\to+\infty} \frac{1}{T}\log \sum_{p\in\mathcal{P}_K(T-c,T)\ ;\ \ell(p\cap T^1K)<\alpha\ell(p)} e^{\int_p F} \\
&= \lim_{\alpha\to 0} \inf_{\substack{K\subset M, K \text{ compact} \\ \overset{\circ}{K}\cap\pi(\Omega)\neq\emptyset}} \limsup_{T\to+\infty} \frac{1}{T}\log \sum_{p\in\mathcal{P}_K(T-c,T)\ ;\ \ell(p\cap T^1K)<\alpha\ell(p)} e^{\int_p F}.
\end{aligned}$$

It does not depend on c.

It is not completely obvious from the definition what happens when one increases a compact set K' to a larger compact set K. Since one may consider orbits that intersect K but not K', one is allowed more orbits. However, the condition $\ell(p \cap T^1K) < \alpha\ell(p)$ becomes more restrictive for K than for K', allowing less orbits. These two effects pull in different directions. It turns out that the latter effect, allowing less orbits, is stronger. We formulate this statement with a third compact set K'' as we will need it later on in this form, but for the previous discussion you may take $K' = K''$.

Proposition 4.8. *Consider three compact sets K'', K', K of M such that the interior of K'' intersects a closed geodesic, and K' is contained in the interior of K. Then, for $\alpha > 0$,*

$$\limsup_{T\to+\infty} \frac{1}{T} \log \sum_{p\in\mathcal{P}_K(T-c,T)\,;\,\ell(p\cap T^1K)<\alpha\ell(p)} e^{\int_p F} \\ \leq \limsup_{T\to+\infty} \frac{1}{T} \log \sum_{p\in\mathcal{P}_{K''}(T-c,T)\,;\,\ell(p\cap T^1K')<2\alpha\ell(p)} e^{\int_p F}.$$

Therefore, the infimum in the definition of the Gurevič pressure may be realized by taking an increasing sequence of balls, just like in Corollary 4.4.

Proof. Consider a periodic orbit p of length $\ell(p)$ starting from $x \in T^1K$, parametrized by $[0, \ell(p)]$. Let also $\varepsilon > 0$. By the Connecting Lemma (Proposition 2.6) there is another periodic orbit p' starting close to x, of length $\ell(p') \in [\ell(p), \ell(p)+C]$ for a constant C depending on K and K'' and ε, parametrized by $[0, \ell(p')]$, following p within ε during the interval of time $[0, \ell(p)]$, and intersecting T^1K''. Lemma 3.1 shows that there exists a constant C' such that $|\int_p F - \int_{p'} F| \leq C'$.

Moreover, still by Proposition 2.6, there exists $C'' = C''(K, C) > 0$ such that the number of closed orbits p with length less than T and which gives the same orbit p' is at most $C'T$.

If ε is such that the ε-neighborhood of K' is included in K, then the times at which p' belongs to T^1K' are of two kind: either they are in $[\ell(p), \ell(p')]$, or they are in $[0, \ell(p)]$ and then the corresponding point on p belongs to T^1K. Hence, $\ell(p''\cap T^1K') \leq C + \ell(p\cap T^1K)$. Taking into account the multiplicity, we obtain

$$\sum_{p\in\mathcal{P}_K(T-c,T)\,;\,\ell(p\cap T^1K)<\alpha\ell(p)} e^{\int_p F} \leq C''T \sum_{p'\in\mathcal{P}_{K'}(T-c,T+C)\,;\,\ell(p''\cap T^1K')<C+\alpha\ell(p')} e^{C'+\int_{p'} F}.$$

When T is large enough, we have $\alpha\ell(p') + C < 2\alpha\ell(p')$. We obtain

$$\limsup_{T\to+\infty} \frac{1}{T} \log \sum_{p\in\mathcal{P}_K(T-c,T)\,;\,\ell(p\cap T^1K)<\alpha\ell(p)} e^{\int_p F} \\ \leq \limsup_{T\to+\infty} \frac{1}{T} \log \sum_{p'\in\mathcal{P}_{K'}(T-c,T+C)\,;\,\ell(p'\cap T^1K')<2\alpha\ell(p')} e^{\int_{p'} F}. \qquad \square$$

4.5 All pressures at infinity coincide

In Sections 5 and 6, we will show Theorem 1.2, that is that the three notions of pressure at infinity coincide:

$$\delta_\Gamma^\infty(F) = P_{\mathrm{var}}^\infty(F) = P_{\mathrm{Gur}}^\infty(F)\,.$$

4.6 Pressure at infinity is invariant under compact perturbations

In this paragraph, we will show that the critical exponent at infinity is invariant under any compact perturbation of the potential or of the underlying metric.

Proposition 4.9. *Let M be a nonelementary complete connected Riemannian manifold with pinched negative curvature and bounded first derivative of the metric. Let $F : T^1M \to \mathbb{R}$ be a Hölder map with finite pressure, let $A : T^1M \to \mathbb{R}$ be a Hölder map, and let $\widetilde{K} \subset \widetilde{M}$ be a compact set such that A vanishes outside of $p_\Gamma(T^1\widetilde{K})$. Then*

$$\delta_{\Gamma_{\widetilde{K}}}(F + A) = \delta_{\Gamma_{\widetilde{K}}}(F).$$

In particular,

$$\delta_\Gamma^\infty(F + A) = \delta_\Gamma^\infty(F)\,.$$

Proof. Set $D = \mathrm{diam}(\widetilde{K})$. By definition, for all $\gamma \in \Gamma_{\widetilde{K}}$, there exist $x, y \in \widetilde{K}$ such that the geodesic segment $[x, \gamma y]$ spends at most a time $2D$ in $\Gamma\widetilde{K}$. We deduce that

$$\left| \int_x^{\gamma y} \widetilde{F} + A - \int_x^{\gamma y} \widetilde{F} \right| \leq 2D\|A\|_\infty .$$

By Lemma 3.1, we deduce that

$$\left| \int_o^{\gamma o} \widetilde{F} + A - \int_o^{\gamma o} \widetilde{F} \right| \leq 2D\|A\|_\infty + 2C(F, D, A) .$$

By definition of $\delta_{\Gamma_{\widetilde{K}}}(F)$ and $\delta_{\Gamma_{\widetilde{K}}}(F + A)$, the result follows immediately. $\square$

In the next proposition, we consider two negatively curved Riemannian metrics g_0 and g on M, and still denote by g_0 and g their lifts to $\widetilde{M}$. For a given potential $F : TM \to \mathbb{R}$, denote by $\delta_{\Gamma_{\widetilde{K}}, g_0}(F)$, $\delta_{\Gamma_{\widetilde{K}}, g}(F)$, $\delta^\infty_{\Gamma, g_0}(F)$, $\delta^\infty_{\Gamma, g}(F)$ the associated critical exponents.

Proposition 4.10. *Let (M, g_0) be a Riemannian manifold with pinched negative curvature, and g be another negatively curved metric on M. Let $F : T^1M \to \mathbb{R}$ be a Hölder potential. Let $\widetilde{K} \subset \widetilde{M}$ be a compact set such that g and g_0 coincide outside of $p_\Gamma(\widetilde{K})$. Then*

$$\delta_{\Gamma_{\widetilde{K}}, g_0}(F) = \delta_{\Gamma_{\widetilde{K}}, g}(F).$$

In particular, $\delta^\infty_{\Gamma, g_0}(F) = \delta^\infty_{\Gamma, g}(F)$.

Proof. When necessary, denote by $[a, b]^g$ or $[a, b]^{g_0}$ the geodesic segment of the metric g (resp. g_0) between a and b. By Proposition 4.2, we have $\Gamma^{g_0}_{\widetilde{K}} = \Gamma^g_{\widetilde{K}}$. Let $\gamma \in \Gamma_{\widetilde{K}}$. There exist $x, y \in \widetilde{K}$ such that $[x, \gamma y]^{g_0} \cap \Gamma\widetilde{K} = \{x, \gamma y\}$.

Outside $\Gamma\widetilde{K}$, the metrics g_0 and g coincide, so that the segments $[x, \gamma y]^g$ and $[x, \gamma y]^{g_0}$ are the same, and the integrals of F coincide: $\int_{[x,\gamma y]^g} \widetilde{F} = \int_{[x,\gamma y]^{g_0}} \widetilde{F}$.

Moreover, by compactness, there exists $D > 0$ depending on $\widetilde{K}$, g_0 and g, such that for both metrics, $d^{g_0}(x, o) \leq D$, $d^g(x, o) \leq D$, $d^{g_0}(y, o) \leq D$, and $d^g(y, o) \leq D$. Therefore, using Lemma 3.1, there exists a constant C depending on D and $\sup_{\widetilde{K}}(\widetilde{F})$ such that for both metrics, we have

$$\left| \int_{[o,\gamma o]^g} \widetilde{F} - \int_{[x,\gamma y]^g} \widetilde{F} \right| \leq C \quad \text{and} \quad \left| \int_{[o,\gamma o]^{g_0}} \widetilde{F} - \int_{[x,\gamma y]^{g_0}} \widetilde{F} \right| \leq C .$$

The result follows by definition of the critical pressure. $\square$

Compact perturbations of a given potential do not change the critical exponent at infinity, but modify the critical pressure, as shown in the next proposition. This kind of statement, very useful, is relatively classical, and similar statements in symbolic dynamics or on geometrically finite manifolds, or for potentials converging to 0 at infinity can be found for example in [IRV18, RV19].

Proposition 4.11. *Let M be a nonelementary complete connected Riemannian manifold with pinched negative curvature and bounded first derivative of the curvature. Let $F : T^1M \to \mathbb{R}$ be a Hölder continuous potential, and $A : T^1M \to [0, +\infty)$ a non-negative Hölder map with compact support. The map*

$$\lambda \in \mathbb{R} \to \delta_\Gamma(F + \lambda A)$$

is continuous, Lipschitz, convex, nondecreasing, and as soon as the interior of A intersects the non-wandering set Ω, we have $\lim_{\lambda \to \infty} \delta_\Gamma(F + \lambda A) = +\infty$.

Proof. The fact that it is Lipschitz-continuous is an immediate consequence of the definition, and that it is nondecreasing is obvious as $A \geq 0$. Convexity follows from the variational principle (Theorem 1.1) because it is a supremum of affine maps.

Now, if the interior of A intersects Ω, there will be at least an invariant probability measure μ with compact support (supported by a periodic orbit intersecting A for example) such that $\int A \,\mathrm{d}\mu > 0$. By the variational principle,

$$\delta_\Gamma(F + \lambda A) \geq h_{KS}(\mu) + \int F \,\mathrm{d}\mu + \lambda \int A \,\mathrm{d}\mu\,,$$

and the latter quantity goes to $+\infty$ when $\lambda \to +\infty$. The result follows. $\square$

The combination of Propositions 4.9 and 4.11 provides the following corollary, which will become relevant in Section 7.

Corollary 4.12. *Let F and $A : T^1M \to \mathbb{R}$ be two Hölder continuous potentials. Assume that A is non-negative, compactly supported, and not everywhere zero on the non-wandering set. Then for $\lambda > 0$ large enough, we have*

$$\delta_\Gamma(F + \lambda A) > \delta_\Gamma^\infty(F + \lambda A).$$

4.7 Infinite pressure

In this paragraph, we prove that if the pressure of a potential is infinite, then its pressure at infinity is also infinite. This is not surprising: everything coming from a compact set is finite, so if the pressure is infinite the major contribution has to come from the complement of compact sets, and therefore the pressure outside any compact set should also be infinite. However, the proof is not completely trivial. It will involve careful splittings of orbits and subadditivity, two themes that will also show up in later proofs. One may think of this proof as a warm-up for the next sections.

Proposition 4.13. *Let M be a nonelementary complete connected Riemannian manifold with pinched negative curvature and bounded first derivative of the curvature. Let $F : T^1M \to \mathbb{R}$ be a Hölder continuous potential with $\delta_\Gamma(F) = +\infty$. Then $\delta_\Gamma^\infty(F) = +\infty$.*

Proof. We will prove the contrapositive, namely, if there exists a compact set $\widetilde{K}$ of $\widetilde{M}$ with $\delta_{\Gamma_{\widetilde{K}}}(F) < \infty$ then $\delta_\Gamma(F) < \infty$. Adding o to $\widetilde{K}$ if necessary, we can assume $o \in \widetilde{K}$. Fix some $s > \delta_{\Gamma_{\widetilde{K}}}(F)$. Let D be the diameter of $\widetilde{K}$.

Let $u_n = \sum_{\gamma : d(o,\gamma o) \in (n-1,n]} e^{\int_o^{\gamma o} \widetilde{F}}$. We claim that there exists $C > 0$ such that, for all n,

$$u_n \leq C \sum_{\substack{1 \leq a,b \leq n-1 \\ |a+b-n| \leq C}} u_a u_b + C e^{sn}. \tag{13}$$

Let us prove the result assuming this inequality, by a subadditivity argument. Extend u_n by 0 on $(-\infty, -1]$, and define a new sequence $v_n = \sum_{n-C}^{n+C} u_i$. It satisfies the inequality

$$v_n \leq C_1 \sum_{\substack{1 \leq a,b \leq n-1 \\ a+b=n}} v_a v_b + C_1 e^{sn}, \tag{14}$$

for some C_1. To get this inequality, bound each u_i appearing in v_n using (13), and notice that the a', b' in the upper bound satisfy $n - 2C \leq a' + b' \leq n + 2C$ and will therefore appear in one of the products $v_a v_b$ for $a + b = n$. We will prove that this sequence v_n grows at most exponentially fast, from which the same result follows for u_n, as desired. For small $z > 0$, define $B(z) = \sum_{n \geq 1} C_1 e^{sn} z^n$ and $V_N(z) = \sum_{n=1}^N v_n z^n$. The inequality (14) gives

$$V_N(z) \leq B(z) + C_1 V_{N-1}(z)^2. \tag{15}$$

The function B is smooth at 0. Let t be strictly larger than its derivative at 0. Fix z positive and small enough so that $B(z) + C_1(tz)^2 < tz$, which is possible since the function on the left has derivative $< t$. We claim that $V_N(z) \leq tz$ for all N. This is obvious for $N = 0$ as $V_0 = 0$, and the choice of z and the inequality (15) imply that, if it holds at $N-1$, then it holds at N, concluding the proof by induction. In particular, $v_n z^n \leq V_n(z) \leq tz$. This proves that v_n grows at most exponentially.

It remains to show (13), using geometry. Let $A > 0$ be large enough ($A > D+1$ will suffice). Take γ with $d(o, \gamma o) \in (n-1, n]$. We consider two different cases: either $[o, \gamma o] \setminus (B(o, A) \cup B(\gamma o, A))$ does intersect $\Gamma \widetilde{K}$ (we say that γ is recurrent – this terminology is local to this proof), or it does not. The former will give rise to the first term in (13), the latter to the second term.

We start with the non-recurrent γ's. Consider the last point x on $[o, \gamma o] \cap B(o, A) \cap \Gamma\widetilde{K}$, and the first point y on $[o, \gamma o] \cap B(\gamma o, A) \cap \Gamma \widetilde{K}$. Take γ_x such that $x \in \gamma_x \widetilde{K}$, and γ_y such that $y \in \gamma\gamma_y \widetilde{K}$. Note that γ_x and γ_y belong to a finite set $\mathcal{F}_A$ (depending on A), made of these elements of Γ that move o by at most $A + D$. Moreover, $\gamma' = \gamma_x^{-1}\gamma\gamma_y$ belongs to $\Gamma_{\widetilde{K}}$ since $[x, y] \cap \Gamma\widetilde{K} = \{x, y\}$ by construction. Applying Lemma 3.1 to the compact set $\bigcup_{g\in\mathcal{F}_A} g\widetilde{K}$, we obtain a constant C such that

$$\int_o^{\gamma o} \widetilde{F} \leq \int_{\gamma_x o}^{\gamma\gamma_y o} \widetilde{F} + C = \int_o^{\gamma' o} \widetilde{F} + C.$$

Finally, the contribution of the non-recurrent γ's to u_n is bounded by

$$\sum_{\gamma_x, \gamma_y \in \mathcal{F}_A} \sum_{\substack{\gamma' \in \Gamma_{\widetilde{K}} \\ d(o,\gamma' o) \in (n-1-2A-2D, n+2A+2D]}} e^{\int_o^{\gamma' o} \widetilde{F} + C}.$$

The sum over γ_x and γ_y gives a finite multiplicity, and the sum over γ' is bounded by $C(A)e^{ns}$ since $s > \delta_{\Gamma_{\widetilde{K}}}(F)$. This is compatible with the second term in the upper bound of (13).

We turn to the contribution to u_n of the recurrent γ's. For such a γ, there is a point x in $[o, \gamma o] \cap \Gamma\widetilde{K} \setminus (B(o, A) \cup B(\gamma o, A))$. Write $x = \gamma' x'$ with $x' \in \widetilde{K}$. Consider the integer a such that $d(o, \gamma' o) \in (a-1, a]$. It satisfies $A - D \leq a$, so if A is large enough one has $a > 0$. Let $\gamma'' = {\gamma'}^{-1}\gamma$, so that $\gamma = \gamma'\gamma''$. The integer b such that $d(o, \gamma'' o) \in (b-1, b]$ satisfies also $b \geq A - D > 0$. Moreover,

$$\begin{aligned} a + b = d(o, \gamma' o) + d(o, \gamma'' o) \pm 2 = d(o, \gamma' o) + d(\gamma' o, \gamma o) \pm 2 \\ = d(o, x) + d(x, \gamma o) \pm (2 + 2D) = d(o, \gamma o) \pm (2 + 2D) = n \pm (3 + 2D). \end{aligned}$$

This shows that $|a + b - n| \leq 3 + 2D$. Finally, applying twice Lemma 3.1, we obtain the existence of a constant C such that

$$\left| \int_o^{\gamma o} \widetilde{F} - \int_o^{\gamma' o} \widetilde{F} - \int_o^{\gamma'' o} \widetilde{F} \right| \leq C.$$

Altogether, this shows that the contribution of recurrent γ's to u_n is bounded by the first term of the right hand side of (13). □

5 Excursions outside compact sets

In this section, we will study and count the possible excursions of periodic orbits outside large compact sets, and deduce the inequalities

$$P_{\mathrm{Gur}}^{\infty}(F) \leq \delta_{\Gamma}^{\infty}(F) \quad \text{and} \quad P_{\mathrm{var}}^{\infty}(F) \leq \delta_{\Gamma}^{\infty}(F).$$

These inequalities are the heart of Theorem 1.2. The reverse inequalities $P_{\mathrm{Gur}}^{\infty}(F) \geq \delta_{\Gamma}^{\infty}(F)$ and $P_{\mathrm{var}}^{\infty}(F) \geq \delta_{\Gamma}^{\infty}(F)$ are simpler, and will be proven in Sections 5.2 and 6.

Let us explain why the above inequalities are the most surprising and difficult. A major difference between the definition of $\delta_{\Gamma}^{\infty}(F)$ and the two others is that $P_{\mathrm{Gur}}^{\infty}(F)$ and $P_{\mathrm{var}}^{\infty}(F)$ take into account trajectories (respectively periodic / typical) that spend most of the time outside a given large compact set, but can however come back inside this compact set several times, whereas $\delta_{\Gamma}^{\infty}(F)$ consider trajectories that start and finish in a given compact set, but never come back in the meantime. Thus, there are apparently much more trajectories considered in the first two definitions. However, in the next two sections, culminating in Corollaries 5.4 and 6.11, we prove that the above inequalities hold.

The strategy developed below is to cut a given trajectory, which comes back several times inside a given compact set, but spends a small proportion of time inside, into several excursions, and to prove precise upper bounds presented below.

5.1 Excursion of closed geodesics outside compact sets

In this section, we study periodic orbits that intersect (the unit tangent bundle of) a fixed compact $K \subset M$, but which spend most of their time away from the R-neighborhood K_R of K.

For all compact sets $K_1 \subset K_2 \subset M$ and $0 < \alpha \leq 1$, we define

$$\mathcal{P}(K_1, K_2, \alpha) = \Big\{p \text{ periodic orbit } ; \; p \cap T^1 K_1 \neq \emptyset, \;\; \ell(p \cap T^1 K_2) \leq \alpha \ell(p)\Big\} \tag{16}$$

and

$$\mathcal{P}(K_1, K_2, \alpha; T, T') = \left\{p \in \mathcal{P}(K_1, K_2, \alpha), \; T \leq \ell(p) \leq T'\right\}. \tag{17}$$

Given a Hölder potential F, we define for all $T, T' > 0$,

$$\mathcal{N}_F(K_1, K_2, \alpha; T, T') = \sum_{p \in \mathcal{P}(K_1, K_2, \alpha, T, T')} e^{\int_p F}.$$

It turns out that it is more efficient for subsequent estimates to bound a slightly larger sum, where the orbit p is weighted by the number of times it meets K_1, defined as follows. As in [PS18] and as in the proof of our Proposition 2.6, we define

$$n_{\widetilde{K}_1}(p) = \#\{\gamma \in \Gamma; \exists x \in \widetilde{K}_1, \; d(x, \gamma x) = \ell(p) \text{ and } p_\Gamma([x, \gamma x]) = \pi(p)\}.$$

As shown in [PS18], $n_{\widetilde{K}_1}$ depends on the choice of $\widetilde{K}_1$ but if $\widetilde{K}_1$ and $\widetilde{K}'_1 \subset \widetilde{M}$ are two compact preimages of K_1 by p_Γ, the ratio $\dfrac{n_{\widetilde{K}'_1}(p)}{n_{\widetilde{K}_1}(p)}$ is uniformly bounded from above and below independently of p. As in [PS18], we consider

$$n_{K_1}(p) = \inf n_{\widetilde{K}_1}(p),$$

the infimum being taken on all compact sets $\widetilde{K}_1$ with $p_\Gamma(\widetilde{K}_1) = K_1$. We think to this quantity as a kind of "number of returns" of p in K_1. Indeed, if $\widetilde{K}_1$ is a closed ball of radius less than the injectivity radius, then $n_{\widetilde{K}_1}$ is the number of connected components of the closed geodesic on M associated to p in K_1.

We define

$$\widehat{\mathcal{N}}_F(K_1, K_2, \alpha; T, T') = \sum_{\substack{p \in \mathcal{P}(K_1, K_2, \alpha) \\ T \leq \ell(p) \leq T'}} n_{K_1}(p) e^{\int_p F}. \tag{18}$$

Theorem 5.1. *Let M be a nonelementary complete connected negatively curved manifold with pinched negative curvature, and bounded first derivative of the curvature. Let $K \subset M$ be a compact set, and $\widetilde{K} \subset \widetilde{M}$ be a compact set such that $p_\Gamma(\widetilde{K}) = K$. Let $T_0 > 0$. Let $F : T^1 \to \mathbb{R}$ be a Hölder potential with $\delta_{\Gamma_{\widetilde{K}}}(F) > -\infty$. Let $\eta > 0$. For all $0 < \alpha \leq 1$ and $R \geq 2$, there exists a positive number $\psi = \psi(\widetilde{K}, F, \eta, \alpha/R)$ such that*

$$\limsup_{T \to +\infty} \frac{1}{T} \log \widehat{\mathcal{N}}_F(K, K_R, \alpha; T, T + T_0) \leq (1 - \alpha)\delta_{\Gamma_{\widetilde{K}}}(F) + \alpha \delta_\Gamma(F) + \eta + \psi.$$

Moreover, when $\widetilde{K}, F$ and η are fixed, $\psi(\widetilde{K}, F, \eta, \alpha/R)$ tends monotonically to 0 when α/R tends to 0.

Remark 5.2. When $\delta_{\Gamma_{\widetilde{K}}}(F) = -\infty$, the statement should be modified, replacing on the right hand side $\delta_{\Gamma_{\widetilde{K}}}(F)$ with an arbitrary real number d, and allowing ψ to depend on d. The same proof applies.

Remark 5.3. It would be interesting to get a lower bound in the above theorem, of the form $\limsup \geq (1 - \alpha)\delta_{\Gamma_{\widetilde{K}}}(F) + \alpha \delta_\Gamma(F) - \eta - \psi$. It is likely that some version in this spirit could hold. However, the attentive reader will observe that most inequalities involved in the proof below, up to some constants, work in both directions, except (24) (where a lower bound could easily be obtained) and Lemma 5.5.

Letting $R \to +\infty$, $\eta \to 0$ and at last K exhaust M and $\alpha \to 0$, we deduce the following corollary.

Corollary 5.4. *Under the same assumptions on M and F, we have*

$$P^{\infty}_{\mathrm{Gur}}(F) \leq \delta^{\infty}_{\Gamma}(F)\,.$$

Proof. If $\delta_\Gamma(F)$ is infinite, then $\delta^\infty_\Gamma(F)$ is also infinite by Proposition 4.13, and the result is obvious. We can therefore assume $\delta_\Gamma(F) < \infty$. We will also assume $\delta^\infty_\Gamma(F) > -\infty$, as the case $\delta^\infty_\Gamma(F) = -\infty$ can be proved similarly using Remark 5.2.

Let $\eta > 0$. We have to find a compact set $\widetilde{L}$ whose interior intersects $\pi(\Omega)$, and $\alpha > 0$, such that the exponential growth rate of $\sum_{p\in\mathcal{P}_L(T,T+1)\,;\,\ell(p\cap T^1L)<\alpha\ell(p)} e^{\int_p F}$ is at most $\delta^\infty_\Gamma(F) + 3\eta$. Fix a large compact set $\widetilde{K}$ with $\delta_{\Gamma_{\widetilde{K}}} \leq \delta^\infty_\Gamma(F) + \eta$. We will use $\widetilde{L} = \widetilde{K}_3$, the neighborhood of size 3 of $\widetilde{K}$.

There is a difficulty that the definition of the Gurevič pressure involves all periodic orbits going through $\widetilde{L}$, while Theorem 5.1 only takes into account those that, additionally, enter $\widetilde{K}$. This difficulty is solved using Proposition 4.8 applied to $K'' = K$, $K' = K_2$ and $K = K_3$: the exponential growth rate of $\sum_{p\in\mathcal{P}_{K_3}(T,T+1)\,;\,\ell(p\cap T^1K_3)<\alpha\ell(p)} e^{\int_p F}$ is bounded by that of $\sum_{p\in\mathcal{P}_K(T,T+1)\,;\,\ell(p\cap T^1K_2)<2\alpha\ell(p)} e^{\int_p F}$. The latter can be estimated thanks to Theorem 5.1 applied to $T_0 = 1$ and 2α: this growth rate is bounded by $(1-2\alpha)\delta_{\Gamma_{\widetilde{K}}}(F) + 2\alpha\delta_\Gamma(F) + \eta + \psi(\alpha)$, where $\psi(\alpha)$ tends to 0 with α. This quantity converges to $\delta_{\Gamma_{\widetilde{K}}}(F) + \eta \leq \delta^\infty_\Gamma(F) + 2\eta$ when α tends to 0, so for some $\alpha > 0$ it is $< \delta^\infty_\Gamma(F) + 3\eta$. □

The strategy of the proof of Theorem 5.1 is as follows. A periodic orbit will be cut into two kinds of segments, those which stay in a given compact set K, and the excursions outside this compact set. The weighted growth of the excursions should be controlled by the exponent $\delta_{\Gamma_K}(F)$ multiplied by the proportion of time spent outside K, and the weighted growth of the segments inside K should be controlled by $\delta_\Gamma(F)$ multiplied by the proportion of time spent in K. However, to succeed to get such a control, we need to avoid the situation with several very short excursions in a very close neighborhood of K. For this reason, we need to play with two compact sets, K and its R-neighborhood K_R.

Proof of Theorem 5.1. Let $\widetilde{K} \subset \widetilde{M}$ be a compact set and $\widetilde{K}_R \subset \widetilde{M}$ be its R-neighborhood, and set $K = p_\Gamma(\widetilde{K})$, $K_R = p_\Gamma(\widetilde{K}_R)$. Let D be the diameter of K. The diameter of K_R is $D + 2R$, so that a geodesic segment joining the boundary of K and the boundary of K_R has length at least R and at most $D + 2R$. Let also $D' = D'(K, T_0)$ be larger than the diameter of $K \cup \{o\}$, 1 and T_0.

Consider a periodic orbit $p \in \mathcal{P}(K, K_R, \alpha)$ with $\ell(p) \in [T, T+T_0]$. By assumption, $\pi(p) \cap K \neq \emptyset$. We will divide it into long excursions, i.e., those excursions outside both K and K_R, of total length at least $(1-\alpha)\ell(p)$ and periods of time of total length at most $\alpha\ell(p)$ where it stays inside K_R.

The closed geodesic $\pi(p)$ of M associated to p admits finitely many lifts $c_1, \dots, c_n$ (in $p_\Gamma^{-1}(\pi(p))$) that intersect $\widetilde{K}$, with $n = n_{\widetilde{K}}(p)$. For each of these geodesics $c_1, \dots, c_n$, let g_j be the hyperbolic isometry whose axis is c_j, and whose translation length is $\ell(p)$, and which translates in the direction given by the orientation of p.

The sequel of the proof concerns each of these axes c_j and isometries g_j. We will omit the index j, and work on the axis c of the isometry g.

Define inductively points a_i, b_i on c as follows. Choose first a point a_0 on c inside $\widetilde{K}$. Consider on the geodesic segment $[a_0, g.a_0]$ of c the first points $b_0, a_1 \in \Gamma\partial\widetilde{K}$ with $(b_0, a_1) \cap \Gamma\widetilde{K} = \emptyset$ and $(b_0, a_1) \cap \widetilde{M} \setminus \Gamma\widetilde{K}_R \neq \emptyset$. The interval (b_0, a_1) projects through p_Γ into a *long excursion*, i.e., an excursion outside K which also goes outside K_R. Inductively, we define $(b_1, a_2), \dots, (b_{N-1}, a_N)$ by the properties that b_i, a_{i+1} are the first points of $[a_i, g.a_0]$ which lie in $\Gamma\partial\widetilde{K}$ and satisfy $(b_i, a_{i+1}) \cap \Gamma\widetilde{K} = \emptyset$ and $(b_i, a_{i+1}) \cap \widetilde{M} \setminus \Gamma\widetilde{K}_R \neq \emptyset$. In other terms, the intervals (b_i, a_{i+1}), $0 \leq i \leq N-1$, are the connected components of $[a_0, g.a_0] \setminus \Gamma\widetilde{K}$ that intersect $\widetilde{M} \setminus \Gamma\widetilde{K}_R$, whereas the segments (a_i, b_i) are included in $\Gamma\widetilde{K}_R$. Finally, set $b_N = g.a_0$.

For all $0 \leq i \leq N$, choose elements $\gamma_i^\pm \in \Gamma$ such that $a_i \in \gamma_i^-\widetilde{K}$ and $b_i \in \gamma_i^+\widetilde{K}$. As $\widetilde{K}$ is compact and the action of Γ is proper, for each i, there are only finitely many choices of such elements $\gamma_i^\pm$. Without loss of generality, set $\gamma_0^- = Id$ and $\gamma_N^+ = g$.

Choose some $\varepsilon > 0$. The following elementary observations are crucial for the sequel.

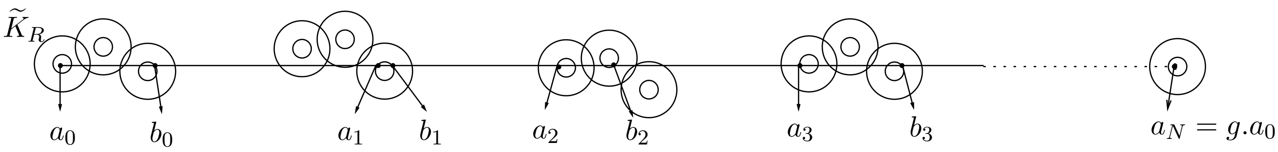

Figure 3 – Long excursions outside $\widetilde{K}$ and $\widetilde{K}_R$

1. As $\bigcup_{0\le i\le N}[a_i,b_i]\subset \Gamma\widetilde{K}_R$, by definition of $\mathcal{P}(K,K_R,\alpha)$ and since $T\le \ell(p)\le T+T_0$, we have
$$\ell(p\cap T^1K)\le \sum_{i=0}^{N} d(a_i,b_i)\le \alpha(T+T_0)\le \alpha T + D'.$$
2. For all $i\in\{0,\dots,N-1\}$, we have $(b_i,a_{i+1})\subset \widetilde{M}\setminus\Gamma\widetilde{K}$. Moreover, the length of $(b_i,a_{i+1})\cap\Gamma\widetilde{K}_R$ is at least $2R$ and $\cup_i[b_i,a_{i+1}]$ does not intersect the interior of $\Gamma\widetilde{K}$, so that by definition of $\mathcal{P}(K,K_R,\alpha)$,
$$(1-\alpha)T+2RN\le \sum_{i=0}^{N-1} d(b_i,a_{i+1})\le T+T_0\le T+D', \tag{19}$$
and therefore
$$N\le \frac{1}{2R}\left(\alpha T+D'\right) =: \nu\,. \tag{20}$$
3. Write $\psi_i=(\gamma_i^-)^{-1}\gamma_i^+\in\Gamma$ for all $i=0,\dots,N$, we have $|d(o,\psi_i o)-d(a_i,b_i)|\le 2D'$, so that
$$\sum_{i=0}^{N} d(o,\psi_i o)\le \alpha(T+T_0)+2(N+1)D'\le \alpha T+5ND'\,.$$
Let s_i be the unique integer such that $d(o,\psi_i o)\le s_i< d(o,\psi_i o)+1$. Then
$$s_0+\dots+s_N\le \alpha T+5ND'+N+1\le \alpha T+7ND'\,. \tag{21}$$
4. By definition of $\Gamma_{\widetilde{K}}$, for all $i=0,\dots,N-1$, we have $\varphi_i=(\gamma_i^+)^{-1}\gamma_{i+1}^-\in\Gamma_{\widetilde{K}}$. Moreover, $|d(o,\varphi_i o)-d(b_i,a_{i+1})|\le 2D'$. Let t_i be the unique integer such that $d(o,\varphi_i o)\le t_i<d(o,\varphi_i o)+1$.
5. As $\sum_{i=0}^{N} d(a_i,b_i)+\sum_{i=0}^{N-1} d(b_i,a_{i+1})=d(a_0,b_N)=\ell(p)\in[T,T+T_0]$, we get
$$\left|\sum_{i=0}^{N} d(o,\psi_i o)+\sum_{i=0}^{N-1} d(o,\varphi_i o)-T\right|\le T_0+(4N+2)D'$$
and therefore
$$\left|\sum_{i=0}^{N} s_i+\sum_{i=0}^{N-1} t_i-T\right|\le T_0+(4N+2)D'+(2N+1)\le 10ND'\,.$$
6. By (19), as $d(a_i,b_i)-D'\le t_i\le d(a_i,b_i)+D'+1$, we get
$$(1-\alpha)T-2ND'\le \sum t_i\le T+4ND'. \tag{22}$$
7. Since M has pinched negative curvatures and F is (β,C_F)-Hölder, by Lemma 3.1 applied to the compact set $\widetilde{K}\cup\{o\}$ there exists a constant $C(F,\widetilde{K})$ depending only on the upper bound of the curvature, on $\widetilde{K}$ and the Hölder constant of F such that for all $i=0,\dots,N$,
$$\left|\int_{a_i}^{b_i}\widetilde{F}-\int_{o}^{\psi_i o}\widetilde{F}\right|\le C(F,\widetilde{K})\,.$$

8. Similarly, for all $i = 0, \dots, N-1$,

$$\left| \int_{b_i}^{a_{i+1}} \widetilde{F} - \int_o^{\varphi_i o} \widetilde{F} \right| \leq C(F, \widetilde{K}) \,.$$

9. As $\int_p F = \int_{a_0}^{g a_0} \widetilde{F}$, and bounding $2N+1$ with 3ν, we deduce

$$\sum_{i=0}^{N} \int_o^{\psi_i o} \widetilde{F} + \sum_{i=0}^{N-1} \int_o^{\varphi_i o} \widetilde{F} - 3C(F,\widetilde{K})\nu \leq \int_p F \leq \sum_{i=0}^{N} \int_o^{\psi_i o} \widetilde{F} + \sum_{i=0}^{N-1} \int_o^{\varphi_i o} \widetilde{F} + 3C(F,\widetilde{K})\nu. \quad (23)$$

eq:int_pl

For all $t \in \mathbb{N}$, set

$$\Gamma(t-1,t) = \{\gamma \in \Gamma \,;\, d(o,\gamma o) \in [t-1,t]\} \quad \text{and} \quad \Gamma_{\widetilde{K}}(t-1,t) = \Gamma(t-1,t) \cap \Gamma_{\widetilde{K}}.$$

We also write

$$Q_{F,\Gamma}(t-1,t) = \sum_{\gamma \in \Gamma(t-1,t)} e^{\int_o^{\gamma o} \widetilde{F}} \quad \text{and} \quad Q_{F,\Gamma_{\widetilde{K}}}(t-1,t) = \sum_{\gamma \in \Gamma_{\widetilde{K}}(t-1,t)} e^{\int_o^{\gamma o} \widetilde{F}} \,.$$

To each periodic orbit $p \in \mathcal{P}(K, K_R, \alpha)$ with $\ell(p) \in [T, T+T_0]$, we have associated a family of hyperbolic isometries $g_1, \dots, g_n \in \Gamma$, with $n = n_{\widetilde{K}}(p)$, those whose axis intersects $\widetilde{K}$ and projects through p_Γ on $\pi(p)$ and with translation length equal to $\ell(p)$. Moreover, for each such $g_j, 1 \leq j \leq n$, the associated periodic orbit is unique.

Then, to each such element g we have associated by the previous construction finite sequences $\varphi_0, \dots, \varphi_{N-1}$ in Γ_K and $\psi_0, \dots, \psi_N \in \Gamma$. As one can recover g from these sequences by the formula $g = \psi_0 \varphi_0 \psi_1 \cdots \varphi_{N-1} \psi_N$, this association is injective.

Let us now bound $\widehat{\mathcal{N}}_F(K, K_R, \alpha; T, T+T_0)$. Bounding $n_K(p)$ with $n_{\widetilde{K}}(p)$, we have for each periodic orbit p the inequality

$$n_K(p) e^{\int_p F} \leq \sum_{i=1}^{n_{\widetilde{K}}(p)} e^{\int_p F}, \quad (24)$$

eqn:majo

where each term $e^{\int_p F}$ can be bounded using the decomposition of g_i as in (eq:int_pF_le 23). Summing over all the periodic orbits, we get the inequality

$$\widehat{\mathcal{N}}_F(K, K_R, \alpha; T, T+T_0) \leq e^{3C(F,\widetilde{K})\nu} \sum_{N=0}^{\nu(\alpha,T,T_0,R)} \sum_{\substack{t_0,\dots,t_{N-1},s_0,\dots,s_N \\ |\sum s_i + \sum t_i - T| \leq 10ND' \\ \sum t_i \geq (1-\alpha)T - 2ND'}} \quad (25)$$

eq:Countl

$$Q_{F,\Gamma}(s_0) \cdot Q_{F,\Gamma_{\widetilde{K}}}(t_0) \cdot Q_{F,\Gamma}(s_1) \cdot Q_{F,\Gamma_K}(t_1) \cdots Q_{F,\Gamma_{\widetilde{K}}}(t_{N-1}) \cdot Q_{F,\Gamma}(s_N).$$

The following lemma is a straightforward consequence of the definition of the critical exponents $\delta_\Gamma(F)$ and $\delta_{\Gamma_{\widetilde{K}}}(F)$.

seulement **Lemma 5.5.** *For all $\eta > 0$, there exists $C_\eta = C_\eta(\widetilde{K}, F, \eta) \geq 1$ such that for all $t > 0$, we have*

$$Q_{F,\Gamma}(t) \leq C_\eta e^{\delta_\Gamma(F)t + \eta t} \quad \text{and} \quad Q_{F,\Gamma_{\widetilde{K}}}(t) \leq C_\eta e^{\delta_{\Gamma_{\widetilde{K}}}(F)t + \eta t}$$

We can write the second bound as $Q_{F,\Gamma_{\widetilde{K}}}(t) \leq e^{(\delta_{\Gamma_{\widetilde{K}}}(F) - \delta_\Gamma(F))t + \delta_\Gamma(F)t + \eta t}$. Multiplying these bounds, we get

$$\begin{aligned}
&Q_{F,\Gamma}(s_0) \cdot Q_{F,\Gamma_{\widetilde{K}}}(t_0) \cdot Q_{F,\Gamma}(s_1) \cdot Q_{F,\Gamma_K}(t_1) \cdots Q_{F,\Gamma_{\widetilde{K}}}(t_{N-1}) \cdot Q_{F,\Gamma}(s_N) \\
&\leq C_\eta^{2N+1} \exp\Big((\delta_\Gamma(F) + \eta)\big(\sum s_i + \sum t_i\big) + (\delta_{\Gamma_{\widetilde{K}}}(F) - \delta_\Gamma(F))\big(\sum t_i\big) \Big) \\
&\leq C_\eta^{3N} \exp\Big((\delta_\Gamma(F) + \eta)T + (|\delta_\Gamma(F)| + \eta)10ND' + (\delta_{\Gamma_{\widetilde{K}}}(F) - \delta_\Gamma(F))((1-\alpha)T - 2ND') \Big) \\
&= C_\eta^{3N} \exp\Big((\alpha\delta_\Gamma(F) + (1-\alpha)\delta_{\Gamma_{\widetilde{K}}}(F) + \eta)T + (|\delta_\Gamma(F)| + \eta + (\delta_\Gamma(F) - \delta_{\Gamma_{\widetilde{K}}}(F)))2ND' \Big).
\end{aligned}$$

Note that this bound does not depend anymore on the choice of the s_i and t_i. To bound (25), one should take into account a multiplicity given by the number of possible choices for these integers.

The following combinatorial standard estimate will control the number of possible choices.

Lemma 5.6. *Let $\tau, \kappa \in \mathbb{N}$ be integers with $\kappa < \tau$. The number of ordered integer decompositions of τ of length κ, i.e., the number of $(u_1, \dots, u_\kappa) \in \mathbb{N}^\kappa$ such that $u_i \geq 0$ and $u_1 + \cdots + u_\kappa \leq \tau$, is equal to*

$$\binom{\tau+\kappa}{\kappa} = \frac{(\tau+\kappa)!}{\kappa!\tau!}.$$

Then $(s_0, t_0, s_1, \dots, s_N)$ forms an ordered partition of $\tau = T + 10ND'$. From the monotonicity properties of binomial coefficients, their number is bounded by $\binom{T+10ND'+2N+1}{2N+1}$. Recall that by (20), we have $N \leq \nu$, which is bounded by $T/2$ for large T, we have $T + 10ND' + 2N + 1 \leq 8D'T$ and $2N + 1 \leq 3\nu \leq 8D'\nu$, we get $\binom{T+10ND'+2N+1}{2N+1} \leq \binom{8D'T}{2N+1} \leq \binom{8D'T}{8D'\nu}$ thanks to monotonicity properties of binomial coefficients. Summing over all the values of N, we obtain the estimate

$$\begin{aligned}\widehat{\mathcal{N}}_F(K, K_R, \alpha; T, T+T_0) \leq \nu \cdot \binom{8D'T}{8D'\nu} e^{3C(F,\widetilde{K})\nu} \cdot C_\eta^{3\nu} \\ \exp\Big((\alpha\delta_\Gamma(F) + (1-\alpha)\delta_{\Gamma_{\widetilde{K}}}(F) + \eta)T + (|\delta_\Gamma(F)| + \eta + (\delta_\Gamma(F) - \delta_{\Gamma_{\widetilde{K}}}(F)))2\nu D'\Big).\end{aligned}$$

To conclude the proof, we should estimate the exponential growth rate of the various terms in this expression when T tends to infinity. Note that $\nu \leq \alpha T/R$. Stirling's formula $n! \sim \sqrt{2\pi n}(n/e)^n$ implies that the exponential growth rate of $\binom{8D'T}{8D'\nu} \leq \binom{8D'T}{8D'T\cdot\alpha/R}$ is bounded by $-\rho\log\rho - (1-\rho)\log(1-\rho)$ for $\rho = \alpha/R$. Finally, the exponential growth rate of $\widehat{\mathcal{N}}_F(K, K_R, \alpha; T, T+T_0)$ is bounded by

$$\begin{aligned}\alpha\delta_\Gamma(F) + (1-\alpha)\delta_{\Gamma_{\widetilde{K}}}(F) + \eta - \rho\log\rho - (1-\rho)\log(1-\rho) \\ + \Big(3C(F,\widetilde{K}) + 3\log C_\eta + 20D'(|\delta_\Gamma(F)| + \eta + (\delta_\Gamma(F) - \delta_{\Gamma_{\widetilde{K}}}(F)))\Big)\frac{\alpha}{R}.\end{aligned}$$

This concludes the proof of the theorem. □

5.2 Gurevič and geometric pressures at infinity coincide

This paragraph is devoted to the proof of the following part of Theorem 1.2.

Theorem 5.7. *Let M be a nonelementary complete connected negatively curved manifold with pinched negative curvature, and bounded first derivative of the curvature. For all Hölder continuous potentials $F : T^1M \to \mathbb{R}$ with finite pressure, we have*

$$P^\infty_{\mathrm{Gur}}(F) = \delta^\infty_\Gamma(F).$$

By Corollary 5.4, it is enough to prove the inequality $P^\infty_{\mathrm{Gur}}(F) \geq \delta^\infty_\Gamma(F)$.

Proof. The set of periodic orbits of the geodesic flow is in $1-1$ correspondence with the set of conjugacy classes of hyperbolic elements of Γ. Let us recall how. Given a periodic orbit $p \subset T^1M$, its preimage $p_\Gamma^{-1}(p) \subset T^1\widetilde{M}$ is a countable union of orbits of the geodesic flow on $T^1\widetilde{M}$. Each of these orbits projects on $\widetilde{M}$ to the axis of a hyperbolic element of Γ, which is unique when requiring that this element translates along the axis with translation length equal to $\ell(p)$, and in the direction given by the direction of $(g^t)_{t>0}$ on this orbit. The hyperbolic elements associated to p in this way are all conjugated.

Let $K \subset M$ be a compact set whose interior intersects a closed geodesic, and containing the projection $p_\Gamma(o)$. Let $\widetilde{K}$ be a compact set of $\widetilde{M}$ which contains o and such that $p_\Gamma(\widetilde{K}) = K$. Let N be the maximal multiplicity of p_Γ on $\widetilde{K}$. Let D be its diameter. Let $\widetilde{K}_R$ be the R-neighborhood of $\widetilde{K}$.

Recall that we have defined in (17) the following sets of periodic orbits:

$$\mathcal{P}(K, \alpha) := \mathcal{P}(K, K, \alpha) = \{p \text{ periodic orbit}\ ;\ 0 < \ell(p \cap K) \leq \alpha\ell(p)\}$$

and

$$\mathcal{P}(K,\alpha,T,T') := \mathcal{P}(K,K,\alpha,T,T') = \left\{p \in \mathcal{P}(K,\alpha) \in \,;\; T \le \ell(p) \le T'\right\}.$$

First, by Lemma 2.1, there exist finitely many elements $g_1,\dots,g_k \in \mathcal{G}$, such that, for all $\gamma \in \Gamma_{\widetilde{K}}$, there exist g_i, g_j (not necessarily unique) such that $g_i^{-1}\gamma g_j$ is hyperbolic with an axis which intersects $\widetilde{K}$. Let p_γ be the associated periodic orbit (it depends on the choice of g_i, g_j but it is not a problem). As the axis of $g_i^{-1}\gamma g_j$ intersects $\widetilde{K}$, we deduce that

$$\left|\ell(p_\gamma) - d(o, g_i^{-1}\gamma g_j o)\right| \le 2D\,.$$

By the triangular inequality, we deduce that

$$|d(o,\gamma o) - \ell(p_\gamma)| \le 2D + 2\max(d(o,g_i o))\,.$$

Similarly, thanks to Lemma 3.1, and using the fact that $\widetilde{F}$ is bounded on the δ-neighborhood of $\Gamma\widetilde{K}$, with $\delta = \max(d(o,g_i o))$, we deduce that there exists a constant $C = C(F,\widetilde{K},g_1,\dots,g_k)$ such that

$$\left|\int_o^{\gamma o}\widetilde{F} - \int_{p_\gamma} F\right| \le C\,.$$

Choose now some $R>1$, and let $\widetilde{K}_R$ be the R-neighborhood of $\widetilde{K}$. Observe that, for $\gamma \in \Gamma_{\widetilde{K}_R}$, the time spent by the geodesic segment $[o,\gamma o]$ in $\widetilde{K}_R$ is bounded by $2D+2R$. Using the above notations, we assume that $g_i^{-1}\gamma g_j$ is hyperbolic with associated periodic orbit p_γ. The point $g_i o$ is at bounded distance δ from o, and the point $g_j o$ is at bounded distance at most δ from $\gamma g_j o$. Therefore, by Lemma 2.2, there exists a constant $T_0>0$ depending on δ and the bounds on the curvature, such that, when removing segments of length T_0 at the beginning and the end of $[g_i o, \gamma g_j o]$, the middle segment is in a neighborhood of radius less than $1/2$ from the geodesic segment $[o,\gamma o]$.

On the other hand, the periodic orbit p_γ associated to $g_i^{-1}\gamma g_j$ admits an axis which intersects $\widetilde{K}$ and $g_i^{-1}\gamma g_j\widetilde{K}$. Let $x\in\widetilde{K}$ be a point on this axis and $g_i^{-1}\gamma g_j x \in g_i^{-1}\gamma g_j\widetilde{K}$ its image by $g_i^{-1}\gamma g_j$. By Lemma 2.2, when removing segments of length T_0 at the beginning and the end of the segment $[x, g_i^{-1}\gamma g_j x]$, the middle segment is in a neighborhood of size less than $1/2$ of the geodesic segment $[o, g_i^{-1}\gamma g_j o]$.

Triangular inequality implies that, after removing segments of length $2T_0$ at the beginning and at the end of the geodesic segment $[g_i x, \gamma g_j x]$, this segment is at distance at most $1/2$ of $[g_i o, \gamma g_j o]$, and therefore, at distance at most 1 from $[o,\gamma o]$. In particular, as $\gamma\in\Gamma_{\widetilde{K}_R}$, and $R\ge 1$, after removing segments of length $2T_0+D+R$ at the beginning and the end of $[x, g_i^{-1}\gamma g_j x]$, this segment spends the rest of the time outside $\widetilde{K}$.

We deduce that the time spent by p_γ inside K is at most $4T_0+2D+2R$. In particular, when $\ell(p_\gamma)\ge \frac{4T_0+2D+2R}{\alpha}$, the periodic orbit p_γ spends a proportion of time at most α inside K. As $|d(o,\gamma o)-\ell(p_\gamma)|\le 2D+2\delta$, it implies that as soon as $d(o,\gamma o)\ge 2D+2\delta+\frac{4T_0+2D+2R}{\alpha}$, p_γ belongs to $\mathcal{P}(K,\alpha)$. In particular, when $T>1+2D+2\delta+\frac{4T_0+2D+2R}{\alpha}$, the above considerations show that for $\gamma\in\Gamma_{\widetilde{K}_R}(T-1,T)$, the associated periodic orbit p_γ belongs to $\mathcal{P}(K,\alpha,T-1-2D-2\delta,T+2D+2\delta)$.

Now, it remains to control the multiplicity of the above map $\gamma\to p_\gamma$. As the cardinality of $\mathcal{G}$ is finite, and the group Γ acts properly discontinuously on $\widetilde{M}$, up to some multiplicative constants, the lack of injectivity of this map comes from the number of hyperbolic elements g with length roughly $\ell(\gamma)$ whose axis stays at bounded distance from a given axis of p_γ. This number is at most linear in $\ell(p_\gamma)$.

All the above considerations imply that there exist constants depending on $K,\widetilde{K},D,\alpha,F$ such that for $T>0$ large enough, and all $R>1$,

$$\sum_{\gamma\in\Gamma_{\widetilde{K}_R},\,T-1\le d(o,\gamma o)\le T} e^{\int_o^{\gamma o}\widetilde{F}} \le (\#\mathcal{G})^2\times C\times T\times \sum_{p\in\mathcal{P}(K,\alpha,T-1-\tau,T+\tau)} e^{\int_p F}\,.$$

Taking $\frac{1}{T}\log$ of the above inequality, and letting $T\to+\infty$, and then letting $R\to+\infty$ and $\alpha\to 0$ gives $P^{\infty}_{\mathrm{Gur}}(F)\ge\delta^{\infty}_{\Gamma}(F)$. $\square$

6 Variational and geometric pressures at infinity coincide

This section is devoted to the proof of the equality between geometric and variational pressures at infinity.

Theorem 6.1. *Let M be a nonelementary complete connected negatively curved manifold with pinched negative curvature, and bounded first derivative of the curvature. Let $F : T^1M \to \mathbb{R}$ be a Hölder potential. Then*

$$\delta_\Gamma^\infty(F) = P_{\mathrm{var}}^\infty(F).$$

The first paragraph contains the proof of the easier inequality $\delta_\Gamma^\infty(F) \le P_{\mathrm{var}}^\infty(F)$.

The inequality $P_{\mathrm{var}}^\infty(F) \le \delta_\Gamma^\infty(F)$ will follow from Section 5, after some reductions. First, in Section 6.2, we introduce a notion of pressure, that we call *Katok pressure* in reference to the Katok entropy introduced in [Kat80]. We show that the variational pressure is bounded from above by this new pressure, involving spanning sets. Using closing lemma, in Section 6.3, we study escape of mass of sequences of probability measures, and relate this new pressure to the Gurevič pressure (which involves weighted growth of periodic orbits), and conclude the proof of the inequality $P_{\mathrm{var}}^\infty(F) \le \delta_\Gamma^\infty(F)$ thanks to Theorem 5.1.

6.1 The first inequality

This paragraph is devoted to the proof of the easier inequality $\delta_\Gamma^\infty(F) \le P_{\mathrm{var}}^\infty(F)$. We deal first with the exceptional situation where $\delta_\Gamma(F) = \infty$.

Lemma 6.2. *Under the assumptions of Theorem 6.1, if we assume $\delta_\Gamma(F) = \infty$, then for any compact set K in T^1M and any $C, \varepsilon > 0$, there exists $\mu \in \mathcal{M}_{1,\mathrm{erg}}^F$ such that $\mu(K) < \varepsilon$ and $h_{KS}(\mu) + \int F \,\mathrm{d}\mu > C$.*

Proof. By Theorem 1.1, we have $P_{\mathrm{var}}(F) = \infty$. For any invariant measure μ, the entropy $h_{KS}(\mu)$ is bounded from below by 0 and from above uniformly thanks to the curvature bounds. Therefore, we can forget about the entropy in the statement, and it suffices to make sure $\int F \,\mathrm{d}\mu > C$.

Choose $R = R(C,K)$ be large enough, and then $C' = C'(C,K)$ large enough. The equality $P_{\mathrm{var}}(F) = \infty$ ensures the existence of a measure $\nu \in \mathcal{M}_1^F$ with $\int F \,\mathrm{d}\nu > C'$. Taking an ergodic component of ν if necessary, we can assume that ν is ergodic. If $\nu(T^1K) = 0$, we are done taking $\mu = \nu$. Otherwise, consider a ν-typical vector v in T^1K. Then $1/T \int_0^T F(g^t v)\,\mathrm{d}t$ converges to $\int F \,\mathrm{d}\nu$, hence it is $> C'$ for large enough T. Consider such a large T with, additionally, $g^T v \in K$: it exists by Poincaré recurrence.

Let K_1 be the neighborhood of size 1 of K. Consider the points $t \in [0,T]$ for which $g^t v \notin K_1$ (this is an open set), and among them the connected components on which $g^t v$ does not always remain in K_R, the neighborhood of size R of K. These components are of length at least $2R$, so there are finitely many of them. If C' is large enough so that $|F| < C'$ on K_R, then there exists such a component (a,b) on which $\int_a^b F(g^t v) > C'(b-a)$: otherwise, one would get $\int_0^T F(g^t v) \le C'T$ by summing the contributions of these big connected components, and integrating the bound $|F| \le C'$ on the remaining points. Restricting the orbit to the interval $[a,b]$ and setting $w = g^t a$, we have found a piece of orbit of length $\tau \ge 2R$ starting and ending in ∂K_1, remaining outside of K_1 in between, and with $\int_0^\tau F(g^t w)\,\mathrm{d}t \ge \tau C'$.

Let us close this orbit using the connecting lemma 2.6 in the compact set K_1: we get a closed orbit $(g^t w')_{0 \le t \le \tau+s}$, which stays at distance at most $1/2$ of the orbit of w for $0 \le t \le \tau$, and with $s \le \tau_0$ depending only on K_1. The measure μ we are looking for will be the uniform measure along this periodic orbit. The only times the orbit of w' can belong to K is for $\tau \le t \le \tau + s$. It follows that, if R is large enough compared to τ_0, the relative mass given by μ to K is smaller than ε. Let us now check that $\int F \,\mathrm{d}\mu$ is large. First, $\left|\int_0^\tau F(g^t w')\,\mathrm{d}t - \int_0^\tau F(g^t w)\right|$ is bounded by a constant C_0 depending only on K, by Lemma 3.1. Second, $\int_\tau^{\tau+s} F(g^t w')$ is bounded below by a constant $-C_1$ depending only on K, as s is bounded by τ_0 and F is bounded on the $\tau_0 + 2$-neighborhood of K. We get

$$\int_0^{\tau+s} F(g^t w')\,\mathrm{d}t \ge \int_0^\tau F(g^t w)\,\mathrm{d}t - C_0 - C_1 \ge C'\tau - C_0 - C_1.$$

If $C' = C'(K, C)$ is large enough, this is at least $C(\tau + s)$, as desired. $\square$

Proposition 6.3. *Under the assumptions of Theorem 6.1, let F be a Hölder continuous map. Then $\delta_\Gamma^\infty(F) \leq P_{\mathrm{var}}^\infty(F)$.*

Proof. If $\delta_\Gamma(F) = \infty$, then Lemma 6.2 shows that one can find a sequence of measures $\mu_n \in \mathcal{M}_1^F$ tending weakly to 0 such that $h_{KS}(\mu_n) + \int_{T^1M} F \,\mathrm{d}\mu_n$ tends to infinity. Therefore, $P_{\mathrm{var}}^\infty(F) = \infty$, and the result is obvious.

Assume now $\delta_\Gamma(F) < \infty$. Choose for every $R \in \mathbb{N}$ a Hölder continuous map $0 \leq \chi_R \leq 1$ which approximates $\mathbf{1}_{T^1 p_\Gamma B(o,R)}$ on T^1M: $\chi_R \equiv 1$ on $T^1(p_\Gamma B(o, R-1))$ and $\chi_R \equiv 0$ outside $T^1(p_\Gamma B(o,R))$. Define $F_{n,R} = F - n\chi_R$, for all $n \in \mathbb{N}$, and note that $F_{n,R} = F$ outside $T^1 p_\Gamma B(o,R)$ so that $\delta_{\Gamma_{B(o,R)}}(F) = \delta_{\Gamma_{B(o,R)}}(F_{n,R})$. As a consequence,

$$\delta_\Gamma(F_{n,R}) \geq \delta_{\Gamma_{B(o,R)}}(F_{n,R}) = \delta_{\Gamma_{B(o,R)}}(F) \geq \delta_\Gamma^\infty(F)\,.$$

By the variational principle [PPS15, Thm 1.1], we can find for all $\varepsilon > 0$ a measure $\mu_{n,R,\varepsilon} \in \mathcal{M}_1^{F_{n,R}}$, such that

$$h_{KS}(\mu_{n,R,\varepsilon}) + \int_{T^1M} F_{n,R} d\mu_{n,R,\varepsilon} > \delta_\Gamma^\infty(F) - \varepsilon\,.$$

Since $F_{n,R} = F$ outside of a compact set, $\mu_{n,R,\delta}$ also belongs to $\mathcal{M}_1^F$. Therefore,

$$\begin{aligned}\delta_\Gamma(F) &\geq h_{KS}(\mu_{n,R,\varepsilon}) + \int_{T^1M} F \,\mathrm{d}\mu_{n,R,\varepsilon} \geq n\mu_{n,R,\varepsilon}(T^1 p_\Gamma B(o, R-1)) + h_{KS}(\mu_{n,R,\varepsilon}) + \int_{T^1M} F_{n,R} \,\mathrm{d}\mu_{n,R,\varepsilon}\\ &\geq n\mu_{n,R,\varepsilon}(T^1 p_\Gamma B(o,R-1)) + \delta_\Gamma^\infty(F) - \varepsilon\,.\end{aligned}$$

Choose any sequence $\varepsilon_k \to 0$, $R_k \to \infty$, $n_k \to \infty$, and $\mu_k = \mu_{n_k, R_k, \varepsilon_k}$. As $\delta_\Gamma(F) < \infty$, we get from the above on the one hand that for all $R > 0$,

$$\limsup \mu_k(T^1 p_\Gamma(o,R)) = 0\,,$$

and on the other hand that

$$\liminf_{k\to\infty} h_{KS}(\mu_k) + \int F \,\mathrm{d}\mu_k \geq \delta_\Gamma^\infty(F)\,.$$

This proves that

$$P_{\mathrm{var}}^\infty(F) \geq \delta_\Gamma^\infty(F)\,. \qquad \square$$

Remark 6.4. *Since the proof only uses ergodic measures, it even proves the slightly stronger result*

$$\delta_\Gamma^\infty(F) \leq P_{\mathrm{var,erg}}^\infty(F) \leq P_{\mathrm{var}}^\infty(F)\,.$$

6.2 Katok pressure

The proof of Theorem 6.10 will rely on the following notion of pressure, extending to general potentials a notion of entropy introduced by A. Katok in [Kat80] in the case $F = 0$.

For all $v \in T^1\widetilde{M}$ and $\varepsilon, T > 0$, the *dynamical ball* $B(v, \varepsilon; -T, T)$ is defined by

$$B(v, \varepsilon; -T, T) = \{w \in T^1\widetilde{M}\ ;\ \forall t \in [-T, T], d(g^t v, g^t w) \leq \varepsilon\}.$$

As in [PPS15], it is more convenient to deal with symmetric dynamical balls. Recall from [PPS15, Lemma 3.14] that for all $0 < \varepsilon \leq \varepsilon'$, there exists $T_{\varepsilon,\varepsilon'} \geq 0$, such that for all $v \in T^1\widetilde{M}$ and $T > 0$, we have

$$B(v, \varepsilon'; -T - T_{\varepsilon,\varepsilon'}, T + T_{\varepsilon,\varepsilon'}) \subset B(v, \varepsilon; -T, T) \subset B(v, \varepsilon'; -T, T) \tag{26}$$

As in [ST19, Rem 3.1], on T^1M, we define two kinds of dynamical balls, the small dynamical ball $B_\Gamma(v, \varepsilon; -T, T) = p_\Gamma(B(\widetilde{v}, \varepsilon; -T, T))$ and the big dynamical ball

$$B_{\mathrm{dyn}}(v, \varepsilon; -T, T) = \{w \in T^1M\ ;\ \forall t \in [-T, T], d(g^t v, g^t w) \leq \varepsilon\} \supseteq B_\Gamma(v, \varepsilon; -T, T). \tag{27}$$

Both balls coincide as soon as the injectivity radius of M is bounded from below and ε is small enough. More generally, if along the geodesic $(g^t v)_{-T\leq t\leq T}$, the injectivity radius at the point $\pi(g^t v)$ is larger than ε, then

$$B_{\rm dyn}(v,\varepsilon;-T,T) = B_\Gamma(v,\varepsilon;-T,T)\,. \tag{28}$$

We will mainly use the small dynamical balls, that are more convenient in our geometric context, but less natural from the dynamical point of view.

Given a probability measure μ on T^1M, $\delta\in(0,1)$ and $\varepsilon, T>0$, we will say that a set $V\subset T^1M$ is $(\mu,\delta,\varepsilon;-T,T)$-*spanning* (respectively *dynamically-*$(\mu,\delta,\varepsilon;-T,T)$*-spanning*) if

$$\mu\left(\bigcup_{v\in V} B_\Gamma(v,\varepsilon;-T,T)\right)\geq\delta\,,\quad\text{respectively}\quad \mu\left(\bigcup_{v\in V} B_{\rm dyn}(v,\varepsilon;-T,T)\right)\geq\delta$$

Of course, a $(\mu,\delta,\varepsilon;-T,T)$-spanning set is also dynamically-$(\mu,\delta,\varepsilon;-T,T)$-spanning.

Let $F: T^1M\to\mathbb{R}$ be a Hölder potential. Let $\mu\in\mathcal{M}^F_{1,\rm erg}$ be an ergodic probability measure on T^1M, invariant under the geodesic flow, such that $\int F^-\,{\rm d}\mu<\infty$.

Definition 6.5. *Set*

$$S_F(\mu,\delta,\varepsilon;-T,T) = \inf\sum_{v\in V} e^{\int_{-T}^T F(g^t v)\,{\rm d}t},$$

where the infimum is taken over all $V\subset T^1M$ that are $(\mu,\delta,\varepsilon;-T,T)$-spanning. Similarly define $S_F^{dyn}(\mu,\delta,\varepsilon;-T,T)$ as the infimum of the same quantity over all dynamically- $(\mu,\delta,\varepsilon;-T,T)$-spanning sets.

The Katok pressure *of F with respect to μ at level δ is defined by*

$$P^\Gamma_{\rm Katok}(\mu,F,\delta) = \limsup_{T\to+\infty}\frac{1}{2T}\log S_F(F,\mu,\delta,\varepsilon;-T,T)\,.$$

Similarly, define

$$P^{dyn}_{\rm Katok}(\mu,F,\delta) = \limsup_{T\to+\infty}\frac{1}{2T}\log S_F^{dyn}(\mu,\delta,\varepsilon;-T,T)\,.$$

The Katok pressure *of F with respect to μ (respectively the* dynamical Katok pressure*) is*

$$P^\Gamma_{\rm Katok}(\mu,F) = \inf_{\delta\in(0,1)}\limsup_{T\to+\infty}\frac{1}{2T}\log S_F(\mu,\delta,\varepsilon;-T,T),$$

respectively

$$P^{dyn}_{\rm Katok}(\mu,F) = \inf_{\delta\in(0,1)}\limsup_{T\to+\infty}\frac{1}{2T}\log S_F^{dyn}(\mu,\delta,\varepsilon;-T,T)\,.$$

By (26), the quantity $P^\Gamma_{\rm Katok}(\mu,F,\delta)$ does not depend on ε.

Comparison between the two kinds of dynamical balls in (27) implies that we have the comparison:

$$P^{dyn}_{\rm Katok}(\mu,F)\leq P^\Gamma_{\rm Katok}(\mu,F)\,.$$

The first and main inequality of Proposition 6.6 was shown in [Kat80]. Compactness was assumed, but his proof [Kat80, (1.4) p. 144] does not use the compactness of the underlying manifold. The second inequality follows obviously from the above considerations.

Proposition 6.6 (Katok [Kat80])**.** *Let $f: X\to X$ be a homeomorphism of a metric space (X,d), and μ be an f-invariant ergodic probability measure. Then for all $\delta>0$,*

$$h_{KS}(\mu)\leq h_{Kat}(f,\mu) = P^{dyn}_{\rm Katok}(\mu,0)\leq P^\Gamma_{\rm Katok}(\mu,0)\,.$$

We provide an appendix by F. Riquelme which shows that these entropies coincide, even in our non-compact setting, cf Theorem A.1.

In the sequel, we will always work with small dynamical balls and the associated Katok pressure $P^{\Gamma}_{\mathrm{Katok}}(\mu, F)$. Assume that μ is ergodic.

For all $A \subset T^1M$, all $\delta \in (0,1)$ and all $\varepsilon, T > 0$, we define

$$S_{F,A}(\mu,\delta,\varepsilon;-T,T) = \inf_{V \subset A\ (\mu,\delta,\varepsilon;-T,T)\text{-spanning}} \sum_{v\in V} e^{\int_{-T}^{T} F(g^t v)\,\mathrm{d}t}$$

and

$$P^{A}_{\mathrm{Katok}}(\mu, F, \delta) = \limsup_{T\to+\infty} \frac{1}{T} \log S_{F,A}(\mu,\delta,\varepsilon,T).$$

The following lemma is elementary but crucial in the sequel.

Lemma 6.7. *Under the assumptions of Theorem 6.1, let $\mu \in \mathcal{M}^F_{1,erg}$ be an ergodic invariant measure. As soon as $\mu(A) > \delta$ we have*

$$P^{\Gamma}_{\mathrm{Katok}}(\mu, F, \delta) \le P^{A}_{\mathrm{Katok}}(\mu, F, \delta). \tag{29}$$

Moreover, if $\mu(A) \ge 1 - \frac{\delta}{6}$, and F is bounded on A, then

$$P^{\Gamma}_{\mathrm{Katok}}(\mu, F, \delta) \ge P^{A}_{\mathrm{Katok}}(\mu, F, \frac{\delta}{2}). \tag{30}$$

Proof. The first inequality is immediate from the definition.

For the second one, let $A' = A \cap g^{-T}A \cap g^{T}A$. It satisfies $\mu(A') \ge 1 - \delta/2$. Consider V a $(\mu,\delta,\varepsilon;-T,T)$-spanning set. As $\mu(\bigcup_{v\in V} B(v,\varepsilon;-T,T)) \ge \delta$, we get $\mu(A' \cap \bigcup_{v\in V} B(v,\varepsilon;-T,T)) \ge \delta/2$. For each $v \in V$ such that $\mu(A' \cap B(v,\varepsilon;-T,T)) > 0$, choose an element v' in the intersection $A' \cap B(v,\varepsilon;-T,T)$, and let V' be the union of all such v'. By construction, $V' \subset A$ is a $(\mu,\delta/2,2\varepsilon;-T,T)$-spanning set.

As F is Hölder continuous, for $v \in V$ such that $\mu(A' \cap B(v,\varepsilon;-T,T)) > 0$ and $v' \in A' \cap B(v,\varepsilon;-T,T)$, the integrals $\int_{-T}^{T} F\circ g^t v\,\mathrm{d}t$ and $\int_{-T}^{T} F\circ g^t v'\,\mathrm{d}t$ differ by an additive constant depending on the Hölder constants of F, and its L^∞-norm on the ε-neighborhood of A, but not on T. This follows from Lemma 3.1 applied to the points $g^{-T}v'$ and $g^{-T}v$ on the one hand (where $g^{-T}v'$ belongs to A thanks to the definition of A', and therefore $g^{-T}v$ belongs to the ε-neighborhood of A), and to $g^{T}v'$ and $g^{T}v$ (with the same argument).

Therefore, up to a multiplicative constant, $\sum_{v\in V} e^{\int_{-T}^{T} F(g^t v)\,\mathrm{d}t}$ is greater than $\sum_{v'\in V'} e^{\int_{-T}^{T} F(g^t v')\,\mathrm{d}t}$. Up to this multiplicative constant, $S_F(\mu,\delta,\varepsilon;-T,T)$ is greater than $S_{F,A}(\mu,\delta/2,2\varepsilon;-T,T)$. Taking the limsup of $1/(2T)\log$ of these quantities leads to the second inequality. □

Since the Katok pressure is defined by taking an infimum over all $(\mu,\delta,\varepsilon;-T,T)$-spanning sets, we deduce the following useful statement.

Lemma 6.8. *Under the assumptions of Theorem 6.1, let $\mu \in \mathcal{M}^F_{1,erg}$ be an ergodic invariant measure. Let $\delta > 0$ be fixed, and for all $T > 0$, let $A_T \subset T^1M$ be a set such that $\mu(A_T) > \delta$. Then*

$$P^{\Gamma}_{\mathrm{Katok}}(\mu, F) \le \limsup_{T\to+\infty} \frac{1}{2T} \log S_{F,A_T}(\mu,\delta,\varepsilon,T).$$

We will use the following analogue of Proposition 6.6 for general potentials.

Proposition 6.9. *Under the assumptions of Theorem 6.1, let $F : T^1M \to \mathbb{R}$ be a Hölder-continuous map, and $\mu \in \mathcal{M}^F_{1,\mathrm{erg}}$ an ergodic probability measure on T^1M such that $\int F^- \,\mathrm{d}\mu < \infty$. Then*

$$h_{KS}(\mu) + \int_{T^1M} F\,\mathrm{d}\mu \le P^{\Gamma}_{\mathrm{Katok}}(\mu, F)\,.$$

Proof. Let μ be an ergodic probability measure and F a Hölder potential. Let $\delta \in (0,1)$ be fixed.

For all $\eta > 0$ and $T > 0$, set

$$G_{T,\eta}(F) = \left\{ v \in T^1M \ ; \ \forall t \geq T, \ \left| \frac{1}{2t} \int_{-t}^{t} F(g^s v) ds - \int F \, \mathrm{d}\mu \right| \leq \eta \right\}.$$

Birkhoff ergodic theorem implies that for all $\eta > 0$, we have $\lim_{T \to +\infty} \mu(G_{T,\eta}(F)) = 1$. Therefore there exist $T_0 > 0$ and a compact set $A_{\delta,\eta} \subset G_{T_0,\eta}(F)$ such that $\mu(A_{\delta,\eta}) > 1 - \frac{\delta}{6}$. Therefore, by (30),

$$P^{\Gamma}_{\mathrm{Katok}}(\mu, F, \delta) \geq P^{A_{\delta,\eta}}_{\mathrm{Katok}}(\mu, F, \frac{\delta}{2}) = \limsup_{T \to +\infty} \frac{1}{2T} \log \inf_{V \subset A_{\delta,\eta} \ (\mu,\delta,\varepsilon;-T,T)\text{-spanning}} \sum_{v \in V} e^{\int_{-T}^{T} F(g^t v) \, \mathrm{d}t}. \quad (31)$$

Let $\mathcal{S}_T \subset A_{\delta,\eta}$ be a finite $(\mu, \delta, \varepsilon; -T, T)$-spanning set which minimizes $\sum_{v \in V} e^{\int_{-T}^{T} F(g^t v) \, \mathrm{d}t}$ among all $(\mu, \delta, \varepsilon; -T, T)$-spanning sets $V \subset A_{\delta,\eta}$. Such a set $\mathcal{S}_T$ exists by compactness of $A_{\delta,\eta}$. Moreover, by definition of $A_{\delta,\eta}$, we have

$$\sum_{v \in \mathcal{S}_T} e^{\int_{-T}^{T} F(g^t v) \, \mathrm{d}t} \geq e^{2T(\int F \, \mathrm{d}\mu - \eta)} \# \mathcal{S}_T \geq e^{2T(\int F \, \mathrm{d}\mu - \eta)} \inf \# V,$$

the infimum being taken over all $(\mu, \delta, \varepsilon, T)$-spanning sets $V \subset A_{\delta,\eta}$.

Proposition 6.6 and Equation (31) lead to

$$P^{\Gamma}_{\mathrm{Katok}}(\mu, F, \delta) \geq \int F \, \mathrm{d}\mu - \eta + h_{KS}(\mu),$$

which concludes the proof of Proposition 6.9 since $\delta \in (0,1)$ and $\eta > 0$ can be arbitrarily small. $\square$

6.3 Escape of mass and pressure at infinity

This paragraph is dedicated to the proof of the following result, of independent interest, which implies Corollary 6.11, a key step in the proof of Theorem 6.1.

Theorem 6.10. *Let M be a nonelementary complete connected negatively curved manifold with pinched negative curvature, and bounded first derivative of the curvature. Let $K \subset M$ be a compact set whose interior intersects $\pi\Omega$, and let $\widetilde{K} \subset \widetilde{M}$ be a compact set such that $p_\Gamma(\widetilde{K}) = K$. Let $F : T^1 \to \mathbb{R}$ be a Hölder potential with $\delta_{\Gamma_{\widetilde{K}}}(F) > -\infty$. Let $\eta > 0$. For all $0 < \alpha \leq 1$ and $R \geq 4$, there exists a positive number $\psi = \psi(\widetilde{K}, F, \eta, \alpha/R)$ with the following property. For every invariant ergodic probability measure $\mu \in \mathcal{M}^F_{1,\mathrm{erg}}$ (i.e., such that $\int F^- \, \mathrm{d}\mu < \infty$) with $\mu(T^1 K_R) \leq \alpha$, we have*

$$h_{KS}(\mu) + \int_{T^1M} F \, \mathrm{d}\mu \leq (1 - \alpha)\delta_{\Gamma_{\widetilde{K}}}(F) + \alpha \delta_\Gamma(F) + \eta + \psi.$$

Moreover, when $\widetilde{K}, F$ and η are fixed, $\psi(\widetilde{K}, F, \eta, \alpha/R)$ tends monotonically to 0 when α/R tends to 0.

Making K grow to exhaust M, we deduce the following corollary, which provides the second half of Theorem 6.1 (the first inequality $\delta^\infty_\Gamma(F) \leq P^i_{var} nfty(F)$ has been proved in proposition 6.3).

Corollary 6.11. *Let M be a nonelementary complete connected negatively curved manifold with pinched negative curvature, and bounded first derivative of the curvature. Let F be a Hölder potential with finite pressure on T^1M. Let $(\mu_n)_{n \geq 0} \in (\mathcal{M}^F_1)^{\mathbb{N}}$ be a sequence of probability measures which converges in the vague topology to a measure μ. Then*

$$\limsup_{n \to +\infty} h_{KS}(\mu_n) + \int F \, \mathrm{d}\mu_n \leq (1 - \|\mu\|)\delta^\infty_\Gamma(F) + \|\mu\|\delta_\Gamma(F).$$

In particular, when $\mu_n \rightharpoonup 0$, then $\limsup_{n \to +\infty} h_{KS}(\mu_n) + \int F \, \mathrm{d}\mu_n \leq \delta^\infty_\Gamma(F)$, so that

$$P^\infty_{\mathrm{var}}(F) \leq \delta^\infty_\Gamma(F).$$

Proof. When $\delta_\Gamma(F) = \infty$, then $\delta_\Gamma^\infty(F) = \infty$ by Proposition 4.13, and the result is obvious. We can therefore assume that $\delta_\Gamma^\infty(F) < \infty$. We will deal with the case $\delta_\Gamma^\infty(F) > -\infty$, as the case $\delta_\Gamma^\infty(F) = -\infty$ can be treated similarly.

Let $\varepsilon > 0$. Let K be a large compact set in M, with a compact lift $\widetilde{K}$ to $\widetilde{M}$ satisfying $\delta_{\Gamma_{\widetilde{K}}}(F) \leq \delta_\Gamma^\infty(F) + \varepsilon$ and $\|\mu\| \leq \mu(T^1K) + \varepsilon$. There are only countably many values of r for which $\mu(\partial T^1 K_r)$ has positive measure as these sets are disjoint. Therefore, we can pick r such that $\mu(\partial T^1 K_r) = 0$. Replacing K with K_r, we can assume $\mu(\partial T^1 K) = 0$.

We apply Theorem 6.10 to $\eta = \varepsilon$, obtaining a function ψ. Let R be large enough so that $\psi(1/R) \leq \varepsilon$. We can also ensure $\mu(\partial T^1 K_R) = 0$. For large enough n, we have $\mu_n(T^1K) \geq \mu(T^1K) - \varepsilon$ and $\mu_n(T^1K_R) \leq \mu(T^1K_R) + \varepsilon \leq \|\mu\| + \varepsilon$. In particular, $\mu_n(T^1K_R) \geq \mu_n(T^1K) \geq \|\mu\| - 2\varepsilon$. Let us estimate $h_{KS}(\mu_n) + \int F \,\mathrm{d}\mu_n$ for such an n, fixed from now on.

We can write μ_n as an average of ergodic measures: $\mu_n = \int_\Omega \mathrm{d}\nu_\omega \,\mathrm{d}\mathbb{P}(\omega)$, where all the ν_ω are invariant probability measures for g_t. Since $\infty > \int F^- \,\mathrm{d}\mu_n = \int (\int F^- \,\mathrm{d}\nu_\omega) \,\mathrm{d}\mathbb{P}(\omega)$, almost all the measures ν_ω belong to $\mathcal{M}^F_{1,\mathrm{erg}}$. We can apply Theorem 6.10 to each of them (with $\alpha = \nu_\omega(T^1K_R)$) and then average with respect to $\mathbb{P}$, yielding

$$
\begin{aligned}
h_{KS}(\mu_n) + \int F \,\mathrm{d}\mu_n &= \int \left(h_{KS}(\nu_\omega) + \int F \,\mathrm{d}\nu_\omega \right) \mathrm{d}\mathbb{P}(\omega) \\
&\leq \int \Big((1 - \nu_\omega(T^1K_R))\delta_{\Gamma_{\widetilde{K}}}(F) + \nu_\omega(T^1K_R)\delta_\Gamma(F) + \varepsilon + \psi(1/R) \Big) \mathrm{d}\mathbb{P}(\omega) \\
&= (1 - \mu_n(T^1K_R))\delta_{\Gamma_{\widetilde{K}}}(F) + \mu_n(T^1K_R)\delta_\Gamma(F) + \varepsilon + \psi(1/R) \\
&\leq (1 - \|\mu\| + 2\varepsilon)(\delta_\Gamma^\infty(F) + \varepsilon) + (\|\mu\| + \varepsilon)\delta_\Gamma(F) + 2\varepsilon.
\end{aligned}
$$

As ε is arbitrary, this gives the conclusion. $\square$

Let us point that when $F = 0$, under the same hypotheses, a stronger version of Corollary 6.11 appears in [Vel19, Thm. 1.1]:

$$
\limsup_{n \to +\infty} h_{KS}(\mu_n) \leq (1 - \|\mu\|)\delta_\Gamma^\infty(0) + \|\mu\| h_{KS}\left(\frac{\mu}{\|\mu\|} \right).
$$

In his PhD [Vel18] (cf also [Vel19]), using a different strategy, Velozo obtains an analogous inequality for pressure in the case of potentials going to 0 at infinity. Our approach is valid for all Hölder potentials, but gives a weaker inequality. However, it provides enough information for our purpose. It is not clear whether the strategy developed in [Vel19] could be adapted to potentials which are not constant at infinity. Our approach could maybe be refined to get his stronger inequality: we will not do it here.

Corollary 6.12. *The pressures $P^\infty_{\mathrm{var}}(F)$ and its modification $P^\infty_{\mathrm{var,erg}}(F)$ are equal.*

Proof. We have obviously the inequality $P^\infty_{\mathrm{var,erg}}(F) \leq P^\infty_{\mathrm{var}}(F)$. Moreover, $P^\infty_{\mathrm{var}}(F) \leq \delta_\Gamma^\infty(F)$ by Corollary 6.11. Finally, Remark 6.4 gives the inequality $\delta_\Gamma^\infty(F) \leq P^\infty_{\mathrm{var,erg}}(F)$. Together, these inequalities show that all these quantities coincide. $\square$

Proof of Theorem 6.10. As the result is obvious if $\delta_\Gamma(F) = \infty$, we may assume $\delta_\Gamma(F) < \infty$. Let $K \subset T^1M$ be a compact set, $R > 0$, and K_R the R-neighborhood of K. Let $F : T^1M \to \mathbb{R}$ be a Hölder potential with finite pressure of F. Let $\eta > 0$.

Let $\mu \in \mathcal{M}^F_{1,\mathrm{erg}}$ be an ergodic probability measure on T^1M, and $0 < \alpha \leq 1$ such that $\mu(K_R) \leq \alpha$. Let $\varepsilon > 0$ be small enough (how small exactly will be prescribed at the end of the proof).

Let A a large compact set containing K and K_R, with $\mu(T^1A) > 1 - \varepsilon$. Define

$$
A_T = \left\{ w \in T^1A,\ \left| \frac{1}{2T} \int_{-T}^{T} F \circ g^t w \,\mathrm{d}t - \int F \,\mathrm{d}\mu \right| \leq \varepsilon \quad \text{and} \frac{1}{2T} \int_{-T}^{T} \mathbf{1}_{K_R}(g^t w) \,\mathrm{d}t \leq \alpha + \varepsilon \right\}.
$$

By Birkhoff ergodic Theorem, there exists $T_1 > 0$ such that for $T \geq T_1$, $\mu(A_T) \geq 1 - \varepsilon$. Then

$$
\mu(A_T \cap g^T A \cap g^{-T} A) \geq 1 - 3\varepsilon.
$$

Let $F : T^1M \to \mathbb{R}$ be Hölder continuous. The strategy is to bound

$$h_{KS}(\mu) + \int F \, \mathrm{d}\mu$$

from above, in terms of periodic orbits, and use Theorem 5.1 to prove Theorem 6.10.

Consider a maximal subset V of $\mathcal{A}_T = A_T \cap g^T A \cap g^{-T} A$ in which all points are at distance at least ε from each other for the dynamical distance (in the universal cover as we are dealing with small dynamical balls) given by $d_T(v, w) = \inf_{p_\Gamma(\widetilde{v})=v,\, p_\Gamma(\widetilde{w})=w} \sup_{|t|\leq T} d(g^t\widetilde{v}, g^t\widetilde{w})$. Then any point in $\mathcal{A}_T$ is within d_T-distance at most ε of a point in V, i.e., $\mathcal{A}_T \subseteq \bigcup_{v\in V} B_\Gamma(v, \varepsilon; -T, T)$. Therefore, V is a $(\mu, \delta, \varepsilon; -T, T)$ spanning set for any $\delta \leq 1/2$, which is additionally ε-separated. Proposition 6.9 and Lemma 6.8 ensure that $h_{KS}(\mu) + \int_{T^1M} F \, \mathrm{d}\mu$ is bounded by the exponential growth rate of the sums $\sum_{p\in V} e^{\int_{-T}^{T} F}$ (where V depends implicitly on T).

Now, to each $v \in V$, we will associate a periodic orbit and bound the above sum in terms of $\widehat{\mathcal{N}}_F(K, K_R, \alpha, T - \tau, T + \tau)$ for some constant $\tau > 0$.

Take $v \in V$. As it belongs to $\mathcal{A}_T$, both points $g^T v$ and $g^{-T} v$ belong to T^1A. By the connecting lemma and the compactness of A, we deduce the existence of a periodic vector v_p, and associated periodic orbit $p(v)$, with $|\ell(p(v)) - 2T| \leq T_0 = T_0(A, \varepsilon)$, and $d(g^t v_p, g^t v) \leq \varepsilon/3$ for all $0 \leq t \leq 2T$. Since the interior of K intersects the nonwandering set, we can also make sure that the orbit $p(v)$ intersects K.

By Lemma 3.1, $\int_0^{\ell(p(v))} F(g^t v_p) \, \mathrm{d}t$ is equal to $\int_0^{2T} F(g^t(g_{-T}v)) \, \mathrm{d}t$ up to a constant depending only on A. Since $v \in A_T$, the latter integral is close to $2T \int F \, \mathrm{d}\mu$, up to $2T\varepsilon$. Altogether, we get

$$\left| \int_0^{\ell(p(v))} F(g^t v_p) \, \mathrm{d}t - \ell(p(v)) \int F \, \mathrm{d}\mu \right| \leq C_0 + \ell(p(v))\varepsilon,$$

for some C_0 depending only on A. In particular, there exists T_3 such that for $T \geq T_3$, $\ell(p(v))$ is also large, so that this inequality becomes

$$\left| \frac{1}{\ell(p(v))} \int_0^{\ell(p(v))} F(g^t v_p) \, \mathrm{d}t - \int F \, \mathrm{d}\mu \right| \leq 2\varepsilon \, .$$

Similarly, we obtain, for T large enough,

$$\ell(p(v) \cap K_{R/2}) \leq \alpha + 2\varepsilon,$$

starting from the same properties for the orbit of v due to the definition of $\mathcal{A}_T$, and using the fact that the orbits of $g^{-T}v$ and v_p remain close to each other up to ε, so the orbit of v_p can be in $K_{R/2}$ only at times when the orbit of $g^{-T}v$ is in K_R.

Moreover, as the set V is $(\varepsilon; -T, T)$ separated, and the periodic orbit $p(v)$ associated to each $v \in V$ is $\varepsilon/3$-close to it, the number of vectors $v \in V$ associated to the same periodic orbit p is bounded by some multiplicative constant times $n_K(p)\ell(p)$.

Therefore, up to some multiplicative constants, $\sum_{v\in V} e^{\int_{-T}^{T} F\circ g^t v \, \mathrm{d}t}$ is bounded by

$$T\widehat{\mathcal{N}}(K, K_{R/2}, \alpha + 2\varepsilon, T - \tau, T + \tau)\,,$$

for some $\tau > 0$ independent of T. Applying Theorem 5.1 with $\eta/2$ and $\widetilde{K}$ and $R/2$, we get that its exponential growth rate is bounded by

$$(1 - \alpha - 2\varepsilon)\delta_{\Gamma_{\widetilde{K}}}(F) + (\alpha + 2\varepsilon)\delta_\Gamma(F) + \eta/2 + \psi((\alpha + 2\varepsilon)/(R/2))$$

where ψ is a function tending to 0 at 0. If ε is small enough, say $\varepsilon \leq \varepsilon_0$, then the error term $2\varepsilon\delta_{\Gamma_{\widetilde{K}}}(F) + 2\varepsilon\delta_\Gamma(F)$ is bounded by $\eta/2$, and we get a bound

$$(1 - \alpha)\delta_{\Gamma_{\widetilde{K}}}(F) + \alpha\delta_\Gamma(F) + \eta + \psi((\alpha + 2\varepsilon)/(R/2)).$$

Finally, we choose $\varepsilon = \alpha\varepsilon_0$, so that $(\alpha + 2\varepsilon)/(R/2)$ is a function of α/R that tends to 0 when α/R tends to 0. This is the desired bound. □

7 Strong positive recurrence

In symbolic dynamics, the notion of strong positive recurrence appeared in several works, as mentioned in the introduction, see for example [Gur69, Gur70, GS98, Sar99, Sar01, Rue03, BBG06, BBG14]. In our geometric context, when $F = 0$, the notion appeared in [ST19, CDST19] under the terminology of "strongly positively recurrent manifold" or strongly positively recurrent action". Independently, it appeared (still in the case $F = 0$) among people interested by geometric group theory, see for example [ACT15, Yan14, Yan19], under the name of "actions with a growth gap" or later "statistically convex-cocompact manifolds". We follow the ergodic terminology of *strong positive recurrence* below, extending the point of view developped in [ST19], in the spirit of the works of symbolic dynamics.

7.1 Different notions of recurrence

Recall some definitions which are classical in symbolic dynamics, and were introduced for the geodesic flow in negative curvature in [PS18, ST19]. Let $K \subset M$ be a compact set, $\widetilde{K} \subset \widetilde{M}$ a compact set such that $p_\Gamma(\widetilde{K}) = K$.

For all $T > 0$ large enough, as in [ST19], we define $U_T(\widetilde{K}) \subset \widetilde{M}$ [4] as the open set

$$U_T(\widetilde{K}) = \{y \in \widetilde{M} \cup \partial\widetilde{M},\ \exists x \in \widetilde{K},\ [x,y)_T \cap \Gamma\widetilde{K} \subset \widetilde{K}\},$$

where $[x,y)_T$ denotes the geodesic segment of length T starting from x on $[x,y)$. In other words, $y \in U_T(\widetilde{K})$ if there exists some geodesic $[x,y)$ starting in $\widetilde{K}$ and arriving at y, which does not meet $\Gamma\widetilde{K} \setminus \widetilde{K}$ until time T.

For technical reasons, we will need to work with the following slightly larger sets:

$$U_{T_0,T}(\widetilde{K}) = \{y \in \widetilde{M} \cup \partial\widetilde{M},\ \exists x \in \widetilde{K},\ [x,y)_{[T_0,T]} \cap \Gamma\widetilde{K} \subset \widetilde{K}\},$$

where $[x,y)_{[T_0,T]}$ denotes the geodesic segment of length $T - T_0$ starting at distance T_0 from x on $[x,y)$. In other words, $y \in U_{T_0,T}(\widetilde{K})$ if there exists some geodesic $[x,y)$ starting in $\widetilde{K}$ and arriving at y, which does not meet $\Gamma\widetilde{K} \setminus \widetilde{K}$ between times T_0 and T.

Let us define $V_T(\widetilde{K}) \subset T^1K$ (resp. $V_{T_0,T}(\widetilde{K}) \subset T^1K$) as the set of unit vectors tangent to K which are images through p_Γ of the unit vector tangent to a geodesic segment $[x,y)$, for some $y \in U_{T_0,T}(\widetilde{K})$ and x associated to y as above.

By definition, the sequences $(U_T(\widetilde{K}))_{T>0}$ $(U_{T_0,T}(\widetilde{K}))_{T>0}$, $(V_{T_0,T}(K))_{T>0}$ and $(V_T(K))_{T>0}$ are decreasing when $T \to \infty$.

Definition 7.1. *A Hölder potential $F : T^1M \to \mathbb{R}$ is said*

1. recurrent *if there exists a compact set $K \subset M$ whose interior intersects the projection $\pi(\Omega)$ of the nonwandering set,*
$$\sum_{p\in\mathcal{P}} n_K(p) e^{\int_p (F-\delta_\Gamma(F))} = +\infty\,;$$

2. positively recurrent *if it is recurrent w.r.t. some compact set $K \subset M$ whose interior intersects $\pi(\Omega)$, and for some $N \geq 1$,*
$$\sum_{p\in\mathcal{P}'_K,\, n_K(p)\leq N} \ell(p) e^{\int_p (F-\delta_\Gamma(F))} < +\infty\,;$$

3. strongly positively recurrent *if its pressure at infinity satisfies*
$$P^\infty_{\mathrm{top}}(F) < P_{\mathrm{top}}(F)\,;$$

4. In [CDST19], the definition has been slightly modified to guarantee that it remains open when $\widetilde{M}$ is a Gromov-hyperbolic metric space

4. exponentially recurrent *w.r.t. an invariant measure* $\mu \in \mathcal{M}_{\leq 1}$ *if there exist a compact set* $K \subset M$ *whose interior intersects* $\pi(\Omega)$, *some compact lift* $\widetilde{K}$ *of* K *with* $p_\Gamma(\widetilde{K}) = K$, $T_0 \geq 0$, $C > 0$ *and* $\alpha > 0$ *such that for* $T \geq T_0$,
$$\mu(V_{T_0,T}(K)) \leq C \exp(-\alpha T)\,.$$

In [PS18, Thms 1.2, 1.4 and 1.6], the following result, reformulated here thanks to Theorem 3.8, is proven.

Theorem 7.2 (Pit-Schapira)**.** *Let* M *be a nonelementary complete connected Riemannian manifold with pinched negative curvature and bounded first derivative of the curvature. Let* $F : T^1M \to \mathbb{R}$ *be a Hölder continuous map.*

1. *The potential* F *is recurrent iff* (Γ, F) *is divergent, iff* m^F *is ergodic and conservative*
2. *The potential* F *is positively recurrent iff* m^F *is finite.*
3. *The potential* F *is positively recurrent iff it is recurrent and there exists a compact set* $K \subset M$ *which intersects at least a closed geodesic, and* $\widetilde{K} \subset \widetilde{M}$ *with* $p_\Gamma(\widetilde{K}) = K$, *such that*

$$\sum_{\gamma \in \Gamma_{\widetilde{K}}} d(o, \gamma o) e^{-\delta_\Gamma(F) d(o,\gamma o) + \int_o^{\gamma o} \widetilde{F}} < +\infty\,.$$

In Section 7.3, we will prove the following result.

Theorem 7.3. *Let* M *be a nonelementary complete connected Riemannian manifold with pinched negative curvature and bounded first derivative of the curvature. Let* $F : T^1M \to \mathbb{R}$ *be a Hölder continuous map. If* $F : T^1M \to \mathbb{R}$ *is strongly positively recurrent, then it is positively recurrent.*

This Theorem has been proven in [ST19] in the case $F \equiv 0$, and the proof is almost the same. We provide it here for the sake of completion and the comfort of the reader.

The contrapositive reformulation is extremely useful:

If the measure m^F is infinite, then $\delta_\Gamma^\infty(F) = \delta_\Gamma(F)$.

It has the following corollary.

Corollary 7.4. *Let* M *be a nonelementary complete connected Riemannian manifold with pinched negative curvature and bounded first derivative of the curvature. Let* $F : T^1M \to \mathbb{R}$ *be a Hölder continuous map. Let* $p : \bar{M} \to M$ *be an infinite Riemannian Galois cover of* M, *and* $H = \pi_1(\bar{M}) \lhd \Gamma = \pi_1(M)$. *Let* $\bar{F} = F \circ dp : T^1\bar{M} \to \mathbb{R}$ *be the lift of* F *to* $T^1\bar{M}$. *Then*
$$\delta_H^\infty(\bar{F}) = \delta_H(\bar{F}) \leq \delta_\Gamma^\infty(F).$$

Proof. The inequality $\delta_H(\bar{F}) \leq \delta_\Gamma^\infty(F)$ is immediate since $H \subset \Gamma$. By contradiction, assume that $\delta_H^\infty(\bar{F}) < \delta_H(\bar{F})$. Then the potential $\bar{F}$ would be strongly positively recurrent. By Theorems 7.3 and 3.7, the associated equilibrium measure m_F is finite and unique. By uniqueness, the measure m_F is invariant under the action of the deck group $G = \Gamma/H$. As G is infinite by hypothesis, it is a contradiction with the finiteness of m_F. □

Remark 7.5. This corollary does not apply to non-regular cover, even for the zero potential. For example, consider the following construction. Given $\Sigma_\Gamma = \mathbb{H}^2/\Gamma$ a compact genus 2 hyperbolic surface, there exists $H < \Gamma$ a non-normal subgroup such that $\Sigma_H = \mathbb{H}^2/H$ is a punctured torus with infinite volume. The (non-regular) covering $p : \Sigma_H = \mathbb{H}^2/H \to \Sigma_\Gamma$ does not satisfy the conclusion of the above corollary. Indeed, Σ_H is convex cocompact, non elementary, with infinite volume. In particular, there exists a large compact set $\widetilde{K} \subset \mathbb{H}^2$ such that $H_{\widetilde{K}}$ is finite, so that
$$\delta_H(0) > 0 \text{ and } \delta_H^\infty(0) = -\infty.$$

Corollary 7.6. *There exists a complete hyperbolic surface* M, *with* $\delta_\Gamma^\infty(0) > 0$, *and a Hölder potential* $F : T^1M \to \mathbb{R}$ *such that* $\delta_\Gamma^\infty(F) = -\infty$.

Observe that if $\delta_\Gamma^\infty(0) > -\infty$, then it is non-negative and every Hölder-continuous potential F which is bounded from below by some constant $-K$ satisfies $\delta_\Gamma^\infty(F) \geq -K$. Therefore examples satisfying Corollary 7.6 must be unbounded from below.

Proof. Let $M = \mathbb{H}^2/\Gamma$ be a $\mathbb{Z}$-cover of a compact hyperbolic surface. By Corollary 7.4, $\delta_\Gamma^\infty(0) = \delta_\Gamma(0) > 0$. It is well known that $\delta_\Gamma(0) = 1$ (it follows for instance from [Bro85], see for instance [CDST19] for details on critical exponents of covers). Choose some compact fundamental domain $D \subset M$ with piecewise smooth boundary for the action of the deck group $G =< g^n \; ; \; n \in \mathbb{Z} >$. For all $n \in \mathbb{Z}$, set $D_n = g^n D$. Build a Hölder continuous map $F : T^1M \to \mathbb{R}$ such that for all $n \in \mathbb{Z}\backslash\{0\}$ and $v \in T^1 D_n$, we have $-|n| \leq F(v) \leq -(|n| - 1)$. Considering compact sets $\widetilde{K}_N$ with $p_\Gamma(\widetilde{K}_N) = \cup_{|n|\leq N} D_n$, we have $\delta_{\Gamma_{\widetilde{K}_N}}(F) = \delta_{\Gamma_{\widetilde{K}_N}}(0) - N$, so that $\delta_\Gamma^\infty(F) = -\infty$. □

The following result is new.

Theorem 7.7. *Let M be a nonelementary complete connected Riemannian manifold with pinched negative curvature and bounded first derivative of the curvature. Let $F : T^1M \to \mathbb{R}$ be a Hölder continuous map. The potential F is strongly positively recurrent iff it is exponentially recurrent w.r.t. the measure m^F given by the Patterson-Sullivan-Gibbs construction.*

The last result that we shall prove provides a very satisfying information on strongly positively recurrent potentials. We will not use it in this paper.

Theorem 7.8. *Let M be a nonelementary complete connected Riemannian manifold with pinched negative curvature and bounded first derivative of the curvature. Let $F : T^1M \to \mathbb{R}$ be a Hölder continuous map. If $F : T^1M \to \mathbb{R}$ is strongly positively recurrent, then for every compact set $\widetilde{K} \subset \widetilde{M}$, whose interior intersects $\pi(\Omega)$, we have*

$$\delta_{\Gamma_{\widetilde{K}}}(F) < \delta_\Gamma(F)\,.$$

It has the following corollary.

Corollary 7.9. *Let M be a nonelementary complete connected Riemannian manifold with pinched negative curvature and bounded first derivative of the curvature. Let $F : T^1M \to \mathbb{R}$ be a Hölder continuous map with finite Gibbs measure m_F. Then the geodesic flow is exponentially recurrent with respect to m_F if and only if for* all *compact set $K \subset M$ whose interior intersects $\pi(\Omega)$ and all compact lift $\widetilde{K}$ of K with $p_\Gamma(\widetilde{K}) = K$, there exists $T_0 \geq 0$, $C > 0$ and $\alpha > 0$ such that for $T \geq T_0$,*

$$\mu(V_{T_0,T}(K)) \leq C \exp(-\alpha T)\,.$$

Before proving these results about strong positive recurrence, we provide in the next paragraph ways of construction of strongly positively recurrent potentials.

7.2 Strong positive recurrence through bumps and wells

Adding a bump λA to a potential F, with A a nonnegative compactly supported Hölder map and $\lambda \to +\infty$, we already proved in Corollary 4.12 the existence of strongly positively recurrent potentials. We restate it below with this terminology.

Corollary 7.10. *On any negatively curved manifold with pinched negative curvature and bounded first derivative of the curvature, there exist Hölder continuous potentials that are strongly positively recurrent.*

It will be convenient to add to a given potential F large bumps of arbitrarily small height. It is what we do in the next proposition.

Proposition 7.11. *Let M be a nonelementary complete connected Riemannian manifold with pinched negative curvature and bounded first derivative of the curvature. Let $F : T^1M \to \mathbb{R}$ be a Hölder potential with finite pressure. For all $\varepsilon > 0$, there exists a compactly supported Hölder map $0 \leq A \leq 1$, such that*

$$\delta_\Gamma^\infty(F + \varepsilon A) = \delta_\Gamma^\infty(F) \leq \delta_\Gamma(F) < \delta_\Gamma(F + \varepsilon A)\,.$$

Proof. For a given $\varepsilon > 0$, by the variational principle for $P_{\mathrm{top}}(F)$, there exists a measure $m_\varepsilon \in \mathcal{M}_1^F$, such that $P_{\mathrm{top}}(F) = \delta_\Gamma(F) = \sup_{m\in\mathcal{M}_1^F} h_{KS}(m) + \int F\,\mathrm{d}m \le h_{KS}(m_\varepsilon) + \int F\,\mathrm{d}m_\varepsilon + \frac{\varepsilon}{2}$.

Choose some compact set K_ε such that $m_\varepsilon(T^1K_\varepsilon) \ge 1-\varepsilon$. Now, choose some Hölder map $0 \le A \le 1$ with compact support such that $A \equiv 1$ on T^1K_ε. Observe that as soon as $0 < \varepsilon < 1/2$, we have

$$\delta_\Gamma(F + \varepsilon A) \ge h_{KS}(m_\varepsilon) + \int F\,\mathrm{d}m_\varepsilon + \varepsilon m(K_\varepsilon) \ge \delta_\Gamma(F) - \frac{\varepsilon}{2} + \varepsilon(1-\varepsilon) > \delta_\Gamma(F)\,.$$

The result follows. □

Adding a bump does not modify the pressure at infinity, and increases the pressure to produce strongly positively recurrent potentials. At the contrary, subtracting a bump, i.e., adding a well, does not modify the pressure at infinity and decreases the pressure towards the pressure at infinity, as shown in the next statement.

Proposition 7.12. *Let M be a nonelementary complete connected Riemannian manifold with pinched negative curvature and bounded first derivative of the curvature. Let $F : T^1M \to \mathbb{R}$ be a Hölder potential with finite pressure. Then for all $\eta > 0$ there exists a compact set $K_\eta \subset M$ and a real $\lambda_\eta > 0$ such that for every Hölder map $A : T^1M \to \mathbb{R}$ with compact support, such that $A \ge 1_{K_\eta}$ and all $\lambda \ge \lambda_\eta$,we have*

$$\delta_\Gamma^\infty(F) = P_{\mathrm{var}}^\infty(F) \le P_{\mathrm{top}}(F - \lambda A) \le P_{\mathrm{var}}^\infty(F) + \eta = \delta_\Gamma^\infty(F) + \eta\,.$$

Proof. By definition of $P_{\mathrm{var}}^\infty(F)$, given $\eta > 0$, there exists a compact set $K_\eta \subset M$ and a real $\lambda_\eta > 0$ such that

$$\delta_\Gamma^\infty(F) = P_{\mathrm{var}}^\infty(F) \le \sup\left\{h_{KS}(\mu) + \int_{T^1M} F\,\mathrm{d}\mu\ ;\ \mu\in\mathcal{M}_1^F \text{ s.t. } \mu(T^1K_\eta)\le\eta\right\} \le P_{\mathrm{var}}^\infty(F) + \eta\,.$$

By Proposition 4.9,

$$\delta_\Gamma^\infty(F) = \delta_\Gamma^\infty(F - \lambda A) \le P_{\mathrm{var}}(F - \lambda A) = \sup_{\mu\in\mathcal{M}_1^F}\left(h_{KS}(\mu) + \int F - \lambda A\,\mathrm{d}\mu\right).$$

We study this supremum by distinguishing measures μ with $\mu(T^1K_\eta)$ greater or smaller than η. On the one hand, we have

$$\sup_{\mu\in\mathcal{M}_1^F,\,\mu(T^1K_\eta)\ge\eta}\left(h_{KS}(\mu) + \int (F - \lambda A)\,\mathrm{d}\mu\right) \le P_{\mathrm{var}}(F) - \lambda\eta\,.$$

If $\lambda \ge \lambda_\eta$ is large enough, this quantity is arbitrarily negative. On the other hand, as $A \ge 0$, we have

$$\sup_{\mu\in\mathcal{M}_1^F,\,\mu(T^1K_\eta)\le\eta}\left(h_{KS}(\mu) + \int F - \lambda A\,\mathrm{d}\mu\right) \le \sup_{\mu\in\mathcal{M}_1^F,\,\mu(T^1K_\eta)\le\eta}\left(h_{KS}(\mu) + \int F\,\mathrm{d}\mu\right) \le P_{\mathrm{var}}^\infty(F) + \eta\,.$$

We deduce the desired result for λ large enough:

$$\delta_\Gamma^\infty(F) \le P_{\mathrm{var}}(F - \lambda A) \le \max(P_{\mathrm{var}}(F) - \lambda\eta, P_{\mathrm{var}}^\infty(F) + \eta) = P_{\mathrm{var}}^\infty(F) + \eta\,.$$

□

7.3 Strong positive recurrence implies positive recurrence

In this section, we shall prove Theorem 7.3. We follow the proof of [ST19] in the case $F = 0$.

Assume that F is strongly positively recurrent. By definition, there exists a compact set $K \subset M$ whose interior intersects at least a closed geodesic, and a compact set $\widetilde{K} \subset \widetilde{M}$ with $p_\Gamma(\widetilde{K}) = K$, such that

$$\delta_{\Gamma_{\widetilde{K}}}(F) < \delta_\Gamma(F)\,.$$

An elementary computation shows that this strict inequality implies the convergence of the series $\sum_{\gamma\in\Gamma_{\widetilde K}} d(o,\gamma o)e^{-\delta_\Gamma(F)d(o,\gamma o)+\int_o^{\gamma o}\widetilde F}$. Therefore, to prove that strong positive recurrence implies positive recurrence, by Theorem 7.2 (point 3), it is enough to show that F is recurrent. By Theorem 3.8, it is equivalent to show that ν_o^F gives full measure to the radial limit set $\Lambda_\Gamma^{\mathrm{rad}}$.

As observed in [ST19], we have

$$\Lambda_\Gamma\setminus\Lambda_\Gamma^{\mathrm{rad}}\subset\Gamma.\left(\bigcap_{T>0}U_T(\widetilde K)\right).$$

The following variant also holds:

$$\Lambda_\Gamma\setminus\Lambda_\Gamma^{\mathrm{rad}}\subset\Gamma.\left(\bigcap_{T>T_0}U_{T_0,T}(\widetilde K)\right)=\bigcup_{T_0>0}\bigcap_{T>T_0}U_{T_0,T}(\widetilde K)\,.$$

Indeed, both sets on the right represent points $y\in\partial\widetilde M$ such that for some $x\in\widetilde K$, the geodesic $[x,y)$ stays a bounded amount of time in $\Gamma.\widetilde K$, whereas the set on the left is the set of $y\in\Lambda_\Gamma$ such that the geodesic $[xy)$ eventually leaves every compact set.

The proof of Theorem 7.3 consists in proving that for some $T_0>0$, we have $\nu_o^F(\cap_{T>0}U_{T_0,T}(\widetilde K))=0$. In [ST19, Eq.29], we used the inclusion

$$\Gamma o\cap U_T(\widetilde K)\subset\bigcup_{\gamma\in\Gamma_{\widetilde K},\mathrm{d}(o,\gamma o)\geq T-2D}\mathcal O_{\widetilde K}(\gamma\widetilde K)\,.$$

We need a refinement of this inclusion. The following lemma is a key step of the proof, and will be useful also in Section 7.4.

Lemma 7.13. *Under the assumptions of Theorem 7.3, for all $\varepsilon>0$, there exist a finite set $\{g_1,\dots,g_N\}$ of elements of Γ and some $T_0>0$ such that for all $T>T_0+2D+\varepsilon$, we have*

$$\bigcup_{\gamma\in\Gamma_{\widetilde K_\varepsilon},\,d(o,\gamma o)\geq T+2D+T_0}\mathcal O_o(\gamma\widetilde K)\quad\subset\quad U_{T_0,T}(\widetilde K)\quad\subset\quad\bigcup_{i=1}^N\ \bigcup_{\gamma\in\Gamma_{\widetilde K},\mathrm{d}(o,\gamma o)\geq T-2D-T_0} g_i.\mathcal O_{\widetilde K}(\gamma\widetilde K)\,.$$

Proof. The first inclusion uses the same kind of arguments as for [ST19, Eq.29]. If $\gamma\in\Gamma_{\widetilde K_\varepsilon}$, the geodesic segment $[o,\gamma o]$ does not intersect $\Gamma\widetilde K_\varepsilon$ outside $\widetilde K_\varepsilon$ and $\gamma\widetilde K_\varepsilon$. And by Lemma 2.3, for every $\varepsilon>0$, there exists $T_0>0$ depending on ε and on the diameter D of $\widetilde K$, such that if $y\in\mathcal O_o(\gamma\widetilde K)$, then the geodesic segments $[o,y]$ and $[o,\gamma o]$ stay ε-close during a time at least $d(o,\gamma o)-T_0$. In particular, if $d(o,\gamma o)\geq T+2D+T_0$, then the geodesic segment $[o,y]_T$ cannot intersect $\Gamma\widetilde K$ outside $\widetilde K_\varepsilon$. It could happen that $[o,y]_T$ intersects $\Gamma\widetilde K\cap\widetilde K_\varepsilon\setminus\widetilde K$. But this can happen only on a segment of length at most $D+\varepsilon$ starting from o. The conclusion follows.

For the right inclusion, let $T_0>0$ be the constant associated to ε and D by Lemma 2.3. Now, introduce the family $(g_i)_{1\leq i\leq N}$ of isometries such that the T_0-neighborhood $\widetilde K_{T_0}$ of $\widetilde K$ is included in $\cup_i g_i\widetilde K$. Consider a point $y\in U_{T_0,T}(\widetilde K)$. Consider on the segment $[o,y]_{T_0}$ the last copy $g_i\widetilde K$ intersected by this short segment, and the first copy $h\widetilde K$ intersected by the segment $[o,y]_{T_0,T}$. By definition, $g_i^{-1}h\in\Gamma_{\widetilde K}$, so that $h\in g_i\Gamma_{\widetilde K}$. The inclusion follows easily. □

Lemmas 7.13 and 3.6 have the following corollary.

Corollary 7.14. *Under the assumptions of Theorem 7.3, for all $0<\eta<\delta_\Gamma(F)-\delta_{\Gamma_{\widetilde K}}(F)$, there exists $T_1>0$ such that for $T\geq T_1$, we have*

$$\nu^F(U_{T_0,T}(\widetilde K))\leq Ce^{-(\delta_\Gamma(F)-\delta_{\Gamma_{\widetilde K}}(F)-\eta)T}.$$

In particular

$$\nu^F(\cap_{T>T_0}U_{T_0,T}(\widetilde K))=0\,.$$

Similar statements appeared in [ST19] and [CDST19], but it appears that some details are welcome on the limit process. We include therefore a detailed (short) argument.

Proof. Choose some $0 < \eta < \delta_\Gamma(F) - \delta_{\Gamma_{\widetilde{K}}}(F)$. By property (10), Lemmas 7.13 and 3.6, for all $s_n > \delta_\Gamma(F)$ close enough to $\delta_\Gamma(F)$, and $T > T_0$ large enough, we have

$$\begin{aligned}
\nu^{F,s_n}(U_{T_0,T}(\widetilde{K})) = \nu^{F,s_n}(\Gamma o \cap U_{T_0,T}(\widetilde{K})) &\leq \sum_{i=1}^{N} \sum_{\gamma\in\Gamma_{\widetilde{K}}, \mathrm{d}(o,\gamma o)\geq T-2D-T_0} \nu^{F,s_n}(g_i.\mathcal{O}_{\widetilde{K}}(\gamma\widetilde{K}))\\
&\leq N \times C \times \sum_{\gamma\in\Gamma_{\widetilde{K}}, \mathrm{d}(o,\gamma o)\geq T-2D-T_0} e^{-s_n d(o,\gamma o)+\int_o^{\gamma o}\widetilde{F}}\\
&\leq \text{Constant} \times e^{(\delta_{\Gamma_{\widetilde{K}}}(F)+\eta-s_n)T}.
\end{aligned}$$

Now, ν^F is the weak limit $\nu^F = \lim_{n\to\infty}\nu^{F,s_n}$. Recall that any Borel probability measure on a metric space is regular, see [Bil99, Thm 1.1]. In particular, we have

$$\nu^F(U_{T_0,T}(\widetilde{K})) = \sup\left\{\int \varphi\,\mathrm{d}\nu^F, \varphi\in C_c(\widetilde{M}\cup\partial\widetilde{M}), 0\leq\varphi\leq 1,\ \mathrm{supp}(\varphi)\subset U_{T_0,T}(\widetilde{K})\right\}.$$

For such a map φ, we have

$$\int\varphi\,\mathrm{d}\nu^F = \lim_{s_n\to\delta_\Gamma(F)}\int\varphi\,\mathrm{d}\nu^{F,s_n} \leq \liminf_{s_n\to\delta_\Gamma(F)} \nu^{F,s_n}(U_{T_0,T}(\widetilde{K})) \leq \text{Constant}\times e^{(\delta_{\Gamma_{\widetilde{K}}}(F)+\eta-\delta_\Gamma(F))T}.$$

Regularity of ν^F leads to

$$\nu^F(U_{T_0,T}(\widetilde{K})) \leq \text{Constant}\times e^{(\delta_{\Gamma_{\widetilde{K}}}(F)+\eta-\delta_\Gamma(F))T}. \tag{32}$$

The result follows. □

Theorem 7.3 follows.

7.4 Strong positive recurrence and exponential recurrence

Let us prove Theorem 7.7.

Proof. The implication "strong positive recurrence implies exponential recurrence w.r.t. m_F" was essentially shown in the above proof of Theorem 7.3, and in particular Equation (32). Indeed, the set $V_{T_0,T}(\widetilde{K})$ is so small that for T large enough, it admits a lift $\widetilde{V}_{T_0,T}(\widetilde{K})$ such that $m_F(V_{T_0,T}(K)) = \widetilde{m}_F(\widetilde{V}_{T_0,T}(\widetilde{K}))$. And on $T^1\widetilde{M}$, the product structure $m_F \sim \nu^F\times\nu^F\times\mathrm{d}t$, see Equation (11), in the Hopf coordinates, see Equation (5), shows that up to some constant c,

$$m_F(V_{T_0,T}(\widetilde{K})) = \widetilde{m}_F(\widetilde{V}_{T_0,T}(\widetilde{K})) \leq c\nu^F(\partial\widetilde{M})\times\nu^F(U_{T_0,T}(\widetilde{K})).$$

Equation (32) concludes. Note that this proof, combined with Theorem 7.8, implies Corollary 7.9.

Conversely, suppose that m_F is exponentially recurrent, so that for some compact set $K\subset M$ whose interior intersects $\pi(\Omega)$, some $T_0 > 0$ and $\alpha > 0$, we have

$$m_F(V_{T_0,T}(K)) = \widetilde{m}_F(\widetilde{V}_{T_0,T}(K)) \leq \exp(-\alpha T).$$

The first step consists in showing that for all $T\geq T_0$, we have

$$\nu^F(U_{T_0,T}(\widetilde{K})) \leq e^{-\alpha T}. \tag{33}$$

By definition, if $v\in\widetilde{V}_{T_0,T}(\widetilde{K})$, then $v^+\in U_{T_0,T}(\widetilde{K})$, and $v^-\in\mathcal{O}_{v^+}(\widetilde{K})$. Recall that m_F is supported in Ω. As above, Equation (11) and (5) show that up to some constant c,

$$m_F(V_{T_0,T}(K)) = \widetilde{m}_F(\widetilde{V}_{T_0,T}(\widetilde{K})) \geq \frac{1}{c}\inf_{v\in\widetilde{\Omega}\cap T^1\widetilde{K}} \nu^F(\mathcal{O}_{v^+}(\widetilde{K})\times\nu^F(U_{T_0,T}(\widetilde{K})).$$

In the above infimum, the vector v varies in the compact set $\widetilde{\Omega} \cap T^1\widetilde{K}$, and ν^F has full support in the limit set, so that this infimum is positive. Therefore, (33) is proven.

In the sequel, we will need to consider a compact set $\widetilde{L}$ large enough to satisfy the lower bound in lemma 3.6. By a standard use of lemma 2.3, for all $\varepsilon > 0$ there exists $\tau > 0$, such that if $\widetilde{L} \supset \widetilde{K}_\varepsilon \supset \widetilde{K}$ contains an ε-neighbourhood of $\widetilde{K}$, uniformly in $T \geq T_0 + 2\tau$, we have

$$U_{T_0,T}(\widetilde{L}) \subset U_{T_0+\tau,T-\tau}(\widetilde{K})$$

In particular, up to changing slightly T_0 and α, the compact set $\widetilde{L}$ also satisfies (33). We omit in the sequel to change the constant, and just assume that $\widetilde{K}$ satisfies the lower bound in lemma 3.6.

As $\nu^F = \lim_{s_n \to \delta_\Gamma(F)} \nu^{F,s_n}$, we deduce from Equation (33) that for some $0 < \beta \leq \alpha$ and all s_n close enough to $\delta_\Gamma(F)$, we have $\nu^{F,s_n}(U_{T_0,T}(\widetilde{K})) \leq e^{-\beta T}$. Now, lemma 7.13 gives

$$\bigcup_{\gamma \in \Gamma_{\widetilde{K}_\varepsilon},\, d(o,\gamma o) \geq T+2D+T_0} \mathcal{O}_o(\gamma \widetilde{K}) \subset U_{T_0,T}(\widetilde{K})\,,$$

so that, as ν^{F,s_n} is supported on Γo,

$$\nu^{F,s_n}\left(\Gamma o \cap \bigcup_{\gamma \in \Gamma_{\widetilde{K}_\varepsilon},\, d(o,\gamma o) \geq T+2D+T_0} \mathcal{O}_o(\gamma \widetilde{K}) \right) \leq e^{-\beta T}\,.$$

As the group Γ acts properly discontinuously on $\widetilde{M}$ and $\widetilde{K}$ is compact, the intersections of shadows in the above union have a bounded multiplicity, say M. Therefore, we deduce that

$$\sum_{\gamma \in \Gamma_{\widetilde{K}_\varepsilon},\, d(o,\gamma o) \geq T+2D+T_0} \nu^{F,s_n}(\mathcal{O}_o(\gamma \widetilde{K})) \leq e^{-\beta T}\,.$$

The orbital shadow lemma 3.6 implies that up to some multiplicative constant, uniformly in s_n, for all $T \geq T_0$ large enough, we have

$$\sum_{\gamma \in \Gamma_{\widetilde{K}_\varepsilon},\, d(o,\gamma o) \geq T+2D+T_0} e^{-s_n d(o,\gamma o) + \int_o^{\gamma o} \widetilde{F}} \leq e^{-\beta T}\,. \tag{34}$$

The series on the left is comparable to the series $\displaystyle\sum_{k=[T+2D+T_0]}^{\infty} e^{-s_n k} \sum_{\gamma \in \Gamma_{\widetilde{K}'_\varepsilon},\, d(o,\gamma o) \in [k,k+1[} e^{\int_o^{\gamma o} \widetilde{F}}$. By definition, the critical pressure satisfies

$$\delta_{\Gamma_{\widetilde{K}'_\varepsilon}}(F) = \limsup_{k \to \infty} \frac{1}{k} \log \sum_{\gamma \in \Gamma_{\widetilde{K}'_\varepsilon},\, d(o,\gamma o) \in [k,k+1[} e^{\int_o^{\gamma o} \widetilde{F}}\,.$$

By contradiction, assume that $\delta_{\Gamma_{\widetilde{K}'_\varepsilon}}(F) = \delta_\Gamma(F)$. Let us fix $\varepsilon \in \left(0, \frac{\beta}{2}\right)$. Then there would exists a sequence $k_j \to \infty$, for k_j large enough, $\displaystyle\sum_{\gamma \in \Gamma_{\widetilde{K}'_\varepsilon},\, d(o,\gamma o) \in [k_j,k_j+1[} e^{\int_o^{\gamma o} \widetilde{F}} \geq e^{(\delta_\Gamma(F)-\varepsilon)K_j}$. This would imply, for $\delta_\Gamma(F) < s_n < \delta_\Gamma(F) + \varepsilon$, that the left hand side in (34) is bounded from below by $\frac{1}{2} e^{-2\varepsilon T}$, which is a contradiction. Therefore $\delta_{\Gamma_{\widetilde{K}'_\varepsilon}}(F) < \delta_\Gamma(F)$ and exponential recurrence implies strong positive recurrence. $\square$

Remark 7.15. *Following carefully the proof shows that, if there exists $C, \alpha > 0$ such that for all T large enough, we have $m_F(V_{T_0,T}(\widetilde{K}) \leq Ce^{-\alpha T}$, then*

$$\delta_{\Gamma_K}(F) \leq \delta_\Gamma(F) - \alpha.$$

7.5 SPR is independent of the compact set

This paragraph is devoted to the proof of Theorem 7.8. Let $F : T^1M \to \mathbb{R}$ be a strongly positively recurrent Hölder potential. Let $K \subset M$ be a compact set whose interior $\overset{\circ}{K}$ intersects $\pi\Omega$, and $\widetilde{K} \subset \widetilde{M}$ a compact set such that $p_\Gamma(\widetilde{K}) = K$. Our proof relies on the following proposition, which provides a convenient upperbound for the growth of $\Gamma_{\widetilde{K}}$.

Proposition 7.16. *Let $A : T^1M \to [0, +\infty)$ be a non-negative Hölder potential whose support is contained in the interior of K. Then*

$$\delta_{\Gamma_{\widetilde{K}}}(F) \le \delta_\Gamma(F - A).$$

Proof. Let $K' \subset \overset{\circ}{K}$ be a compact set containing $\pi(\mathrm{Supp}(A))$ and $\varepsilon > 0$ such that the 2ε-neighbourhood of K' is contained in K. By definition, for all $T > 0$ and $\gamma \in \Gamma_{\widetilde{K}}(T-1, T)$, there exist $x, y \in \partial\widetilde{K}$ such that $[x, \gamma y] \cap \Gamma \cdot \widetilde{K} \subset \{x, \gamma y\}$ and $d(x, \gamma y) \in [T-1, T]$. By the Connecting Lemma 2.6, there exists $T_0 > 0$ depending only on K and ε, a periodic orbit $p_\gamma \subset T^1M$ of length $\ell(p_\gamma) \in [T-1, T+T_0]$ with a lift $\widetilde{p}_\gamma \subset T^1\widetilde{M}$ such that the geodesic segment $[x, \gamma y]$ is contained in the ε-neighbourhood of $\widetilde{p}_\gamma$ except maybe inside $B(x, T_0) \cup B(\gamma y, T_0)$. In particular, we have

$$\ell(p_\gamma \cap K') \le 5T_0.$$

Moreover, still by Lemma 2.6, there exists $C_0 > 0$ depending only on $\widetilde{K}$ such that the number of $\gamma \in \Gamma_{\widetilde{K}}$ leading as above to the same periodic orbit p_γ is at most C_0T. Set

$$\|F\|_{\infty, T_0} = \max\{|F(v)\ ;\ d(\pi v, K) \le T_0\} \quad \text{and} \quad \|A\|_\infty = \max\{|A(v)|, v \in T^1M\}\,.$$

By lemma 3.1, we deduce that there exists $C > 0$ such that for all $s \in [\delta_{\Gamma_{\widetilde{K}}(F)}, 2\delta_\Gamma(F)]$, we have

$$\begin{aligned}
\sum_{\gamma \in \Gamma_{\widetilde{K}}(T-1,T)} e^{-sd(o,\gamma o) + \int_o^{\gamma o} F} &\le C \sum_{\gamma \in \Gamma_{\widetilde{K}}(T-1,T)} e^{-sT + \int_x^{\gamma y} F} \\
&\le C_0TCe^{-sT} e^{5T_0\|F\|_{\infty,T_0}} \sum_{\gamma \in \mathcal{P}_K(T-1,T+T_0),\ \ell(p\cap K') \le 5T_0} e^{\int_p F} \\
\text{[as } A \equiv 0 \text{ outside } K] \qquad &\le C_0TCe^{-sT} e^{5T_0\|F\|_{\infty,T_0}} \sum_{\gamma \in \mathcal{P}_K(T-1,T+T_0),\ \ell(p\cap K') \le 5T_0} e^{\int_p (F-A)} \\
&\le C_0TCe^{-sT} e^{5T_0(\|F\|_{\infty,T_0} + \|A\|_\infty)} \sum_{\gamma \in \mathcal{P}_K(T-1,T+T_0)} e^{\int_p (F-A)},
\end{aligned}$$

where $\mathcal{P}_K(T-1, T_0)$ is the set of periodic orbits with length in $[T-1, T_0]$ whose projection intersects K.

By Theorem 1.2,

$$\limsup_{T\to+\infty} \frac{1}{T} \log \sum_{\gamma \in \mathcal{P}_K(T-1,T+T_0)} e^{\int_p (F-A)} = \delta_\Gamma(F - A).$$

Therefore, $\delta_{\Gamma_{\widetilde{K}}}(F) \le \delta_\Gamma(F - A)$. □

We will also need the following proposition.

Proposition 7.17. *Let $F_1, F_2 : T^1M \to \mathbb{R}$ be two Hölder potentials with finite pressure that satisfy $F_2 \le F_1$ and $F_2(w) < F_1(w)$ for some $w \in \Omega$. IF F_2 admits a finite Gibbs measure m_{F_2}, then their pressures satisfy*

$$P_{\mathrm{top}}(F_2) < P_{\mathrm{top}}(F_1)\,.$$

Proof. For $i = 1, 2$, we have

$$P_{\mathrm{top}}(F_i) = P_{\mathrm{var}}(F_i) \sup\{\int F_i dm + h_{KS}(m)\ ;\ m \text{ invariant probability measure with } \int F_i^- d\mu_i < +\infty\}.$$

As $F_2 \le F_1$, we have $\int F_1^- dm \le \int F_2^- dm$ for any invariant probability measure m. Therefore, when $m = m_{F_2}$,

$$P_{\mathrm{var}}(F_2) = \int F_2 dm_{F_2} + h_{KS}(m_{F_2}) \le \int F_1 dm_{F_2} + h_{KS}(m_{F_2}) = P_{\mathrm{var}}(F_1).$$

Assume by contradiction that $P_{\mathrm{var}}(F_1) = P_{\mathrm{var}}(F_2)$. Then by the previous inequalities,

$$\int F_1 dm_{F_2} = \int F_2 dm_{F_2}\,.$$

It implies that $F_1 = F_2$ m_{F_2}-almost surely. As $F_2 \le F_1$ and $F_2 < F_1$ on a neighbourhood of w, this contradicts the fact that m_{F_2} has full support in Ω. Therefore $P_{\mathrm{var}}(F_2) < P_{\mathrm{var}}(F_1)$. $\square$

Let us conclude the proof of Theorem 7.8.

Proof of Theorem 7.8. Choose some $w \in \Omega \cap T^1K$ and $\varepsilon > 0$ such that $B(w, 2\varepsilon) \subset T^1K$. Let $A : T^1M \to [0, +\infty)$ be a non-negative Hölder continuous potential supported in $B(w, \varepsilon)$ with $A(w) > 0$. By Proposition 4.9, for all $\eta > 0$, $\delta_\Gamma^\infty(F - \eta A) = \delta_\Gamma^\infty(F)$. Moreover, the map $\eta \mapsto \delta_\Gamma(F - \eta A)$ is Lipschitz continuous. As F is strongly positively recurrent, for $\eta > 0$ small enough, the map $F - \eta A$ is still strongly positively recurrent. In particular, by Theorem 1.4, it admits a finite Gibbs measure. Therefore, Propositions 7.16 and 7.17 give the inequalities

$$\delta_{\Gamma_{\widetilde{K}}}(F) \le \delta_\Gamma(F - \eta A) < \delta_\Gamma(F)\,.$$

Theorem 7.8 follows. $\square$

A Entropies for geodesic flows, by Felipe Riquelme

In this appendix, we prove that three important notions of entropies of an invariant probability measure for the dynamic of the geodesic flow coincide, namely the Kolmogorov-Sinai, the Katok and the Brin-Katok entropies. These results were firstly proved for dynamical systems defined on compact metric spaces in [Kat80] and [BK83], and generalized for Lipschitz maps on noncompact manifolds in [Riq18] taking only in consideration ergodic measures. This appendix treats the case of non-ergodic measures as well as the one of Katok and local (Brin-Katok) entropies relative to small dynamical balls.

A.1 Different notions of entropy

Let (M, g) be a smooth Riemannian manifold with pinched negative sectional curvatures $-b^2 \le K_g \le -a^2$, for some $0 < a \le b$. Let $\widetilde{M}$ be its universal cover, $\Gamma = \pi_1(M)$ its fundamental group, and $p_\Gamma : T^1\widetilde{M} \to T^1M$ the differential of the quotient map $\widetilde{M} \to M$. Using abuse of notation, we will denote by (g^t) the geodesic flow on T^1M and the corresponding one on $T^1\widetilde{M}$.

For all definitions of entropy, the entropy of the geodesic flow (g^t) with respect to an invariant probability measure μ on T^1M is defined as the entropy of its time 1-map $g := g^1$ with respect to μ. If μ is ergodic w.r.t. the flow, it is not necessarily ergodic w.r.t. this time one map g^1. However, in this case, a.e. time $\tau \in \mathbb{R}$ is ergodic, so that the relation $h(g^\tau) = |\tau| h(g^1)$ allows us to assume, without loss of generality, that μ is ergodic w.r.t. g^1.

A.1.1 The Kolmogorov-Sinai entropy

Let $\mu \in \mathcal{M}_1$ be an invariant probability measure on T^1M. Let $\mathcal{P}$ be a finite or countable measurable partition of T^1M. The entropy of $\mathcal{P}$ is defined by

$$H(\mu, \mathcal{P}) = -\sum_{P \in \mathcal{P}} \mu(P) \log \mu(P)\,.$$

The join $\mathcal{P}^n = \bigvee_{i=0}^{n} g^{-i}\mathcal{P}$ is the partition whose atoms are of the form $P_0 \cap g^{-1}P_1 \cap \cdots g^{-n}P_n$, where the sets P_i are in $\mathcal{P}$. The entropy of μ w.r.t. $\mathcal{P}$ is the limit

$$h(\mu, \mathcal{P}) = \lim_{n\to\infty} \frac{1}{n} H(\mu, \mathcal{P}^n)\,.$$

The Kolmogorov-Sinai entropy of μ is the supremum

$$h_{KS}(\mu) := \sup_{\mathcal{P}} h(\mu, \mathcal{P})$$

over all partitions $\mathcal{P}$ with finite entropy.

A.1.2 The Katok entropies

For completeness, let us recall the following definitions. Let d be any metric on $T^1\widetilde{M}$ equivalent to the Sasaki metric. Using abuse of notation, we will denote d the corresponding induced metric on T^1M.

Let $\widetilde{v} \in T^1\widetilde{M}$ and $\varepsilon, T > 0$. The *dynamical ball* $B(\widetilde{v}, \varepsilon; T)$ on the universal cover is defined by

$$B(\widetilde{v}, \varepsilon; T) = \{\widetilde{w} \in T^1\widetilde{M} \ ;\ \forall t \in [0, T], d(g^t\widetilde{v}, g^t\widetilde{w}) \leq \varepsilon\}.$$

As in [ST19, Rem 3.1], we consider on T^1M the small dynamical ball $B_\Gamma(v, \varepsilon; T) = p_\Gamma(B(\widetilde{v}, \varepsilon; T)$ and the big dynamical ball

$$B_{dyn}(v, \varepsilon; T) = \{w \in T^1M \ ;\ \forall t \in [0, T], d(g^tv, g^tw) \leq \varepsilon\} \supset B_\Gamma(v, \varepsilon; T). \tag{35}$$

Both balls coincide as soon as the injectivity radius of M is bounded from below away from zero and ε small enough uniformly on T^1M. More generally, if along the orbit $(g^tv)_{0\leq t\leq T}$, the injectivity radius at the point $\pi(g^tv)$ is larger than ε, then

$$B_{dyn}(v, \varepsilon; T) = B_\Gamma(v, \varepsilon; T)\,. \tag{36}$$

Given a probability measure μ on T^1M, $\delta \in (0, 1)$ and $\varepsilon, T > 0$, a set $V \subset T^1M$ is $(\mu, \delta, \varepsilon; T)$-*spanning* (respectively *dynamically-*$(\mu, \delta, \varepsilon; T)$*-spanning*) if

$$\mu\left(\bigcup_{v\in V} B_\Gamma(v, \varepsilon; T)\right) \geq \delta\,, \quad \text{respectively} \quad \mu\left(\bigcup_{v\in V} B_{dyn}(v, \varepsilon; T)\right) \geq \delta.$$

Of course, a $(\mu, \delta, \varepsilon; T)$-spanning set is also dynamically-$(\mu, \delta, \varepsilon; T)$-spanning.

Let $S_\Gamma(\mu, \delta, \varepsilon; T)$ (resp. $S_{dyn}(\mu, \delta, \varepsilon; T)$) be the minimal cardinality of a $(\mu, \delta, \varepsilon; T)$-spanning set (resp. of a dynamically-$(\mu, \delta, \varepsilon; T)$-spanning set).

The Katok entropy of μ w.r.t the small (resp. big) dynamical balls is defined as

$$h^\Gamma_{\mathrm{Kat}}(\mu) = \inf_{\delta>0} \limsup_{T\to\infty} \frac{1}{T} \log S_\Gamma(\mu, \delta, \varepsilon; T)\,, \quad \text{resp.} \quad h^{dyn}_{\mathrm{Kat}}(\mu) = \inf_{\delta>0} \limsup_{T\to\infty} \frac{1}{T} \log S_{dyn}(\mu, \delta, \varepsilon; T)\,.$$

Note that, in both definitions above, the supremum limits are independent of ε (see for instance [PPS15, Lemma 3.14]).

A.1.3 The Brin-Katok entropies

Given a compact set $\mathcal{K} \subset T^1M$, we define the local entropies on $\mathcal{K}$ relative respectively to small and big dynamical balls as

$$\bar{h}^\Gamma_{loc}(\mu, \mathcal{K}) = \operatorname*{sup\,ess}_{v\in\mathcal{K}} \limsup_{T\to\infty,\ g^Tv\in\mathcal{K}} -\frac{1}{T} \log \mu(B_\Gamma(v, \varepsilon; T))\,,$$

and

$$\bar{h}^{dyn}_{loc}(\mu, \mathcal{K}) = \operatorname*{sup\,ess}_{v\in\mathcal{K}} \limsup_{T\to\infty,\ g^Tv\in\mathcal{K}} -\frac{1}{T} \log \mu(B_{dyn}(v, \varepsilon; T))\,.$$

Taking the supremum over compact sets $\mathcal{K}$ leads to the definition of the upper Brin-Katok local entropies

$$\bar{h}^\Gamma_{BK}(\mu) = \sup_{\mathcal{K}} \bar{h}^\Gamma_{loc}(\mu, \mathcal{K}) \quad \text{and} \quad \bar{h}^{dyn}_{BK}(\mu) = \sup_{\mathcal{K}} \bar{h}^{dyn}_{loc}(\mu, \mathcal{K})\,.$$

As in the case of the Katok entropies, the supremum limits above do not depend on ε.

A.2 All entropies coincide

The main result of this appendix is stated below. Despite of being expected, the relevance of it lies on its many potential applications. For example, in [ST19, Theorem 1.4], a formula relating local entropies of invariant measures through a change of the Riemannian metric has been established, which brings as consequence such a formula for Kolmogorov-Sinai entropies. In particular, it also gives a relation between topological entropies of geodesic flows coming from perturbations of a given Riemannian metric by the use of measures of maximal entropies on the corresponding dynamics.

Theorem A.1. *Let (M, g) be a Riemannian manifold with pinched negative curvatures $-b^2 \leq K_g \leq -a^2 < 0$. Let $\mu \in \mathcal{M}_1$ be an ergodic invariant probability measure for the geodesic flow on T^1M. Then*

$$h_{KS}(\mu) = \bar{h}^{\Gamma}_{BK}(\mu) = \bar{h}^{dyn}_{BK}(\mu) = h^{\Gamma}_{\mathrm{Kat}}(\mu) = h^{dyn}_{\mathrm{Kat}}(\mu)\,.$$

We will prove Theorem A.1 in two steps. The first step is to prove that the Kolmogorov-Sinai entropy coincides with the local entropies, and the second one is the analogue with the Katok entropies.

Step 1. Note that inequality $h_{KS}(\mu) \leq \bar{h}^{dyn}_{BK}(\mu)$ is due to Brin-Katok [BK83]. In this reference, equality is proved on a compact manifold, but this inequality does not use compactness. Inequality $\bar{h}^{dyn}_{BK}(\mu) \leq \bar{h}^{\Gamma}_{BK}(\mu)$ is immediate from (35). Therefore, we just need to prove that $\bar{h}^{\Gamma}_{BK}(\mu) \leq h_{KS}(\mu)$.

The proof relies on a crucial geometric property : as the curvature is bounded from below, the injectivity radius along a geodesic decays at most exponentially. More precisely, for every compact set $C \subset M$, there exists a positive constant $c > 0$ such that for all vectors $w \in T^1C$, and all $t \in \mathbb{R}$, we have

$$r_{\mathrm{inj}}(g^t w) \geq e^{-c|t|}\,. \tag{37}$$

This geometric inequality follows from [CGT82, Thm 4. 7], see also [CCG$^+$07, Prop 4.19].

Observe now that if $r_{\mathrm{inj}}(\pi(g^t v)) \geq \varepsilon$ for all $0 \leq t \leq T$, then

$$B_\Gamma(v, \varepsilon; T) = B_{dyn}(v, \varepsilon; T) = \{w \in T^1M,\ \forall 0 \leq t \leq T,\ d(g^t v, g^t w) \leq \varepsilon\}.$$

For the next proposition we do not need the ergodicity of μ. In particular, the corollary stated after its proof is satisfied for any invariant probability measure.

Proposition A.2. *For every compact set $\mathcal{K} \subset T^1M$ with $\mu(\mathcal{K}) > 0$, for every $0 < \varepsilon \leq 1$ small enough, there exists a partition $\mathcal{P}_{\mathcal{K}}$ of $\mathcal{K}$ with finite entropy such that, if $\mathcal{P} = \mathcal{P}_K \sqcup T^1M \setminus \mathcal{K}$, for μ-a.e.$v \in \mathcal{K}$, the sequence of return times $n_k \to \infty$ of $(g^n v)_{n\in\mathbb{N}}$ satisfies*

$$\mathcal{P}^{n_k}(v) \subset B_\Gamma(v, \varepsilon; n_k)\,.$$

In particular, for every compact set $\mathcal{K} \in T^1M$, for μ-a.e. $v \in \mathcal{K}$,

$$\limsup_{n\to\infty, g^n v \in \mathcal{K}} -\frac{1}{n} \log \mu(B_\Gamma(v, \varepsilon; n)) \leq \limsup_{n\to\infty, g^n v \in \mathcal{K}} -\frac{1}{n} \log \mu\left(\mathcal{P}^n(x)\right)\,. \tag{38}$$

Proof. By [Riq16, Proposition 1.34], for every compact set $\mathcal{K} \subset T^1M$, for all $\delta > 0$, there exists a partition $\mathcal{P}_\delta$ of $\mathcal{K}$ such that $\mathrm{diam}(\mathcal{P}_\delta(v)) \leq \delta$, $\mu(\partial\mathcal{P}_\delta(v)) = 0$, and $\#\mathcal{P}_\delta \leq C\delta^{-d}$. As $\mu(\mathcal{K}) > 0$, by Poincaré recurrence Theorem, we know that for μ-a.e. $v \in \mathcal{K}$, infinitely often $g^n v \in \mathcal{K}$. Divide the set $\mathcal{K}$ into the return time partition : for all $k \geq 1$, let

$$A_k = \{v \in \mathcal{K}, g^k v \in \mathcal{K}, \text{ and } \quad g^i v \notin \mathcal{K} \quad \text{for all} \quad 1 \leq i \leq k-1\}\,.$$

For all $k \geq 1$, set $\delta_k = \frac{\varepsilon}{(Le^c)^k}$, where L is the Lipschitz-constant for the time one map $g = g^1$ of the geodesic flow, $c > 0$ is the constant associated to the compact set $\pi(\mathcal{K}) \subset M$ from equation (37). For $v \in A_k$, define $\mathcal{P}(v)$ as $\mathcal{P}(v) := \mathcal{P}_{\delta_k}(v) \cap A_k$. For $v \notin \mathcal{K}$, set $\mathcal{P}(v) = T^1M \setminus \mathcal{K}$.

Thanks to the choice of δ_k, an immediate verification shows that for $v \in A_k$, we have $\mathcal{P}(v) \subset B_{dyn}(v, \frac{\varepsilon}{e^{ck}}; k)$. By equations (28) and (37), in fact, we have in this case

$$\mathcal{P}(v) \subset B_\Gamma(v, \frac{\varepsilon}{e^{ck}}; k) = B_{dyn}(v, \frac{\varepsilon}{e^{ck}}; k)\,.$$

Recall the notation
$$\mathcal{P}^n(v) = \mathcal{P}(v) \cap g^{-1}\mathcal{P}(gv) \cap \cdots g^{-(n-1)}\mathcal{P}(g^{n-1}v)\,.$$
Now, let $n_k \to \infty$ be the sequence of return times of $(g^n v)_{n\geq 0}$ inside $\mathcal{K}$ (with $n_0 = 0$). By construction of $\mathcal{P}$, and by the above, we have

$$\begin{aligned}
\mathcal{P}^{n_k}(v) &\subseteq \mathcal{P}(v) \cap g^{-n_1}\mathcal{P}(g^{n_1}v) \cap \cdots g^{-n_{k-1}}\mathcal{P}(g^{n_{k-1}}v) \\
&\subseteq \bigcap_{i=0}^{k-1} g^{-n_i} B_{dyn}(g^{n_i}v, \frac{\varepsilon}{e^{c(n_{i+1}-n_i)}}; n_{i+1}-n_i) \\
&= \bigcap_{i=0}^{k-1} g^{-n_i} B_{\Gamma}(g^{n_i}v, \frac{\varepsilon}{e^{c(n_{i+1}-n_i)}}; n_{i+1}-n_i) \\
&\subseteq \bigcap_{i=0}^{k-1} g^{-n_i} B_{\Gamma}(g^{n_i}v, \varepsilon; n_{i+1}-n_i) \\
&= \bigcap_{i=0}^{k-1} g^{-n_i} p_{\Gamma}\left(B(g^{n_i}\widetilde{v}, \varepsilon; n_{i+1}-n_i)\right) \\
&= \bigcap_{i=0}^{k-1} p_{\Gamma}\left(g^{-n_i} B(g^{n_i}\widetilde{v}, \varepsilon; n_{i+1}-n_i)\right) \\
&= p_{\Gamma}\left(\bigcap_{i=0}^{k-1} g^{-n_i} B(g^{n_i}\widetilde{v}, \varepsilon; n_{i+1}-n_i)\right) \\
&= p_{\Gamma}\left(B(\widetilde{v}, \varepsilon; n_k)\right) \\
&= B_{\Gamma}(v, \varepsilon; n_k)\,.
\end{aligned}$$

It remains to prove that $\mathcal{P}$ is a partition of finite entropy.

$$\begin{aligned}
H_\mu(\mathcal{P}) &= -\sum_{P\in\mathcal{P}} \mu(P)\log\mu(P) \\
&= -\mu(\mathcal{K}^c)\log\mu(\mathcal{K}^c) - \sum_{k=1}^{\infty}\sum_{P\in\mathcal{P}\cap A_k} \mu(P)\log\mu(P) \\
&= -\mu(\mathcal{K}^c)\log\mu(\mathcal{K}^c) - \sum_{k=1}^{\infty}\mu(A_k)\log\mu(A_k) + \sum_{k=1}^{\infty}\mu(A_k)\log\frac{1}{\#\mathcal{P}\cap A_k} \\
&= -\mu(\mathcal{K}^c)\log\mu(\mathcal{K}^c) - \sum_{k=1}^{\infty}\mu(A_k)\log\mu(A_k) \\
&\quad + \left(\sum_{k=1}^{\infty}\mu(A_k)\right)\times\log r^d + \sum_{k=1}^{\infty}\mu(A_k)\times k\log(Le^c)d
\end{aligned}$$

The first term is some finite constant. The third term is bounded from above by a constant times $\mu(\mathcal{K})$ and is therefore finite. By Kac lemma, the last term, up to a constant, is equal to $\sum_{k=1}^{\infty} k\mu(A_k) = \mu(\mathcal{K})$ which is finite. The second term is finite since Lemma 1.35 in [Riq16] together with $\sum_k k\mu(A_k) < \infty$ imply $\sum_k \mu(A_k)\log\mu(A_k) < \infty$.Therefore, $\mathcal{P}$ has finite entropy. □

Integrating (38) over $\mathcal{K}$ on the left, and over T^1M on the right, Proposition A.2 leads to the following corollary.

Corollary A.3. *Under the same assumptions, we have*

$$\int_K \limsup_{n\to\infty,\, g^n v\in\mathcal{K}} -\frac{1}{n}\log\mu(B_\Gamma(v,n,\varepsilon))\,d\mu \leq \int_{T^1M} \limsup_{n\to\infty, g^n v\in\mathcal{K}} -\frac{1}{n}\log\mathcal{P}^n(x)\,d\mu(x) \leq h_{KS}(\mu)\,. \tag{39}$$

If we consider the essential supremum on the left and on the right in (38), using the ergodicity of μ and Shannon-McMillan-Breiman Theorem, we get

$$\bar{h}^{\Gamma}_{loc}(\mu, \mathcal{K}) \leq h(\mu, \mathcal{P}).$$

This already implies $\bar{h}^{\Gamma}_{BK}(\mu) \leq h_{KS}(\mu)$ since the RHS of the inequality is less than $h_{KS}(\mu)$ and $\mathcal{K} \subset T^1M$ is arbitrary.

Step 2. The goal now is to prove equality between Katok entropies and the Kolmogorov-Sinai entropy. Inequality $h_{KS}(\mu) \leq h^{dyn}_{\mathrm{Kat}}(\mu)$ is due to Katok [Kat80]. In this reference, equality is proved on a compact manifold, but the proof of this inequality does not use compactness. Inequality $h^{dyn}_{\mathrm{Kat}}(\mu) \leq h^{\Gamma}_{\mathrm{Kat}}(\mu)$ is immediate from (35). Hence, by Step 1 we just need to prove that $h^{\Gamma}_{\mathrm{Kat}}(\mu) \leq \bar{h}^{\Gamma}_{BK}(\mu)$.

Let $h := \bar{h}^{\Gamma}_{BK}(\mu)$. By definition of local entropy, there exists a compact set $\mathcal{K} \subset T^1M$ such that $\mu(\mathcal{K}) > 4/5$ and for μ-a.e. $v \in \mathcal{K}$, we have

$$\limsup_{T\to\infty,\ g^T v \in \mathcal{K}} -\frac{1}{T}\log \mu(B_\Gamma(v, \varepsilon/2; T)) \leq h.$$

Fix $\rho > 0$ and set

$$\mathcal{K}_\tau := \{v \in \mathcal{K} : \mu(B_\Gamma(v, \varepsilon/2; T)) \geq \exp(-T(h+\rho)),\ \forall T \geq \tau,\ g^T v \in \mathcal{K}\}.$$

Then there exists $\tau_0 > 0$ such that $\mu(\mathcal{K}_{\tau_0}) > 3/4$. Note that $\mu(Y_T) > 1/2$ for every $T \geq \tau_0$, where $Y_T = \mathcal{K}_{\tau_0} \cap g^{-T}\mathcal{K}_{\tau_0}$. Let $0 < \delta < 1/2$. Then

$$h^{\Gamma}_{\mathrm{Kat}}(\mu) \leq \limsup_{T\to\infty} \frac{1}{T}\log S_\Gamma(\mu, \delta, \varepsilon; T) \leq \limsup_{T\to\infty} \frac{1}{T}\log S_\Gamma(Y_T, \varepsilon; T),$$

where $S_\Gamma(Y_T, \varepsilon, T)$ is the minimal cardinality of a (ε, T)-spanning set of Y_T.

Choose a maximal $(\varepsilon/2, T)$-separated set $\mathcal{E}$ in Y_T, and denote by $\Sigma_\Gamma(Y_T, \varepsilon/2, T)$ its cardinality. By maximality, $\mathcal{E}$ is also (ε, T)-spanning, so that $S_\Gamma(Y_T, \varepsilon, T) \leq \Sigma_\Gamma(Y_T, \varepsilon/2, T)$. By construction, we have

$$e^{-T(h+\rho)}\Sigma_\Gamma(Y_T, \varepsilon/2, T) \leq \sum_{y\in\mathcal{E}} \mu(B_\Gamma(y, \varepsilon/2; T)) \leq 1\,.$$

With the above inequalities, we deduce that

$$h^{\Gamma}_{\mathrm{Kat}}(\mu) \leq h + \rho\,.$$

As ρ is arbitrary, the result follows.

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