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Some existence results for a quasilinear problem with source term in Zygmund-space

Boussad Hamour

Abstract. In this paper we study the existence of solution to the problem

$$\begin{cases} u \in H_0^1(\Omega), \\ -\operatorname{div}(A(x)Du) = H(x, u, Du) + f(x) + a_0(x)u \quad \text{in } \mathcal{D}'(\Omega), \end{cases}$$

where Ω is an open bounded set of $\mathbb{R}^2$, $A(x)$ a coercive matrix with coefficients in $L^\infty(\Omega)$, $H(x, s, \xi)$ a Carathéodory function satisfying, for some $\gamma > 0$,

$$-c_0 A(x) \xi \xi \leq H(x, s, \xi) \operatorname{sign}(s) \leq \gamma A(x) \xi \xi \quad \text{a.e. } x \in \Omega, \quad \forall s \in \mathbb{R}, \quad \forall \xi \in \mathbb{R}^2.$$

Here f belongs to $L^1(\log L^1)(\Omega)$ and $a_0 \geq 0$ to $L^q(\Omega)$, $q > 1$. For f and a_0 sufficiently small, we prove the existence of at least one solution u of this problem which is such that $e^{\delta_0|u|} - 1$ belongs to $H_0^1(\Omega)$ for some $\delta_0 \geq \gamma$ and satisfies an *a priori* estimate.

Keywords. Quasilinear problems; Existence; Zygmund-space.

1. Introduction

In this paper, we consider the quasilinear problem

$$\begin{cases} u \in H_0^1(\Omega), \\ -\operatorname{div}(A(x)Du) = H(x, u, Du) + f(x) + a_0(x)u \quad \text{in } \mathcal{D}'(\Omega), \end{cases} \quad (1.1)$$

where Ω is a bounded open subset of $\mathbb{R}^2$, A a coercive matrix with bounded measurable coefficients, $H(x, s, \xi)$ a Carathéodory function having quadratic growth in ξ , more precisely, for some $\gamma > 0$ and $c_0 \geq 0$, one has

$$-c_0 A(x) \xi \cdot \xi \leq H(x, s, \xi) \operatorname{sign}(s) \leq \gamma A(x) \xi \cdot \xi \quad \text{a.e. } x \in \Omega, \quad \forall s \in \mathbb{R}, \quad \forall \xi \in \mathbb{R}^2, \quad (1.2)$$

where

$$f \in L^1(\log L^1)(\Omega), f \neq 0, \quad \text{and} \quad 0 \leq a_0 \in L^q(\Omega), q > 1 \quad \text{with} \quad a_0 \not\equiv 0. \quad (1.3)$$

For f and a_0 are sufficiently small, in the sense that f and a_0 satisfy the two smallness conditions (3.7) and (3.8), we prove that problem (1.1) has at least one solution, which is such that

$$e^{\delta_0|u|} - 1 \in H_0^1(\Omega), \quad (1.4)$$

with

$$\left\| \frac{e^{\delta_0|u|} - 1}{\delta_0} \right\|_{H_0^1(\Omega)} \leq Z_{\delta_0}, \quad (1.5)$$

where $\delta_0 \geq \gamma$ and Z_{δ_0} are two constants (see (7.9), (7.10), and (7.11) for their definitions) depending only on the data of the problem.

The main originality of our result is the fact that we assume that a_0 satisfies (1.3), namely that a_0 is a nonnegative function and $f \in L^1(\log L^1)(\Omega)$, with $f \neq 0$. The originality of this paper is twofold. First, we consider the case of source term f in a class of Zygmund-space and no more in Lebesgue space [13]. Second, the fact that we assume that a_0 satisfies (1.3), namely that a_0 is a nonnegative function.

Let us begin with vast literature concerned with problems like (1.1) which has been studied in many papers in the case where $a_0 \leq 0$.

Among these papers is a series of papers [8], [9], [10], [11] and [12] by L. Boccardo, F. Murat and J.-P. Puel (see also the paper [26] by J.-M. Rokotoson), which are concerned with the case where

$$a_0(x) \leq -\alpha_0 < 0. \quad (1.6)$$

In these papers, the authors prove that when a_0 satisfies (1.6) and when f belongs to $L^q(\Omega)$, $q > \frac{N}{2}$, there exists at least one solution of (1.1) belonging to $L^\infty(\Omega)$ and satisfying some *a priori* estimates.

The case where

$$a_0 = 0 \quad (1.7)$$

was considered, among others, by A. Alvino, P.-L. Lions and G. Trombetti in [1], by C. Maderna, C. Pagani and S. Salsa in [23], by V. Ferone and M.-R. Posteraro in [18], and by N. Grenon-Isselkou and J. Mossino in [19]. In these papers (which consider also nonlinear monotone operators), the authors prove that if $a_0 \equiv 0$ and f belonging to $L^q(\Omega)$, $q > \frac{N}{2}$, with $\|f\|_{L^q(\Omega)}$ sufficiently small, there exists at least one $L^\infty(\Omega)$ -solution of (1.1) which moreover belongs to $L^\infty(\Omega)$ and which satisfies some *a priori* estimates.

The case where a_0 satisfies (1.7) but f belongs only to $L^{N/2}(\Omega)$ for $N \geq 3$ (and no more to $L^q(\Omega)$ with $q > \frac{N}{2}$) was considered by V. Ferone and F. Murat in [15] (and in [16] in the nonlinear monotone case). These authors proved that when $\|f\|_{L^{N/2}(\Omega)}$ is sufficiently small, there exists at least one solution of (1.1) such that

$e^{\delta|u|} - 1 \in H_0^1(\Omega)$ for some $\delta > \gamma$, and that such a solution satisfies an *a priori* estimate. Similar results were obtained in the case of nonlinear monotone operators where $f \in L^1(\log L)^{N-1}(\Omega)$ and a_0 satisfies (1.7) by F. Chiacchio in [13]. Finally, in [17], V. Ferone and F. Murat considered (also in the case of nonlinear monotone operators) the case where a_0 satisfies $a_0 \leq 0$ and f belongs to the Zygmund-space $L^{N/2,\infty}(\Omega)$; in this case two smallness conditions should be fulfilled.

To finish with the case where a_0 satisfies $a_0 \leq 0$ and f belongs to $H^{-1}(\Omega)$, let us quote the papers [7] by L. Boccardo, F. Murat and J.-P. Puel and [8] by A. Bensoussan, L. Boccardo and F. Murat, where these authors proved the existence of at least one solution of (1.1).

In contrast with the cases (1.6) and (1.7), the present paper is concerned with the case (1.3) where $a_0 \geq 0$, and $a_0 \neq 0$. In this setting we are only aware of four recent papers. In [20], B. Hamour and F. Murat proved a similar result to the one of the present paper, for $\|f\|_{L^{N/2}(\Omega)}$ and $\|a_0\|_q$ are sufficiently small, with $N \geq 3$ and $q > \frac{N}{2}$. In [22], L. Jeanjean and B. Sirakov proved too a result similar to the one of the present paper when $f \in L^q(\Omega)$, $q > \frac{N}{2}$, they also proved the existence of at least two solutions of (1.1). In [3], D. Arcoya, C. De Coster, L. Jeanjean and K. Tanaka proved the existence of a continuum (u, λ) of solutions (with u belonging moreover to $L^\infty(\Omega)$) when $A(x) = Id$, $H(x, s, \xi) = \mu(x)|\xi|^2$, with $\mu \in L^\infty(\Omega)$, $\mu(x) \geq \mu > 0$, $f \in L^q(\Omega)$, $q > \frac{N}{2}$, $f \geq 0$, $f \neq 0$ and $a_0(x) = \lambda a_0^*(x)$ with $a_0^* \in L^q(\Omega)$, $a_0^* \geq 0$ and $a_0^* \neq 0$; moreover, under some further conditions on f , and that there are at least two nonnegative solutions of (1.1) when $\lambda > 0$ is sufficiently small.

In [27], in a similar setting, assuming only that $\mu(x) \geq 0$ but that the supports of μ and of a_0^* have a nonempty intersection and that $N \leq 5$, P. Souplet proved the existence of a continuum (u, λ) of solutions, and that there are at least two nonnegative solutions of (1.1) when $\lambda > 0$ is sufficiently small.

With respect to the results obtained in the four latest papers, we prove in this paper, as said above, the existence of (only) one solution of (1.1) in the case $a_0 \geq 0$ when a_0 and f satisfy the two smallness conditions (3.7) and (3.8), but our result is obtained in the general case of a nonlinearity $H(x, s, \xi)$ which satisfies only (1.2), with $f \in L^1(\log L^1)(\Omega)$ and with $a_0 \in L^q(\Omega)$, $q > 1$.

The paper is organized as follows:

The (short) Section 2 deals with preliminaries results about Zygmund-spaces. The precise statement of our result is given in Section 3 (Theorem 3.1), as well as the precise assumptions under which we are able to prove it. These assumptions include in particular the two smallness conditions (3.7) and (3.8).

Our method for proving Theorem 3.1 is based on an equivalence result (see Theorem 4.2) stated in Section 4 once we have introduced the functions $K_\delta(x, s, \zeta)$ and $g_\delta(s)$ (see (4.5) and (4.6)) and made some technical remarks on them. This equivalence result is very close to that given in the paper [16] by V. Ferone and F. Murat.

Our equivalence result permits us to reduce the proof of existence of a solution u of (1.1), which satisfies (1.4) and (1.5), to the existence (see Theorem 4.3) of a function w related to u by

$$w = \frac{1}{\delta_0}(e^{\delta_0|u|} - 1) \operatorname{sign}(u),$$

which solve the problem (see (4.13))

$$\begin{cases} w \in H_0^1(\Omega), \\ -\operatorname{div}(A(x)Dw) + K_{\delta_0}(x, w, Dw) \operatorname{sign}(w) \\ \quad = (1 + \delta_0|w|)f(x) + a_0(x)w + a_0(x)g_{\delta_0}(w) \operatorname{sign}(w) \quad \text{in } \mathcal{D}'(\Omega), \end{cases}$$

and satisfies the estimate see (4.14)

$$\|w\|_{H_0^1(\Omega)} \leq Z_{\delta_0}.$$

Our goal thus becomes to prove Theorem 4.3, namely to prove the existence of a solution w which satisfies (4.13) and (4.14).

Problem (4.13) is very similar to Problem (1.1), since it involves a term $-K_{\delta_0}(x, w, Dw) \operatorname{sign}(w)$ which has quadratic growth in Dw , as well as a zeroth order term is given by $\delta_0|w|f(x) + a_0(x)w + a_0(x)g_{\delta_0}(w) \operatorname{sign}(w)$. But this problem is also very different from (1.1), since the term $-K_{\delta_0}(x, w, Dw) \operatorname{sign}(w)$ with quadratic growth has the “good sign property”, since $K_{\delta_0}(x, s, \xi)$ satisfies

$$K_{\delta_0}(x, s, \xi) \geq 0,$$

while the zeroth order term is now no more a linear but a semilinear term with $|s|^2$ growth due to presence of the term $a_0(x)g_{\delta_0}(w) \operatorname{sign}(w)$.

We will prove Theorem 4.3 essentially by applying Schauder’s fixed point theorem. But there are some difficulties to do it directly, since the term with quadratic growth $K_{\delta_0}(x, w, Dw) \operatorname{sign}(w)$ only belongs to $L^1(\Omega)$ in general. We therefore begin by defining an approximate problem (see (5.1)) where $K_\delta(x, w, Dw)$ is replaced by its truncation at height k , namely $T_k(K_\delta(x, w, Dw))$, and we prove (see Theorem 5.1) that if f and a_0 satisfy the two smallness conditions (3.7) and (3.8),

this approximate problem has at least one solution w_k which satisfies the a priori estimate

$$\|w_k\|_{H_0^1(\Omega)} \leq Z_{\delta_0}. \quad (1.8)$$

This result is obtained in Section 5 by applying Schauder's fixed point theorem in a classical way.

We then pass to the limit as k tends to infinity and we prove in Section 6 that (for a subsequence of k) w_k tend to some w^* strongly in $H_0^1(\Omega)$ and that this w^* is a solution of (4.13) which also satisfies (4.14) (see End of the proof of Theorem 4.3).

This completes the proof of Theorem 4.3, and therefore proves Theorem 3.1, as announced.

This proof follows along the lines of the proof used by V. Ferone and F. Murat in [15] in the case where $a_0 = 0$. As mentioned above, this method can be applied to the nonlinear case where the linear operator $-\operatorname{div}(A(x)Du)$ is replaced by a Leray-Lions operator $-\operatorname{div}(a(x, u, Du))$ working in $W_0^{1,p}(\Omega)$ for some $1 < p < N$ and $p = N$, and where the quasilinear term $H(x, u, Du)$ has p -growth in $|Du|$, as it was done in [16] by V. Ferone and F. Murat in this nonlinear setting when $a_0 = 0$. This will be the goal of our next paper [21].

2. Preliminary result

In this section we recall some definitions and classical properties about rearrangements and introduce the Zygmund-spaces. Let Ω be an open bounded subset of $\mathbb{R}^N$ equipped with the Lebesgue's measure and let $\phi : \Omega \rightarrow \mathbb{R}$ be a measurable function, we define the distribution function μ_ϕ by

$$\mu_\phi(t) = \operatorname{meas}\{x \in \Omega : |\phi(x)| > t\}, \quad \forall t \geq 0,$$

and the decreasing rearrangement of ϕ by

$$\phi^*(s) = \sup\{t > 0 : \mu_\phi(t) > s\}, \quad \forall s \in (0, |\Omega|) \quad (|\Omega| = \operatorname{meas}(\Omega)).$$

We recall only the following Hardy-Littlewood inequality (see [24])

$$\int_{\Omega} |g(x)h(x)|dx \leq \int_0^{|\Omega|} g^*(s)h^*(s)ds, \quad (2.1)$$

for any g and h real measurable functions defined in Ω .

The Zygmund-space $L^p(\log L)^q$, $p > 0$ and $q \in \mathbb{R}$, consists of all measurable functions $\phi : \Omega \rightarrow \mathbb{R}$ such that the following quantity is finite.

$$\|\phi\|_{L^p(\log L)^q} = \left(\int_0^{|\Omega|} \left(\left(\log \frac{|\Omega|}{s} \right)^q \phi^*(s) \right)^p ds \right)^{1/p}. \quad (2.2)$$

Finally, we recall the following Theorem (see [2]).

Theorem 2.1. *If u is a real function in $W_0^{1,N}(\Omega)$, then*

$$u^*(s) \leq \frac{\|Du\|_N}{N\omega_N^{1/N}} \left(\log \frac{|\Omega|}{s} \right)^{\frac{N-1}{N}} \quad \forall s \in (0, |\Omega|), \quad (2.3)$$

where w_N is the the measure of unit sphere of $\mathbb{R}^N$ such that

$$w_N = \frac{\pi^{N/2}}{\Gamma(1 + N/2)}.$$

In particular, for $N = 2$ one has

$$N\omega_N^{1/N} = (2\pi)^{1/2}.$$

3. The main result

In this paper we consider the following quasilinear problem

$$\begin{cases} u \in H_0^1(\Omega), \\ -\operatorname{div}(A(x)Du) = H(x, u, Du) + a_0(x)u + f(x) \quad \text{in } \mathcal{D}'(\Omega), \end{cases} \quad (3.1)$$

where Ω is bounded open subset of $\mathbb{R}^2$, where A is a coercive matrix with bounded measurable coefficients, i.e.,

$$\begin{cases} A \in (L^\infty(\Omega))^{2 \times 2}, \\ \exists \alpha > 0, \quad A(x)\xi\xi \geq \alpha|\xi|^2 \quad \text{a.e. } x \in \Omega, \quad \forall \xi \in \mathbb{R}^2, \end{cases} \quad (3.2)$$

where $H : \Omega \times \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is a Carathéodory function which satisfies the condition:

$$-c_0 A(x)\xi\xi \leq H(x, s, \xi) \operatorname{sign}(s) \leq \gamma A(x)\xi\xi, \quad \text{a.e. } x \in \Omega, \quad \forall s \in \mathbb{R}, \quad \forall \xi \in \mathbb{R}^2, \quad (3.3)$$

for a given $\gamma > 0$ and $c_0 \geq 0$, where the function $\operatorname{sign}(s)$ is defined by:

$$\operatorname{sign}(s) = \begin{cases} +1 & \text{if } s > 0, \\ 0 & \text{if } s = 0, \\ -1 & \text{if } s < 0, \end{cases} \quad (3.4)$$

where the coefficient a_0 satisfies

$$a_0 \in L^q(\Omega) \quad \text{for some } q > 1, \quad a_0 \geq 0, \quad a_0 \neq 0, \quad (3.5)$$

and finally where

$$f \in L^1(\log L^1)(\Omega), \quad f \neq 0. \quad (3.6)$$

Since Ω is bounded, we equip the space $H_0^1(\Omega)$ with the gradient norm

$$\|u\|_{H_0^1(\Omega)} = \|Du\|_{L^2(\Omega)^2}.$$

We finally assume that f and a_0 are sufficiently small, and more precisely that

$$\alpha - C_1(q)\|a_0\|_q - \gamma\|f\|_{L^1(\log L)(\Omega)} > 0, \quad (3.7)$$

$$\|f\|_{H^{-1}(\Omega)} \leq \frac{(2\pi\alpha - C_1(q)\|a_0\|_q - \delta\|f\|_{L^1(\log L)(\Omega)})^2}{16\pi^2\delta_1 C_2(q)\|a_0\|_q}, \quad (3.8)$$

where the constant δ_1 is defined by (7.5) and where some constants $C_1(q) > 0$ and $C_2(q) > 0$, as well as

$$C_1(q) = \left(\int_0^{|\Omega|} \left(\log \frac{|\Omega|}{s} \right)^{q'} ds \right)^{1/q'}, \quad C_2(q) = \left(\int_0^{|\Omega|} \left(\log \frac{|\Omega|}{s} \right)^{\frac{3q'}{2}} ds \right)^{1/q'} \quad (3.9)$$

where q' the Hölder's conjugate of the exponent q , i.e.

$$\frac{1}{q} + \frac{1}{q'} = 1.$$

Our aim is to prove the following Theorem:

Theorem 3.1. *Assume that (3.2), (3.3), (3.4), (3.5), (3.6) hold true. Assume moreover that the two smallness conditions (3.7) and (3.8) hold true.*

Then there exist a constant δ_0 with $\delta_0 \geq \gamma$ and a constant Z_{δ_0} , such that there exists at least one solution u of (3.1) which further satisfies

$$(e^{\delta_0|u|} - 1) \in H_0^1(\Omega), \quad (3.10)$$

with

$$\|e^{\delta_0|u|} Du\|_{L^2(\Omega)^2} = \left\| \frac{e^{\delta_0|u|} - 1}{\delta_0} \right\|_{H_0^1(\Omega)} \leq Z_{\delta_0}. \quad (3.11)$$

Remark 3.2. The definitions of the constants δ_0 and Z_{δ_0} which appear in Theorem 3.1 are given in Appendix 7 (see Lemma 7.2). These definitions are based on the properties of the family of functions ϕ_δ which look like convex parabolas (see Fig. 2). The constant δ_0 is the unique value of the parameter δ_0 for which the function ϕ_{δ_0} admits to double zero and Z_{δ_0} is the value such that $\phi_\delta(Z_{\delta_0}) = 0$.

4. An equivalence result

The main results of this Section are Theorems 4.2 and 4.3.

Indeed, as said above, the proof of Theorem 3.1 is based on the equivalence result of Theorem 4.2 that we state and we prove in this section. Here we always assume that

$\delta > 0$ and let us first proceed with a formal computation. If u is a solution of the following problem:

$$\begin{cases} -\operatorname{div}(A(x)Du) = H(x, u, Du) + f(x) + a_0(x)u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (4.1)$$

and if we formally define the function w_δ by

$$w_\delta = \frac{1}{\delta} \left(e^{\delta|u|} - 1 \right) \operatorname{sign}(u), \quad (4.2)$$

where the function sign is defined by (3.4), we have, at least formally,

$$\begin{cases} e^{\delta|u|} = 1 + \delta|w_\delta|, & |u| = \frac{1}{\delta} \log(1 + \delta|w_\delta|), & \operatorname{sign}(u) = \operatorname{sign}(w_\delta), \\ Dw_\delta = e^{\delta|u|} Du, \\ -\operatorname{div}(A(x)Dw_\delta) = -\delta e^{\delta|u|} A(x)DuDu \operatorname{sign}(u) - e^{\delta|u|} (\operatorname{div}(A(x)Du)), \end{cases} \quad (4.3)$$

and therefore w_δ is, at least formally, a solution of

$$\begin{cases} -\operatorname{div}(A(x)Dw_\delta) = -K_\delta(x, w_\delta, Dw_\delta) \operatorname{sign}(w_\delta) \\ \quad + (1 + \delta|w_\delta|) f(x) + a_0(x) w_\delta \\ \quad + a_0(x) g_\delta(w_\delta) \operatorname{sign}(w_\delta) & \text{in } \Omega, \\ w_\delta = 0 & \text{on } \partial\Omega, \end{cases} \quad (4.4)$$

where the functions $K_\delta : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ and $g_\delta : \mathbb{R} \rightarrow \mathbb{R}$ are defined by the following formulas:

$$\begin{cases} K_\delta(x, t, \zeta) = \\ = \frac{\delta}{1 + \delta|t|} A(x)\zeta\zeta - (1 + \delta|t|)H\left(x, \frac{1}{\delta} \log(1 + \delta|t|) \operatorname{sign}(t), \frac{\zeta}{1 + \delta|t|}\right) \operatorname{sign}(t), \\ \text{a.e. } x \in \Omega, \quad \forall t \in \mathbb{R}, \quad \forall \zeta \in \mathbb{R}^N, \end{cases} \quad (4.5)$$

and

$$g_\delta(t) = -|t| + \frac{1}{\delta}(1 + \delta|t|) \log(1 + \delta|t|), \quad \forall t \in \mathbb{R}. \quad (4.6)$$

Conversely, if w_δ is a solution of (4.4), and if we formally define the function by

$$u = \frac{1}{\delta} \log(1 + \delta|w_\delta|) \operatorname{sign}(w_\delta), \quad (4.7)$$

the some formal computation easily shows that u is solution of 4.1.

Remark 4.1. Observe that the functions $K_\delta(x, w, Dw)$ and $K_\delta(x, w, Dw)\operatorname{sign}(w)$ are correctly defined and are mesurable functions when $w \in H^1(\Omega)$, and continuous with respect to the almost everywhere convergence of w and Dw (see Lemma 3.3 in [20])

Theorem 4.2. *Assume that (3.2), (3.3), (3.4), (3.5), (3.6) hold true, and let $\delta > 0$ be fixed. Let the functions K_δ and g_δ be defined by (4.5) and (4.6).*

If u is any solution of (3.1) which satisfies

$$(e^{\delta|u|} - 1) \in H_0^1(\Omega), \quad (4.8)$$

then the function w_δ defined by (4.2), namely by

$$w_\delta = \frac{1}{\delta} (e^{\delta|u|} - 1) \operatorname{sign}(u),$$

satisfies

$$\begin{cases} w_\delta \in H_0^1(\Omega), \\ -\operatorname{div}(A(x)Dw_\delta) + K_\delta(x, w_\delta, Dw_\delta) \operatorname{sign}(w_\delta) \\ = (1 + \delta|w_\delta|) f(x) + a_0(x) w_\delta + a_0(x) g_\delta(w_\delta) \operatorname{sign}(w_\delta) \quad \text{in } \mathcal{D}'(\Omega). \end{cases} \quad (4.9)$$

Conversely, if w_δ is any solution of (4.9), then the function u defined by

$$u = \frac{1}{\delta} \log(1 + \delta|w_\delta|) \operatorname{sign}(w_\delta),$$

is a solution of (3.1) which satisfies (4.8).

Proof of Theorem 4.2. Define the function $\hat{f}$ by

$$\hat{f}(x) = f(x) + a_0(x) u(x), \quad (4.10)$$

In view of (4.2) and of the definition (4.6) of $g_\delta(s)$, one has

$$\begin{cases} (1 + \delta|w_\delta|) f(x) + a_0(x) w_\delta + a_0(x) g_\delta(w_\delta) \operatorname{sign}(w_\delta) \\ = (1 + \delta|w_\delta|) (f(x) + a_0(x) \frac{1}{\delta} \log(1 + \delta|w_\delta|) \operatorname{sign}(w_\delta)) \\ = (1 + \delta|w_\delta|) (f(x) + a_0(x) u(x)) = (1 + \delta|w_\delta|) \hat{f}(x). \end{cases} \quad (4.11)$$

Then Theorem 4.2 becomes an immediate application of Proposition 1.8 of [15], once one observes that

$$\hat{f} \in L^1(\Omega); \quad (4.12)$$

Indeed $a_0 u \in L^1(\Omega)$, since a_0 is assumed to belong to $L^q(\Omega)$, $q > 1$, while $u \in L^r(\Omega)$ for every $r < \infty$, since $(e^{\delta|u|} - 1) \in H_0^1(\Omega)$, hence in particular to $L^1(\Omega)$, which implies that $e^{\delta|u|}$ belongs to $L^1(\Omega)$. Theorem 4.2 is therefore proved. $\square$

From the equivalence Theorem 4.2 one immediately deduces that Theorem 3.1 is equivalent to the following Theorem.

Theorem 4.3. *Assume that (3.2), (3.3), (3.4), (3.5), (3.6) hold true. Assume moreover that the two smallness conditions (3.7) and (3.8) hold true.*

Then there exist a constant δ_0 with $\delta_0 \geq \gamma$, and a constant Z_{δ_0} , such that there exists at least one solution w of

$$\begin{cases} w \in H_0^1(\Omega), \\ -\operatorname{div}(A(x)Dw) + K_{\delta_0}(x, w, Dw) \operatorname{sign}(w) \\ = (1 + \delta_0|w|)f(x) + a_0(x)w + a_0(x)g_{\delta_0}(w) \operatorname{sign}(w) \quad \text{in } \mathcal{D}'(\Omega), \end{cases} \quad (4.13)$$

which satisfies

$$\|w\|_{H_0^1(\Omega)} \leq Z_{\delta_0}. \quad (4.14)$$

The rest of this paper will therefore be devoted to the proof of Theorem 4.3. This will be done in two steps: first, we will prove the existence of a solution for an approximate problem; second, we will pass to the limit in this approximate problem.

Remark 4.4. We assume that (3.2), (3.3), (3.5), (3.6) hold true. Assume moreover that the two smallness conditions (3.7) and (3.8) hold true, we try to explain how these two conditions come from an “a priori estimate” that one can obtain on the solutions of (4.9).

If w_δ is any solution of (4.9), using $T_k(w_\delta) \in H_0^1(\Omega) \cap L^\infty(\Omega)$ as test function, where $T_k : \mathbb{R} \rightarrow \mathbb{R}$ is the usual truncation at height k defined by

$$T_k(s) = \begin{cases} -k & \text{if } s \leq -k, \\ s & \text{if } -k \leq s \leq +k, \\ +k & \text{if } +k \leq s, \end{cases} \quad (4.15)$$

one has

$$\begin{cases} \int_{\Omega} A(x)Dw_\delta DT_k(w_\delta) dx + \int_{\Omega} K_\delta(x, w_\delta, Dw_\delta) |T_k(w_\delta)| dx \\ = \int_{\Omega} f(x) T_k(w_\delta) dx + \int_{\Omega} \delta f(x) |w_\delta| T_k(w_\delta) dx \\ + \int_{\Omega} a_0(x) w_\delta T_k(w_\delta) dx + \int_{\Omega} a_0(x) g_\delta(w_\delta) |T_k(w_\delta)| dx. \end{cases} \quad (4.16)$$

We assume that δ satisfies

$$\gamma \leq \delta \leq \delta_1, \quad (4.17)$$

where δ_1 is defined by (7.5).

Since $\delta \geq \gamma$ by (4.17), we deduce that $K_\delta(x, s, \zeta) \geq 0$ (see Remark 3.1 in [20]). Using this fact and passing to the limit as k tends to $+\infty$, we obtain

$$\begin{cases} \int_{\Omega} A(x)Dw_\delta Dw_\delta dx \leq \int_{\Omega} f(x) w_\delta dx + \int_{\Omega} \delta f(x) |w_\delta|^2 dx \\ + \int_{\Omega} a_0(x) |w_\delta|^2 dx + \int_{\Omega} a_0(x) g_\delta(w_\delta) |w_\delta| dx. \end{cases} \quad (4.18)$$

In (4.18) we use the left-hand side the fact that the matrix is coercive, in all terms of the right-hand side the fact that $f \in L^1(\log L)(\Omega)$, Holder’s inequality, Hardy-Littlewood’s

inequality, classical properties about rearrangements and the fact

$$0 \leq g_\delta(t) \leq \delta |t|^2, \quad \forall t \in \mathbb{R}, \quad t \neq 0, \quad \forall \delta, \quad 0 < \delta \leq \delta_1.$$

This allows us to obtain an estimate on $T_k(w_\delta)$, in which we pass to the limit in k to get

$$\begin{cases} \alpha \|Dw_\delta\|_2^2 < \|f\|_{H^{-1}(\Omega)} \|Dw_\delta\|_2 + \frac{\delta \|f\|_{L^1(\log L)(\Omega)}}{2\pi} \|Dw_\delta\|_2^2 \\ + \frac{C_1(q)\|a_0\|_q}{2\pi} \|Dw_\delta\|_2^2 + \frac{\delta C_2(q)\|a_0\|_q}{(2\pi)^{3/2}} |Dw_\delta|_2^3 \quad \text{if } w_\delta \neq 0, \end{cases} \quad (4.19)$$

where these constants $C_1(q)$ and $C_2(q)$ depend only on q (see (3.9)).

Dividing by $\|Dw_\delta\|_2$ this implies that

$$\begin{cases} \alpha \|Dw_\delta\|_2 < \|f\|_{H^{-1}(\Omega)} + \frac{\delta \|f\|_{L^1(\log L)(\Omega)}}{2\pi} \|Dw_\delta\|_2 \\ + \frac{C_1(q)\|a_0\|_q}{2\pi} \|Dw_\delta\|_2 + \frac{\delta C_2(q)\|a_0\|_q}{(2\pi)^{3/2}} |Dw_\delta|_2^2 \quad \text{if } w_\delta \neq 0, \end{cases} \quad (4.20)$$

Using (4.17) in the fourth term of the right-hand side of (4.20), we have

$$\begin{cases} \alpha \|Dw_\delta\|_2 < \|f\|_{H^{-1}(\Omega)} + \frac{\delta \|f\|_{L^1(\log L)(\Omega)}}{2\pi} \|Dw_\delta\|_2 \\ + \frac{C_1(q)\|a_0\|_q}{2\pi} \|Dw_\delta\|_2 + \frac{\delta_1 C_2(q)\|a_0\|_q}{(2\pi)^{3/2}} |Dw_\delta|_2^2 \quad \text{if } w_\delta \neq 0, \end{cases} \quad (4.21)$$

In view of the definition (7.7) of the function Φ_δ (see also Fig. 2), we have proved that if w_δ is any solution of (4.9), one has

$$\Phi_\delta(\|Dw_\delta\|_2) > 0 \quad \text{if } \gamma \leq \delta \leq \delta_1. \quad (4.22)$$

But by the definition of δ_0 , one has

$$\Phi_\delta(X) > 0, \quad \forall X \geq 0, \quad \forall \delta, \quad 0 < \delta \leq \delta_1,$$

and therefore inequality (4.22) does not imply anything on $\|Dw_\delta\|_2$ when δ satisfies $\delta_0 < \delta \leq \delta_1$. In contrast, when $\delta < \delta_0$, the strict inequality (4.22) implies that

$$\text{either } \|Dw_\delta\|_2 < Y_\delta^- \quad \text{or} \quad \|Dw_\delta\|_2 > Y_\delta^+ \quad \text{if } \delta < \delta_0, \quad (4.23)$$

where $Y_\delta^- < Y_\delta^+$ are the two distinct zeros of the function Φ_δ (see Remarks 7.3 and Fig. 2), while when $\delta = \delta_0$, the strict inequality (4.22) implies that

$$\text{either } \|Dw_{\delta_0}\|_2 < Z_{\delta_0} \quad \text{or} \quad \|Dw_{\delta_0}\|_2 > Z_{\delta_0} \quad \text{if } \delta = \delta_0. \quad (4.24)$$

Inequalities (4.23) and (4.24) are not a priori estimates, since they do not imply any bound on $\|Dw_\delta\|_2$. Nevertheless these inequalities exclude the closed interval $[Y_\delta^-, Y_\delta^+]$ or the point Z_{δ_0} for $\|Dw_\delta\|_2$, and they give the hope to prove the existence of a fixed point in the set $\|Dw_\delta\|_2 \leq Y_\delta^-$, when $\delta < \delta_0$, or in the set $\|Dw_{\delta_0}\|_2 \leq Z_{\delta_0}$, when $\delta = \delta_0$.

These inequalities also explain where the two smallness conditions (3.7) and (3.8) come from. Indeed (see Remark 7.3), these two smallness conditions imply that the value δ_0 of δ for which Φ_δ has a double zero satisfies $\delta_0 \geq \gamma$, which is the case where, as said just above, some hope is allowed.

5. Existence of a solution for an approximate problem

In this Section we introduce an approximation of problem (4.13). Under the two smallness conditions (3.7) and (3.8), we prove by applying Schauder's fixed point theorem that this approximate problem has at least one solution which satisfies the estimate (4.14).

Let δ_0 be defined by (7.9), (7.10) and (7.11). For any $k > 0$, we consider the following approximate problem.

$$\begin{cases} w_k \in H_0^1(\Omega), \\ -\operatorname{div}(A(x)Dw_k) + T_k(K_{\delta_0}(x, w_k, Dw_k)) \operatorname{sign}_k(w_k) \\ = (1 + \delta_0|w_k|)f(x) + a_0(x)w_k + a_0(x)g_{\delta_0}(w_k) \operatorname{sign}(w_k) \quad \text{in } \mathcal{D}'(\Omega), \end{cases} \quad (5.1)$$

where T_k is the usual truncation at height k defined by (4.15) and where $\operatorname{sign}_k : \mathbb{R} \rightarrow \mathbb{R}$ is the approximation of the function sign which is defined by

$$\operatorname{sign}_k(s) = \begin{cases} ks & \text{if } |s| \leq \frac{1}{k}, \\ \operatorname{sign}(s) & \text{if } |s| \geq \frac{1}{k}. \end{cases} \quad (5.2)$$

Theorem 5.1. *Assume that (3.2), (3.3), (3.5) and (3.6) hold true. Assume moreover that the two smallness conditions (3.7) and (3.8) hold true. Let δ_0 be defined in Lemma 7.2 (see (7.9) and (7.10)), and let $k > 0$ be fixed.*

Then there exists at least one solution of (5.1) such that

$$\|w_k\|_{H_0^1(\Omega)} \leq Z_{\delta_0}, \quad (5.3)$$

where Z_{δ_0} is defined in Lemma 7.2 (see (7.9), (7.10) and (7.11)).

The proof of Theorem 5.1 consists in applying Schauder's fixed point theorem. First we prove the two following lemmas.

Lemma 5.2. *Assume that (3.2), (3.3), (3.4), (3.5), (3.6) hold true. Let $k > 0$ be fixed.*

Then, for any $w \in H_0^1(\Omega)$, there exists a unique solution W of the following semilinear problem

$$\begin{cases} W \in H_0^1(\Omega), \\ -\operatorname{div}(A(x)DW) + T_k(K_{\delta_0}(x, w, Dw)) \operatorname{sign}_k(W) \\ = (1 + \delta_0|w|)f(x) + a_0(x)w + a_0(x)g_{\delta_0}(w) \operatorname{sign}(w) \quad \text{in } \mathcal{D}'(\Omega). \end{cases} \quad (5.4)$$

Moreover W satisfies

$$\begin{cases} \alpha \|DW\|_2 < \|f\|_{H^{-1}(\Omega)} + \frac{\delta_0 \|f\|_{L^1(\log L)(\Omega)}}{2\pi} \|Dw\|_2 \\ + \frac{C_1(q)\|a_0\|_q}{2\pi} \|Dw\|_2 + \frac{\delta_0 C_2(q)\|a_0\|_q}{(2\pi)^{3/2}} \|Dw\|_2^2 \quad \text{if } w \neq 0, \end{cases} \quad (5.5)$$

where $C_1(q)$ and $C_2(q)$ are the constants given by (3.9).

Proof. Problem (5.4) is of the form

$$\begin{cases} W \in H_0^1(\Omega), \\ -\operatorname{div}(A(x)DW) + \hat{b}(x) \operatorname{sign}_k(W) = \hat{F}(x) \quad \text{in } \mathcal{D}'(\Omega), \end{cases} \quad (5.6)$$

where $\hat{b}(x)$ and $\hat{F}(x)$ are given. Since $\hat{b}(x) = T_k(K_{\delta_0}(x, w, Dw))$ belongs to $L^\infty(\Omega)$ and is nonnegative (see Remark 3.1 in [20]) and $\delta_0 \geq \gamma$ (see (7.9)), since the function sign_k is continuous and nondecreasing, and since $\hat{F}$ belong to $H^{-1}(\Omega)$ (see e.g. the computation which allows one to obtain (5.8) below), this problem has a unique solution.

Since $W \in H_0^1(\Omega)$, the use of W as a test function in (5.4) is licit. Since the function $T_k(K_\delta(x, t, \zeta))$ is nonnegative, this gives

$$\begin{cases} \int_{\Omega} A(x) DW DW \, dx \\ \leq \int_{\Omega} (1 + \delta_0 |w|) f(x) W \, dx \\ + \int_{\Omega} a_0(x) w W \, dx + \int_{\Omega} a_0(x) g_{\delta_0}(w) \operatorname{sign}(w) W \, dx. \end{cases} \quad (5.7)$$

As in the computation made in Remark 4.4 to obtain the inequality (4.19), we use in (5.7) the coerciveness (3.3) of the matrix A , and classical properties about rearrangements. We obtain

$$\begin{cases} \alpha \|DW\|_2^2 \leq \|f\|_{H^{-1}(\Omega)} \|Dw_\delta\|_2 \|DW\|_2 + \frac{\delta_0 \|f\|_{L^1(\log L)(\Omega)}}{2\pi} \|Dw\|_2 \|DW\|_2 \\ + \frac{C_1(q) \|a_0\|_q}{2\pi} \|Dw\|_2 \|DW\|_2 + \frac{C_2(q) \delta_0 \|a_0\|_q}{(2\pi)^{3/2}} \|Dw\|_2^2 \|DW\|_2, \end{cases} \quad (5.8)$$

which implies (5.5). $\square$

Lemma 5.3. *Assume (3.2), (3.3), (3.4), (3.5), (3.6) hold true. Assume moreover that the two smallness conditions (3.7) and (3.8) hold true. Let $k > 0$ be fixed.*

Let w_n be a sequence such that

$$w_n \rightharpoonup w \quad \text{in } H_0^1(\Omega) \quad \text{weakly and a.e. in } \Omega. \quad (5.9)$$

Define W_n as the unique solution of (5.4) for $w = w_n$, i.e.

$$\begin{cases} W_n \in H_0^1(\Omega), \\ -\operatorname{div}(A(x)DW_n) + T_k(K_{\delta_0}(x, w_n, Dw_n)) \operatorname{sign}_k(W_n) \\ = (1 + \delta_0 |w_n|) f(x) + a_0(x) w_n + a_0(x) g_{\delta_0}(w_n) \operatorname{sign}(w_n) \quad \text{in } \mathcal{D}'(\Omega). \end{cases} \quad (5.10)$$

Assume moreover that for a subsequence, still denoted by n , and for some $W^ \in H_0^1(\Omega)$, W_n satisfies*

$$W_n \rightharpoonup W^* \quad \text{in } H_0^1(\Omega) \quad \text{weakly and a.e. in } \Omega. \quad (5.11)$$

Then for the same subsequence one has

$$W_n \rightarrow W^* \quad \text{in } H_0^1(\Omega) \quad \text{strongly.} \quad (5.12)$$

Proof. Since $W_n - W^* \in H_0^1(\Omega)$, the use of $(W_n - W^*)$ as test function in (5.10) is licit. This gives

$$\left\{ \begin{array}{l} \int_{\Omega} A(x) D(W_n - W^*) D(W_n - W^*) dx \\ = - \int_{\Omega} A(x) D W^* D(W_n - W^*) dx \\ - \int_{\Omega} T_k(K_{\delta_0}(x, w_n, Dw_n)) \operatorname{sign}_k(W_n) (W_n - W^*) dx \\ + \int_{\Omega} (1 + \delta_0 |w_n|) f(x) (W_n - W^*) dx + \int_{\Omega} a_0(x) w_n (W_n - W^*) dx \\ + \int_{\Omega} a_0(x) g_{\delta_0}(w_n) \operatorname{sign}(w_n) (W_n - W^*) dx. \end{array} \right. \quad (5.13)$$

We claim that every term of the right-hand of (5.13) tends to zero as n tends to infinity.

For the first term, we just use the fact that $W_n - W^*$ tends to zero in $H_0^1(\Omega)$ weakly.

For the second term, we use the fact $T_k(K_{\delta_0}(x, w_n, Dw_n)) \operatorname{sign}_k(W_n)$ is bounded in $L^\infty(\Omega)$, since k is fixed, while $W_n - W^*$ tends to zero in $L^1(\Omega)$ strongly.

For the last three terms we observe that, since w_n and W_n respectively converge almost everywhere to w and to W^* (see (5.9) and (5.11)), we have

$$\left\{ \begin{array}{ll} (1 + \delta_0 |w_n|) f(x) (W_n - W^*) \rightarrow 0 & \text{a.e. in } \Omega, \\ a_0(x) w_n (W_n - W^*) \rightarrow 0 & \text{a.e. in } \Omega, \\ a_0(x) g_{\delta_0}(w_n) \operatorname{sign}(w_n) (W_n - W^*) \rightarrow 0 & \text{a.e. in } \Omega. \end{array} \right. \quad (5.14)$$

We will now prove that each of the three sequences which appear in (5.14) are equi-integrable. Together with (5.14), this will imply that these sequences converge to zero in $L^1(\Omega)$ strongly, and this will prove that the three last terms of the right-hand side of (5.13) tend to zero as n tends to infinity.

In order to prove that the sequence $(1 + \delta_0 |w_n|) f(x) (W_n - W^*)$ is equiintegrable, we use the classical properties about rearrangements (2.1), (2.2) and (2.3).

For any measurable set E , $E \subset \Omega$, we have

$$\left\{ \begin{aligned} & \int_E |(1 + \delta|w_n|) f(x) (W_n - W^*)| dx \\ &= \int_E f(x) (W_n - W^*) dx + \delta \int_E |w_n| f(x) (W_n - W^*) dx \\ &\leq \frac{\|f\|_{L^1(\Omega)}^{1/2} \|f\|_{L^1(\log L)(E)}^{1/2}}{(2\pi)^{1/2}} \|D(W_n - W^*)\|_2 + \frac{\delta \|f\|_{L^1(\log L)(E)}}{2\pi} \|Dw_n\|_2 \|D(W_n - W^*)\|_2 \\ &\leq c_1 \|f\|_{L^1(\log L)(E)}^{1/2} + c_2 \|f\|_{L^1(\log L)(E)}. \end{aligned} \right.$$

where c_1, c_2 denote the constants which are independents of n .

Proving that the sequence $a_0(x) w_n (W_n - W^*)$ is equiintegrable is similar, since for any measurable set E , $E \subset \Omega$, we have

$$\left\{ \begin{aligned} & \int_E |a_0(x) w_n (W_n - W^*)| dx \\ &\leq \left(\int_E |a_0(x)|^q dx \right)^{1/q} \frac{C_1(q)}{2\pi} \|Dw_n\|_2 \|D(W_n - W^*)\|_2 \\ &\leq c \left(\int_E |a_0(x)|^q dx \right)^{1/q}. \end{aligned} \right.$$

Finally, in order to prove that the sequence $a_0(x) g_{\delta_0}(w_n) \text{sign}(w_n) (W_n - W^*)$ is equiintegrable, we use as in (5.8), Hardy-Littlewood inequality; for any measurable set E , $E \subset \Omega$, we have

$$\left\{ \begin{aligned} & \int_E |a_0(x) g_{\delta_0}(w_n) \text{sign}(w_n) (W_n - W^*)| dx \\ &\leq \int_E |a_0(x)| \delta_0 |w_n|^2 |W_n - W^*| dx \\ &\leq \left(\int_E |a_0(x)|^q dx \right)^{1/q} \delta_0 \frac{C_2(q)}{(2\pi)^{3/2}} \|Dw_n\|_2^2 \|D(W_n - W^*)\|_2 \leq c \left(\int_E |a_0(x)|^q dx \right)^{1/q}. \end{aligned} \right.$$

We have proved that the right-hand side of (5.13) tends to zero. Since the matrix A is coercive (see (3.3)), this proves that W_n tends to W^* in $H_0^1(\Omega)$ strongly. Lemma 5.3 is proved.

Proof of Theorem 5.1. Recall that in this Theorem $k > 0$ is fixed.

Consider the ball B of $H_0^1(\Omega)$ defined by

$$B = \{w \in H_0^1(\Omega) : \|Dw\|_2 \leq Z_{\delta_0}\}, \quad (5.15)$$

where Z_{δ_0} is defined from δ_0 by (7.11).

Consider also the mapping $S : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$ defined by

$$S(w) = W, \quad (5.16)$$

where for every $w \in H_0^1(\Omega)$, W is the unique solution of (5.4) (see Lemma 5.2).

We will apply Schauder's fixed point theorem in the Hilbert space $H_0^1(\Omega)$ to the mapping S and to the ball B .

First step : In this step we prove that S maps B into itself.

Indeed by Lemma 5.2, $W = S(w)$ satisfies (5.5); therefore, when $\|Dw\|_2 \leq Z_{\delta_0}$, one has, in view of the definition (7.7) the function Φ_δ and of the property (7.10) of Z_{δ_0} ,

$$\left\{ \begin{array}{l} \alpha \|DW\|_2 \leq \|f\|_{H^{-1}(\Omega)} + \frac{\delta_0 \|f\|_{L^1(\log L)(\Omega)}}{2\pi} \|Dw\|_2 \\ \quad + \frac{C_1(q) \|a_0\|_q}{2\pi} \|Dw\|_2 + \frac{\delta_1 C_2(q) \|a_0\|_q}{(2\pi)^{3/2}} \|Dw\|_2^2 \\ \leq \|f\|_{H^{-1}(\Omega)} + \frac{\delta_0 \|f\|_{L^1(\log L)(\Omega)}}{2\pi} Z_{\delta_0} \\ \quad + \frac{C_1(q) \|a_0\|_q}{2\pi} Z_{\delta_0} + \frac{\delta_1 C_2(q) \|a_0\|_q}{(2\pi)^{3/2}} Z_{\delta_0}^2 \\ = \alpha Z_{\delta_0} + \Phi_{\delta_0}(Z_{\delta_0}) = \alpha Z_{\delta_0}, \end{array} \right. \quad (5.17)$$

i.e. $\|DW\|_2 \leq Z_{\delta_0}$, or in other terms $W \in B$, which proves that $S(B) \subset B$.

Second step : In this step we prove that S is continuous from $H_0^1(\Omega)$ strongly into $H_0^1(\Omega)$ strongly.

For this we consider a sequence such that

$$w_n \in B, \quad w_n \rightarrow w \quad \text{in } H_0^1(\Omega) \quad \text{strongly}, \quad (5.18)$$

and we define W_n as $W_n = S(w_n)$, i.e. as the solution of (5.10).

The functions w_n belong to B , and therefore the functions W_n belong to B in view of the first step. We can therefore extract a subsequence, still denoted by n , such that for some $W^* \in H_0^1(\Omega)$,

$$W_n \rightharpoonup W^* \quad \text{in } H_0^1(\Omega) \quad \text{weakly and a.e. in } \Omega. \quad (5.19)$$

We can moreover assume that for a further subsequence, still denoted by n ,

$$w_n \rightarrow w \quad \text{a.e. in } \Omega \quad \text{and} \quad Dw_n \rightarrow Dw \quad \text{a.e. in } \Omega. \quad (5.20)$$

Since the assumptions of Lemma 5.3 are satisfied by the subsequences w_n and W_n , the subsequence W_n converges to W^* in $H_0^1(\Omega)$ strongly.

We now pass to the limit in equation (5.10) as n tends to infinity by using the fact that $\text{sign}_k(s)$ and $g_{\delta_0}(s) \text{sign}(s)$ are Carathéodory functions, and the result of almost everywhere convergence (see Lemma 3.3 in [20]) as far as $T_k(K_{\delta_0}(x, w_n, Dw_n))$ is concerned (this point is the only point of the proof of Theorem 5.1 where the assumption

of strong $H_0^1(\Omega)$ convergence in (5.18), or more exactly its consequence (5.20), is used). This implies that W^* is a solution of (5.4). Since the solution of (5.4) is unique, one has $W^* = S(w)$.

In view of the fact that W^* is uniquely determined, we conclude that it was not necessary to extract a subsequence in (5.19) and (5.20), and that the whole sequence $W_n = S(w_n)$ converges in $H_0^1(\Omega)$ strongly to $W^* = S(w)$. This proves the continuity of the application S .

Third step : In this step we prove that $S(B)$ is precompact in $H_0^1(\Omega)$.

For this we consider a sequence $w_n \in B$ and we define W_n as $W_n = S(w_n)$; in other terms W_n is the solution of (5.10). Since w_n and W_n belong to B , they are bounded in $H_0^1(\Omega)$, and we can extract a subsequence, still denoted by n , such that

$$w_n \rightharpoonup w \quad \text{in } H_0^1(\Omega) \text{ weakly and a.e. in } \Omega,$$

$$W_n \rightharpoonup W^* \quad \text{in } H_0^1(\Omega) \text{ weakly and a.e. in } \Omega.$$

Since w_n and W_n satisfies the assumptions of Lemma 5.3, we have

$$W_n \rightarrow W^* \quad \text{in } H_0^1(\Omega) \text{ strongly.}$$

This proves that $S(B)$ is precompact in $H_0^1(\Omega)$ (note that in contrast with the second step, we do not need here to prove that $W^* = S(w)$).

End of the proof of Theorem 5.1 We have proved that the application S and the ball B satisfy the assumptions of Schauder's fixed point theorem. Therefore there exists at least one $w_k \in B$ such that $S(w_k) = w_k$. This proves Theorem 5.1. $\square$

6. Proof of Theorem 4.3

Theorem 5.1 asserts that for every $k > 0$ fixed there exists at least a solution w_k of (5.1) which satisfies (5.3). We can therefore extract a subsequence, still denoted by k , such that for some $w^* \in H_0^1(\Omega)$

$$w_k \rightharpoonup w^* \quad \text{in } H_0^1(\Omega) \text{ weakly and a.e. in } \Omega, \quad (6.1)$$

where w^* satisfies

$$\|w^*\|_{H_0^1(\Omega)} \leq Z_{\delta_0}, \quad (6.2)$$

i.e. (4.14).

In this Section we will first prove that for this subsequence

$$w_k \rightarrow w^* \quad \text{in } H_0^1(\Omega) \text{ strongly,} \quad (6.3)$$

and then that w^* is a solution of (4.13) (which satisfies (4.14)). This will prove Theorem 4.3.

To prove (6.3), we use a technique which traces back to [7] (see also [15]).

For $n > 0$, we define $G_n : \mathbb{R} \rightarrow \mathbb{R}$ as the remainder of the truncation at height n , namely

$$G_n(s) = s - T_n(s), \quad \forall s \in \mathbb{R} \quad (6.4)$$

where T_n is the truncation at height n defined by (4.15), or in other terms

$$G_n(s) = \begin{cases} s + n & \text{if } s \leq -n, \\ 0 & \text{if } -n \leq s \leq n, \\ s - n & \text{if } s \geq n. \end{cases} \quad (6.5)$$

First we use the two following Lemmas.

Lemma 6.1. [15] *Assume that (3.2), (3.3), (3.4), (3.5), (3.6) hold true. Assume moreover that the two smallness conditions (3.7) and (3.8) hold true. Let w_k be a solution of (5.1). Assume finally that the subsequence w_k satisfies (6.1).*

Then for this subsequence we have

$$\limsup_{k \rightarrow +\infty} \int_{\Omega} |DG_n(w_k)|^2 dx \rightarrow 0 \quad \text{as } n \rightarrow +\infty. \quad (6.6)$$

Lemma 6.2. [15] *Assume that (3.2), (3.3), (3.4), (3.5), (3.6) hold true. Assume moreover that the two smallness conditions (3.7) and (3.8) hold true. Let w_k be a solution of (5.1). Assume finally that the subsequence w_k satisfies (6.1).*

Then for this subsequence we have for every $n > 0$ fixed

$$T_n(w_k) \rightarrow T_n(w^*) \quad \text{in } H_0^1(\Omega) \quad \text{strongly as } k \rightarrow +\infty. \quad (6.7)$$

End of the proof of Theorem 4.3

First step Since we have

$$w_k - w^* = T_n(w_k) + G_n(w_k) - T_n(w^*) - G_n(w^*),$$

and since by Lemma 6.2 (see (6.7)) we have

$$\|T_n(w_k) - T_n(w^*)\|_{H_0^1(\Omega)} \rightarrow 0 \quad \text{as } k \rightarrow +\infty \quad \text{for every } n > 0 \text{ fixed},$$

while by Lemma 6.1 (see (6.6)) we have

$$\limsup_{n \rightarrow +\infty} \limsup_{k \rightarrow +\infty} \|G_n(w_k)\|_{H_0^1(\Omega)} = 0,$$

and while we have

$$\limsup_{n \rightarrow +\infty} \|G_n(w^*)\|_{H_0^1(\Omega)} = 0,$$

since $w^* \in H_0^1(\Omega)$, we conclude that

$$w_k \rightarrow w^* \quad \text{in } H_0^1(\Omega) \quad \text{strongly as } k \rightarrow +\infty, \quad (6.8)$$

which is nothing but (6.3).

Second step Let us now pass to the limit in (5.1) as k tends to infinity. This is easy for the first term of the left-hand side of (5.1) as well as for the three terms of the right-hand

side of (5.1), which is similar to the one that we used in the proof of Lemma 5.3.

It remains to pass to the limit in the second term of the left-hand side of (5.1), namely in

$$T_k(K_{\delta_0}(x, w_k, Dw_k)) \operatorname{sign}_k(w_k).$$

We first observe that $K_{\delta}(x, s, \zeta) > 0$ and $\delta_0 \geq \gamma$ (see (7.9)), we have

$$\begin{cases} |T_k(K_{\delta_0}(x, w_k, Dw_k)) \operatorname{sign}_k(w_k)| \leq |K_{\delta_0}(x, w_k, Dw_k)| \\ \leq (c_0 + \delta_0) \|A\|_{\infty} |Dw_k|^2 \quad \text{a.e. in } \Omega, \end{cases}$$

which implies that the functions $T_k(K_{\delta_0}(x, w_k, Dw_k)) \operatorname{sign}_k(w_k)$ are equiintegrable since Dw_k converges strongly to Dw^* in $L^2(\Omega)^2$.

Extracting if necessary a subsequence, still denoted by k , such that

$$Dw_k \rightarrow Dw^* \quad \text{a.e. in } \Omega,$$

we claim that

$$T_k(K_{\delta_0}(x, w_k, Dw_k)) \operatorname{sign}_k(w_k) \rightarrow K_{\delta_0}(x, w^*, Dw^*) \operatorname{sign}(w^*) \quad \text{a.e. in } \Omega. \quad (6.9)$$

On the first hand we use the following almost everywhere convergence, which asserts that

$$K_{\delta_0}(x, w_k, Dw_k) \rightarrow K_{\delta_0}(x, w^*, Dw^*) \quad \text{a.e. in } \Omega,$$

and the fact that for every $s \in \mathbb{R}$

$$T_k(s_k) \rightarrow s \quad \text{if } k \rightarrow +\infty \quad \text{when } s_k \rightarrow s,$$

to deduce that

$$T_k(K_{\delta_0}(x, w_k, Dw_k)) \rightarrow K_{\delta_0}(x, w^*, Dw^*) \quad \text{a.e. in } \Omega. \quad (6.10)$$

On the other hand we use the fact that

$$\operatorname{sign}_k(w_k) \rightarrow \operatorname{sign}(w^*) \quad \text{a.e. in } \{y \in \Omega : w^*(y) \neq 0\},$$

which together with (6.10) proves convergence (6.9) in the set $\{y \in \Omega : w^*(y) \neq 0\}$.

Finally, as far as convergence in the set $\{y \in \Omega : w^*(y) = 0\}$ is concerned, convergence (6.10), the fact that

$$K_{\delta_0}(x, w^*, Dw^*) = 0 \quad \text{a.e. in } \{y \in \Omega : w^*(y) = 0\},$$

and the fact that $|\operatorname{sign}_k(s)| \leq 1$ for every $s \in \mathbb{R}$ together prove that

$$\begin{cases} T_k(K_{\delta_0}(x, w_k, Dw_k)) \operatorname{sign}_k(w_k) \rightarrow 0 = K_{\delta_0}(x, w^*, Dw^*) \operatorname{sign}(w^*) \\ \text{a.e. in } \{y \in \Omega : w^*(y) = 0\}. \end{cases}$$

This completes the proof of (6.9).

The equiintegrability and the almost everywhere convergence of the functions $T_k(K_{\delta_0}(x, w_k, Dw_k)) \operatorname{sign}_k(w_k)$ then imply that

$$T_k(K_{\delta_0}(x, w_k, Dw_k)) \operatorname{sign}_k(w_k) \rightarrow K_{\delta_0}(x, w^*, Dw^*) \operatorname{sign}(w^*) \quad \text{in } L^1(\Omega) \quad \text{strongly.}$$

This proves that w^* satisfies (4.13). Since w^* also satisfies (4.14) (see (6.2)), Theorem 4.3 is proved.

7. Appendix

In this Appendix, we give an estimate of the function g_δ defined by (4.6) (see Lemma 7.1), and the definitions of the constants δ_0 and Z_{δ_0} which appear in Theorem 3.1 (see Lemma 7.2).

7.1. An estimate for the function g_δ .

Lemma 7.1. *For $\delta > 0$, let $g_\delta : \mathbb{R} \rightarrow \mathbb{R}$ be the function defined by (4.6), i.e. by*

$$g_\delta(t) = -|t| + \frac{1}{\delta}(1 + \delta|t|) \log(1 + \delta|t|), \quad \forall t \in \mathbb{R}. \quad (7.1)$$

Then

$$0 \leq g_\delta(t) \leq \delta|t|^2, \quad \forall t \in \mathbb{R}, \quad \forall \delta, \quad 0 < \delta \leq \delta_1. \quad (7.2)$$

Proof. Let $g : \mathbb{R}^+ \rightarrow \mathbb{R}$ be the function defined by

$$g(\tau) = -\tau + (1 + \tau) \log(1 + \tau), \quad \forall \tau \geq 0.$$

Since $g(0) = 0$ and $g'(\tau) \geq 0$, one has

$$g(\tau) \geq 0, \quad \forall \tau \geq 0. \quad (7.3)$$

Since $\log(1 + \tau) < \tau$ for $\tau > 0$, one has

$$g(\tau) < \tau^2, \quad \forall \tau, \quad \tau > 0. \quad (7.4)$$

Since

$$g_\delta(t) = \frac{1}{\delta}g(\delta|t|), \quad \forall t \in \mathbb{R},$$

one deduces from (7.4) that g_δ satisfies (7.2). $\square$

7.2. Definition of δ_0 and Z_{δ_0} . The goal of this Subsection is to define the constants δ_0 and Z_{δ_0} which appear in Theorem 3.1. We will prove the following result.

Lemma 7.2. *Assume that (3.2), (3.3), (3.4), (3.5) and (3.6) hold true. Assume moreover that the two smallness conditions (3.7) and (3.8) hold true.*

Let δ_1 be the number defined by

$$\delta_1 = \frac{2\pi\alpha - C_1(q)\|a_0\|_q}{\|f\|_{L^1(\log L)(\Omega)}}. \quad (7.5)$$

One has

$$\delta_1 > \gamma. \quad (7.6)$$

For $\delta \geq 0$, let $\Phi_\delta : \mathbb{R}^+ \rightarrow \mathbb{R}$ (see Fig. 2) be the function defined by

$$\begin{aligned} \Phi_\delta(X) = & \delta_1 C_2(q) \|a_0\|_q X^2 \\ & - \frac{2\pi\alpha - C_1(q)\|a\|_q - \delta\|f\|_{L^1(\log L)(\Omega)}}{2\pi} X + \|f\|_{H^{-1}(\Omega)}, \end{aligned} \quad (7.7)$$

where $C_1(q)$ and $C_2(q)$ are the constants given in (3.9).

Then, for $0 \leq \delta \leq \delta_1$, the function Φ_δ has a unique minimizer Z_δ on $\mathbb{R}^+$, which is given by

$$Z_\delta = \frac{2\pi\alpha - C_1(q)\|a_0\|_q - \delta\|f\|_{L^1(\log L)(\Omega)}}{4\pi\delta_1 C_2(q)\|a_0\|_q}, \quad \text{for } 0 \leq \delta \leq \delta_1. \quad (7.8)$$

Moreover, there exists a unique number δ_0 such that

$$\gamma \leq \delta_0 < \delta_1, \quad (7.9)$$

and

$$\Phi_{\delta_0}(Z_{\delta_0}) = 0. \quad (7.10)$$

This number is the number δ_0 which appear in Theorem 3.1, and Z_{δ_0} is then defined from δ_0 through formula (7.8), namely by

$$Z_{\delta_0} = \frac{2\pi\alpha - C_1(q)\|a_0\|_q - \delta_0\|f\|_{L^1(\log L)(\Omega)}}{4\pi\delta_1 C_2(q)\|a_0\|_q}. \quad (7.11)$$

Remark 7.3. Let us explain the meaning of the results stated in Lemma 7.2.

As we will see in the proof of Lemma 7.2 (see also Fig. 2), the function Φ_δ is the restriction to $\mathbb{R}^+$ of a function which looks like a convex parabola. This function attains its minimum at a unique point Z_δ , and for δ which satisfies $\delta < \delta_1$ with δ_1 given by (7.5), one has $Z_\delta > 0$.

The smallness condition (3.7) is equivalent to the fact that $\delta_1 > \gamma$, and the smallness condition (3.8) to the fact that the minimum $\Phi_\gamma(Z_\gamma)$ of Φ_γ is nonpositive. For $\delta = \delta_1$, the minimum $\Phi_{\delta_1}(Z_{\delta_1})$ of Φ_{δ_1} is equal to $\|f\|_{H^{-1}(\Omega)}$, which is strictly positive. Therefore it can be proved that there exists some δ_0 with $\gamma \leq \delta_0 < \delta_1$ (see (7.9)) such that the minimum $\Phi_{\delta_0}(Z_{\delta_0})$ of Φ_{δ_0} is equal to zero (see (7.10)), or in other terms such that the function Φ_{δ_0} has a double zero in Z_{δ_0} . Moreover, when $\gamma < \delta_0$, for every δ with $\gamma \leq \delta < \delta_0$, the function Φ_δ has two distinct zeros Y_δ^- and Y_δ^+ with $Y_\delta^- < Y_\delta^+$ which satisfy $0 < Y_\delta^- < Z_{\delta_0} < Y_\delta^+$.

Fig. 1: The graph of the straight line L_δ

Fig. 2: The graphs of the functions $\Phi_\delta(X)$

Proof of Lemma 7.2 For $\delta \geq 0$, let L_δ be the constant defined by (see Fig. 1)

$$L_\delta = \frac{1}{2\pi} (2\pi\alpha - C_1(q)\|a_0\|_q - \delta\|f\|_{L^1(\log L)(\Omega)}), \quad (7.12)$$

where $C_1(q)$ and $C_2(q)$ are two positives constants satisfy (3.9). Note that L_δ is decreasing with respect to δ .

Since δ_1 is defined by (7.5), one has $L_{\delta_1} = 0$. On the other hand, the first smallness condition (3.7) is nothing but $L_\gamma > 0$. Since L_δ is decreasing in δ , one has $\delta_1 > \gamma$, i.e. (7.6).

Let us now study the family of functions $\Phi_\delta : \mathbb{R}^+ \rightarrow \mathbb{R}$ defined by (7.7), i.e., in view of the definition (7.12) of L_δ , by

$$\Phi_\delta(X) = \delta_1 C_2(q)\|a_0\|_q X^2 - L_\delta X + \|f\|_{H^{-1}(\Omega)}, \quad \forall X \geq 0, \quad (7.13)$$

(see Fig. 2).

Since $a_0 \neq 0$ (see (3.6)), each function Φ_δ looks like the restriction to $\mathbb{R}^+$ of a convex parabola. When $0 \leq \delta \leq \delta_1$, one has $L_\delta \geq 0$, and this convex parabola has a unique minimizer Z_δ on $\mathbb{R}^+$ which is also the minimizer of the function Φ_δ . A simple computation shows that Z_δ is given by

$$Z_\delta = \frac{L_\delta}{2\delta_1 C_2(q)\|a_0\|_q} = \frac{2\pi\alpha - C_1(q)\|a_0\|_q - \delta\|f\|_{L^1(\log L)(\Omega)}}{4\pi\delta_1 C_2(q)\|a_0\|_q}, \quad (7.14)$$

i.e (7.8), and that the minimum of Φ_δ , namely $\Phi_\delta(Z_\delta)$, is given by

$$\begin{aligned} \Phi_\delta(Z_\delta) &= \|f\|_{H^{-1}(\Omega)} - \frac{L_\delta^2}{4\delta_1 C_2(q)\|a_0\|_q} \\ &= \|f\|_{H^{-1}(\Omega)} - \frac{(2\pi\alpha - C_1(q)\|a_0\|_q - \delta\|f\|_{L^1(\log L)(\Omega)})^2}{16\pi^2\delta_1 C_2(q)\|a_0\|_q}. \end{aligned} \quad (7.15)$$

When $0 \leq \delta \leq \delta_1$, the function L_δ is nonnegative, continuous and decreasing with respect to δ . Therefore Z_δ is continuous and decreasing with respect to δ , while $\Phi_\delta(Z_\delta)$ is continuous and increasing with respect to δ .

When $\delta = \delta_1$, one has $L_{\delta_1} = 0$, the function Φ_{δ_1} attains its minimum in $Z_{\delta_1} = 0$, and $\Phi_{\delta_1}(Z_{\delta_1}) = \|f\|_{H^{-1}(\Omega)}$, while the second smallness condition (3.8) is nothing but $\Phi_\gamma(Z_\gamma) \leq 0$. Therefore there exists a unique δ_0 with $\gamma \leq \delta_0 < \delta_1$ such that $\Phi_{\delta_0}(Z_{\delta_0}) = 0$. This is the definition of δ_0 given by (7.9) and (7.10) in Lemma 7.2.

Lemma 7.2 is proved. $\square$

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