Bisimilar Booleanization of multivalued networks

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Abstract

Discrete modelling frameworks of Biological networks can be divided in two distinct categories: Boolean and multivalued. Although multivalued networks are more expressive for qualifying the regulatory behaviours modelled by more than two values, the ability to automatically convert them to Boolean network with an equivalent behaviour breaks down the fundamental borders between the two approaches. Theoretically investigating the conversion process provides relevant insights into bridging the gap between them. Basically, the conversion aims at finding a Boolean network *bisimulating* a multivalued one. In this article, we investigate the bisimilar conversion where the Boolean integer coding is a parameter that can be freely modified. Based on this analysis, we define a computational method automatically inferring a bisimilar Boolean network from a given multivalued one.

Keywords: Boolean Network, multivalued network, Bisimulation, Biological network modelling, Automatic conversion inference

1. Introduction

Discrete network based modelling frameworks, seminally introduced by S. Kauffman (Glass and Kauffman, 1973; Kauffman, 1969) and R. Thomas (Thieffry and Thomas, 1995; Thomas et al., 1995) for regulation network modelling can be divided in two distinct categories: Boolean networks and multivalued networks. In the former, the states of genes are modelled by Boolean values, with propositional logic as the modelling framework, whereas in the

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latter the state is extended to the integer domain, also called multivalued, using Presburger arithmetic as modelling framework. It is often admitted that multivalued networks provide more expressiveness for modelling gene expression behaviour by distinguishing between more than two states (*i.e.*, OFF or ON) for specifying the regulatory activity. However, the ability to automatically convert a multivalued network to a Boolean one with the same dynamical behaviour weakens this distinction from an analytical standpoint since the analysis of the dynamics can be performed on the Boolean network directly.

More generally, the Boolean conversion of a multivalued network offers the opportunity to bridge the gap between the two modelling formalisms that enables to inherit, adapt and extend the theoretical results defined in a framework to the other (Tonello, 2019). Moreover, this allows the use of software based on propositional logic that could prove computationally more efficient than the algorithms developed for Presburger arithmetic for the same problem. In particular, a wide spectrum of problems in modelling regulatory networks by symbolic characterization of stable states can be formalized as problems of logical valuation of variables satisfying a formula in the Boolean case (the SAT problem) or finding solutions complying to a set of linear constraints for the integer case (integer linear programming, ILP). (Aloul et al., 2002) provide an experimental comparison of ILP and SAT solvers applied to the SAT problem.

By considering these opportunities, the issue is thus to investigate methods for converting multivalued networks to Boolean while preserving the dynamical behaviour. This conversion is primarily based on an encoding of integers by Boolean profiles, establishing the equivalence between the two kinds of values. The challenge is to extend this equivalence to state transitions in order to certify the behavioural integrity.

In (Didier et al., 2011), G. Didier, E. Remy and C. Chaouiya extensively study the conditions for the conversion of multivalued networks to Boolean ones using Van Ham code (Van Ham, 1979) (Section 4). To overcome the potential limitation of Van Ham code restraining the dynamics to a sub-region of the Boolean state space, A. Fauré and S. Kaji study the conversion based on *Summing code* (Section 4), which provides several alternative Boolean profiles for encoding an integer, such that the resulting Boolean dynamics is deployed on the whole Boolean state space (Fauré and Kaji, 2018). Following similar motivations, E. Tonello also studies the conversion based on this code (Tonello, 2019). Regarding conversion tool, A. Naldi developed the Java library bioLQM (Naldi, 2018), which is a toolkit dedicated to qualitative analysis of biological regulatory networks, and which includes a multivalued to Boolean converter. The converter is based on Van Ham coding and ensures that the trajectories always converge to the region of the valid Boolean codings, called the admissible region (Section 3), thereby avoiding potential spurious equilibria outside this region. This converter is also available in GINsim 3.0 environment (Chaouiya et al., 2012), which relies on bioLQM.

These research works elegantly pave the formal foundation of the multivalued to Boolean network conversion. However, the results are intrinsically dependent on a specific coding, the Summing and Van Ham code. Moreover they are mainly designed for the asynchronous mode. Therefore, it appears interesting to generalise this approach by distinguishing the properties that purely relate to the conversion process from those depending on the code for highlighting the foundations of this process.

The behavioural equivalence is formally defined by the *reachability preservation* property, namely: whenever an integer state is reachable from another one, the equivalent Boolean state of the former is also reachable by the equivalent Boolean state of the latter, and conversely. Reachability preservation relies on the existence of a bisimulation (Sangiorgi, 2011) between both networks, parametrised by the Boolean-to-integer coding. While preserving the reachability is essential, it also appears desirable to extend the preservation to structural properties of the interactions and other properties related to equilibrium. In this article, we study the network conversion by regarding it as a bisimulation process applied to any Boolean coding of the integers. Based on this study, we propose an algorithm inferring the formulas of a Boolean network behaviourally equivalent to the input multivalued network.

After recalling the main notions of multivalued networks (Section 2), we examine the bisimulation properties between the dynamics of the networks and the admissibility conditions for stating a bisimulation between a multivalued and Boolean networks (Section 3) with regard to different codings (Section 4). Then, we study the extension of the properties preserved by conversion (Section 5). Finally we define a method inferring a Boolean network bisimilar to the multivalued one whatever the coding procedure (Section 6).

We use the following notations:

Set. The *complement* of a subset $E \backslash E', E' \subseteq E$ is denoted by $-E'$. A *singleton* $\{e\}$ is denoted by its element e. The *set of parts* of E is noted $\mathbf{2}^E = \{E' \mid E' \subseteq E\}$.

State. A *state* s is an application from variables Y to a domain of values $\mathbb{D}$, *i.e.*, $s = \{y_1 \mapsto d_1, \ldots, y_n \mapsto d_n\}$ and $\mathbb{D}_Y = (Y \to \mathbb{D})$ denotes the *state space* defined on variables Y. We define $[\![L]\!]_Y$ as the state domain defined on the integral interval between 0 and L, *i.e.*, $[\![L]\!]_Y = (Y \to \{d_i \mid 0 \leqslant d_i \leqslant L\})$. The *restriction/projection* of a state $s \in \mathbb{D}_Y$ on $W \subseteq Y$ is denoted $s_W \in \mathbb{D}_W$. This notation also holds for function on states, *i.e.*, $\mathbf{dom}\, g_W = \mathbb{D}_W$. A *substitution* within a state s is the replacement of the value of a variable of s by another value, formally defined as: $s_{[y \mapsto v]} = s \backslash \{y \mapsto s_y\} \cup \{y \mapsto v\}$. The *distance* on states is defined as: $d(s, s') = \sum_{i=1}^{n} |s_{y_i} - s'_{y_i}|$.

2. multivalued networks

A *multivalued network* $\langle g, Y \rangle$ defined on a set of variables Y is a dynamical system characterized by an *evolution function* g.

Let $\mathcal{L} = \{L_i \in \mathbb{N} \mid 1 \leqslant i \leqslant n\}, n = |Y|$ be an indexed set of integral values (levels), we define: $[\![\mathcal{L}]\!]_Y = \times_{i=1}^{n} [\![L_i]\!]_{y_i}$ as the product of finite multivalued state domains. The evolution function on this domain $g : [\![\mathcal{L}]\!]_Y \to [\![\mathcal{L}]\!]_Y$ is composed of a collection of *local* evolution functions $g = (g_1, \cdots, g_n)$ such that $g_i : [\![\mathcal{L}]\!]_Y \to [\![L_i]\!]_{y_i}$ is defined as follows (see Figure 1 for an example):

$$g_i(s) = \begin{cases} 1 & \textit{if } C^1(s) \\ 2 & \textit{if } C^2(s) \\ & \cdots \\ l & \textit{if } C^l(s) \\ & \cdots \\ L_i & \textit{if } C^L(s) \\ 0 & \textit{otherwise} \end{cases} \tag{1}$$

where C^l is the *guard* of level l such that all the guards are *mutually exclusive*, namely:

$$\forall s \in [\![\mathcal{L}]\!]_Y, \forall\, 1 \leqslant l, l' \leqslant L_i : l \neq l' \implies \neg(C^l(s) \wedge C^{l'}(s)).$$

Hence, the application $g_i(s)$ equals level l if and only if the guard $C^l(s)$ is satisfied, and by convention 0 is returned when no guards are satisfied.

Model of dynamics. The *model of dynamics* of a multivalued network $\langle g, Y\rangle$ is formalized by a labelled transition system $\langle [\![\mathcal{L}]\!]_Y, M, \longrightarrow_g\rangle$ where the labels are sets of variables that determine which variables are updated jointly during a transition. The *mode* $M \subseteq \mathbf{2}^Y$ describes the organization of the joint updates per transition. For example, in the *asynchronous* mode, $\mathbf{1}_Y = \{\{y_i\}\}_{y_i \in Y}$, the state of one variable only is updated per transition and in the *parallel* or *synchronous* mode $\{Y\}$, all the variables are updated together. The mode is also introduced in the network specification if needed, *i.e.*, $\langle g, Y, M\rangle$.

Thus, only the state of the variables in $m \in M$ can be updated by a transition $s \xrightarrow{m}_g s'$ whereas the state of the other variables remains unchanged *i.e.*, $s' = g_m(s) \cup s_{-m}$. A transition that does not change the state, $s \xrightarrow{m}_g s$, is called a *self-loop.* The global transition relation corresponds to the union of all transition relations labelled by the components of the mode: $\longrightarrow_g = \bigcup_{m \in M} \xrightarrow{m}_g$.

Hereafter, $f : \mathbb{B}_X \to \mathbb{B}_X, \mathbb{B} = \{0, 1\}$, always stands for a Boolean function, $g : [\![\mathcal{L}]\!]_Y \to [\![\mathcal{L}]\!]_Y$ designates a multivalued/integer function, and Y corresponds to a set of integer variables. $w \in \mathbb{B}_X$ represents a Boolean state whereas $s \in [\![\mathcal{L}]\!]_Y$ a multivalued one.

Multivalued network is subjected to properties distinguishing the evolution behaviour capabilities. Specifically, the dynamics in which transitions modify the current level by 1 only ($i.e.$, $\forall s \longrightarrow s', \forall y_i \in Y : d(s_{y_i}, s'_{y_i}) \leqslant 1$) is said *unitary stepwise.*

Equilibrium. A state s is an *equilibrium*, if it can be reached[1] infinitely once met:

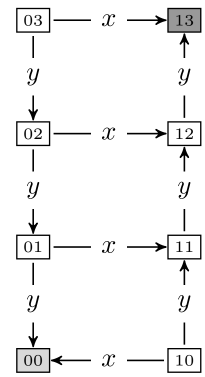
$$\forall s' \in \mathbb{N}_Y : s \longrightarrow^* s' \implies s' \longrightarrow^* s. \tag{2}$$

An *attractor* is a set of equilibria that are mutually reachable and a *stable state* is an attractor of cardinality 1.

Figure 1 shows an example of a multivalued network and the resulting dynamics for the asynchronous mode with two stable states that are respectively 13 and 00.

[1] $\longrightarrow^*$ denotes the reflexive and transitive closure of $\longrightarrow$.

$$g = \begin{cases} x = \begin{cases} 1 & y \geqslant 1 \\ 0 & \textit{otherwise} \end{cases} \\ \\ y = \begin{cases} 3 & x = 1 \wedge y \geqslant 2 \\ 2 & (x = 1 \wedge y = 1) \vee (x = 0 \wedge y = 3) \\ 1 & (x = 1 \wedge y = 0) \vee (x = 0 \wedge y = 2) \\ 0 & \textit{otherwise} \end{cases} \end{cases}$$



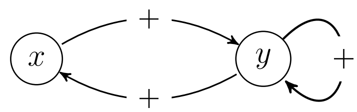


Figure 1: A multivalued network with the interaction graph (below) and the asynchronous dynamics (right), with the self-loops removed.

Interaction graph. An *interaction graph* $\langle Y, \longrightarrow\!\!\triangleright \rangle$ portrays the interdependencies of the variables in the network $\langle g, Y \rangle$. An interaction $y_i \longrightarrow\!\!\triangleright\, y_j$ exists whenever changing the value of y_i may lead to a change in the value of y_j:

$$y_i \longrightarrow\!\!\triangleright\, y_j \overset{\text{def}}{=} \exists s, s' \in \mathbb{N}_Y : s_{y_i} \neq s'_{y_i} \wedge s_{-y_i} = s'_{-y_i} \wedge g_j(s) \neq g_j(s'). \tag{3}$$

The *signed interaction graph* $\langle Y, \longrightarrow\!\!\triangleright, \sigma \rangle$ refines the nature of the interactions by signing the arcs with $\sigma : (\longrightarrow\!\!\triangleright) \to \{-1, 0, 1\}$ to represent a monotone relation between the source and target variables of the interaction (4); either increasing (label 1, denoted $'+'$), or decreasing (label -1, denoted $'-'$), or neither (label 0, denoted $'\pm'$), and formally defined as:

$$\begin{aligned} y_i \overset{+}{\longrightarrow\!\!\triangleright}\ y_j &\overset{\text{def}}{=} y_i \longrightarrow\!\!\triangleright\, y_j \wedge \\ &\quad \forall s, s' \in \mathbb{N}_Y : s_{y_i} \leqslant s'_{y_i} \wedge s_{-y_i} = s'_{-y_i} \implies g_j(s) \leqslant g_j(s') \\ y_i \overset{-}{\longrightarrow\!\!\triangleright}\ y_j &\overset{\text{def}}{=} y_i \longrightarrow\!\!\triangleright\, y_j \wedge \\ &\quad \forall s, s' \in \mathbb{N}_Y : s_{y_i} \leqslant s'_{y_i} \wedge s_{-y_i} = s'_{-y_i} \implies g_j(s) \geqslant g_j(s') \end{aligned} \tag{4}$$

3. Network bisimulation

By definition (Sangiorgi, 2011), bisimulation between the dynamics of networks preserves the reachability, thereby maintaining the trajectories and the attractors in both ways. Definition 1 illustrated in Figure 2 formally defines functional bisimulation, which depends on a partial function $\psi : \mathbb{B}_X \to [\![\mathcal{L}]\!]_Y$ decoding a Boolean state to an integer state.

Definition 1. *Given a Boolean network $B = \langle f, X, M_X \rangle$ and a multivalued network $N = \langle g, Y, M_Y \rangle$, a pair of functions (ψ, μ), with $\psi : \mathbb{B}_X \to [\![\mathcal{L}]\!]_Y$ a partial function and $\mu : M_X \to M_Y$ a total function, form a bisimulation if and only if the following properties hold:*

1. *(forward simulation) for any two Boolean states $w, w' \in \boldsymbol{dom}\,\psi$ and $m \in M_X$, $w \xrightarrow{m}_f w'$ implies $\psi(w) \xrightarrow{\mu(m)}_g \psi(w')$:*

$$\forall w, w' \in \boldsymbol{dom}\,\psi,\ \forall m \in M_X : w \xrightarrow{m}_f w' \implies \psi(w) \xrightarrow{\mu(m)}_g \psi(w');$$

2. *(backward simulation) for any two multivalued states $s, s' \in [\![\mathcal{L}]\!]_Y$, for any $w \in \mathbb{B}_X$ such that $\psi(w) = s$, and for any $n \in M_Y$, $s \xrightarrow{n}_g s'$ implies that there exists a $w' \in \mathbb{B}_X$ and an $m \in M_X$ such that $\psi(w') = s'$, $\mu(m) = n$ and $w \xrightarrow{m}_f w'$:*

$$\forall s, s' \in [\![\mathcal{L}]\!]_Y,\ \forall w \in \mathbb{B}_X,\ \forall n \in M_Y : \psi(w) = s \wedge s \xrightarrow{n}_g s' \implies \\ \big(\exists w' \in \mathbb{B}_X,\ \exists m \in M_X : \mu(m) = n \wedge \psi(w') = s' \wedge w \xrightarrow{m}_f w'\big).$$

$$\begin{tikzcd} w \arrow[r, dashed] \arrow[d, "m"] & \psi(w) \arrow[d, "\mu(m)"] \\ w' \arrow[r, dashed] & \psi(w') \end{tikzcd}$$

(a) Forward simulation

$$\begin{tikzcd} s \arrow[r, dashed] \arrow[d, "n"] & \forall w \in \psi^{-1}(s) \arrow[d, "\exists m \in \mu^{-1}(n)"] \\ s' \arrow[r, dashed] & \exists w' \in \psi^{-1}(s') \end{tikzcd}$$

(b) Backward simulation

Figure 2: Illustration of a bisimulation (ψ, μ) between a Boolean and a multivalued network. $\psi^{-1}(s)$ and $\mu^{-1}(n)$ denote the preimages of s and n under ψ and μ respectively.

Two networks B and N complying with Definition 1 with respect to ψ are said *bisimilar*, noted $B \sim_\psi N$. Although, (1.2) and (1.1) are similar in their definition, it is worth noticing that they however differ in the following point: all the transitions on the integer state space should fulfill (1.2) whereas only the transitions defined on the domain of ψ, $\mathbf{dom}\,\psi$, should comply with (1.1). $\mathbf{dom}\,\psi$ circumscribes the *admissible region* (Fauré and Kaji, 2018; Tonello, 2019), where each Boolean state encodes an integer state and each transition is bisimilar to a multivalued one. Hence, no transitions from a state located in the admissible region can escape from this region, thus avoiding aberrant cases exemplified in (Tonello, 2019). From (1.2), we deduce that ψ is a surjective partial function defined on $\mathbb{B}_X$ but it is not necessary injective and thus not bijective. Hence, the preimage of an integer state is a set: $\psi^{-1}(s) = \{w \in \mathbb{B}_X \mid \psi(w) = s\}$.

The issue is to determine the conditions on a Boolean network enabling a bisimulation with a multivalued network. These conditions depend on a general relation between the integer and Boolean function including the mode.

3.1. From global to local bisimulation discovery

Integer states are coded by the Boolean states in which the Boolean variables storing the code constitute the *support* of the integer variables. The *support function* associates each subset of integer variables to its support: $\widehat{\cdot} : \mathbf{2}^Y \to \mathbf{2}^X$. This function has the following properties: 1) the Boolean variables are exactly the supports of the integer variables, 2) the supports are pairwise disjoint, and 3) they are modular in the sense that the union of the supports is the support of the union of the integer variables:

$$\begin{array}{lll} 1) & X = \widehat{Y} & \text{and,} \\ 2) & \forall y_i, y_j \in Y : y_i \neq y_j \iff \widehat{y_i} \cap \widehat{y_j} = \varnothing & \text{and,} \\ 3) & \forall Y', Y" \subseteq Y : \widehat{Y' \cup Y"} = \widehat{Y'} \cup \widehat{Y"}. & \end{array} \tag{5}$$

For example, the state $s_{(y_1,y_2)} = (0,1)$ is encoded by $w = (00, 01)$ by using the classical binary code or the Gray code. The variables of the Boolean network will be therefore the variables supporting the Boolean code of the integer variables of Y ($X = \widehat{Y}$). The states of Boolean variables are respectively: $w_{\widehat{y_1}_1} = 0, w_{\widehat{y_1}_2} = 0, w_{\widehat{y_2}_1} = 0, w_{\widehat{y_2}_2} = 1$. Note that there are two kinds of indices: one for the multivalued variables, and the other for the Boolean variables of the corresponding supports.

We consider henceforth that ψ fits all supports, *i.e.*,

$$\psi \in \bigcup_{W \subseteq Y} \left(\mathbb{B}_{\widehat{W}} \to [\![\mathcal{L}]\!]_W\right).$$

The function ψ transforms the Boolean state of the support into an integer state in a modular manner, by decoding distinct sub-parts of a Boolean state separately, so that the decoding of the whole integer state is the union of the local decoding results:

$$\forall W \subseteq Y, \forall w \in \mathbb{B}_{\widehat{Y}} : \psi(w_{\widehat{W}}) = \psi(w)_W. \tag{6}$$

From (5,6) we deduce the following relation on two disjoint sets of variables representing the modularity of the decoding:

$$\psi(w_{\widehat{W \cup W'}}) = \psi(w_{\widehat{W}}) \cup \psi(w_{\widehat{W'}}) = \psi(w)_W \cup \psi(w)_{W'}, W \cap W' = \varnothing, W, W' \subseteq Y.$$

Moreover, the mode of the converted Boolean network must be compatible with the modularity of the coding. A mode is *local-to-support* when the parallel updates of the Boolean variables operate inside supports only, namely M is local-to-support if and only if: $\forall m \in M, \exists y_i \in Y : m \subseteq \widehat{y_i}$. The asynchronous mode is always local-to-support and the parallel local-to-support mode update all the support variable in parallel: $\{\widehat{y_i}\}_{y_i \in Y}$. The parallel mode for the Boolean network is not local-to-support since the joined update is accross all the supports.

Within the framework, inferring a Boolean network bisimilar to a multivalued one is reduced to the discovery of a Boolean network in bisimulation with a *local multivalued network* $\langle g_i, Y \rangle$ where only the state of a single variable evolves. A Boolean network in bisimulation with the entire multivalued network results from the union of Boolean networks in bisimulation with local multivalued networks (Proposition 1). Hence, for each g_i, we focus on the discovery of the appropriate evolution function of the support $f_{\widehat{y_i}}$ and the determination of the admissible modes for enabling the bisimulation.

Proposition 1. *Consider the multivalued network* $N = \langle g, Y, \mathbf{1}_Y \rangle$ *and the family* $(B_i)_{y_i \in Y}$ *of Boolean networks over the supports of the variables in* Y*:* $B_i = \langle f_{\widehat{y_i}}, X, M_{y_i} \rangle$, $M_{y_i} \subseteq \mathbf{2}^{\widehat{y_i}}$*. Then the following holds:*

$$\big(\forall y_i \in Y : \langle f_{\widehat{y_i}}, X, M_{y_i} \rangle \sim \langle g_i, Y, \mathbf{1}_{y_i} \rangle\big) \implies \langle f, X, \bigcup_{y_i \in Y} M_{y_i} \rangle \sim \langle g, Y, \mathbf{1}_Y \rangle.$$

where $f = (f_{\widehat{y_i}})_{y_i \in Y}$ *and* $g = (g_i)_{y_i \in Y}$ *are the global evolution functions collecting their respective local evolution functions.*

According to the definition of a transition (Section 2) and to Proposition 1, for establishing a bisimulation relation between a multivalued transition function f and a Boolean transition function g, it suffices that the *local* integer evolution function applied to the decoding $g_i \circ \psi(w)$ coincide with the Boolean evolution function $\psi \circ f_{\widehat{y}_i}(w)$ or, more generally, with the evolution taken under a local-to-support mode:

$$\begin{array}{lll} \psi \circ f & = g \circ \psi & \text{Global function} \\ \psi(f_m(w) \cup w_{\widehat{y}_i \setminus m}) & = g_i \circ \psi(w), m \subseteq \widehat{y}_i & \text{Local-to-support mode} \end{array} \tag{7}$$

If ψ is a bijective function then (7) is expressed as $f = \psi^{-1} \circ g \circ \psi$, which corresponds to the conjugated evolution function defined in Didier et al. (2011).

Theorem 1 shows that Property (7) is necessary and sufficient to ascertain that a multivalued network bisimulates a Boolean network with a local-to-support mode.

Theorem 1. *Let $N = \langle g_i, Y, \mathbf{1}_{y_i} \rangle$ be a multivalued network, $B = \langle f_{\widehat{y}_i}, \widehat{Y}, M \rangle$ a Boolean network with M a local-to-support mode, and $\psi : \mathbb{B}_{\widehat{y}_i} \to [\![L]\!]_{y_i}$, a surjective function, Property 7 is met between the evolution functions of B and N if and only if $B \sim_\psi N$.*

3.2. Bisimulation admisibility

The bisimulation necessitates to comply with conditions involving the mode and the coding. In this section we focus on the *admissibility condition* for the bisimulation extending the result of Theorem 1.

Admissibility based on mode and coding. The mode may prevent the bisimulation by forbidding the implementation of a Boolean transition simulating a multivalued transition. For example, assume that $g_i(0) = 2$ leading to the multivalued transition $0 \xrightarrow{y_i} 2$, the Boolean transition simulating it is naturally $00 \xrightarrow{w_{\hat{y}_1}, w_{\hat{y}_2}} 11$ using the Gray code or any coding procedures detailed in Section 4. However, this Boolean transition cannot be implemented in the asynchronous mode because both variables of the support should be updated jointly which is prohibited by the mode policy. Now, assume that the binary code is used leading to the Boolean transition $00 \xrightarrow{w_{\hat{y}_1}, w_{\hat{y}_2}} 10$ then the asynchronous Boolean transition is enabled by updating the first Boolean support variable ($w_{\widehat{y}_1}$) only. Hence, the mode selection strongly depends on a relation between the mode and the coding. Therefore the primary admissibility condition for the bisimulation lies on the mode selection of the Boolean network with regards to the coding that should enable the simulation for all multivalued transitions considered independently. Formally, a transition of a Boolean network $B = \langle f, W, M \rangle$ always fulfils:

$$\forall w, w' \in \mathbb{B}_W, \forall m \in M : w \xrightarrow{m} w' \implies d(w, w') \leqslant |m|.$$

Otherwise, the transition cannot be implemented due to the number of modifications exceeding the permitted capacity offered by the mode. The permissible distance for enabling a transition has connexion with the distance on Boolean coded integers. All the codings ψ described in Section 4 fulfil the condition stipulating that the distance between multivalued states is always lower or equal to the (Hamming) distance of their Boolean code:

$$\forall s_{y_i}, s'_{y_i} \in [\![L_i]\!]_{y_i} : \psi(w_{\widehat{y}_i}) = s_{y_i}, \psi(w'_{\widehat{y}_i}) = s'_{y_i} \implies d(s_{y_i}, s'_{y_i}) \leqslant d(w_{\widehat{y}_i}, w'_{\widehat{y}_i}).$$

Thereby, if a Boolean network B bisimulates a multivalued network $N = \langle g_i, Y, \mathbf{1}_{y_i} \rangle$, we deduce from these two previous properties that:

$$\begin{gathered}\forall s \in [\![\mathcal{L}]\!]_Y, \forall w, w' \in \mathbb{B}_{\widehat{Y}}, \forall m \in M : \\ \psi(w_{\widehat{y}_i}) = s_{y_i} \wedge \psi(w'_{\widehat{y}_i}) = g_i(s) \wedge w \xrightarrow{m} w' \implies d(s_{y_i}, s'_{y_i}) \leqslant d(w, w') \leqslant |m|.\end{gathered}$$

This property characterizes the unique necessary bisimulation admissibility condition with regards to a local-to-support mode used for Boolean network and the codings of Section 4. An admissibility test can be finally derived establishing a condition between the multivalued network and the mode of the Boolean network directly:

$$\forall s \in [\![\mathcal{L}]\!]_Y, \forall y_i \in Y : d(s_{y_i}, g_i(s)) \leqslant \min\{|m| \mid m \in M\}. \tag{8}$$

As practical consequence, the local-to-support parallel/synchronous mode enables the bisimulation with any multivalued network and only the unitary stepwise multivalued network (Section 2) can be bisimulated by asynchronous Boolean network since the cardinality of the modalities is always 1.

Family of admissible bisimilar Boolean networks. Theorem 1 states the equivalence between bisimilarity and Property 7 for local-to-support modes. we extend this result to a larger family of modes that are *admissible* with respect to the parallel mode. Informally, a modality m is m_0-admissible if, for any Boolean state w, running f on w under m or under m_0 yields (possibly different) states belonging to the same preimage under ψ. Definitions 2 and 3 detail this compatibility formally. Both definitions assume the Boolean network $B = \langle f, \widehat{Y}, M_0 \rangle$ operating in mode M_0 and the multivalued network $N = \langle g_i, Y, \mathbf{1}_{y_i} \rangle$.

Definition 2. *A mode component* $m \in M$ *is* m_0-admissible with respect to the functional bisimulation $B_0 \sim_\psi N$ *denoted by* $\mathrm{adm}_\psi(m, m_0)$*, if the following holds:*

$$\forall w \in \mathbb{B}_{\hat{Y}} : \psi(f_{m_0}(w) \cup w_{-m_0}) = \psi(f_m(w) \cup w_{-m}).$$

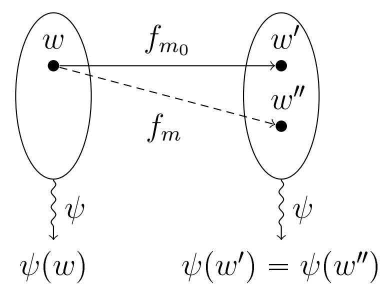


Figure 3: Illustration of m_0-admissibility $\mathrm{adm}_\psi(m, m_0)$ of a mode component m with respect to the bisimulation $B \sim_\psi N$.

Figure 3 illustrates the implications of Definition 2. Essentially, two mode components are admissible with respect to the bisimulation relation $\sim_\psi$ if the image of any Boolean state under these two mode components yields Boolean states which are mapped by ψ to the same multivalued state. Notice that it follows directly from the definition that admissibility is an equivalence relation on mode components. Admissibility can be naturally lifted from mode components to modes:

Definition 3. *A mode* M *is* M_0-admissible with respect to the functional bisimulation $B \sim_\psi N$*, denoted by* $\mathrm{adm}_\psi(M, M_0)$*, iff the following conditions hold:*

1. $\forall m_0 \in M_0, \exists m \in M \ : \ \mathrm{adm}_\psi(m, m_0)$;
2. $\forall m \in M, \exists m_0 \in M_0 \ : \ \mathrm{adm}_\psi(m, m_0)$.

According to the definition, a mode M is M_0-admissible if, for every modality $m \in M$, there exists a modality $m_0 \in M_0$ such that m is m_0-admissible. Note that this requirement does not imply the existence of a bijection between M and M_0: two functions $M_0 \to M$ and $M \to M_0$ are indeed required by, respectively, clauses (1) and (2) of Definition 3, but they may not be each other's inverses.

Lemma 1. *The relation of admissibility with respect to the functional bisimulation $B \sim_\psi N$, defined on all possible modes of B, is an equivalence relation.*

Intuitively, two modes are admissible with respect to a functional bisimulation $\sim_\psi$ if the application of ψ to the transitions in both dynamics leads to the same dynamics that corresponds to the dynamics of N. This implies that admissible modes cannot be distinguished with respect to the bisimulation relation. The following theorem formally captures this observation.

Theorem 2. *Given the functional bisimulation $B \sim_\psi N$ between the Boolean network $B = \langle f, \widehat{Y}, M_0 \rangle$ and the multivalued network $N = \langle g_i, Y, \mathbf{1}_{y_i} \rangle$, any Boolean network $B' = \langle f, \widehat{Y}, M \rangle$ with $\mathrm{adm}_\psi(M, M_0)$ functionally bisimulates N as well:*

$$\forall M \subseteq \mathbf{2}^{\widehat{Y}} \; : \; B \sim_\psi N \wedge \mathrm{adm}_\psi(M, M_0) \implies B' \sim_\psi N.$$

$$\begin{tikzcd}
 & N = \langle g_i, Y, \mathbf{1}_{y_i} \rangle & \\
B = \langle f, \widehat{Y}, M_0 \rangle \arrow[ur, "\sim_\psi"] \arrow[rr, leftrightarrow, "\mathrm{adm}_\psi(M, M_0)"] & & B' = \langle f, \widehat{Y}, M \rangle \arrow[ul, dashed, "\sim_\psi"']
\end{tikzcd}$$

Figure 4: Diagrammatic illustration of Theorem 2.

Figure 4 illustrates the statement of the previous theorem diagrammatically. Given a boolean network B which is functionally bisimilar to a multivalued network N under the mapping ψ, any other Boolean network B' with the same variables and evolution functions as B', and with an admissible mode with respect to ψ, is also functionally bisimilar to the multivalued network N. Theorem 2 allows us to prove the bisimilarity of a network with another mode provided that Property (7) holds and the mode is admissible.

Corollary 1. *Let* $B = \langle f, \widehat{Y}, M\rangle$, $B' = \langle f, \widehat{Y}, M'\rangle$ *be two Boolean networks,* $\psi : \mathbb{B}_{\widehat{Y}} \to [\![\mathcal{L}]\!]_Y$ *a surjective function, and* $N = \langle g_i, Y, \mathbf{1}_{y_i}\rangle$ *a multivalued network. If Property 7 holds for* B, N, *and* ψ, *and if* $\mathrm{adm}_\psi(M, M')$, *then* $B' \sim_\psi N$.

4. Boolean coding

The coding procedure characterizes a function ψ mapping a Boolean profile to an integer. We study two fundamental codes that are suitable for asynchronous Boolean dynamics: the *Summing code* and the *Gray code*. Table 1 shows both codings for encoding levels ranging from 0 to 3.

Summing code. For the Summing code, the integer is the sum of the states of the Boolean support variables:

$$\psi(w_{\widehat{y_i}}) = \sum_{\widehat{y_i}_k \in \widehat{y_i}} w_{\widehat{y_i}_k}.$$

The size of the support is linear in the maximal level, $|\widehat{y_i}| = L$, and different encodings are possible for the same integer. The number of different codes for an integer $0 \leqslant l \leqslant L$ is $\binom{L}{l}$. Van Ham code (Van Ham, 1979) is a sub-case of the Summing code in which the unitary stepwise evolution restricts the filling of 1 from left to right. The van Ham code is emphasized in bold in Table 1.

Gray code. The Gray code associates Boolean states differing in only one position to consecutive integers. The coding function is bijective and constructs the integer value from a Boolean state by first transforming a Gray code profile into its equivalent binary code and then by computing the integer from this coding[2]:

$$\psi(w_{\widehat{y_i}}) = \sum_{k=1}^{|\widehat{y_i}|} 2^{|\widehat{y_i}|-k} . \bigoplus_{j=1}^{k} w_{\widehat{y_i}_j}.$$

The support size is logarithmic in the maximal level: $|\widehat{y_i}| = \lceil \log_2(L+1) \rceil$.

The Summing code is defined on the whole Boolean state space (*i.e.*, $\mathbf{dom}\,\psi = \mathbb{B}_{\widehat{Y}}$). The Gray code can be also defined on the whole Boolean space

[2] $\oplus$ is the exclusive OR, XOR.

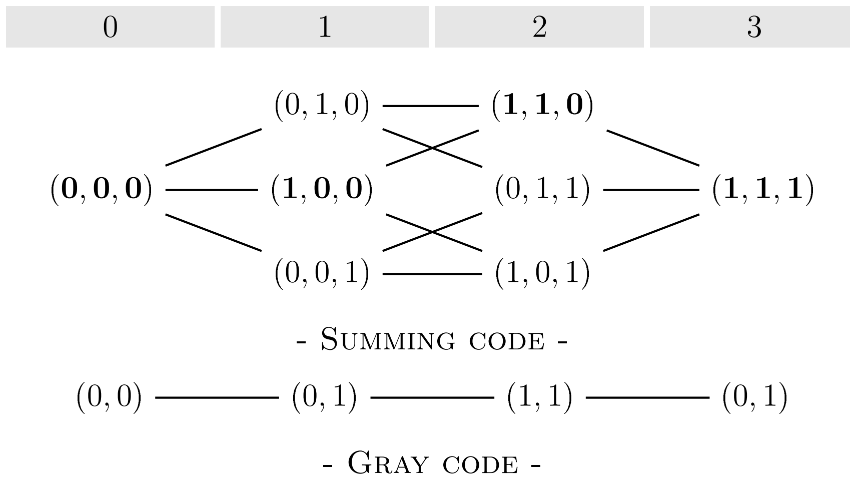


Table 1: Example of codes for levels ranging from 0 to 3. The states correspond to the variable profiles $(\hat{y}_{i1}, \hat{y}_{i2}, \hat{y}_{i3})$. The links connect codes differing by 1, and the codes in bold correspond to Van Ham sequence.

when the maximal number of levels is $L = 2^k - 1$. The Van Ham code, on the other hand, never covers the entire Boolean space, except when the maximal level is 1 corresponding to Boolean. All these codings associate the integer 0 to the 0 Boolean profile. Furthermore, they all fulfil the *neighbourhood preserving* property (9) defined in Didier et al. (2011) and stressing that the distance of 1 between two integer states should map to a distance of 1 between the corresponding Boolean states, and conversely:

$$\begin{aligned} &\forall s, s' \in [\![\mathcal{L}]\!]_Y : d(s, s') = 1 \implies \exists w \in \psi^{-1}(s), \exists w' \in \psi^{-1}(s') : d(w, w') = 1 \wedge \\ &\forall w, w' \in \mathbf{dom}\, \psi : d(w, w') = 1 \implies d(\psi(w), \psi(w')) = 1. \end{aligned} \tag{9}$$

These codes are individual representatives of families of linear and log-size codes which can be obtained by a permutation π on the integer states, *i.e.*, $\psi' = \pi \circ \psi$. This permutation may notably relax the neighbourhood preserving property. In literature, the study of the multivalued-to-Boolean network conversion has been carried out extensively for Summing and Van Ham codes (Didier et al., 2011; Fauré and Kaji, 2018; Tonello, 2019) but not Gray code which provides the most compact binary representation of integers and may be bijective.

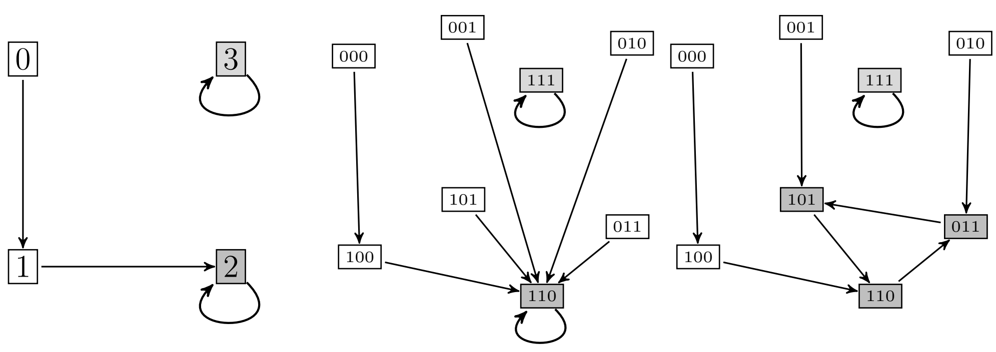
5. Extensions of property preservation

Although bisimulation preserves the essential property of reachability, it appears desirable to preserve additional properties for performing an accurate analysis of dynamics on the Boolean network directly. These additional properties pertain to the nature of equilibria and the interaction graph.

5.1. Preservation of stability of the equilibria

By definition of the bisimulation, the equilibria of a multivalued network match with the equilibria of a bisimilar Boolean network, and conversely. However, when some equilibria are stable states, their nature may differ: a stable state of the multivalued network can be represented by a cyclic attractor over Boolean profiles, all coding for the same integer (Figure 5). Nevertheless, any cyclic attractor will still be bisimulated by a cyclic attrac-

$$y = \begin{cases} 3 & y = 3 \\ 2 & 1 \leqslant y \leqslant 2 \\ 1 & y = 0 \\ 0 & \textit{otherwise} \end{cases} \qquad \begin{cases} \widehat{y_1} = & 1 \\ \widehat{y_2} = & \widehat{y_1} \vee \widehat{y_2} \vee \widehat{y_3} \\ \widehat{y_3} = & \widehat{y_1} \wedge \widehat{y_2} \wedge \widehat{y_3} \end{cases} \qquad \begin{cases} \widehat{y_1} = & \neg\widehat{y_2} \vee \widehat{y_3} \\ \widehat{y_2} = & \widehat{y_1} \vee (\widehat{y_2} \wedge \neg\widehat{y_3}) \\ \widehat{y_3} = & (\neg\widehat{y_1} \wedge \widehat{y_3}) \vee \widehat{y_2} \end{cases}$$



multivalued network *Bisimilar Boolean network preserving stability* *Bisimilar Boolean network losing stability*

Figure 5: Stability loss during bisimulation – synchronous mode.

tor since, by definition of coding, a transition between two different integer states is always simulated by a transition with two different Boolean profiles. Figure 5 shows an example where the self-loop of stable state 2 is simulated by a cyclic attractor over the three Boolean profiles coding for it (right-hand side network). Indeed, for any integer level, the synchronous dynamics allows reaching any of its codings from any other one. This case is however not encountered for stable state 3 coded by a single Boolean profile. The occurrence of such situations also depends on the concrete Boolean function, as shown by the middle network that preserves the stability. Even though in the former case the stable state is represented by a cyclic attractor, it is worth noticing that the original level 2 can be recovered from the states encompassed by the attractor since $\{\psi(101), \psi(011), \psi(110)\} = \{2\}$.

Maintaining the stability matters for the analysis performed on the Boolean dynamics. In particular, the symbolic computation of stable states will fail to find state 2 as equilibrium from the Boolean network since this state is represented by a cyclic attractor. Therefore, such cases should be ruled out to ensure a matching analysis of the dynamics on the two networks. To preserve the stability of equilibria, we basically have to prevent reaching a code of an integer level l from another code also encoding for l. This depends on the mode and on the Boolean function (*cf.* Figure 5). The expected outcome can be expressed as follows for a mode M:

$$\forall w \in \mathbf{dom}\,\psi, \forall m \in M : w_m \neq f_m(w) \implies \psi(w) \neq \psi(f_m(w) \cup w_{-m}). \quad (10)$$

We examine two effective conditions for satisfying (10) that are independent of the specification of the Boolean network. A simple one working whatever the mode and the Boolean function is to remove the self-loops, and thus establish bisimulation between reflexive reductions of the state graphs of both networks, instead of operating on the original state graphs. The equilibrium stability then remains preserved since no circuits can simulate a self-loop and the important features of the reachability are not altered. Another more explicit condition, based on the code and the cardinality of modalities, forbids the access by a transition to another Boolean profile coding for the same integer.

Proposition 2. *Let $B = \langle f, \widehat{Y}, M\rangle$ be a Boolean network bisimilar to a multivalued network $N = \langle g, Y, \mathbf{1}_Y\rangle$, with M a local-to-support mode. If the following holds:*

$$\forall y_i \in Y, \forall s_{y_i} \in [\![L]\!]_{y_i}, \forall w, w' \in \psi^{-1}(s_{y_i}) : \\ w \neq w' \implies d(w, w') > \max\{|m| \mid m \in M\},$$

then the equilibrium stability (10) is preserved.

Consequently, if ψ is a bijection, the stability of equilibria is preserved since every integer level is coded by a single Boolean profile. On the other hand, the asynchronous mode preserves the stability under the Summing code, since distances between two Boolean profiles coding for the same integer are at least 2.

5.2. Interaction preserving

A Boolean network bisimulating a multivalued network is *regulatory-preserving* if it is possible to unambiguously recover the signed interaction graph of the multivalued network (MIGS), $\langle Y, \longrightarrow, \sigma\rangle$, from the signed Boolean interaction graph of the bisimilar Boolean network (BIGS), $\langle \widehat{Y}, \longrightarrow, \sigma_{\mathbb{B}}\rangle$. Retrieving MIGS from BIGS is divided in two steps: retrieving the interaction graph and finding the signs.

Interaction graph retrieval. The structure of MIGS is retrieved from the quotient graph of BIGS defined on the support of the integer variables, called the *support interaction graph* (SIG) $\langle\{\widehat{y}_i\}_{y_i\in Y}, \longrightarrow\rangle$, where an interaction between two Boolean support variables induces an interaction between the supports they belong:

$$\widehat{y}_i \longrightarrow \widehat{y}_i \overset{\text{def}}{=} \exists \widehat{y}_{ik} \in \widehat{y}_i, \exists \widehat{y}_{j_r} \in \widehat{y}_j : \widehat{y}_{ik} \longrightarrow \widehat{y}_{j_r}. \tag{11}$$

As a consequence, the topological structure of MIGS is the same as that of SIG by merely replacing the supports by the integer variables they support (Proposition 3). In fact, SIG essentially provides an intermediary representation used for recovering the interactions of MIGS and their signs.

Proposition 3. *Let N be a multivalued network and B a Boolean network. If N is bisimilar to B then* MIGS(N) *is isomorphic to* SIG(B).

Sign retrieval. The sign of an interaction is determined by BIGS once the conversion is achieved (see Figures 6, 7). Therefore the issue is to deduce from the signs of the interactions between the Boolean variables the signs of the corresponding interactions in MIGS. The recovery procedure is based on a set of reference Boolean variables, considered as *markers of sign*, covering all the supports such that the signs of the interactions between these variables are the same as the signs of the interactions between the integer variables they support. Hence the set of markers $\mathcal{M}_{\widehat{Y}}$ is a subset of Boolean variables of $\widehat{Y}$ defined by:

Definition 4 (Markers of sign)**.** *Let $\langle g, Y\rangle$ a multivalued network bisimulating a Boolean network $\langle f, \widehat{Y}\rangle$ with $\langle Y, \longrightarrow, \sigma\rangle$ and $\langle \widehat{Y}, \longrightarrow, \sigma_{\mathbb{B}}\rangle$ as their respective signed interaction graphs. $\mathcal{M}_{\widehat{Y}} \subseteq \widehat{Y}$ is a set of* markers of sign *if and only if:*

1. *The sign σ of an interaction between any two Boolean variables in $\mathcal{M}_{\widehat{Y}} \subseteq \widehat{Y}$ is the same as the sign of the interaction between the integer variables that they support:*

$$\forall \widehat{y}_{ik}, \widehat{y}_{j_r} \in \mathcal{M}_{\widehat{Y}} : \widehat{y}_{ik} \xrightarrow{\sigma} \widehat{y}_{j_r} \iff y_i \xrightarrow{\sigma} y_j.$$

2. *All integer variables have markers:*

$$\forall y_i \in Y : \mathcal{M}_{\widehat{Y}} \cap \widehat{y}_i \neq \varnothing.$$

To operationally identify the markers from a code, we define a code-based marker condition (12) directly linking the markers to the code for the asynchronous mode. This condition asserts the monotony of the coding for markers with respect to the integer and Boolean orders by stipulating that an integer coded by a Boolean profile is less than another coded by this Boolean profile where a marker value is substituted by 1 (Lemma 2).

Lemma 2. *Let $N = \langle g, Y\rangle$ be a multivalued network in bisimulation with an asynchronous Boolean network $B = \langle f, \widehat{Y}, \mathbf{1}_{\widehat{Y}}\rangle$, and $\mathcal{M}_{\widehat{Y}} \subseteq \widehat{Y}$ be a set of Boolean variables complying to (4.2). If:*

$$\forall \widehat{y}_{ik} \in \mathcal{M}_{\widehat{Y}}, \forall w \in \boldsymbol{dom}\,\psi : \psi(w) \leqslant \psi(w_{[\widehat{y}_{ik} \mapsto 1]}) \tag{12}$$

then $\mathcal{M}_{\widehat{Y}}$ fulfils Definition (4.1) and $\mathcal{M}_{\widehat{Y}}$ is a set of markers.

Therefore, the goal is to determine for each integer variable the set of markers by checking (12) for a given coding. For the Summing code all the Boolean variables are markers, and for the Gray code the variables storing the most significant bit indexed by 1 ($\widehat{y}_{i1}$) are the markers.

Theorem 3. *Let $N = \langle g, Y\rangle$ be a multivalued network in bisimulation with an asynchronous Boolean network $B = \langle f, \widehat{Y}, \mathbf{1}_{\widehat{Y}}\rangle$. The sets of markers $\mathcal{M}_{\widehat{Y}}$ are respectively for the codes:*

- *Summing code:* $\mathcal{M}_{\widehat{Y}} = \widehat{Y}$*;*
- *Van Ham code:* $\mathcal{M}_{\widehat{Y}} = \widehat{Y}$*;*
- *Gray code:* $\mathcal{M}_{\widehat{Y}} = \{\widehat{y}_{i1} \mid y_i \in Y\}$.

6. Inference of Boolean formulas

An analytical definition of the Boolean network function is given by (7). Although the function $\psi^{-1} \circ g \circ \psi$ is closed on Boolean states when ψ is bijective characterizing a Boolean network, the Boolean formulas are not explicitly defined. The lack of Boolean formulas makes the analysis harder in practice, notably by preventing the characterization of the interaction graph directly from formula specifications. Moreover, the analytical definition does not hold when ψ is not bijective, since ψ^{-1} returns a set of Boolean profiles. To circumvent this limitation, the objective is to infer the Boolean network bisimilar to a multivalued network directly from the specification of the latter (1). As it is sufficient to find a bisimilar Boolean network for each local multivalued evolution function g_i (Proposition 1), the algorithm will act on each function of integer variables independently. In this section we define a method inferring the formulas $f_{i,k}$ for each support variable $\widehat{y}_{ik}$ of y_i such that the reflexive reduction of the resulting Boolean network is bisimilar to the reflexive reduction of the initial multivalued network where the code is a parameter of this method. Due to the reflexive reductions, this method preserves the nature of the stable states (Section 5.1). For simplicity, the inference is presented for the asynchronous mode with a unitary stepwise multivalued network as input. However the inference can be applied to any local-to-support mode. This point is discussed at the end of the section.

The definition of a formula $f_{i,k}$ for a support variable is divided in two stages: The *conversion of the guard into a Boolean form*, and the *derivation of the admissibility condition for guard validation*. The examples use the Summing code which is the most complex coding for the inference.

Boolean conversion of the guard. Basically, the guard of level l' must also be satisfied in the Boolean network to simulate a transition shifting the current level l to l'. The conversion of a multivalued guard to a *Boolean guard* gathers the codes of the state profiles fulfilling the conditions of level l', *i.e.*, $\mathcal{C}^{l'}_{\star \twoheadrightarrow y_i} = \{s_{(\star \twoheadrightarrow y_i)} \mid C^{l'}(s)\}$ where $(\star \twoheadrightarrow y_i)$ is the set of regulators of y_i. As all these integer states satisfy the guard $C^{l'}$, their Boolean codes should also satisfy the Boolean guard $C_{\mathbb{B}}^{l'}$ defined as[3]:

$$C_{\mathbb{B}}^{l'} = \bigvee_{s \in \mathcal{C}_{\star}^{l'} \rightarrow y_i} \left(\bigwedge_{y_j \in (\star \rightarrow y_i)} \bigvee_{w_{\widehat{y_j}} \in \psi^{-1}(s_{y_j})} \text{minterm}(w_{\widehat{y_j}}) \right). \tag{13}$$

For example, in the case of the multivalued network from Figure 1, the states fulfilling the conditions to reach level 2 for y are $(x = 1, y = 1)$ for the transition from level 1 to 2 for y, or $(x = 0, y = 3)$ for the transition from level 3 to 2. The code for x is $\psi_x^{-1}(0) = \{(0)\}$, $\psi_x^{-1}(1) = \{(1)\}$ and the codes for y are respectively for 1 and 3: $\psi_y^{-1}(1) = \{(0,0,1),(0,1,0),(1,0,0)\}$ and $\psi_y^{-1}(3) = \{(1,1,1)\}$. Hence, the Boolean guard of level 2 for y is:

$$C_{\mathbb{B}}^{2} = \overbrace{\left(\underbrace{\widehat{x}}_{\text{minterm}_x(1)} \wedge \left(\underbrace{(\wedge \widehat{y}_1 \wedge \neg\widehat{y}_2 \wedge \neg\widehat{y}_3)}_{\text{minterm}_y(1,0,0)} \vee \underbrace{(\neg\widehat{y}_1 \wedge \widehat{y}_2 \wedge \neg\widehat{y}_3)}_{\text{minterm}_y(0,1,0)} \vee \underbrace{(\neg\widehat{y}_1 \wedge \neg\widehat{y}_2 \wedge \widehat{y}_3)}_{\text{minterm}_y(0,0,1)} \right)\right)}^{(x=1,y=1)}$$
$$\vee \overbrace{\left(\underbrace{\neg\widehat{x}}_{\text{minterm}_x(0)} \wedge \underbrace{(\widehat{y}_1 \wedge \widehat{y}_2 \wedge \widehat{y}_3)}_{\text{minterm}_y(1,1,1)} \right)}^{(x=0,y=3)}$$

Guard admissibility condition. The generation of the Boolean guard is however insufficient for obtaining the final formula because some support variables shift to 0 during the transition even though the guard is satisfied, meaning that a direct evaluation of the Boolean guard would shift them to 1. For example, in Figure 1, shifting from 3 to 2 for y is bisimilar to $(1,1,1) \xrightarrow{\widehat{y}_2} (1,0,1)$. In this case, we need to shift the state of $\widehat{y}_2$ to 0 although the guard is satisfied with $s_x = 0$. We thus need to characterize the situations in which the transition necessarily shifts the value of a support

[3]The minterm of a state is a conjunction of the variables such that the unique interpretation satisfying it is the state itself, *e.g.*, $\text{minterm}(x_1 = 0, x_2 = 1) = \neg x_1 \wedge x_2$.

variable to 1. This restricts the set of admissible encodings triggering the guard, outside of which the transition always shifts the support variable state to 0.

Let $s \xrightarrow{y_i} s'$ be an integer unitary stepwise transition with $s_{y_i} = l$ and $s'(y_i) = l'$ such that $|l - l'| = 1$, and $\widehat{y}_i$ be the support of y_i ($\widehat{y}_{ik} \in \widehat{y}_i$), we denote by $w \longrightarrow w'$ the asynchronous Boolean transition bisimilar to $s \xrightarrow{y_i} s'$. Two cases where $\widehat{y}_{i,k} = 1$ should be considered depending on the encoding of the levels: $\widehat{y}_{i,k}$ is shifted from 0 to 1 during the transition (*i.e.*, $w_{\widehat{y}_{i,k}} = 0$ and $w'(\widehat{y}_{i,k}) = 1$), or $\widehat{y}_{i,k}$ remains as 1 (*i.e.*, $w_{\widehat{y}_{i,k}} = 1$ and $w'(\widehat{y}_{i,k}) = 1$).

In both cases, we characterize for each Boolean variable the set of codes corresponding to the initial level l such that $\widehat{y}_{ik}$ is either shifted to or remains at 1. The initial level l is determined from the target level l' by considering that it is either $l'-1, l'$ or $l'+1$ by definition of an unitary stepwise transition.

We define the set of codes for the initial level such that $\widehat{y}_{ik}$ is shifted from 0 to 1 during the transition (ψ is implicitly restricted to $\psi : \mathbb{B}_{\widehat{y}_i} \to [\![L]\!]_{y_i}$):

$$\Psi_{0\to 1}(l', \widehat{y}_{ik}) = \{w_{\widehat{y}_i} \in \mathbf{dom}\,\psi \mid \exists\ \max(0, l'-1) \leqslant l \leqslant \min(l'+1, L) : \\ \psi(w_{\widehat{y}_i}) = l \wedge w_{\widehat{y}_{ik}} = 0 \wedge \psi(w_{\widehat{y}_i[\widehat{y}_{ik} \mapsto 1]}) = l'\}.$$

Similarly, we define the set of codes for which a shift from 1 to 0 occurs:

$$\Psi_{1\to 0}(l', \widehat{y}_{ik}) = \{w_{\widehat{y}_i} \in \mathbf{dom}\,\psi \mid \exists\ \max(0, l'-1) \leqslant l \leqslant \min(l'+1, L) : \\ \psi(w_{\widehat{y}_i}) = l \wedge w_{\widehat{y}_{ik}} = 1 \wedge \psi(w_{\widehat{y}_i[\widehat{y}_{ik} \mapsto 0]}) = l'\}.$$

Finally, we define the set of codes where $\widehat{y}_{ik}$ is 1 in both l and l':

$$\Psi_{1\to 1}(l', \widehat{y}_{ik}) = \{w_{\widehat{y}_i} \in \mathbf{dom}\,\psi \mid \exists\ \max(0, l'-1) \leqslant l \leqslant \min(l'+1, L), \\ \exists w'_{\widehat{y}_{ik}} \in \psi^{-1}(l') : \psi(w_{\widehat{y}_i}) = l \wedge w_{\widehat{y}_{ik}} = 1 \wedge w'_{\widehat{y}_{ik}} = 1\}.$$

The set of Boolean states coding for level l, always reaching state 1 and never a state 0 for $\widehat{y}_{ik}$ in a transition to a code of level l', is defined as:

$$\Psi_{\star\to 1}(l', \widehat{y}_{ik}) = \Psi_{0\to 1}(l', \widehat{y}_{ik})\ \cup \left(\Psi_{1\to 1}(l', \widehat{y}_{ik}) \backslash \Psi_{1\to 0}(l', \widehat{y}_{ik})\right).$$

Note that the set difference in the previous equation is not necessarily empty. Indeed, there may exist a pair of states $w_{\widehat{y}_i}$ and $w'_{\widehat{y}_i}$, with $\psi(w_{\widehat{y}_i}) = l$ and $\psi(w'_{\widehat{y}_i}) = l'$, such that $w_{\widehat{y}_{ik}} = w'_{\widehat{y}_{ik}} = 1$, but for which $\psi(w_{\widehat{y}_i[\widehat{y}_{ik} \mapsto 0]}) = l'$ also holds. We need to exclude such states $w_{\widehat{y}_i}$ from $\Psi_{\star\to 1}(l', \widehat{y}_{ik})$, because they still allow reaching a Boolean profile coding for l' by setting $\widehat{y}_{i_k}$ to 0. The resulting transition is bisimilar to an integer transition, and thus must be kept.

The *guard admissibility condition* G of $C_{\mathbb{B}}^{l'}$ is thus defined as the disjunction of the minterms of the admissible codes:

$$G_{\Psi_{\star\to 1}(l',\widehat{y}_{i_k})} = \bigvee_{c \in \Psi_{\star\to 1}(l',\widehat{y}_{i_k})} \mathrm{minterm}(c). \tag{14}$$

In our running example, consider the levels that potentially reach level 2 in a unitary stepwise transition (levels 1, 2, and 3). The final simplified formulas of the code admissibility conditions $G_{\Psi_{\star\to 1}(2,\widehat{y}_k)}, 1 \leqslant k \leqslant 3$, for each support variable are detailed in Table 2.

From these conditions (Table 2), we deduce that the asynchronous transitions from level 3 coded by $(1,1,1)$ to level 2 all set to 0 one of the Boolean support variables. Indeed, the update of $\widehat{y}_1$ to 0 leads to $(0,1,1)$ and similarly for $\widehat{y}_2$, $(1,0,1)$ and $\widehat{y}_3$, $(1,1,0)$ that all represent a Summing code of level 2.

$$\begin{aligned}
G_{\Psi_{\star\to 1}(2,\widehat{y}_1)} &= (y_1 \wedge \neg y_2) \vee (y_2 \wedge \neg y_3) \vee (\neg y_2 \wedge y_3) \\
G_{\Psi_{\star\to 1}(2,\widehat{y}_2)} &= (y_1 \wedge \neg y_3) \vee (\neg y_1 \wedge y_3) \vee (y_2 \wedge \neg y_3) \\
G_{\Psi_{\star\to 1}(2,\widehat{y}_3)} &= (y_1 \wedge \neg y_2) \vee (\neg y_1 \wedge y_2) \vee (\neg y_2 \wedge y_3)
\end{aligned}$$

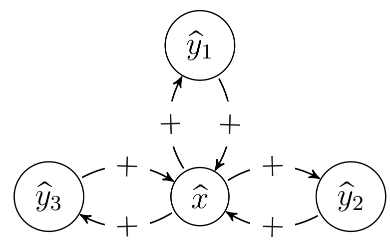
Table 2: Guard admissibility condition for level 2 of the support variables of y.

Boolean formula of a support variable. The final formula $f_{i,k}$ for a support variable $\widehat{y}_{i,k}$ can be understood as the Boolean version of the guards restricted to the codings admissible for their triggering, defined as:

$$f_{i,k} = \bigvee_{1 \leqslant l \leqslant L} \left(C_{\mathbb{B}}^{l} \wedge G_{\Psi_{\star\to 1}(l,\widehat{y}_{i_k})} \right). \tag{15}$$

The Boolean network gathers the formulas defined by (15) for each support variable. For the running example (Figure 1), the final Boolean network provides a clean description of the formulas once simplified for the Summing code (Figure 6) and the Gray code (Figure 7), that differ due to the codings. Theorem 4 demonstrates the correction of the conversion method.

$$f = \begin{cases} \widehat{x} \ = \widehat{y}_1 \vee \widehat{y}_2 \vee \widehat{y}_3 \\ \widehat{y}_1 = \widehat{x} \\ \widehat{y}_2 = \widehat{x} \\ \widehat{y}_3 = \widehat{x} \end{cases}$$



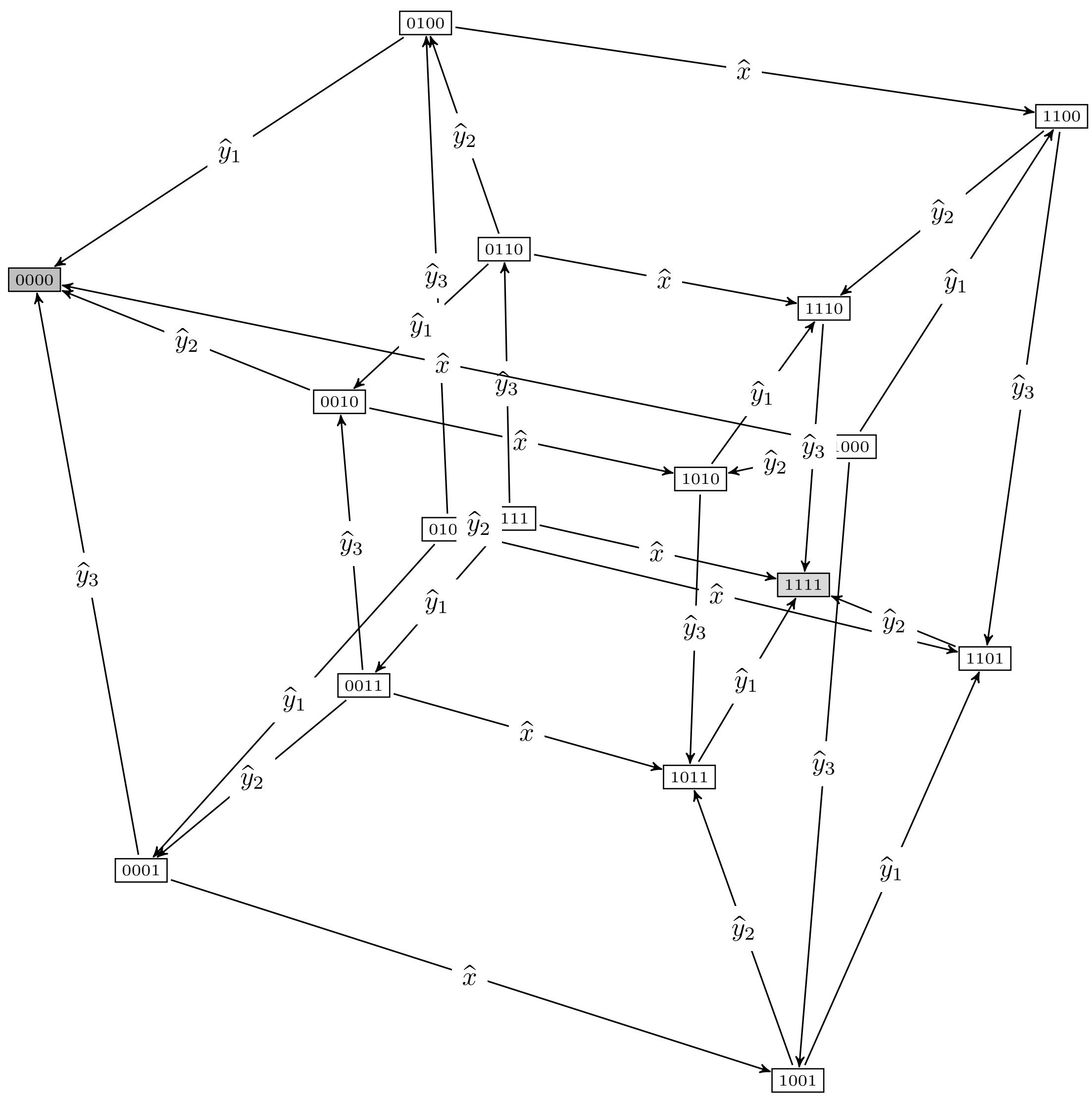
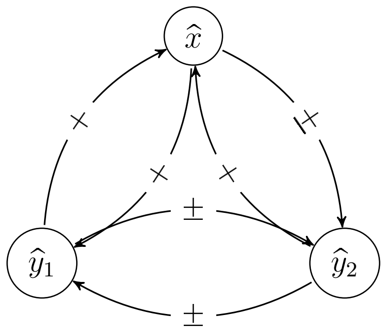


Figure 6: A Boolean network bisimilar to the multivalued network in Figure 1, its interaction graph (right), and its asynchronous dynamics for the Summing code without the self-loops (below).

$$\begin{cases} \widehat{x} = \widehat{y}_1 \vee \widehat{y}_2 \\ \widehat{y}_1 = (\widehat{x} \wedge \widehat{y}_2) \vee (\widehat{y}_1 \wedge \neg \widehat{y}_2) \\ \widehat{y}_2 = (\widehat{x} \wedge \neg \widehat{y}_1) \vee (\neg \widehat{x} \wedge \widehat{y}_1) \end{cases}$$



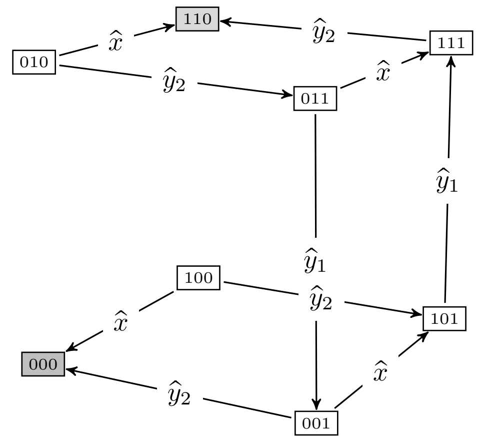


Figure 7: A Boolean network bisimilar to the multivalued network in Figure 1, its interaction graph (below), and its asynchronous dynamics for the Gray code without the self-loops (right).

Bisimilar reflexive reduction. Under the asynchronous mode, some support variables may maintain their value inducing self-loops that are not bisimilar to any transition in the integer dynamics. In the running example, shifting y from 2 to 3 is bisimilar to $(1, 0, 1) \xrightarrow{\widehat{y}_2} (1, 1, 1)$, which modifies the value of $\widehat{y}_2$. However, any of the Boolean variables may be updated in asynchronous dynamics leading to two self loops on $\widehat{y}_1$ and $\widehat{y}_3$ for maintaining the state of these variables at 1. Obviously, these self-loops are not bisimilar to the integer state transition since the variation of the integer state from 2 to 3 is carried out by one transition only. No alternatives preventing these additional self-loops in the Boolean network are possible since any one of the Boolean variables may be updated, but the state must not change for $\widehat{y}_1$ and $\widehat{y}_3$. This situation explains why our method operates on reflexive reductions of the networks, effectively discarding these extra self-loops. Also notice that this transformation can only be performed if the integer level 0 is coded by the 0 Boolean profile, meaning that the behaviours of the multivalued and Boolean networks match when no guards are satisfied. The reflexive reduction of a network N is denoted $N^{\neq}$.

Theorem 4. *Given a neighbourhood preserving Boolean coding ψ such that* 0 *is coded by the* 0 *Boolean profile, the inference by (15) from a multivalued unitary stepwise network $N = \langle g, Y, \mathbf{1}_Y \rangle$ produces a Boolean network $B = \langle f, \widehat{Y}, \mathbf{1}_{\widehat{Y}} \rangle$ such that the reflexive reductions of both networks are bisimilar: $N^{\neq} \sim_{\psi} B^{\neq}$.*

Extension to other modes. The method can be applied to any local-to-support mode but the admissible region may be reduced compared to the asynchronous mode. This reduction is caused by the decrease of the update capacity allowed by the mode. Hence, by Definition 1, this implies selecting the appropriate codes for always reaching a code supporting an unitary stepwise transition for each update according to the mode components. For the running example, by using the parallel local-to-support mode with the Summing code, $M = \{\{x\}, \{\widehat{y}_1, \widehat{y}_2, \widehat{y}_3\}\}$, all the Boolean variables of $\widehat{y}$ are updated jointly allowing to reach a single code instead of reaching the different codes of the same integer by separate updates of variables. Thus, the coding is reduced to Van Ham coding.

Moreover, the trajectories starting from a state located outside the admissible region always end in the admissible region and no supplementary equilibra are thus created (Proposition 4). This property generally holds for any coding that partially covers the Boolean state space.

In conclusion, the domain of ψ may thus be reduced for a surjective decoding function such as the Summing code without altering the asymptotic dynamics, but remains unchanged for a bijective decoding.

Proposition 4. *Let f be an evolution function defined according to (15) from a multivalued unitary stepwise network $N = \langle g, Y, \mathbf{1}_Y \rangle$. Let $B = \langle f, \widehat{Y}, M \rangle$ be the corresponding Boolean network with M a local-to-support mode, and ψ a decoding function such that $N^{\neq} \sim_{\psi} B^{\neq}$. All the states of the Boolean space eventually reach a state in the admissible region:*

$$\forall w \in \mathbb{B}_{\widehat{Y}}, \exists w' \in \boldsymbol{dom}\, \psi : w \longrightarrow_f^{\star} w'.$$

Complexity of the algorithm. Assume that the multivalued network has n variables reaching at most level L, the upper bound on the number of regulators for a variable is r, the maximal number of support variables is m, and the maximal bound of code variants is c. Then the complexity of the

Boolean guard is in $\mathcal{O}(nL^r rcm)$ and the complexity of computing the guard admissibility condition is in $\mathcal{O}(nLcm)$. Thus the complexity of the algorithm is dominated by the complexity of the Boolean guard computation. Accordingly, the computation time mainly depends on the maximal level and the number of regulators. The algorithm is efficient in practice since the maximal level and the number of regulators often remain tractable on real biological network models.

7. Conclusion

The conversion of multivalued networks to Boolean networks bridges these formalisms by providing a better understanding of the theoretical differences and similarities between these frameworks, and by extending the use of analytical tools dedicated to Boolean networks to their multivalued counterpart. The major property to be preserved at conversion is reachability, ensuring the equivalence between the dynamics. Accordingly, the conversion is underpinned by bisimulation (Section 3) which guarantees the preservation of reachability. The fundamental analysis concludes with an original algorithm that automatically generates a Boolean network bisimulating a multivalued network. The input multivalued network operates under the asynchronous mode, while the resulting Boolean network operates under a local-to-support mode. The proposed framework explicitly distinguishes the conversion process from choosing a Boolean encoding, which separates the properties related to the bisimulation from those related to the encoding, or both.

A number of results relate to the asynchronous mode, which is standard for discrete biological network modelling (Thieffry and Thomas, 1995; Thomas et al., 1995). Accordingly, our analysis promotes local-to-support modes, which are a natural decomposition of the updates of Boolean variables. Indeed, the joint updates of Boolean support variables should always correspond to a single transition in the multivalued dynamics. Hence, local-to-support modes fit well with asynchronous updates of the multivalued variables, facilitating the conversion process.

Even though the preservation of reachability is central for conversion, this property is not sufficient for analysis. Additional properties are also required pertaining to the nature of the equilibria (Section 5.1) and to the interaction graph (Section 5.2). The preservation of stable states ensures that both networks have attractors of the same kind. In particular, this enables a fast symbolic discovery of stable states using SAT solvers. The preservation of stable states needs to depend on the mode and the coding, in order to prevent reciprocal transitions between Boolean profiles coding for the same integer. The asynchronous mode fulfils this property for all codings.

The regulation graph of the multivalued network must also be retrieved from the bisimilar Boolean network. All the codings comply with this property. This property also addresses the issue of monotonicity preservation, because preserving the sign means preserving the transition variation. For Summing and Van Ham codes, the interactions on Boolean variables are all identical to the interactions between the multivalued variables they support. For Gray code, the interaction graph is retrieved from the support variable carrying the most significant bit. Therefore Gray code possibly induces non-monotonicity for the other support variables as shown in Figure 7.

Accordingly, the choice of the most suitable coding for the conversion depends on several factors which are: the compactness of the coding, the coverage of the Boolean space by the admissible region, the bijection of the (de)coding function ψ, and the monotonicity preservation. A log-size coding with a bijective (de)coding function and a full coverage of the Boolean space preserving the monotonicity would provide a Boolean simulation with the most compact representation, a simple state conversion represented by a conjugated function, without spurious trajectories outside the admissible region, and following the same state variation in each trajectory. However, none of these codings complies with all these requirements at the same time, which introduces a trade-off between the desired preservation properties. Table 3 summarizes these properties for each coding.

Code	Code size of level l	Full Boolean space coverage	(De)coding Bijection	IG retrieval	Monotony preserving
Van Ham	l	No	Yes	Yes	Yes
Summing	l	Yes	No	Yes	Yes
Gray	$\log_2 l$	if $L = 2^k - 1$	Yes	Yes	No

Table 3: Summary of the properties in a conversion of the codings.

Such automatic conversion sketches a pipeline where the multivalued network becomes an input specification for modelling while the bulk of the analysis is performed on the Boolean network. Such pipeline suggests that the Boolean framework is central and sufficient for biological network modelling, thus calling to focus theoretical efforts on this framework since the results will benefit to both categories of discrete models via this pipeline.

A research perspective would concern the extension of the bisimulation to other modes for the multivalued network, while considering families of local-to-support modes for the Boolean network to fit the multivalued mode. Another perspective may focus on the bisimulation between Boolean networks themselves. As bisimulation formally represents behavioural equivalence, we could investigate the global properties of families of bisimilar Boolean networks in order to discover general rules governing their behaviour. Moreover, given a Boolean network, we may exhibit a simpler bisimilar Boolean network on which the analysis will be performed, and that will potentially improve the efficiency of the dynamics analysis.

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Appendix

Proposition 1. The global Boolean transition relation is the union of the local transition relations that are bisimilar to the local multivalued relations. As the union of bisimilar relations is bisimilar to the union relation, we deduce that the global Boolean relation is bisimilar to the global multivalued relation. □

Theorem 1. We prove that Property (7) is met if and only if $N \sim_\psi B$. We first prove the implication and next the reciprocal. Before, we prove the following property for any mode component $m \in M$ used in the proofs:

$$-m = (\widehat{y}_i \backslash m \cup -\widehat{y}_i) \tag{T1}$$

Proof.

$$\begin{aligned}
-m &= \widehat{Y} \backslash m && \text{by definition of } -m;\\
&= (\widehat{y}_i \cup -\widehat{y}_i) \backslash m && \text{as } -\widehat{y}_i = \widehat{Y} \backslash \widehat{y}_i \text{ by definition;}\\
&= \widehat{y}_i \backslash m \cup -\widehat{y}_i \backslash m &&\\
&= (\widehat{y}_i \backslash m \cup -\widehat{y}_i) && \text{since } m \subseteq \widehat{y}_i \text{ by definition of the local-to-support mode, meaning that } -\widehat{y}_i \backslash m = -\widehat{y}_i.
\end{aligned}$$

□

($\Longrightarrow$) Assume that Property (7) is met for the local-to-support mode M, *i.e.*, $\forall m \in M : m \subseteq \widehat{y}_i \wedge \psi(f_m(w) \cup w_{\widehat{y}_i \backslash m}) = g_i \circ \psi(w)$.

- *N simulates B.* Let $w \xrightarrow{m}_f w', m \in M$, be a transition in the model of B such that $w, w' \in \mathbf{dom}\,\psi$. We define the transition $\psi(w) \longrightarrow \psi(w')$ by application of ψ on w and w'. We have:

  $$\begin{aligned}
  \psi(w') &= \psi(f_m(w) \cup w_{-m}) && \text{by definition of a transition (Section 2);}\\
  &= \psi(f_m(w) \cup w_{\widehat{y}_i \backslash m \cup -\widehat{y}_i}) && \text{by (T1);}\\
  &= \psi(f_m(w) \cup w_{\widehat{y}_i \backslash m} \cup w_{-\widehat{y}_i}) && \text{from (5)}\\
  &= \psi(f_m(w) \cup w_{\widehat{y}_i \backslash m}) \cup \psi(w_{-\widehat{y}_i}) && \text{from (5) and (6);}\\
  &= g_i \circ \psi(w) \cup \psi(w_{-\widehat{y}_i}) && \text{from (7), true by hypothesis.}
  \end{aligned}$$

  Set $s = \psi(w)$ and $s' = \psi(w')$. Then $s_{-y_i} = \psi(w_{-\widehat{y}_i})$, because $w_{-\widehat{y}_i}$ is the Boolean encoding of the rest of the state s_{-y_i}. We finally have:

  $$\psi(w) \longrightarrow \psi(w') = s \longrightarrow g_i(s) \cup s_{-y_i} = s \xrightarrow{y_i}_{g_i} s',$$

  which defines a transition of $\longrightarrow_{g_i}$ with the asynchronous mode $\mathbf{1}_{y_i}$.

- *B simulates N.* Let $s \xrightarrow{y_i}_g s'$ be a transition in the model of N. As $\psi : \mathbb{B}_{\widehat{Y}} \to [\![\mathcal{L}]\!]_Y$ is surjective, there exist two Boolean states $w, w' \in \mathbb{B}_{\widehat{Y}}$ such that: $\psi(w) = s$ and $\psi(w') = s'$. We prove that we can select w' in the preimage of s' so that a transition $w \xrightarrow{m}_f w'$ exists in the model of the Boolean network B.

  Firstly, let us characterize s' based on w.

  $$
  \begin{aligned}
  s' &= g_i(s) \cup s_{-y_i} && \text{by definition of transition (Section 2);}\\
  &= g_i \circ \psi(w) \cup s_{-y_i} && \text{as } \psi(w) = s \text{ by hypothesis;}\\
  &= \psi(f_m(w) \cup w_{\widehat{y_i} \setminus m}) \cup s_{-y_i} && \text{from (7), true d by hypothesis;}\\
  &= \psi(f_m(w) \cup w_{\widehat{y_i} \setminus m}) \cup \psi(w_{\widehat{-y_i}}) && \text{from (6) and } \psi(w) = s;\\
  &= \psi(f_m(w) \cup w_{\widehat{y_i} \setminus m}) \cup \psi(w_{-\widehat{y_i}}) && \text{by definition of the support (5);}\\
  &= \psi(f_m(w) \cup w_{\widehat{y_i} \setminus m}) \cup w_{-\widehat{y_i}}) && \text{from (6);}\\
  &= \psi(f_m(w) \cup w_{\widehat{y_i} \setminus m \cup -\widehat{y_i}} && \text{by definition of the support (5);}\\
  &= \psi(f_m(w) \cup w_{-m}) && \text{by (T1).}
  \end{aligned}
  $$

  Thus, we conclude that $\psi(w') = s'$ implies that $w' = f_m(w) \cup w_{\widehat{y_i} \setminus m}$. Hence, by definition of a transition, we have $w \xrightarrow{m}_f w'$, meaning that B simulates N.

In conclusion, if Property 7 is verified then networks N and B are bisimilar with respect to ψ.

($\Longleftarrow$) Assume that $N \sim_\psi B$. Hence, for all transitions $w \xrightarrow{m}_f w'$ such that $w, w' \in \mathbf{dom}\,\psi$, there exist $s, s' \in [\![\mathcal{L}]\!]_Y$ such that $s \longrightarrow_{g_i} s'$ and $s = \psi(w), s' = \psi(w')$.

From the bisimulation, we deduce that:

$$
\begin{aligned}
s' &= \psi(w') && \text{by hypothesis;}\\
&= \psi(f_m \cup w_{-m}) && \text{by definition of } w \xrightarrow{m}_f w';\\
&= \psi(f_m \cup w_{\widehat{y_i} \setminus m \cup -\widehat{y_i}}) && \text{by (T1);}\\
&= \psi(f_m \cup w_{\widehat{y_i} \setminus m} \cup w_{-\widehat{y_i}}) && \text{from (5);}\\
&= \psi(f_m \cup w_{\widehat{y_i} \setminus m}) \cup \psi(w_{-\widehat{y_i}}) && \text{by (5), (6);}\\
&= \psi(f_m \cup w_{\widehat{y_i} \setminus m}) \cup s_{-y_i} && \text{as } s = \psi(w) \text{ by hypothesis.}
\end{aligned}
$$

From the definition of a transition, we deduce the following:

$$
\begin{aligned}
s' &= g_i(s) \cup s_{-y_i} && \text{as } s \longrightarrow_{g_i} s' \text{ by hypothesis;}\\
&= g_i \circ \psi(w) \cup s_{-y_i} && \text{as } s = \psi(w) \text{ by the bisimulation hypothesis.}
\end{aligned}
$$

As $-y_i \cap y_i = \varnothing$ because $-y_i = Y \backslash y_i$, we can simplify the equation by removing s_{-y_i} in both part, leading to:

$$\psi(f_m \cup w_{\widehat{y_i} \backslash m}) = g_i \circ \psi(w) \text{ for all } w \in \mathbf{dom}\, \psi,$$

which defines Property (7). □

Lemma 1. That admissibility for modes is reflexive and symmetric follows directly from Definition 3. To show transitivity of admissibility for modes, consider three arbitrary modes $M_1, M_2, M_3 \subseteq 2^{\widehat{Y}}$, such that both $adm_\psi(M_1, M_2)$ and $adm_\psi(M_2, M_3)$ (with respect to the functional bisimulation $B \sim_\psi N$). We can show that clause (1) of Definition 3 is satisfied for modes M_1 and M_3 in the following way:

$$\begin{array}{rll} & adm_\psi(M_1, M_2) \wedge adm_\psi(M_2, M_3) & \\ \Longrightarrow & \big(\forall m_2 \in M_2, \exists m_1 \in M_1 \,:\, adm_\psi(m_1, m_2)\big) & \\ \wedge & \big(\forall m_3 \in M_3, \exists m_2 \in M_2 \,:\, adm_\psi(m_2, m_3)\big) & \text{Definition 3 (1)} \\ \Longrightarrow & \forall m_3 \in M_3, \exists m_2 \in M_2, \exists m_1 \in M_1 \,: & \\ & \quad adm_\psi(m_2, m_3) \wedge adm_\psi(m_1, m_2) & \\ \Longrightarrow & \forall m_3 \in M_3, \exists m_1 \in M_1 \,:\, adm_\psi(m_1, m_3), & \end{array}$$

where the last transition is done by the symmetricity and transitivity of admissibility for modalities. Showing that clause (2) of Definition 3 is satisfied for M_1 and M_3 can be done symmetrically, which implies $adm_\psi(M_1, M_3)$ and the transitivity of admissibility for modes. □

Theorem 2. Consider the Boolean network $B = \langle f, \widehat{Y}, M_0 \rangle$ and the multivalued network $N = \langle g_i, Y, \mathbf{1}_{y_i} \rangle$, related by the bisimulation $B \sim_\psi N$. Pick an M_0-admissible mode $M \subseteq 2^{\widehat{Y}}$ and consider the Boolean network $B' = \langle f, \widehat{Y}, M \rangle$. We will show that B' bisimulates N, $B' \sim_\psi N$, by directly checking clauses (1) and (2) of the definition of bisimulation (Definition 1).

Clause (1) (forward simulation): Take two Boolean states $w, w' \in \mathbf{dom}\, \psi$ such that $w \xrightarrow{m}_f w'$ for some $m \in M$. We can then write the following deduction:

$$
\begin{aligned}
& w \xrightarrow{m}_f w' & \\
\Longrightarrow\ & \exists m_0 \in M_0,\ \exists w'' \in \mathbf{dom}\,\psi & \\
& \quad w \xrightarrow{m_0}_f w'' \wedge \psi(w') = \psi(w'') & M \text{ is } M_0\text{-admissible} \\
\Longrightarrow\ & \exists m_0 \in M_0,\ \exists w'' \in \mathbf{dom}\,\psi & \\
& \quad \psi(w) \xrightarrow{\mu(m_0)}_g \psi(w'') \wedge \psi(w') = \psi(w'') & B \sim_\psi N \\
\Longrightarrow\ & \exists m_0 \in M_0,\ \exists w'' \in \mathbf{dom}\,\psi & \\
& \quad \psi(w) \xrightarrow{\mu(m_0)}_g \psi(w'). &
\end{aligned}
$$

Remark that since N is only allowed to update one variable, y_i, $\mu(m_0)$ can only be equal to $\{y_i\}$.

Clause (2) (backward simulation): Take any two integer states $s, s' \in [\![\mathcal{L}]\!]_Y$ and an arbitrary Boolean state $w \in \mathbb{B}_X$. Since N is only allowed to update y_i, we can carry out the following deduction:

$$
\begin{aligned}
& \psi(w) = s \wedge s \xrightarrow{y_i}_g s' & \\
\Longrightarrow\ & \exists w' \in \mathbb{B}_X,\ \exists m_0 \in M_0\ : \mu(m_0) = \{y_i\} \wedge \psi(w') = & B \sim_\psi N \\
& s' \wedge w \xrightarrow{m_0}_f w' & \\
\Longrightarrow\ & \exists w' \in \mathbb{B}_X,\ \exists m_0 \in M_0, \exists w'' \in \mathbb{B}_X, \exists m \in M\ : & M \text{ is } M_0\text{-admissible} \\
& \mu(m_0) = \{y_i\} \wedge \psi(w') = s' \wedge w \xrightarrow{m_0}_f & \\
& w' \wedge\ \psi(w'') = \psi(w') \wedge w \xrightarrow{m}_f w'' & \\
\Longrightarrow\ & \exists w'' \in \mathbb{B}_X,\ \exists m \in M\ : \psi(w'') = s' \wedge w \xrightarrow{m}_f w''. &
\end{aligned}
$$

The two previous paragraphs show that the clauses of the definition of bisimulation (Definition 1) are satisfied for the Boolean network B', running under mode M, and for the multivalued network N, meaning that $B' \sim_\psi N$. The associated function mapping the modalities of B' to those of N is the unique total function $2^{\hat{Y}} \to \{y_i\}$ (i.e., the same as for the bisimulation $B \sim_\psi N$). $\square$

Proposition 2. By definition of a transition (Section 3.2) we have:

$$\forall w, w' \in \mathbb{B}_{\hat{Y}}, \forall m \in M : w \xrightarrow{m} w' \implies d(w, w') \leqslant |m|,$$

Therefore if the different coding of multivalued state complies to:

$$
\begin{aligned}
&\forall y_i \in Y, \forall s_{y_i} \in [\![L]\!]_{y_i}, \forall w, w' \in \psi^{-1}(s_{y_i}) : \\
&\qquad\qquad w \neq w' \implies d(w, w') > \max\{|m| \mid m \in M\},
\end{aligned}
$$

We deduce that the transition cannot be achieved between codes of the same integer, thus leading to:

$$\forall w, w' \in \mathbb{B}_{\widehat{Y}}, \forall m \in M : w \neq w' \wedge w \xrightarrow{m} w' \implies \psi(w) \neq \psi(w'),$$

As $w' = f_m(w) \cup w_{-m}$ by definition of a transition, this statement is equivalent to (10), concluding that the equilibrium stability is preserved. □

Lemma 2. let N be an asynchronous multivalued network bisimulating an asynchronous Boolean network B with:
$\text{MIGS}(N) = \langle Y, \longrightarrow, \sigma \rangle$, and $\text{BIGS}(B) = \langle \widehat{Y}, \longrightarrow, \sigma_{\mathbb{B}} \rangle$, as their respective signed interaction graphs; let $\mathcal{M}_{\widehat{Y}} \subseteq \widehat{Y}$ be a set of Boolean variables complying to (12), we prove Statement (4.1) by considering that Statement (4.2) holds.

First we demonstrate two properties (L2.a) and (L2.b) used in the proof:

$$\forall w, w' \in \mathbf{dom}\, \psi, \forall \widehat{y}_{ik} \in \mathcal{M}_{\widehat{Y}} : \quad w_{\widehat{y}_{ik}} \leqslant w'_{\widehat{y}_{ik}} \wedge w_{-\widehat{y}_{ik}} = w'_{-\widehat{y}_{ik}} \implies \psi(w) \leqslant \psi(w'). \quad \text{(L2.a)}$$

Proof. Assume that:
$\forall w, w' \in \mathbf{dom}\, \psi : w_{\widehat{y}_{ik}} \leqslant w'_{\widehat{y}_{ik}} \wedge w_{-\widehat{y}_{ik}} = w'_{-\widehat{y}_{ik}}$ for $\widehat{y}_{ik} \in \mathcal{M}_{\widehat{Y}}$.
Two cases occur:

1. $w_{\widehat{y}_{ik}} = w'_{\widehat{y}_{ik}}$: in this case $w = w'$ leading to $\psi(w) = \psi(w')$ since ψ is a function, thus satisfying $\psi(w) \leqslant \psi(w')$.
2. $w_{\widehat{y}_{ik}} < w'_{\widehat{y}_{ik}}$: as only two values are possible, 0 or 1, we deduce that $w'_{\widehat{y}_{ik}} = 1$. Hence, w' can be defined as $w' = w_{[\widehat{y}_{ik} \mapsto 1]}$. As $\widehat{y}_{ik} \in \mathcal{M}_{\widehat{Y}}$ by hypothesis, we conclude from (12) that $\psi(w) \leqslant \psi(w_{[\widehat{y}_{ik} \mapsto 1]})$. This inequality is equivalent to $\psi(w) \leqslant \psi(w')$. □

$$\forall w, w' \in \mathbf{dom}\, \psi : w_{\widehat{y}_{ik}} \leqslant w'_{\widehat{y}_{ik}} \wedge w_{-\widehat{y}_{ik}} = w'_{-\widehat{y}_{ik}} \implies \psi(w_{\widehat{y}_i}) \leqslant \psi(w'_{\widehat{y}_i}) \wedge \psi(w_{-\widehat{y}_i}) = \psi(w'_{-\widehat{y}_i}). \quad \text{(L2.b)}$$

Proof. Assume that: $\forall w, w' \in \mathbf{dom}\, \psi : w_{\widehat{y}_{ik}} \leqslant w'_{\widehat{y}_{ik}} \wedge w_{-\widehat{y}_{ik}} = w'_{-\widehat{y}_{ik}}$.
As $w_{-\widehat{y}_i} \subseteq w_{-\widehat{y}_{ik}}$ and since $\widehat{y}_{ik} \in \widehat{y}_i$, we have: $w_{-\widehat{y}_{ik}} = w'_{-\widehat{y}_i} \implies w_{-\widehat{y}_i} = w'_{-\widehat{y}_i}$, thus implying that: $\forall w, w' \in \mathbf{dom}\, \psi : w_{\widehat{y}_{ik}} \leqslant w'_{\widehat{y}_{ik}} \wedge w_{-\widehat{y}_i} = w'_{-\widehat{y}_i}$.

Hence, from Property (L2.a) applied to $w_{\widehat{y_i}}, w'_{\widehat{y_i}}$, we deduce that:

$$\forall w, w' \in \mathbf{dom}\,\psi : w_{\widehat{y_i}_k} \leqslant w'_{\widehat{y_i}_k} \wedge w_{-\widehat{y_i}} = w'_{-\widehat{y_i}} \implies \psi(w_{\widehat{y_i}}) \leqslant \psi(w'_{\widehat{y_i}}),$$

Moreover, as ψ is a function defined on supports, we have:

$$\forall w, w \in \mathbf{dom}\,\psi : w_{-\widehat{y_i}} = w'_{-\widehat{y_i}} \implies \psi(w_{-\widehat{y_i}}) = \psi(w'_{-\widehat{y_i}})$$

In conclusion, the following statement holds:

$$\psi(w_{\widehat{y_i}}) \leqslant \psi(w'_{\widehat{y_i}}) \wedge \psi(w_{-\widehat{y_i}}) = \psi(w'_{-\widehat{y_i}}).$$

□

Now we prove that Statement 4.1 is satisfied. The proof is given for positive interaction.

($\implies$) By definition (3), a positive interaction, $\widehat{y}_{ik} \xrightarrow{+} \widehat{y}_{jr}$ is defined as:

$$\forall w, w' \in \mathbf{dom}\,\psi : w_{\widehat{y_i}_k} \leqslant w'_{\widehat{y_i}_k} \wedge w_{-\widehat{y_i}_k} = w'_{-\widehat{y_i}_k} \implies f_{j,r}(w) \leqslant f_{j,r}(w').$$

From (L2.b), we can rewrite this statement as:

$$\forall w, w' \in \mathbf{dom}\,\psi : \psi(w_{\widehat{y_i}}) \leqslant \psi(w'_{\widehat{y_i}}) \wedge \psi(w_{-\widehat{y_i}}) = \psi(w'_{-\widehat{y_i}}) \implies f_{j,r}(w) \leqslant f_{j,r}(w').$$

Let $v = f_{j,r}(w) \cup w_{-\widehat{y_j}_r}$ and $v' = f_{j,r}(w') \cup w'_{-\widehat{y_j}_r}$, as $f_{j,r}(w) \leqslant f_{j,r}(w')$ by hypothesis, we conclude that: $\psi(v) \leqslant \psi(v')$ from (L2.a), thus leading to:

$$\forall w, w' \in \mathbf{dom}\,\psi : \psi(w_{\widehat{y_i}}) \leqslant \psi(w'_{\widehat{y_i}}) \wedge \psi(w_{-\widehat{y_i}}) = \psi(w'_{-\widehat{y_i}}) \implies \\ \psi(f_{j,r}(w) \cup w_{-\widehat{y_j}_r}) \leqslant \psi(f_{j,r}(w') \cup w'_{-\widehat{y_j}_r}).$$

As N and B are bisimilar, Property (7) holds. By application of this property we have: $\psi(f_{j,r}(w) \cup w_{-\widehat{y_j}_r}) = g_j \circ \psi(w)$ and similarly for w'. Thus we deduce that:

$$\forall w, w' \in \mathbf{dom}\,\psi : \psi(w_{\widehat{y_i}}) \leqslant \psi(w'_{\widehat{y_i}}) \wedge \psi(w_{-\widehat{y_i}}) = \psi(w'_{-\widehat{y_i}}) \implies \\ g_j \circ \psi(w) \leqslant g_j \circ \psi(w').$$

Finally, as $\mathbf{codom}\,\psi = [\![\mathcal{L}]\!]_Y$ by definition, we can rewrite the previous statement as follows by setting, $s = \psi(w), s' = \psi(w')$:

$$\forall s, s' \in [\![\mathcal{L}]\!]_Y : s_i \leqslant s'_i \wedge s_{-i} = s'_{-i} \implies g_j(s) \leqslant g_j(s'),$$

that defines the positive interaction on MIGS(N): $y_i \xrightarrow{+} y_j$.

($\Longleftarrow$) Assume that an interaction $y_i \xrightarrow{+} y_j$ exists and there exist two Boolean variables $\widehat{y}_{ik} \in \mathcal{M}_{\hat{Y}} \cap \widehat{y}_i$ and $\widehat{y}_{j_r} \in \mathcal{M}_{\hat{Y}} \cap \widehat{y}_j$ with no positive interactions between these variables, *i.e.*, $\widehat{y}_{ik} \xrightarrow{\sigma} \widehat{y}_{j_r} \implies \sigma \neq +$. We give a proof for the case $\sigma = -$; the proof for $\sigma = 0$ is similar.

From definition of the interactions (3), we deduce that:

$$\exists w, w' \in \mathbf{dom}\,\psi : w_{\widehat{y}_{ik}} \leqslant w'_{\widehat{y}_{ik}} \wedge w_{-\widehat{y}_{ik}} = w'_{-\widehat{y}_{ik}} \wedge f_{j,r}(w) > f_{j,r}(w').$$

From Property (L2.b), we can rewrite the previous statement as:

$$\exists w, w' \in \mathbf{dom}\,\psi : \psi(w_{\widehat{y}_i}) \leqslant \psi(w'_{\widehat{y}_i}) \wedge \psi(w_{-\widehat{y}_i}) = \psi(w'_{-\widehat{y}_i}) \wedge f_{j,r}(w) > f_{j,r}(w').$$

Let $v = f_{j,r}(w) \cup w_{-\widehat{y_j}_r}$ and $v' = f_{j,r}(w') \cup w'_{-\widehat{y_j}_r}$, as $f_{j,r}(w) > f_{j,r}(w')$ by hypothesis, we conclude that: $\psi(v) > \psi(v')$ from (L2.a), thus leading to:

$$\forall w, w' \in \mathbf{dom}\,\psi : \psi(w_{\widehat{y}_i}) \leqslant \psi(w'_{\widehat{y}_i}) \wedge \psi(w_{-\widehat{y}_i}) = \psi(w'_{-\widehat{y}_i}) \implies \psi(f_{j,r}(w) \cup w_{-\widehat{y_j}_r}) > \psi(f_{j,r}(w') \cup w'_{-\widehat{y_j}_r}).$$

As N and B are bisimilar, Property (7) holds. By application of this property we have: $\psi(f_{j,r}(w) \cup w_{-\widehat{y_j}_r}) = g_j \circ \psi(w)$ and similarly for w'. Thus we have:

$$\exists w, w' \in \mathbf{dom}\,\psi : \psi(w_{\widehat{y}_i}) \leqslant \psi(w'_{\widehat{y}_i}) \wedge \psi(w_{-\widehat{y}_i}) = \psi(w'_{-\widehat{y}_i}) \wedge g_j \circ \psi(w) > g_j \circ \psi(w').$$

As $\mathbf{codom}\,\psi = [\![\mathcal{L}]\!]_Y$ by definition, we can rewrite the previous statement as follows by setting, $s = \psi(w), s' = \psi(w')$:

$$\exists s, s' \in [\![\mathcal{L}]\!]_Y : s_i \leqslant s'_i \wedge s_{-i} = s'_{-i} \wedge g_j(s) > g_j(s'),$$

that contradicts the existence of a positive interaction $y_i \xrightarrow{+} y_j$, which is false by hypothesis.

The proof for negative interaction follows the same scheme. Thus, we conclude that Statement 4.1 is satisfied. $\square$

Corollary 1. If Property 7 holds then we deduce that $B \sim_\psi N$ from Theorem 1. As $B \sim_\psi N$ and M is an admissible mode we conclude from Theorem 2 that $B' \sim_\psi N$. $\square$

Theorem 3. We prove that (12) holds for a set of Boolean variables belonging to a support $\widehat{y}_i$.

Let $\widehat{y}_{ik}$ be a Boolean variable of this set, two cases occur: either $w_{\widehat{y}_{ik}} = 0$, or $w_{\widehat{y}_{ik}} = 1$. For the latter, w is left untouched by substitution leading to $\psi(w) = \psi(w_{[\widehat{y}_{ik} \mapsto 1]})$ since ψ is a function, thus fulfilling (12). Hence, we address the case when $w_{\widehat{y}_{ik}} = 0$ in the proofs.

Summing code. The following property holds when $w_{\widehat{y}_{ik}} = 0$:

$$\sum_{\widehat{y}_{ij} \in \widehat{y}_i \setminus \widehat{y}_{ik}} w_{\widehat{y}_{ij}} = \sum_{\widehat{y}_{ij} \in \widehat{y}_i} w_{\widehat{y}_{ij}},$$

thus, we have:

$$\psi(w_{[\widehat{y}_{ik} \mapsto 1]}) = \sum_{\widehat{y}_{ij} \in \widehat{y}_i \setminus \widehat{y}_{ik}} w_{\widehat{y}_{ij}} + 1 = \sum_{\widehat{y}_{ij} \in \widehat{y}_i} w_{\widehat{y}_{ij}} + 1 = \psi(w) + 1.$$

We conclude that: $\psi(w) < \psi(w_{[\widehat{y}_{ik} \mapsto 1]}$.

Van Ham code. Van Ham code is a sub-code of the Summing code, thus complying to its results.

Gray code. Let $\widehat{y}_{i1}$ be a Boolean variable carrying the most significant digit, we separate $\widehat{y}_{i1}$ from the other variables in the definition of ψ:

$$\psi(w) = \sum_{k=1}^{|\widehat{y}_i|} 2^{|\widehat{y}_i|-k} . \bigoplus_{j=1}^{k} w_{\widehat{y}_{ij}} = 2^{|\widehat{y}_i|-1} . w_{\widehat{y}_{i1}} + \sum_{k=2}^{|\widehat{y}_i|} 2^{|\widehat{y}_i|-k} . \bigoplus_{j=1}^{k} w_{\widehat{y}_{ij}}.$$

Hence, when $w_{\widehat{y}_{i1}} = 0$, we deduce that $\psi(w_{[\widehat{y}_{i1} \mapsto 1]}) = 2^{|\widehat{y}_i|-1} + \psi(w)$, leading to $\psi(w) < \psi(w_{[\widehat{y}_{i1} \mapsto 1]})$.

Thus we conclude that $\widehat{Y}$ is the set of markers for the Summing and Van Ham code, while $\{\widehat{y}_{i1} \mid y_i \in Y\}$ are the markers for the Gray code by application of Lemma 2. $\square$

Theorem 4. We first show that the computation of the Boolean function f defined by (15) is correct with respect to the integer function g and the asynchronous mode (A). Next (B), we examine the satisfaction of Property (7). Finally we demonstrate the bisimulation of the reflexive reduction for both networks (C).

A) The construction of f is correct.

Let $s \xrightarrow{y_i} s'$ be a multivalued transition, such that $s'_{y_i} = g_i(s) = l'$ and $s'_{y_j} = s_{y_j}$ for all $1 \leqslant j \leqslant n, j \neq i$, by definition of the asynchronous dynamics. We have: $\max(l'-1, 0) \leqslant s_{y_i} \leqslant \min(l'+1, L_i)$ since the evolution is unitary stepwise. There exist two Boolean states $w, w' \in \mathbb{B}_{\widehat{Y}}$ such that $\psi(w) = s$ as **codom** $\psi = [\![\mathcal{L}]\!]_Y$. We check that for all $\widehat{y}_{ik} \in \widehat{y}_i$ if $f_{i,k}(w) = w'_{\widehat{y}_{ik}}$ then $\psi(w') = s'$ and $w_{-\widehat{y}_{ik}} = w'_{-\widehat{y}_{ik}}$, thus proving the correction of $f_{i,k}$. The fact that $w_{-\widehat{y}_{ik}} = w'_{-\widehat{y}_{ik}}$ is a direct consequence an asynchronous transition updating one variable only.

Two cases are considered qualifying whether $s'_{y_i} \neq 0$ or $s'_{y_i} = 0$. For each, we examine whether the target state of the support variable $\widehat{y}_{ik}$ is 0 or 1. Let us consider the following cases:

1. $s'_{y_i} \neq 0$: By definition of a multivalued network (1) $C_{l'}(s)$ is necessary satisfied as $l' = s'_{y_i} \neq 0$. Let $R(y_i)$ be the set of regulators of y_i, we have: $s_{R(y_i)} \in \mathcal{C}_{R(y_i),l'}$. Hence, we deduce that $w_{\widehat{y}_i}$ satisfies the Boolean version of the condition, $C^{\mathbb{B}}_{l'}$, by construction of the Boolean condition (13). Now we examine, the possible target states of the support variable $\widehat{y}_{ik}$, $w'_{\widehat{y}_{ik}}$:
   - $w'_{\widehat{y}_{ik}} = 1$: in this case $w_{\widehat{y}_i}$ belongs to $\Psi_{\star\to 1}(l', \widehat{y}_{ik})$ by definition, meaning that w admissible for the guard. Thus we have:
   $$f_{i,k}(w) = C^{\mathbb{B}}_{l'}(w) \wedge C_{\Psi_{\star\to 1}(l',\widehat{y}_{ik})}(w) = 1.$$
   - $w'_{\widehat{y}_{ik}} = 0$: in this case $w_{\widehat{y}_i}$ does not belong to $\Psi_{\star\to 1}(l', \widehat{y}_{ik})$ by definition meaning that w is not admissible for the guard. Thus we have:
   $$f_{i,k}(w) = C^{\mathbb{B}}_{l'}(w) \wedge C_{\Psi_{\star\to 1}(l',\widehat{y}_{ik})}(w) = 0.$$

   In both cases, $f_{i,k}$ provides the expected result.
2. $s'_{y_i} = 0$: By definition of the multivalued dynamics (1), no guards are satisfied. The conjunction of the guards for all levels is unsatisfiable, thus by definition of the part related to the guard in $f_{i,k}$ (13), we deduce

that $f_{i,k}(w) = 0$ by (15), which is the expected result as 0 is encoded by a Boolean profile filled with 0 leading to $w'_{\widehat{y}_{i_k}} = 0$ for all k.

f returns the appropriate result regarding a pair w, w' encoding the pair s, s'. If $s \neq s'$ then there exists a Boolean support variable $\widehat{y}_{ik}, 1 \leqslant k \leqslant |\widehat{y}_i|$ such that: $\psi(f_{i,k}(w) \cup w_{-\widehat{y}_{i_k}}) = s'$, corresponding to the following condition: $f_{i,k}(w) \neq w_{\widehat{y}_{i_k}}$. Otherwise ($s = s'$) any index k satisfies $\psi(f_{i,k}(w) \cup w_{-\widehat{y}_{i_k}}) = s'$. Notice that this part is not sufficient for proving bisimulation, since we may have $f_{i,j}(w) = w'_{\widehat{y}_{i_j}} = w_{\widehat{y}_{i_j}}$ by definition of the asynchronous dynamics, thus also leading to a transition $w \xrightarrow{\widehat{y}_{i_j}} w$ by definition. This transition does not simulate the transition $s \xrightarrow{y_i} s'$ when $s \neq s$, motivating the proof of the bisimulation restricted to the reflexive reduction. However a multivalued self-loop ($s = s'$) is simulated by a self-loop in the Boolean network by construction of f.

B) Property (7) is satisfied.

Let $s \xrightarrow{y_i} s'$ be a multivalued asynchronous transition such that $s \neq s'$, then there exist $w, w' \in \mathbb{B}_{\widehat{Y}}$ such that $\psi(w) = s, \psi(w') = s'$, by construction of f (A). Moreover, we have: $\psi(w) \neq \psi(w')$, leading to $w \neq w'$, as ψ is a function. Thus, a Boolean support variable $\widehat{y}_{ik}$ verifies that $w_{\widehat{y}_{i_k}} \neq w'_{\widehat{y}_{i_k}}$, as g and f are neighbourhood preserving (9). In this case, we have: $w' = f_{i,k}(w) \cup w_{-\widehat{y}_{i_k}}$ from (A). Thus, we have the following equalities:

$$\begin{aligned}
s' &= \psi(w') && \text{by definition of } \psi;\\
&= \psi(f_{i,k}(w) \cup w_{-\widehat{y}_{i_k}}) && \text{by construction of } f \text{ (A)};\\
&= \psi(f_{i,k} \cup w_{\widehat{y}_i \setminus \widehat{y}_{i_k} \cup -\widehat{y}_i}) && \text{from (T1)};\\
&= \psi(f_{i,k} \cup w_{\widehat{y}_i \setminus \widehat{y}_{i_k}} \cup w_{-\widehat{y}_i}) && \text{from (5)};\\
&= \psi(f_{i,k} \cup w_{\widehat{y}_i \setminus \widehat{y}_{i_k}}) \cup \psi(w_{-\widehat{y}_i}) && \text{by (5), (6)};\\
&= \psi(f_{i,k} \cup w_{\widehat{y}_i \setminus \widehat{y}_{i_k}}) \cup s_{-y_i} && \text{as } s = \psi(w).
\end{aligned}$$

As $s' = g_i(s) \cup s_{-y_i}$ by definition of a transition, we deduce by simplification of s_{-y_i} that: $g_i(s) = g_i \circ \psi(w) = \psi(f_{i,k} \cup w_{\widehat{y}_i \setminus \widehat{y}_{i_k}})$, concluding that Property (7) holds.

C) Bisimulation between reflexive reductions. It follows from (B), that we can set that Property (7) holds whenever $s_{y_i} \neq g_i(s)$. As the asynchronous mode is local-to-support, and we always have $s \xrightarrow{y_i} s' \implies s_{y_i} \neq s'_{y_i} = g_i(s)$ by reflexive reduction, we conclude by application of Theorem 1 that the reflexive reduction of the multivalued and Boolean networks are bisimilar with respect to ψ.

□

Proposition 4. We denote: $\mathbf{0}_m$ a Boolean state with 0 for all variables in $m \subseteq \widehat{Y}$.

If a state is in the admissible region then it always reaches states in the admissible region, and only in the admissible region, by definition of bisimulation.

If the Boolean state $w \in \mathbb{B}_{\widehat{Y}}$ is outside of the admissible region, $w \notin \mathbf{dom}\,\psi$, then it is not accounted for by the computation of admissibility of the guard condition, by definition of $\Psi_{\star\to 1}$. Therefore we have: $f_m(w) = \mathbf{0}_m, \forall m \in M$. Thus, all the trajectories starting from w successively cancel (set to 0) the states of the variables of $m \in M$ whenever the result of the cancellation leads to a state outside the admissible region; otherwise the proposition holds. As $\bigcup_{m\in M} m = \widehat{Y}$ by definition of a mode, the cancellation process terminates at state $\mathbf{0}_{\widehat{Y}}$, which is always in the admissible region since $\mathbf{0}_{\widehat{y_i}}, \forall y_i \in Y$, is the sole code for the integer value 0, regardless of the variable y_i.

In conclusion, the trajectories starting from any $w \in \mathbb{B}_{\widehat{Y}}$ eventually end up in a state in the admissible region $\mathbf{dom}\,\psi$. □