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Abstract

Reliability properties associated to the classic models of systems with age replacement have been a usual topic of research. Most previous works have checked aging properties of the lifetime of the working units using stochastic comparisons between the systems with age replacement at different times. However, from a practical point of view, it would be also interesting to deduce the belonging to aging classes of the lifetime of the system from the aging properties of the lifetime of its working units. The first part of this article deals with this problem. Later, stochastic orderings are established between the two systems with replacement at the same time using several stochastic comparisons among the lifetimes of their working units. In addition, the lifetimes of two systems with age replacement are also compared assuming stochastic orderings between the number of replacement until failure and the lifetimes of their working units conditioned to be less or equal than the replacement time. Similar comparisons are accomplished considering two systems with age replacement where the replacements occur at random time. This last case is very interesting for real-life applications. Illustrative examples are also presented.

Keywords: age replacement; random time replacement; aging classes; stochastic orders; stationary pointwise availability.

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1 Introduction

A natural way to improve the reliability of systems is implementing a replacement policy for a unit after it had been working for a period of time. Acting in this way it is possible to avoid system failures and periods of inactivity. Stochastic properties of lifetimes of systems with planned replacement policies have been widely studied and the age replacement policy comes up as one of the most studied kind of replacement. Under an age replacement policy it is supposed that a single unit works upon failure or upon a specified age T, when a replacement by a new unit occurs, whichever comes first. The times of replacement are assumed as instantaneous. We assume that lifetimes of all units to be placed in service are independent and equally distributed with finite mean and survival function $\overline{F} = 1 - F$, where F is the distribution function of X. Let us denote by $\tau_{X,T}$ the lifetime of the system with age replacement planned at $T > 0$. It is well known (see e.g. [2]) that the survival function of $\tau_{X,T}$, denoted by $\overline{F}_{X,T}$, satisfies

$$\overline{F}_{X,T}(t) = [\overline{F}(T)]^{\lfloor t/T \rfloor} \overline{F}\left(t - \left\lfloor \frac{t}{T} \right\rfloor T\right), \tag{1}$$

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for all $t \geq 0$, where $\lfloor x \rfloor$ is the greater integer less or equal than x.

Systems with age replacement are important for practical and theoretical reasons. These systems have been used to characterize some aging classes. For example, Barlow and Proschan [2, 3] prove that the lifetime of a system with age replacement, $\tau_{X,T}$, is stochastically decreasing (increasing) in T if and only if $X \in \mathrm{IFR}$ ($X \in \mathrm{DFR}$). They also consider a model where the time until replacement is random and all lifetimes and times until replacement are assumed independent.

Marshall and Proschan [17] provide characterizations of NBU (NWU) and NBUE (NWUE) aging classes. They show that $X \leq_{\mathrm{st}} (\geq_{\mathrm{st}}) \tau_{X,T}$ for all $T > 0$ if and only is $X \in \mathrm{NBU}$ ($X \in \mathrm{NWU}$). Furthermore, $\mu = \mathbb{E}[X] \leq (\geq) \mathbb{E}[\tau_{X,T}]$ for all $T > 0$ if and only is $X \in \mathrm{NBUE}$ ($X \in \mathrm{NWUE}$). See also [21] for these and other results associated to maintenance policies and stochastic orders and aging classes. Similar characterizations are deduced by Belzunce et al. [5] for the increasing convex order and its associated *new better than used* class. In Proposition 2.5 we analyze the hazard rate and likelihood ratio orderings between X and $\tau_{X,T}$ using the belonging of X to IFR (DFR) and ILR (DLR) aging classes, respectively.

These previous results characterize the aging properties of the random variable X using the random variable $\tau_{X,T}$, for $T > 0$. However, in practical situations it would be interesting to deduce some properties of $\tau_{X,T}$ using the aging class to which X belongs. Theorems 2.1 and 2.2 give conditions on the aging properties of X for $\tau_{X,T}$ belonging to NBU (NWU) and DMLR (IMLR) aging classes, respectively.

It is well known, see e.g. [2], that

$$\mathbb{E}\left[\tau_{X,T}\right] = \frac{\int_0^T \overline{F}(x)\,\mathrm{d}x}{F(T)}. \tag{2}$$

Marshall and Proschan [17] point out the lack of relation between the monotony of $\mathbb{E}\left[\tau_{X,T}\right]$ in T and the most common aging classes. As they noted, $X \in \mathrm{IFR}$ implies that $\mathbb{E}\left[\tau_{X,T}\right]$ is decreasing in T, and that implies $X \in \mathrm{NBUE}$, furthermore both implications are false in the opposite sense. Due to this fact, some authors, as Klefsjö [12], Knopik [13, 14], Kayid et al. [10] and Nair et al. [20], study the *Decreasing Mean Time to Failure until Replacement* (DMTFR) aging class. We say that X belongs to the DMTFR class (denoted by $X \in \mathrm{DMTFR}$) if $\mathbb{E}\left[\tau_{X,T}\right]$ is decreasing in $T > 0$. It is well known that $\mathrm{DMTFR} \subset \mathrm{NBUE}$ (see [2] and [12]) and $\mathrm{IFR} \subset \mathrm{IFRA} \subset \mathrm{DMTFR}$ (see [14]). Kayid et al. [11] and Izadi et al. [7] consider generalizations of this aging class. In Corollary 2.4 we prove that $X \in \mathrm{DMTFR}$ if and only if $\tau_{X,T} \in \mathrm{NBUE}$ for all $T > 0$.

In practice it does not always have sense to implement a replacement. For example, if $X \in \mathrm{NWU}$ then the lifetime of the working unit, i.e. the working time of a single unit without replacement, is greater in the usual stochastic order sense than the lifetime of the system with replacement after T, for all $T > 0$. However, in Example 1 we show that, even when $X \in \mathrm{NWU}$, we could have $\tau_{X,T} \in \mathrm{NBUE}$ for some values of $T > 0$ [17]. Thus, deciding if it is better to implement a replacement or not and set the optimum time until replacements could be non intuitive problems, which strongly depend on the chosen optimization criterion.

Consider we have two systems with age replacement whose units have lifetimes X_1 and X_2, respectively, and with time until replacement $T > 0$ for both systems. It is natural to ask for the relation between the stochastic orderings that X_1, X_2 and $\tau_{X_1,T}$, $\tau_{X_2,T}$ satisfy. For example, Asha and Unnikrishnan Nair [1] and Kayid et al. [10] study the comparisons of the mean lifetimes of two systems with age replacement. Additionally, Block et al. [6] prove that $X_1 \geq_{\mathrm{st}} X_2$ if and only if $\tau_{X_1,T} \geq_{\mathrm{st}} \tau_{X_2,T}$, for all $T > 0$. Jain [8] extends these results proving that $X_1 \succ X_2$ if and only if $\tau_{X_1,T} \succ \tau_{X_2,T}$, for all $T > 0$; where $\succ$ denotes the *hazard rate order* or the *likelihood ratio order*. In Theorems 2.8 and 2.9 we obtain similar results for the *reversed hazard rate order*, the *mean residual lifetime order* and the *increasing convex and concave orders*. We remark that the use of the increasing concave order has not been so common in reliability. The recent work of Mercier and Castro [19] use it for comparing the lifetimes of systems with imperfect maintenance actions modelled as a gamma process. This model is, in some sense, a generalization of the classical system with age replacement.

The last part of the present work deals with the system with replacement occurring at random time. Lifetimes of two systems with this kind of replacement are compared using the usual stochastic order and Laplace transform order. Furthermore, the mean lifetimes and the pointwise stationary availabilities of two systems are compared as other measure of their reliability. Recently, Park et al. [22] use the pointwise stationary availabilities for comparing some generalized age replacement models among them and with the age replacement model we consider in this paper.

The rest of the paper is organized as follows. Section 1.1 provides the definitions of aging classes and stochastic orders we use in the sequel. Section 2 deals with the system with age replacement where the time until replacement is constant. Section 2.1 mainly deals with the results associated to aging classes and Section 2.2 is devoted to the results associated to stochastic orderings. Finally, Section 3 studies a system with replacement at random time. In particular, Section 3.1 focusses on the stationary pointwise availability of systems with replacement at random time.

1.1 Background on stochastic orders and aging classes

During this work we assume all lifetimes and times until replacements are non-negative absolutely continuous random variables with finite means. Moreover, for every random variable X we assume that $F(x) > 0$, for all $x > 0$. The following definitions introduce some well known concepts related to stochastic orders and aging classes. Throughout the article increasing means nondecreasing and decreasing means nonincreasing.

Definition 1.1 (Aging classes). *Let X be a non-negative random variable with density function f, survival function $\overline{F}$ and finite mean μ. We say that X belongs to the aging class*

a) *New Better (Worse) than Used in Expectation, denoted by* NBUE (NWUE), *if* $\int_t^\infty \overline{F}(x)\,\mathrm{d}x \le (\ge)\, \mu\, \overline{F}(t)$ *for all* $t \ge 0$,

b) *Decreasing (Increasing) Mean Time to Failure with Replacement, denoted by* DMTFR (IMTFR), *if the function* $m_X(t) = \int_0^t \overline{F}(x)\mathrm{d}x / F(t)$ *is decreasing (increasing) in* $t > 0$,

c) *New Better (Worse) than Used, denoted by* NBU (NWU), *if* $\overline{F}(x+t) \le (\ge) \overline{F}(x)\overline{F}(t)$ *for all* $x, t \in \mathbb{R}_+$,

d) *Decreasing (Increasing) Mean Residual Life, denoted by* DMRL (IMRL), *if* $\mu_X(t) = \int_t^\infty \overline{F}(x)\mathrm{d}x / \overline{F}(t)$ *is decreasing (increasing) in* $\{t \ge 0 : \overline{F}(t) > 0\}$,

e) *Increasing (Decreasing) Failure Rate, denoted by* IFR (DFR), *if* $\overline{F}(x+t)/\overline{F}(x)$ *is decreasing (increasing) in* $\{x \ge 0 : \overline{F}(x) > 0\}$ *for all* $t \ge 0$,

f) *Increasing (Decreasing) Likelihood Rate, denoted by* ILR (DLR), *if* $f(x+t)/f(x)$ *is decreasing (increasing) in* $\{x \ge 0 : f(x) > 0\}$ *for all* $t \ge 0$.

These aging classes are related as follows

$$\begin{array}{ll} \mathrm{ILR} \subset \mathrm{IFR} \subset \mathrm{NBU} \subset \mathrm{NBUE}, & \mathrm{DLR} \subset \mathrm{DFR} \subset NWU \subset NWUE, \\ \mathrm{IFR} \subset \mathrm{DMRL} \subset \mathrm{NBUE}, & \mathrm{DFR} \subset \mathrm{IMRL} \subset \mathrm{NWUE}, \\ \mathrm{IFR} \subset \mathrm{DMTFR} \subset \mathrm{NBUE}, & \mathrm{DFR} \subset \mathrm{IMTFR} \subset \mathrm{NWUE}. \end{array}$$

Let us denote the hazard rate of X by $\lambda = f/\overline{F}$, which is defined for all $t \ge 0$ such that $\overline{F}(t) > 0$. It is well known that $X \in$ IFR ($X \in$ DFR) if and only if λ is increasing (decreasing) in its domain. See [3], [15] and [20] for more details about these aging notions.

Definition 1.2 (Stochastic orders). *Let X_1 and X_2 be two non-negative random variables with finite means and with density functions f_1 and f_2, distribution functions F_1 and F_2 and survival functions $\overline{F}_1$ and $\overline{F}_2$, respectively. We say that X_1 is greater than X_2 in the*

a) *Laplace transform order, denoted by* $X_1 \ge_{\mathrm{Lt}} X_2$, *if* $\int_0^\infty \overline{F}_1(t)\,\mathrm{e}^{-st}\mathrm{d}t \ge \int_0^\infty \overline{F}_2(t)\,\mathrm{e}^{-st}\mathrm{d}t$ *for all* $s \ge 0$,

b) *Increasing convex order, denoted by* $X_1 \ge_{\mathrm{icx}} X_2$, *if* $\int_t^\infty \overline{F}_1(x)\mathrm{d}x \ge \int_t^\infty \overline{F}_2(x)\mathrm{d}x$, *for all* $t \ge 0$,

c) *Increasing concave order, denoted by* $X_1 \ge_{\mathrm{icv}} X_2$, *if* $\int_0^t \overline{F}_1(x)\mathrm{d}x \ge \int_0^t \overline{F}_2(x)\mathrm{d}x$, *for all* $t \ge 0$,

d) *Mean residual lifetime order, denoted by* $X_1 \ge_{\mathrm{mrl}} X_2$, *if* $\int_t^\infty \overline{F}_1(x)\mathrm{d}x / \int_t^\infty \overline{F}_2(x)\mathrm{d}x$ *is increasing for all* $t \ge 0$ *such that* $\int_t^\infty \overline{F}_2(x)\mathrm{d}x > 0$,

e) *Usual stochastic order, denoted by* $X_1 \ge_{\mathrm{st}} X_2$, *if* $\overline{F}_1(t) \ge \overline{F}_2(t)$ *for all* $t \ge 0$,

f) *Reversed hazard rate order, denoted by* $X_1 \ge_{\mathrm{rh}} X_2$, *if* $F_1(t)/F_2(t)$ *is decreasing for all* $t \ge 0$,

g) *Hazard rate order, denoted by* $X_1 \ge_{\mathrm{hr}} X_2$, *if* $\overline{F}_1(t)/\overline{F}_2(t)$ *is increasing for all* $t \ge 0$ *such that* $\overline{F}_2(t) > 0$,

h) *Likelihood ratio order, denoted by* $X_1 \ge_{\mathrm{lr}} X_2$, *if* $f_1(t)/f_2(t)$ *is increasing for all* $t \ge 0$ *such that* $f_2(t) > 0$.

The following relations among these orders are well known:

$$h \Rightarrow g \Rightarrow e \Rightarrow c \Rightarrow a, \;\; h \Rightarrow f \Rightarrow e \Rightarrow b \;\; and \;\; g \Rightarrow d \Rightarrow b.$$

Denote by λ_1 and λ_2 the hazard rates of X_1 and X_2, respectively. The ordering $X_1 \geq_{\mathrm{hr}} X_2$ holds if and only if $\lambda_1(t) \leq \lambda_2(t)$, for all t where λ_1 and λ_2 are defined. It is also well known that $X_1 \geq_{\mathrm{rh}} X_2$ if and only $r_1(t) \geq r_2(t)$, for all $t \geq 0$, where $r_i = f_i/F_i$ is the reversed hazard rate of X_i, for $i = 1,2$. A deeply treatment of these and other stochastic orders can be found in [24] and [4].

Consider the following sets of functions:

$$\begin{aligned} \mathscr{G}_{\mathrm{rh}} &= \{g : \mathbb{R}^2 \to \mathbb{R} \text{ such that } g(x,y) - g(y,x) \text{ is increasing in } x \text{ for all } x \leq y\}, \\ \mathscr{G}_{\mathrm{lr}} &= \{g : \mathbb{R}^2 \to \mathbb{R} \text{ such that } g(x,y) \geq g(y,x) \text{ for all } x \geq y\}. \end{aligned}$$

These sets of functions can be used to characterize the reversed hazard rate and likelihood ratio orders.

Proposition 1.1 (Characterization of rh and lr orders)**.** *Let X_1 and X_2 be two independent absolutely continuous random variables. Then,*

a) $X_1 \geq_{\mathrm{rh}} X_2$ *if and only if* $\mathbb{E}[\phi(X_1,X_2)] \geq \mathbb{E}[\phi(X_2,X_1)]$*, for all* $\phi \in \mathscr{G}_{\mathrm{rh}}$*,*

b) $X_1 \geq_{\mathrm{lr}} X_2$ *if and only if* $\mathbb{E}[\phi(X_1,X_2)] \geq \mathbb{E}[\phi(X_2,X_1)]$*, for all* $\phi \in \mathscr{G}_{\mathrm{lr}}$*.*

See Theorem 2.3 of [25] and Theorem 1.B.48 of [24] for the proofs of these results.

A non-negative function $h : \mathbb{R}^2 \to [0,\infty)$ is said to be *totally positive of order* 2 (TP$_2$) if $h(x_1,y_1)h(x_2,y_2) \geq h(x_2,y_1)h(x_1,y_2)$, for all $x_1 \leq x_2$ and $y_1 \leq y_2$, see [9]. The following characterizations of IFR and DFR aging classes can be found in Propositions B.8 and B.9 of [16].

Proposition 1.2 (Characterization of IFR and DFR aging classes)**.** *Let X be an absolutely continuous random variable. Then,*

a) $X \in$ IFR *if and only if* $\overline{F}(y-x)$ *is a* TP$_2$ *function in* (x,y)*,*

b) $X \in$ DFR *if and only if* $\overline{F}(x+y)$ *is a* TP$_2$ *function in* (x,y)*.*

2 Results on systems with age replacement

This section is devoted to the study of the lifetime of systems with age replacement at age $T > 0$ using aging classes and stochastic orders. In Section 2.1 we study the relation between the aging classes to which X and $\tau_{X,T}$ belong. In Section 2.2 we establish stochastic orderings between the lifetimes of two systems with age replacement using stochastic orderings between the lifetimes of the working units.

2.1 Aging classes

As we said, the relation between τ_{X,T_1} and τ_{X,T_2} when X belongs to an aging class have been widely studied. However, as far as we know, it has not been analyzed the aging classes to which $\tau_{X,T}$ belongs, using aging properties of X. The following results deal with this problem.

Theorem 2.1. *If $X \in$ IFR ($X \in$ DFR), then $\tau_{X,T} \in$ NBU ($\tau_{X,T} \in$ NWU), for all $T > 0$.*

Proof. To prove $\tau_{X,T} \in$ NBU we have to check the inequality $\overline{F}_{X,T}(x)\overline{F}_{X,T}(t) \geq \overline{F}_{X,T}(x+t)$ for all non-negative x and t, where $\overline{F}_{X,T}(t)$ is defined as in (1). This inequality is equivalent to

$$[\overline{F}(T)]^{\lfloor x/T \rfloor + \lfloor t/T \rfloor}\overline{F}\left(x - \left\lfloor \frac{x}{T} \right\rfloor T\right)\overline{F}\left(t - \left\lfloor \frac{t}{T} \right\rfloor T\right) \geq [\overline{F}(T)]^{\lfloor (x+t)/T \rfloor}\overline{F}\left(x+t - \left\lfloor \frac{x+t}{T} \right\rfloor T\right). \tag{3}$$

As $\left\lfloor \frac{x+t}{T} \right\rfloor = \left\lfloor \frac{t}{T} \right\rfloor + \left\lfloor \frac{x}{T} \right\rfloor$ or $\left\lfloor \frac{x+t}{T} \right\rfloor = \left\lfloor \frac{t}{T} \right\rfloor + \left\lfloor \frac{x}{T} \right\rfloor + 1$, we can focus our attention on the following two excluding cases:

Case 1: $\left\lfloor \frac{x+t}{T} \right\rfloor = \left\lfloor \frac{t}{T} \right\rfloor + \left\lfloor \frac{x}{T} \right\rfloor$

In this case, (3) can be written as

$$\overline{F}\left(x - \left\lfloor \frac{x}{T} \right\rfloor T\right)\overline{F}\left(t - \left\lfloor \frac{t}{T} \right\rfloor T\right) \geq \overline{F}\left(x - \left\lfloor \frac{x}{T} \right\rfloor T + t - \left\lfloor \frac{t}{T} \right\rfloor T\right). \tag{4}$$

Then, for (4) to hold it is sufficient that $\overline{F}(w)\overline{F}(z) \geq \overline{F}(w+z)$ for all $w \geq 0$ and $z \geq 0$ satisfying $w+z \in [0,T]$. So, the inequality (3) holds because $X \in$ NBU.

Case 2: $\left\lfloor \frac{x+t}{T} \right\rfloor = \left\lfloor \frac{t}{T} \right\rfloor + \left\lfloor \frac{x}{T} \right\rfloor + 1$

Now, (3) is equivalent to

$$\overline{F}\left(x - \left\lfloor \frac{x}{T} \right\rfloor T\right) \overline{F}\left(t - \left\lfloor \frac{t}{T} \right\rfloor T\right) \geq \overline{F}(T)\overline{F}\left(x + t - \left\lfloor \frac{x}{T} \right\rfloor T - \left\lfloor \frac{t}{T} \right\rfloor T - T\right). \tag{5}$$

Let us suppose, without lost of generality, that $t - \left\lfloor \frac{t}{T} \right\rfloor T \leq x - \left\lfloor \frac{x}{T} \right\rfloor T \leq T$. Then the following inequalities hold

$$T \geq x - \left\lfloor \frac{x}{T} \right\rfloor T \geq t - \left\lfloor \frac{t}{T} \right\rfloor T \geq x - \left\lfloor \frac{x}{T} \right\rfloor T + t - \left\lfloor \frac{t}{T} \right\rfloor T - T.$$

Hence (5) is true when $\overline{F}(x_1)\overline{F}(x_4) \leq \overline{F}(x_2)\overline{F}(x_3)$ for $x_1 \geq x_2 \geq x_3 \geq x_4$ such that $x_i \in [0,T]$, $i = 1,2,3,4$. So, (5) is satisfied when $\overline{F}(y-x)$ is a TP_2 function in (x,y), or equivalently by part $a)$ of Proposition 1.2, when $X \in$ IFR.

The proof when $X \in$ DFR is analogous using part $b)$ of Proposition 1.2. □

Remark 2.1. *It is easy to see that if λ is increasing (decreasing) in $[0,T]$, then $\tau_{X,T} \in$ NBU ($\tau_{X,T} \in$ NWU). Thus, $\tau_{X,T}$ could belong to NBU even if $X \notin$ IFR.*

When X has probability density, so does $\tau_{X,T}$ and its hazard rate, denoted by $\lambda_{X,T}$, satisfies $\lambda_{X,T}(t) = \lambda\left(t - \left\lfloor \frac{t}{T} \right\rfloor T\right)$, for all $t \geq 0$. As $\lambda_{X,T}$ is a periodic function, it is monotonic if and only if it is constant. So, $\tau_{X,T} \in$ IFR (DFR) for all $T > 0$ if and only if it is exponentially distributed, i.e. $\tau_{X,T} \notin$ IFR(DFR) except when there exists a $\lambda > 0$ such that $\overline{F}(t) = \mathrm{e}^{-\lambda t}$, for all $t \in [0,T]$, which implies $\overline{F}_{X,T}(t) = \mathrm{e}^{-\lambda t}$, for all $t \geq 0$. In Theorem 2.2 we set a stronger result considering the DMRL and IMRL aging classes.

First, let us denote by $\mu_{X,T}$ the *mean residual lifetime* of $\tau_{X,T}$ and denote $x = nT + h$, where $h \in [0,T)$ and $n = \lfloor x/T \rfloor \geq 0$. Next, we will obtain an expression for $\mu_{X,T}$. Note that

$$\mu_{X,T}(x) = \int_0^\infty \frac{\overline{F}_{X,T}(x+t)}{\overline{F}_{X,T}(x)} \mathrm{d}t = \frac{1}{\overline{F}_{X,T}(x)} \int_x^\infty \overline{F}_{X,T}(t)\mathrm{d}t. \tag{6}$$

Furthermore,

$$\begin{aligned}
\int_x^\infty \overline{F}_{X,T}(t)\mathrm{d}t &= \int_x^{(n+1)T} \overline{F}_{X,T}(t)\mathrm{d}t + \int_{(n+1)T}^\infty \overline{F}_{X,T}(t)\mathrm{d}t \\
&= \int_x^{(n+1)T} \overline{F}_{X,T}(t)\mathrm{d}t + \sum_{k=n+1}^\infty \int_{kT}^{(k+1)T} \overline{F}_{X,T}(t)\mathrm{d}t \\
&= \left[\overline{F}(T)\right]^n \int_{nT+h}^{(n+1)T} \overline{F}(t-nT)\,\mathrm{d}t + \sum_{k=n+1}^\infty \left[\overline{F}(T)\right]^k \int_{kT}^{(k+1)T} \overline{F}(t-kT)\,\mathrm{d}t.
\end{aligned}$$

Then,

$$\int_x^\infty \overline{F}_{X,T}(t)\mathrm{d}t = \left[\overline{F}(T)\right]^n \int_h^T \overline{F}(u)\mathrm{d}u + \frac{\left[\overline{F}(T)\right]^{n+1}}{1-\overline{F}(T)} \int_0^T \overline{F}(u)\mathrm{d}u. \tag{7}$$

Moreover,

$$\overline{F}_{X,T}(x) = \left[\overline{F}(T)\right]^n \overline{F}(h). \tag{8}$$

Thus, plugging (7) and (8) in (6) we get

$$\mu_{X,T}(x) = \frac{1}{\overline{F}(h)} \left(\int_h^T \overline{F}(u)\mathrm{d}u + \frac{\overline{F}(T)}{1-\overline{F}(T)} \int_0^T \overline{F}(u)\mathrm{d}u \right), \tag{9}$$

for $x = nT + h$, with $h \in [0,T)$, and $n = \lfloor x/T \rfloor \geq 0$.

Theorem 2.2. $\tau_{X,T} \in$ DMRL (IMRL) *if and only if it is exponentially distributed.*

Proof. Suppose $\mu_{X,T}$ is monotonic. As $\mu_{X,T}$ is a periodic function, it must be constant. Let us define $c = \mathbb{E}[\tau_{X,T}] = \mu_{X,T}(0)$. Using (9), we get $\mu_{X,T}$ is a constant function if and only if

$$\int_h^T \overline{F}(u)du = c[\overline{F}(h) - \overline{F}(T)], \text{ for } h \in [0,T].$$

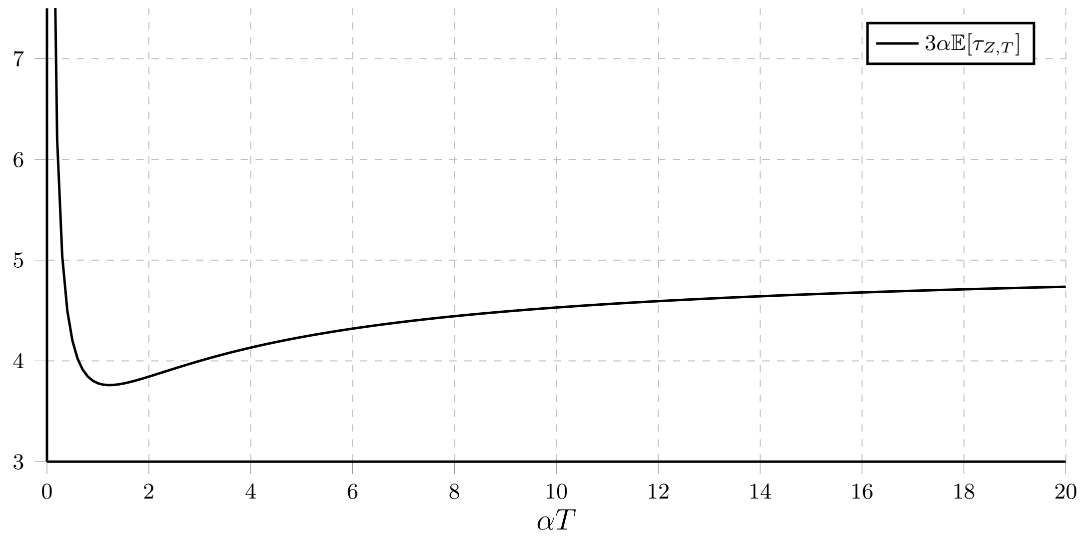

Figure 1: Plot of $3\alpha\mathbb{E}[\tau_{Z,T}]$ as a function of αT.

Solving the previous integral equation we have $\overline{F}(h) = \mathrm{e}^{-h/c}$, for $h \in [0,T]$. Consequently, using Eq. (1), we obtain that $\tau_{X,T}$ is exponentially distributed. The converse implication is trivially true. $\square$

Proposition 2.3. $\tau_{X,T} \in$ NBUE ($\tau_{X,T} \in$ NWUE) *if and only if* $\mathbb{E}[\tau_{X,h}] \geq (\leq) \mathbb{E}[\tau_{X,T}]$ *for all* $h \in [0,T]$.

Proof. Using (2) and (9) we know that $\tau_{X,T} \in$ NBUE if and only if

$$\int_h^T \overline{F}(u)\mathrm{d}u + \frac{\overline{F}(T)}{1-\overline{F}(T)} \int_0^T \overline{F}(u)\mathrm{d}u \leq \frac{\overline{F}(h)}{1-\overline{F}(T)} \int_0^T \overline{F}(u)\mathrm{d}u,$$

for $h \in [0,T)$. This inequality is equivalent to

$$\int_0^T \overline{F}(u)\mathrm{d}u - \int_0^h \overline{F}(u)\mathrm{d}u = \int_h^T \overline{F}(u)\mathrm{d}u \leq \frac{\overline{F}(h)-\overline{F}(T)}{1-\overline{F}(T)} \int_0^T \overline{F}(u)\mathrm{d}u.$$

Rearranging conveniently we obtain the equivalent inequality

$$\mathbb{E}[\tau_{X,T}] = \frac{1}{F(T)} \int_0^T \overline{F}(u)\mathrm{d}u \leq \frac{1}{F(h)} \int_0^h \overline{F}(u)\mathrm{d}u = \mathbb{E}[\tau_{X,h}],$$

which proves the proposition. The result when $\tau_{X,T} \in$ NWUE can be analogously obtained. $\square$

Example 1. *Let us consider a random variable Z with survival function*

$$\overline{F}_Z(t) = 1 - \left[1 - \frac{1}{(1+\alpha t)^2}\right]^2,$$

where $\alpha > 0$. Then, the expected value of $\tau_{Z,T}$ can be computed using (2) and is equal to

$$\mathbb{E}[\tau_{Z,T}] = \frac{1}{3\alpha} \frac{(\alpha T+1)(5\alpha^2T^2+9\alpha T+3)}{\alpha T(\alpha T+2)^2}, \text{ for } T > 0.$$

This example was considered first by Weiss [26]. In Figure 1 we can see that $3\alpha\mathbb{E}[\tau_{Z,T}]$ is not monotonic as a function of αT. Also, as α is fixed, the global minimum of $\mathbb{E}[\tau_{Z,T}]$ is attained at $\alpha T^\star = \sqrt{6}/2$. Thus, by Proposition 2.3 we have that $\tau_{Z,T} \in$ NBUE if and only if $T \leq \sqrt{6}/2\alpha$.

It can be proved that the random variable Z in Example 1 belongs to NWU. So, from a theoretical point of view, even when the random variable Z is NWU, it can be set up a replacement policy at age $T \in \left(0, \sqrt{6}/2\alpha\right)$ such that $\tau_{Z,T}$ is NBUE. However, from a practical point of view, as Weiss [26] points out, it is inadvisable to set a replacement policy up unless $\mathbb{E}[\tau_{Z,T}] \geq \lim_{T\to\infty} \mathbb{E}[\tau_{Z,T}] = \mathbb{E}[Z] = 5/3\alpha$, or equivalently $\alpha T \leq (\sqrt{34}-4)/6 \approx 0.3052$. Indeed, for a planned replacement at T such that $\mathbb{E}[\tau_{Z,T}] \leq \mathbb{E}[Z]$ it is better not to implement a replacement in the sense of the expected value.

The following result is a consequence of Proposition 2.3.

Corollary 2.4. *$X \in$ DMTFR (IMTFR) if and only if $\tau_{X,T} \in$ NBUE (NWUE) for all $T \geq 0$.*

In Theorem 2.1 we proved that $X \in$ IFR is a sufficient condition for $\tau_{X,T} \in$ NBU to hold. One could ask if $X \in$ NBU is a sufficient condition for $\tau_{X,T} \in$ NBU. The following example give us a negative answer to this question showing that $X \in$ NBU is not even a sufficient condition for $\tau_{X,T} \in$ NBUE to hold.

Example 2. *Let us consider the function*

$$h(x) = \begin{cases} x^2 & \text{if} \quad x \in [0, \frac{1}{2}] \\ -(x-1)^2 + \frac{1}{2} & \text{if} \quad x \in (\frac{1}{2}, 1] \\ \dfrac{x^2}{2} & \text{if} \quad x \in (1, \infty). \end{cases}$$

Note that $h(x)$ is increasing and continuous. Figure 2 (a) shows a plot of this function.

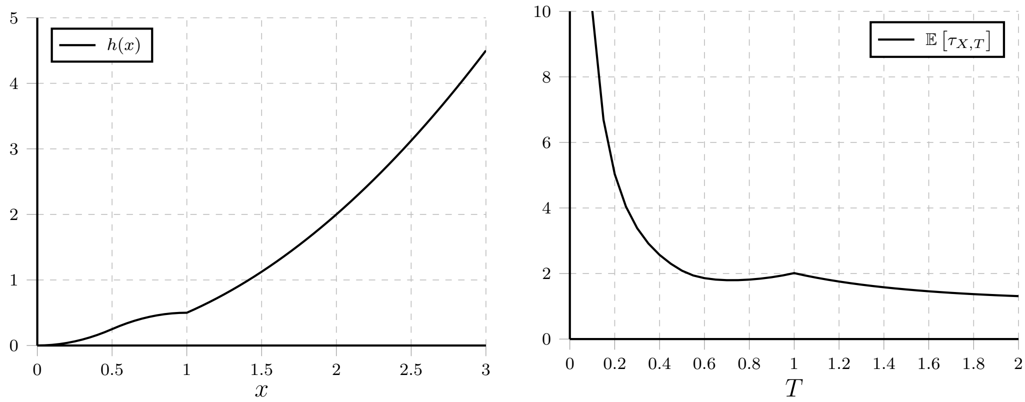

Figure 2: Plots of function $h(x)$ (left) and the expectation $\mathbb{E}[\tau_{X,T}]$ (right) as a function of T.

Let X be a random variable with survival function $\overline{F}(x) = \mathrm{e}^{-h(x)}$. It can be shown that $h(x+t) \geq h(x) + h(t)$ for all x and t in $\mathbb{R}_+$, thus $X \in$ NBU. Note that $h(x)$ is not a convex function, so $X \notin$ IFR. Also, as $\mathbb{E}[\tau_{X,T}]$ is not non-increasing in T (see Figure 2 (b)). Then, from Proposition 2.3, there are values of T such that $\tau_{X,T} \notin$ NBUE. For example, taking $T_1 = \frac{3}{4}$ and $T = 1$, we have

$$\mathbb{E}[\tau_{X,T_1}] = \frac{\int_0^{1/2} \mathrm{e}^{-x^2} \mathrm{d}x + \int_{1/2}^{3/4} \mathrm{e}^{(x-1)^2 - 1/2} \mathrm{d}x}{1 - \mathrm{e}^{-7/16}} \approx 1.7976,$$

$$\mathbb{E}[\tau_{X,T}] = \frac{\int_0^{1/2} \mathrm{e}^{-x^2} \mathrm{d}x + \int_{1/2}^{1} \mathrm{e}^{(x-1)^2 - 1/2} \mathrm{d}x}{1 - \mathrm{e}^{-1/2}} \approx 2.012.$$

Thus, even if $X \in$ NBU and $T_1 < T$, it holds that $\mathbb{E}[\tau_{X,T_1}] \leq \mathbb{E}[\tau_{X,T}]$ and consequently $\tau_{X,T} \notin$ NBUE.

Marshall and Proschan [17] prove that $X \leq_{\mathrm{st}} \tau_{X,T}$ ($X \geq_{\mathrm{st}} \tau_{X,T}$) for all $T > 0$ if and only if $X \in$ NBU ($X \in$ NWU). In a similar way, the following proposition provides necessary and sufficient conditions for stochastic orderings in the hazard rate and the likelihood ratio senses between the lifetime of a system with replacement and the lifetime of its units.

Proposition 2.5. *Let X be an absolutely continuous random variable. Then*

i) *$\tau_{X,T} \geq_{\mathrm{hr}} X$ ($\tau_{X,T} \leq_{\mathrm{hr}} X$), for all $T \geq 0$, if and only if $\dfrac{\overline{F}(x+y)}{\overline{F}(x)}$ is decreasing (increasing) for $x \in [0, y]$ and for all $y \geq 0$.*

ii) *$\tau_{X,T} \geq_{\mathrm{lr}} X$ ($\tau_{X,T} \leq_{\mathrm{lr}} X$), for all $T \geq 0$, if and only if $\dfrac{f(x+y)}{f(x)}$ is decreasing (increasing) for $x \in [0, y]$ and for all $y \geq 0$.*

Proof.

i) We will prove the result for $\tau_{X,T} \geq_{\text{hr}} X$. The proof for $\tau_{X,T} \leq_{\text{hr}} X$ is analogous.

Due to the continuity of $\overline{F}(t)$, the function

$$\frac{\overline{F}_{X,T}(t)}{\overline{F}(t)} = \frac{\overline{F}(T)^{\lfloor t/T \rfloor}\overline{F}\left(t - \left\lfloor \frac{t}{T} \right\rfloor T\right)}{\overline{F}(t)} \tag{10}$$

is also continuous. Then, to prove that (10) is increasing is equivalent to check the function $\dfrac{\overline{F}(t-(n-1)T)}{\overline{F}(t)}$ is increasing in $t \in [(n-1)T, nT)$, for all natural n. Let $x = t-(n-1)T$. Thus $\tau_{X,T} \geq_{\text{hr}} X$ if and only if $\dfrac{\overline{F}(x)}{\overline{F}(x+(n-1)T)}$ is increasing in $x \in [0,T]$ for all $T > 0$ and $n \geq 1$. Setting $n = 2$ and $T = y$, it is clear that $\dfrac{\overline{F}(x)}{\overline{F}(x+y)}$ is increasing for $x \in [0,y]$ and for all $y \geq 0$.

Now, we will prove the converse implication. Assume $\dfrac{\overline{F}(x)}{\overline{F}(x+y)}$ is increasing in $x \in [0,y]$, for all $y \geq 0$, and let m be a natural number. Then $\dfrac{\overline{F}(x)}{\overline{F}(x+mT)}$ is increasing for $x \in [0,T]$ because it is for $x \in [0,mT] \supset [0,T]$.

ii) The proof is very similar to the previous case.

□

The next two corollaries follow straightforward from Proposition 2.5.

Corollary 2.6.

a) If $X \in$ IFR*, then* $\tau_{X,T} \geq_{\text{hr}} X$ *for all* $T > 0$.

b) If $X \in$ ILR*, then* $\tau_{X,T} \geq_{\text{lr}} X$ *for all* $T > 0$.

Corollary 2.7. *If* $\sup_{t\in[0,T)} \lambda(t) \leq \inf_{t\in[T,\infty)} \lambda(t)$*, then* $\tau_{X,T} \geq_{\text{hr}} X$.

Corollary 2.7 shows that the ordering $\tau_{X,T} \geq_{\text{hr}} X$ can hold even when $X \notin$ IFR. For instance, consider a random variable X with cumulative hazard rate h as in Example 2. It can be proved that $\tau_{X,T} \geq_{\text{hr}} X$ for all $T \geq 1$, since the minimum is attained for $T = 1$.

2.2 Stochastic orderings

Let us consider two systems with age replacement whose units have lifetimes with the same distribution of the random variables X_1 and X_2, respectively. Block et al. [6] and Jain [8] prove the existence stochastic orderings between $\tau_{X_1,T}$ and $\tau_{X_2,T}$ when the same ordering hold between X_1 and X_2. They considered the usual stochastic order, the hazard rate order and the likelihood ratio order. The next theorem expands these results considering other stochastic orders.

Theorem 2.8. *The following statements hold,*

a) If $X_1 \geq_{\text{rh}} X_2$*, then* $\tau_{X_1,T} \geq_{\text{rh}} \tau_{X_2,T}$*, for all* $T \geq 0$.

b) If $\int_h^T \overline{F}_1(u)\mathrm{d}u / \int_h^T \overline{F}_2(u)\mathrm{d}u$ *is increasing in* $h \in (0,T)$*, then* $\tau_{X_1,T} \geq_{\text{mrl}} \tau_{X_2,T}$.

c) If $\overline{F}_1(T) \geq \overline{F}_2(T)$ *and*

$$\int_0^t \overline{F}_1(x)\mathrm{d}x \geq \int_0^t \overline{F}_2(x)\mathrm{d}x, \tag{11}$$

for all $t \in [0,T]$*, then* $\tau_{X_1,T} \geq_{\text{icv}} \tau_{X_2,T}$.

Proof.

a) The reversed hazard rate of $\tau_{X_i,T}$, denoted by $r_{X_i,T}(t)$, is equal to

$$\begin{aligned} r_{X_i,T}(t) &= \frac{\overline{F}_i(T)^{\lfloor t/T \rfloor} f_i(t-\lfloor t/T \rfloor T)}{1-\overline{F}_i(T)^{\lfloor t/T \rfloor} + \overline{F}_i(T)^{\lfloor t/T \rfloor} F_i(t-\lfloor t/T \rfloor T)} \\ &= \frac{r_i(t-\lfloor t/T \rfloor T)}{\frac{1-\overline{F}_i(T)^{\lfloor t/T \rfloor}}{\overline{F}_i(T)^{\lfloor t/T \rfloor} F_i(t-\lfloor t/T \rfloor T)} + 1}, \end{aligned} \tag{12}$$

where $r_i(t)$ is the reversed hazard rate of X_i, for $i = 1,2$.

It is well known that $\tau_{X_1,T} \geq_{\text{rh}} \tau_{X_2,T}$ is equivalent to $r_{X_1,T}(t) \geq r_{X_2,T}(t)$, for $t \geq 0$, which using (12) becomes equivalent to

$$r_1(t - \lfloor t/T \rfloor T) \left[\frac{1 - \overline{F}_2(T)^{\lfloor t/T \rfloor}}{\overline{F}_2(T)^{\lfloor t/T \rfloor} F_2(t - \lfloor t/T \rfloor T)} + 1 \right] \geq r_2(t - \lfloor t/T \rfloor T) \left[\frac{1 - \overline{F}_1(T)^{\lfloor t/T \rfloor}}{\overline{F}_1(T)^{\lfloor t/T \rfloor} F_1(t - \lfloor t/T \rfloor T)} + 1 \right].$$

As $r_1(t - \lfloor t/T \rfloor T) \geq r_2(t - \lfloor t/T \rfloor T)$ for all $t \geq 0$, it is sufficient to check the inequality

$$\frac{1 - \overline{F}_2(T)^{\lfloor t/T \rfloor}}{\overline{F}_2(T)^{\lfloor t/T \rfloor} F_2(t - \lfloor t/T \rfloor T)} \geq \frac{1 - \overline{F}_1(T)^{\lfloor t/T \rfloor}}{\overline{F}_1(T)^{\lfloor t/T \rfloor} F_1(t - \lfloor t/T \rfloor T)},$$

which can be written in the way

$$\frac{F_2(T)\left[1 + \overline{F}_2(T) + \cdots + \overline{F}_2(T)^{\lfloor t/T \rfloor - 1}\right]}{F_2(t - \lfloor t/T \rfloor T)\overline{F}_2(T)^{\lfloor t/T \rfloor}} \geq \frac{F_1(T)\left[1 + \overline{F}_1(T) + \cdots + \overline{F}_1(T)^{\lfloor t/T \rfloor - 1}\right]}{F_1(t - \lfloor t/T \rfloor T)\overline{F}_1(T)^{\lfloor t/T \rfloor}}. \tag{13}$$

Since $\overline{F}_1(T) \geq \overline{F}_2(T)$, a sufficient condition for the inequality (13) to hold is

$$\frac{F_2(T)}{F_2(t - \lfloor t/T \rfloor T)} \geq \frac{F_1(T)}{F_1(t - \lfloor t/T \rfloor T)}.$$

But this last inequality is equivalent to

$$\frac{F_2(T)}{F_1(T)} \geq \frac{F_2(t - \lfloor t/T \rfloor T)}{F_1(t - \lfloor t/T \rfloor T)},$$

and it is true because, as $X_1 \geq_{\text{rh}} X_2$, the function $\dfrac{F_2(t)}{F_1(t)}$ is increasing for all $t \geq 0$.

b) Using (7) it can be seen the ordering $\tau_{X_1,T} \geq_{\text{mrl}} \tau_{X_2,T}$ is equivalent to

$$\left[\frac{\overline{F}_1(T)}{\overline{F}_2(T)}\right]^n l(h) \tag{14}$$

being increasing in $h \in [0,T]$ and in $n \in \mathbb{N}$, where

$$l(h) = \left(\frac{\displaystyle\int_h^T \frac{\overline{F}_1(u)}{\overline{F}_1(T)} \mathrm{d}u + \mathbb{E}[\tau_{X_1,T}]}{\displaystyle\int_h^T \frac{\overline{F}_2(u)}{\overline{F}_2(T)} \mathrm{d}u + \mathbb{E}[\tau_{X_2,T}]} \right).$$

Observe that

$$\frac{\overline{F}_1(T)}{\overline{F}_2(T)} \geq \frac{\displaystyle\int_h^T \overline{F}_1(u)\mathrm{d}u}{\displaystyle\int_h^T \overline{F}_2(u)\mathrm{d}u} \geq \frac{\overline{F}_1(h)}{\overline{F}_2(h)}, \tag{15}$$

where the first inequality holds taking limits when h tends to T and the second inequality is true because by hypothesis the derivative of $\int_h^T \overline{F}_1(u)\mathrm{d}u / \int_h^T \overline{F}_2(u)\mathrm{d}u$ with respect to h is non-negative.

Taking $h = 0$ in (15) we have $\dfrac{\overline{F}_1(T)}{\overline{F}_2(T)} \geq \dfrac{\overline{F}_1(0)}{\overline{F}_2(0)} = 1$ and thus $\left[\dfrac{\overline{F}_1(T)}{\overline{F}_2(T)}\right]^n$ is increasing in n. Consequently, it only remains to prove that $l(h)$ is increasing in $h \in (0,T)$.

Deriving $l(h)$ we get it is increasing if and only if

$$\frac{\overline{F}_2(h)\displaystyle\int_h^T \overline{F}_1(u)\mathrm{d}u}{\overline{F}_1(T)\overline{F}_2(T)} + \frac{\overline{F}_2(h)}{\overline{F}_2(T)}\mathbb{E}[\tau_{X_1,T}] \geq \frac{\overline{F}_1(h)\displaystyle\int_h^T \overline{F}_2(u)\mathrm{d}u}{\overline{F}_1(T)\overline{F}_2(T)} + \frac{\overline{F}_1(h)}{\overline{F}_1(T)}\mathbb{E}[\tau_{X_2,T}]. \tag{16}$$

Note that the inequalities $\overline{F}_2(h)\int_h^T \overline{F}_1(u)\mathrm{d}u \geq \overline{F}_1(h)\int_h^T \overline{F}_2(u)\mathrm{d}u$ and $\dfrac{\overline{F}_2(h)}{\overline{F}_2(T)} \geq \dfrac{\overline{F}_1(h)}{\overline{F}_1(T)}$ hold due to (15).

Taking $h = 0$ in (15) we have $\int_0^T \overline{F}_1(u)\mathrm{d}u \geq \int_0^T \overline{F}_2(u)\mathrm{d}u$ and using $F_1(T) \leq F_2(T)$ we get $\mathbb{E}[\tau_{X_1,T}] \geq \mathbb{E}[\tau_{X_2,T}]$. Thus, the inequality (16) holds.

c) $\tau_{X_1,T} \geq_{\rm icv} \tau_{X_2,T}$ holds when

$$\int_0^t \overline{F}_{X_1,T}(u)\mathrm{d}u \geq \int_0^t \overline{F}_{X_2,T}(u)\mathrm{d}u,$$

for all $t \geq 0$. This inequality is equivalent to

$$\mathbb{E}[\tau_{X_1,T}] - \int_t^\infty \overline{F}_{X_1,T}(u)\mathrm{d}u \geq \mathbb{E}[\tau_{X_2,T}] - \int_t^\infty \overline{F}_{X_2,T}(u)\mathrm{d}u. \tag{17}$$

Using (2) and (7), the inequality (17) becomes

$$\frac{\int_0^T \overline{F}_1(u)\mathrm{d}u}{F_1(T)} - \left[\overline{F}_1(T)\right]^{n-1} \int_h^T \overline{F}_1(u)\mathrm{d}u - \frac{[\overline{F}_1(T)]^n}{F_1(T)} \int_0^T \overline{F}_1(u)\mathrm{d}u \geq$$

$$\frac{\int_0^T \overline{F}_2(u)\mathrm{d}u}{F_2(T)} - \left[\overline{F}_2(T)\right]^{n-1} \int_h^T \overline{F}_2(u)\mathrm{d}u - \frac{[\overline{F}_2(T)]^n}{F_2(T)} \int_0^T \overline{F}_2(u)\mathrm{d}u,$$

which is equivalent to

$$\frac{\int_0^T \overline{F}_1(u)\mathrm{d}u}{F_1(T)} \left[1 - \overline{F}_1(T)^{n-1}\right] + \left[\overline{F}_1(T)\right]^{n-1} \int_0^h \overline{F}_1(u)\mathrm{d}u \geq$$

$$\frac{\int_0^T \overline{F}_2(u)\mathrm{d}u}{F_2(T)} \left[1 - \overline{F}_2(T)^{n-1}\right] + \left[\overline{F}_2(T)\right]^{n-1} \int_0^h \overline{F}_2(u)\mathrm{d}u.$$

Note that

$$\frac{1-\overline{F}_i(T)^{n-1}}{F_i(T)} = 1 + \overline{F}_i(T) + \overline{F}_i(T)^2 + \cdots + \overline{F}_i(T)^{n-2}, \text{ for } i = 1,2.$$

Thus, the result comes from (11) and the inequality $\overline{F}_1(T)^k \geq \overline{F}_2(T)^k$ for all $k \geq 1$.

□

Let $X^{(1)}, X^{(2)}, \ldots$ be independent random variables with the same distribution of X such that $X^{(n)}$ is the lifetime of the unit which starts to work after the $(n-1)$th replacement. The number of replacements until the failure of the system is the random variable $\nu_X = \inf\{n \geq 1 : X^{(n)} < T\}$, which has geometric distribution with parameter $\mathbb{P}[X < T]$. Note that the lifetime $\tau_{X,T}$ can be expressed as a random sum of independent random variables as follows

$$\begin{aligned} \tau_{X,T} &= \sum_{n=1}^{\nu_X} \min\{X^{(n)}, T\} \\ &=_{\rm st} (\nu_X - 1)T + (X \mid X < T), \end{aligned} \tag{18}$$

where $=_{\rm st}$ stands for the equality in distribution. When we observe the operation of a system with replacement, it gives us information about the lifetime of its units conditioned to be smaller than T and about the number of replacements until failure, not about the (unconditional) lifetime of the units. Suppose we have two different kinds of units with lifetimes X_1 and X_2, respectively. Moreover, X_1 and X_2 have survival functions $\overline{F}_1$ and $\overline{F}_2$ and density functions f_1 and f_2, respectively. From a practical point of view it would be more useful to take advantage of the stochastic relations between $X_1' = (X_1 \mid X_1 < T)$ and $X_2' = (X_2 \mid X_2 < T)$ to decide which system has a greater lifetime, instead of the relations between the lifetimes of the units. The next result is similar to Theorem 2.8 but under this different approach.

Theorem 2.9. *The following statements hold*

a) If $\nu_{X_1} \geq_{\rm st} \nu_{X_2}$ and $(X_1 \mid X_1 < T) \geq_{\rm icx} (X_2 \mid X_2 < T)$, then $\tau_{X_1,T} \geq_{\rm icx} \tau_{X_2,T}$,

b) If $\nu_{X_1} \geq_{\rm st} \nu_{X_2}$ and $(X_1 \mid X_1 < T) \geq_{\rm icv} (X_2 \mid X_2 < T)$, then $\tau_{X_1,T} \geq_{\rm icv} \tau_{X_2,T}$,

c) If $\nu_{X_1} \geq_{\rm st} \nu_{X_2}$ and $(X_1 \mid X_1 < T) \geq_{\rm st} (X_2 \mid X_2 < T)$, then $\tau_{X_1,T} \geq_{\rm st} \tau_{X_2,T}$,

d) If $f_1(t) \leq f_2(t)$ for all $t \in [0,T]$ and $(X_1 \mid X_1 < T) \geq_{\rm hr} (X_2 \mid X_2 < T)$, then $\tau_{X_1,T} \geq_{\rm hr} \tau_{X_2,T}$,

e) If $f_1(t) \leq f_2(t)$ for all $t \in [0,T]$ and $(X_1 \mid X_1 < T) \geq_{\rm rh} (X_2 \mid X_2 < T)$, then $\tau_{X_1,T} \geq_{\rm rh} \tau_{X_2,T}$,

f) $(X_1 \mid X_1 < T) \geq_{\rm lr} (X_2 \mid X_2 < T)$ if and only if $\tau_{X_1,T} \geq_{\rm lr} \tau_{X_2,T}$.

Proof. Note that as ν_{X_1} and ν_{X_2} are geometric random variables, the ordering $\nu_{X_1} \geq_{\rm st} \nu_{X_2}$ is equivalent to $\overline{F}_1(T) \geq \overline{F}_2(T)$. In fact, the ordering $\nu_{X_1} \geq_{\rm st} \nu_{X_2}$ is also equivalent to the same ordering in the sense of all the stochastic orders defined in Definition 1.1, and to the inequality $\mathbb{E}[\nu_{X_1}] \geq \mathbb{E}[\nu_{X_2}]$.

It is well known, see [24], that the increasing concave, the increasing convex and the usual stochastic orders remain valid under sums of independent random variables. So, the results in *a*), *b*) and *c*) come from (18).

Let us denote by $\overline{F}_{X_i'}$ and $f_{X_i'}$ the survival and the density functions of $X_i' = (X_i \mid X_i < T)$, respectively, for $i = 1,2$. Using (18) we have

$$\begin{aligned} \overline{F}_{X_i,T}(t) &= \mathbb{P}\left[\nu_{X_i} = \left\lfloor \frac{t}{T} \right\rfloor + 1\right] \overline{F}_{X_i'}\left(t - \left\lfloor \frac{t}{T} \right\rfloor\right) + \mathbb{P}\left[\nu_{X_i} > \left\lfloor \frac{t}{T} \right\rfloor + 1\right] \\ &= p_i(1-p_i)^{\lfloor t/T \rfloor} \overline{F}_{X_i'}\left(t - \left\lfloor \frac{t}{T} \right\rfloor\right) + (1-p_i)^{\lfloor t/T \rfloor + 1}, \end{aligned} \tag{19}$$

where $p_i = F_i(T)$, for $i = 1,2$. From (19) we get

$$f_{X_i,T}(t) = p_i(1-p_i)^{\lfloor t/T \rfloor} f_{X_i'}\left(t - \left\lfloor \frac{t}{T} \right\rfloor\right). \tag{20}$$

It remains to prove *d*) and *e*). The hazard rate function of $\tau_{X_i,T}$ can be written as

$$\lambda_{X_i,T}(t) = \frac{p_i(1-p_i)^{\lfloor \frac{t}{T} \rfloor} f_{X_i'}\left(t - \lfloor \frac{t}{T} \rfloor\right)}{p_i(1-p_i)^{\lfloor t/T \rfloor} \overline{F}_{X_i'}\left(t - \lfloor \frac{t}{T} \rfloor\right) + (1-p_i)^{\lfloor t/T \rfloor + 1}} = \frac{p_i f_{X_i'}\left(t - \lfloor \frac{t}{T} \rfloor\right)}{p_i \overline{F}_{X_i'}\left(t - \lfloor \frac{t}{T} \rfloor\right) + (1-p_i)},$$

for $i = 1,2$.

The inequality $\lambda_{X_1,T}(t) \leq \lambda_{X_2,T}(t)$ is equivalent to

$$p_1 p_2 f_{X_1'}(x) \overline{F}_{X_2'}(x) + p_1(1-p_2) f_{X_1'}(x) \leq p_1 p_2 f_{X_2'}(x) \overline{F}_{X_1'}(x) + p_2(1-p_1) f_{X_2'}(x),$$

for all $x \in [0,T]$. Note that $p_1 p_2 f_{X_1'}(x) \overline{F}_{X_2'}(x) \leq p_1 p_2 f_{X_2'}(x) \overline{F}_{X_1'}(x)$ holds due to $X_1' \geq_{\rm hr} X_2'$. Also, as $p_1 \leq p_2$, a sufficient condition for $p_1(1-p_2) f_{X_1'}(x) \leq p_2(1-p_1) f_{X_2'}(x)$ to hold is $f_1(x) = p_1 f_{X_1'}(x) \leq p_2 f_{X_2'}(x) = f_2(x)$, for all $x \in [0,T]$, which is true by hypothesis. Consequently, $\tau_{X_1,T} \geq_{\rm hr} \tau_{X_2,T}$. Part *e*) is analogously proved.

Moreover, *f*) is trivially proved using (20). □

The ordering $(X_1 \mid X_1 < T) \geq_{\rm icv} (X_2 \mid X_2 < T)$ is equivalent to the inequality

$$\frac{\int_0^t F_1(x)\mathrm{d}x}{F_1(T)} \leq \frac{\int_0^t F_2(x)\mathrm{d}x}{F_2(T)} \tag{21}$$

for all $t \in [0,T]$. If $\overline{F}_1(T) \geq \overline{F}_2(T)$ and (11) hold, so does (21). Consequently, part *b*) of Theorem 2.9 is a particular case of part c) of Theorem 2.8. Also, note that $X_1 \geq_{\rm icv} X_2$ is sufficient for (11) to hold.

Weibull distribution with increasing hazard rate is one of the most commonly used distribution to model lifetimes of components in a system, see e.g. [18]. Consider the following example.

Example 3. *Suppose the lifetimes of the units of two systems with age replacements have Weibull distribution. Namely, X_i has Weibull distribution with survival function $\overline{F}_i(t) = \mathrm{e}^{-\lambda_i t^{\alpha_i}}$, for $t \geq 0$, where $\lambda_i > 0$ and $\alpha_i > 1$, for $i = 1,2$. If $\alpha_1 \neq \alpha_2$, then there is not a usual stochastic ordering between X_1 and X_2. The condition $(X_1 \mid X_1 < T) \geq_{\rm lr} (X_2 \mid X_2 < T)$ becomes equivalent to the function*

$$\beta(t) = \frac{\overline{F}_1'(t)}{\overline{F}_2'(t)} = \frac{\lambda_1 \alpha_1 t^{\alpha_1} \mathrm{e}^{-\lambda_1 t^{\alpha_1}}}{\lambda_2 \alpha_2 t^{\alpha_2} \mathrm{e}^{-\lambda_2 t^{\alpha_2}}}$$

being increasing for $t \in [0,T]$. Deriving $\beta(t)$ we get

$$\beta'(t) =_{sg} g(t) = \alpha_1 - \alpha_2 + \alpha_2 \lambda_2 t^{\alpha_2} - \alpha_1 \lambda_1 t^{\alpha_1}, \tag{22}$$

where $h_1(x) =_{sg} h_2(x)$ means there exists a real positive function $h(t)$ such that $h_1(t) = h(t) h_2(t)$, for all $t \geq 0$. Assume $\alpha_1 \geq \alpha_2$, then a sufficient condition for β being increasing in $[0,T]$ is that $T^{\alpha_1 - \alpha_2} \leq \frac{\alpha_2 \lambda_2}{\alpha_1 \lambda_1}$.

Now, suppose that $T^{\alpha_1 - \alpha_2} \leq \frac{\lambda_2}{\lambda_1}$. Note that

$$\begin{aligned} g(t) = (\alpha_1 - \alpha_2) + \alpha_2 \lambda_2 t^{\alpha_1}\left(t^{\alpha_2 - \alpha_1} - \frac{\alpha_1 \lambda_1}{\alpha_2 \lambda_2}\right) &\geq (\alpha_1 - \alpha_2) + \lambda_1 \alpha_2 t^{\alpha_1}\left(1 - \frac{\alpha_1}{\alpha_2}\right) \\ &\geq (\alpha_1 - \alpha_2)(1 - \lambda_1 T^{\alpha_1}), \end{aligned}$$

which is non-negative if $T^{\alpha_1} \leq \frac{1}{\lambda_1}$. So, using Theorem 2.9 we have $\tau_{X_1,T} \geq_{\rm lr} \tau_{X_2,T}$ if $\alpha_1 \geq \alpha_2$ and one of the following two conditions holds:

a) $T^{\alpha_1-\alpha_2} \leq \frac{\alpha_2\lambda_2}{\alpha_1\lambda_1}$,

b) $T^{\alpha_1-\alpha_2} \leq \frac{\lambda_2}{\lambda_1}$ *and* $T^{\alpha_1} \leq \frac{1}{\lambda_1}$,

even though $X_1 \not\geq_{\mathrm{st}} X_2$.

3 Results on systems with replacement at random time

Consider a non-negative absolutely continuous random variable Y, with finite mean, distribution function G, survival function $\overline{G}$ and density function g. Let us suppose that $X^{(n)}$ is the lifetime of the unit which starts to work after the $(n-1)$th replacement and $Y^{(n)}$ the random time between the $(n-1)$th and the nth replacements which has the same distribution of Y, for $n \geq 1$. Assume that $X^{(n)}$ and $Y^{(n)}$ are independent for all $n \geq 1$. Thus, the lifetime of the system with replacement at random time, denoted by $\tau_{X,Y}$, satisfies

$$\tau_{X,Y} =_{\mathrm{st}} \sum_{n=0}^{\nu_{X,Y}-1} \left(Y^{(n)} \mid X^{(n)} \geq Y^{(n)}\right) + \left(X^{(n)} \mid X^{(n)} < Y^{(n)}\right), \tag{23}$$

where $Y_0 = 0$, $X^{(n)} =_{\mathrm{st}} X$ and $Y^{(n)} =_{\mathrm{st}} Y$, for all $n \geq 1$, and $\nu_{X,Y} = \min\{n \geq 1 : X^{(n)} < Y^{(n)}\}$ has geometric distribution with parameter $\mathbb{P}[X < Y]$. Observe that the summands in (23) and $\nu_{X,Y}$ are independent.

From Theorem 1.A.4 in [24] we know that the usual stochastic ordering between random variables is preserved by random sums if the random number of summands also satisfy the usual stochastic ordering in the same sense, the summands are independent among them and independent of the number of summands. Thus, we are interested in comparing random variables with the same distribution as $(Y \mid X \geq Y)$ and $(X \mid X < Y)$. In order to do that we have the following result which has an independent interest.

Lemma 3.1. *Consider X_1, X_2, Y_1 and Y_2 absolutely continuous random variables such that X_i and Y_i are independent for $i = 1,2$. Let us suppose $Y_1 =_{\mathrm{st}} Y_2 =_{\mathrm{st}} Y$. The following two statements hold*

1. *If $X_1 \geq_{\mathrm{hr}} X_2$, then $(Y_1 \mid X_1 \geq Y_1) \geq_{\mathrm{st}} (Y_2 \mid X_2 \geq Y_2)$.*
2. *If $X_1 \geq_{\mathrm{rh}} X_2$, then $(X_1 \mid X_1 < Y_1) \geq_{\mathrm{st}} (X_2 \mid X_2 < Y_2)$.*

Proof. 1. It must be proved that for all $t \geq 0$

$$\mathbb{P}[Y_1 > t \mid X_1 \geq Y_1] \geq \mathbb{P}[Y_2 > t \mid X_2 \geq Y_2],$$

or equivalently

$$\frac{\mathbb{P}[Y_1 > t; X_1 \geq Y_1]}{\mathbb{P}[X_1 \geq Y_1]} \geq \frac{\mathbb{P}[Y_2 > t; X_2 \geq Y_2]}{\mathbb{P}[X_2 \geq Y_2]}.$$

This inequality can be written as

$$\frac{\int_t^\infty \overline{F}_1(x)\mathrm{d}G(x)}{\int_0^t \overline{F}_1(x)\mathrm{d}G(x) + \int_t^\infty \overline{F}_1(x)\mathrm{d}G(x)} \geq \frac{\int_t^\infty \overline{F}_2(x)\mathrm{d}G(x)}{\int_0^t \overline{F}_2(x)\mathrm{d}G(x) + \int_t^\infty \overline{F}_2(x)\mathrm{d}G(x)}.$$

Using Fubini's Theorem it is obtained the equivalent inequality

$$\int_t^\infty \int_0^t \overline{F}_1(x)\overline{F}_2(y)\mathrm{d}G(y)\mathrm{d}G(x) \geq \int_t^\infty \int_0^t \overline{F}_1(y)\overline{F}_2(x)\mathrm{d}G(y)\mathrm{d}G(x). \tag{24}$$

Thus, (24) holds if $\overline{F}_1(x)\overline{F}_2(y) \geq \overline{F}_1(y)\overline{F}_2(x)$, for $x \geq t$ y $0 \leq y \leq t$, which is true due to $X_1 \geq_{\mathrm{hr}} X_2$.

2. In a similar way to the previous case, it must be proved

$$\frac{\int_t^\infty \overline{G}(x)\mathrm{d}F_1(x)}{\int_0^\infty \overline{G}(x)\mathrm{d}F_1(x)} \geq \frac{\int_t^\infty \overline{G}(x)\mathrm{d}F_2(x)}{\int_0^\infty \overline{G}(x)\mathrm{d}F_2(x)}.$$

That is equivalent to

$$\int_t^\infty \int_0^t \overline{G}(x)\overline{G}(y)f_1(x)f_2(y)\,\mathrm{d}x\,\mathrm{d}y \geq \int_t^\infty \int_0^t \overline{G}(x)\overline{G}(y)f_2(x)f_1(y)\,\mathrm{d}x\,\mathrm{d}y. \tag{25}$$

Denote $\phi(x,y) = 1_{[t,\infty)}(x)1_{(-\infty,t]}(y)\overline{G}(x)\overline{G}(y)$. The inequality (25) is equivalent to $\mathbb{E}[\phi(X_1,X_2)] \geq \mathbb{E}[\phi(X_2,X_1)]$. When $x \leq y$ we have

$$\phi(x,y) - \phi(y,x) = -\overline{G}(x)\overline{G}(y)1_{(-\infty,t]}(x)1_{[t,\infty)}(y)$$

which is increasing in x. Thus, $\mathbb{E}[\phi(X_1,X_2)] \geq \mathbb{E}[\phi(X_2,X_1)]$ holds due to $\phi \in \mathscr{G}_{\mathrm{rh}}$ and part $a)$ of Proposition 1.1.

□

In the following theorem two systems with replacement at random time are compared in the usual stochastic order.

Theorem 3.2. *If $X_1 \geq_{\mathrm{hr}} X_2$, $X_1 \geq_{\mathrm{rh}} X_2$ and $Y_1 =_{\mathrm{st}} Y_2$, then $\tau_{X_1,Y_1} \geq_{\mathrm{st}} \tau_{X_2,Y_2}$.*

Proof. Note that $\nu_{X_1,Y_1} \geq_{\mathrm{st}} \nu_{X_2,Y_2}$ due to $\mathbb{P}[X_1 < Y_1] \leq \mathbb{P}[X_2 < Y_2]$. Now, using (23), Lemma 3.1 and Theorem 1.A.4 in [24] we get the desired result. □

Remark 3.1. *The conditions in Theorem 3.2 hold, for instance, when $X_1 \geq_{\mathrm{lr}} X_2$ and $Y_1 =_{\mathrm{st}} Y_2$. As an example, consider X_1 and X_2 with gamma distribution with shape parameter a_i and scale parameter b_i, for $i = 1,2$. Assume $a_1 \geq a_2$ and $b_1 \leq b_2$. In this case $X_1 \geq_{\mathrm{lr}} X_2$ holds (see e.g. Lemma 1 in [19]). Thus, $\tau_{X_1,Y_1} \geq_{\mathrm{st}} \tau_{X_2,Y_2}$.*

Let $\Phi_{X,Y}$ be the survival function of $\tau_{X,Y}$ and consider the Laplace transform of $\Phi_{X,Y}$, denoted by $\hat{\Phi}_{X,Y}$, defined by

$$\hat{\Phi}_{X,Y}(s) = \int_0^\infty \Phi(t)\mathrm{e}^{-st}\mathrm{d}t,$$

for all $s \geq 0$. It is not difficult to see that

$$\hat{\Phi}_{X,Y}(s) = \frac{\displaystyle\int_0^\infty \overline{F}(x)\overline{G}(x)\mathrm{e}^{-sx}\mathrm{d}x}{\displaystyle\int_0^\infty f(x)\overline{G}(x)\mathrm{e}^{-sx}\mathrm{d}x + s\int_0^\infty \overline{F}(x)\overline{G}(x)\mathrm{e}^{-sx}\mathrm{d}x}. \tag{26}$$

Theorem 3.3. *If $X_1 =_{\mathrm{st}} X_2$, $Y_1 \leq_{\mathrm{hr}} Y_2$ ($Y_1 \geq_{\mathrm{hr}} Y_2$) and $X_1, X_2 \in$ IFR ($X_1, X_2 \in$ DFR), then $\tau_{X_1,Y_1} \geq_{\mathrm{Lt}} \tau_{X_2,Y_2}$.*

Proof. Let $\hat{\Phi}_{X_1,Y_1}$ and $\hat{\Phi}_{X_2,Y_2}$ be the Laplace transforms of Φ_{X_1,Y_1} and Φ_{X_2,Y_2}, respectively. Let $\overline{G}_1$ and $\overline{G}_2$ be the survival functions of Y_1 and Y_2, respectively. We need to prove that, $\hat{\Phi}_{X_1,Y_1}(s) \geq \hat{\Phi}_{X_2,Y_2}(s)$, for all $s \geq 0$ which, due to (26), is equivalent to

$$\frac{\displaystyle\int_0^\infty f(x)\overline{G}_1(x)\mathrm{e}^{-sx}\mathrm{d}x}{\displaystyle\int_0^\infty \overline{F}(x)\overline{G}_1(x)\mathrm{e}^{-sx}\mathrm{d}x} \leq \frac{\displaystyle\int_0^\infty f(x)\overline{G}_2(x)\mathrm{e}^{-sx}\mathrm{d}x}{\displaystyle\int_0^\infty \overline{F}(x)\overline{G}_2(x)\mathrm{e}^{-sx}\mathrm{d}x}.$$

Rearranging the terms and using Fubini's Theorem, the previous expression becomes equivalent to

$$\iint\limits_{\mathbb{R}_+^2} \mathrm{e}^{-s(x+y)}\overline{G}_1(x)\overline{G}_2(y)\overline{F}(x)f(y)\mathrm{d}x\mathrm{d}y \leq \iint\limits_{\mathbb{R}_+^2} \mathrm{e}^{-s(x+y)}\overline{G}_1(x)\overline{G}_2(y)\overline{F}(y)f(x)\mathrm{d}x\mathrm{d}y,$$

which is equivalent to

$$I = \iint\limits_{\mathbb{R}_+^2} \mathrm{e}^{-s(x+y)}\overline{G}_1(x)\overline{G}_2(y)\overline{F}(x)\overline{F}(y)[\lambda(x) - \lambda(y)]\mathrm{d}x\mathrm{d}y \geq 0 \tag{27}$$

where λ is the hazard rate of X_1 and X_2. Let us define

$$h(x,y) = \mathrm{e}^{-s(x+y)}\overline{G}_1(x)\overline{G}_2(y)\overline{F}(x)\overline{F}(y)[\lambda(x) - \lambda(y)].$$

Then,

$$\begin{aligned}
I &= \iint\limits_{x\geq y} h(x,y)\mathrm{d}x\mathrm{d}y + \iint\limits_{x\leq y} h(x,y)\mathrm{d}x\mathrm{d}y \\
&= \iint\limits_{x\geq y} [h(x,y) + h(y,x)]\mathrm{d}x\mathrm{d}y \\
&= \iint\limits_{x\geq y} \mathrm{e}^{-s(x+y)}\overline{F}(x)\overline{F}(y)[\lambda(x) - \lambda(y)] \times [\overline{G}_1(x)\overline{G}_2(y) - \overline{G}_1(y)\overline{G}_2(x)]\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Finally, the last expression is positive when $Y_1 \leq_{\mathrm{hr}} Y_2$ and $X_1 \in$ IFR or $Y_1 \geq_{\mathrm{hr}} Y_2$ and $X_1 \in$ DFR. □

As a consequence of Theorems 3.2 and 3.3 we have the following result.

Corollary 3.4. *If $X_1 \geq_{\rm hr} X_2$, $X_1 \geq_{\rm rh} X_2$ and one of the following conditions holds*

a) $Y_1 \leq_{\rm hr} Y_2$ *and* $X_1 \in \mathrm{IFR}$ *or* $X_2 \in \mathrm{IFR}$,

b) $Y_1 \geq_{\rm hr} Y_2$ *and* $X_1 \in \mathrm{DFR}$ *or* $X_2 \in \mathrm{DFR}$,

then $\tau_{X_1,Y_1} \geq_{\rm Lt} \tau_{X_2,Y_2}$.

Example 4. *Let us suppose that $X_1 \geq_{\rm hr} X_2$, $X_1 \geq_{\rm rh} X_2$ and $X_2 \in \mathrm{DFR}$. Then, using Corollary 3.4 we get $X_1 \geq_{\rm Lt} \tau_{X_2,Y_2}$, for every random variable Y_2. From a practical point of view this means that it is better, in the sense of Laplace order, the system formed by a unit with lifetime X_1 and without replacement than the system formed by units with lifetimes distributed as X_2 and replacement after a random time Y_2, for any Y_2 independent of X_2.*

Taking $s = 0$ in (26) we obtain

$$\mathbb{E}[\tau_{X,Y}] = \frac{\mathbb{E}[X \wedge Y]}{\mathbb{P}[Y > X]}. \tag{28}$$

Consider the following result,

Lemma 3.5. *Let X and Y be independent random variables with survival functions $\overline{F}$ and $\overline{G}$, respectively. Then,*

$$\begin{aligned} \mathbb{E}[X \wedge Y] &= \mathbb{E}\left[\mathscr{H}\overline{G}(X)\right]. & (29)\\ &= \mathbb{E}\left[\mathscr{H}\overline{F}(Y)\right], & (30) \end{aligned}$$

where $\mathscr{H}$ is the operator satisfying

$$\mathscr{H}h(x) = \int_0^x h(u)\mathrm{d}u, \quad \textit{for all } x \geq 0.$$

Proof. Note that

$$\begin{aligned} \mathbb{E}[X \wedge Y] &= \int_0^\infty \overline{G}(x)\overline{F}(x)\mathrm{d}x = \int_0^\infty \overline{F}(x)\,\mathrm{d}\left(\int_0^x \overline{G}(u)\mathrm{d}u\right)\\ &= \int_0^\infty \left(\int_0^x \overline{G}(u)\mathrm{d}u\right)\mathrm{d}F(x) = \mathbb{E}\left[\mathscr{H}\overline{G}(X)\right]. \end{aligned}$$

Equality (30) is analogously proved. □

In order to compare the mean values of τ_{X_1,Y_1} and τ_{X_2,Y_2} we have the following proposition.

Proposition 3.6. *If $X_1 \geq_{\rm st} X_2$, $Y_1 \leq_{\rm lr} Y_2$ ($Y_1 \geq_{\rm lr} Y_2$) and $X_1 \in \mathrm{DMTFR}$ or $X_2 \in \mathrm{DMTFR}$ ($X_1 \in \mathrm{IMTFR}$ or $X_2 \in \mathrm{IMTFR}$), then $\mathbb{E}\left[\tau_{X_1,Y_1}\right] \geq \mathbb{E}\left[\tau_{X_2,Y_2}\right]$.*

Proof. Suppose first that $X_1 =_{\rm st} X_2 =_{\rm st} X \in \mathrm{DMTFR}$ and $Y_1 \leq_{\rm lr} Y_2$. Using (30) we get

$$\mathbb{E}\left[\tau_{X_i,Y_i}\right] = \frac{\mathbb{E}\left[\mathscr{H}\overline{F}(Y_i)\right]}{\mathbb{E}[F(Y_i)]}.$$

Thus, the inequality $\mathbb{E}[\tau_{X_1,Y_1}] \geq \mathbb{E}[\tau_{X_2,Y_2}]$ is equivalent to

$$\mathbb{E}\left[F(Y_1)F(Y_2)\frac{\mathscr{H}\overline{F}(Y_1)}{F(Y_1)}\right] \geq \mathbb{E}\left[F(Y_1)F(Y_2)\frac{\mathscr{H}\overline{F}(Y_2)}{F(Y_2)}\right]. \tag{31}$$

Consider $\phi(x,y) = F(x)F(y)\dfrac{\int_0^y \overline{F}(u)\mathrm{d}u}{F(y)}$. As $X_1 \in \mathrm{DMTFR}$ we have that

$$\phi(x,y) - \phi(y,x) = F(x)F(y)\left(\frac{\int_0^y \overline{F}(u)\mathrm{d}u}{F(y)} - \frac{\int_0^x \overline{F}(u)\mathrm{d}u}{F(x)}\right) \geq 0,$$

for all $x \geq y$. Thus, the inequality (31) is equivalent to $\mathbb{E}[\phi(Y_1,Y_2)] \leq \mathbb{E}[\phi_1(Y_2,Y_1)]$ for $\phi \in \mathscr{G}_{\rm lr}$, which holds from part b) of Proposition 1.1.

Suppose now that $Y_1 =_{\rm st} Y_2$. A sufficient condition for the inequality $\mathbb{E}[\tau_{X_1,Y_1}] \geq \mathbb{E}[\tau_{X_2,Y_2}]$ to hold is $\mathbb{E}[X_1 \wedge Y_1] \geq \mathbb{E}[X_2 \wedge Y_2]$ and $\mathbb{P}[Y_1 > X_1] \leq \mathbb{P}[Y_2 > X_2]$. It is not difficult to check that these inequalities hold when $X_1 \geq_{\rm st} X_2$.

In the general case assume, without lost of generality, that $X_1 \in \mathrm{DMTFR}$. Then $\mathbb{E}[\tau_{X_1,Y_1}] \geq \mathbb{E}[\tau_{X_1,Y_2}]$ using a), and $\mathbb{E}[\tau_{X_1,Y_2}] \geq \mathbb{E}[\tau_{X_2,Y_2}]$ using b). Consequently, $\mathbb{E}[\tau_{X_1,Y_1}] \geq \mathbb{E}[\tau_{X_2,Y_2}]$. □

3.1 Stationary pointwise availability

Let us assume we only can detect if the system is not working when a replacement is done. After a system failure, when the next replacement takes place, a renewal occurs and the system starts to work. Thus, at every instant there is a probability the system is working or not. We define the *pointwise availability* of the system at instant t, denoted by $A_{X,Y}(t)$ as the probability that it is working at the instant t. The *stationary pointwise availability*, denoted by $A_{X,Y}$, is defined as the limit of $A_{X,Y}(t)$ as t tends to infinity. Let us denote by $\overline{F}$ and $\overline{G}$ the survival functions of the random variables X and Y, respectively, and assume X and Y have finite means. The following integral equation holds,

$$A_{X,Y}(t) = \overline{F}(t)\overline{G}(t) + \int_0^t A_{X,Y}(t-x)\,\mathrm{d}G(x). \tag{32}$$

Thus, using the well known *Key Renewal Theorem* (see e.g. Theorems 3.4.2 and 3.4.4 in [23]) we get

$$A_{X,Y} = \frac{\int_0^\infty \overline{F}(x)\overline{G}(x)\mathrm{d}x}{\int_0^\infty \overline{G}(x)\mathrm{d}x} = \frac{\mathbb{E}[X \wedge Y]}{\mathbb{E}[Y]}. \tag{33}$$

Consider the random variable *asymptotic equilibrium age*, denoted by Y^e, associated to a random variable Y with finite mean, which has distribution function

$$G^e(t) = \frac{\int_0^t \overline{G}(x)\mathrm{d}x}{\mathbb{E}[Y]}.$$

See e.g. [24] for more details about the asymptotic equilibrium age. The asymptotic equilibrium age random variable has been used to study the DMTFR aging class. Kayid et al. [10] prove that if $Y \in$ DMLR then $Y^e \in$ DMTFR. Also, they prove that the reverse implication holds when $\int_0^t \overline{G}(x)\mathrm{d}x/t\overline{G}(t)$ is decreasing in t.

The following theorem deals with the comparison of the pointwise stationary availabilities of two systems with different distributions of lifetimes and replacement times.

Theorem 3.7. *If $X_1 \geq_{\text{icv}} X_2$ and $Y_1^e \leq_{\text{st}} Y_2^e$, then $A_{X_1,Y_1} \geq A_{X_2,Y_2}$.*

Proof. Consider $X_1 =_{\text{st}} X_2 =_{\text{st}} X$ and $Y_1 \leq_{\text{icx}} Y_2$. By (33), the inequality $A_{X_1,Y_1} \geq A_{X_2,Y_2}$ is equivalent to

$$\mathbb{E}\left[\overline{F}\left(Y_1^e\right)\right] = \int_0^\infty \overline{F}(x)\frac{\overline{G}_1(x)}{\mathbb{E}[Y_1]}\mathrm{d}x \geq \int_0^\infty \overline{F}(x)\frac{\overline{G}_2(x)}{\mathbb{E}[Y_2]}\mathrm{d}x = \mathbb{E}\left[\overline{F}\left(Y_2^e\right)\right]. \tag{34}$$

It is well know that the stochastic ordering $Y_1^e \leq_{\text{st}} Y_2^e$ is equivalent to $\mathbb{E}\left[\phi(Y_1^e)\right] \leq \mathbb{E}\left[\phi(Y_2^e)\right]$ for all increasing function ϕ such that the expectations exist (cf. [24]). As $\overline{F}$ is decreasing, the inequality (34) holds because $Y_1^e \leq_{\text{st}} Y_2^e$.

Now assume $Y_1 =_{\text{st}} Y_2$. In this case the inequality $A_{X_1,Y_1} \geq A_{X_2,Y_2}$, using (33), is equivalent to $\mathbb{E}[X_1 \wedge Y_1] \geq \mathbb{E}[X_2 \wedge Y_2]$. From the proof of Proposition 3.6 we know that this inequality holds when

$$\mathbb{E}[X_1 \wedge Y_1] = \mathbb{E}[\mathscr{H}\overline{G}(X_1)] \geq \mathbb{E}[\mathscr{H}\overline{G}(X_2)] = \mathbb{E}[X_2 \wedge Y_2].$$

Note that $\mathscr{H}\overline{G}(x)$ is increasing and concave for $x \geq 0$. Moreover, $X_1 \geq_{\text{icv}} X_2$ is equivalent to $\mathbb{E}[\phi(X_1)] \geq \mathbb{E}[\phi(X_2)]$ for every real function ϕ increasing and concave such that the expectation exists (cf. [24]). Thus, $A_{X_1,Y_1} \geq A_{X_2,Y_2}$ holds. The general result is obtained using the transitivity property, i.e. $A_{X_1,Y_1} \geq A_{X_2,Y_1} \geq A_{X_2,Y_2}$. □

As an example of ordering $X_1 \geq_{\text{icv}} X_2$, consider X_i with gamma distribution with shape parameter a_i and scale parameter b_i, where $a_i, b_i > 0$, for $i = 1,2$. We have that the ordering $X_1 \geq_{\text{icv}} X_2$ holds when it is satisfied one of the following two conditions:

- $a_1 \geq a_2$ and $b_1 \leq b_2$, or
- $a_1 \geq a_2$, $b_1 \geq b_2$ and $a_1/b_1 \leq a_2/b_2$.

See e.g. Lemma 1 in [19]. On the other hand, the ordering $Y_1^e \leq_{\text{st}} Y_2^e$ holds, for example, when

- $\mathbb{E}[Y_1] = \mathbb{E}[Y_2]$ and $Y_1 \leq_{\text{icx}} Y_2$, or

- $Y_1 \leq_{\rm mrl} Y_2$, which is equivalent to $Y_1^e \leq_{\rm hr} Y_2^e$ (Theorem 2.A.4 in [24]), or
- $Y_1 \leq_{\rm hr} Y_2$, which is equivalent to $Y_1^e \leq_{\rm lr} Y_2^e$.

Using Proposition 3.6 and Theorem 3.7 we can compare two systems with replacement at random time in the sense of the mean lifetime and the stationary availability under weaker conditions that those of Corollary 3.4, which assures a stochastic ordering in the Laplace sense. This is one of the reasons why this comparison becomes interesting. Note also that when $X_1 \geq_{\rm st} X_2$, $Y_1 \leq_{\rm lr} Y_2$ and $X_1 \in$ DMTFR or $X_2 \in$ DMTFR we get both $\mathbb{E}\left[\tau_{X_1,Y_1}\right] \geq \mathbb{E}\left[\tau_{X_2,Y_2}\right]$ and $A_{X_1,Y_1} \geq A_{X_2,Y_2}$ but we cannot assure that $\tau_{X_1,Y_1} \geq_{\rm Lt} \tau_{X_2,Y_2}$ holds.

4 Conclusions

Standing out as a novelty with respect to previous research on this topic, and, consequently filling a gap on the matter, this paper continues the research on the aging properties of the lifetime of the classical system with age replacement using properties of the lifetimes of the working units. We mainly focussed on the aging class to which the lifetime of the system belongs, according to the aging properties of the lifetime of the working unit. As a matter of fact, Theorem 2.8 extends the comparisons of the lifetimes of systems with age replacement to other notions of stochastic orders than the previous considered. Moreover, in Theorem 2.9 we also compare the lifetimes of the systems with age replacement using comparisons between the number of replacements and the lifetimes of the working units conditioned to be less or equal than the replacement time, which seems interesting from the practical point of view.

Finally, we have additionally worked upon the lifetimes of systems with age replacement, studying the case where the replacement time is random. In this concern, our results include usual stochastic orderings, Laplace transform orderings and comparisons of the expected lifetimes and the stationary availabilities. The existence of stronger stochastic orderings for this model is a topic of further research.

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