Multi-dimensional backward stochastic differential equations of diagonally quadratic generators: the general result

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Multi-dimensional backward stochastic differential equations of diagonally quadratic generators: the general result[☆]

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Abstract

This paper is devoted to a general solvability of a multi-dimensional backward stochastic differential equation (BSDE) of a diagonally quadratic generator $g(t, y, z)$, by relaxing the assumptions of Hu and Tang [15] on the generator and terminal value. More precisely, the generator $g(t, y, z)$ can have more general growth and continuity in y in the local solution; while in the global solution, the generator $g(t, y, z)$ can have a skew sub-quadratic but in addition "strictly and diagonally" quadratic growth in the second unknown variable z, or the terminal value can be unbounded but the generator $g(t, y, z)$ is "diagonally dependent" on the second unknown variable z (i.e., the i-th component g^i of the generator g only depends on the i-th row z^i of the variable z for each $i = 1, \cdots, n$). Three new results are established on the local and global solutions when the terminal value is bounded and the generator g is subject to some general assumptions. When the terminal value is unbounded but is of exponential moments of arbitrary order, an existence and uniqueness result is given under the assumptions that the generator $g(t, y, z)$ is Lipschitz continuous in the first unknown variable y, and varies with the second unknown variable z in a "diagonal" , "component-wisely convex or concave", and "quadratically growing" way, which seems to be the first general solvability of system of quadratic BSDEs with unbounded terminal values. This generalizes and strengthens some existing results via some new ideas.

Keywords: Multi-dimensional BSDE, diagonally quadratic generator, convex generator,

BMO martingale, unbounded terminal value.

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1. Introduction

Fix a terminal time $T \in (0, +\infty)$ and two positive integers n and d. Let $(B_t)_{t \in [0,T]}$ be a d-dimensional standard Brownian motion defined on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $(\mathcal{F}_t)_{t \in [0,T]}$ be the augmented natural filtration generated by the standard Brownian motion B. Consider the following backward stochastic differential equation (BSDE in short):

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s) \mathrm{d}s - \int_t^T Z_s \mathrm{d}B_s, \quad t \in [0, T], \tag{1.1}$$

where the terminal value ξ is an $\mathcal{F}_T$-measurable n-dimensional random vector, the generator function $g(\omega, t, y, z) : \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \to \mathbb{R}^n$ is $(\mathcal{F}_t)$-progressively measurable for each pair (y, z), and the solution $(Y_t, Z_t)_{t \in [0,T]}$ is a pair of $(\mathcal{F}_t)$-progressively measurable processes with values in $\mathbb{R}^n \times \mathbb{R}^{n \times d}$ which almost surely verifies BSDE (1.1). The history of BSDEs (1.1) can be dated back to Bismut [2] for the linear case, and to Bismut [3] for a specifically structured matrix-valued nonlinear case where the matrix-valued generator contains a quadratic form of the second unknown. In 1990, Pardoux and Peng [24] established the existence and uniqueness result for a multidimensional ($n \geq 1$) nonlinear BSDE with a uniformly Lipschitz continuous generator. Subsequently, there has been an increasing interest in BSDEs with applications in various fields such as stochastic control, mathematical finance, partial differential equations (PDEs).

The class of BSDEs, with generators having a quadratic growth in the state variable z, has attracted a lot of attention in recent years. On the one hand, the existence and uniqueness theory is well developed in the scalar ($n = 1$) case. Kobylanski [19] established the first existence and uniqueness result for scalar-valued quadratic BSDEs with bounded terminal values, and some subsequent intensive efforts can be founded in Tevzadze [26], Briand and Elie [4], Fan [10] and Luo and Fan [22] for the bounded terminal value case, and in Briand and Hu [5, 6], Delbaen et al. [8, 9], Barrieu and El Karoui [1] and Fan et al. [11] for the unbounded terminal value case. On the other hand, Frei and Dos Reis [13] constructed an example of multidimensional quadratic BSDE with a simple generator and a bounded terminal value to show that the equation might fail to have a global bounded solution on $[0, T]$, which illustrates the difficulty of the quadratic part contributing to the underlying scalar generator as an unbounded process. Moreover, it is well known that some tools used when $n = 1$, like Girsanov's transform and monotone convergence, can no longer be applied when $n > 1$ in most cases. Consequently, multidimensional quadratic BSDEs, the focus of the present paper, pose a great challenge. Solutions of multidimensional quadratic BSDEs with unbounded terminal values have been listed as an open problem in Peng [25, Section 5, page 270].

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Nevertheless, motivated by their intrinsic mathematical interest and especially by diverse applications in various fields, such as nonzero-sum risk-sensitive stochastic differential games, financial price-impact models, financial market equilibrium problems for several interacting agents, and stochastic equilibria problems in incomplete financial markets, many scholars have studied systems of quadratic BSDEs in recent years. First of all, by the theory of BMO (bounded in mean oscillation) martingales and using the contract mapping argument, Tevzadze [26] proved a general existence and uniqueness result for multi-dimensional quadratic BSDEs when the terminal value is small enough in the supremum norm, which has inspired subsequent works under some different types of "smallness" assumptions on the terminal value and the generator, see for example Frei [12], Kardaras et al. [17], Jamneshan et al. [16], Kramkov and Pulido [21, 20] and Harter and Richou [14]. We also note that some different ideas and methods have been applied in these works mentioned above. Secondly, in the Markovian setting, Cheridito and Nam [7] proved the solvability for a special system of quadratic BSDEs, and Xing and Žitković [27] obtained, by virtue of analytic PDE methods, the global solvability for a large class of multidimensional quadratic BSDEs with weak regularity assumptions on the terminal value and the generator. Finally, by utilizing the Girsanov transform and adopting a distinct idea from the above works, which is to search for some sufficient conditions on the generator such that the corresponding system of quadratic BSDEs admits a (unique) local or global solution for any bounded terminal values rather than some certain terminal values, Cheridito and Nam [7], Hu and Tang [15] and Luo [23] respectively established several existence and uniqueness results of local and global solutions for systems of BSDEs with some structured quadratic generators. More specifically, Cheridito and Nam [7] investigated system of BSDEs with projectable quadratic generators and subquadratic generators, Hu and Tang [15] addressed a kind of multi-dimensional BSDEs with diagonally quadratic generators, in which the ith $(i = 1, \cdots , n)$ component g^i of the generator g has a quadratic growth only on the ith row of the matrix z, and Luo [23] considered a class of multi-dimensional BSDEs with triangularly quadratic generators. We would like to mention that all of these results mentioned in this paragraph are obtained under the bounded terminal value condition, and up to our best knowledge, in existing literature there seems to be no positive general solvability result on the system of quadratic BSDEs with unbounded terminal values.

The present paper is the continuation and extension of Hu and Tang [15]. Under more general assumptions on the generator and the terminal condition than those used in [15], we are devoted to the general solvability of multidimensional diagonally quadratic BSDEs. A local solution is first constructed by virtue of uniform a priori estimates on the solution of scalar-valued BSDEs

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and the fixed-point argument, where the terminal value is bounded and the generator g can have a general growth in the variable y. We note that a simpler and more direct idea than that used in the proof of [15, Theorem 2.2] is used to obtain the radius of the centered ball within which the constructed mapping is stable. Then, by stitching local solutions we proved two existence and uniqueness results on global solution of system of diagonally quadratic BSDEs with bounded terminal values, where the generator g needs to satisfy an additional one-sided growth condition with respect to the variable y. In particular, we eliminate the restriction condition used in [15] that the ith component of the generator g is bounded with respect to the jth ($j \neq i$) row of the matrix z by imposing a strictly quadratic condition on the generator g (see assumption (H5) in Section 2). Finally, assuming that for each $i = 1, \cdots, n$, the ith component g^i of the generator g is Lipschitz continuous in the state variable y, depends only on the ith row z^i of the state variable z, and is either convex or concave with quadratic growth in z^i, utilizing the iterative algorithm together with uniform a priori estimates and the θ-method, we prove existence and uniqueness of the global solution to the multidimensional diagonally quadratic BSDE with the terminal value of exponential moments of arbitrary order, which seems to be the first result on the general solvability of system of quadratic BSDEs with unbounded terminal values.

The rest of the paper is organized as follows. In Section 2, we introduce some notations used later, and state the main results of this paper. In Sections 3-5, we respectively prove our existence and uniqueness results on the local and global solution for our multi-dimensional diagonally quadratic BSDEs with bounded and unbounded terminal values. Finally, in the Appendix we present some auxiliary results for scalar-valued quadratic BSDEs with bounded and unbounded terminal values, including existence, uniqueness and several a priori estimates.

2. Notations and statement of main results

2.1. Notations

Let $a \wedge b$ and $a \vee b$ be the minimum and maximum of two real numbers a and b, respectively. Set $a^+ := a \vee 0$ and $a^- := -(a \wedge 0)$. Denote by $\mathbf{1}_A(\cdot)$ the indicator of set A, and $\mathrm{sgn}(x) := \mathbf{1}_{x>0} - \mathbf{1}_{x \leq 0}$.

Throughout this paper, all the processes are assumed to be $(\mathcal{F}_t)_{t \in [0,T]}$-progressively measurable, and all equalities and inequalities between random variables and processes are understood in the sense of $\mathbb{P}-a.s.$ and $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, respectively. The Euclidean norm is always denoted by $|\cdot|$, and $\|\cdot\|_\infty$ denotes the L^∞-norm for one-dimensional or multidimensional random variable defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

We define the following four Banach spaces of stochastic processes. By $\mathcal{S}^p(\mathbb{R}^n)$ for $p \geq 1$, we denote the totality of all $\mathbb{R}^n$-valued continuous adapted processes $(Y_t)_{t \in [0,T]}$ such that

$$\|Y\|_{\mathcal{S}^p} := \left(\mathbb{E}[\sup_{t \in [0,T]} |Y_t|^p] \right)^{1/p} < +\infty.$$

By $\mathcal{S}^\infty(\mathbb{R}^n)$, we denote the totality of all $Y \in \bigcap_{p \geq 1} \mathcal{S}^p(\mathbb{R}^n)$ such that

$$\|Y\|_{\mathcal{S}^\infty} := \operatorname*{ess\,sup}_{(\omega,t)} |Y_t(\omega)| = \left\| \sup_{t \in [0,T]} |Y_t| \right\|_\infty < +\infty.$$

By $\mathcal{H}^p(\mathbb{R}^{n \times d})$ for $p \geq 1$, we denote the totality of all $\mathbb{R}^{n \times d}$-valued $(\mathcal{F}_t)_{t \in [0,T]}$-progressively measurable processes $(Z_t)_{t \in [0,T]}$ such that

$$\|Z\|_{\mathcal{H}^p} := \left\{ \mathbb{E} \left[\left(\int_0^T |Z_s|^2 \mathrm{d}s \right)^{p/2} \right] \right\}^{1/p} < +\infty.$$

By $\mathrm{BMO}(\mathbb{R}^{n \times d})$, we denote the totality of all $Z \in \mathcal{H}^2(\mathbb{R}^{n \times d})$ such that

$$\|Z\|_{\mathrm{BMO}} := \sup_\tau \left\| \mathbb{E}_\tau \left[\int_\tau^T |Z_s|^2 \mathrm{d}s \right] \right\|_\infty^{1/2} < +\infty.$$

Here and hereafter the supremum is taken over all $(\mathcal{F}_t)$-stopping times τ with values in $[0,T]$, and $\mathbb{E}_\tau$ denotes the conditional expectation with respect to $\mathcal{F}_\tau$.

The spaces $\mathcal{S}^p_{[a,b]}(\mathbb{R}^n)$, $\mathcal{S}^\infty_{[a,b]}(\mathbb{R}^n)$, $\mathcal{H}^p_{[a,b]}(\mathbb{R}^{n \times d})$, and $\mathrm{BMO}_{[a,b]}(\mathbb{R}^{n \times d})$ are identically defined for stochastic processes over the time interval $[a,b]$. We note that for $Z \in \mathrm{BMO}(\mathbb{R}^{n \times d})$, the process $\int_0^t Z_s \mathrm{d}B_s, t \in [0,T]$, is an n-dimensional BMO martingale. For the theory of BMO martingales, we refer the reader to the monograph Kazamaki [18].

For $i = 1, \cdots, n$, denote by z^i, y^i and g^i respectively the ith row of matrix $z \in \mathbb{R}^{n \times d}$, the ith component of the vector $y \in \mathbb{R}^n$ and the generator g.

Finally, we write $Y \in \mathcal{E}(\mathbb{R}^n)$ if

$$\exp\left(|Y|\right) \in \bigcap_{p \geq 1} \mathcal{S}^p(\mathbb{R}^n),$$

and $Z \in \mathcal{M}(\mathbb{R}^{n \times d})$ if

$$Z \in \bigcap_{p \geq 1} \mathcal{H}^p(\mathbb{R}^{n \times d}).$$

2.2. Statement of the main results

Throughout the paper, we always fix an $(\mathcal{F}_t)$-progressively measurable scalar-valued non-negative process $(\alpha_t)_{t \in [0,T]}$, a deterministic nondecreasing continuous function $\phi(\cdot) : [0, +\infty) \to [0, +\infty)$ with $\phi(0) = 0$ and several real constants $\beta \geq 0$, $0 < \bar{\gamma} \leq \gamma$, $\lambda \geq 0$ and $\delta \in [0,1)$.

The first main result of this paper concerns local solutions for the bounded terminal value case. We need the following three assumptions.

(H1) For $i = 1, \cdots, n$, g^i satisfies that $d\mathbb{P} \times dt - a.e.$, for each $(y, z) \in \mathbb{R}^n \times \mathbb{R}^{n \times d}$,

$$|g^i(\omega, t, y, z)| \leq \alpha_t(\omega) + \phi(|y|) + \frac{\gamma}{2}|z^i|^2 + \lambda \sum_{j \neq i} |z^j|^{1+\delta};$$

(H2) For $i = 1, \cdots, n$, g^i satisfies that $d\mathbb{P} \times dt - a.e.$, for each $(y, \bar{y}, z, \bar{z}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \times \mathbb{R}^{n \times d}$,

$$|g^i(\omega, t, y, z) - g^i(\omega, t, \bar{y}, \bar{z})|$$
$$\leq \phi(|y| \vee |\bar{y}|) \left[(1 + |z| + |\bar{z}|) \left(|y - \bar{y}| + |z^i - \bar{z}^i| \right) + \left(1 + |z|^\delta + |\bar{z}|^\delta \right) \sum_{j \neq i} |z^j - \bar{z}^j| \right];$$

(H3) There exists two non-negative constants C_1 and C_2 such that

$$\|\xi\|_\infty \leq C_1 \quad \text{and} \quad \left\| \int_0^T \alpha_t dt \right\|_\infty \leq C_2.$$

In the first two assumptions (H1) and (H2), it creates no essential difference to replace both terms $\sum_{j \neq i} |z^j|^{1+\delta}$ and $\sum_{j \neq i} |z^j - \bar{z}^j|$ with $|z|^{1+\delta}$ and $|z - \bar{z}|$, respectively. The underlying way of formulation is more convenient for subsequent exposition.

Theorem 2.1. *Let assumptions (H1)-(H3) hold. Then, there exist a real $\varepsilon > 0$ (depending only on constants $(n, \gamma, \lambda, \delta, C_1, C_2)$ and function $\phi(\cdot)$) and a bounded subset $\mathcal{B}_\varepsilon$ of the product space $\mathcal{S}^\infty_{[T-\varepsilon,T]}(\mathbb{R}^n) \times \text{BMO}_{[T-\varepsilon,T]}(\mathbb{R}^{n \times d})$ such that BSDE (1.1) has a unique local solution (Y, Z) on the time interval $[T - \varepsilon, T]$ with $(Y, Z) \in \mathcal{B}_\varepsilon$.*

Remark 2.2. *Assumptions (H1) and (H2) of Theorem 2.1 are more general than those of Hu and Tang [15, Theorem 2.2, p. 1072] in that the former relaxes the growth and continuity of the generator in the first unknown variable y. For example, the following generator g satisfies the former, while not the latter:*

$$g^i(\omega, t, y, z) = (|y|^2 + \sin |z^i|)|z| + |z|^{\frac{3}{2}} + |z^i|^2, \quad i = 1, \cdots, n.$$

The second and third main results of this paper concern global solutions of quadratic BSDEs with bounded terminal values. The following two assumptions are further required.

(H4) For $i = 1, \cdots, n$, g^i satisfies that $d\mathbb{P} \times dt - a.e.$, for each $(y, z) \in \mathbb{R}^n \times \mathbb{R}^{n \times d}$,

$$\text{sgn}(y^i)g^i(\omega, t, y, z) \leq \alpha_t(\omega) + \beta|y| + \lambda|z|^{1+\delta} + \frac{\gamma}{2}|z^i|^2;$$

(H5) For $i = 1, \cdots, n$, it holds that $d\mathbb{P} \times dt - a.e.$, for each $(y, z) \in \mathbb{R}^n \times \mathbb{R}^{n \times d}$,

$$g^i(\omega, t, y, z) \geq \frac{\bar{\gamma}}{2}|z^i|^2 - \alpha_t(\omega) - \beta|y| - \lambda|z|^{1+\delta} \tag{2.1}$$

or

$$g^i(\omega, t, y, z) \leq -\frac{\bar{\gamma}}{2}|z^i|^2 + \alpha_t(\omega) + \beta|y| + \lambda|z|^{1+\delta}. \tag{2.2}$$

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Remark 2.3. *Assumption (H5) holds for the generator g if some components of g satisfy (2.1), and the others satisfy (2.2).*

Theorem 2.4. *Let assumptions (H1)-(H4) be satisfied. If the constant λ in (H4) vanishes, then BSDE (1.1) admits a unique global solution $(Y, Z) \in \mathcal{S}^\infty(\mathbb{R}^n) \times \mathrm{BMO}(\mathbb{R}^{n \times d})$ on $[0, T]$.*

Theorem 2.5. *Let assumptions (H1)-(H5) hold. Then BSDE (1.1) admits a unique global solution $(Y, Z) \in \mathcal{S}^\infty(\mathbb{R}^n) \times \mathrm{BMO}(\mathbb{R}^{n \times d})$ on $[0, T]$.*

Remark 2.6. *Assumption (H4) is some kind of one-sided linear growth condition of the generator g with respect to the variable y, and assumption (H5) can be regarded as some kind of strictly quadratic condition of g^i with respect to z^i. A generator g satisfying assumptions (H1)-(H5) can still have a general growth in the variable y. For example, the following generator g satisfies all these assumptions:*

$$g^i(\omega, t, y, z) = (\mathrm{e}^{-y^i} + \cos |z^i|)|z| - |z|^{\frac{4}{3}} + (-1)^i |z^i|^2, \quad i = 1, \cdots, n.$$

Note that this g does not satisfy the corresponding assumptions used in Hu and Tang [15].

For the sake of studying global solutions of multidimensional diagonally quadratic BSDEs with unbounded terminal values, we introduce the following four assumptions on the data (g, ξ) of BSDEs.

(B1) For $i = 1, \cdots, n$, $g^i(\omega, t, y, z)$ varies with (ω, t, y) and the ith row z^i of the matrix $z \in \mathbb{R}^{n \times d}$ only, and grows linearly in y and quadratically in z^i, i.e., $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$,

$$|g^i(\omega, t, y, z)| \leq \alpha_t(\omega) + \beta |y| + \frac{\gamma}{2}|z|^2 \quad \text{for each } (y, z) \in \mathbb{R}^n \times \mathbb{R}^{1 \times d};$$

(B2) g is uniformly Lipschitz continuous in y, i.e., $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$,

$$|g(\omega, t, y, z) - g(\omega, t, \bar{y}, z)| \leq \beta |y - \bar{y}| \quad \text{for each } (y, \bar{y}, z) \in (\mathbb{R}^n)^2 \times \mathbb{R}^{1 \times d};$$

(B3) $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, for each $i = 1, \cdots, n$ and $y \in \mathbb{R}^n$, $g^i(\omega, t, y, \cdot)$ is either convex or concave;

(B4) The terminal value ξ is of exponential moments of arbitrary order as well as $\int_0^T \alpha_t \mathrm{d}t$. That is, we have for each $p \geq 1$,

$$\mathbb{E}\left[\exp\left\{p\left(|\xi| + \int_0^T \alpha_t \mathrm{d}t\right)\right\}\right] < +\infty.$$

Remark 2.7. *Assumption (B3) holds for the generator g if some components of g are convex in z, and the others are concave in z.*

The following theorem seems to be the first result on the general solvability of systems of quadratic BSDEs with unbounded terminal values, and constitutes the last main result of the paper.

Theorem 2.8. *Let assumptions (B1)-(B4) be in force. Then BSDE (1.1) admits a unique global solution $(Y, Z) \in \mathcal{E}(\mathbb{R}^n) \times \mathcal{M}(\mathbb{R}^{n \times d})$ on $[0, T]$.*

Remark 2.9. *In the case of unbounded terminal values, the martingale part of the first known process Y goes beyond the space of BMO martingales, and some delicate and technical computations are developed in the proof of Theorem 2.8, in which a priori estimates on one-dimensional quadratic BSDEs, the θ-method for convex functions and Doob's maximal inequality for martingales play a crucial role.*

3. Local solution with bounded terminal value: proof of Theorem 2.1

For each $i = 1, \cdots, n$, $H \in \mathbb{R}^{n \times d}$ and $z \in \mathbb{R}^{1 \times d}$, define by $H(z; i)$ the matrix in $\mathbb{R}^{n \times d}$ whose ith row is z and whose jth row is H^j for any $j \neq i$.

Let assumptions (H1)-(H3) hold. For a pair of processes $(U, V) \in \mathcal{S}^\infty(\mathbb{R}^n) \times \text{BMO}(\mathbb{R}^{n \times d})$, we consider the following decoupled system of quadratic BSDEs:

$$Y_t^i = \xi^i + \int_t^T g^i(s, U_s, V_s(Z_s^i; i)) \mathrm{d}s - \int_t^T Z_s^i \mathrm{d}B_s, \quad t \in [0, T]; \quad i = 1, \cdots, n. \qquad (3.1)$$

For each fixed $i = 1, \cdots, n$, in view of assumptions (H1) and (H2), it is not difficult to verify that $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, for each $(z, \bar{z}) \in (\mathbb{R}^{1 \times d})^2$,

$$|g^i(t, U_t, V_t(z; i))| \leq \alpha_t + \phi(|U_t|) + n\lambda|V_t|^{1+\delta} + \frac{\gamma}{2}|z|^2$$

and

$$|g^i(t, U_t, V_t(z; i)) - g^i(t, U_t, V_t(\bar{z}; i))| \leq \phi(|U_t|) \left(1 + 2|V_t| + |z| + |\bar{z}|\right) |z - \bar{z}|.$$

This means that the generator $g^i(t, U_t, V_t(z; i))$ satisfies assumptions (A1) and (A2) defined in Appendix. Then, in view of assumption (H3), it follows from Lemma A.1 that for each $i = 1, \cdots, n$, one-dimensional BSDE with the terminal value ξ^i and the generator $g^i(t, U_t, V_t(z; i))$ has a unique solution (Y^i, Z^i) such that Y^i is (essentially) bounded and $Z^i \cdot B := \left(\int_0^t Z_s^i \mathrm{d}B_s\right)_{t \in [0, T]}$ is a BMO martingale. That is to say, the system of BSDEs (3.1) has a unique solution $(Y, Z) \in \mathcal{S}^\infty(\mathbb{R}^n) \times \text{BMO}(\mathbb{R}^{n \times d})$.

Now, define the solution map $\Gamma : (U, V) \mapsto \Gamma(U, V)$ as follows:

$$\Gamma(U, V) := (Y, Z), \quad \forall (U, V) \in \mathcal{S}^\infty(\mathbb{R}^n) \times \text{BMO}(\mathbb{R}^{n \times d}).$$

It is a transformation in the space $\mathcal{S}^\infty(\mathbb{R}^n) \times \mathrm{BMO}(\mathbb{R}^{n\times d})$. Moreover, it follows from Lemma A.1 that for each $i = 1, \cdots, n$, $t \in [0, T]$ and stopping time τ with values in $[t, T]$,

$$
\begin{aligned}
|Y_t^i| \;\leq\; & \frac{1}{\gamma}\ln 2 + \|\xi^i\|_\infty + \left\|\int_0^T \alpha_s \mathrm{d}s\right\|_\infty \\
& + \phi\left(\|U\|_{\mathcal{S}^\infty_{[t,T]}}\right)(T-t) + \gamma^{\frac{1+\delta}{1-\delta}} C_{\delta,\lambda,n}\|V\|^{2\frac{1+\delta}{1-\delta}}_{\mathrm{BMO}_{[t,T]}}(T-t)
\end{aligned}
$$

and

$$
\begin{aligned}
& \mathbb{E}_\tau\left[\int_\tau^T |Z_s^i|^2 \mathrm{d}s\right] \\
\leq\; & \frac{1}{\gamma^2}\exp(2\gamma\|\xi^i\|_\infty) + \frac{1}{\gamma}\exp\left(2\gamma\left\|\sup_{s\in[t,T]}|Y_s^i|\right\|_\infty\right) \\
& \cdot\left(1 + 2\left\|\int_0^T \alpha_s \mathrm{d}s\right\|_\infty + 2\phi\left(\|U\|_{\mathcal{S}^\infty_{[t,T]}}\right)(T-t) + 2C_{\delta,\lambda,n}\|V\|^{2\frac{1+\delta}{1-\delta}}_{\mathrm{BMO}_{[t,T]}}(T-t)\right),
\end{aligned}
$$

where the constant $C_{\delta,\lambda,n}$ is defined in (A.4) of Appendix. Therefore, in view of assumption (H3), for each $t \in [0, T]$, we have

$$
\begin{aligned}
\|Y\|_{\mathcal{S}^\infty_{[t,T]}} \;\leq\; & \frac{n}{\gamma}\ln 2 + n(C_1 + C_2) \\
& + n\phi\left(\|U\|_{\mathcal{S}^\infty_{[t,T]}}\right)(T-t) + n\gamma^{\frac{1+\delta}{1-\delta}}C_{\delta,\lambda,n}\|V\|^{2\frac{1+\delta}{1-\delta}}_{\mathrm{BMO}_{[t,T]}}(T-t)
\end{aligned} \tag{3.2}
$$

and

$$
\begin{aligned}
\|Z\|^2_{\mathrm{BMO}_{[t,T]}} \;\leq\; & \frac{n}{\gamma^2}\exp(2\gamma C_1) + \frac{n}{\gamma}\exp\left(2\gamma\|Y\|_{\mathcal{S}^\infty_{[t,T]}}\right) \\
& \cdot\left(1 + 2C_2 + 2\phi\left(\|U\|_{\mathcal{S}^\infty_{[t,T]}}\right)(T-t) + 2C_{\delta,\lambda,n}\|V\|^{2\frac{1+\delta}{1-\delta}}_{\mathrm{BMO}_{[t,T]}}(T-t)\right).
\end{aligned} \tag{3.3}
$$

Define

$$
K_1 := \frac{n}{\gamma}\ln 2 + n(C_1 + C_2),
$$

$$
K_2 := \frac{n}{\gamma^2}\exp(2\gamma C_1) + \frac{n}{\gamma}\exp\left(4\gamma K_1\right)\left(1 + 2C_2\right)
$$

and

$$
\varepsilon_0 := \left(\frac{K_1}{n\phi\left(2K_1\right) + n\gamma^{\frac{1+\delta}{1-\delta}}C_{\delta,\lambda,n}(2K_2)^{\frac{1+\delta}{1-\delta}}}\right)\bigwedge\left(\frac{\frac{\gamma}{n}\exp\left(-4\gamma K_1\right)K_2}{2\phi\left(2K_1\right) + 2C_{\delta,\lambda,n}(2K_2)^{\frac{1+\delta}{1-\delta}}}\right) > 0.
$$

By virtue of (3.2) and (3.3), we can verify directly that for each $\varepsilon \in (0, \varepsilon_0]$, if

$$
\|U\|_{\mathcal{S}^\infty_{[T-\varepsilon,T]}} \leq 2K_1 \quad\text{and}\quad \|V\|^2_{\mathrm{BMO}_{[T-\varepsilon,T]}} \leq 2K_2,
$$

then

$$
\|Y\|_{\mathcal{S}^\infty_{[T-\varepsilon,T]}} \leq 2K_1 \quad\text{and}\quad \|Z\|^2_{\mathrm{BMO}_{[T-\varepsilon,T]}} \leq 2K_2.
$$

This means that

$$
\Gamma(U, V) \in \mathcal{B}_\varepsilon, \quad \forall\, (U, V) \in \mathcal{B}_\varepsilon,
$$

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where

$$\mathcal{B}_\varepsilon := \Big\{ (U,V) \in \mathcal{S}^\infty(\mathbb{R}^n) \times \mathrm{BMO}(\mathbb{R}^{n\times d}) :$$

$$\|U\|_{\mathcal{S}^\infty_{[T-\varepsilon,T]}} \le 2K_1 \quad \text{and} \quad \|V\|^2_{\mathrm{BMO}_{[T-\varepsilon,T]}} \le 2K_2 \Big\} \qquad (3.4)$$

is a Banach space with the following norm

$$\|(U,V)\|_{\mathcal{B}_\varepsilon} := \sqrt{\|U\|^2_{\mathcal{S}^\infty_{[T-\varepsilon,T]}} + \|V\|^2_{\mathrm{BMO}_{[T-\varepsilon,T]}}}, \quad \forall\, (U,V) \in \mathcal{B}_\varepsilon.$$

That is, the mapping Γ is stable in the Banach space $\mathcal{B}_\varepsilon$ for each $\varepsilon \in (0,\varepsilon_0]$.

It remains to show that there exists a real $\varepsilon \in (0,\varepsilon_0]$ depending only on constants $(n,\gamma,\lambda,\delta,C_1,C_2)$ and function $\phi(\cdot)$ such that Γ is a contraction in $\mathcal{B}_\varepsilon$. Indeed, for any fixed $\varepsilon \in (0,\varepsilon_0]$, and $(U,V) \in \mathcal{B}_\varepsilon$ and $(\widetilde{U},\widetilde{V}) \in \mathcal{B}_\varepsilon$, we set

$$(Y,Z) := \Gamma(U,V), \quad (\widetilde{Y},\widetilde{Z}) := \Gamma(\widetilde{U},\widetilde{V}).$$

That is, for $i = 1,\cdots,n$ and $t \in [0,T]$,

$$Y^i_t = \xi^i + \int_t^T g^i(s,U_s,V_s(Z^i_s;i))\mathrm{d}s - \int_t^T Z^i_s \mathrm{d}B_s,$$

$$\widetilde{Y}^i_t = \xi^i + \int_t^T g^i(s,\widetilde{U}_s,\widetilde{V}_s(\widetilde{Z}^i_s;i))\mathrm{d}s - \int_t^T \widetilde{Z}^i_s \mathrm{d}B_s.$$

Define for $i = 1,\cdots,n$ and $s \in [0,T]$,

$$\Delta^{1,i}_s := g^i(s,U_s,V_s(Z^i_s;i)) - g^i(s,U_s,V_s(\widetilde{Z}^i_s;i)), \quad \Delta^{2,i}_s := g^i(s,U_s,V_s(\widetilde{Z}^i_s;i)) - g^i(s,\widetilde{U}_s,\widetilde{V}_s(\widetilde{Z}^i_s;i)).$$

Then,

$$Y^i_t - \widetilde{Y}^i_t + \int_t^T \left(Z^i_s - \widetilde{Z}^i_s \right) \mathrm{d}B_s - \int_t^T \Delta^{1,i}_s \mathrm{d}s = \int_t^T \Delta^{2,i}_s \mathrm{d}s, \quad t \in [0,T]. \qquad (3.5)$$

It follows from assumption (H2) that $\mathrm{d}\mathbb{P} \times \mathrm{d}s - a.e.$, for each $i = 1,\cdots,n$,

$$|\Delta^{1,i}_s| \le \phi(|U_s|)\left(1 + 2|V_s| + |Z_s| + |\widetilde{Z}_s| \right) |Z^i_s - \widetilde{Z}^i_s| \qquad (3.6)$$

and

$$\begin{aligned}
|\Delta^{2,i}_s| \;\le\;\; & \phi(|U_s| \vee |\widetilde{U}_s|) \Big[\left(1 + |V_s| + |\widetilde{V}_s| + |Z_s| + |\widetilde{Z}_s| \right) |U_s - \widetilde{U}_s| \\
& + \sqrt{n}\left(1 + |V_s|^\delta + |\widetilde{V}_s|^\delta + 2|\widetilde{Z}_s|^\delta \right) |V_s - \widetilde{V}_s| \Big].
\end{aligned} \qquad (3.7)$$

For $i = 1,\cdots,n$, by (3.6) we can define the $\mathbb{R}^d$-valued process $G(i)$ in an obvious way such that

$$\Delta^{1,i}_s = \left(Z^i_s - \widetilde{Z}^i_s \right) G_s(i) \quad \text{and} \quad |G_s(i)| \le \phi(|U_s|)\left(1 + 2|V_s| + |Z_s| + |\widetilde{Z}_s| \right), \quad s \in [0,T]. \qquad (3.8)$$

Finally, in view of the fact that all pairs of processes $(U,V),(\widetilde{U},\widetilde{V})$ and $(Y,Z),(\widetilde{Y},\widetilde{Z})$ are in $\mathcal{B}_\varepsilon$ together with definition (3.4) of $\mathcal{B}_\varepsilon$ and inequalities (3.5)-(3.8), using Girsanov's transform, Hölder's inequality, the energy inequality for BMO martingale and Lemma A.4 in Hu and Tang [15] we can follow the argument in pages 1078-1079 of Hu and Tang [15] to get the desired conclusion. Theorem 2.1 is then proved.

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4. Global solution with bounded terminal value: proof of Theorems 2.4 and 2.5

We need the following lemma.

Lemma 4.1. *Let assumption (H3) hold. Assume that for some $h \in (0, T]$, the BSDE (1.1) has a solution $(Y, Z) \in \mathcal{S}_{[T-h,T]}^\infty \times \mathrm{BMO}_{[T-h,T]}$ on time interval $[T-h, T]$. We have*

(i) If the generator g satisfies assumption (H4) with $\lambda = 0$, then

$$\|Y\|_{\mathcal{S}_{[T-h,T]}^\infty} \le (2n)^{[2n\beta T]+2}C_1 + \left(2n + (2n)^2 + \cdots + (2n)^{[2n\beta T]+2}\right)C_2,$$

where and hereafter $[x]$ denotes the maximum of integers smaller than or equal to x.

(ii) If the generator g satisfies assumptions (H4) and (H5), then

$$\|Y\|_{\mathcal{S}_{[T-h,T]}^\infty} \le (4n)^{[4n\beta T]+2}C_1 + \left(4n + (4n)^2 + \cdots + (4n)^{[4n\beta T]+2}\right)C_5,$$

where C_5 is a positive constant depending only on $(n, \beta, \gamma, \bar{\gamma}, \lambda, \delta, T, C_2)$.

Proof. For $i = 1, \cdots, n$, it follows from assumption (H4) that

$$\mathrm{sgn}(Y_t^i(\omega))g^i\left(\omega, t, Y_t(\omega), Z_t(\omega)\right) \le \alpha_t(\omega) + \beta|Y_t(\omega)| + \lambda|Z_t(\omega)|^{1+\delta} + \frac{\gamma}{2}|Z_t^i(\omega)|^2, \quad t \in [T-h, T],$$

which means that the generator of ith equation in system of BSDEs (1.1) satisfies (A.8) in Appendix (by letting $f(\omega, t, z) \equiv g^i\left(\omega, t, Y_t(\omega), Z_t(\omega)\right)$). It then follows from (i) of Lemma A.2 and assumption (H3) that for each $i = 1, \cdots, n$,

$$\begin{aligned}
\exp\left(\gamma|Y_t^i|\right) &\le \exp\left(\gamma(C_1 + C_2) + \beta\gamma\|Y\|_{\mathcal{S}_{[t,T]}^\infty}(T-t)\right) \\
&\quad \cdot \mathbb{E}_t\left[\exp\left(\lambda\gamma\int_t^T |Z_s|^{1+\delta}\mathrm{d}s\right)\right], \quad t \in [T-h, T].
\end{aligned} \tag{4.1}$$

(i) In the case of $\lambda = 0$, it follows from (4.1) that for $i = 1, \cdots, n$,

$$|Y_t^i| \le C_1 + C_2 + \beta\|Y\|_{\mathcal{S}_{[t,T]}^\infty}(T-t), \quad t \in [T-h, T].$$

Therefore,

$$\|Y\|_{\mathcal{S}_{[t,T]}^\infty} \le n(C_1 + C_2) + n\beta\|Y\|_{\mathcal{S}_{[t,T]}^\infty}(T-t), \quad t \in [T-h, T]. \tag{4.2}$$

For $\beta = 0$, it is clear that

$$\|Y\|_{\mathcal{S}_{[T-h,T]}^\infty} \le n(C_1 + C_2). \tag{4.3}$$

Otherwise, let m_0 is the unique positive integer satisfying

$$T - h \in [T - m_0\varepsilon, T - (m_0 - 1)\varepsilon)$$

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or, equivalently,

$$2n\beta h = \frac{h}{\varepsilon} \le m_0 < \frac{h}{\varepsilon} + 1 \le 2n\beta T + 1 \qquad (4.4)$$

with

$$\varepsilon := \frac{1}{2n\beta} > 0.$$

If $m_0 = 1$, then $n\beta(T - t) \le n\beta h \le n\beta\varepsilon = 1/2$ for $t \in [T - h, T]$, and it follows from (4.2) that

$$\|Y\|_{\mathcal{S}^\infty_{[t,T]}} \le 2n(C_1 + C_2), \quad t \in [T - h, T],$$

which yields that

$$\|Y\|_{\mathcal{S}^\infty_{[T-h,T]}} \le 2n(C_1 + C_2). \qquad (4.5)$$

If $m_0 = 2$, then $n\beta(T - t) \le n\beta\varepsilon = 1/2$ for $t \in [T - \varepsilon, T]$, and it follows from (4.2) that

$$\|Y\|_{\mathcal{S}^\infty_{[t,T]}} \le 2n(C_1 + C_2), \quad t \in [T - \varepsilon, T],$$

which yields that

$$\|Y_{T-\varepsilon}\|_\infty \le \|Y\|_{\mathcal{S}^\infty_{[T-\varepsilon,T]}} \le 2n(C_1 + C_2). \qquad (4.6)$$

Now, consider the following system of BSDEs

$$Y_t = Y_{T-\varepsilon} + \int_t^{T-\varepsilon} g(s, Y_s, Z_s)\mathrm{d}s - \int_t^{T-\varepsilon} Z_s\mathrm{d}B_s, \quad t \in [T - h, T - \varepsilon].$$

In view of (4.6), identically as in obtaining (4.5) , we have

$$\|Y\|_{\mathcal{S}^\infty_{[T-h,T-\varepsilon]}} \le 2n\left(2n(C_1 + C_2) + C_2\right),$$

and therefore,

$$\|Y\|_{\mathcal{S}^\infty_{[T-h,T]}} \le (2n)^2 C_1 + \left(2n + (2n)^2\right) C_2.$$

Proceeding the above computation gives that if m_0 satisfies (4.4), then

$$\|Y\|_{\mathcal{S}^\infty_{[T-h,T]}} \le (2n)^{m_0} C_1 + \left(2n + (2n)^2 + \cdots + (2n)^{m_0}\right) C_2,$$

which together with (4.4) and (4.3) yields assertion (i) immediately.

(ii) For $i = 1, \cdots, n$, it follows from assumption (H5) that

$$g^i\left(\omega, t, Y_t(\omega), Z_t(\omega)\right) \ge \frac{\bar{\gamma}}{2}|Z_t^i(\omega)|^2 - \alpha_t(\omega) - \beta|Y_t(\omega)| - \lambda|Z_t(\omega)|^{1+\delta}$$

or

$$g^i\left(\omega, t, Y_t(\omega), Z_t(\omega)\right) \le -\frac{\bar{\gamma}}{2}|Z_t^i(\omega)|^2 + \alpha_t(\omega) + \beta|Y_t(\omega)| + \lambda|Z_t(\omega)|^{1+\delta},$$

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which means that the generator of ith equation in system of BSDEs (1.1) satisfies (A.9) or (A.10) in Appendix (by letting $f(\omega, t, z) \equiv g^i(\omega, t, Y_t(\omega), Z_t(\omega))$). It then follows from (ii) of Lemma A.2 and assumption (H3) that for each $i = 1, \cdots, n$,

$$
\begin{aligned}
&\mathbb{E}_t\left[\exp\left(\frac{\bar{\gamma}}{2}\varepsilon_0 \int_t^T |Z_s^i|^2 \mathrm{d}s\right)\right] \\
\leq\ &\mathbb{E}_t\left[\exp\left(6\varepsilon_0 \left\|\sup_{s\in[t,T]}|Y_s^i|\right\|_\infty + 3\varepsilon_0 \left\|\int_0^T \alpha_s \mathrm{d}s\right\|_\infty + 3\varepsilon_0\beta T\|Y\|_{\mathcal{S}_{[t,T]}^\infty} + 3\varepsilon_0\lambda \int_t^T |Z_s|^{1+\delta}\mathrm{d}s\right)\right] \\
\leq\ &\exp\left(3\varepsilon_0 C_2 + 3\varepsilon_0(2+\beta T)\|Y\|_{\mathcal{S}_{[t,T]}^\infty}\right) \mathbb{E}_t\left[\exp\left(3\varepsilon_0\lambda \int_t^T |Z_s|^{1+\delta}\mathrm{d}s\right)\right], \quad t \in [T-h, T],
\end{aligned}
$$

where

$$
\varepsilon_0 := \left(\frac{\bar{\gamma}}{9}\right) \bigwedge \left(\frac{\gamma}{12(\beta T + 2)}\right) > 0.
$$

Thus, by Hölder's inequality we get that for $t \in [T-h, T]$,

$$
\begin{aligned}
&\mathbb{E}_t\left[\exp\left(\frac{\bar{\gamma}\varepsilon_0}{2n}\int_t^T |Z_s|^2 \mathrm{d}s\right)\right] \\
\leq\ &\exp\left(3\varepsilon_0 C_2 + 3\varepsilon_0(2+\beta T)\|Y\|_{\mathcal{S}_{[t,T]}^\infty}\right) \mathbb{E}_t\left[\exp\left(3\varepsilon_0\lambda \int_t^T |Z_s|^{1+\delta}\mathrm{d}s\right)\right].
\end{aligned} \tag{4.7}
$$

Note by Young's inequality that for each pair of $a, b > 0$,

$$
ab^{1+\delta} = \left(\left(\frac{1+\delta}{2}\right)^{\frac{1+\delta}{1-\delta}} a^{\frac{2}{1-\delta}}\right)^{\frac{1-\delta}{2}} \left(\frac{2}{1+\delta}b^2\right)^{\frac{1+\delta}{2}} \leq b^2 + \frac{1-\delta}{2}\left(\frac{1+\delta}{2}\right)^{\frac{1+\delta}{1-\delta}} a^{\frac{2}{1-\delta}}. \tag{4.8}
$$

By letting $a = 12n\lambda/\bar{\gamma}$ and $b = |Z_s|$ in (4.8), we have

$$
3\varepsilon_0\lambda |Z_s|^{1+\delta} = \frac{\bar{\gamma}\varepsilon_0}{4n}\left(\frac{12n\lambda}{\bar{\gamma}}|Z_s|^{1+\delta}\right) \leq \frac{\bar{\gamma}\varepsilon_0}{4n}|Z_s|^2 + C_3, \quad s \in [0, T], \tag{4.9}
$$

where

$$
C_3 := \frac{\bar{\gamma}\varepsilon_0(1-\delta)}{8n}\left(\frac{1+\delta}{2}\right)^{\frac{1+\delta}{1-\delta}}\left(\frac{12n\lambda}{\bar{\gamma}}\right)^{\frac{2}{1-\delta}}.
$$

Coming back to (4.7), by (4.9) and Hölder's inequality we deduce that for $t \in [T-h, T]$,

$$
\mathbb{E}_t\left[\exp\left(\frac{\bar{\gamma}\varepsilon_0}{2n}\int_t^T |Z_s|^2 \mathrm{d}s\right)\right] \leq \exp\left(6\varepsilon_0 C_2 + 2C_3 T + 6\varepsilon_0(2+\beta T)\|Y\|_{\mathcal{S}_{[t,T]}^\infty}\right). \tag{4.10}
$$

On the other hand, it follows from (4.1) and Jensen's inequality that

$$
\begin{aligned}
\exp\left(\gamma|Y_t|\right) \leq\ &\exp\left(n\gamma(C_1 + C_2) + n\beta\gamma\|Y\|_{\mathcal{S}_{[t,T]}^\infty}(T-t)\right) \\
&\cdot\mathbb{E}_t\left[\exp\left(n\lambda\gamma \int_t^T |Z_s|^{1+\delta}\mathrm{d}s\right)\right], \quad t \in [T-h, T].
\end{aligned} \tag{4.11}
$$

By letting $a = 2n^2\lambda\gamma/\bar{\gamma}\varepsilon_0$ and $b = |Z_s|$ in (4.8), we have

$$
n\lambda\gamma|Z_s|^{1+\delta} = \frac{\bar{\gamma}\varepsilon_0}{2n}\left(\frac{2n^2\lambda\gamma}{\bar{\gamma}\varepsilon_0}|Z_s|^{1+\delta}\right) \leq \frac{\bar{\gamma}\varepsilon_0}{2n}|Z_s|^2 + C_4, \quad s \in [0, T], \tag{4.12}
$$

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where

$$C_4 := \frac{\bar{\gamma}\varepsilon_0(1-\delta)}{4n}\left(\frac{1+\delta}{2}\right)^{\frac{1+\delta}{1-\delta}}\left(\frac{2n^2\lambda\gamma}{\bar{\gamma}\varepsilon_0}\right)^{\frac{2}{1-\delta}}.$$

Combining (4.10)-(4.12) yields that

$$\begin{aligned}
|Y_t| &\leq n(C_1+C_2) + \frac{C_4 T}{\gamma} + \frac{6\varepsilon_0 C_2 + 2C_3 T}{\gamma} + \frac{6\varepsilon_0(2+\beta T)}{\gamma}\|Y\|_{\mathcal{S}^\infty_{[t,T]}} \\
&\quad + n\beta\|Y\|_{\mathcal{S}^\infty_{[t,T]}}(T-t), \qquad t \in [T-h, T].
\end{aligned}$$

And, from the definition of ε_0 it follows that

$$\|Y\|_{\mathcal{S}^\infty_{[t,T]}} \leq 2n(C_1+C_5) + 2n\beta\|Y\|_{\mathcal{S}^\infty_{[t,T]}}(T-t), \quad t \in [T-h,T], \tag{4.13}$$

where

$$C_5 := C_2 + \frac{6\varepsilon_0 C_2 + 2C_3 T}{n\gamma} + \frac{C_4 T}{n\gamma}.$$

Finally, observing that (4.13) is almost the same as (4.2), we can use the same computation as in (i) to obtain the desired conclusion of (ii). The proof of Lemma 4.1 is then complete. $\square$

Remark 4.2. *Observe that the term* $\|U\|_{\mathcal{S}^\infty_{[t,T]}}(T-t)$ *in the conclusion (i) of Lemma A.1 can be replaced with* $\int_t^T \|U\|_{\mathcal{S}^\infty_{[s,T]}}\mathrm{d}s$. *Then the term* $\|Y\|_{\mathcal{S}^\infty_{[t,T]}}(T-t)$ *in inequalities (4.1) and (4.2) can be replaced with* $\int_t^T \|Y\|_{\mathcal{S}^\infty_{[s,T]}}\mathrm{d}s$. *Therefore, by Gronwall's inequality, we obtain the following better upper bound under the assumptions of (i) in Lemma 4.1:*

$$\|Y\|_{\mathcal{S}^\infty_{[T-h,T]}} \leq n(C_1+C_2)\exp(n\beta h) \leq n(C_1+C_2)\exp(n\beta T).$$

The same computation yields the following estimate under the assumptions of (ii) in Lemma 4.1:

$$\|Y\|_{\mathcal{S}^\infty_{[T-h,T]}} \leq 2n(C_1+C_5)\exp(2n\beta h) \leq 2n(C_1+C_5)\exp(2n\beta T).$$

Using Theorem 2.1 and Lemma 4.1, we can follow the proof of Cheridito and Nam [7, Theorem 4.1] to derive our Theorems 2.4 and 2.5. All the details are omitted here.

5. Global solution with unbounded terminal value: proof of Theorem 2.8

Let assumptions (B1)-(B4) be in force. For a pair of processes $(U,V) \in \mathcal{E}(\mathbb{R}^n) \times \mathcal{M}(\mathbb{R}^{n\times d})$, we consider the following decoupled system of quadratic BSDEs:

$$Y_t^i = \xi^i + \int_t^T g^i(s, U_s, Z_s^i)\mathrm{d}s - \int_t^T Z_s^i \mathrm{d}B_s, \quad t \in [0,T]; \quad i = 1, \cdots, n. \tag{5.1}$$

For each fixed $i = 1, \cdots, n$, in view of assumptions (B1) and (B3), it is clear that $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$,

$$|g^i(t, U_t, z)| \leq \alpha_t + \beta|U_t| + \frac{\gamma}{2}|z|^2, \quad \forall z \in \mathbb{R}^{1\times d},$$

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and $g^i(t, U_t, z)$ is convex or concave in z. Furthermore, in view of assumption (B4) and the fact that $U \in \mathcal{E}(\mathbb{R}^n)$, by Hölder's inequality we have

$$\forall\, q > 1, \quad \mathbb{E}\left[\exp\left\{q\left(|\xi^i| + \int_0^T (\alpha_t + \beta|U_t|)\,\mathrm{d}t\right)\right\}\right] < +\infty, \quad i = 1, \cdots, n.$$

It then follows from Corollary 6 in Briand and Hu [6] that for each $i = 1, \cdots, n$, the BSDE

$$Y_t^i = \xi^i + \int_t^T g^i(s, U_s, Z_s^i)\mathrm{d}s - \int_t^T Z_s^i \mathrm{d}B_s, \quad t \in [0, T]$$

has a unique adapted solution (Y^i, Z^i) such that for each $q > 1$,

$$\mathbb{E}\left[\exp\left(q \sup_{t\in[0,T]} |Y_t^i|\right) + \left(\int_0^T |Z_t^i|^2 \mathrm{d}t\right)^{\frac{q}{2}}\right] < +\infty,$$

which means that, in view of Hölder's inequality, the system of BSDEs (5.1) admits a unique solution (Y, Z) in the space of processes $\mathcal{E}(\mathbb{R}^n) \times \mathcal{M}(\mathbb{R}^{n\times d})$.

Based on the above argument, we can set $(Y^{(0)}, Z^{(0)}) = (0, 0)$ and define, recursively, the sequence of processes $\{(Y^{(m)}, Z^{(m)})\}_{m=1}^\infty$ in the space of processes $\mathcal{E}(\mathbb{R}^n) \times \mathcal{M}(\mathbb{R}^{n\times d})$ by the unique adapted solution of system of BSDEs:

$$Y_t^{(m+1);i} = \xi^i + \int_t^T g^i(s, Y_s^{(m)}, Z_s^{(m+1);i})\mathrm{d}s - \int_t^T Z_s^{(m+1);i}\mathrm{d}B_s, \quad t \in [0, T]; \quad i = 1, \cdots, n, \quad (5.2)$$

where for sake of convenience, we denote by $Y^{(m);i}$ and $Z^{(m);i}$, respectively, the ith component of $Y^{(m)}$ and the ith row of $Z^{(m)}$. In the sequel, we will show that $\{(Y^{(m)}, Z^{(m)})\}_{m=1}^\infty$ is a Cauchy sequence in the space $\mathcal{S}^q(\mathbb{R}^n) \times \mathcal{H}^q(\mathbb{R}^{n\times d})$ for each $q \geq 1$, and then converges to a pair of adapted processes (Y, Z) in $\mathcal{E}(\mathbb{R}^n) \times \mathcal{M}(\mathbb{R}^{n\times d})$, which is the unique desired solution of system of BSDE (1.1).

We first prove that

$$\forall\, q > 1, \quad \sup_{m \geq 0} \mathbb{E}\left[\exp\left(q\gamma \sup_{t\in[0,T]} |Y_t^{(m)}|\right)\right] \leq K(q), \quad (5.3)$$

where

$$K(q) := (A(2q)A(8nq))^{[2n\beta T]+1} \mathbb{E}\left[\exp\left(4n(8n)^{[2n\beta T]+1} q\gamma|\xi|\right)\right]$$
$$\cdot \mathbb{E}\left[\exp\left(4n(16n)^{[2n\beta T]+1} q\gamma \int_0^T \alpha_s \mathrm{d}s\right)\right] < +\infty$$

with

$$A(q) := \left(\frac{q}{q-1}\right)^{2q}. \quad (5.4)$$

In fact, for each $i = 1, \cdots, n$ and $m \geq 0$, it follows from (B1) and (B3) that $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$,

$$\forall\, z \in \mathbb{R}^{1\times d}, \quad |g^i(t, Y_t^{(m)}, z)| \leq \alpha_t + \beta|Y_t^{(m)}| + \frac{\gamma}{2}|z|^2, \quad (5.5)$$

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and $g^i(t, Y_t^{(m)}, z)$ is convex or concave in z. Furthermore, in view of assumption (B4) and the fact that $Y^{(m)} \in \mathcal{E}(\mathbb{R}^n)$, by Hölder's inequality we have

$$\forall\, p \geq 1, \quad \mathbb{E}\left[\exp\left\{p\left(\sup_{t \in [0,T]} |Y_t^{(m+1);i}| + \int_0^T \left(\alpha_t + \beta|Y_t^{(m)}|\right) dt\right)\right\}\right] < +\infty.$$

Then, we can use Lemma A.3 in Appendix to get that for $i = 1, \cdots, n$ and $m \geq 0$,

$$\exp\left(\gamma|Y_t^{(m+1);i}|\right) \leq \mathbb{E}_t\left[\exp\left(\gamma|\xi^i| + \gamma\int_t^T \left(\alpha_s + \beta|Y_s^{(m)}|\right) ds\right)\right], \quad t \in [0,T].$$

Consequently, by Jensen's inequality we have, for each $m \geq 0$,

$$\exp\left(\gamma|Y_t^{(m+1)}|\right) \leq \mathbb{E}_t\left[\exp\left(n\gamma|\xi| + n\gamma\int_t^T \left(\alpha_s + \beta|Y_s^{(m)}|\right) ds\right)\right], \quad t \in [0,T]. \tag{5.6}$$

In view of (5.6), Doob's maximal inequality for martingales together with Hölder's inequality yields that for each $q > 1$, $m \geq 0$ and $t \in [0,T]$, we have

$$\mathbb{E}\left[\exp\left(q\gamma \sup_{s \in [t,T]} |Y_s^{(m+1)}|\right)\right]$$

$$\leq \left(\frac{q}{q-1}\right)^q \mathbb{E}\left[\exp\left(nq\gamma|\xi| + nq\gamma\int_t^T \alpha_s ds + nq\gamma\beta \sup_{s \in [t,T]} |Y_s^{(m)}|(T-t)\right)\right] \tag{5.7}$$

$$\leq [C(q)]^{1/2}\left\{\mathbb{E}\left[\exp\left(2nq\gamma\beta \sup_{s \in [t,T]} |Y_s^{(m)}|(T-t)\right)\right]\right\}^{1/2},$$

where, in view of assumption (B4),

$$C(q) := A(q)\mathbb{E}\left[\exp\left(2nq\gamma|\xi| + 2nq\gamma\int_0^T \alpha_s ds\right)\right] < +\infty$$

with $A(q)$ being defined in (5.4).

For $\beta = 0$, it is clear from (5.7) that

$$\forall\, q > 1, \quad \sup_{m \geq 0} \mathbb{E}\left[\exp\left(q\gamma \sup_{t \in [0,T]} |Y_t^{(m)}|\right)\right] \leq \sqrt{C(q)}. \tag{5.8}$$

Otherwise, let m_0 is the unique positive integer satisfying $T - m_0\varepsilon \leq 0 < T - (m_0 - 1)\varepsilon$ or, equivalently,

$$2n\beta T = \frac{T}{\varepsilon} \leq m_0 < 2n\beta T + 1 \tag{5.9}$$

with

$$\varepsilon := \frac{1}{2n\beta} > 0.$$

If $m_0 = 1$, then $2nq\gamma\beta(T-t) \leq 2nq\gamma\beta T \leq 2nq\gamma\beta\varepsilon = q\gamma$ for $t \in [0,T]$, and it follows from (5.7) that for each $m \geq 0$, $q > 1$ and $t \in [0,T]$,

$$\mathbb{E}\left[\exp\left(q\gamma \sup_{s \in [t,T]} |Y_s^{(m+1)}|\right)\right] \leq \sqrt{C(q)}\left(\mathbb{E}\left[\exp\left(q\gamma \sup_{s \in [t,T]} |Y_s^{(m)}|\right)\right]\right)^{1/2},$$

and, by induction,

$$\mathbb{E}\left[\exp\left(q\gamma \sup_{s\in[t,T]} |Y_s^{(m+1)}|\right)\right] \leq \sqrt{C(q)}^{1+\frac{1}{2}+\cdots+\frac{1}{2^m}} \left(\mathbb{E}\left[\exp\left(q\gamma \sup_{s\in[t,T]} |Y_s^{(0)}|\right)\right]\right)^{\frac{1}{2^{m+1}}} \leq C(q).$$

Consequently, by Hölder's inequality we have, for each $q > 1$,

$$\sup_{m\geq 0} \mathbb{E}\left[\exp\left(q\gamma \sup_{t\in[0,T]} |Y_t^{(m)}|\right)\right] \leq A(q)\mathbb{E}\left[\exp\left(4nq\gamma|\xi|\right)\right] \mathbb{E}\left[\exp\left(4nq\gamma \int_0^T \alpha_s \mathrm{d}s\right)\right] < +\infty. \tag{5.10}$$

If $m_0 = 2$, then $2nq\gamma\beta(T - t) \leq 2nq\gamma\beta\varepsilon = q\gamma$ for $t \in [T - \varepsilon, T]$, and from the argument in the case of $m_0 = 1$ it follows that for each $q > 1$,

$$\sup_{m\geq 0} \mathbb{E}\left[\exp\left(q\gamma \sup_{t\in[T-\varepsilon,T]} |Y_t^{(m)}|\right)\right] \leq A(q)\mathbb{E}\left[\exp\left(4nq\gamma|\xi|\right)\right] \mathbb{E}\left[\exp\left(4nq\gamma \int_0^T \alpha_s \mathrm{d}s\right)\right] < +\infty, \tag{5.11}$$

which yields that

$$\sup_{m\geq 0} \mathbb{E}\left[\exp\left(q\gamma|Y_{T-\varepsilon}^{(m)}|\right)\right] \leq A(q)\mathbb{E}\left[\exp\left(4nq\gamma|\xi|\right)\right] \mathbb{E}\left[\exp\left(4nq\gamma \int_0^T \alpha_s \mathrm{d}s\right)\right] < +\infty. \tag{5.12}$$

Now, consider the following system of BSDEs: for $i = 1, \cdots, n$,

$$Y_t^{(m+1);i} = Y_{T-\varepsilon}^{(m+1);i} + \int_t^{T-\varepsilon} g^i(s, Y_s^{(m)}, Z_s^{(m+1);i})\mathrm{d}s - \int_t^{T-\varepsilon} Z_s^{(m+1);i}\mathrm{d}B_s, \quad t \in [0, T - \varepsilon].$$

In view of (5.12), a similar argument as that obtaining (5.10) yields that for each $q > 1$,

$$\sup_{m\geq 0} \mathbb{E}\left[\exp\left(q\gamma \sup_{t\in[0,T-\varepsilon]} |Y_t^{(m)}|\right)\right]$$
$$\leq A(q) \sup_{m\geq 0} \mathbb{E}\left[\exp\left(4nq\gamma|Y_{T-\varepsilon}^{(m)}|\right)\right] \mathbb{E}\left[\exp\left(4nq\gamma \int_0^T \alpha_s \mathrm{d}s\right)\right],$$
$$\leq A(q)A(4nq)\mathbb{E}\left[\exp\left(16n^2 q\gamma|\xi|\right)\right] \mathbb{E}\left[\exp\left(32n^2 q\gamma \int_0^T \alpha_s \mathrm{d}s\right)\right] < +\infty,$$

and then, in view of (5.11) and Hölder's inequality,

$$\sup_{m\geq 0} \mathbb{E}\left[\exp\left(q\gamma \sup_{t\in[0,T]} |Y_t^{(m)}|\right)\right]$$
$$\leq A(2q)A(8nq)\mathbb{E}\left[\exp\left(32n^2 q\gamma|\xi|\right)\right] \mathbb{E}\left[\exp\left(64n^2 q\gamma \int_0^T \alpha_s \mathrm{d}s\right)\right] < +\infty.$$

Proceeding the above computation gives that if m_0 satisfies (5.9), then for each $q > 1$,

$$\sup_{m\geq 0} \mathbb{E}\left[\exp\left(q\gamma \sup_{t\in[0,T]} |Y_t^{(m)}|\right)\right]$$
$$\leq (A(2q)A(8nq))^{m_0-1} \mathbb{E}\left[\exp\left(4n(8n)^{m_0-1}q\gamma|\xi|\right)\right] \mathbb{E}\left[\exp\left(4n(16n)^{m_0-1}q\gamma \int_0^T \alpha_s \mathrm{d}s\right)\right] < +\infty,$$

which together with (5.9) and (5.8) yields the desired conclusion (5.3).

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In the sequel, we show that

$$\forall\, q > 1, \quad \sup_{m \geq 0} \mathbb{E}\left[\left(\int_0^T |Z_s^{(m)}|^2 \mathrm{d}s\right)^{q/2}\right] < +\infty. \tag{5.13}$$

In fact, using Itô-Tanaka's formula to compute $\exp(2\gamma |Y_t^{(m+1);i}|)$ and utilizing inequality (5.5), we can deduce that for each $i = 1, \cdots, n$ and $m \geq 0$,

$$\gamma^2 \int_0^T \exp(2\gamma |Y_s^{(m+1);i}|) |Z_s^{(m+1);i}|^2 \mathrm{d}s$$

$$\leq \exp(2\gamma |\xi^i|) + 2\gamma \int_0^T \exp(2\gamma |Y_s^{(m+1);i}|) \left(\alpha_s + \beta |Y_s^m|\right) \mathrm{d}s$$

$$-2\gamma \int_0^T \exp(2\gamma |Y_s^{(m+1);i}|) \mathrm{sgn}(Y_s^{(m+1);i}) Z_s^{(m+1);i} \mathrm{d}B_s.$$

In view of the previous inequality, we can use the BDG inequality and Young's inequality to derive that for each $q > 1$, $m \geq 0$ and $i = 1, \cdots, n$,

$$\mathbb{E}\left[\left(\int_0^T |Z_s^{(m+1);i}|^2 \mathrm{d}s\right)^{q/2}\right]$$

$$\leq C\left(\mathbb{E}\left[\exp\left(2q\gamma \sup_{s \in [0,T]} |Y_s^{(m+1);i}|\right)\right] + \mathbb{E}\left[\left(\int_0^T \alpha_s + \beta |Y_s^{(m)}| \mathrm{d}s\right)^q\right]\right) \tag{5.14}$$

$$\leq \bar{C}\left(\mathbb{E}\left[\exp\left(\int_0^T \alpha_s \mathrm{d}s\right)\right] + \sup_{m \geq 0} \mathbb{E}\left[\exp\left(2q\gamma \sup_{s \in [0,T]} |Y_s^{(m)}|\right)\right]\right),$$

where C and $\bar{C}$ are two positive constants depending only on (q, γ) and (q, γ, β, T), respectively. Then the inequality (5.13) follows immediately from (5.14) and (5.3).

Next, without loss of generality, we assume that the generator g is component-wisely convex in (B3), that is, $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, $g^i(\omega, t, y, \cdot)$ is convex for each $i = 1, \cdots, n$ and $y \in \mathbb{R}^n$. Otherwise, if $g^i(\omega, t, y, \cdot)$ is concave for some integer i, it is sufficient to replace the primary unknown (y^i, z^i) with the new pair of unknown variables $(-y^i, -z^i)$ in the underlying system of BSDEs.

For each fixed $m, p \geq 1$ and $\theta \in (0, 1)$, define

$$\delta_\theta Y^{(m,p)} := \frac{Y^{(m+p)} - \theta Y^{(m)}}{1 - \theta} \quad \text{and} \quad \delta_\theta Z^{(m,p)} := \frac{Z^{(m+p)} - \theta Z^{(m)}}{1 - \theta}.$$

Then $(\delta_\theta Y^{(m,p)}, \delta_\theta Z^{(m,p)}) \in \mathcal{E}(\mathbb{R}^n) \times \mathcal{M}(\mathbb{R}^{n \times d})$ solves the following system of BSDEs: for each $i = 1, \cdots, n$,

$$\delta_\theta Y_t^{(m,p);i} = \xi^i + \int_t^T \delta_\theta g^{(m,p);i}(s, \delta_\theta Z_s^{(m,p);i}) \mathrm{d}s - \int_t^T \delta_\theta Z_s^{(m,p);i} \mathrm{d}B_s, \quad t \in [0, T], \tag{5.15}$$

where $\mathrm{d}\mathbb{P} \times \mathrm{d}s - a.e.$, for each $z \in \mathbb{R}^{1 \times d}$,

$$\delta_\theta g^{(m,p);i}(s, z) := \frac{1}{1 - \theta}\left(g^i(s, Y_s^{(m+p-1)}, (1-\theta)z + \theta Z_s^{(m);i}) - \theta g^i(s, Y_s^{(m-1)}, Z_s^{(m);i})\right). \tag{5.16}$$

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It follows from (5.16) and assumptions (B2), (B3) and (B1) that $d\mathbb{P}\times ds-a.e.$, for each $z\in\mathbb{R}^{1\times d}$,

$$
\begin{aligned}
\delta_\theta g^{(m,p);i}(s,z) &\leq \beta|\delta_\theta Y_s^{(m-1,p)}| + \beta|Y_s^{(m-1)}| + g^i(s, Y_s^{(m-1)}, z) \\
&\leq \alpha_s + \beta|\delta_\theta Y_s^{(m-1,p)}| + 2\beta|Y_s^{(m-1)}| + \frac{\gamma}{2}|z|^2,
\end{aligned}
$$

which together with (5.3) means that all the conditions in Lemma A.4 are satisfied for BSDE (5.15), and then for $i = 1, \cdots, n$, we have

$$
\begin{aligned}
&\exp\left(\gamma\left(\delta_\theta Y_t^{(m,p);i}\right)^+\right) \\
&\leq \mathbb{E}_t\left[\exp\left(\gamma(\xi^i)^+ + \gamma\int_t^T\left(\alpha_s + \beta|\delta_\theta Y_s^{(m-1,p)}| + 2\beta|Y_s^{(m-1)}|\right)ds\right)\right], \quad t\in[0,T].
\end{aligned}
\tag{5.17}
$$

On the other hand, define

$$
\delta_\theta\widetilde{Y}^{(m,p)} := \frac{Y^{(m)} - \theta Y^{(m+p)}}{1-\theta} \quad \text{and} \quad \delta_\theta\widetilde{Z}^{(m,p)} := \frac{Z^{(m)} - \theta Z^{(m+p)}}{1-\theta}.
$$

The same computation as above yields that for $i = 1, \cdots, n$,

$$
\begin{aligned}
&\exp\left(\gamma\left(\delta_\theta\widetilde{Y}_t^{(m,p);i}\right)^+\right) \\
&\leq \mathbb{E}_t\left[\exp\left(\gamma(\xi^i)^+ + \gamma\int_t^T\left(\alpha_s + \beta|\delta_\theta\widetilde{Y}_s^{(m-1,p)}| + 2\beta|Y_s^{(m+p-1)}|\right)ds\right)\right], \quad t\in[0,T].
\end{aligned}
\tag{5.18}
$$

Furthermore, observe that for $i = 1, \cdots, n$ and $t\in[0,T]$, we have

$$
\begin{aligned}
\left(\delta_\theta Y_t^{(m,p);i}\right)^- &= \frac{\left(Y_t^{(m+p);i} - \theta Y_t^{(m);i}\right)^-}{1-\theta} = \frac{\left(\theta Y_t^{(m);i} - Y_t^{(m+p);i}\right)^+}{1-\theta} \\
&\leq \frac{\theta\left(Y_t^{(m);i} - \theta Y_t^{(m+p);i}\right)^+ + (1-\theta^2)|Y_t^{(m+p);i}|}{1-\theta} \\
&\leq \left(\delta_\theta\widetilde{Y}_t^{(m,p);i}\right)^+ + 2|Y_t^{(m+p)}|
\end{aligned}
$$

and, similarly,

$$
\left(\delta_\theta\widetilde{Y}_t^{(m,p);i}\right)^- \leq \left(\delta_\theta Y_t^{(m,p);i}\right)^+ + 2|Y_t^{(m)}|.
$$

It follows from (5.17) and (5.18) together with Jensen's inequality that for each $i = 1, \cdots, n$ and $t\in[0,T]$,

$$
\begin{aligned}
&\exp\left(\gamma|\delta_\theta Y_t^{(m,p);i}|\right) = \exp\left(\gamma\left(\delta_\theta Y_t^{(m,p);i}\right)^+\right) \cdot \exp\left(\gamma\left(\delta_\theta Y_t^{(m,p);i}\right)^-\right) \\
&\leq \mathbb{E}_t\left[\exp\left(2\gamma|\xi| + 2\gamma|Y_t^{(m+p)}| + 2\gamma\int_t^T\left(\alpha_s + 2\beta|Y_s^{(m-1)}| + 2\beta|Y_s^{(m+p-1)}|\right)ds\right.\right. \\
&\qquad\qquad\left.\left. +2\gamma\beta\int_t^T\left(|\delta_\theta Y_s^{(m-1,p)}| + |\delta_\theta\widetilde{Y}_s^{(m-1,p)}|\right)ds\right)\right]
\end{aligned}
$$

and

$$
\begin{aligned}
&\exp\left(\gamma|\delta_\theta\widetilde{Y}_t^{(m,p);i}|\right) = \exp\left(\gamma\left(\delta_\theta\widetilde{Y}_t^{(m,p);i}\right)^+\right) \cdot \exp\left(\gamma\left(\delta_\theta\widetilde{Y}_t^{(m,p);i}\right)^-\right) \\
&\leq \mathbb{E}_t\left[\exp\left(2\gamma|\xi| + 2\gamma|Y_t^{(m)}| + 2\gamma\int_t^T\left(\alpha_s + 2\beta|Y_s^{(m-1)}| + 2\beta|Y_s^{(m+p-1)}|\right)ds\right.\right. \\
&\qquad\qquad\left.\left. +2\gamma\beta\int_t^T\left(|\delta_\theta Y_s^{(m-1,p)}| + |\delta_\theta\widetilde{Y}_s^{(m-1,p)}|\right)ds\right)\right].
\end{aligned}
$$

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Consequently, by Jensen's inequality again we have for each $t \in [0, T]$,

$$
\begin{aligned}
&\exp\left(\gamma\left(|\delta_\theta Y_t^{(m,p);i}| + |\delta_\theta \widetilde{Y}_t^{(m,p);i}|\right)\right) \\
&\leq \ \mathbb{E}_t\left[\exp\left\{4\gamma\left(|\xi| + |Y_t^{(m)}| + |Y_t^{(m+p)}| + \int_t^T \left(\alpha_s + 2\beta|Y_s^{(m-1)}| + 2\beta|Y_s^{(m+p-1)}|\right)\mathrm{d}s\right)\right.\right. \\
&\qquad\qquad \left.\left. + 4\gamma\beta\int_t^T \left(|\delta_\theta Y_s^{(m-1,p)}| + |\delta_\theta \widetilde{Y}_s^{(m-1,p)}|\right)\mathrm{d}s\right\}\right], \quad i = 1, \cdots, n,
\end{aligned}
$$

and then

$$
\begin{aligned}
&\exp\left(\gamma\left(|\delta_\theta Y_t^{(m,p)}| + |\delta_\theta \widetilde{Y}_t^{(m,p)}|\right)\right) \\
&\leq \ \mathbb{E}_t\left[\exp\left\{4n\gamma\left(|\xi| + |Y_t^{(m)}| + |Y_t^{(m+p)}| + \int_t^T \left(\alpha_s + 2\beta|Y_s^{(m-1)}| + 2\beta|Y_s^{(m+p-1)}|\right)\mathrm{d}s\right)\right.\right. \\
&\qquad\qquad \left.\left. + 4n\gamma\beta\int_t^T \left(|\delta_\theta Y_s^{(m-1,p)}| + |\delta_\theta \widetilde{Y}_s^{(m-1,p)}|\right)\mathrm{d}s\right\}\right].
\end{aligned}
\tag{5.19}
$$

In view of (5.19), Doob's maximal inequality for martingales together with Hölder's inequality yields that for each $q > 1$ and $t \in [0, T]$, we have

$$
\begin{aligned}
&\mathbb{E}\left[\exp\left(q\gamma \sup_{s \in [t,T]}\left(|\delta_\theta Y_s^{(m,p)}| + |\delta_\theta \widetilde{Y}_s^{(m,p)}|\right)\right)\right] \\
&\leq \ \left(\frac{q}{q-1}\right)^q \mathbb{E}\left[\exp\left\{4nq\gamma\left(|\xi| + \sup_{s \in [t,T]}\left(|Y_s^{(m)}| + |Y_s^{(m+p)}|\right)\right)\right.\right. \\
&\qquad\qquad + 4nq\gamma\int_t^T \left(\alpha_s + 2\beta|Y_s^{(m-1)}| + 2\beta|Y_s^{(m+p-1)}|\right)\mathrm{d}s \\
&\qquad\qquad \left.\left. + 4nq\gamma\beta \sup_{s \in [t,T]}\left(|\delta_\theta Y_s^{(m-1,p)}| + |\delta_\theta \widetilde{Y}_s^{(m-1,p)}|\right)(T-t)\right\}\right] \\
&\leq \ [\bar{C}(q)]^{1/2}\left\{\mathbb{E}\left[\exp\left(8nq\gamma\beta \sup_{s \in [t,T]}\left(|\delta_\theta Y_s^{(m-1,p)}| + |\delta_\theta \widetilde{Y}_s^{(m-1,p)}|\right)(T-t)\right)\right]\right\}^{1/2},
\end{aligned}
\tag{5.20}
$$

where, in view of (5.3),

$$
\begin{aligned}
\bar{C}(q) \ := \ A(q) \sup_{m,p \geq 1} \mathbb{E}&\left[\exp\left\{8nq\gamma\left(|\xi| + \sup_{t \in [0,T]}|Y_t^{(m)}| + \sup_{t \in [0,T]}|Y_t^{(m+p)}|\right)\right.\right. \\
&\left.\left. + 8nq\gamma\int_0^T \left(\alpha_s + 2\beta|Y_s^{(m-1)}| + 2\beta|Y_s^{(m+p-1)}|\right)\mathrm{d}s\right\}\right] < +\infty
\end{aligned}
$$

with $A(q)$ being defined in (5.4).

Based on the above analysis, we now can prove that $\{(Y^{(m)}, Z^{(m)})\}_{m=1}^\infty$ is a Cauchy sequence in the space $\mathcal{S}^q(\mathbb{R}^n) \times \mathcal{H}^q(\mathbb{R}^{n \times d})$ for each $q > 1$. In fact, observing the similarity of (5.20) and (5.7), we can induce with respect to m and use a similar argument as that obtaining (5.3) to derive the existence of a positive constant $\bar{K}(q)$ depending on q and being independent of θ

such that for each $q > 1$,

$$\mathbb{E}\left[\exp\left(q\gamma \sup_{t\in[0,T]}\left(|\delta_\theta Y_t^{(m,p)}| + |\delta_\theta \widetilde{Y}_t^{(m,p)}|\right)\right)\right]$$

$$\leq \bar{K}(q)\left(\mathbb{E}\left[\exp\left(q\gamma \sup_{t\in[0,T]}\left(|\delta_\theta Y_t^{(1,p)}| + |\delta_\theta \widetilde{Y}_t^{(1,p)}|\right)\right)\right]\right)^{\frac{2[2n\beta T]+1}{2^m}},$$

from which together with (5.3) it follows that for each $q > 1$ and $\theta \in (0,1)$,

$$\limsup_{m\to\infty}\sup_{p\geq 1}\mathbb{E}\left[\exp\left(q\gamma \sup_{t\in[0,T]}\left(|\delta_\theta Y_t^{(m,p)}| + |\delta_\theta \widetilde{Y}_t^{(m,p)}|\right)\right)\right] \leq \bar{K}(q). \tag{5.21}$$

Thus, for each $\theta \in (0,1)$,

$$\limsup_{m\to\infty}\sup_{p\geq 1}\mathbb{E}\left[\sup_{t\in[0,T]}|Y_t^{(m+p)} - \theta Y_t^{(m)}|\right] \leq (1-\theta)\frac{\bar{K}(2)}{2\gamma},$$

and then, in view of (5.3),

$$\limsup_{m\to\infty}\sup_{p\geq 1}\mathbb{E}\left[\sup_{t\in[0,T]}|Y_t^{(m+p)} - Y_t^{(m)}|\right] \leq (1-\theta)\left(\frac{\bar{K}(2)}{2\gamma} + \sup_{m\geq 1}\mathbb{E}\left[\sup_{t\in[0,T]}|Y_t^{(m)}|\right]\right) < +\infty.$$

Sending θ to 1, in view of (5.3), we see that there is an adapted process $Y \in \mathcal{E}(\mathbb{R}^n)$ such that for each $q > 1$,

$$\lim_{m\to\infty}\mathbb{E}\left[\sup_{t\in[0,T]}|Y_t^{(m)} - Y_t|^q\right] = 0 \tag{5.22}$$

and

$$\lim_{m\to\infty}\mathbb{E}\left[\exp\left(q\sup_{t\in[0,T]}|Y_t^{(m)} - Y_t|\right)\right] = 1. \tag{5.23}$$

Furthermore, it follows from Itô's formula that for each $m, p \geq 1$,

$$\mathbb{E}\left[\int_0^T |Z_s^{(m+p)} - Z_s^{(m)}|^2 \mathrm{d}s\right] \leq 2\mathbb{E}\left[\sup_{t\in[0,T]}\left|Y_t^{(m+p)} - Y_t^{(m)}\right|\right.$$

$$\left. \cdot \int_0^T \sum_{i=1}^n \left|g^i(s, Y_s^{(m+p-1)}, Z_s^{(m+p);i}) - g^i(s, Y_s^{(m-1)}, Z_s^{(m);i})\right| \mathrm{d}s\right]. \tag{5.24}$$

And, by virtue of (B1), (5.3) and (5.13) we get that

$$\sup_{m,p\geq 1}\mathbb{E}\left[\left(\int_0^T \sum_{i=1}^n \left|g^i(s, Y_s^{(m+p-1)}, Z_s^{(m+p);i}) - g^i(s, Y_s^{(m-1)}, Z_s^{(m);i})\right| \mathrm{d}s\right)^2\right] < +\infty. \tag{5.25}$$

Then, applying Hölder's inequality to (5.24) and using (5.22) and (5.25) leads to that

$$\lim_{m\to\infty}\sup_{p\geq 1}\mathbb{E}\left[\int_0^T |Z_s^{(m+p)} - Z_s^{(m)}|^2 \mathrm{d}s\right] = 0,$$

from which together with (5.13) it follows that there exists a process $Z \in \mathcal{M}(\mathbb{R}^{n\times d})$ such that

$$\forall\, q > 1, \quad \lim_{m\to\infty}\mathbb{E}\left[\left(\int_0^T |Z_s^{(m)} - Z_s|^2 \mathrm{d}s\right)^{q/2}\right] = 0. \tag{5.26}$$

Finally, in view of (5.22), (5.23) and (5.26), by sending m to infinity in (5.2) we can deduce that (Y, Z) is a desired solution of system of BSDE (1.1).

It remains to show the uniqueness part for completing the proof of Theorem 2.8. For this, let $(\widetilde{Y}, \widetilde{Z})$ be also a solution of system of BSDE (1.1) in the space $\mathcal{E}(\mathbb{R}^n) \times \mathcal{M}(\mathbb{R}^{n \times d})$, and for $\theta \in (0, 1)$, define

$$\delta_\theta U := \frac{Y - \theta \widetilde{Y}}{1 - \theta}, \qquad \delta_\theta V := \frac{Z - \theta \widetilde{Z}}{1 - \theta},$$

and

$$\delta_\theta \widetilde{U} := \frac{\widetilde{Y} - \theta Y}{1 - \theta}, \qquad \delta_\theta \widetilde{V} := \frac{\widetilde{Z} - \theta Z}{1 - \theta}.$$

Using a similar argument as that from (5.15) to (5.20), we can deduce that for each $q > 1$,

$$
\begin{aligned}
&\mathbb{E}\left[\exp\left(q\gamma \sup_{s \in [t,T]} \left(|\delta_\theta U_s| + |\delta_\theta \widetilde{U}_s|\right)\right)\right] \\
&\leq \left(\frac{q}{q-1}\right)^q \mathbb{E}\left[\exp\left\{4nq\gamma \left(|\xi| + \sup_{s \in [t,T]} \left(|Y_s| + |\widetilde{Y}_s|\right) + \int_t^T \left(\alpha_s + 2\beta|Y_s| + 2\beta|\widetilde{Y}_s|\right) ds\right)\right.\right. \\
&\qquad \left.\left. + 4nq\gamma\beta \sup_{s \in [t,T]} \left(|\delta_\theta U_s| + |\delta_\theta \widetilde{U}_s|\right)(T - t)\right\}\right] \\
&\leq \left[\widetilde{C}(q)\right]^{1/2} \left\{\mathbb{E}\left[\exp\left(8nq\gamma\beta \sup_{s \in [t,T]} \left(|\delta_\theta U_s| + |\delta_\theta \widetilde{U}_s|\right)(T - t)\right)\right]\right\}^{1/2}, \quad t \in [0, T],
\end{aligned}
$$
(5.27)

where

$$
\begin{aligned}
\widetilde{C}(q) &:= A(q)\mathbb{E}\left[\exp\left\{8nq\gamma\left(|\xi| + \sup_{t \in [0,T]}\left(|Y_t| + |\widetilde{Y}_t|\right) + \int_0^T \left(\alpha_s + 2\beta|Y_s| + 2\beta|\widetilde{Y}_s|\right) ds\right)\right\}\right] \\
&< +\infty
\end{aligned}
$$

with $A(q)$ being defined in (5.4). For $\beta = 0$, it is clear from (5.27) that

$$\mathbb{E}\left[2\gamma \sup_{t \in [0,T]} |\delta_\theta U_t|\right] \leq \mathbb{E}\left[\exp\left(2\gamma \sup_{t \in [0,T]} |\delta_\theta U_t|\right)\right] \leq \sqrt{\widetilde{C}(2)},$$

and then

$$
\begin{aligned}
\mathbb{E}\left[\sup_{t \in [0,T]} |Y_t - \widetilde{Y}_t|\right] &\leq \mathbb{E}\left[\sup_{t \in [0,T]} |Y_t - \theta \widetilde{Y}_t|\right] + (1 - \theta)\mathbb{E}\left[\sup_{t \in [0,T]} |\widetilde{Y}_t|\right] \\
&\leq (1 - \theta)\left(\frac{\sqrt{\widetilde{C}(2)}}{2\gamma} + \mathbb{E}\left[\exp\left(\sup_{t \in [0,T]} |\widetilde{Y}_t|\right)\right]\right),
\end{aligned}
$$

in which letting $\theta \to 1$ yields that $Y = \widetilde{Y}$ and then $Z = \widetilde{Z}$ on the time interval $[0, T]$. Otherwise, in the case of $T \leq \bar{\varepsilon} := 1/8n\beta$, it follows from (5.27) that

$$\mathbb{E}\left[2\gamma \sup_{t \in [0,T]} |\delta_\theta U_t|\right] \leq \mathbb{E}\left[\exp\left(2\gamma \sup_{t \in [0,T]} |\delta_\theta U_t|\right)\right] \leq \widetilde{C}(2),$$

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and then $Y = \widetilde{Y}$ and $Z = \widetilde{Z}$ on the time interval $[0, T]$. Similarly, if m_0 is the unique positive integer such that $(m_0 - 1)\bar{\varepsilon} < T \le m_0\bar{\varepsilon}$, then we can successively prove the uniqueness on the time intervals $[T - \bar{\varepsilon}, T]$, $[T - 2\bar{\varepsilon}, T - \bar{\varepsilon}]$, $\cdots$, $[T - (m_0 - 1)\bar{\varepsilon}, T - (m_0 - 2)\bar{\varepsilon}]$ and $[0, T - (m_0 - 1)\bar{\varepsilon}]$.

The proof of Theorem 2.8 is then complete.

Appendix A. Several auxiliary results on one-dimensional quadratic BSDEs

We consider the following one-dimensional BSDE:

$$Y_t = \eta + \int_t^T f(s, Z_s)\, \mathrm{d}s - \int_t^T Z_s\, \mathrm{d}B_s, \quad t \in [0, T], \tag{A.1}$$

where the terminal value η is a real-valued $\mathcal{F}_T$-measurable random variable, and the generator function $f(\cdot, \cdot, z) : \Omega \times [0, T] \to \mathbb{R}$ is $(\mathcal{F}_t)$-progressively measurable for each $z \in \mathbb{R}^{1 \times d}$. Here, the solution $(Y_t, Z_t)_{t \in [0, T]}$ is defined as a pair of $(\mathcal{F}_t)$-progressively measurable processes taking values in $\mathbb{R} \times \mathbb{R}^{1 \times d}$, such that (A.1) is satisfied.

Assume that there exists a pair of processes $(U, V) \in \mathcal{S}^\infty(\mathbb{R}^n) \times \mathrm{BMO}(\mathbb{R}^{n \times d})$ such that the generator f satisfies the following assumptions.

(A1) $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, we have

$$|f(\omega, t, z)| \le \alpha_t(\omega) + \phi(|U_t(\omega)|) + n\lambda|V_t(\omega)|^{1+\delta} + \frac{\gamma}{2}|z|^2 \quad \text{for each } z \in \mathbb{R}^{1 \times d};$$

(A2) $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, we have

$$|f(\omega, t, z) - f(\omega, t, \bar{z})| \le \phi(|U_t(\omega)|)\left(1 + 2|V_t(\omega)| + |z| + |\bar{z}|\right)|z - \bar{z}|$$

for each $(z, \bar{z}) \in (\mathbb{R}^{1 \times d})^2$.

The following lemma slightly generalizes Hu and Tang [15, Lemma 2.1].

Lemma A.1. *Let the generator f satisfy assumptions (A1) and (A2), and both $|\eta|$ and $\int_0^T \alpha_t \mathrm{d}t$ be (essentially) bounded. Then, BSDE (A.1) admits a unique solution (Y, Z) such that Y is (essentially) bounded and $Z \cdot B := \left(\int_0^t Z_s \mathrm{d}B_s\right)_{t \in [0, T]}$ is a BMO martingale. Moreover, for each $t \in [0, T]$ and each stopping time τ with values in $[t, T]$, we have*

$$\begin{aligned}
|Y_t| &\le \frac{1}{\gamma}\ln 2 + \|\eta\|_\infty + \left\|\int_0^T \alpha_s \mathrm{d}s\right\|_\infty \\
&\quad + \phi\left(\|U\|_{\mathcal{S}^\infty_{[t,T]}}\right)(T - t) + \gamma^{\frac{1+\delta}{1-\delta}} C_{\delta, \lambda, n}\|V\|_{\mathrm{BMO}_{[t,T]}}^{2\frac{1+\delta}{1-\delta}}(T - t)
\end{aligned} \tag{A.2}$$

and

$$\mathbb{E}_\tau \left[\int_\tau^T |Z_s|^2 \mathrm{d}s \right]$$

$$\leq \frac{1}{\gamma^2} \exp(2\gamma \|\eta\|_\infty) + \frac{1}{\gamma} \exp\left(2\gamma \left\| \sup_{s \in [t,T]} |Y_s| \right\|_\infty \right) \tag{A.3}$$

$$\cdot \left(1 + 2 \left\| \int_0^T \alpha_s \mathrm{d}s \right\|_\infty + 2\phi \left(\|U\|_{\mathcal{S}^\infty_{[t,T]}} \right) (T-t) + 2C_{\delta,\lambda,n} \|V\|_{\mathrm{BMO}_{[t,T]}}^{2\frac{1+\delta}{1-\delta}} (T-t) \right),$$

where

$$C_{\delta,\lambda,n} := \frac{1-\delta}{2}(1+\delta)^{\frac{1+\delta}{1-\delta}} (n\lambda)^{\frac{2}{1-\delta}}. \tag{A.4}$$

Proof. Since $V \in \mathrm{BMO}(\mathbb{R}^{n \times d})$, it follows from Young's inequality that for each real $k > 0$,

$$kn\lambda |V_s|^{1+\delta} = \left((1+\delta)^{\frac{1+\delta}{1-\delta}} \|V\|_{\mathrm{BMO}_{[t,T]}}^{2\frac{1+\delta}{1-\delta}} (kn\lambda)^{\frac{2}{1-\delta}} \right)^{\frac{1-\delta}{2}} \left(\frac{|V_s|^2}{(1+\delta)\|V\|_{\mathrm{BMO}_{[t,T]}}^2} \right)^{\frac{1+\delta}{2}}$$

$$\leq \frac{1}{2\|V\|_{\mathrm{BMO}_{[t,T]}}^2} |V_s|^2 + k^{\frac{2}{1-\delta}} C_{\delta,\lambda,n} \|V\|_{\mathrm{BMO}_{[t,T]}}^{2\frac{1+\delta}{1-\delta}}, \quad 0 \leq t \leq s \leq T, \tag{A.5}$$

where the constant $C_{\delta,\lambda,n}$ is defined in (A.4). On the other hand, it follows from John-Nirenberg inequality for BMO martingale (see for example Lemma A.1 in Hu and Tang [15]) that for each $t \in [0,T]$ and each stopping time τ with values in $[t,T]$,

$$\mathbb{E}_\tau \left[\exp\left(\frac{1}{2\|V\|_{\mathrm{BMO}_{[t,T]}}^2} \int_\tau^T |V_s|^2 \mathrm{d}s \right) \right] \leq \frac{1}{1 - \frac{1}{2}} = 2, \quad t \in [0,T]. \tag{A.6}$$

Thus, combining previous inequality and inequality (A.5) with $k := p\gamma$ yields that for each $p \geq 1$ and $t \in [0,T]$,

$$\mathbb{E}_t \left[\exp\left(p\gamma n\lambda \int_t^T |V_s|^{1+\delta} \mathrm{d}s \right) \right] \leq 2 \exp\left((p\gamma)^{\frac{2}{1-\delta}} C_{\delta,\lambda,n} \|V\|_{\mathrm{BMO}_{[t,T]}}^{2\frac{1+\delta}{1-\delta}} (T-t) \right) < +\infty, \tag{A.7}$$

and then, in view of $U \in \mathcal{S}^\infty(\mathbb{R}^n)$ and the boundedness of $|\eta|$ and $\int_0^T \alpha_s \mathrm{d}s$,

$$\mathbb{E}_t \left[\exp\left(p\gamma |\eta| + p\gamma \int_t^T \left(\alpha_s + \phi(|U_s|) + n\lambda |V_s|^{1+\delta} \right) \mathrm{d}s \right) \right]$$

$$\leq 2 \exp\left(p\gamma \|\eta\|_\infty + p\gamma \left\| \int_0^T \alpha_s \mathrm{d}s \right\|_\infty + p\gamma \phi\left(\|U\|_{\mathcal{S}^\infty_{[t,T]}} \right) (T-t) \right.$$

$$\left. + (p\gamma)^{\frac{2}{1-\delta}} C_{\delta,\lambda,n} \|V\|_{\mathrm{BMO}_{[t,T]}}^{2\frac{1+\delta}{1-\delta}} (T-t) \right) < +\infty.$$

In view of assumptions (A1)-(A2) and with previous inequality in the hand, we can apply Theorem 2 in Briand and Hu [6] to see that BSDE (A.1) admits a solution (Y, Z) such that

$$\mathbb{E} \left[\int_0^T |Z_s|^2 \mathrm{d}s \right] < +\infty$$

and

$$\exp\left(\gamma|Y_t|\right) \leq 2\exp\left(\gamma\|\eta\|_\infty + \gamma\left\|\int_0^T \alpha_s \mathrm{d}s\right\|_\infty + \gamma\phi\left(\|U\|_{\mathcal{S}_{[t,T]}^\infty}\right)(T-t)\right.$$
$$\left. +\gamma^{\frac{2}{1-\delta}} C_{\delta,\lambda,n}\|V\|_{\mathrm{BMO}_{[t,T]}}^{2\frac{1+\delta}{1-\delta}}(T-t)\right).$$

This shows that (A.2) holds and Y is bounded.

We now show that (A.3) holds and $Z \cdot B$ is a BMO martingale. Using Itô-Tanaka's formula to compute $\exp(2\gamma|Y_t|)$ and utilizing assumption (A1), we have, for each $t \in [0,T]$ and each stopping time τ with values in $[t,T]$,

$$\exp(2\gamma|Y_\tau|) + 2\gamma^2\mathbb{E}_\tau\left[\int_\tau^T \exp(2\gamma|Y_s|)|Z_s|^2\mathrm{d}s\right]$$
$$\leq \mathbb{E}_\tau\left[\exp(2\gamma|\eta|)\right] + 2\gamma\mathbb{E}_\tau\left[\int_\tau^T \exp(2\gamma|Y_s|)\left(\alpha_s + \phi(|U_s|) + n\lambda|V_s|^{1+\delta} + \frac{\gamma}{2}|Z_s|^2\right)\mathrm{d}s\right].$$

Therefore, in view of (A.5) with $k := 1$,

$$\gamma^2\mathbb{E}_\tau\left[\int_\tau^T |Z_s|^2\mathrm{d}s\right]$$
$$\leq \exp(2\gamma\|\eta\|_\infty) + 2\gamma\exp\left(2\gamma\left\|\sup_{s\in[t,T]}|Y_s|\right\|_\infty\right)$$
$$\cdot\left(\left\|\int_0^T \alpha_s\mathrm{d}s\right\|_\infty + \phi\left(\|U\|_{\mathcal{S}_{[t,T]}^\infty}\right)(T-t) + \frac{1}{2} + C_{\delta,\lambda,n}\|V\|_{\mathrm{BMO}_{[t,T]}}^{2\frac{1+\delta}{1-\delta}}(T-t)\right) < +\infty,$$

from which the desired conclusions follows immediately.

Finally, in view of assumption (A2) with $(U,V) \in \mathcal{S}^\infty(\mathbb{R}^n) \times \mathrm{BMO}(\mathbb{R}^{n\times d})$, by a similar argument to that in Hu and Tang [15, Lemma 2.1], we can use the Girsanov transform to prove a comparison result on the solutions of BSDE (A.1), which yields the desired uniqueness. $\square$

Lemma A.2. *Assume that $(U,V) \in \mathcal{S}^\infty(\mathbb{R}^n) \times \mathrm{BMO}(\mathbb{R}^{n\times d})$, both $|\eta|$ and $\int_0^T \alpha_t\mathrm{d}t$ are (essentially) bounded, and (Y,Z) is a solution of BSDE (A.1) such that Y is (essentially) bounded.*

(i) If $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, it holds that

$$\mathrm{sgn}(Y_t(\omega))f\left(\omega,t,Z_t(\omega)\right) \leq \alpha_t(\omega) + \beta|U_t(\omega)| + \lambda|V_t(\omega)|^{1+\delta} + \frac{\gamma}{2}|Z_t(\omega)|^2, \qquad (\mathrm{A.8})$$

then for each $t \in [0,T]$, we have

$$\exp\left(\gamma|Y_t|\right) \leq \mathbb{E}_t\left[\exp\left(\gamma\|\eta\|_\infty + \gamma\left\|\int_0^T \alpha_s\mathrm{d}s\right\|_\infty\right.\right.$$
$$\left.\left. +\beta\gamma\|U\|_{\mathcal{S}_{[t,T]}^\infty}(T-t) + \lambda\gamma\int_t^T |V_s|^{1+\delta}\mathrm{d}s\right)\right];$$

(ii) If $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, it holds that

$$f\left(\omega,t,Z_t(\omega)\right) \geq \frac{\bar{\gamma}}{2}|Z_t(\omega)|^2 - \alpha_t(\omega) - \beta|U_t(\omega)| - \lambda|V_t(\omega)|^{1+\delta} \qquad (\mathrm{A.9})$$

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or

$$f(\omega, t, Z_t(\omega)) \le -\frac{\bar{\gamma}}{2}|Z_t(\omega)|^2 + \alpha_t(\omega) + \beta|U_t(\omega)| + \lambda|V_t(\omega)|^{1+\delta}, \tag{A.10}$$

then for each $\varepsilon \in (0, \frac{\bar{\gamma}}{9}]$ and $t \in [0, T]$, we have

$$\mathbb{E}_t\left[\exp\left(\frac{\bar{\gamma}}{2}\varepsilon\int_t^T |Z_s|^2\mathrm{d}s\right)\right] \le \mathbb{E}_t\left[\exp\left(6\varepsilon\left\|\sup_{s\in[t,T]}|Y_s|\right\|_\infty + 3\varepsilon\left\|\int_0^T \alpha_s\mathrm{d}s\right\|_\infty\right.\right.$$
$$\left.\left. +3\varepsilon\beta\|U\|_{\mathcal{S}^\infty_{[t,T]}}(T-t) + 3\varepsilon\lambda\int_t^T |V_s|^{1+\delta}\mathrm{d}s\right)\right].$$

Proof. In view of (A.7), using Itô-Tanaka's formula to compute

$$\exp\left(\gamma|Y_t| + \gamma\int_0^t \left(\alpha_s(\omega) + \beta|U_s(\omega)| + \lambda|V_s(\omega)|^{1+\delta}\right)\mathrm{d}s\right)$$

one can easily obtain (i). And, in view of (A.7) again, we can apply a similar argument as in the proof of Fan et al. [11, Proposition 2] to get (ii). The detailed proof is omitted here. □

The following two lemmas provide some bounds on the (possibly unbounded) solutions of one-dimensional quadratic BSDEs, which can be derived from (i) of Fan et al. [11, Proposition 1]. We omit the detailed proof here.

Lemma A.3. *Assume that there exists an $(\mathcal{F}_t)$-progressively measurable scalar-valued non-negative process $(\bar{\alpha}_t)_{t\in[0,T]}$ such that $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$,*

$$\forall z \in \mathbb{R}^{1\times d}, \quad |f(\omega, t, z)| \le \bar{\alpha}_t(\omega) + \frac{\gamma}{2}|z|^2.$$

Then, for any solution (Y, Z) of BSDE (A.1) satisfying

$$\mathbb{E}\left[\exp\left(2\gamma\sup_{t\in[0,T]}|Y_t| + 2\gamma\int_0^T \bar{\alpha}_s\mathrm{d}s\right)\right] < +\infty,$$

we have

$$\exp\left(\gamma|Y_t|\right) \le \mathbb{E}_t\left[\exp\left(\gamma|\eta| + \gamma\int_t^T \bar{\alpha}_s\mathrm{d}s\right)\right], \quad t \in [0, T]$$

and

$$\mathbb{E}\left[\int_0^T |Z_s|^2\mathrm{d}s\right] \le \frac{1}{\gamma^2}\mathbb{E}\left[\exp\left(2\gamma|\eta| + 2\gamma\int_0^T \bar{\alpha}_s\mathrm{d}s\right)\right].$$

Lemma A.4. *Assume that there exists an $(\mathcal{F}_t)$-progressively measurable scalar-valued non-negative process $(\bar{\alpha}_t)_{t\in[0,T]}$ such that $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$,*

$$\forall z \in \mathbb{R}^{1\times d}, \quad f(\omega, t, z) \le \bar{\alpha}_t(\omega) + \frac{\gamma}{2}|z|^2.$$

Then, for any solution (Y, Z) of BSDE (A.1) satisfying

$$\mathbb{E}\left[\exp\left(2\gamma\sup_{t\in[0,T]} Y_t^+ + 2\gamma\int_0^T \bar{\alpha}_s\mathrm{d}s\right)\right] < +\infty,$$

we have

$$\exp\left(\gamma Y_t^+\right) \le \mathbb{E}_t\left[\exp\left(\gamma\eta^+ + \gamma\int_t^T \bar{\alpha}_s\mathrm{d}s\right)\right], \quad t \in [0, T].$$

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