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Distributed event-triggered formation control for multi-Agent systems in presence of packet losses

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Abstract

This paper proposes an event-triggered approach for the distributed formation control problem of an Euler-Lagrange multi-agent system with state perturbations, when communications between agents are prone to losses. To evaluate its control input, each agent maintains estimators of its own state and of the states of its neighbors. Each estimator accounts for a different packet-loss hypothesis. Each agent is then able to compute the expected value of the estimation error of its own state as evaluated by its neighbors. A communication triggering condition (CTC) exploiting this expected error is then proposed. An analysis of the behavior of the system with this CTC is performed using stochastic Lyapunov functions. Simulations confirm the effectiveness of the proposed approach.

Key words: Communication constraints, event-triggered control, packet losses, formation stabilization, multi-Agent system.

1 Introduction

Distributed control with event-triggered communication is an efficient method to coordinate Multi-Agent Systems (MAS) with a reduced amount of communications between agents. The Communication Triggering Condition (CTC) is instrumental in these approaches to limit communications, while allowing enough information to be exchanged between agents to complete the task assigned to the MAS [16,20,29,24,15]. Designing a suitable CTC when communications between agents are prone to packet losses is challenging. With event-triggered control, a message is transmitted only when required. A loss of information may thus have a critical impact on the performance and even stability of the MAS.

Packet losses may result from collisions between packets simultaneously transmitted from different agents, from occlusions by obstacles, or from interference with other communications systems. Considering two packet-loss models, [7] has shown that event-triggered control schemes are more vulnerable to packet losses than time-triggered control strategies. Acknowledgment mechanisms are helpful to detect and possibly re-transmit lost messages. Nevertheless, acknowledgments or re-transmitted messages may also be lost, which increases communication delays, risk of packet collisions, and may lead to desynchronization between agents. In [2,6,7,28,25] packet losses are addressed by combining an H_∞ control and event-triggered communications. For agents with linear dynamics, sufficient conditions are established to ensure the global exponential stability of the system. In [6], communication delays and packet losses are considered simultaneously. In [2], the focus is on a MAS where agents follow several leaders. Each agent maintains observers of the state of other agents. These observers account for the last received message from the other agents and for models of their dynamics. In [26], two types of networked controller design methods are proposed. The first one ensures that the system is asymptotically stable in the presence of an arbitrary bounded number of packet losses. The second one provides mean square stability in presence of Markovian packet losses. In [5], communication delays are also considered. Two communication protocols are proposed, and the convergence of the MAS is guaranteed if the delay and the number of consecutive packet losses are bounded. All previous works consider only linear dynamics.

Nonlinear dynamics are studied in [3,4]. In [3], packet losses are taken into account in the estimator models but not in the CTC: new distributed estimators are designed to guarantee the exponential stability of the estimation errors. To update

the estimate of the state of other agents, each agent uses its own innovation and the innovation of its neighbors obtained from received packets. This improves the accuracy of the estimates at the cost of an increased sensitivity to losses. The control of a *single* Agent in presence of measurements losses is considered in [4]. An event-triggered strategy is proposed along with two communication protocols, with and without acknowledgments. With acknowledgments, the most recently received measurement can be identified. Without acknowledgment, this information is no longer available. A set of estimators is used to estimate the measurement lost during transmission. Each estimator uses a different hypothesis of the last packet received. As previously stated, the case of a MAS is not considered.

This paper addresses the distributed formation control of a MAS consisting of agents with nonlinear Euler-Lagrange dynamics, affected by state perturbations, and communications with losses. An event-triggered control strategy is proposed extending that presented in [21] to account for packet losses. Each agent maintains several estimators of its *own* state to mimic the estimators of its state maintained by its neighbors. Each estimator considers a different hypothesis of packet reception by these neighbors. This extends the idea of [4], where only two estimators are maintained. Contrary to most studies accounting for packet losses, no explicit feedback mechanism is considered here. Nevertheless, packets received from neighbors provide some (usually delayed) implicit feedback which is exploited to reduce the number of considered loss hypotheses, without requiring additional communications. This reduces the amount of estimators of its own state maintained by each agent. The CTC proposed in [21] is then updated to explicitly account for the potential loss of transmitted packets. The asymptotic convergence of the MAS to the target formation, as well as the absence of Zeno behavior have been proved.

Assumptions and the formation parameterization are introduced in Section 2 and 3. The distributed control law is described in Section 3. State estimators to replace missing information in control law and evaluate the CTC are proposed in Section 4.1. Influence of packet losses on estimator is presented in Section 4.2, to evaluate an expected value of the estimation error. Knowledge of this error is improved using a feedback information from other agents, as described in Section 4.4. The distributed CTC is presented in Section 5. A simulation example is presented in Section 6 to illustrate the reduction of the number of communications obtained by the proposed approach. Finally, Section 7 presents conclusions and perspectives for future work.

2 Notations and hypotheses

For a vector $x = [x_1, \ldots, x_n]^T \in \mathbb{R}^n$, $x \geqslant 0$ indicates that $x_i \geqslant 0$, $\forall i \in \{1, \ldots, n\}$. The absolute value of the i-th component of x is $|x_i|$ and $|x| = [|x_1|, \ldots, |x_n|]^T$. Table 1 gathers the main notations used in this paper.

2.1 Multi-Agent system

Consider a MAS consisting of N communicating agents with indexes in the set $\mathcal{N} = \{1, \ldots, N\}$. In a global fixed reference frame $\mathcal{F}$, let $q_i \in \mathbb{R}^n$ be the vector of *coordinates* of Agent i and $q = \left[q_1^T, \ldots, q_N^T\right]^T \in \mathbb{R}^{Nn}$ be the *configuration* of the MAS. The relative coordinate vector between two Agents i and j is $r_{ij} = q_i - q_j$.

The evolution of the state $x_i = \left[q_i^T, \dot{q}_i^T\right]^T$ of Agent i is assumed to be described by the Euler-Lagrange model

$$M_i(q_i)\,\ddot{q}_i + C_i(q_i,\,\dot{q}_i)\,\dot{q}_i + G = u_i + d_i, \tag{1}$$

where $u_i \in \mathbb{R}^n$ is the control input, $M_i(q_i) \in \mathbb{R}^{n\times n}$ is the inertia matrix, $C_i(q_i, \dot{q}_i) \in \mathbb{R}^{n\times n}$ is the matrix of the Coriolis and centripetal terms for Agent i, G accounts for gravitational acceleration supposed to be known and constant, and $d_i \in \mathbb{R}^n$ is a time-varying state perturbation satisfying $\|d_i(t)\| \leqslant D_{\max}$ with known $D_{\max}$.

For each Agent i of the MAS, we consider the following assumptions.

A1) $M_i(q_i)$ is symmetric positive and there exists $k_M > 0$ satisfying $\forall x \in \mathbb{R}^n$, $x^T M_i x \leqslant k_M x^T x$.

A2) $\dot{M}_i(q_i) - 2C_i(q_i, \dot{q}_i)$ is skew symmetric or negative definite and there exists $k_C > 0$ satisfying $\forall x \in \mathbb{R}^n$, $x^T C_i(q_i, \dot{q}_i)\, x \leqslant k_C \|\dot{q}_i\| x^T x$.

A3) For all $\xi_1 \in \mathbb{R}^n$ and $\xi_2 \in \mathbb{R}^n$, the left side of (1) can be linearly parametrized as

$$M_i(q_i)\,\xi_1 + C_i(q_i,\,\dot{q}_i)\,\xi_2 = Y_i(q_i,\,\dot{q}_i,\,\xi_1,\,\xi_2)\,\theta_i, \tag{2}$$

see [18]. $Y_i(q_i,\,\dot{q}_i,\,\xi_1,\,\xi_2)$ is a regressor matrix with known structure, identical for all agents, and $\theta_i \in \mathbb{R}^p$ is a vector of constant parameters known by Agent i via, *e.g.*, an offline identification. Modeling and estimation errors for Y_i and θ_i may be incorporated in $d_i(t)$.

N	number of agents
q_i	coordinates vector of Agent i
q	configuration vector, $q = \left[q_1^T, \ldots, q_N^T\right]^T$
$\dot{q}_i^*$	target reference velocity of Agent i
r_{ij}	relative coordinate vector between Agents i and j, with $r_{ij} = q_i - q_j$
r_{ij}^*	target relative coordinate vector between Agents i and j
r^*	target relative configuration vector with $r^* = [r_{11}^*, r_{12}^*, ..., r_{1N}^*]^T$
x_i	state of Agent i, $x_i = \left[q_i^T, \dot{q}_i^T\right]^T$
$\hat{x}_i^j$	estimate of x_i by Agent j with $\left(\hat{x}_i^j\right)^T = \left[\left(\hat{q}_i^j\right)^T, \left(\dot{\hat{q}}_i^j\right)^T\right]$
$\hat{q}_i^{i,\ell}$	estimate of q_i performed by Agent i using the information in its ℓ-th transmitted message and not in the following one
e_i^j	error between q_i and $\hat{q}_i^j$
$\dot{e}_i^j$	error between $\dot{q}_i$ and $\dot{\hat{q}}_i^j$
$\bar{r}_{ij}$	estimated relative coordinate vector between Agents i and j as evaluated by Agent i with $\bar{r}_{ij} = q_i - \hat{q}_j^i$
k_i	index of k_i-th message sent by Agent i
t_{i,k_i}	transmission time of the k_i-th message sent by Agent i
π	packet loss probability
κ	maximum number of estimators of its own state maintained by each agent
δ_{i,k_i}^j	indicates whether the k_i-th message sent by Agent i has been received by Agent j ($\delta_{i,k_i}^j = 1$) or lost ($\delta_{i,k_i}^j = 0$)
k_i^j	index of the last message received by Agent j among those sent by Agent i
$k_i^{j,i}$	index managed by Agent i of the last message received by Agent j among those sent by Agent i
m_{ij}	potential energy coefficient between Agents i and j
α_i	sum of coefficients m_{ij} for $j \in \mathcal{N}_i$

Table 1
Main notations

A4) x_i is measured without error.

Assumptions A1, A2, A3 and A4 have been previously considered in [11–13,18,19]. The following assumptions are considered for each Agent i:

A5) An estimate $\hat{x}_i^j(0) = \left[\dot{\hat{q}}_i^j(0)^T, \dot{\hat{q}}_i^j(0)^T\right]$ of the state $x_i(0)$ is known by all its neighbors $j \in \mathcal{N}_i$ and the squared norm of the estimation errors $||q_i(0) - \hat{q}_i^j(0)||^2$ and $||\dot{q}_i(0) - \dot{\hat{q}}_i^j(0)||^2$ are bounded with bounds described in Proposition 8, see Section 5.

A6) The velocity $\dot{q}_i$ and acceleration $\ddot{q}_i$ are bounded,

$$||\dot{q}_i(t)|| \leqslant \dot{q}_{\max}, \tag{3}$$
$$||\ddot{q}_i(t)|| \leqslant \ddot{q}_{\max}. \tag{4}$$

Moreover, $\dot{q}_i$ and Lipschitz, *i.e.*, there exists $K_d > 0$ such that $\forall t, \forall \Delta t$,

$$||\dot{q}_i(t + \Delta t) - \dot{q}_i(t)|| \leqslant K_d|\Delta t|. \tag{5}$$

In what follows, the notations M_i and C_i are used in place of $M_i(q_i)$ and $C_i(q_i, \dot{q}_i)$.

2.2 Communication model

When its CTC is satisfied, Agent i broadcasts a message to its neighbors. The packet containing the message is either received without error or is lost. Usually, packet losses are due *i*) to collisions (packets are transmitted at the same time instants by different agents), *ii*) to occlusions by obstacles (two agents are not in line of sight), *iii*) to a signal-to-noise ratio below a certain threshold (agents are too far away).

The packet loss probability between Agents i and j is denoted $\pi_{ij} = \pi_{ji}$. One considers that a communication link exists between two agents i and j if π_{ij} is less than $\pi \leqslant 0.5$. From this hypothesis, the communication topology of the MAS can be described by a undirected graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$, where $\mathcal{E} \subset \mathcal{N} \times \mathcal{N}$ is the set of edges of the graph. Agent i can communicate with its N_i one-hop neighbors with indexes in $\mathcal{N}_i = \{j \in \mathcal{N} | \, (i,j) \in \mathcal{E},\, i \neq j\}$. For each Agent $j \in \mathcal{N}_i$, one has therefore $\pi_{ij} \leqslant \pi$.

In this paper, we assume that $\mathcal{G}$ is connected and invariant with the time. Moreover, to simplify analysis, we assume that $\forall (i,j) \in \mathcal{E},\, \pi_{ij} = \pi$.

Communication delays are neglected: When Agent i broadcasts its k_i-th message at time t_{i,k_i}, Agent $j \in \mathcal{N}_i$ either receives this message without error at time t_{i,k_i} or does not receive it.

Consider a pairs of neighboring agents $(i,j) \in \mathcal{E}$. Let $\{\delta^j_{i,k_i}\}_{k_i \geqslant 1}$ be a sequence of binary variables such that $\delta^j_{i,k_i} = 1$ if the k_i-th message sent by Agent i has been received by Agent j and $\delta^j_{i,k_i} = 0$ else. The δ^j_{i,k_i}s are modeled as realizations of time-invariant memoryless Bernoulli processes with

$$\Pr\left(\delta^j_{i,k_i} = 1\right) = 1 - \pi \tag{6}$$

$$\Pr\left(\delta^j_{i,k_i} = 0\right) = \pi. \tag{7}$$

The model (6)-(7) captures relatively accurately situation *i*). Packet loss events due to collisions are often independent from one communication trial to the next one, provided that there is no synchronization between agents (as in the ALOHA protocol [1]). The considered packet loss model can also represent situation *ii*) provided that obstacles are small or agents move fast enough to experience only very short occlusions. Situation *iii*) is more difficult to represent. Adjusting the transmission power periodically, so as to reach farther agents (even less frequently), may partly address the problem. Nevertheless, this would lead to a time-varying probability π of packet loss. For situations *ii*) and *iii*), one may alternatively consider a modification of the agent communication topology, which is out of the scope of this paper. Some works use feedback to partially solve the problem, but feedback requires extra communications and so increases the risk of collision between packets, as described in situation *i*). This is why, here, the only feedback information considered is that received from packets sent by other agents, when this own CTC is satisfied.

2.3 Message content

Let $k^i_j \leqslant k_j$ be the index of the last message Agent i has received from its neighbor j. When a communication is triggered at time t_{i,k_i}, Agent i broadcasts a message containing k_i, t_{i,k_i}, $x_i\left(t_{i,k_i}\right)$, θ_i, and $\{k^i_j\}, j \in \mathcal{N}_i$. By sending $k^i_j \leqslant k_j$ for all $j \in \mathcal{N}_i$, Agent i indicates the index of the last message received from each of its neighbors. This can be considered as an implicit acknowledgment mechanism for the neighbors $j \in \mathcal{N}_i$.

When Agent j receives a message from Agent i, it updates k^j_i to k_i. Moreover, $x_i\left(t_{i,k_i}\right)$ and θ_i are used to update its estimator of the state of Agent i, as detailed in Section 4.1. Finally, Agent j keeps track in the variables $k^{i,j}_j$ of the value of k^i_j which represents the index of the last message sent by Agent j and which has been actually received by Agent i. The index $k^{i,j}_j$ is used by Agent j to evaluate the knowledge Agent i has about x_j (see the example in Figure 1).

2.4 Target formation

A potentially time-varying target formation is defined by the set $\mathcal{R}(t) = \left\{r^*_{ij}\left(t\right), (i,j) \in \mathcal{N} \times \mathcal{N}\right\}$, where $r^*_{ij}\left(t\right)$ is the target relative coordinate vector between Agents i and j. Without loss of generality, the first agent is considered as the reference agent. Any target relative coordinate vector r^*_{ij} can be expressed as $r^*_{ij}\left(t\right) = r^*_{i1}\left(t\right) - r^*_{j1}\left(t\right)$. The target relative *configuration* vector is $r^*\left(t\right) = \left[\, r^{*T}_{11}\left(t\right) \;\ldots\; r^{*T}_{1N}\left(t\right) \,\right]^T$. Each Agent i is assumed to only know the relative coordinate vector with its direct neighbors $r^*_{ij}\left(t\right)$, $j \in \mathcal{N}_i$. Additionally, a constant target reference velocity $\dot{q}^*_1$ known by all agents is imposed to the MAS. The reference velocities $\dot{q}^*_i$ are expressed as $\dot{q}^*_i = \dot{q}^*_1 + \dot{r}^*_{i1}$ and are assumed to satisfy

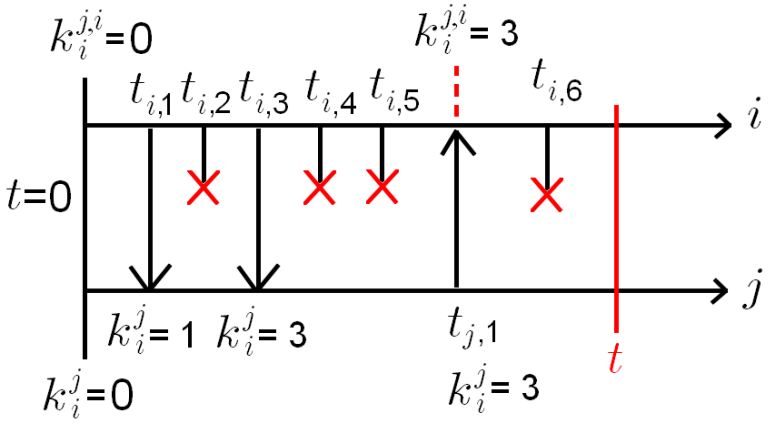


Fig. 1. Communication instants between Agents i and j and evolution of the indexes k_i^j and $k_i^{j,i}$ of last message received; from the packet received at time $t_{j,1}$, Agent i can deduce that Agent j has received the packet sent at time $t_{i,3}$ from the content of the packet it receives from Agent j at time $t_{j,1}$.

A7) *For all Agents i, the target velocity $\dot{q}_i^*$ and acceleration $\ddot{q}_i^*$ are bounded such that*

$$||\dot{q}_i^*(t)|| < \dot{q}_{\max}, \tag{8}$$
$$||\ddot{q}_i^*(t)|| < \ddot{q}_{\max}, \tag{9}$$

for $i = 1, \ldots, N$. Moreover, $\dot{q}_i^*$ is Lipschitz with constant $K_d^* \leqslant K_d$, i.e. $\forall t\ \Delta t$,

$$\|\dot{q}_i^*(t + \Delta t) - \dot{q}_i^*(t)\| \leqslant K_d^* |\Delta t|. \tag{10}$$

Our aim is to evaluate the control input for each Agent in a distributed way so that the MAS converges to $\mathcal{R}(t)$, while limiting the number of communications between agents and accounting for losses. For that purpose, the control input of each Agent has to provide an asymptotic convergence of the MAS to the target configuration vector with a bounded Mean-Square Error (MSE).

Definition 1 *The MAS asymptotically mean-square converges to the target formation with a bounded MSE (bounded average asymptotic convergence) iff there exists some $\varepsilon_1 > 0$ such that*

$$\forall (i,j) \in \mathcal{N}^2, \lim_{t \to \infty} \mathbb{E}\left(\left\|r_{ij}(t) - r_{ij}^*(t)\right\|^2\right) \leqslant \varepsilon_1, \tag{11}$$

where the expectation is evaluated considering the packet loss events.

2.5 Overview of the proposed approach

A control law is introduced in Section 3 to drive the MAS to its target formation and reference speed in a distributed way. This requires the knowledge by each Agent of the state vector of its neighbors. Since the state vector of a neighbor Agent j is only available at Agent i when Agent j broadcasts its state, Agent i has to maintain an estimator of the state of each of its neighbors. This estimator is described in Section 4.

To determine the quality of the estimate of x_i evaluated by its neighbors, Agent i has also to estimate its own state x_i with the information it has transmitted to its neighbors. As soon as a function of the error between this estimate and x_i reaches some threshold, Agent i triggers a communication to allow its neighbors to refresh their estimate of x_i. The main difficulty, compared to [19,21], lies in the fact that estimators have to account for packet losses. In the solution proposed here, each Agent maintains several estimates of its own state accounting for different packet loss hypotheses, and an estimate of the state of its neighbors with the last information received. As will be seen in Section 4.4, the number of hypotheses can be limited to a manageable amount determined by the last received packet from Agent i.

Usually, a CTC relies on the error between the states of agents and the state estimates evaluated by neighboring agents. Here, since this error cannot be exactly obtained due to packet losses, the CTC involves the MSE between the state of an agent and its estimate evaluated by its neighbors, see Section 5. This paper proposes different methods to evaluate or upper-bound this MSE, which is then used to analyze the convergence and the stability of the MAS.

3 Distributed control inputs

Section 3.1 introduces the potential energy $P(q,\,t)$ of the MAS to quantify the discrepancy between the current and target formations. A control input minimising $P(q,\,t)$ by exploiting the agent state estimators is presented in Section 3.2.

3.1 Potential energy of the MAS

In [14,27], consider the *potential energy* of the MAS

$$P(q,\,t) = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} m_{ij}\left\|r_{ij} - r_{ij}^{*}\right\|^{2}, \tag{12}$$

is introduced, where $m_{ij} = m_{ji}$ are some positive or null coefficients. $P(q,t)$ quantifies the discrepancy between the actual and target relative coordinate vectors. We take $m_{ii} = 0$, $m_{ij} = 0$ if $(i,j) \notin \mathcal{E}$, and $m_{ij} > 0$ if $(i,j) \in \mathcal{E}$. Since $\mathcal{G}$ is connected, the minimum number of non-zero coefficients m_{ij} to properly define a target formation is $N-1$.

Proposition 2 *The MAS asymptotically converges to the target formation with a bounded MSE iff there exists some $\varepsilon_2 > 0$ such that*

$$\lim_{t\to\infty} \mathbb{E}\left(P(q,\,t)\right) \leqslant \varepsilon_2, \tag{13}$$

where the expectation is evaluated considering the packet loss events.

The proof of Proposition 2 is provided in Appendix A.1.

3.2 Control input with Agent state estimators

In what follows, a control law is designed for each agent so that the MAS asymptotically converges with a bounded MSE. The control law requires only local knowledge of the agent and can therefore be implemented in a distributed way. It has to make $P(q,\,t)$ decrease. One introduces, as in [27],

$$g_i = \frac{\partial P(q,\,t)}{\partial q_i} = \sum_{j\in\mathcal{N}_i} m_{ij}\left(r_{ij} - r_{ij}^{*}\right), \tag{14}$$

$$\dot{g}_i = \sum_{j\in\mathcal{N}_i} m_{ij}\left(\dot{r}_{ij} - \dot{r}_{ij}^{*}\right), \tag{15}$$

$$s_i = \dot{q}_i - \dot{q}_i^{*} + k_p g_i, \tag{16}$$

where $\dot{q}_i^{*} = \dot{q}_1^{*} - \dot{r}_{1i}^{*}$ is the reference velocity of Agent i. The vectors g_i and $\dot{g}_i$ characterize the evolution with q_i and $\dot{q}_i$ of the discrepancy between the actual and target relative coordinate vectors. In (16), $k_p > 0$ is a scalar design parameter. When the agents are far from the target formation, g_i is large and determines the direction for Agent i to get closer to the target formation.

To make $P(q,\,t)$ decrease, Agent i has to evaluate (14). The control input of Agent i requires r_{ij}, and thus q_j, $j \in \mathcal{N}_i$. Nevertheless, q_j is only available to Agent i when it receives a packet from Agent j containing q_j, see Section 2.2. Between the reception of two packets from Agent j, an estimate $\hat{q}_j^{i}$ of $q_j, j \in \mathcal{N}_i$ needs to be evaluated, see Section 4.1. Using estimates $\hat{q}_j^{i}$ and $\hat{\dot{q}}_j^{i}$ of q_j and $\dot{q}_j$ for all $j \in \mathcal{N}_i$, Agent i is able to evaluate the discrepancies $\bar{r}_{ij} = q_i - \hat{q}_j^{i}$, $\dot{\bar{r}}_{ij} = \dot{q}_i - \hat{\dot{q}}_j^{i}$ between its own state and the estimate of the state of the neighbors, as well as

$$\bar{g}_i = \sum_{j\in\mathcal{N}_i} m_{ij}\left(\bar{r}_{ij} - r_{ij}^{*}\right), \tag{17}$$

$$\dot{\bar{g}}_i = \sum_{j\in\mathcal{N}_i} m_{ij}\left(\dot{\bar{r}}_{ij} - \dot{r}_{ij}^{*}\right), \tag{18}$$

$$\bar{s}_i = \dot{q}_i - \dot{q}_i^{*} + k_p \bar{g}_i. \tag{19}$$

Then, the following control input (to be used in (1)) can be evaluated in a distributed way by each Agent $i \in \mathcal{N}$, *i.e.*, using only vectors available locally

$$u_i = -k_s \bar{s}_i - k_g \bar{g}_i + G - Y_i\left(q_i,\,\dot{q}_i,\,\dot{\bar{p}}_i,\,\bar{p}_i\right)\theta_i, \tag{20}$$

where $\bar{p}_i = k_p \bar{g}_i - \dot{q}_i^{*}$ and $\dot{\bar{p}}_i = k_p \dot{\bar{g}}_i - \ddot{q}_i^{*}$ with the additional design parameters $k_g > 0$ and $k_s \geqslant 1 + k_p\left(k_M + 1\right)$. In (20), $\bar{s}_i$ maintain the formation at the reference velocity, $\bar{g}_i$ drives the agent to the target formation, G compensates the action of the gravity, and $Y_i\left(q_i,\,\dot{q}_i,\,\dot{\bar{p}}_i,\,\bar{p}_i\right)\theta_i$ compensates the inertia, Coriolis and centripetal terms of the dynamic of the agents. The convergence properties of the MAS when each agent applies (20) is analyzed and ensured in Section 5.

4 State estimators and packet losses

Section 4.1 introduces the estimators involved in the control input (20) applied by each agent. Section 4.2 describes the way Agent i estimates its own state x_i, with the information transmitted to its neighbors, to determine the quality of their estimates of x_i. In Section 4.3, the MSE between the current state x_i and its remote estimates $\hat{x}_i^j, j \in \mathcal{N}_i$ is evaluated. In Section 4.4, packets received from other agents are exploited to improve the evaluation of the MSE of the estimate $\hat{x}_i^j$ of x_i.

4.1 Estimation of the state of other agents

To evaluate (20), Agent i has to maintain an estimate $\hat{x}_j^i$ of the state x_j of all its neighbors $j \in \mathcal{N}_i$. Assume that Agent j broadcasts its k-th message at time $t_{j,k}$. Then, since communication delays are neglected, depending on whether this message has been received by Agent i, $\hat{x}_j^i$ is updated as follows, see [3]

$$\hat{x}_j^i\left(t_{j,k}^+\right) = \delta_{j,k}^i x_j\left(t_{j,k}\right) + \left(1-\delta_{j,k}^i\right)\hat{x}_j^i\left(t_{j,k}^-\right), \tag{21}$$

where $x_j(t_{j,k})$ is obtained from the received packet, where $\hat{x}_j^i\left(t_{j,k}^-\right)$ is the value of the state estimate at t_{i,k_i} before the update and $\hat{x}_j^i\left(t_{j,k}^+\right)$ is its value after the update. For all $t \geqslant t_{j,k}$ and up to the time instant of reception of the next packet sent by Agent j, the components $\hat{q}_j^i$ and $\dot{\hat{q}}_j^i$ of $\hat{x}_j^i$ evolve as

$$M_j\left(\hat{q}_j^i\right)\ddot{\hat{q}}_j^i + C_j\left(\hat{q}_j^i, \dot{\hat{q}}_j^i\right)\dot{\hat{q}}_j^i + G = \hat{u}_j^i. \tag{22}$$

where M_j and C_j are evaluated using (2) with Y_j and $\hat{\theta}_j^i = \theta_j$, where the structure of Y_j and θ_j are initially known by Agent i or have been transmitted by Agent j at time $t = 0$. The estimator (22) maintained by Agent i requires itself an estimate $\hat{u}_j^i$ of the control input u_j evaluated by Agent j. This estimate $\hat{u}_j^i$, used by Agent i, is chosen as

$$\hat{u}_j^i = -k_s\dot{\hat{\varepsilon}}_j^i + G - Y_j\left(\hat{q}_j^i, \dot{\hat{q}}_j^i, -\ddot{q}_j^*, -\dot{q}_j^*\right)\hat{\theta}_j^i, \tag{23}$$

with $\dot{\hat{\varepsilon}}_j^i = \dot{\hat{q}}_j^i - \dot{q}_j^*$. The control input (23) only depends on information available to Agent i Therefore, (23) has been built from (20) by removing all terms unknown by the neighbors of Agent j. Consequently $\bar{g}_j$ and $\dot{\bar{g}}_j$ cannot be used, and x_j is replaced by $\hat{x}_j^i$. Since (23) differs from (20), $\hat{x}_j^i$ will progressively diverge from x_j. Since each agent runs an estimator of its own state with the information available to its neighbors, it can trigger a communication when the discrepancy is too large. When the MAS is close to the target formation, the term $\dot{\bar{g}}_j$ becomes negligible and (20) and (23) get closer. This choice limits the number of state estimate hypotheses to consider, see Section 4.2.

We consider Assumptions A8 and A9 on the components of $\hat{x}_j^i$:

A8) The velocity $\dot{\hat{q}}_j^i$ and acceleration $\ddot{\hat{q}}_j^i$ are bounded

$$||\dot{\hat{q}}_j^i|| \leqslant \dot{q}_{\max} \tag{24}$$
$$||\ddot{\hat{q}}_j^i|| \leqslant \ddot{q}_{\max}. \tag{25}$$

Moreover, $\dot{\hat{q}}_j^i$ is Lipschitz on all intervals $[t_{j,k}, t_{j,k+1}[$, *i.e* $\forall t \in [t_{j,k}, t_{j,k+1}[$ and $(t+\Delta t) \in [t_{j,k}, t_{j,k+1}[$ such that

$$\left\|\dot{\hat{q}}_j^i\left(t+\Delta t\right) - \dot{\hat{q}}_j^i\left(t\right)\right\| \leqslant \hat{K}_d|\Delta t|. \tag{26}$$

with $\hat{K}_d > 0$.

This assumtion is consistent with that considered for $\dot{q}_j$, *i.e* Assumption A6, since between two communication time instants, (22) is similar to (1).

A9) There exists $e_{\max} > 0$ such that the norm of the estimation error satisfies

$$||q_i\left(t\right) - \hat{q}_i^j\left(t\right)|| \leqslant e_{\max}. \tag{27}$$

This assumption is reasonable for MAS evolving in some limited geographical area.

4.2 Multi-hypothesis state estimates

The estimate $\hat{q}_i^j$ of the state of Agent i, evaluated by Agent j, only depends on the information provided by Agent i. The estimate $\hat{q}_i^j$ is reset to q_i as soon as a message sent by Agent i is received by Agent j, see (21). Consequently, when Agent i has sent k_i messages, and wants to evaluate an image of its own state as computed by one of its neighbors, k_i different hypotheses have to be considered, each of which is associated to a different estimator of q_i at time $t \in [t_{i,k_i}, t_{i,k_i+1}[$:

- A first estimator considers the k_i-th packet as received,
- A second estimator considers the k_i-th packet as lost, but the $k_i - 1$-th packet as received,
- ...
- A last estimator considers that no packet has been received, and uses the initial state estimate $\hat{x}_i^j(0)$.

At time $t \in [t_{i,k_i},\ t_{i,k_i+1}[$, the state estimates corresponding to these hypotheses are denoted as

$$\hat{x}_i^{i,\ell}(t) = \left[\hat{q}_i^{i,\ell}(t), \dot{\hat{q}}_i^{i,\ell}(t)\right], \tag{28}$$

with $\ell = 0, \ldots, k_i$ and $\hat{x}_i^{i,k_i} = \hat{x}_i^i$.

Since k_i can become very large, we impose that Agent i maintains at most κ estimates of x_i, denoted $\hat{x}_i^{(1)}(t), ..., \hat{x}_i^{(\kappa)}(t)$. For all $t \in [t_{i,k_i}, t_{i,k_i+1}[$, $k_i \geqslant \kappa$, one has $\hat{x}_i^{(1)}(t) = \hat{x}_i^{i,k_i}(t), \ldots, \hat{x}_i^{(\kappa)}(t) = \hat{x}_i^{i,k_i-\kappa+1}(t)$. These estimates evolve according to the dynamics (22)-(23) introduced in Section 4.1. When a new packet is sent by Agent i at time t_{i,k_i+1}, the estimates are updated as

$$\hat{x}_i^{(1)}\left(t_{i,k_i+1}^+\right) = x_i\left(t_{i,k_i+1}^-\right). \tag{29}$$

$$\hat{x}_i^{(\ell+1)}\left(t_{i,k_i+1}^+\right) = \hat{x}_i^{(\ell)}\left(t_{i,k_i+1}^-\right),\ \ell = 1, \ldots, \kappa - 1. \tag{30}$$

4.3 Expected value of the estimation error of $x_i(t)$

At time $t \in [t_{i,k_i},\ t_{i,k_i+1}[$, Agent i has sent k_i packets. Let

$$p_{k_i,\ell}^j = \Pr\left(\delta_{i,\ell}^j = 1, \delta_{i,\ell+1}^j = 0, \ldots, \delta_{i,k_i}^j = 0\right) \tag{31}$$

with $0 \leqslant \ell \leqslant k_i$, be the probability that the ℓ-th packet has been received by a given neighbor j and that all packets from the $\ell+1$-th to the k_i-th have been lost. By convention,

$$p_{k_i,0}^j = \Pr\left(\delta_{i,1}^j = 0, \ldots, \delta_{i,k_i}^j = 0\right)$$

and

$$p_{k_i,k_i}^j = \Pr\left(\delta_{i,k_i}^j = 1\right). \tag{32}$$

Note that $p_{k_i,\ell}^j$ only depends on the packet loss probability π of the packet loss model (6)-(7), and does not depend on the neighbor index j, which is omitted in what follows.

Proposition 3 *One has*

$$p_{1,1} = 1 - \pi \tag{33}$$

$$p_{1,0} = \pi. \tag{34}$$

For all $k_i > 0$ and $\ell < k_i$,

$$p_{k_i,\ell} = \pi p_{k_i-1,\ell}. \tag{35}$$

Moreover

$$p_{k_i,k_i} = 1 - \pi. \tag{36}$$

The proof of Proposition 3 is in Appendix A.2.

At time $t \in [t_{i,k_i},\, t_{i,k_i+1}[$, the estimation error of the coordinates of Agent i, as evaluated by Agent j, is

$$e_i^j\,(t) = \hat{q}_i^j\,(t) - q_i\,(t)\,. \tag{37}$$

Since Agent i does not know the index of the last packet received by Agent j among those it has sent, Agent i cannot evaluate (37). Alternatively, Agent i can evaluate the mean square value of $e_i^j(t)$ considering the estimates $\hat{q}_i^{i,\ell}$ and the associated probabilities $p_{k_i,\ell}$

$$\mathbb{E}\left(||e_i^j\,(t)\,||^2\right) = \sum_{\ell=0}^{k_i} p_{k_i,\ell}||\hat{q}_i^{i,\ell}\,(t) - q_i\,(t)\,||^2. \tag{38}$$

Since Agent i maintains only κ estimators of x_i, it cannot evaluate (38) when $k_i > \kappa$. Nevertheless, using Assumption A9, the MSE (38) can be upper-bounded.

Proposition 4 *The MSE (38) can be expressed or upper-bounded as*

$$\begin{cases} \mathbb{E}\left(||e_i^j\,(t)\,||^2\right) = \sum_{\ell=0}^{k_i} p_{k_i,\ell}||\hat{q}_i^{i,\ell}\,(t) - q_i\,(t)\,||^2 & \text{if } k_i \leqslant \kappa \\ \mathbb{E}\left(||e_i^j\,(t)\,||^2\right) \leqslant \overline{\mathbb{E}}\left(\left\|e_i^j\,(t)\right\|^2\right) & \text{else,} \end{cases} \tag{39}$$

where

$$\overline{\mathbb{E}}\left(\left\|e_i^j\,(t)\right\|^2\right) = \sum_{\ell=\ell_{\min}}^{k_i} p_{k_i,\ell}\left\|\hat{q}_i^{i,\ell}\,(t) - q_i\,(t)\right\|^2 + \min\left(\left[\overline{e}_{i,k_i}^j\,(t)\,,\, \sigma_i^j\,(k_i,\kappa)e_{\max}^2\right]\right) \tag{40}$$

with $\sigma_i^j\,(k_i,\kappa) = 1 - \sum_{\ell=k_i-\kappa+1}^{k_i} p_{k_i,\ell}$, $\ell_{\min} = \max\,(0, k_i - \kappa + 1)$ *and*

$$\overline{e}_{i,k_i}^j\,(t) = 4\dot{q}_{\max}e_{\max}\sigma_i^j\,(k_i,\kappa)\,(t - t_{i,k_i}) + \pi\overline{\mathbb{E}}\left(\left\|e_i^j\left(t_{i,k_i}^-\right)\right\|^2\right) - \sum_{\ell=\ell_{\min}}^{k_i} p_{k_i,\ell}^j\left\|\hat{q}_i^{i,\ell}\left(t_{i,k_i}^+\right) - q_i\left(t_{i,k_i}^+\right)\right\|^2. \tag{41}$$

Similar bounds can be obtained for

$$\overline{\mathbb{E}}\left(\left\|\dot{e}_i^j\,(t)\right\|^2\right) = \sum_{\ell=\ell_{\min}}^{k_i} p_{k_i,\ell}\left\|\dot{\hat{q}}_i^{i,\ell}\,(t) - \dot{q}_i\,(t)\right\|^2 + \min\left(\left\{\dot{\overline{e}}_{i,k_i}^j\,(t)\,,\, 4\sigma_i^j\,(k_i,\kappa)\dot{q}_{\max}^2\right\}\right) \tag{42}$$

$$\overline{\mathbb{E}}\left(\left\|e_i^j\,(t)\right\|^4\right) = \sum_{\ell=\ell_{\min}}^{k_i} p_{k_i,\ell}\left\|\hat{q}_i^{i,\ell}\,(t) - q_i\,(t)\right\|^4 + \min\left(\left\{\overline{e}_{i,k_i}^{j,2}\,(t)\,,\, \sigma_i^j\,(k_i,\kappa)e_{\max}^4\right\}\right) \tag{43}$$

$$\overline{\mathbb{E}}\left(\left\|\dot{e}_i^j\,(t)\right\|^4\right) = \sum_{\ell=\ell_{\min}}^{k_i} p_{k_i,\ell}\left\|\dot{\hat{q}}_i^{i,\ell}\,(t) - \dot{q}_i\,(t)\right\|^4 + \min\left(\left\{\dot{\overline{e}}_{i,k_i}^{j,2}\,(t)\,,\, 16\sigma_i^j\,(k_i,\kappa)\dot{q}_{\max}^4\right\}\right) \tag{44}$$

with

$$\dot{\overline{e}}_{i,k_i}^j\,(t) = 8\dot{q}_{\max}\ddot{q}_{\max}\sigma_i^j\,(k_i,\kappa)\,(t - t_{i,k_i}) + \pi\overline{\mathbb{E}}\left(\left\|\dot{e}_i^j\left(t_{i,k_i}^-\right)\right\|^2\right) - \sum_{\ell=\ell_{\min}}^{k_i} p_{k_i,\ell}^j\left\|\dot{\hat{q}}_i^{i,\ell}\left(t_{i,k_i}^+\right) - \dot{q}_i\left(t_{i,k_i}^+\right)\right\|^2 \tag{45}$$

$$\overline{e}_{i,k_i}^{j,2}\,(t) = 8\dot{q}_{\max}e_{\max}^3\sigma_i^j\,(k_i,\kappa)\,(t - t_{i,k_i}) + \pi\overline{\mathbb{E}}\left(\left\|e_i^j\left(t_{i,k_i}^-\right)\right\|^4\right) - \sum_{\ell=\ell_{\min}}^{k_i} p_{k_i,\ell}^j\left\|\hat{q}_i^{i,\ell}\left(t_{i,k_i}^+\right) - q_i\left(t_{i,k_i}^+\right)\right\|^4 \tag{46}$$

$$\dot{\overline{e}}_{i,k_i}^{j,2}\,(t) = 32\ddot{q}_{\max}\dot{q}_{\max}^3\sigma_i^j\,(k_i,\kappa)\,(t - t_{i,k_i}) + \pi\overline{\mathbb{E}}\left(\left\|\dot{e}_i^j\left(t_{i,k_i}^-\right)\right\|^4\right) - \sum_{\ell=\ell_{\min}}^{k_i} p_{k_i,\ell}^j\left\|\dot{\hat{q}}_i^{i,\ell}\left(t_{i,k_i}^+\right) - \dot{q}_i\left(t_{i,k_i}^+\right)\right\|^4. \tag{47}$$

The proofs of Proposition 4 is provided in Appendix A.7. In the condition of Proposition 4 without implicit feed-back, one always has $\sigma_i^j(k_i,\kappa)=\sum_{\ell=0}^{k_i-\kappa}p_{k_i,\ell}=\pi^\kappa$.

Using (39)-(47), Agent i is able to determine the quality of the estimate of x_i evaluated by its neighbors.

Using (39), Agent i is able to determine the quality of the estimate of x_i evaluated by its neighbors. The choice of κ impacts the upper bound of $\mathbb{E}\left(||e_i^j(t)||^2\right)$ and $\mathbb{E}\left(||\dot{e}_i^j(t)||^2\right)$ in (39). A large value of κ reduces the influence of $e_{\max}$ and $\dot{q}_{\max}$ on the MSE. Nevertheless, κ should not be chosen too large to limit the number of estimators of its own state maintained by Agent i.

4.4 Estimates accounting for received packets (implicit feed-back)

Consider Agent i, the time interval $[t_{i,k_i},t_{i,k_i+1}[$, and assume that $[t_{i,\rho_i^j},t_{i,\rho_i^j+1}[$ is the time interval during which the last packet has been received from Agent j. This packet contains the index k_i^j of the last message received by Agent j and sent by Agent i, as illustrated by Figure 1. This index is kept by Agent i in $k_i^{j,i}$, see Section 2.2. This implicit acknowledgment can significantly improve the evaluation of the mean-square values of $e_i^j(t)$ and $\dot{e}_i^j(t)$. From this message, Agent i knows that all packets sent in the time interval $[t_{i,k_i^j+1},t_{i,\rho_i^j+1}[$ have not been received by Agent j.

Consider again the example in Figure 1. The packet received in $[t_{i,k_i},t_{i,k_i+1}[$ with $k_i^j=k_i-2$ indicates that packet k_i-2 has been received, but neither packet k_i-1 nor k_i.

Using this knowledge, Agent i can evaluate the probability

$$p^j_{k_i,\ell|k_i^{j,i},\rho_i^j}=\Pr\left(\delta^j_{i,\ell}=1,\sum\nolimits_{m=\ell+1}^{k_i}\delta^j_{i,m}=0\,\middle|\,\delta^j_{i,k_i^j}=1,\sum\nolimits_{m=k_i^{j,i}+1}^{\rho_i^j}\delta^j_{i,m}=0\right) \tag{48}$$

that the ℓ-th message sent by Agent i (with $k_i^j\leqslant\ell\leqslant k_i$) has been received by Agent j and that all following messages, including the k_i-th have been lost. By convention,

$$p^j_{k_i,k_i|k_i^{j,i},\rho_i^j}=\Pr\left(\delta^j_{i,k_i}=1|\delta^j_{i,k_i^{j,i}}=1,\sum\nolimits_{m=k_i^{j,i}+1}^{\rho_i^j}\delta^j_{i,m}=0\right). \tag{49}$$

Proposition 5 *As long as Agent i has not received any message from Agent j, $p^j_{k_i,\ell|0,0}$ is evaluated for all $k_i>0$ and $\ell\leqslant k_i$ as*

$$p^j_{k_i,\ell|0,0}=(1-\pi)\,\pi^{k_i-\ell}\quad \textit{if } \ell>0, \tag{50}$$
$$p^j_{k_i,0|0,0}=\pi^{k_i}\quad \textit{else.} \tag{51}$$

If Agent i receives a message from Agent j at $t_{j,k_j}\in[t_{i,k_i},t_{i,k_i+1}[$ containing k_i^j, then $k_i^{j,i}=k_i^j$, $\rho_i^j=k_i$, and

$$p^j_{k_i,k_i^{j,i}|k_i^{j,i},k_i}=1 \tag{52}$$
$$p^j_{k_i,\ell|k_i^{j,i},k_i}=0\quad\forall\ell\leqslant k_i,\ell\neq k_i^{j,i}. \tag{53}$$

Consider $t\in[t_{i,k_i+n},t_{i,k_i+n+1}[$ with $n>0$ and assume that the last message received by Agent i from Agent j has been at time $t_{j,k_j}\in[t_{i,k_i},t_{i,k_i+1}[$. Consequently, $k_i^{j,i}\leqslant k_i$, and one has still $\rho_i^j=k_i$. Then $p^j_{k_i+n,\ell|k_i^{j,i},k_i}$ is evaluated recursively for all $\ell=0,\ldots,k_i+n$ as

$$p^j_{k_i+n,k_i^{j,i}|k_i^{j,i},k_i}=\pi p^j_{k_i+n-1,k_i^{j,i}|k_i^{j,i},k_i} \tag{54}$$
$$\begin{aligned}p^j_{k_i+n,\ell|k_i^{j,i},k_i}&=\pi p^j_{k_i+n-1,\ell|k_i^{j,i},k_i}\quad \textit{if } k_i<\ell<k_i+n\\&=0\quad \textit{if } \ell<k_i \textit{ and } \ell\neq k_i^{j,i}\end{aligned} \tag{55}$$
$$p^j_{k_i+n,k_i+n|k_i^{j,i},k_i}=1-\pi. \tag{56}$$

		k_i+n					
		5	message from Agent j	6	7	8	9
ℓ	0	π^5	0	0	0	0	0
	1	$(1-\pi)\,\pi^4$	0	0	0	0	0
	2	$(1-\pi)\,\pi^3$	0	0	0	0	0
	3	$(1-\pi)\,\pi^2$	1	π	π^2	π^3	π^4
	4	$(1-\pi)\,\pi$	0	0	0	0	0
	5	$1-\pi$	0	0	0	0	0
	6	$*$	$*$	$1-\pi$	$(1-\pi)\,\pi$	$(1-\pi)\,\pi^2$	$(1-\pi)\,\pi^3$
	8	$*$	$*$	$*$	$1-\pi$	$(1-\pi)\,\pi$	$(1-\pi)\,\pi^2$
	9	$*$	$*$	$*$	$*$	$1-\pi$	$(1-\pi)\,\pi$
	10	$*$	$*$	$*$	$*$	$*$	$1-\pi$

Table 2
Probabilities $p_{k_i+n,\ell|k_i^{j,i},k_i}$ that the ℓ-th message sent by Agent i has been received by Agent j and all following messages including the k_i-th one have been lost, for $n \in [0,\ldots,5]$, $k_i = 5$; $*$ represents probabilities not defined. In the time interval $[t_{k_i,5},\, t_{k_i,6}[$, a message is received from Agent j indicating that the last message it has received from Agent i is message $k_i = 3$.

The proof of Proposition 5 is in Appendix A.3.

Table 2 illustrates the evolution of $p_{k_i+n,\ell|k_i^{j,i},k_i}$ as a function of n when $\kappa = 3$, $k_i = 5$, and $k_i^j = 3$.

Proposition 5 can be used by Agent i to evaluate $\mathbb{E}\left(||e_i^j(t)||^2\right)$, taking into account the implicit acknowledgement provided by neighbors as follows.

Proposition 6 *Consider some Agent i and $k_i > 0$. Assume that Agent i knows the index k_i^j of the last message sent by Agent i and received by some neighbor Agent j. At time $t \in [t_{i,k_i+n},\, t_{i,k_i+n+1}[$, one has*

$$\mathbb{E}\left(\left\|e_i^j(t)\right\|^2 |k_i^{j,i}\right) \leqslant \overline{\mathbb{E}}\left(\left\|e_i^j(t)\right\|^2 |k_i^{j,i}\right), \tag{57}$$

where

$$\begin{aligned}\overline{\mathbb{E}}\left(\left\|e_i^j(t)\right\|^2 |k_i^{j,i}\right) = &\sum_{\ell=\ell_{\min}}^{k_i+n} p^j_{k_i+n,\ell|k_i^{j,i},k_i} \left\|\hat{q}_i^{i,\ell}(t) - q_i(t)\right\|^2 \\ &+ 1_{k_i>\kappa} 1_{k_i-k_i^{j,i}>\kappa} \times \min\left(\left\{\bar{e}^j_{i,k_i+n}(t),\, \sigma_i^j\left(k_i+n,\kappa|k_i^{j,i},k_i\right)e^2_{\max}\right\}\right)\end{aligned} \tag{58}$$

with $\ell_{\min} = \max\left(\{0, k_i+n-\kappa+1\}\right)$, $\sigma_i^j\left(k_i+n,\kappa|k_i^{j,i},k_i\right) = 1 - \sum_{\ell=k_i-\kappa+1}^{k_i} p^j_{k_i+n,\ell|k_i^{j,i},k_i}$ *and*

$$1_{k_i>\kappa} = \begin{cases} 1 & \text{if } k_i > \kappa \\ 0 & \text{else,} \end{cases} \tag{59}$$

$$\begin{aligned}\bar{e}^j_{i,k_i+n}(t) = &\, 4\dot{q}_{\max} e_{\max} \sigma_i^j\left(k_i+n,\kappa|k_i^{j,i},k_i\right)(t - t_{i,k_i+n}) \\ &+ \pi\overline{\mathbb{E}}\left(\left\|e_i^j\left(t^-_{i,k_i+n}\right)\right\|^2 |k_i^{j,i}\right) - \sum_{\ell=\ell_{\min}}^{k_i+n} p^j_{k_i+n,\ell|k_i^{j,i},k_i}\left\|\hat{q}_i^{i,\ell}\left(t^+_{i,k_i+n}\right) - q_i\left(t^+_{i,k_i+n}\right)\right\|^2.\end{aligned} \tag{60}$$

Contrary to (39), (58) depends now on the index of the neighbor Agent j via $k_i^{j,i}$, and so is updated by its neighbor when Agent i receives a packet, in addition to the update made each time Agent i broadcast a message as in (39). Note that, in opposition with Proposition 4 without implicit acknowledgement, the value of σ_i^j mostly depend of the messages received from Agent j.

Similar results can be obtained for $\mathbb{E}\left(||\dot{e}_i^j(t)||^2 |k_i^{j,i}\right)$. In what follows, the notation $\mathbb{E}(||e_i^j(t)||^2)$ is used in place of $\mathbb{E}(||e_i^j(t)||^2 |k_i^{j,i})$.

5 Event-triggered communications accounting for packet losses

This section presents a CTC which may involve one of the state estimators introduced in Section 4.

Let $m_{\min} = \min_{i,j=1,\ldots,N} \{m_{ij} \neq 0\}$, $m_{\max} = \max_{i,j=1,\ldots,N} \{m_{ij}\}$, $N_{\min} = \min_{i=1\ldots N}\{N_i\}$, $\alpha_i = \sum_{j=1}^{N} m_{ij}$, and $\alpha_{\mathrm{M}} = \max_{i=1,\ldots,N} \alpha_i$. The distributed CTC (61) presented in Theorem 7 is designed to ensure an asymptotic convergence of the MAS to the target formation with a bounded MSE.

Theorem 7 *Consider a MAS with agent dynamics given by (1), the communication protocol defined in Section 2.2, and the control law (20). In absence of communication delays, and with a packet loss model satisfying (6)-(7), if the communications are triggered by each Agent i of the MAS when the following condition is satisfied*

$$\begin{aligned}
&\alpha_M \left[\sum_{j=1}^{N} m_{ij} \left(k_e \overline{\mathbb{E}} \left(\left\|e_i^j\right\|^2\right) + k_p k_M \overline{\mathbb{E}} \left(\left\|\dot{e}_i^j\right\|^2\right)\right) + k_p k_C^2 \right. \\
&\left. \times \sum_{j=1}^{N} m_{ij} \left(2\overline{\mathbb{E}} \left(\left\|e_i^j\right\|^2\right) \left\|\dot{\hat{q}}_j^i\right\|^2 + \overline{\mathbb{E}} \left(\left\|e_i^j\right\|^4\right) + \overline{\mathbb{E}} \left(\left\|\dot{e}_i^j\right\|^4\right)\right)\right] \\
&+ k_g b_i \left\|\dot{q}_i - \dot{q}_i^*\right\|^2 \geqslant k_s \bar{s}_i^T \bar{s}_i + k_p k_g \bar{g}_i^T \bar{g}_i + \eta
\end{aligned} \tag{61}$$

where $k_e = k_s k_p^2 + k_g k_p + \frac{k_g}{b_i}$, η and b_i are design parameters such that

$$\eta > 4 k_g b_i \dot{q}_{\max}^2 \tag{62}$$

for some $0 < b_i < \frac{k_s}{k_s k_p + k_g}$, then

(a) the MAS asymptotically converges to the target formation with a bounded MSE such that

$$\lim_{t\to\infty} \mathbb{E}\left(\frac{1}{2} P\left(q,t\right)\right) \leqslant \xi, \text{‘} \tag{63}$$

where $\xi = \frac{N}{k_g c_3}\left[D_{\max}^2 + \eta\right]$,

$$c_3 = \frac{\min\left\{k_1, k_p\right\} \min\left(1, \frac{N_{\min} m_{\min}}{m_{\max}}\right)}{\max\left\{1, k_M\right\}} \tag{64}$$

and $k_1 = k_s - \left(1 + k_p\left(k_M + 1\right)\right)$.

(b) one has $t_{i,k_i+1} - t_{i,k_i} > \tau_{\min}$ for some $\tau_{\min} > 0$.

The proof of (a) in Theorem 7 is given in Appendix A. The absence of Zeno behavior is shown by the existence of a minimum inter-event time $\tau_{\min}$ (Theorem 7(b)) as shown in Appendix B . Each Agent i has to evaluate the expected values of $||e_i^j||^2$, $||\dot{e}_i^j||^2$, $||e_i^j||^4$ and $||\dot{e}_i^j||^4$ for all $j \in \mathcal{N}_i$. This can be done using (39) or (58) as detailed in Sections 4.3 and 4.4.

The CTC proposed in Theorem 7 is analyzed considering that the state estimators and the communication protocol are such that for all $i \in \mathcal{N}$ and for all agent $j \in \mathcal{N}_i$ that has received its last message from Agent i at $t_\ell \in [t_{i,k_i-\kappa}, t_{i,k_i}[$ one has

$$\widehat{x}_i^{i,\ell}\left(t\right) = \widehat{x}_i^j\left(t\right) \tag{65}$$

all $t \in [t_\ell, t_{i,k_i}[$. If Agent j has received the ℓ-th message from Agent i, its estimation $\widehat{x}_i^j\left(t\right)$ of $x_i\left(t\right)$ is equal to $\widehat{x}_i^{i,\ell}\left(t\right)$, one of the κ estimators maintained by Agent i. This property is actually satisfied by the communication protocol described in Section 2.2 and the state estimator described in Section 4. Alternative estimators can be used.

The CTC (61) is satisfied for Agent i mainly when $\overline{\mathbb{E}}(||e_i^j||^2)$ and $\overline{\mathbb{E}}(||\dot{e}_i^j||^2)$ become large. Thus, it is preferable to use the knowledge of $k_i^{j,i}$ provided by the proposed implicit feedback mechanism to calculate (58) rather than using (39).

A large packet loss probability π results in large values of $\overline{\mathbb{E}}(||e_i^j\left(t\right)||^2)$ and $\overline{\mathbb{E}}(||\dot{e}_i^j\left(t\right)||^2)$, and therefore leads to an increase in the number of communications to compensate for the losses.

When π is a conservative upper-bound of the packet loss probabilities π_{ij}, $(i,j) \in \mathcal{E}$, the upper bounds of $\mathbb{E}(||e_i^j(t)||^2)$ and $\mathbb{E}(||\dot{e}_i^j(t)||^2)$ evaluated by Agent i will be conservative. This leads to more communications than necessary. This effect may be limited by exploiting the implicit acknowledgement.

The right hand side of the CTC (61) is proportional to $\bar{g}_i(t)$ and $\bar{s}_i(t)$, *i.e.*, to the potential energy of the formation $P(q,t)$, which is large when agents are far from the target formation. When agents are far from the target formation, the discrepancy between (20) and (23) is large. This lead to a fast increase of $\overline{\mathbb{E}}(||e_i^j(t)||^2)$ and $\overline{\mathbb{E}}(||\dot{e}_i^j(t)||^2)$. The fact that the right hand side of the CTC (61) is large too prevents the CTC from being satisfied too often. When agents are close to the target formation, even if the right hand side of the CTC (61) is small, the fact that (23) is close to (20) leads to $\overline{\mathbb{E}}(||e_i^j(t)||^2)$ and $\overline{\mathbb{E}}(||\dot{e}_i^j(t)||^2)$ increasing slowly. Consequently, less communications will be required. See Section 6 for an illustration.

An analysis of the impact of the values of the parameters on the reduction of communications has been presented in [21] in absence of packet losses. These results can be extended to the case with packet losses. The choice of the parameters α_{M}, k_g, k_p and b_i also determines the number of broadcast messages. Choosing the coefficients m_{ij} such that $\alpha_i = \sum_{j=1}^{N} m_{ij}$ is small, leads to a reduction in the number of communications triggered resulting from the satisfaction of (61), at the cost of a less precise formation.

The following proposition introduces a condition on the initial estimate of agent states to guarantee that (61) in Theorem 7 is not satisfied at $t = 0$.

Proposition 8 *If a common initial estimate $\hat{x}_i^j(0)$ is known by all the neighbors $j \in \mathcal{N}_i$ of each Agent i such that*

$$||\dot{\hat{q}}_i^j(0)|| = 0,\ ||e_i^j(0)||^2 \leqslant H_i, ||\dot{e}_i^j(0)||^2 \leqslant H_i \tag{66}$$

where $H_i \geqslant 0$ is defined for each Agent i as

$$H_i = \frac{\sqrt{(k_e + k_p k_M)^2 + k_p k_C^2 \xi_i} - (k_e + k_p k_M)}{2 k_p k_C^2} \tag{67}$$

$$\xi_i = \frac{k_p (k_s + k_g)}{\alpha_M \alpha_i} \bar{g}_i^T \bar{g}_i(0) + \frac{\eta}{\alpha_M \alpha_i}, \tag{68}$$

then the condition (61) in Theorem 7 is guaranteed not to be satisfied at $t = 0$.

The proof of Proposition 8 is given in Appendix A.8.

$H_i = 0$ corresponds to the case where the initial state $x_i(0)$ is known by all neighbors of Agent i, *i.e.* $\hat{x}_i^j(0) = x_i(0)\ \forall j \in \mathcal{N}_i$. The value of the bound H_i is proportional to $\bar{g}_i(0)$, *i.e.* the initial value of the potential energy of the formation. Thus, the most distant from the target formation agents are, the largest the initial error of the estimate $\hat{x}_i^j(0)$ can be tolerated.

We have assumed in Proposition 8 that all neighbors of Agent i share the same estimate $\hat{x}_i^j(0)$ of $x_i(0)$. This allows Agent i to initialize the estimator of its own state by $\hat{x}_i^j(0)$ and avoids using a different estimator for each of its neighbors. When this hypothesis is not satisfied initially, in practice, the local estimators of x_i and those performed by neighbors are likely to coincide after few communications. In practice, the formation can still be achieved even if the initial conditions do not satisfy Proposition 8.

6 Example

Consider the dynamical model of N identical surface ships with coordinate vectors $q_i = [\, x_i \ y_i \ \psi_i \,]^T \in \mathbb{R}^3$, $i = 1 \ldots N$, in a local Earth-fixed frame. For Agent i, (x_i, y_i) represents its position and ψ_i its heading angle. The Agent dynamics are expressed in the body frame as

$$M_{\mathrm{b},i} \dot{\mathrm{v}}_i + C_{\mathrm{b},i}(\mathrm{v}_i)\,\mathrm{v}_i + D_{\mathrm{b},i}\mathrm{v}_i = \tau_{\mathrm{b},i} + d_{\mathrm{b},i}, \tag{69}$$

where v_i is the velocity vector in the body frame. The values of $M_{\mathrm{b},i}$, $D_{\mathrm{b},i}$, and $C_{\mathrm{b},i}(\mathrm{v}_i)$ are found in [10].

One takes $N = 6$. The model (69) may be expressed as (1) with $G = 0$ using an appropriate change of variables detailed in [10]. The parameters of (20) are $k_M = \|M_i\| = 33.8$, $k_C = \|C_i(1_N)\| = 43.96$, $k_p = 2$, $k_g = 20$, $k_s = 1 + 6(k_M + 1)$, $b_i = \frac{1}{k_g}$, $e_{\max} = 20$, $\dot{q}_{\max} = 2$ and $\ddot{q}_{\max} = 1$. In the simulations, the following state estimator $\widehat{x}_j^i$ is used for all $t \in [t_{j,k}, t_{j,k+1}[$ with components

$$\left[\dot{\hat{q}}_j^i(t)\,,\ \ddot{\hat{q}}_j^i(t)\right] = \left[\dot{\hat{q}}_j^i(t_{i,k_i})\,,\ 0\right] \tag{70}$$

$$\hat{q}_j^i(t) = \hat{q}_j^i(t_{j,k}) + (t - t_{j,k})\,\dot{\hat{q}}_j^i(t_{j,k})\,, \tag{71}$$

$\widehat{x}_j^i$ is updated using (21).

6.1 Parameters

The initial value of the configuration vector is $q(0) = [x(0)^T, y(0)^T, \psi(0)^T]^T$, $\dot{q}(0) = 0_{3N\times 1}$, with $x(0) = [-0.35, 4.59, 4.72, 0.64, 3.53, -1.26]$, $y(0) = [-1.11, -4.59, 2.42, 1.36, 1.56, 3.36]$ and $\psi(0) = 0_N$. An hexagonal target formation is considered with $r^*(0) = [\, r^*_{(1)}(0)^T \; r^*_{(2)}(0)^T \; r^*_{(3)}(0)^T \,]^T$ where $r^*_{(1)}(0) = [0, 2, 3, 2, 0, -1]$, $r^*_{(2)}(0) = \left[0, 0, \sqrt{3}, \;\; 2\sqrt{3}, 2\sqrt{3}, \sqrt{3}\right]$, and $r^*_{(3)}(0) = 0_N$. Moreover, the target MAS velocity is $\dot{q}^*_1 = [1,\, 1,\, 0]^T$. Each Agent communicates with $N/2 = 3$ other agents. From [27], one obtains the coefficients matrix $S = [m_{ij}]_{i,j=1\ldots N}$ such

$$S = 0.1 \begin{bmatrix} 0 & 1.85 & 0 & 0.926 & 0 & 1.85 \\ 1.85 & 0 & 1.85 & 0 & 0.926 & 0 \\ 0 & 1.85 & 0 & 1.85 & 0 & 0.926 \\ 0.926 & 0 & 1.85 & 0 & 1.85 & 0 \\ 0 & 0.926 & 0 & 1.85 & 0 & 1.85 \\ 1.85 & 0 & 0.926 & 0 & 1.85 & 0 \end{bmatrix}.$$

One has $\alpha_i = \sum_{j=1}^{N} m_{ij} = 0.463$, for all $i = 1, \ldots, N$ and $\alpha_{\mathrm{M}} = 0.463$.

The simulation duration is $T = 2\,\mathrm{s}$, taken sufficiently large to reach a steady-state behavior, with an integration step size $\Delta t = 0.01$ s. Since time has been discretized, the minimum delay between the transmission of two messages by the same Agent is set to Δt. The perturbation $d_i(t)$ is assumed constant over each interval $[k\Delta t,\, (k+1)\,\Delta t[$. The components of $d_i(t)$ are independent realizations of zero-mean uniformly distributed noise $U\left(-D_{\max}/\sqrt{3},\, D_{\max}/\sqrt{3}\right)$ and are thus such that $\|d_i(t)\| \leqslant D_{\max}$. Let N_{m} be the total number of messages transmitted during a simulation. The performance of the proposed approach is evaluated with

$$R_{\mathrm{com}} = N_{\mathrm{m}} / \overline{N}_{\mathrm{m}} \tag{72}$$

where $\overline{N}_{\mathrm{m}} = NT/\Delta t \geqslant N_{\mathrm{m}}$. R_{com} is the ratio between the number of communications required using the proposed approach and the number of communications that would be obtained with a communication triggered at each sampling time instant. One takes $\kappa = 6$.

6.2 Simulations results

Figure 2 shows the performance of the proposed approach with the CTC (61) for different values of the packet loss probability π and disturbance bound $D_{\max}$. Results are averaged over 50 independent realizations of the noise and of the packet loss events. As expected, the number of communications required for the MAS to converge increases with π and $D_{\max}$.

The influence of η on the number of communication is detailed in [22]. Increasing η leads to a reduction of R_{com} but increases the potential energy $P(q, T)$, and thus the discrepancy with respect to the target formation at $t = T$.

Figure 3 compares results of the proposed approach obtained without (a) and with (b) the exploitation of the index $k_i^{j,i}$ of the last message sent by Agent i and received by some neighbor Agent j. Using the implicit acknowledgement from neighbors, and thus $\overline{\mathbb{E}}\left(\left\|e_i^j(t)\right\|^2 |k_j^i\right)$ instead of $\overline{\mathbb{E}}\left(\left\|e_i^j(t)\right\|^2\right)$ in the CTC, convergence is obtained with 75% less messages.

Figure 4 shows the influence of κ on the number of communication R_{com}. One observes that increasing κ reduces R_{com}, until R_{com} reaches a minimum value (when $k = 3$). Increasing further κ does not reduce R_{com}.

When κ is small, $\overline{\mathbb{E}}\left(\left\|e_i^j(t)\right\|^2 |k_j^{i,j}\right)$ and $\overline{\mathbb{E}}\left(\left\|\dot{e}_i^j(t)\right\|^2 |k_j^{i,j}\right)$ are conservative, which leads to more communications than necessary. When $\kappa \geqslant 3$, the additional terms in the upper bound have a negligible impact. R_{com} is large due to $e_{\max}$, $\dot{q}_{\max}$ans $\ddot{q}_{\max}$ which influence the value of $\overline{\mathbb{E}}\left(\left\|e_i^j(t)\right\|^2 |k_j^{i,j}\right)$.

7 Conclusion

This paper addresses the problem of communication reduction in distributed formation control of a MAS with Euler-Lagrange dynamics in presence of packet losses and state perturbations.

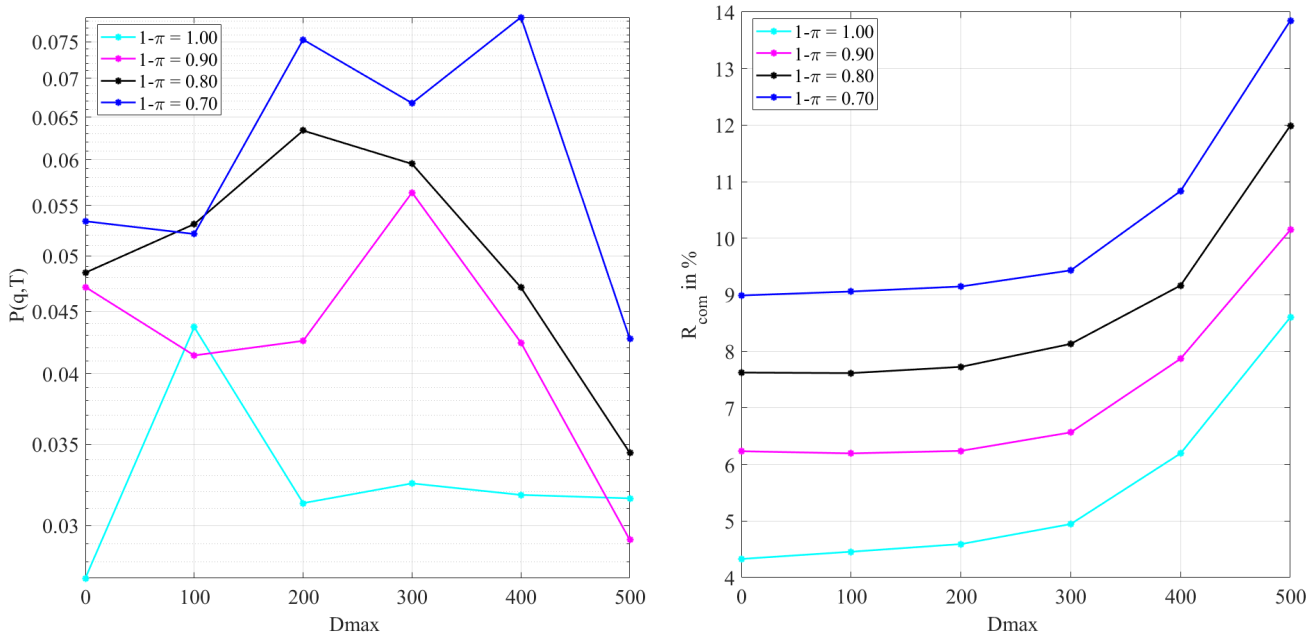
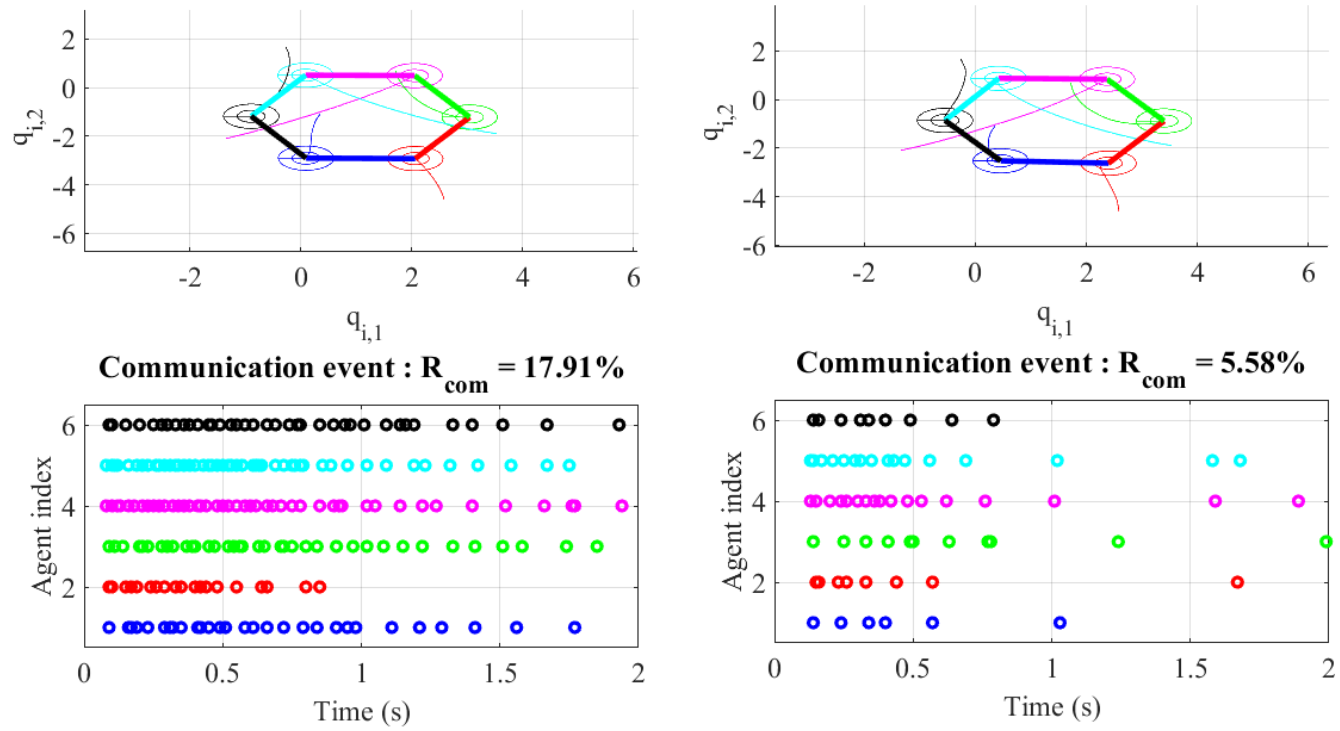


Fig. 2. Evolution of $P(q,T)$ and R_{com} for different values of $D_{\max}$ and π, when $\eta = 100$, the estimator (22) is considered for the dynamics of neighbors agents, as well as $\mathbb{E}\left(\left\|e_i^j(t)\right\|^2 |k_j^{i,j}\right)$ from (58).



(a) without implicit acknowledgement

(b) with implicit acknowledgement

Fig. 3. Results of the proposed method using an estimated of the state error (a) via (39) (implicit acknowledgement not used), (b) via (58) (implicit acknowledgement used) when $D_{\max} = 200$ and $\pi = 0.2$

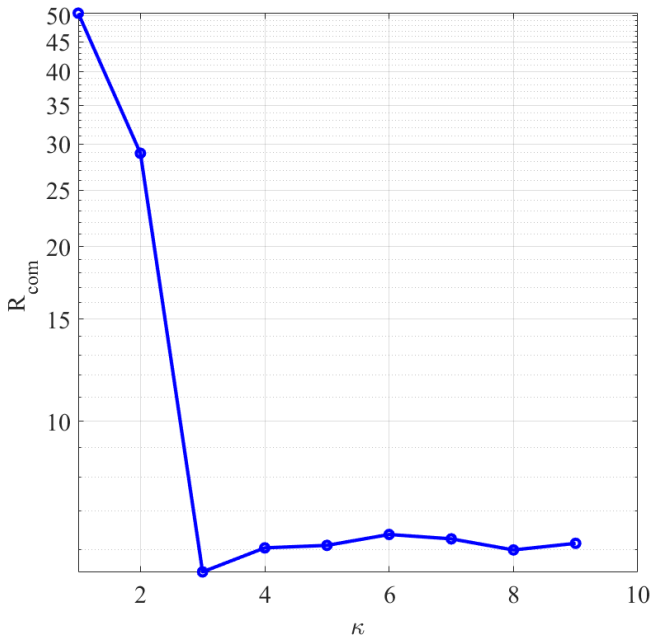


Fig. 4. Evolution of R_{com} as a function of κ when $\eta = 100$, $\pi = 0.2$ and $D_{\max} = 100$. $\overline{\mathbb{E}}\left(\left\|e_i^j(t)\right\|^2 |k_j^{i,j}\right)$ given by (58) is considered in the CTC.

To evaluate its control input, each agent maintains estimators of the states of its neighbors as well as multiple estimators of its own state accounting for different packet loss hypotheses in the communications with its neighbors. Using these estimators, each agent is then able to compute the expected value of the estimation error of its own state as evaluated by its neighbors. An implicit acknowledgement from other agents may be used to evaluate more accurately the estimation error. A distributed CTC is then proposed, involving these estimation errors. The behavior of the MAS is analyzed using stochastic Lyapunov functions in [23]. The convergence to the target formation and the absence of Zeno behavior have been proven. Simulations illustrate the effectiveness of the proposed approach. In future work, communication delays will also be considered along with packet losses.

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A Appendix

A.1 Proof of Proposition 2

Assume that there exists $\varepsilon_2 > 0$ such that

$$\lim_{t\to\infty} \mathbb{E}\left(P\left(q,\, t\right)\right) \leqslant \varepsilon_2, \tag{A.1}$$

then

$$\lim_{t\to\infty} \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} m_{ij}\mathbb{E}\left(\left\|r_{ij}\left(t\right) - r_{ij}^{*}\left(t\right)\right\|^2\right) \leqslant \varepsilon_2. \tag{A.2}$$

Since

$$m_{ij}\mathbb{E}\left(\left\|r_{ij}\left(t\right) - r_{ij}^{*}\left(t\right)\right\|^2\right) \leqslant \sum_{i=1}^{N}\sum_{j=1}^{N} m_{ij}\mathbb{E}\left(\left\|r_{ij}\left(t\right) - r_{ij}^{*}\left(t\right)\right\|^2\right). \tag{A.3}$$

and $m_{ij} \geqslant 0$ for all (i,j) such that $m_{ij} > 0$, this implies

$$\lim_{t\to\infty} \mathbb{E}\left(\left\|r_{ij}(t) - r^*_{ij}(t)\right\|^2\right) \leqslant \frac{2\varepsilon_2}{m_{ij}}. \tag{A.4}$$

Consider now a pair (i,j) such that $m_{ij} = 0$. The communication graph has been assumed connected. Consequently, there exists a sequence $\left(i_1, \ldots, i_{N_{ij}}\right)$ of $N_{ij} \leqslant N$ nodes with $i_1 = i$ and $i_{N_{ij}} = j$ and such that $m_{i_k i_{k+1}} > 0$ for all $k = 1, \ldots, N_{ij} - 1$. Then

$$\lim_{t\to\infty} \mathbb{E}\left(\left\|r_{ij}(t) - r^*_{ij}(t)\right\|^2\right) \leqslant \lim_{t\to\infty} \sum_{k=1}^{N_{ij}-1} \mathbb{E}\left(\left\|r_{i_k i_{k+1}}(t) - r^*_{i_k i_{k+1}}(t)\right\|^2\right).$$

Using (A.4), one gets

$$\lim_{t\to\infty} \mathbb{E}\left(\left\|r_{ij}(t) - r^*_{ij}(t)\right\|^2\right) \leqslant \sum_{k=1}^{N_{ij}-1} \frac{2\varepsilon_2}{m_{i_k i_{k+1}}}. \tag{A.5}$$

Then introduce

$$\varepsilon_{11} = \max_{(i,j)\in\mathcal{N}^2, m_{ij}=0} \sum_{k=1}^{N_{ij}-1} \frac{2\varepsilon_2}{m_{i_k i_{k+1}}}, \tag{A.6}$$

$$\varepsilon_{12} = \max_{(i,j)\in\mathcal{N}^2, m_{ij}>0} \frac{2\varepsilon_2}{m_{ij}} \tag{A.7}$$

and

$$\varepsilon_1 = \max\left\{\varepsilon_{11}, \varepsilon_{12}\right\}. \tag{A.8}$$

Combining (A.4) and (A.5), one has for all $(i,j) \in \mathcal{N}^2$,

$$\lim_{t\to\infty} \mathbb{E}\left(\left\|r_{ij}(t) - r^*_{ij}(t)\right\|^2\right) \leqslant \varepsilon_1, \tag{A.9}$$

The converse is immediate: if there exists $\varepsilon_1 > 0$ such that (A.9) is satisfied for all $(i,j) \in \mathcal{N}^2$, then

$$\begin{aligned}\lim_{t\to\infty} \mathbb{E}\left(P(q,t)\right) &= \lim_{t\to\infty} \frac{1}{2} \sum_{i=1}^{N}\sum_{j=1}^{N} m_{ij} \mathbb{E}\left(\left\|r_{ij}(t) - r^*_{ij}(t)\right\|^2\right) \\ &\leqslant \varepsilon_2,\end{aligned}$$

with $\varepsilon_2 = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} m_{ij}\varepsilon_1$.

A.2 Proof of Proposition 3

Consider first

$$p^j_{k_i,\ell} = \Pr\left(\delta^j_{i,\ell} = 1, \delta^j_{i,\ell+1} = 0, \ldots, \delta^j_{i,k_i} = 0\right). \tag{A.10}$$

Since the channel is memoryless,

$$p^j_{k_i,\ell} = \Pr\left(\delta^j_{i,k_i} = 0\right) \Pr\left(\delta^j_{i,\ell} = 1, \delta^j_{i,\ell+1} = 0, \ldots, \delta^j_{i,k_i-1} = 0\right) \tag{A.11}$$

so, using (6)-(7), one gets

$$p^j_{k_i,\ell} = \pi p^j_{k_i-1,\ell}. \tag{A.12}$$

Consider now $p^j_{k_i+n,k_i+n}$. Using (6)-(7), one gets

$$\begin{aligned}p^j_{k_i+n,k_i+n} &= 1 - \Pr\left(\delta^j_{i,k_i+n} = 0\right) \\ &= 1 - \pi.\end{aligned} \tag{A.13}$$

Note that $p^j_{k_i+n,k_i+n}$ is independent of j because no implicit acknowledgment is exploited in Proposition 3.

A.3 Proof of proposition 5

Before any reception from a packet for Agent j, (51) is evaluated as in Proposition 3.

Consider $t \in [t_{i,k_i+n}, t_{i,k_i+n+1}[$ with $n \geqslant 0$ and assume that the last message received by Agent i from Agent j was at time $t_{j,k_j} \in [t_{i,k_i}, t_{i,k_i+1}[$. Thus $k_i^{j,i} = k_i^j \leqslant k_i$ and Agent i knows that Agent j has received the $k_i^{j,i}$-th message sent by Agent i, and has not received the following ones with index between $k_i^{j,i}$ and k_i. Agent i has no information about the reception by Agent j of the $k_i+1, \ldots, k_i+n$-th messages. Then, by definition of $p^j_{k_i,\ell|k_i^{j,i},k_i}$, one has

$$p^j_{k_i,k_i^{j,i}|k_i^{j,i},k_i} = \Pr\left(\delta^j_{i,k_i^{j,i}} = 1 | \delta^j_{i,k_i^{j,i}} = 1, \sum_{m=k_i^{j,i}+1}^{k_i} \delta^j_{i,m} = 0\right) = 1 \tag{A.14}$$

and for all $\ell \leqslant k_i$ with $\ell \neq k_i^{j,i}$

$$p^j_{k_i,\ell|k_i^{j,i},k_i} = \Pr\left(\sum_{m=\ell+1}^{k_i} \delta^j_{i,m} = 0, \delta^j_{i,\ell} = 1 | \delta^j_{i,k_i^{j,i}} = 1, \sum_{m=k_i^{j,i}+1}^{k_i} \delta^j_{i,m} = 0\right) = 0. \tag{A.15}$$

Consider now the evaluation of $p^j_{k_i+n,\ell|k_i^{j,i},k_i}$ with $n > 0$. One has

$$p^j_{k_i+n,\ell|k_i^{j,i},k_i} = \Pr\left(\sum_{m=\ell+1}^{k_i+n} \delta^j_{i,m} = 0, \delta^j_{i,\ell} = 1 | \delta^j_{i,k_i^{j,i}} = 1, \sum_{m=k_i^{j,i}+1}^{k_i} \delta^j_{i,m} = 0\right). \tag{A.16}$$

If $\ell \leqslant k_i$ with $\ell \neq k_i^{j,i}$, one has clearly $p^j_{k_i+n,\ell|k_i^{j,i},k_i} = 0$. In what follows, we consider thus $\ell = k_i^{j,i}$ or $k_i < \ell < k_i + n$ and, since the channel is memoryless, one has

$$p^j_{k_i+n,\ell|k_i^{j,i},k_i} = \Pr\left(\delta^j_{i,k_i+n} = 0\right) \Pr\left(\sum_{m=\ell+1}^{k_i+n-1} \delta^j_{i,m} = 0, \delta^j_{i,\ell} = 1 | \delta^j_{i,k_i^{j,i}} = 1, \sum_{m=k_i^{j,i}+1}^{k_i} \delta^j_{i,m} = 0\right) \tag{A.17}$$

Using (6)-(7), one gets

$$p^j_{k_i+n,\ell|k_i^{j,i},k_i} = \pi p^j_{k_i+n-1,\ell|k_i^{j,i},k_i}. \tag{A.18}$$

Consider now the evaluation of $p^j_{k_i+n,k_i+n|k_i^{j,i},k_i}$ with $n > 0$. Since the channel is memoryless, one has

$$\begin{aligned} p^j_{k_i+n,k_i+n|k_i^{j,i},k_i} &= \Pr\left(\delta^j_{i,k_i+n} = 1\right) \\ &= 1 - \pi. \end{aligned} \tag{A.19}$$

A.4 Proof of convergence with packet losses

To prove Theorem 7 a) one shows first that the MAS is converging with a bounded mean-square error. For that purpose, one will introduce a candidate Lyapunov function and show that it satisfies the conditions introduced in the Definition 1.

Consider some $D_{\max} \geqslant 0$, $\eta \geqslant 0$, and realizations $d_i(t)$, $i = 1, \ldots, N$ of the state perturbations.

Inspired by the proof developed in [3,17], consider the continuous positive-definite candidate Lyapunov function

$$V(t) = \mathbb{E}\left(\frac{1}{2}\sum_{i=1}^{N}\left(s_i(q(t,\delta))^T M_i s_i(q(t,\delta))\right) + \frac{k_g}{4} P(q(t,\delta),t)\right) \tag{A.20}$$

where the expectation is evaluated considering the random losses described by δ.

A.4.1 Continuity of the Lyapunov function

Assume that the first message is transmitted at time t_1, without loss of generality, by Agent 1 to N_1 neighbors. Consider some $t \in [t_1,\, t_2[$, where t_2 is the time at which the second message is transmitted, whatever the Agent. There are 2^{N_1} possible reception scenario, from no reception by all agents to a reception by all agents. Let σ represent the index of the σ-th scenario, $0 \leqslant \sigma \leqslant 2^{N_1}$ and $p_{\sigma,1}$ be the associated probability for the first communication. One may write

$$\begin{aligned} V(t) &= \mathbb{E}\left(\frac{1}{2}\sum_{i=1}^{N}\left(s_i\left(q\left(t,\delta\right)\right)^T M_i s_i\left(q\left(t,\delta\right)\right)\right) + \frac{k_g}{4}P\left(q\left(t,\delta\right),t\right)\right) \\ &= \frac{1}{2}\sum_{\sigma=1}^{2^{N_1}} p_{\sigma,1}\left(\sum_{i=1}^{N} s_i\left(q\left(t,\delta_\sigma\right)\right)^T M_i s_i\left(q\left(t,\delta_\sigma\right)\right) + \frac{k_g}{4}P\left(q\left(t,\delta_\sigma\right),t\right)\right) \end{aligned} \tag{A.21}$$

where $\sum_{\sigma=1}^{2^{N_1}} p_{\sigma,1} = 1$.

For a given reception scenario σ of the first message, the time instant $t_{\sigma,2}$ of transmission of the second message and the index $i_{\sigma,2}$ of the transmitting Agent both depend on σ. More generally, at time $\underline{t}$, $S_{\underline{t}}$ different transmission and reception scenarios have to be considered. For a given scenario σ, let n_σ be the number of communications that have occurred. The associated loss vector is $\delta_\sigma = (\delta_{\sigma,1}, \ldots, \delta_{\sigma,n_\sigma})$, where $\delta_{\sigma,k}$ is the loss vector for the k-th communication. The probability associated to δ_σ is p_σ. The next communication time instant is $t_{\sigma,n_\sigma+1} > \underline{t}$ and the communicating Agent is $i_{\sigma,n_\sigma+1}$. Let

$$\bar{t} = \min_{\sigma=1,\ldots,S_{\underline{t}}} t_{\sigma,n_\sigma+1}$$

$$\overline{\sigma} = \arg\min_{\sigma=1,\ldots,S_{\underline{t}}} t_{\sigma,n_\sigma+1}$$

and $\bar{i}$ denote the index of the associated communicating Agent.

For all $t \in [\underline{t}, \bar{t}[$, one has

$$V(t) = \frac{1}{2}\sum_{\sigma=1}^{S_{\underline{t}}} p_\sigma\left(\sum_{i=1}^{N} s_i\left(q\left(t,\delta_\sigma\right)\right)^T M_i s_i\left(q\left(t,\delta_\sigma\right)\right) + \frac{k_g}{4}P\left(q\left(t,\delta_\sigma\right),t\right)\right).$$

In the scenario $\overline{\sigma}$, at time $\bar{t}$, Agent $\bar{i}$ is communicating. Consequently

$$\begin{aligned} V(\bar{t}) = &\frac{1}{2}\sum_{\sigma=1,\ldots,S_{\underline{t}},\sigma\neq\overline{\sigma}} p_\sigma\left(\sum_{i=1}^{N} s_i\left(q\left(\bar{t},\delta_\sigma\right)\right)^T M_i s_i\left(q\left(\bar{t},\delta_\sigma\right)\right) + \frac{k_g}{4}P\left(q\left(\bar{t},\delta_\sigma\right),\bar{t}\right)\right) \\ &+ \frac{1}{2}\sum_{\mu=1}^{2^{N_{\bar{i}}}} p_{(\overline{\sigma},\mu)}\left(\sum_{i=1}^{N} s_i\left(q\left(\bar{t},\delta_{(\overline{\sigma},\mu)}\right)\right)^T M_i s_i\left(q\left(\bar{t},\delta_{(\overline{\sigma},\mu)}\right)\right) + \frac{k_g}{4}P\left(q\left(\bar{t},\delta_{(\overline{\sigma},\mu)}\right),\bar{t}\right)\right) \end{aligned} \tag{A.22}$$

where $p_{(\overline{\sigma},\mu)}$ denotes the probability of the μ-th loss scenario associated to the $n_{\overline{\sigma}}+1$ communication performed by Agent $\bar{i}$ at time $\bar{t}$, when the previous loss scenario is $\overline{\sigma}$. One has

$$\sum_{\mu=1}^{2^{N_{\bar{i}}}} p_{(\overline{\sigma},\mu)} = p_{\overline{\sigma}}. \tag{A.23}$$

Upon reception at time $\bar{t}^+$ of a message sent at time $\bar{t}$, only the estimators are updated according to (21). The state of agents receiving a message at time $\bar{t}^+$ from a neighbor is continuous, *i.e.*, $q_i\left(\bar{t}^+, \delta_{(\overline{\sigma},\mu)}\right) = q_i\left(\bar{t}^-, \delta_{\overline{\sigma}}\right)$, where $\bar{t}^-$ is a time instant immediately before transmission. This is also true for agents which do not receive the message sent at time $\bar{t}$. Thus, one gets $g_i\left(\bar{t}^+, \delta_{(\overline{\sigma},\mu)}\right) = g_i\left(\bar{t}^-, \delta_{\overline{\sigma}}\right)$, $s_i\left(q\left(\bar{t}^+, \delta_{(\overline{\sigma},\mu)}\right)\right) = s_i\left(q\left(\bar{t}^-, \delta_{\overline{\sigma}}\right)\right)$ for $i = 1, \ldots, N$, and consequently, $P\left(q\left(\bar{t}^+, \delta_{(\overline{\sigma},\mu)}\right), \bar{t}^+\right) =$

$P\left(q\left(\bar{t}^{-},\delta_{\overline{\sigma}}\right),\bar{t}^{-}\right)$ for all μ. Thus, at time $\bar{t}^{+}$, (A.22) becomes

$$\begin{aligned}
V\left(\bar{t}^{+}\right) &= \frac{1}{2}\sum_{\sigma=1,\ldots,S_{\underline{t}},\sigma\neq\overline{\sigma}} p_{\sigma}\left(\sum_{i=1}^{N} s_i\left(q\left(\bar{t}^{+},\delta_{\sigma}\right)\right)^{T} M_i s_i\left(q\left(\bar{t}^{+},\delta_{\sigma}\right)\right)+\frac{k_g}{4}P\left(q\left(\bar{t}^{+},\delta_{\sigma}\right),\bar{t}^{+}\right)\right)\\
&+\frac{1}{2}\sum_{\mu=1}^{2^{N_{\bar{i}}}} p_{(\overline{\sigma},\mu)}\left(\sum_{i=1}^{N} s_i\left(q\left(\bar{t}^{+},\delta_{(\overline{\sigma},\mu)}\right)\right)^{T} M_i s_i\left(q\left(\bar{t}^{+},\delta_{(\overline{\sigma},\mu)}\right)\right)+\frac{k_g}{4}P\left(q\left(\bar{t}^{+},\delta_{(\overline{\sigma},\mu)}\right),\bar{t}^{+}\right)\right)\\
&= \frac{1}{2}\sum_{\sigma=1,\ldots,S_{\underline{t}},\sigma\neq\overline{\sigma}} p_{\sigma}\left(\sum_{i=1}^{N} s_i\left(q\left(\bar{t}^{-},\delta_{\sigma}\right)\right)^{T} M_i s_i\left(q\left(\bar{t}^{-},\delta_{\sigma}\right)\right)+\frac{k_g}{4}P\left(q\left(\bar{t}^{-},\delta_{\sigma}\right),\bar{t}^{-}\right)\right)\\
&+\frac{1}{2}\sum_{\mu=1}^{2^{N_{\bar{i}}}} p_{(\overline{\sigma},\mu)}\left(\sum_{i=1}^{N} s_i\left(q\left(\bar{t}^{-},\delta_{\overline{\sigma}}\right)\right)^{T} M_i s_i\left(q\left(\bar{t}^{-},\delta_{\overline{\sigma}}\right)\right)+\frac{k_g}{4}P\left(q\left(\bar{t}^{-},\delta_{\overline{\sigma}}\right),\bar{t}^{-}\right)\right)
\end{aligned}$$

and using (A.23), one gets

$$\begin{aligned}
V\left(\bar{t}^{+}\right) &= \frac{1}{2}\sum_{\sigma=1,\ldots,S_{\underline{t}},\sigma\neq\overline{\sigma}} p_{\sigma}\left(\sum_{i=1}^{N} s_i\left(q\left(\bar{t}^{-},\delta_{\sigma}\right)\right)^{T} M_i s_i\left(q\left(\bar{t}^{-},\delta_{\sigma}\right)\right)+\frac{k_g}{4}P\left(q\left(\bar{t}^{-},\delta_{\sigma}\right),\bar{t}^{-}\right)\right)\\
&+\frac{1}{2}p_{\overline{\sigma}}\left(\sum_{i=1}^{N} s_i\left(q\left(\bar{t}^{-},\delta_{\overline{\sigma}}\right)\right)^{T} M_i s_i\left(q\left(\bar{t}^{-},\delta_{\overline{\sigma}}\right)\right)+\frac{k_g}{4}P\left(q\left(\bar{t}^{-},\delta_{\overline{\sigma}}\right),\bar{t}^{-}\right)\right)\\
&= V\left(\bar{t}^{-}\right).
\end{aligned}$$

Consequently, $V(t)$ is continuous at $\bar{t}$.

A.4.2 Differential inequality satisfied by the Lyapunov function

Using (A.22) from the previous section, the time derivative of V exists and can be evaluated for each $t\in[\underline{t},\bar{t}[$ as follows

$$\begin{aligned}
\dot{V}(t) = \sum_{\sigma=1}^{S_{\underline{t}}} p_{\sigma}\left(\sum_{i=1}^{N}\left(\frac{1}{2}s_i^{T}\left(q\left(t,\delta_{\sigma}\right)\right)\dot{M}_i s_i\left(q\left(t,\delta_{\sigma}\right)\right)+s_i^{T}\left(q\left(t,\delta_{\sigma}\right)\right)M_i\dot{s}_i\left(q\left(t,\delta_{\sigma}\right)\right)\right)\right.\\
\left.+\frac{k_g}{4}\frac{d}{dt}P\left(q\left(t,\delta_{\sigma}\right),t\right)\right).
\end{aligned}\tag{A.24}$$

which may be written more concisely as

$$\dot{V}=\mathbb{E}\left(\sum_{i=1}^{N}\left[\frac{1}{2}s_i^{T}\dot{M}_i s_i+s_i^{T}M_i\dot{s}_i\right]+\frac{k_g}{4}\frac{d}{dt}P(q,t)\right),\tag{A.25}$$

where the expectation is to be taken over all possible transmission loss events.

Our aim, in what follows is to obtain a differential inequality satisfied by V. One starts considering the two terms in the right hand side of (A.25).

In (A.24), one has

$$\begin{aligned}
&\frac{1}{4}\frac{d}{dt}P(q,t)\\
&=\frac{1}{4}\frac{d}{dt}\sum_{i=1}^{N}\sum_{j=1}^{N} m_{ij}\left\|r_{ij}-r_{ij}^{*}\right\|^{2}\\
&=\sum_{i=1}^{N}\left[\frac{1}{2}\sum_{j=1}^{N} m_{ij}\left(\dot{r}_{ij}-\dot{r}_{ij}^{*}\right)^{T}\left(r_{ij}-r_{ij}^{*}\right)\right]\\
&=\sum_{i=1}^{N}\frac{1}{2}\sum_{j=1}^{N} m_{ij}\left[\left(\dot{q}_i-\dot{q}_i^{*}\right)^{T}\left(r_{ij}-r_{ij}^{*}\right)-\left(\dot{q}_j-\dot{q}_j^{*}\right)^{T}\left(r_{ij}-r_{ij}^{*}\right)\right]\\
&=\sum_{i=1}^{N}\frac{1}{2}\sum_{j=1}^{N} m_{ij}\left[\left(\dot{q}_i-\dot{q}_i^{*}\right)^{T}\left(r_{ij}-r_{ij}^{*}\right)-\left(\dot{q}_i-\dot{q}_i^{*}\right)^{T}\left(r_{ji}-r_{ji}^{*}\right)\right]
\end{aligned}\tag{A.26}$$

Since $r_{ji} = -r_{ij}$, one gets

$$\begin{aligned}\frac{1}{4}\frac{d}{dt}P(q,\,t) &= \sum_{i=1}^{N}\left(\dot{q}_i - \dot{q}_i^*\right)^T \sum_{j=1}^{N} m_{ij}\left(r_{ij} - r_{ij}^*\right) \\ &= \sum_{i=1}^{N}\left(\dot{q}_i - \dot{q}_i^*\right)^T g_i. \end{aligned} \tag{A.27}$$

Combining (A.24) and (A.27), one obtains

$$\dot{V} = \mathbb{E}\left(\sum_{i=1}^{N}\left[\frac{1}{2}s_i^T\dot{M}_i s_i + s_i^T M_i \dot{s}_i + k_g\left(\dot{q}_i - \dot{q}_i^*\right)^T g_i\right]\right) \tag{A.28}$$

One focuses now on the term $M_i\dot{s}_i$. Using (16), one may write

$$M_i\dot{s}_i + C_i s_i = M_i\left(\ddot{q}_i - \ddot{q}_i^* + k_p\dot{g}_i\right) + C_i\left(\dot{q}_i - \dot{q}_i^* + k_p g_i\right)$$

and using (1), one gets

$$M_i\dot{s}_i + C_i s_i = u_i + d_i - G + M_i\left(k_p\dot{g}_i - \ddot{q}_i^*\right) + C_i\left(k_p g_i - \dot{q}_i^*\right). \tag{A.29}$$

Now, introducing (20), one gets

$$\begin{aligned}M_i\dot{s}_i + C_i s_i = &-k_s\bar{s}_i - k_g\bar{g}_i - Y_i\left(q_i,\,\dot{q}_i,\,k_p\dot{\bar{g}}_i - \ddot{q}_i^*, k_p\bar{g}_i - \dot{q}_i^*\right)\theta_i \\ &+M_i\left(k_p\dot{g}_i - \ddot{q}_i^*\right) + C_i\left(k_p g_i - \dot{q}_i^*\right) + d_i\end{aligned} \tag{A.30}$$

In what follows, one uses Y_i to represent $Y_i\left(q_i,\,\dot{q}_i,\,k_p\dot{\bar{g}}_i - \ddot{q}_i^*, k_p\bar{g}_i - \dot{q}_i^*\right)$. Assumption A3 leads to

$$-s_i^T Y_i\theta_i = -s_i^T\left(M_i\left(k_p\dot{\bar{g}}_i - \ddot{q}_i^*\right) + C_i\left(k_p\bar{g}_i - \dot{q}_i^*\right)\right). \tag{A.31}$$

Considering (2) and (A.30) in (A.28), one gets

$$\begin{aligned}\dot{V} = \mathbb{E}\Bigg(\sum_{i=1}^{N}\Big[&\frac{1}{2}s_i^T\dot{M}_i s_i - k_s s_i^T\bar{s}_i - k_g s_i^T\bar{g}_i - s_i^T C_i s_i + s_i^T\left(M_i\left(k_p\dot{g}_i - \ddot{q}_i^*\right) + C_i\left(k_p g_i - \dot{q}_i^*\right)\right) \\ &-s_i^T\left(M_i\left(k_p\dot{\bar{g}}_i - \ddot{q}_i^*\right) + C_i\left(k_p\bar{g}_i - \dot{q}_i^*\right)\right) + k_g\left(\dot{q}_i - \dot{q}_i^*\right)^T g_i + s_i^T d_i\Big]\Bigg).\end{aligned} \tag{A.32}$$

Now, introduce (14) in (16) to get

$$s_i = \dot{q}_i - \dot{q}_i^* + k_p\sum_{i=1}^{N} m_{ij}\left(q_i - q_j - r_{ij}^*\right). \tag{A.33}$$

Since $e_j^i = \hat{q}_j^i - q_j$, one gets

$$\begin{aligned}s_i &= \dot{q}_i - \dot{q}_i^* + k_p\sum_{i=1}^{N} m_{ij}\left(q_i - \hat{q}_j^i + e_j^i - r_{ij}^*\right) \\ &= \dot{q}_i - \dot{q}_i^* + k_p\sum_{i=1}^{N} m_{ij}\left(\bar{r}_{ij} - r_{ij}^*\right) + k_p\sum_{\substack{j=1 \\ j\neq i}}^{N} m_{ij}e_j^i \\ &= \bar{s}_i + k_p E^i\end{aligned} \tag{A.34}$$

with

$$E^i = \sum_{i=1}^{N} m_{ij} e_j^i, \tag{A.35}$$

since $m_{ii} = 0$. Using similar derivations, one may show that

$$g_i = \bar{g}_i + E^i. \tag{A.36}$$

Replacing (A.34) and (A.36) in (A.32), one gets

$$\begin{aligned}\dot{V} = \mathbb{E}\Bigg(\sum_{i=1}^{N}\Bigg[s_i^T\left[\frac{1}{2}\dot{M}_i - C_i\right]s_i - k_s s_i^T\bar{s}_i - k_g\left(\dot{q}_i - \dot{q}_i^* + k_p g_i\right)^T\bar{g}_i \\ + k_p s_i^T\left(M_i\dot{E}^i + C_i E^i\right) + k_g\left(\dot{q}_i - \dot{q}_i^*\right)^T g_i + s_i^T d_i\Bigg]\Bigg).\end{aligned} \tag{A.37}$$

Let $\dot{V}_1 = \sum_{i=1}^{N} 2k_p s_i^T\left(M_i\dot{E}^i + C_i E^i\right)$. Using Assumption A2, $\frac{1}{2}\dot{M}_i - C_i$ is skew symmetric or definite negative thus $s_i^T\left[\frac{1}{2}\dot{M}_i - C_i\right]s_i \leqslant 0$. For all $b > 0$ and all vectors x and y of similar size, one has

$$x^T y \leqslant \frac{1}{2}\left(bx^T x + \frac{1}{b}y^T y\right). \tag{A.38}$$

Using (A.38) with $b = 1$, and the fact that $d_i^T d_i \leqslant D_{\max}^2$, one deduces that $d_i^T s_i \leqslant \frac{1}{2}\left(D_{\max}^2 + s_i^T s_i\right)$ and that

$$\begin{aligned}\dot{V} \leqslant \mathbb{E}\Bigg(\sum_{i=1}^{N}\Bigg[-k_s s_i^T\bar{s}_i - k_g k_p g_i^T\bar{g}_i + \frac{1}{2}s_i^T s_i + \frac{1}{2}D_{\max}^2 \\ + k_g\left(\dot{q}_i - \dot{q}_i^*\right)^T\left(g_i - \bar{g}_i\right)\Bigg] + \frac{1}{2}\dot{V}_1\Bigg).\end{aligned} \tag{A.39}$$

One notices that $r_{ij} = q_i - q_j = q_i - \hat{q}_j^i + e_j^i = \bar{r}_{ij} + e_j^i$, thus

$$\begin{aligned}\left\|s_i - \bar{s}_i\right\|^2 &= s_i^T s_i - 2s_i^T\bar{s}_i + \bar{s}_i^T\bar{s}_i \\ \left\|k_p E^i\right\|^2 &= s_i^T s_i - 2s_i^T\bar{s}_i + \bar{s}_i^T\bar{s}_i \\ s_i^T\bar{s}_i &= -\frac{1}{2}\left\|k_p E^i\right\|^2 + \frac{1}{2}s_i^T s_i + \frac{1}{2}\bar{s}_i^T\bar{s}_i\end{aligned} \tag{A.40}$$

Using similar derivations, from (A.40), one shows that $g_i^T\bar{g}_i = -\frac{1}{2}\left\|E^i\right\|^2 + \frac{1}{2}g_i^T g_i + \frac{1}{2}\bar{g}_i^T\bar{g}_i$. Injecting the latter expression in (A.39), one gets

$$\begin{aligned}\dot{V} &\leqslant \mathbb{E}\Bigg(\sum_{i=1}^{N}\Bigg[\frac{k_s}{2}\left(k_p^2\left\|E^i\right\|^2 - s_i^T s_i - \bar{s}_i^T\bar{s}_i\right) + k_p k_g\frac{1}{2}\left(\left\|E^i\right\|^2 - g_i^T g_i - \bar{g}_i^T\bar{g}_i\right) + \frac{1}{2}s_i^T s_i + \frac{1}{2}D_{\max}^2 \\ &\quad + k_g\left(\dot{q}_i - \dot{q}_i^*\right)^T\left(g_i - \bar{g}_i\right)\Bigg] + \frac{1}{2}\dot{V}_1\Bigg) \\ &\leqslant \mathbb{E}\Bigg(\sum_{i=1}^{N}\Bigg[-\frac{(k_s-1)}{2}s_i^T s_i - \frac{k_s}{2}\bar{s}_i^T\bar{s}_i + \frac{k_s k_p^2 + k_g k_p}{2}\left\|E^i\right\|^2 - \frac{1}{2}k_p k_g\left(g_i^T g_i + \bar{g}_i^T\bar{g}_i\right) + \frac{1}{2}D_{\max}^2 \\ &\quad + k_g\left(\dot{q}_i - \dot{q}_i^*\right)^T\left(g_i - \bar{g}_i\right)\Bigg] + \frac{1}{2}\dot{V}_1\Bigg).\end{aligned} \tag{A.41}$$

Using (A.38) with $b = b_i > 0$, one shows that $2\dot{q}_i^T\left(g_i - \bar{g}_i\right) \leqslant \left(b_i\left\|\dot{q}_i\right\|^2 + \frac{1}{b_i}\left\|E^i\right\|^2\right)$. Using this result in (A.41), one gets

$$\begin{aligned}\dot{V} \leqslant \frac{1}{2}\sum_{i=1}^{N}\Bigg[-\left(k_s - 1\right)\mathbb{E}\left(s_i^T s_i\right) - k_s\mathbb{E}\left(\bar{s}_i^T\bar{s}_i\right) + \left(k_s k_p^2 + k_g k_p + \frac{k_g}{b_i}\right)\mathbb{E}\left(\left\|E^i\right\|^2\right) + b_i k_g\mathbb{E}\left(\left\|\dot{q}_i - \dot{q}_i^*\right\|^2\right) \\ -k_p k_g\mathbb{E}\left(g_i^T g_i + \bar{g}_i^T\bar{g}_i\right) + D_{\max}^2\Bigg] + \frac{1}{2}\mathbb{E}\left(\dot{V}_1\right)\end{aligned} \tag{A.42}$$

Consider now $\dot{V}_1$. Using (A.38) with $b = 1$ and Assumption A1, one obtains

$$\begin{aligned}\sum_{i=1}^{N} 2k_p s_i^T \left(M_i \dot{E}^i + C_i E^i\right) &\leqslant \sum_{i=1}^{N} k_p \left(s_i^T M_i s_i + s_i^T s_i + \left[\dot{E}^{iT} M_i \dot{E}^i + E^{iT} C_i^T C_i E^i\right]\right) \\ &\leqslant \sum_{i=1}^{N} k_p \left((k_M + 1)\, s_i^T s_i + \left[k_M \dot{E}^{iT} \dot{E}^i + E^{iT} C_i^T C_i E^i\right]\right) \end{aligned} \tag{A.43}$$

Focus now on the terms $E^{iT} C_i^T C_i E^i$. Using Assumption A2, one has

$$\begin{aligned}\sum_{i=1}^{N} E^{iT} C_i^T C_i E^i &= \sum_{i=1}^{N} \left(\sum_{j=1}^{N} m_{ij} e_j^i\right)^T C_i^T C_i \left(\sum_{\ell=1}^{N} m_{i\ell} e_\ell^i\right) \\ &\leqslant \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{\ell=1}^{N} m_{i\ell} m_{ij} \left\|C_i\right\|^2 e_j^{iT} e_\ell^i. \end{aligned} \tag{A.44}$$

Using again (A.38) with $b = 1$, one gets

$$\begin{aligned}\sum_{i=1}^{N} E^{iT} C_i^T C_i E^i &\leqslant \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{\ell=1}^{N} m_{i\ell} m_{ij} \left\|C_i\right\|^2 \left(e_j^{iT} e_j^i + e_\ell^{iT} e_\ell^i\right) \\ &\leqslant \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{\ell=1}^{N} m_{i\ell} m_{ij} \left\|C_i\right\|^2 \left(e_j^{iT} e_j^i\right) \\ &\leqslant \sum_{i=1}^{N} \alpha_i \sum_{j=1}^{N} m_{ij} \left\|C_i\right\|^2 \left(e_j^{iT} e_j^i\right). \end{aligned} \tag{A.45}$$

Since $m_{ij} = m_{ji}$ and $m_{ij} = 0$ if $(i, j) \notin \mathcal{N}_j$, one may write

$$\sum_{i=1}^{N} \sum_{j=1}^{N} m_{ij} \left\|e_j^i\right\|^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} m_{ji} \left\|e_i^j\right\|^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} m_{ij} \left\|e_i^j\right\|^2 \tag{A.46}$$

and using Assumption A2, one gets

$$\begin{aligned}\sum_{i=1}^{N} E^{iT} C_i^T C_i E^i &\leqslant \sum_{i=1}^{N} \left(\alpha_{\mathrm{M}} \sum_{j=1}^{N} \left[m_{ij} \left\|e_i^j\right\|^2 \left\|C_j\right\|^2\right]\right) \\ &\leqslant \sum_{i=1}^{N} \left(\alpha_{\mathrm{M}} \sum_{j=1}^{N} \left[m_{ij} \left\|e_i^j\right\|^2 k_C^2 \left\|\dot{q}_j\right\|^2\right]\right). \end{aligned} \tag{A.47}$$

Observing that

$$\begin{aligned}\left\|\dot{q}_j\right\|^2 &= \left\|\dot{\hat{q}}_j^i + \dot{e}_j^i\right\|^2 \\ &= \left\|\dot{\hat{q}}_j^i\right\|^2 + \left\|\dot{e}_j^i\right\|^2 + 2\dot{\hat{q}}_j^{iT} \dot{e}_j^i \\ &\leqslant 2\left\|\dot{\hat{q}}_j^i\right\|^2 + 2\left\|\dot{e}_j^i\right\|^2, \end{aligned}$$

(A.46) can be rewritten as

$$\begin{aligned}\sum_{i=1}^{N} E^{iT} C_i^T C_i E^i &\leqslant 2\alpha_{\mathrm{M}} k_C^2 \sum_{i=1}^{N} \sum_{j=1}^{N} m_{ij} \left\|e_i^j\right\|^2 \left(\left\|\dot{\hat{q}}_j^i\right\|^2 + \left\|\dot{e}_j^i\right\|^2\right) \\ &\leqslant 2\alpha_{\mathrm{M}} k_C^2 \sum_{i=1}^{N} \sum_{j=1}^{N} m_{ij} \left(\left\|e_i^j\right\|^2 \left\|\dot{\hat{q}}_j^i\right\|^2 + \left\|e_i^j\right\|^2 \left\|\dot{e}_j^i\right\|^2\right). \end{aligned}$$

Using (A.38) with $b=1$ and $m_{ij}=m_{ji}$, one gets

$$\sum_{i=1}^{N} E^{iT} C_i^T C_i E^i \leqslant 2\alpha_{\mathrm{M}} k_C^2 \sum_{i=1}^{N}\sum_{j=1}^{N} m_{ij} \left(\left\| e_i^j \right\|^2 \left\| \dot{\hat{q}}_j^i \right\|^2 + \frac{1}{2} \left\| e_i^j \right\|^4 + \frac{1}{2} \left\| \dot{e}_j^i \right\|^4 \right)$$
$$\leqslant 2\alpha_{\mathrm{M}} k_C^2 \sum_{i=1}^{N}\sum_{j=1}^{N} m_{ij} \left(\left\| e_i^j \right\|^2 \left\| \dot{\hat{q}}_j^i \right\|^2 + \frac{1}{2} \left\| e_i^j \right\|^4 + \frac{1}{2} \left\| \dot{e}_i^j \right\|^4 \right). \tag{A.48}$$

Similarly, one shows that $\sum_{i=1}^{N} E^{iT} E^i \leqslant \sum_{i=1}^{N} \alpha_{\mathrm{M}} \sum_{j=1}^{N} m_{ij} \left\| e_i^j \right\|^2$ and $\sum_{i=1}^{N} \dot{E}^{iT} \dot{E}^i \leqslant \sum_{i=1}^{N} \alpha_{\mathrm{M}} \sum_{j=1}^{N} m_{ij} \left\| \dot{e}_i^j \right\|^2$.

Injecting (A.43) and (A.48) in (A.42), one gets

$$\begin{aligned}
\dot{V} \leqslant \frac{1}{2} \sum_{i=1}^{N} \Big[& -\left(k_s - 1 - k_p\left(k_M+1\right)\right) \mathbb{E}\left(s_i^T s_i\right) - k_s \mathbb{E}\left(\bar{s}_i^T \bar{s}_i\right) + D_{\max}^2 \\
& - k_p k_g \mathbb{E}\left(g_i^T g_i\right) - k_p k_g \mathbb{E}\left(\bar{g}_i^T \bar{g}_i\right) + k_g b_i \mathbb{E}\left(\left\|\dot{q}_i - \dot{q}_i^*\right\|^2\right) + k_p k_M \sum_{j=1}^{N} \alpha_{\mathrm{M}} m_{ij} \mathbb{E}\left(\left\|\dot{e}_i^j\right\|^2\right) \\
& + \left(k_s k_p^2 + k_g k_p + \frac{k_g}{b_i}\right) \sum_{j=1}^{N} \alpha_{\mathrm{M}} m_{ij} \mathbb{E}\left(\left\|e_i^j\right\|^2\right) \\
& + 2\alpha_{\mathrm{M}} k_p k_C^2 \sum_{j=1}^{N} m_{ij} \left(\mathbb{E}\left(\left\|e_i^j\right\|^2\right) \left\|\dot{\hat{q}}_j^i\right\|^2 + \frac{1}{2}\mathbb{E}\left(\left\|e_i^j\right\|^4\right) + \frac{1}{2}\mathbb{E}\left(\left\|\dot{e}_i^j\right\|^4\right) \right) \Big]
\end{aligned} \tag{A.49}$$

The CTC (61) leads to

$$\dot{V} \leqslant \frac{1}{2} \sum_{i=1}^{N} \mathbb{E}\left[-\left(k_s - 1 - k_p\left(k_M+1\right)\right) s_i^T s_i - k_g k_p g_i^T g_i + D_{\max}^2 + \eta \right]$$
$$\dot{V} \leqslant \frac{1}{2} \sum_{i=1}^{N} \mathbb{E}\left[-k_1 s_i^T s_i - k_g k_p g_i^T g_i + D_{\max}^2 + \eta \right] \tag{A.50}$$

with $k_1 = k_s - 1 - k_p\left(k_M+1\right)$.

Introducing $k_{\mathrm{m}} = \min\left\{k_1, k_p\right\}$, from (A.50), one gets

$$\dot{V} \leqslant \frac{1}{2} \sum_{i=1}^{N} \mathbb{E}\left[-k_{\mathrm{m}} \left(s_i^T s_i + k_g g_i^T g_i \right) + D_{\max}^2 + \eta \right]. \tag{A.51}$$

A lower bound of $\sum_{i=1}^{N} g_i^T g_i$ has now to be introduced using the following lemma, which proof is given in Appendix A.5.1.

Lemma 9 *For all t, one has*

$$\sum_{i=1}^{N} g_i^T g_i \geqslant \frac{N_{\min} m_{\min}}{m_{\max}} P\left(q,t\right), \tag{A.52}$$

where $m_{\min} = \min_{i,j=1\ldots N}\left(m_{ij} \neq 0\right)$, $m_{\max} = \max_{i,\,\ell\,=\,1\ldots N}\left(m_{ij}\right)$ and $N_{\min} = \min_{i=1\ldots N}\left(N_i\right)$.

Using Lemma 9 and introducing $k_3 = \frac{N_{\min} m_{\min}}{m_{\max}}$, one may write

$$\begin{aligned}
\dot{V} &\leqslant \mathbb{E}\left(-\frac{k_{\mathrm{m}}}{2} \left[\sum_{i=1}^{N} s_i^T s_i + \frac{k_3 k_g}{4} P\left(q,t\right) \right] + \frac{N}{2}\left(D_{\max}^2 + \eta\right) \right) \\
&\leqslant \mathbb{E}\left(-\frac{k_{\mathrm{m}}}{k_M^*} \left[\frac{1}{2} \sum_{i=1}^{N} \left(k_M s_i^T s_i \right) + \frac{k_3 k_g}{4} P\left(q,t\right) \right] + \frac{N}{2}\left(D_{\max}^2 + \eta\right) \right) \\
&\leqslant \mathbb{E}\left(-\frac{k_4}{k_M^*} \left[\frac{1}{2} \sum_{i=1}^{N} \left(k_M s_i^T s_i \right) + \frac{k_g}{4} P\left(q,t\right) \right] + \frac{N}{2}\left(D_{\max}^2 + \eta\right) \right)
\end{aligned} \tag{A.53}$$

with $k_M^* = 1$ if $k_M < 1$ and $k_M^* = k_M$ else, and $k_4 = k_\mathrm{m} \min(1, k_3)$. Introducing $c_3 = \frac{k_4}{k_M^*}$, one gets

$$\begin{aligned}\dot{V} &\leqslant \mathbb{E}\left(-c_3\left[\frac{1}{2}\sum_{i=1}^{N}\left(s_i^T M_i s_i\right) + \frac{k_g}{4}P(q,t)\right] + \frac{N}{2}\left[D_{\max}^2 + \eta\right]\right)\\ \dot{V} &\leqslant -c_3 V + \frac{N}{2}\left[D_{\max}^2 + \eta\right].\end{aligned} \tag{A.54}$$

A.4.3 Upper bound of the Lyapunov function

Consider $t \in [\underline{t}, \bar{t}[$ and the function W satisfying

$$\dot{W} = -c_3 W + \frac{N}{2}\left[D_{\max}^2 + \eta\right]. \tag{A.55}$$

The solution of (A.55) with initial condition $W(\underline{t}) = V(\underline{t})$ is

$$W(t) = \exp\left(-c_3(t-\underline{t})\right)V(\underline{t}) + \left(1 - \exp\left(-c_3(t-\underline{t})\right)\right)\frac{N}{2c_3}\left[D_{\max}^2 + \eta\right]. \tag{A.56}$$

Then, using [9, Lemma 3.4] (Comparison lemma), one has $V(t) \leqslant W(t)\ \forall t \in [\underline{t}, \bar{t}[$, so

$$V(t) \leqslant \exp\left(-c_3(t-\underline{t})\right)V(\underline{t}) + \left(1 - \exp\left(-c_3(t-\underline{t})\right)\right)\frac{N}{2c_3}\left[D_{\max}^2 + \eta\right] \tag{A.57}$$

$$\leqslant \exp\left(-c_3(t-\underline{t})\right)\left[V(\underline{t}) - \frac{N}{2c_3}\left[D_{\max}^2 + \eta\right]\right] + \frac{N}{2c_3}\left[D_{\max}^2 + \eta\right] \tag{A.58}$$

Then, since $V(\underline{t}) > \frac{N}{2c_3}\left[D_{\max}^2 + \eta\right]$, $V(t)$ is decreasing over the interval $[\underline{t}, \bar{t}[$.

Using (A.57), one may write $\forall t > 0$

$$V(t) \leqslant \exp(-c_3 t)V(0) + \left(1 - \exp(-c_3 t)\right)\frac{N}{2c_3}\left[D_{\max}^2 + \eta\right] \tag{A.59}$$

and from (A.59), one has

$$\begin{aligned}\lim_{t\to\infty} V(t) &\leqslant \frac{N}{2c_3}\left[D_{\max}^2 + \eta\right]\\ \lim_{t\to\infty}\mathbb{E}\left(\frac{1}{2}\sum_{i=1}^{N}\left(s_i^T M_i s_i\right) + \frac{k_g}{4}P(q,t)\right) &\leqslant \frac{N}{2c_3}\left[D_{\max}^2 + \eta\right]\\ \lim_{t\to\infty}\mathbb{E}\left(\frac{1}{2}P(q,t)\right) &\leqslant \frac{N}{k_g c_3}\left[D_{\max}^2 + \eta\right].\end{aligned} \tag{A.60}$$

Asymptotically, the formation error is bounded and according to Definition 1, the system is asymptotically converging to the target formation with a bounded mean-square error.

A.5 Additional proof elements

A.5.1 Upper-bound on $\sum_{i=1}^{N} g_i^T g_i$

From (14), one may write

$$\begin{aligned}\sum_{i=1}^{N} g_i^T g_i &= \sum_{i=1}^{N}\left(\sum_{j=1}^{N} m_{ij}\left(r_{ij} - r_{ij}^*\right)\right)^T\left(\sum_{\ell=1}^{N} m_{i\ell}\left(r_{i\ell} - r_{i\ell}^*\right)\right)\\ &= \sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} m_{i\ell}m_{ij}\left(r_{ij} - r_{ij}^*\right)^T\left(r_{i\ell} - r_{i\ell}^*\right)\end{aligned} \tag{A.61}$$

Using the fact that

$$(a-b)^T(a-b) = a^T a + b^T b - 2a^T b, \tag{A.62}$$

one gets

$$\sum_{i=1}^{N} g_i^T g_i = \sum_{i=1}^{N}\left[\frac{1}{2}\sum_{\ell=1}^{N}\sum_{j=1}^{N} m_{i\ell}m_{ij}\left[\left\|r_{ij}-r_{ij}^*\right\|^2 + \left\|r_{i\ell}-r_{i\ell}^*\right\|^2 - \left\|r_{ij}-r_{ij}^* - (r_{i\ell}-r_{i\ell}^*)\right\|^2\right]\right]. \tag{A.63}$$

One has

$$\begin{aligned}\left(r_{ij}-r_{ij}^*\right) - (r_{i\ell}-r_{i\ell}^*) &= (r_{ij}-r_{i\ell}) - \left(r_{ij}^*-r_{i\ell}^*\right)\\ &= r_{\ell j} - r_{\ell j}^*\end{aligned}$$

Injecting this result in (A.63) leads to

$$\sum_{i=1}^{N} g_i^T g_i = \sum_{i=1}^{N}\left[\frac{1}{2}\sum_{\ell=1}^{N}\sum_{j=1}^{N} m_{i\ell}m_{ij}\left[\left\|r_{ij}-r_{ij}^*\right\|^2 + \left\|r_{i\ell}-r_{i\ell}^*\right\|^2 - \left\|r_{\ell j}-r_{\ell j}^*\right\|^2\right]\right] \tag{A.64}$$

with $m_{\max} = \max_{i,j=1\ldots N}(m_{ij})$

$$\begin{aligned}
m_{\max}\sum_{i=1}^{N} g_i^T g_i &\geqslant \sum_{i=1}^{N}\left[\frac{1}{2}\sum_{\ell=1}^{N}\sum_{j=1}^{N} m_{i\ell}m_{ij}m_{\ell j}\left[\left\|r_{ij}-r_{ij}^*\right\|^2 + \left\|r_{i\ell}-r_{i\ell}^*\right\|^2 - \left\|r_{\ell j}-r_{\ell j}^*\right\|^2\right]\right]\\
m_{\max}\sum_{i=1}^{N} g_i^T g_i &\geqslant \frac{1}{2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} m_{i\ell}m_{ij}m_{\ell j}\left\|r_{ij}-r_{ij}^*\right\|^2 + \frac{1}{2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} m_{i\ell}m_{ij}m_{\ell j}\left\|r_{i\ell}-r_{i\ell}^*\right\|^2\\
&\quad - \frac{1}{2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} m_{i\ell}m_{ij}m_{\ell j}\left\|r_{\ell j}-r_{\ell j}^*\right\|^2\\
m_{\max}\sum_{i=1}^{N} g_i^T g_i &\geqslant \frac{1}{2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} m_{i\ell}m_{ij}m_{\ell j}\left\|r_{ij}-r_{ij}^*\right\|^2 + \frac{1}{2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} m_{i\ell}m_{ij}m_{\ell j}\left\|r_{ij}-r_{ij}^*\right\|^2\\
&\quad - \frac{1}{2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} m_{i\ell}m_{ij}m_{\ell j}\left\|r_{ij}-r_{ij}^*\right\|^2\\
m_{\max}\sum_{i=1}^{N} g_i^T g_i &\geqslant \frac{1}{2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} m_{i\ell}m_{ij}m_{\ell j}\left\|r_{ij}-r_{ij}^*\right\|^2.
\end{aligned} \tag{A.65}$$

According $m_{i\ell} = 0$ if $\ell \notin \mathcal{N}_i$, one gets

$$\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} m_{i\ell}m_{ij}m_{\ell j}\left\|r_{ij}-r_{ij}^*\right\|^2 = \sum_{i=1}^{N}\sum_{\ell\in\mathcal{N}_i} m_{i\ell}\sum_{j=1}^{N} m_{ij}m_{\ell j}\left\|r_{ij}-r_{ij}^*\right\|^2$$

Let $m_{\min} = \min_{i,j=1\ldots N}(m_{ij} \neq 0)$ and $N_{\min} = \min_{i=1\ldots N}(N_i)$. One may write

$$\begin{aligned}
&\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} m_{i\ell}m_{ij}m_{\ell j}\left\|r_{ij}-r_{ij}^*\right\|^2 = \sum_{i=1}^{N}\sum_{\ell\in\mathcal{N}_i} m_{\min}\sum_{j=1}^{N} m_{ij}\left\|r_{ij}-r_{ij}^*\right\|^2.\\
&\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} m_{i\ell}m_{ij}m_{\ell j}\left\|r_{ij}-r_{ij}^*\right\|^2 \geqslant N_{\min}m_{\min}\sum_{i=1}^{N}\sum_{j=1}^{N} m_{ij}\left\|r_{ij}-r_{ij}^*\right\|^2
\end{aligned}$$

and so

$$\begin{aligned}
m_{\max}\sum_{i=1}^{N} g_i^T g_i &\geqslant N_{\min}m_{\min}\sum_{i=1}^{N}\sum_{j=1}^{N} m_{ij}\left\|r_{ij}-r_{ij}^*\right\|^2\\
\sum_{i=1}^{N} g_i^T g_i &\geqslant \frac{N_{\min}m_{\min}}{m_{\max}}P(q,t)
\end{aligned} \tag{A.66}$$

A.5.2 Evaluation of c_3

$$\begin{aligned} c_3 &= \frac{k_4}{k_M^*} \\ &= \frac{k_{\mathrm{m}} \min\left(1, k_3\right)}{\max\left\{1, k_M\right\}} \\ &= \frac{\min\left\{k_1, k_p\right\} \min\left(1, \frac{N_{\min} m_{\min}}{m_{\max}}\right)}{\max\left\{1, k_M\right\}}. \end{aligned} \tag{A.67}$$

where $k_1 = k_s - 1 - k_p\left(k_M + 1\right)$.

A.5.3 Evaluation of $\mathbb{E}\left(\left\|e_i^j\left(t_{i,k_i+1}^+\right)\right\|^2\right)$ and $\mathbb{E}\left(\left\|\dot{e}_i^j\left(t_{i,k_i+1}^+\right)\right\|^2\right)$

$\mathbb{E}\left(\left\|e_i^j\left(t_{i,k_i+1}^+\right)\right\|^2\right)$ and $\mathbb{E}\left(\left\|\dot{e}_i^j\left(t_{i,k_i+1}^+\right)\right\|^2\right)$ are evaluated assuming that the implicit feedback is not employed. A similar evaluation may be performed considering this information. Using (39), after the transmission of the k_i+1 message by Agent i, one gets

$$\begin{aligned} \mathbb{E}\left(\left\|e_i^j\left(t_{i,k_i+1}^+\right)\right\|^2\right) &= \sum_{\ell=k_i-\kappa+2}^{k_i+1} p_{k_i+1,\ell} \left\|\hat{q}_i^{i,\ell}\left(t_{i,k_i+1}^+\right) - q_i\left(t_{i,k_i+1}^+\right)\right\|^2 \\ &+ \sum_{\ell=0}^{k_i-\kappa+1} p_{k_i+1,\ell} \left\|\hat{q}_i^{i,\ell}\left(t_{i,k_i+1}^+\right) - q_i\left(t_{i,k_i+1}^+\right)\right\|^2 \\ &= \sum_{\ell=k_i-\kappa+2}^{k_i} \pi p_{k_i,\ell} \left\|\hat{q}_i^{i,\ell}\left(t_{i,k_i+1}^+\right) - q_i\left(t_{i,k_i+1}^+\right)\right\|^2 \\ &+ \sum_{\ell=0}^{k_i-\kappa+1} \pi p_{k_i,\ell} \left\|\hat{q}_i^{i,\ell}\left(t_{i,k_i+1}^+\right) - q_i\left(t_{i,k_i+1}^+\right)\right\|^2 \\ &+ p_{k_i+1,k+1} \left\|\hat{q}_i^{i,k_i+1}\left(t_{i,k_i+1}^+\right) - q_i\left(t_{i,k_i+1}^+\right)\right\|^2, \end{aligned} \tag{A.68}$$

using $p_{k_i+1,\ell} = \pi p_{k_i,\ell}$, see (54). Since $\hat{q}_i^{i,\ell}\left(t_{i,k_i+1}^+\right) = \hat{q}_i^{i,\ell}\left(t_{i,k_i+1}^-\right)$ for all $\ell = k_i - \kappa + 1, \ldots, k_i$ and $q_i\left(t_{i,k_i+1}^+\right) = q_i\left(t_{i,k_i+1}^-\right)$, one deduces

$$\begin{aligned} \mathbb{E}\left(\left\|e_i^j\left(t_{i,k_i+1}^+\right)\right\|^2\right) &= \sum_{\ell=k_i-\kappa+2}^{k_i} \pi p_{k_i,\ell} \left\|\hat{q}_i^{i,\ell}\left(t_{i,k_i+1}^-\right) - q_i\left(t_{i,k_i+1}^-\right)\right\|^2 \\ &+ \sum_{\ell=0}^{k_i-\kappa+1} \pi p_{k_i,\ell} \left\|\hat{q}_i^{i,\ell}\left(t_{i,k_i+1}^-\right) - q_i\left(t_{i,k_i+1}^-\right)\right\|^2 \\ &+ p_{k_i+1,k+1} \left\|\hat{q}_i^{i,k_i+1}\left(t_{i,k_i+1}^+\right) - q_i\left(t_{i,k_i+1}^+\right)\right\|^2, \end{aligned} \tag{A.69}$$

Moreover, at $t = t_{k+1}^+$, $\hat{q}_i^{i,k_i+1}\left(t_{i,k_i+1}^+\right) = q_i\left(t_{i,k_i+1}^+\right)$. Consequently

$$\begin{aligned} \mathbb{E}\left(\left\|e_i^j\left(t_{i,k_i+1}^+\right)\right\|^2\right) &= \sum_{\ell=k_i-\kappa+2}^{k_i} \pi p_{k_i,\ell} \left\|\hat{q}_i^{i,\ell}\left(t_{i,k_i+1}^-\right) - q_i\left(t_{i,k_i+1}^-\right)\right\|^2 \\ &+ \sum_{\ell=0}^{k_i-\kappa+1} \pi p_{k_i,\ell} \left\|\hat{q}_i^{i,\ell}\left(t_{i,k_i+1}^-\right) - q_i\left(t_{i,k_i+1}^-\right)\right\|^2 \\ &= \pi \mathbb{E}\left(\left\|e_i^j\left(t_{i,k_i+1}^-\right)\right\|^2\right). \end{aligned} \tag{A.70}$$

One shows similarly that

$$\mathbb{E}\left(\left\|\dot{e}_i^j\left(t_{i,k_i+1}^+\right)\right\|^2\right) = \pi \mathbb{E}\left(\left\|\dot{e}_i^j\left(t_{i,k_i+1}^-\right)\right\|^2\right). \tag{A.71}$$

A.6 *Upper-bound on* $\mathbb{E}\left(\left\|e_i^j(t)\right\|^2\right)$

Immediately after the transmission of the k_i-th message by Agent i, using (39), one gets for $t \in [t_{i,k_i}, t_{i,k_i+1}[$

$$\begin{aligned}\mathbb{E}\left(\left\|e_i^j(t)\right\|^2\right) &= \sum_{\ell=k_i-\kappa+1}^{k_i} p_{k_i,\ell}\left\|\hat{q}_i^{i,\ell}(t) - q_i(t)\right\|^2 \\ &\quad + \sum_{\ell=0}^{k_i-\kappa} p_{k_i,\ell}\left\|\hat{q}_i^{i,\ell}(t) - q_i(t)\right\|^2. \end{aligned} \tag{A.72}$$

Since at most κ estimators are run in parallel, the estimators $\hat{q}_i^{i,\ell}(t)$ for $\ell \leqslant k_i - \kappa$ are no more available at Agent i. Nevertheless, from Assumption A9, one has $\left\|\hat{q}_i^{i,\ell}(t) - q_i(t)\right\| \leqslant e_{\max}$ and

$$\mathbb{E}\left(\left\|e_i^j(t)\right\|^2\right) \leqslant \sum_{\ell=k_i-\kappa+1}^{k_i} p_{k_i,\ell}\left\|\hat{q}_i^{i,\ell}(t) - q_i(t)\right\|^2 + \sum_{\ell=0}^{k_i-\kappa} p_{k_i,\ell} e_{\max}^2. \tag{A.73}$$

Introducing $\sigma_i^j(k_i,\kappa) = \sum_{\ell=0}^{k_i-\kappa} p_{k_i,\ell} = 1 - \sum_{\ell=k_i-\kappa+1}^{k_i} p_{k_i,\ell}$, as $\sum_{\ell=0}^{k_i} p_{k_i,\ell} = 1$, one gets

$$\mathbb{E}\left(\left\|e_i^j(t)\right\|^2\right) \leqslant \sum_{\ell=k_i-\kappa+1}^{k_i} p_{k_i,\ell}\left\|\hat{q}_i^{i,\ell}(t) - q_i(t)\right\|^2 + \sigma_i^j(k_i,\kappa)\, e_{\max}^2. \tag{A.74}$$

When the implicit (and delayed) feedback information introduced in Section 4.4 is exploited, then (A.74) becomes

$$\mathbb{E}\left(\left\|e_i^j(t)\right\|^2 | k_i^{j,i}, \rho_i^j\right) \leqslant \sum_{\ell=k_i-\kappa+1}^{k_i} p^j_{k_i,\ell|k_i^{j,i},\rho_i^j}\left\|\hat{q}_i^{i,\ell}(t) - q_i(t)\right\|^2 + \sigma_i^j\left(k_i,\kappa|k_i^{j,i},\rho_i^j\right) e_{\max}^2 \tag{A.75}$$

where Agent i is aware from a packet received from Agent j in the time interval $[t_{i,\rho_i^j}, t_{i,\rho_i^j+1}[$ that Agent j has received packet $k_i^{j,i}$ sent by Agent i and

$$\sigma_i^j\left(k_i,\kappa|k_i^{j,i},\rho_i^j\right) = 1 - \sum_{\ell=k_i-\kappa+1}^{k_i} p^j_{k_i,\ell|k_i^{j,i},\rho_i^j}.$$

When the implicit (and delayed) feedback information is not exploited, one has

$$\begin{aligned}\sigma_i^j(k_i,\kappa) &= \sum_{\ell=0}^{k_i-\kappa} p_{k_i,\ell} \\ &= \sum_{\ell=1}^{k_i-\kappa} (1-\pi)\,\pi^{k_i-\ell} + \pi^{k_i} \\ &= (1-\pi)\sum_{\ell=1}^{k_i-\kappa} \pi^{k_i-\ell} + \pi^{k_i} \\ &= (1-\pi)\sum_{\ell=\kappa}^{k_i-1} \pi^{\ell} + \pi^{k_i} \\ &= (1-\pi)\,\frac{\pi^{\kappa} - \pi^{k_i}}{(1-\pi)} + \pi^{k_i} \\ &= \pi^{\kappa}. \end{aligned} \tag{A.76}$$

The same method can be used to upper-bound $\mathbb{E}\left(\left\|\dot{e}_i^j(t)\right\|^2\right)$ using the fact that $\left\|\dot{\hat{q}}_i^{i,\ell}(t) - \dot{q}_i(t)\right\| \leqslant 2\dot{q}_{\max}$ (see Assumptions A6 and A8) to get

$$\mathbb{E}\left(\left\|\dot{e}_i^j(t)\right\|^2\right) \leqslant \sum_{\ell=k_i-\kappa+1}^{k_i} p_{k_i,\ell}\left\|\dot{\hat{q}}_i^{i,\ell}(t) - \dot{q}_i(t)\right\|^2 + 4\sigma_i^j(k_i,\kappa)\,\dot{q}_{\max}^2. \tag{A.77}$$

The upper-bounds (A.74) and (A.77) will be used in Appendix A.7 to get more accurate upper bounds $\overline{\mathbb{E}}\left(\left\|e_i^j(t)\right\|^2\right)$ and $\overline{\mathbb{E}}\left(\left\|\dot{e}_i^j(t)\right\|^2\right)$ for $\mathbb{E}\left(\left\|e_i^j(t)\right\|^2\right)$ and $\mathbb{E}\left(\left\|\dot{e}_i^j(t)\right\|^2\right)$. In what follows, one focuses on the case where the feedback information is not exploited. The following proofs can be translated to the case where the implicit feedback information is exploited by replacing $\mathbb{E}\left(\left\|e_i^j(t)\right\|^2\right)$, $p_{k_i,\ell}$, and $\sigma_i^j(k_i,\kappa)$ by $\mathbb{E}\left(\left\|e_i^j(t)\right\|^2|k_i^{j,i}\right)$, $p^j_{k_i,\ell|k_i^{j,i},\rho_i^j}$, and $\sigma_i^j\left(k_i,\kappa|k_i^{j,i},\rho_i^j\right)$.

A.7 Study of $\overline{\mathbb{E}}\left(\left\|e_i^j(t)\right\|^2\right)$

Our aim in this appendix is to build an upper bound $\overline{\mathbb{E}}\left(\left\|e_i^j(t)\right\|^2\right)$ of $\mathbb{E}\left(\left\|e_i^j(t)\right\|^2\right)$ such that

$$\mathbb{E}\left(\left\|e_i^j(t)\right\|^2\right)\leqslant\overline{\mathbb{E}}\left(\left\|e_i^j(t)\right\|^2\right)\quad\forall t\geqslant 0 \tag{A.78}$$

$$\overline{\mathbb{E}}\left(\left\|e_i^j\left(t_{i,k_i+1}^+\right)\right\|^2\right)=\pi\overline{\mathbb{E}}\left(\left\|e_i^j\left(t_{i,k_i+1}^-\right)\right\|^2\right)\quad\forall k_i>0. \tag{A.79}$$

A.7.1 Case 1: when $k_i\leqslant\kappa$

In this case, one has

$$\mathbb{E}\left(\left\|e_i^j(t)\right\|^2\right)=\sum_{\ell=0}^{k_i}p_{k_i,\ell}\left\|-q_i(t)\right\|^2. \tag{A.80}$$

Since $k_i\leqslant\kappa$, all $\hat{q}_i^{i,\ell}(t)$, $\ell\leqslant k_i$ are available at Agent i and one can take

$$\begin{aligned}\overline{\mathbb{E}}\left(\left\|e_i^j(t)\right\|^2\right)&=\sum_{\ell=0}^{k_i}p_{k_i,\ell}\left\|\hat{q}_i^{i,\ell}(t)-q_i(t)\right\|^2\\&=\mathbb{E}\left(\left\|e_i^j(t)\right\|^2\right).\end{aligned} \tag{A.81}$$

From Appendix A.5.3, one has

$$\mathbb{E}\left(\left\|e_i^j\left(t_{i,k_i}^+\right)\right\|^2\right)=\pi\mathbb{E}\left(\left\|e_i^j\left(t_{i,k_i}^-\right)\right\|^2\right) \tag{A.82}$$

so (A.78) and (A.79) are satisfied for all $k_i\leqslant\kappa$ and $t\leqslant t_{\kappa+1}$.

A.7.2 Case 2: $k_i>\kappa$

Consider $k_i>\kappa$ and introduce

$$\overline{\mathbb{E}}\left(\left\|e_i^j(t)\right\|^2\right)=\sum_{\ell=k_i-\kappa+1}^{k_i}p_{k_i,\ell}\left\|\hat{q}_i^{i,\ell}(t)-q_i(t)\right\|^2+\min\left(\left\{\overline{e}_{i,k_i}^j(t),\sigma_i^j(k_i,\kappa)e_{\max}^2\right\}\right) \tag{A.83}$$

where

$$\begin{aligned}\overline{e}_{i,k_i}^j(t)=&\pi\overline{\mathbb{E}}\left(\left\|e_i^j\left(t_{i,k_i}^-\right)\right\|^2\right)+4\dot{q}_{\max}e_{\max}\sigma_i^j(k_i,\kappa)\left(t-t_{i,k_i}\right)\\&-\sum_{\ell=k_i-\kappa+1}^{k_i}p_{k_i,\ell}\left\|\hat{q}_i^{i,\ell}\left(t_{i,k_i}^+\right)-q_i\left(t_{i,k_i}^+\right)\right\|^2.\end{aligned} \tag{A.84}$$

Our main in what follows is to show that (A.83)-(A.84) satisfy the following conditions for $t \in [t_{i,k_i},\, t_{i,k_i+1}[$:

$$\mathbb{E}\left(\left\|e_i^j(t)\right\|^2\right) \leqslant \overline{\mathbb{E}}\left(\left\|e_i^j(t)\right\|^2\right) \tag{A.85}$$

$$\overline{\mathbb{E}}\left(\left\|e_i^j\left(t_{i,k_i}^+\right)\right\|^2\right) \leqslant \pi\overline{\mathbb{E}}\left(\left\|e_i^j\left(t_{i,k_i}^-\right)\right\|^2\right) \tag{A.86}$$

$$\overline{\mathbb{E}}\left(\left\|e_i^j(t)\right\|^2\right) \leqslant \sum_{\ell=k_i-\kappa+1}^{k_i} p_{k_i,\ell}\left\|\hat{q}_i^{i,\ell}(t) - q_i(t)\right\|^2 + \sigma_i^j(k_i,\kappa)\, e_{\max}^2. \tag{A.87}$$

The proof is by induction. Assume that (A.85)-(A.87) is true for $k_i - 1$ and all $t \in [t_{i,k_i-1}, t_{k_i}[$. Consider now the time interval $[t_{i,k_i},\, t_{i,k_i+1}[$.

1) From (A.83), (A.87) is obviously satisfied.

2) At $t = t_{i,k_i}^+$, one has

$$\begin{aligned}\overline{\mathbb{E}}\left(\left\|e_i^j\left(t_{i,k_i}^+\right)\right\|^2\right) &= \sum_{\ell=k_i-\kappa+1}^{k_i} p_{k_i,\ell}\left\|\hat{q}_i^{i,\ell}\left(t_{i,k_i}^+\right) - q_i\left(t_{i,k_i}^+\right)\right\|^2 \\ &\quad + \min\left(\left\{\pi\overline{\mathbb{E}}\left(\left\|e_i^j\left(t_{i,k_i}^-\right)\right\|^2\right) - \sum_{\ell=k_i-\kappa+1}^{k_i} p_{k_i,\ell}^j\left\|\hat{q}_i^{i,\ell}\left(t_{i,k_i}^+\right) - q_i\left(t_{i,k_i}^+\right)\right\|^2,\, \sigma_i^j(k_i,\kappa)\, e_{\max}^2\right\}\right).\end{aligned} \tag{A.88}$$

If $\pi\overline{\mathbb{E}}\left(\left\|e_i^j\left(t_{i,k_i}^-\right)\right\|^2\right) - \sum_{\ell=k_i-\kappa+1}^{k_i} p_{k_i,\ell}^j\left\|\hat{q}_i^{i,\ell}\left(t_{i,k_i}^+\right) - q_i\left(t_{i,k_i}^+\right)\right\|^2 \leqslant \sigma_i^j(k_i,\kappa)\, e_{\max}^2$, one has

$$\begin{aligned}\overline{\mathbb{E}}\left(\left\|e_i^j\left(t_{i,k_i}^+\right)\right\|^2\right) &= \sum_{\ell=k_i-\kappa+1}^{k_i} p_{k_i,\ell}\left\|\hat{q}_i^{i,\ell}\left(t_{i,k_i}^+\right) - q_i\left(t_{i,k_i}^+\right)\right\|^2 + \pi\overline{\mathbb{E}}\left(\left\|e_i^j\left(t_{i,k_i}^-\right)\right\|^2\right) - \sum_{\ell=k_i-\kappa+1}^{k_i} p_{k_i,\ell}^j\left\|\hat{q}_i^{i,\ell}\left(t_{i,k_i}^+\right) - q_i\left(t_{i,k_i}^+\right)\right\|^2 \\ &= \pi\overline{\mathbb{E}}\left(\left\|e_i^j\left(t_{i,k_i}^-\right)\right\|^2\right).\end{aligned} \tag{A.89}$$

Else, one has

$$\sigma_i^j(k_i,\kappa)\, e_{\max}^2 < \pi\overline{\mathbb{E}}\left(\left\|e_i^j\left(t_{i,k_i}^-\right)\right\|^2\right) - \sum_{\ell=k_i-\kappa+1}^{k_i} p_{k_i,\ell}^j\left\|\hat{q}_i^{i,\ell}\left(t_{i,k_i}^+\right) - q_i\left(t_{i,k_i}^+\right)\right\|^2 \tag{A.90}$$

so

$$\begin{aligned}\overline{\mathbb{E}}\left(\left\|e_i^j\left(t_{i,k_i}^+\right)\right\|^2\right) &= \sum_{\ell=k_i-\kappa+1}^{k_i} p_{k_i,\ell}\left\|\hat{q}_i^{i,\ell}\left(t_{i,k_i}^+\right) - q_i\left(t_{i,k_i}^+\right)\right\|^2 + \sigma_i^j(k_i,\kappa)\, e_{\max}^2 \\ &\leqslant \sum_{\ell=k_i-\kappa+1}^{k_i} p_{k_i,\ell}\left\|\hat{q}_i^{i,\ell}\left(t_{i,k_i}^+\right) - q_i\left(t_{i,k_i}^+\right)\right\|^2 + \pi\overline{\mathbb{E}}\left(\left\|e_i^j\left(t_{i,k_i}^-\right)\right\|^2\right) - \sum_{\ell=k_i-\kappa+1}^{k_i} p_{k_i,\ell}^j\left\|\hat{q}_i^{i,\ell}\left(t_{i,k_i}^+\right) - q_i\left(t_{i,k_i}^+\right)\right\|^2 \\ &\leqslant \pi\overline{\mathbb{E}}\left(\left\|e_i^j\left(t_{i,k_i}^-\right)\right\|^2\right).\end{aligned} \tag{A.91}$$

From (A.89) and (A.91), one deduces that (A.86) is satisfied. Moreover, using (A.71), and the induction hypothesis $\mathbb{E}\left(\left\|e_i^j\left(t_{i,k_i}^-\right)\right\|^2\right) \leqslant \overline{\mathbb{E}}\left(\left\|e_i^j\left(t_{i,k_i}^-\right)\right\|^2\right)$, then (A.85) is satisfied at $t = t_{i,k_i}^+$.

3) Since $\overline{e}_{i,k_i}^j(t)$, introduced in (A.84), is an increasing function of t. Two situations have to be considered.

First, if for all $t \in [t_{i,k_i}, t_{i,k_i+1}[$ one has $\overline{e}^j_{i,k_i}(t) \leqslant \sigma^j_i(k_i,\kappa)\, e^2_{\max}$, then

$$\overline{\mathbb{E}}\left(\left\|e^j_i(t)\right\|^2\right) = \sum_{\ell=k_i-\kappa+1}^{k_i} p_{k_i,\ell}\left\|\hat{q}^{i,\ell}_i(t) - q_i(t)\right\|^2 + \overline{e}^j_{i,k_i}(t).$$

Second, if there exist $\bar{t}_{i,k_i} \in [t_{i,k_i}, t_{i,k_i+1}[$ such that $\overline{e}^j_{i,k_i}(\bar{t}_{i,k_i}) = \sigma^j_i(k_i,\kappa)\, e^2_{\max}$, then $\forall t \in [\bar{t}_{i,k_i}, t_{i,k_i+1}[$, $\overline{e}^j_{i,k_i}(t) > \sigma^j_i(k_i,\kappa)\, e^2_{\max}$ and

$$\overline{\mathbb{E}}\left(\left\|e^j_i(t)\right\|^2\right) = \sum_{\ell=k_i-\kappa+1}^{k_i} p_{k_i,\ell}\left\|\hat{q}^{i,\ell}_i(t) - q_i(t)\right\|^2 + \sigma^j_i(k_i,\kappa)\, e^2_{\max}. \tag{A.92}$$

Then, from (A.74) and (A.92), for all $t > \bar{t}_{i,k_i}$ one has $\mathbb{E}\left(\left\|e^j_i(t)\right\|^2\right) \leqslant \overline{\mathbb{E}}\left(\left\|e^j_i(t)\right\|^2\right)$.

Consider now the time interval $[t_{i,k_i},\, \min\{\bar{t}_{i,k_i}, t_{i,k_i+1}\}[$. To show that (A.85) is satisfied for all $t \in [t_{i,k_i},\, \min\{\bar{t}_{i,k_i}, t_{i,k_i+1}\}[$, we first show that

$$\frac{d}{dt}\overline{\mathbb{E}}\left(\left\|e^j_i(t)\right\|^2\right) - \frac{d}{dt}\mathbb{E}\left(\left\|e^j_i(t)\right\|^2\right) \geqslant 0. \tag{A.93}$$

One has

$$\begin{aligned}\frac{d}{dt}\overline{\mathbb{E}}\left(\left\|e^j_i(t)\right\|^2\right) &= \frac{d}{dt}\sum_{\ell=k_i-\kappa+1}^{k_i} p^j_{k_i,\ell}\left\|\hat{q}^{i,\ell}_i(t) - q_i(t)\right\|^2 + \frac{d}{dt}\overline{e}^j_{i,k_i}(t)\\ &= \frac{d}{dt}\sum_{\ell=k_i-\kappa+1}^{k_i} p^j_{k_i,\ell}\left\|\hat{q}^{i,\ell}_i(t) - q_i(t)\right\|^2 + 4\dot{q}_{\max} e_{\max}\sigma^j_i(k_i,\kappa)\end{aligned}$$

and

$$\frac{d}{dt}\mathbb{E}\left(\left\|e^j_i(t)\right\|^2\right) = \frac{d}{dt}\sum_{\ell=k_i-\kappa+1}^{k_i} p^j_{k_i,\ell}\left\|\hat{q}^{i,\ell}_i(t) - q_i(t)\right\|^2 + \frac{d}{dt}\sum_{\ell=0}^{k_i-\kappa} p^j_{k_i,\ell}\left\|\hat{q}^{i,\ell}_i(t) - q_i(t)\right\|^2. \tag{A.94}$$

For all $\ell \in [0,\, k_i]$, one has

$$\begin{aligned}\frac{d}{dt}\left\|\hat{q}^{i,\ell}_i(t) - q_i(t)\right\|^2 &= \frac{d}{dt}\left(\left(\hat{q}^{i,\ell}_i(t) - q_i(t)\right)^T\left(\hat{q}^{i,\ell}_i(t) - q_i(t)\right)\right)\\ &= 2\left(\dot{\hat{q}}^{i,\ell}_i(t) - \dot{q}_i(t)\right)^T\left(\hat{q}^{i,\ell}_i(t) - q_i(t)\right)\\ &\leqslant 2\left\|\hat{q}^{i,\ell}_i(t) - q_i(t)\right\|\left\|\dot{\hat{q}}^{i,\ell}_i(t) - \dot{q}_i(t)\right\|.\end{aligned} \tag{A.95}$$

Using Assumptions A6, A8 and A9, one has $\left\|\dot{\hat{q}}^{i,\ell}_i(t) - \dot{q}_i(t)\right\| \leqslant 2\dot{q}_{\max}$ and $\left\|\dot{\hat{q}}^{i,\ell}_i(t) - \dot{q}_i(t)\right\| \leqslant e_{\max}$, so

$$\frac{d}{dt}\left\|\hat{q}^{i,\ell}_i(t) - q_i(t)\right\|^2 \leqslant 4\dot{q}_{\max} e_{\max}. \tag{A.96}$$

Then,

$$\begin{aligned}\frac{d}{dt}\mathbb{E}\left(\left\|e^j_i(t)\right\|^2\right) &\leqslant \frac{d}{dt}\sum_{\ell=k_i-\kappa+1}^{k_i} p_{k_i,\ell}\left\|\hat{q}^{i,\ell}_i(t) - q_i(t)\right\|^2 + 4\dot{q}_{\max} e_{\max}\sum_{\ell=0}^{k_i-\kappa} p_{k_i,\ell}\\ &\leqslant \frac{d}{dt}\sum_{\ell=k_i-\kappa+1}^{k_i} p_{k_i,\ell}\left\|\hat{q}^{i,\ell}_i(t) - q_i(t)\right\|^2 + 4\dot{q}_{\max} e_{\max}\sigma^j_i(k_i,\kappa)\\ &\leqslant \frac{d}{dt}\overline{\mathbb{E}}\left(\left\|e^j_i(t)\right\|^2\right).\end{aligned} \tag{A.97}$$

Thus, (A.93) is satisfied and since $\mathbb{E}\left(\left\|e_i^j\left(t_{i,k_i}^+\right)\right\|^2\right) \leqslant \overline{\mathbb{E}}\left(\left\|e_i^j\left(t_{i,k_i}^+\right)\right\|^2\right)$, (A.85) is satisfied for all $t \in \left[t_{i,k_i}^+, t_{i,k_i+1}\right[$.

4) The condition $\mathbb{E}\left(\left\|e_i^j\left(t_{i,k_i}^-\right)\right\|^2\right) \leqslant \overline{\mathbb{E}}\left(\left\|e_i^j\left(t_{i,k_i}^-\right)\right\|^2\right)$ is satisfied for $k_i = \kappa$, see Case 1. Then, from (A.71) and (A.91), $\mathbb{E}\left(\left\|e_i^j\left(t_{i,k_i}^+\right)\right\|^2\right) \leqslant \overline{\mathbb{E}}\left(\left\|e_i^j\left(t_{i,k_i}^+\right)\right\|^2\right)$ is true for $k_i = \kappa + 1$. Using the previous proof by induction, (A.85) is satisfied for $k_i = \kappa + 1$ and for all $k_i > \kappa$.

A.7.3 General expression of $\overline{\mathbb{E}}\left(\left\|e_i^j(t)\right\|^2\right)$

Considering Case 1 and Case 2 studied in Sections A.7.1 and A.7.2, one may write a single expression of $\overline{\mathbb{E}}\left(\left\|e_i^j(t)\right\|^2\right)$ valid in both cases for all $t \in [t_{i,k_i},\, t_{i,k_i+1}[$ as

$$\begin{aligned}\overline{\mathbb{E}}\left(\left\|e_i^j(t)\right\|^2\right) = &\sum_{\ell=\ell_{\min}}^{k_i} p_{k_i,\ell}\left\|\hat{q}_i^{i,\ell}(t) - q_i(t)\right\|^2 \\ &+ 1_{k_i>\kappa}\min\left(\left[\overline{e}_{i,k_i}^j(t)\,,\, \sigma_i^j(k_i,\kappa)\, e_{\max}^2\right]\right)\end{aligned} \tag{A.98}$$

with $\ell_{\min} = \max\left([0, k_i - \kappa + 1]\right)$,

$$1_{k_i>\kappa} = \begin{cases} 1 & \text{if } k_i > \kappa \\ 0 & \text{else,}\end{cases}$$

$$\overline{e}_{i,k_i}^j(t) = 4\dot{q}_{\max} e_{\max}\sigma_i^j(k_i,\kappa)\left(t - t_{i,k_i}\right) + \pi\overline{\mathbb{E}}\left(\left\|e_i^j\left(t_{i,k_i}^-\right)\right\|^2\right) - \sum_{\ell=k_i-\kappa+1}^{k_i} p_{k_i,\ell}^j\left\|\hat{q}_i^{i,\ell}\left(t_{i,k_i}^+\right) - q_i\left(t_{i,k_i}^+\right)\right\|^2 \tag{A.99}$$

and $\overline{\mathbb{E}}\left(\left\|e_i^j(0)\right\|^2\right) = \left\|\hat{q}_i^i(0) - q_i(0)\right\|^2$.

Note that (A.98) guarantees that $\overline{\mathbb{E}}\left(\left\|e_i^j\left(t_{i,k_i}^+\right)\right\|^2\right) = \pi\overline{\mathbb{E}}\left(\left\|e_i^j\left(t_{i,k_i}^-\right)\right\|^2\right)$ for all communication instants t_{i,k_i}.

Following the same steps, (A.98) can be extended to the case where the implicit feedback information is exploited as

$$\begin{aligned}\overline{\mathbb{E}}\left(\left\|e_i^j(t)\right\|^2 | k_i^{j,i}, \rho_i^j\right) = &\sum_{\ell=\ell_{\min}}^{k_i} p_{k_i,\ell|k_i^{j,i},\rho_i^j}^j\left\|\hat{q}_i^{i,\ell}(t) - q_i(t)\right\|^2 \\ &+ 1_{k_i>\kappa} 1_{k_i - k_i^{j,i} > \kappa}\min\left(\left[\overline{e}_{i,k_i}^j(t)\,,\, \sigma_i^j\left(k_i,\kappa|k_i^{j,i},\rho_i^j\right) e_{\max}^2\right]\right)\end{aligned} \tag{A.100}$$

with

$$\begin{aligned}\overline{e}_{i,k_i}^j(t) = &\, 4\dot{q}_{\max} e_{\max}\sigma_i^j\left(k_i,\kappa|k_i^{j,i},\rho_i^j\right)\left(t - t_{i,k_i}\right) + \pi\overline{\mathbb{E}}\left(\left\|e_i^j\left(t_{i,k_i}^-\right)|k_i^{j,i},\rho_i^j\right\|^2\right) \\ &- \sum_{\ell=k_i-\kappa+1}^{k_i} p_{k_i,\ell|k_i^{j,i},\rho_i^j}^j\left\|\hat{q}_i^{i,\ell}\left(t_{i,k_i}^+\right) - q_i\left(t_{i,k_i}^+\right)\right\|^2.\end{aligned} \tag{A.101}$$

A.7.4 Evaluation of $\overline{\mathbb{E}}\left(\left\|\dot{e}_i^j(t)\right\|^2\right)$, $\overline{\mathbb{E}}\left(\left\|e_i^j(t)\right\|^4\right)$, *and* $\overline{\mathbb{E}}\left(\left\|\dot{e}_i^j(t)\right\|^4\right)$

Following the same lines of Appendix A.7, we can obtain an upper bound $\overline{\mathbb{E}}\left(\left\|\dot{e}_i^j(t)\right\|^2\right)$ of $\mathbb{E}\left(\left\|\dot{e}_i^j(t)\right\|^2\right)$ using the fact that $\left\|\dot{\hat{q}}_i^{i,\ell}(t) - \dot{q}_i(t)\right\| \leqslant 2\dot{q}_{\max}$ and $\left\|\ddot{\hat{q}}_i^{i,\ell}(t) - \ddot{q}_i(t)\right\| \leqslant 2\ddot{q}_{\max}$ as follows

$$\overline{\mathbb{E}}\left(\left\|\dot{e}_i^j(t)\right\|^2\right) = \sum_{\ell=\ell_{\min}}^{k_i} p_{k_i,\ell}\left\|\dot{\hat{q}}_i^{i,\ell}(t) - \dot{q}_i(t)\right\|^2 + 1_{k_i>\kappa}\min\left(\left[\dot{\overline{e}}_{i,k_i}^j(t)\,,\, 4\sigma_i^j(k_i,\kappa)\,\dot{q}_{\max}^2\right]\right) \tag{A.102}$$

with

$$\dot{\bar{e}}_{i,k_i}^{j}(t) = 8\dot{q}_{\max}\ddot{q}_{\max}\sigma_i^j(k_i,\kappa)(t-t_{i,k_i}) + \pi\overline{\mathbb{E}}\left(\left\|\dot{e}_i^j\left(t_{i,k_i}^-\right)\right\|^2\right) - \sum_{\ell=k_i-\kappa+1}^{k_i} p_{k_i,\ell}^j\left\|\dot{\hat{q}}_i^{i,\ell}\left(t_{i,k_i}^+\right) - \dot{q}_i\left(t_{i,k_i}^+\right)\right\|^2. \tag{A.103}$$

Consider now the upper bound $\overline{\mathbb{E}}\left(\left\|e_i^j(t)\right\|^4\right)$ of $\mathbb{E}\left(\left\|e_i^j(t)\right\|^4\right)$. Following the idea used for $\overline{\mathbb{E}}\left(\left\|e_i^j(t)\right\|^2\right)$, we study $\frac{d}{dt}\left\|\hat{q}_i^{i,\ell}(t) - q_i(t)\right\|^4$ for all $\ell \geqslant 0$

$$\begin{aligned}
\frac{d}{dt}\left\|\hat{q}_i^{i,\ell}(t) - q_i(t)\right\|^4 &= \frac{d}{dt}\left(\left(\hat{q}_i^{i,\ell}(t) - q_i(t)\right)^T\left(\hat{q}_i^{i,\ell}(t) - q_i(t)\right)\right)^2 \\
&= 4\left(\dot{\hat{q}}_i^{i,\ell}(t) - \dot{q}_i(t)\right)^T\left(\hat{q}_i^{i,\ell}(t) - q_i(t)\right)\left\|\hat{q}_i^{i,\ell}(t) - q_i(t)\right\|^2 \\
&\leqslant 4\left\|\dot{\hat{q}}_i^{i,\ell}(t) - \dot{q}_i(t)\right\|\left\|\hat{q}_i^{i,\ell}(t) - q_i(t)\right\|^3 \\
&\leqslant 8\dot{q}_{\max}e_{\max}^3.
\end{aligned} \tag{A.104}$$

Then, following the same steps as those used for the evaluation of $\overline{\mathbb{E}}\left(\left\|e_i^j(t)\right\|^2\right)$ in Appendix A.7, one gets

$$\overline{\mathbb{E}}\left(\left\|e_i^j(t)\right\|^4\right) = \sum_{\ell=\ell_{\min}}^{k_i} p_{k_i,\ell}\left\|\hat{q}_i^{i,\ell}(t) - q_i(t)\right\|^4 + 1_{k_i>\kappa}\min\left(\left[\bar{e}_{i,k_i}^{j,2}(t),\ \sigma_i^j(k_i,\kappa)\,e_{\max}^4\right]\right) \tag{A.105}$$

with

$$\bar{e}_{i,k_i}^{j,2}(t) = 8\dot{q}_{\max}e_{\max}^3\sigma_i^j(k_i,\kappa)(t-t_{i,k_i}) + \pi\overline{\mathbb{E}}\left(\left\|e_i^j\left(t_{i,k_i}^-\right)\right\|^4\right) - \sum_{\ell=k_i-\kappa+1}^{k_i} p_{k_i,\ell}^j\left\|\hat{q}_i^{i,\ell}\left(t_{i,k_i}^+\right) - q_i\left(t_{i,k_i}^+\right)\right\|^4 \tag{A.106}$$

The same method can be used to obtain an upper bound $\overline{\mathbb{E}}\left(\left\|\dot{e}_i^j(t)\right\|^4\right)$ of $\mathbb{E}\left(\left\|\dot{e}_i^j(t)\right\|^4\right)$:

$$\overline{\mathbb{E}}\left(\left\|\dot{e}_i^j(t)\right\|^4\right) = \sum_{\ell=\ell_{\min}}^{k_i} p_{k_i,\ell}\left\|\dot{\hat{q}}_i^{i,\ell}(t) - \dot{q}_i(t)\right\|^4 + 1_{k_i>\kappa}\min\left(\left[\dot{\bar{e}}_{i,k_i}^{j,2}(t),\ 16\sigma_i^j(k_i,\kappa)\,\dot{q}_{\max}^4\right]\right) \tag{A.107}$$

with

$$\dot{\bar{e}}_{i,k_i}^{j,2}(t) = 32\ddot{q}_{\max}\dot{q}_{\max}^3\sigma_i^j(k_i,\kappa)(t-t_{i,k_i}) + \pi\overline{\mathbb{E}}\left(\left\|\dot{e}_i^j\left(t_{i,k_i}^-\right)\right\|^4\right) - \sum_{\ell=k_i-\kappa+1}^{k_i} p_{k_i,\ell}^j\left\|\dot{\hat{q}}_i^{i,\ell}\left(t_{i,k_i}^+\right) - \dot{q}_i\left(t_{i,k_i}^+\right)\right\|^4. \tag{A.108}$$

Here again, these results can be extended to the case where the implicit feedback information is used by replacing in the expressions the notation $\mathbb{E}\left(\left\|e_i^j(t)\right\|^2\right)$, $p_{k_i,\ell}$, $1_{k_i>\kappa}$, and $\sigma_i^j(k_i,\kappa)$ by $\mathbb{E}\left(\left\|e_i^j(t)\right\|^2|k_i^{j,i},\rho_i^j\right)$, $p^j_{k_i,\ell|k_i^{j,i},\rho_i^j}$, $1_{k_i>\kappa}1_{k_i-k_i^{j,i}>\kappa}$, and $\sigma_i^j\left(k_i,\kappa|k_i^{j,i},\rho_i^j\right)$.

A.8 Proof of the upper-bound H_i

The CTC (61) is not triggering at $t=0$ if

$$\begin{aligned}
k_s\bar{s}_i^T\bar{s}_i(0) + k_pk_g\bar{g}_i^T\bar{g}_i(0) + \eta > \alpha_{\mathrm{M}}\Bigg[&\sum_{j=1}^N m_{ij}\left(k_e\left\|e_i^j\right\|^2 + k_pk_M\left\|\dot{e}_i^j\right\|^2\right) \\
&+2k_pk_C^2\sum_{j=1}^N m_{ij}\left(\left\|e_i^j\right\|^2\left\|\dot{\hat{q}}_j^i\right\|^2 + \frac{1}{2}\left\|e_i^j\right\|^4 + \frac{1}{2}\left\|\dot{e}_i^j\right\|^4\right)\Bigg] \\
&+ k_gb_i\left\|\dot{q}_i - \dot{q}_i^*\right\|^2.
\end{aligned}$$

Remind $\bar{s}_i^T \bar{s}_i = k_p \bar{g}_i^T \bar{g}_i + \|\dot{q}_i - \dot{q}_i^*\|^2$, so

$$\begin{aligned} k_s \|\dot{q}_i - \dot{q}_i^*\|^2 (0) + k_p (k_s + k_g) \bar{g}_i^T \bar{g}_i (0) + \eta > \alpha_{\mathrm{M}} \Bigg[& \sum_{j=1}^{N} m_{ij} \left(k_e \left\| e_i^j \right\|^2 + k_p k_M \left\| \dot{e}_i^j \right\|^2 \right) \\ & + 2 k_p k_C^2 \sum_{j=1}^{N} m_{ij} \left(\left\| e_i^j \right\|^2 \left\| \dot{\hat{q}}_j^i \right\|^2 + \frac{1}{2} \left\| e_i^j \right\|^4 + \frac{1}{2} \left\| \dot{e}_i^j \right\|^4 \right) \Bigg] \\ & + k_g b_i \|\dot{q}_i - \dot{q}_i^*\|^2 . \end{aligned} \tag{A.109}$$

Since $b_i < \frac{k_s}{k_s k_p + k_g}$, one has

$$\begin{aligned} k_s - k_g \left(\frac{k_s}{k_s k_p + k_g} \right) &= k_s \frac{k_s k_p + k_g - k_g}{k_s k_p + k_g} \\ &= k_s \frac{k_s k_p}{k_s k_p + k_g} > 0 \end{aligned}$$

so (A.109) is satisfied if

$$\begin{aligned} k_p (k_s + k_g) \bar{g}_i^T \bar{g}_i (0) + \eta > \alpha_{\mathrm{M}} \Bigg[& \sum_{j=1}^{N} m_{ij} \left(k_e \left\| e_i^j \right\|^2 + k_p k_M \left\| \dot{e}_i^j \right\|^2 \right) \\ & + 2 k_p k_C^2 \sum_{j=1}^{N} m_{ij} \left(\left\| e_i^j \right\|^2 \left\| \dot{\hat{q}}_j^i \right\|^2 + \frac{1}{2} \left\| e_i^j \right\|^4 + \frac{1}{2} \left\| \dot{e}_i^j \right\|^4 \right) \Bigg] . \end{aligned} \tag{A.110}$$

We choose the condition $\left\| \dot{\hat{q}}_j^i \right\|^2 = 0$, $\left\| e_i^j \right\|^2 < H_i$ and $\left\| \dot{e}_i^j \right\|^2 < H_i$ where $H_i \geqslant 0$ is a constant. Remind $\alpha_i = \sum_{j=1}^{N} m_{ij}$. Thus, (A.110) may be rewritten as

$$\begin{aligned} k_p (k_s + k_g) \bar{g}_i^T \bar{g}_i (0) + \eta > \alpha_{\mathrm{M}} \Bigg[& \sum_{j=1}^{N} m_{ij} (k_e + k_p k_M) H_i \\ & + 2 k_p k_C^2 \sum_{j=1}^{N} m_{ij} \left(\frac{1}{2} H_i^2 \right) \Bigg] \\ k_p (k_s + k_g) \bar{g}_i^T \bar{g}_i (0) + \eta > \alpha_{\mathrm{M}} \alpha_i & \left[(k_e + k_p k_M) H_i + k_p k_C^2 H_i^2 \right] \end{aligned} \tag{A.111}$$

and so

$$\left(\frac{k_p (k_s + k_g)}{\alpha_{\mathrm{M}} \alpha_i} \bar{g}_i^T \bar{g}_i (0) + \frac{\eta}{\alpha_{\mathrm{M}} \alpha_i} \right) - (k_e - k_p k_M) H_i - \left(k_p k_C^2 \right) H_i^2 > 0. \tag{A.112}$$

By solving (A.112), we find

$$H_i < \frac{\sqrt{(k_e + k_p k_M)^2 + k_p k_C^2 \xi_i} - (k_e + k_p k_M)}{2 k_p k_C^2}$$

where $\xi_i = \frac{k_p \left(k_s + k_g \right)}{\alpha_{\mathrm{M}} \alpha_i} \bar{g}_i^T \bar{g}_i (0) + \frac{\eta}{\alpha_{\mathrm{M}} \alpha_i}$.

B Proof of absence of Zeno behavior

To prove the absence of Zeno behavior, we have to define a minimum inter-event time $\tau_{\min}$ such $t_{i,k_i+1} - t_{i,k_i} \geqslant \tau_{\min}$. To obtain it, we study the evolution of the right-hand side $C_R(t)$ and the left-hand side $C_L(t)$ of the CTC exposed in Theorem 7 to find a condition such $C_R(t_{i,k_i} + \Delta t) < C_L(t_{i,k_i} + \Delta t)$ for $\Delta t < \tau_{\min}$. Lipschitzien hypotheses described in Assumptions A6, A7 and A8 will be used to obtain an estimation of $C_R(t_{i,k_i} + \Delta t)$ and $C_L(t_{i,k_i} + \Delta t)$ before be used to estimate $\tau_{\min}$.

B.1 Proof that $t_{i,k} < t_{i,k+1}$

In a first time, let show the CTC is not satisfied immediately after a communication, *i.e.* $t_{i,k} < t_{i,k+1}$, and find conditions on design parameters to guarantee it. Consider

$$C_{\mathrm{L}}(t) = k_s \bar{s}_i^T(t)\,\bar{s}_i(t) + k_p k_g \bar{g}_i^T(t)\,\bar{g}_i(t) + \eta$$

and

$$\begin{aligned} C_{\mathrm{R}}(t) = \alpha_{\mathrm{M}} &\left[\sum_{j=1}^{N} m_{ij}\left(k_e\overline{\mathbb{E}}\left(\left\|e_i^j(t)\right\|^2\right) + k_p k_{\mathrm{M}}\overline{\mathbb{E}}\left(\left\|\dot{e}_i^j(t)\right\|^2\right)\right)\right. \\ &\left. +2k_p k_C^2\sum_{j=1}^{N} m_{ij}\left(\overline{\mathbb{E}}\left(\left\|e_i^j(t)\right\|^2\right)\left\|\hat{\dot{q}}_j^i(t)\right\|^2 + \frac{1}{2}\overline{\mathbb{E}}\left(\left\|e_i^j(t)\right\|^4\right) + \frac{1}{2}\overline{\mathbb{E}}\left(\left\|\dot{e}_i^j(t)\right\|^4\right)\right)\right] \\ &+ k_g b_i \left\|\dot{q}_i(t) - \dot{q}_i^*(t)\right\|^2, \end{aligned}$$

with $k_e = \left(k_s k_p^2 + k_g k_p + \frac{k_g}{b_i}\right)$.

According to (61), no communication is triggered as long as $C_{\mathrm{L}}(t) > C_{\mathrm{R}}(t)$. A communication is triggered at $t = t_{i,k}$ when

$$C_{\mathrm{L}}(t_{i,k}) = C_{\mathrm{R}}(t_{i,k}). \tag{B.1}$$

The message sent at time $t = t_{i,k}$ by Agent i implies an update of the estimates $\widehat{q}_i^j$, $j \in \mathcal{N}_i$ of that state q_i run by the neighbors of Agent i. Introduce t_{i,k_i}^- as the instant t_{i,k_i} before the update and t_{i,k_i}^+ be the instant t_{i,k_i} after the update. Consequently, the expected state estimation error will be such that

$$\mathbb{E}\left(\left\|e_i^j\left(t_{i,k_i}^+\right)\right\|^2\right) = \pi\mathbb{E}\left(\left\|e_i^j\left(t_{i,k_i}^-\right)\right\|^2\right) \tag{B.2}$$

$$\overline{\mathbb{E}}\left(\left\|e_i^j\left(t_{i,k_i}^+\right)\right\|^2\right) = \pi\overline{\mathbb{E}}\left(\left\|e_i^j\left(t_{i,k_i}^-\right)\right\|^2\right), \tag{B.3}$$

see in Appendix A.5.3 and A.7. Nevertheless, $k_s\bar{s}_i^T\bar{s}_i + k_p k_g \bar{g}_i^T\bar{g}_i + \eta$ and $\|\dot{q}_i - \dot{q}_i^*\|^2$, which are not updated by the communication, stay unchanged. Consequently

$$C_{\mathrm{L}}\left(t_{i,k_i}^+\right) = C_{\mathrm{L}}\left(t_{i,k_i}^-\right). \tag{B.4}$$

To prove that $t_{i,k} < t_{i,k+1}$, one has to show that $C_{\mathrm{L}}\left(t_{i,k_i}^+\right) > C_{\mathrm{R}}\left(t_{i,k_i}^+\right)$.

Using Appendix A.5.3 and the continuity of $\dot{q}_i(t)$ and $\dot{q}_i^*$, one may write

$$\begin{aligned} C_{\mathrm{R}}\left(t_{i,k_i}^+\right) = k_g b_i \left\|\dot{q}_i\left(t_{i,k_i}^-\right) - \dot{q}_i^*\left(t_{i,k_i}^-\right)\right\|^2 + \pi\alpha_{\mathrm{M}} &\left[\sum_{j=1}^{N} m_{ij}\left(k_e\overline{\mathbb{E}}\left(\left\|e_i^j\left(t_{i,k_i}^-\right)\right\|^2\right) + k_p k_M\overline{\mathbb{E}}\left(\left\|\dot{e}_i^j\left(t_{i,k_i}^-\right)\right\|^2\right)\right)\right. \\ &\left. +2k_p k_C^2\sum_{j=1}^{N} m_{ij}\left(\overline{\mathbb{E}}\left(\left\|e_i^j\left(t_{i,k_i}^-\right)\right\|^2\right)\left\|\hat{\dot{q}}_j^i\right\|^2 + \frac{1}{2}\overline{\mathbb{E}}\left(\left\|e_i^j\left(t_{i,k_i}^-\right)\right\|^4\right) + \frac{1}{2}\overline{\mathbb{E}}\left(\left\|\dot{e}_i^j\left(t_{i,k_i}^-\right)\right\|^4\right)\right)\right] \end{aligned}$$

Using (B.1) and (B.4), one gets

$$\begin{aligned} C_{\mathrm{R}}\left(t_{i,k_i}^+\right) &= k_g b_i \left\|\dot{q}_i\left(t_{i,k_i}^-\right) - \dot{q}_i^*\left(t_{i,k_i}^-\right)\right\|^2 + \pi\left[k_s\bar{s}_i^T\bar{s}_i\left(t_{i,k_i}^-\right) + k_p k_g\bar{g}_i^T\bar{g}_i\left(t_{i,k_i}^-\right) + \eta - k_g b_i\left\|\dot{q}_i\left(t_{i,k_i}^-\right) - \dot{q}_i^*\left(t_{i,k_i}^-\right)\right\|^2\right] \\ &= (1-\pi)\,k_g b_i\left\|\dot{q}_i\left(t_{i,k_i}^-\right) - \dot{q}_i^*\left(t_{i,k_i}^-\right)\right\|^2 + \pi\left[k_s\bar{s}_i^T\bar{s}_i\left(t_{i,k_i}^-\right) + k_p k_g\bar{g}_i^T\bar{g}_i\left(t_{i,k_i}^-\right) + \eta\right]. \end{aligned} \tag{B.5}$$

The CTC is not satisfied if

$$\begin{aligned} C_{\mathrm{L}}\left(t_{i,k_i}^{+}\right) &> C_{\mathrm{R}}\left(t_{i,k_i}^{+}\right) \\ k_s \bar{s}_i^T \bar{s}_i\left(t_{i,k_i}^{-}\right) + k_p k_g \bar{g}_i^T \bar{g}_i\left(t_{i,k_i}^{-}\right) + \eta &> (1-\pi)\, k_g b_i \left\|\dot{q}_i\left(t_{i,k_i}^{-}\right) - \dot{q}_i^*\left(t_{i,k_i}^{-}\right)\right\|^2 \\ &\quad + \pi\left[k_s \bar{s}_i^T \bar{s}_i\left(t_{i,k_i}^{-}\right) + k_p k_g \bar{g}_i^T \bar{g}_i\left(t_{i,k_i}^{-}\right) + \eta\right] \qquad \text{(B.6)} \\ (1-\pi)\left(k_s \bar{s}_i^T \bar{s}_i\left(t_{i,k_i}^{-}\right) + k_p k_g \bar{g}_i^T \bar{g}_i\left(t_{i,k_i}^{-}\right) + \eta\right) &> (1-\pi)\, k_g b_i \left\|\dot{q}_i\left(t_{i,k_i}^{-}\right) - \dot{q}_i^*\left(t_{i,k_i}^{-}\right)\right\|^2 \\ k_s \bar{s}_i^T \bar{s}_i\left(t_{i,k_i}^{-}\right) + k_p k_g \bar{g}_i^T \bar{g}_i\left(t_{i,k_i}^{-}\right) + \eta &> k_g b_i \left\|\dot{q}_i\left(t_{i,k_i}^{-}\right) - \dot{q}_i^*\left(t_{i,k_i}^{-}\right)\right\|^2 \qquad \text{(B.7)} \end{aligned}$$

and let show now that (B.7) is always satisfied.

Using the property $x^T y \geqslant -\frac{1}{2}\left(b_{i2} x^T x + \frac{1}{b_{i2}} y^T y\right)$ for some $b_{i2} > 0$, one deduces that

$$\begin{aligned} \bar{s}_i^T \bar{s}_i &= k_p^2 \bar{g}_i^T \bar{g}_i + \left\|\dot{q}_i - \dot{q}_i^*\right\|^2 + 2 k_p \bar{g}_i^T \left(\dot{q}_i - \dot{q}_i^*\right) \\ &\geqslant \left(k_p^2 - k_p b_{i2}\right) \bar{g}_i^T \bar{g}_i + \left(1 - \frac{k_p}{b_{i2}}\right) \left\|\dot{q}_i - \dot{q}_i^*\right\|^2. \end{aligned} \qquad \text{(B.8)}$$

Using (B.8), a sufficient condition for (B.7) to be satisfied is

$$\begin{aligned} k_s\left(k_p^2 - k_p b_{i2}\right) \bar{g}_i^T \bar{g}_i + k_s\left(1 - \frac{k_p}{b_{i2}}\right) \left\|\dot{q}_i - \dot{q}_i^*\right\|^2 + k_p k_g \bar{g}_i^T \bar{g}_i + \eta &> k_g b_i \left\|\dot{q}_i - \dot{q}_i^*\right\|^2 \\ k_s\left(1 - \frac{k_p}{b_{i2}}\right) \left\|\dot{q}_i - \dot{q}_i^*\right\|^2 + \left[k_p k_g + k_s\left(k_p^2 - k_p b_{i2}\right)\right] \bar{g}_i^T \bar{g}_i + \eta &> k_g b_i \left\|\dot{q}_i - \dot{q}_i^*\right\|^2 \\ k_1 \bar{g}_i^T \bar{g}_i + \eta &> k_2 \left\|\dot{q}_i - \dot{q}_i^*\right\|^2 \end{aligned} \qquad \text{(B.9)}$$

where $k_1 = \left[k_p k_g + k_s\left(k_p^2 - k_p b_{i2}\right)\right]$ and $k_2 = \left[k_g b_i - k_s\left(1 - \frac{k_p}{b_{i2}}\right)\right]$. To ensure that the inequality (B.9) is satisfied independently of the values of $\bar{g}_i$ and $\dot{q}_i$, it is sufficient to find b_i and b_{i2} such that $k_1 > 0$ and $k_2 < 0$. Consider first k_1.

$$\begin{aligned} k_p k_g + k_s\left(k_p^2 - k_p b_{i2}\right) &> 0 \\ \frac{k_g}{k_s} &> \left(-k_p + b_{i2}\right) \\ \frac{k_s k_p + k_g}{k_s} &> b_{i2}. \end{aligned} \qquad \text{(B.10)}$$

Focus now on k_2

$$\begin{aligned} k_g b_i - k_s\left(1 - \frac{k_p}{b_{i2}}\right) &< 0 \\ \frac{k_g b_i}{k_s} &< 1 - \frac{k_p}{b_{i2}} \end{aligned} \qquad \text{(B.11)}$$

Since $b_{i2} > 0$, one has $\frac{k_g b_i}{k_s} < 1$ and so $b_i < \frac{k_s}{k_g}$. Then

$$\frac{k_s k_p}{k_s - k_g b_i} < b_{i2}. \qquad \text{(B.12)}$$

Finally, one has to find a condition on b_i such that (B.10) and (B.11) can be satisfied simultaneously

$$\frac{k_s k_p + k_g}{k_s} > b_{i2} > \frac{k_s k_p}{k_s - k_g b_i}. \qquad \text{(B.13)}$$

One may find b_{i2} if

$$\begin{aligned} k_s - k_g b_i &> \frac{k_s^2 k_p}{k_s k_p + k_g} \\ \frac{1}{k_g}\left(k_s - \frac{k_s^2 k_p}{k_s k_p + k_g}\right) &> b_i \\ b_i &< \frac{k_s}{k_s k_p + k_g}. \end{aligned} \tag{B.14}$$

which also ensures that $b_i < \frac{k_s}{k_g}$. Thus, once $b_i < \frac{k_s}{k_s k_p + k_g}$, there exists some b_{i2} such that (B.13) is satisfied. As a consequence, (61) stops to be satisfied when $t = t_{i,k}^+$.

B.2 Study of upper-bound

From the Lipschitzien hypotheses of Assumptions A6, A7 and A8, we can deduce there exist K, $\hat{K}$ and K^* such

$$\left\| q_i\left(t + \Delta t\right) - q_i\left(t\right) \right\| \leqslant K \Delta t \tag{B.15}$$

$$\left\| \hat{q}_i^j\left(t + \Delta t\right) - \hat{q}_i^j\left(t\right) \right\| \leqslant \hat{K} \Delta t \tag{B.16}$$

$$\left\| q_i^*\left(t + \Delta t\right) - q_i^*\left(t\right) \right\| \leqslant K^* \Delta t \tag{B.17}$$

These conditions are now used to find an upper bound for all the terms inside the CTC.

B.2.1 Upper-bound on $\left\| \dot{\hat{q}}_j^i\left(t_{i,k_i}^+ + \Delta t\right) \right\|$, $\mathbb{E}\left(\left\| e_i^j\left(t_{i,k_i}^+ + \Delta t\right) \right\|^2 \right)$ *and* $\mathbb{E}\left(\left\| \dot{e}_i^j\left(t_{i,k_i}^+ + \Delta t\right) \right\|^2 \right)$

Define $I = \left[t_{i,k_i}^+,\, t_{i,k_i}^+ + \Delta t \right]$ such that Agent i broadcasts no communication inside I. Consider first the MSE

$$\mathbb{E}\left(\left\| e_i^j\left(t_{i,k_i}^+ + \Delta t\right) \right\|^2 \right) = \sum_{\ell=0}^{k_i} p_{k_i,\ell|k_i^{j,i},\rho_i^j} \left\| \hat{q}_i^{i,\ell}\left(t_{i,k_i}^+ + \Delta t\right) - q_i\left(t_{i,k_i}^+ + \Delta t\right) \right\|^2 \tag{B.18}$$

Let study $\frac{d}{dt}\mathbb{E}\left(\left\| e_i^j\left(t\right) \right\|^2 \right)$:

$$\begin{aligned} \frac{d}{dt}\mathbb{E}\left(\left\| e_i^j\left(t\right) \right\|^2 \right) &= \sum_{\ell=0}^{k_i} p_{k_i,\ell|k_i^{j,i},\rho_i^j} \frac{d}{dt} \left\| \hat{q}_i^{i,\ell}\left(t\right) - q_i\left(t\right) \right\|^2 \\ &= 2 \sum_{\ell=0}^{k_i} p_{k_i,\ell|k_i^{j,i},\rho_i^j} \left(\dot{\hat{q}}_i^{i,\ell}\left(t\right) - \dot{q}_i\left(t\right) \right)^T \left(\hat{q}_i^{i,\ell}\left(t\right) - q_i\left(t\right) \right) \\ &\leqslant 2 \sum_{\ell=0}^{k_i} p_{k_i,\ell|k_i^{j,i},\rho_i^j} \left\| \dot{\hat{q}}_i^{i,\ell}\left(t\right) - \dot{q}_i\left(t\right) \right\| \left\| \hat{q}_i^{i,\ell}\left(t\right) - q_i\left(t\right) \right\| \\ &\leqslant 4 \dot{q}_{\max} e_{\max} \sum_{\ell=0}^{k_i} p_{k_i,\ell|k_i^{j,i},\rho_i^j} \end{aligned} \tag{B.19}$$

and since $\sum_{\ell=0}^{k_i} p_{k_i,\ell|k_i^{j,i},\rho_i^j} = 1$, one has

$$\frac{d}{dt}\mathbb{E}\left(\left\| e_i^j\left(t\right) \right\|^2 \right) \leqslant 4 \dot{q}_{\max} e_{\max}. \tag{B.20}$$

Thus, one gets

$$\left\|\hat{q}_i^{i,\ell}\left(t_{i,k_i}^+ + \Delta t\right) - q_i\left(t_{i,k_i}^+ + \Delta t\right)\right\|^2 \leqslant \left\|\hat{q}_i^{i,\ell}\left(t_{i,k_i}^+\right) - q_i\left(t_{i,k_i}^+\right)\right\|^2 + 4\dot{q}_{\max} e_{\max} \Delta t \tag{B.21}$$

$$\mathbb{E}\left(\left\|e_i^j\left(t_{i,k_i}^+ + \Delta t\right)\right\|^2\right) \leqslant \mathbb{E}\left(\left\|e_i^j\left(t_{i,k_i}^+\right)\right\|^2\right) + 4\dot{q}_{\max} e_{\max} \Delta t. \tag{B.22}$$

In practice, $\mathbb{E}\left(\left\|e_i^j(t)\right\|^2\right)$ is upper-bound by $\overline{\mathbb{E}}\left(\left\|e_i^j(t)\right\|^2\right)$ in the CTC. If $\boldsymbol{p}_{k_i+n,k_i^{j,i},\kappa} = 0$, one has $\overline{\mathbb{E}}\left(\left\|e_i^j\left(t_{i,k_i}^+\right)\right\|^2\right) = \mathbb{E}\left(\left\|e_i^j\left(t_{i,k_i}^+\right)\right\|^2\right)$ and so the previous upper-bound is also valid for $\overline{\mathbb{E}}\left(\left\|e_i^j(t)\right\|^2\right)$. Else, suppose I is defined such for $t \in I$, one has $\overline{\mathbb{E}}\left(\left\|e_i^j(t)\right\|^2\right) \leqslant \sum_{\ell=\ell_{\min}}^{k_i} p_{k_i,\ell} \left\|\hat{q}_i^{i,\ell}(t) - q_i(t)\right\|^2 + \sigma_i^j(t_{i,k_i}) e_{\max}^2$. Thus, one gets

$$\begin{aligned}
\overline{\mathbb{E}}\left(\left\|e_i^j\left(t_{i,k_i}^+ + \Delta t\right)\right\|^2\right) &= \sum_{\ell=\ell_{\min}}^{k_i} p_{k_i,\ell} \left\|\hat{q}_i^{i,\ell}\left(t_{i,k_i}^+ + \Delta t\right) - q_i\left(t_{i,k_i}^+ + \Delta t\right)\right\|^2 + 4\dot{q}_{\max} e_{\max} \sigma_i^j(t_{i,k_i}) \Delta t + \pi \overline{\mathbb{E}}\left(\left\|e_i^j\left(t_{i,k_i}^-\right)\right\|^2\right) \\
&\quad - \sum_{\ell=\ell_{\min}}^{k_i} p_{k_i,\ell}^j \left\|\hat{q}_i^{i,\ell}\left(t_{i,k_i}^+\right) - q_i\left(t_{i,k_i}^+\right)\right\|^2 \\
&\leqslant \sum_{\ell=\ell_{\min}}^{k_i} p_{k_i,\ell|k_i^{j,i},\rho_i^j} \left[\left\|\hat{q}_i^{i,\ell}\left(t_{i,k_i}^+\right) - q_i\left(t_{i,k_i}^+\right)\right\|^2 + 4\dot{q}_{\max} e_{\max} \Delta t\right] + 4\dot{q}_{\max} e_{\max} \sigma_i^j(t_{i,k_i}) \Delta t \\
&\quad + \overline{\mathbb{E}}\left(\left\|e_i^j\left(t_{i,k_i}^+\right)\right\|^2\right) - \sum_{\ell=\ell_{\min}}^{k_i} p_{k_i,\ell}^j \left\|\hat{q}_i^{i,\ell}\left(t_{i,k_i}^+\right) - q_i\left(t_{i,k_i}^+\right)\right\|^2
\end{aligned} \tag{B.23}$$

and since $\sigma_i^j(t_{i,k_i}) = \sum_{\ell=0}^{k_i-\kappa} p_{k_i,\ell}^j = 1 - \sum_{\ell=k_i-\kappa+1}^{k_i} p_{k_i,\ell}^j$ with $k_i - \kappa = \ell_{\min} - 1$, one has

$$\begin{aligned}
\overline{\mathbb{E}}\left(\left\|e_i^j\left(t_{i,k_i}^+ + \Delta t\right)\right\|^2\right) &\leqslant 4\dot{q}_{\max} e_{\max} \Delta t \left(\sum_{\ell=\ell_{\min}}^{k_i} p_{k_i,\ell|k_i^{j,i},\rho_i^j} + \sum_{\ell=0}^{\ell_{\min}-1} p_{k_i,\ell|k_i^{j,i},\rho_i^j} \Delta t\right) + \overline{\mathbb{E}}\left(\left\|e_i^j\left(t_{i,k_i}^+\right)\right\|^2\right) \\
&\leqslant \overline{\mathbb{E}}\left(\left\|e_i^j\left(t_{i,k_i}^+\right)\right\|^2\right) + \left[4\dot{q}_{\max} e_{\max} \sum_{\ell=0}^{k_i} p_{k_i,\ell|k_i^{j,i},\rho_i^j}\right] \Delta t \\
&\leqslant \overline{\mathbb{E}}\left(\left\|e_i^j\left(t_{i,k_i}^+\right)\right\|^2\right) + 4\dot{q}_{\max} e_{\max} \Delta t.
\end{aligned} \tag{B.24}$$

In similar way, one can obtain

$$\left\|\dot{\hat{q}}_j^i\left(t_{i,k_i}^+ + \Delta t\right)\right\| \leqslant \left\|\dot{\hat{q}}_j^i\left(t_{i,k_i}^+\right)\right\| + \ddot{q}_{\max} \Delta t. \tag{B.25}$$

and

$$\mathbb{E}\left(\left\|\dot{e}_i^j\left(t_{i,k_i}^+ + \Delta t\right)\right\|^2\right) \leqslant \mathbb{E}\left(\left\|\dot{e}_i^j\left(t_{i,k_i}^+\right)\right\|^2\right) + 8\dot{q}_{\max} \ddot{q}_{\max} \Delta t, \tag{B.26}$$

$$\overline{\mathbb{E}}\left(\left\|\dot{e}_i^j\left(t_{i,k_i}^+ + \Delta t\right)\right\|^2\right) \leqslant \overline{\mathbb{E}}\left(\left\|\dot{e}_i^j\left(t_{i,k_i}^+\right)\right\|^2\right) + 8\dot{q}_{\max} \ddot{q}_{\max} \Delta t. \tag{B.27}$$

B.2.2 Upper-bound on $\mathbb{E}\left(\left\|e_i^j\left(t_{i,k_i}^+ + \Delta t\right)\right\|^4\right)$ *and* $\mathbb{E}\left(\left\|\dot{e}_i^j\left(t_{i,k_i}^+ + \Delta t\right)\right\|^4\right)$

Using results of Appendix B.2.1, since the structure of $\mathbb{E}\left(\left\|e_i^j\left(t_{i,k_i}^+ + \Delta t\right)\right\|^4\right)$ and $\mathbb{E}\left(\left\|\dot{e}_i^j\left(t_{i,k_i}^+ + \Delta t\right)\right\|^4\right)$ are similar to $\mathbb{E}\left(\left\|e_i^j\left(t_{i,k_i}^+ + \Delta t\right)\right\|^2\right)$ and $\mathbb{E}\left(\left\|\dot{e}_i^j\left(t_{i,k_i}^+ + \Delta t\right)\right\|^2\right)$, one gets

$$\mathbb{E}\left(\left\|e_i^j\left(t_{i,k_i}^+ + \Delta t\right)\right\|^4\right) \leqslant \mathbb{E}\left(\left\|e_i^j\left(t_{i,k_i}^+\right)\right\|^4\right) + 8\dot{q}_{\max} e_{\max} \Delta t \tag{B.28}$$

$$\mathbb{E}\left(\left\|\dot{e}_i^j\left(t_{i,k_i}^+ + \Delta t\right)\right\|^4\right) \leqslant \mathbb{E}\left(\left\|\dot{e}_i^j\left(t_{i,k_i}^+\right)\right\|^4\right) + 32\dot{q}_{\max}\ddot{q}_{\max} \Delta t, \tag{B.29}$$

and

$$\overline{\mathbb{E}}\left(\left\|e_i^j\left(t_{i,k_i}^+ + \Delta t\right)\right\|^4\right) \leqslant \overline{\mathbb{E}}\left(\left\|e_i^j\left(t_{i,k_i}^+\right)\right\|^4\right) + 8\dot{q}_{\max} e_{\max} \Delta t \tag{B.30}$$

$$\overline{\mathbb{E}}\left(\left\|\dot{e}_i^j\left(t_{i,k_i}^+ + \Delta t\right)\right\|^4\right) \leqslant \overline{\mathbb{E}}\left(\left\|\dot{e}_i^j\left(t_{i,k_i}^+\right)\right\|^4\right) + 32\dot{q}_{\max}\ddot{q}_{\max} \Delta t. \tag{B.31}$$

B.2.3 Upper-bound on $\left\|\dot{\hat{q}}_j^i\left(t_{i,k_i}^+ + \Delta t\right)\right\|^2 \mathbb{E}\left(\left\|e_i^j\left(t_{i,k_i}^+ + \Delta t\right)\right\|^2\right)$

Using results obtained in Appendix B.2.1, one gets

$$\begin{aligned}
\left\|\dot{\hat{q}}_j^i\left(t_{i,k_i}^+ + \Delta t\right)\right\|^2 \mathbb{E}\left(\left\|e_i^j\left(t_{i,k_i}^+ + \Delta t\right)\right\|^2\right) &\leqslant \left(\left\|\dot{\hat{q}}_j^i\left(t_{i,k_i}^+\right)\right\| + \Delta t\right)\left(\overline{\mathbb{E}}\left(\left\|e_i^j\left(t_{i,k_i}^+\right)\right\|^2\right) + 4\dot{q}_{\max} e_{\max} \Delta t\right) \\
&\leqslant \left\|\dot{\hat{q}}_j^i\left(t_{i,k_i}^+\right)\right\| \overline{\mathbb{E}}\left(\left\|e_i^j\left(t_{i,k_i}^+\right)\right\|^2\right) + \Delta t\left(\ddot{q}_{\max} + 4\dot{q}_{\max} e_{\max}\right) \\
&\quad + 4\ddot{q}_{\max}\dot{q}_{\max} e_{\max} \left(\Delta t\right)^2
\end{aligned} \tag{B.32}$$

B.2.4 Upper-bound on $\|\bar{g}_i\left(t + \Delta t\right)\|$ *and* $\|\bar{s}_i\left(t + \Delta t\right)\|$

Consider now $\|\bar{g}_i\left(t\right)\|$

$$\begin{aligned}
\|\bar{g}_i\left(t\right)\| &= \left\|\sum_{j=1}^N m_{ij}\left(\left(q_i\left(t\right) - \hat{q}_j^i\left(t\right)\right) - \left(q_i^*\left(t\right) - q_j^*\left(t\right)\right)\right)\right\| \\
&= \left\|\sum_{j=1}^N m_{ij}\left(\left(q_i\left(t\right) - \hat{q}_j^i\left(t\right)\right) - \left(q_i^*\left(t\right) - q_j^*\left(t\right)\right)\right)\right. \\
&\quad - \sum_{j=1}^N m_{ij}\left(\left(q_i\left(t + \Delta t\right) - \hat{q}_j^i\left(t + \Delta t\right)\right) - \left(q_i^*\left(t + \Delta t\right) - q_j^*\left(t + \Delta t\right)\right)\right) \\
&\quad \left. + \sum_{j=1}^N m_{ij}\left(\left(q_i\left(t + \Delta t\right) - \hat{q}_j^i\left(t + \Delta t\right)\right) - \left(q_i^*\left(t + \Delta t\right) - q_j^*\left(t + \Delta t\right)\right)\right)\right\| \\
&= \left\|\sum_{j=1}^N m_{ij}\left(\left(q_i\left(t\right) - q_i\left(t + \Delta t\right)\right) - \left(q_i^*\left(t\right) - q_i^*\left(t + \Delta t\right)\right)\right)\right. \\
&\quad - \sum_{j=1}^N m_{ij}\left(\left(\hat{q}_j^i\left(t\right) - \hat{q}_j^i\left(t + \Delta t\right)\right) - \left(q_j^*\left(t\right) - q_j^*\left(t + \Delta t\right)\right)\right) \\
&\quad \left. + \bar{g}_i\left(t + \Delta t\right)\right\|
\end{aligned} \tag{B.33}$$

In (B.33), using (B.15) and Assumption A7, one may write

$$\begin{aligned}\|\bar{g}_i(t)\| &\leqslant \|\bar{g}_i(t+\Delta t)\| + \sum_{j=1}^{N} m_{ij}\left(\|q_i(t+\Delta t) - q_i(t)\| + \|q_i^*(t+\Delta t) - q_i^*(t)\|\right.\\ &\left. + \left\|\hat{q}_j^i(t+\Delta t) - \hat{q}_j^i(t)\right\| + \left\|q_j^*(t+\Delta t) - q_j^*(t)\right\|\right)\\ &\leqslant \|\bar{g}_i(t+\Delta t)\| + \Delta t \sum_{j=1}^{N} m_{ij}\left(K + \hat{K} + 2K^*\right)\\ &\leqslant \|\bar{g}_i(t+\Delta t)\| + \alpha_i \Delta t\left(K + \hat{K} + 2K^*\right). \end{aligned} \tag{B.34}$$

Using the same steps of (B.34), it can be shown than

$$\|\bar{g}_i(t+\Delta t)\| \leqslant \|\bar{g}_i(t)\| + \alpha_i \Delta t\left(K + \hat{K} + 2K^*\right). \tag{B.35}$$

Thus, one can deduce from (B.34) and (B.35) that

$$\left|\|\bar{g}_i(t)\| - \|\bar{g}_i(t+\Delta t)\|\right| \leqslant \alpha_i \Delta t\left(K + \hat{K} + 2K^*\right). \tag{B.36}$$

From (B.36), one may write

$$\begin{aligned}\left|\|\bar{g}_i(t)\| - \|\bar{g}_i(t+\Delta t)\|\right| &\leqslant \alpha_i \Delta t\left(K + \hat{K} + 2K^*\right)\\ \left(\|\bar{g}_i(t)\| - \|\bar{g}_i(t+\Delta t)\|\right)^2 &\leqslant \left(\alpha_i \Delta t\left(K + \hat{K} + 2K^*\right)\right)^2\\ \|\bar{g}_i(t+\Delta t)\|^2 - 2\|\bar{g}_i(t)\|\|\bar{g}_i(t+\Delta t)\| + \|\bar{g}_i(t)\|^2 &\leqslant \left(\alpha_i \Delta t\left(K + \hat{K} + 2K^*\right)\right)^2\\ -\left(\alpha_i \Delta t\left(K + \hat{K} + 2K^*\right)\right)^2 + \|\bar{g}_i(t)\|^2 &\leqslant -\|\bar{g}_i(t+\Delta t)\|^2 + 2\|\bar{g}_i(t)\|\|\bar{g}_i(t+\Delta t)\| \end{aligned} \tag{B.37}$$

Young relation $xy \leqslant \frac{1}{2\boldsymbol{d}}x^2 + \frac{\boldsymbol{d}}{2}y^2$ with $\boldsymbol{d} > 1$, one gets

$$\begin{aligned}-\left(\alpha_i \Delta t\left(K + \hat{K} + 2K^*\right)\right)^2 + \|\bar{g}_i(t)\|^2 &\leqslant -\|\bar{g}_i(t+\Delta t)\|^2 + \frac{1}{\boldsymbol{d}}\|\bar{g}_i(t)\|^2 + \boldsymbol{d}\|\bar{g}_i(t+\Delta t)\|^2\\ -\left(\alpha_i \Delta t\left(K + \hat{K} + 2K^*\right)\right)^2 + \left(1 - \frac{1}{\boldsymbol{d}}\right)\|\bar{g}_i(t)\|^2 &\leqslant (\boldsymbol{d}-1)\|\bar{g}_i(t+\Delta t)\|^2\\ -\frac{1}{(\boldsymbol{d}-1)}\left(\alpha_i \Delta t\left(K + \hat{K} + 2K^*\right)\right)^2 + \frac{\left(1 - \frac{1}{\boldsymbol{d}}\right)}{(\boldsymbol{d}-1)}\|\bar{g}_i(t)\|^2 &\leqslant \|\bar{g}_i(t+\Delta t)\|^2. \end{aligned} \tag{B.38}$$

By taking $\boldsymbol{d} = \frac{1}{1-\boldsymbol{b}}$ with $1 > \boldsymbol{b} > 0$, one gets $\frac{\left(1-\frac{1}{\boldsymbol{b}}\right)}{(\boldsymbol{b}-1)} = 1 - \boldsymbol{b}$ and $\frac{1}{(\boldsymbol{d}-1)} = \frac{1-\boldsymbol{b}}{\boldsymbol{b}}$:

$$\left(1 - \frac{1}{\boldsymbol{b}}\right)(\Delta t)^2\left(\alpha_i\left(K + \hat{K} + 2K^*\right)\right)^2 + (1-\boldsymbol{b})\|\bar{g}_i(t)\|^2 \leqslant \|\bar{g}_i(t+\Delta t)\|^2 \tag{B.39}$$

Using the same steps that for $\|\bar{g}_i(t)\|$, one gets for $\|\bar{s}_i(t)\|$ the following expression:

$$\begin{aligned}
\|\bar{s}_i(t)\| &= \|\dot{q}_i(t) - \dot{q}_i^*(t) + k_p \bar{g}_i(t)\| \\
\|\bar{s}_i(t)\| &= \Bigg\| k_p \sum_{j=1}^{N} m_{ij} \left((q_i(t) - q_i(t+\Delta t)) - (q_i^*(t) - q_i^*(t+\Delta t)) \right) \\
&\quad - k_p \sum_{j=1}^{N} m_{ij} \left(\left(\hat{q}_j^i(t) - \hat{q}_j^i(t+\Delta t) \right) - \left(q_j^*(t) - q_j^*(t+\Delta t) \right) \right) \\
&\quad + \dot{q}_i(t) - \dot{q}_i(t+\Delta t) - \dot{q}_i^*(t) + \dot{q}_i^*(t+\Delta t) + \bar{s}_i(t+\Delta t) \| \\
\|\bar{s}_i(t)\| &\leqslant \|\bar{s}_i(t+\Delta t)\| + \sum_{j=1}^{N} m_{ij} \left(\|q_i(t+\Delta t) - q_i(t)\| + \|q_i^*(t+\Delta t) - q_i^*(t)\| \right. \\
&\quad \left. + \left\| \hat{q}_j^i(t+\Delta t) - \hat{q}_j^i(t) \right\| + \left\| q_j^*(t+\Delta t) - q_j^*(t) \right\| \right) \\
&\quad + \|\dot{q}_i(t) - \dot{q}_i(t+\Delta t)\| + \|\dot{q}_i^*(t) - \dot{q}_i^*(t+\Delta t)\| \\
\|\bar{s}_i(t)\| &\leqslant \|\bar{s}_i(t+\Delta t)\| + \Delta t \left[\alpha_i \left(K + \hat{K} + 2K^* \right) + 2K_d^* \right]
\end{aligned} \tag{B.40}$$

In a similar way, it can be shown that

$$\|\bar{s}_i(t+\Delta t)\| \leqslant \|\bar{s}_i(t)\| + \Delta t \left[\alpha_i \left(K + \hat{K} + 2K^* \right) + 2K_d^* \right] \tag{B.41}$$

Using (B.40) and (B.41), one can deduce

$$\left| \|\bar{s}_i(t+\Delta t)\| - \|\bar{s}_i(t)\| \right| \leqslant \Delta t \left[\alpha_i \left(K + \hat{K} + 2K^* \right) + 2K_d^* \right]$$

Following the same steps from (B.37) to (B.39), one gets

$$\left(1 - \frac{1}{\boldsymbol{b}} \right) (\Delta t)^2 \left[\alpha_i \left(K + \hat{K} + 2K^* \right) + 2K_d^* \right]^2 + (1 - \boldsymbol{b}) \|\bar{s}_i(t)\|^2 \leqslant \|\bar{s}_i(t+\Delta t)\|^2 \tag{B.42}$$

with $1 > \boldsymbol{b} > 0$.

B.3 Minimum inter-event time

Let's consider $t = t_{i,k_i}^+$, where t_{i,k_i}^+ correspond to the instant t_{i,k_i} after the update of the estimate values. The left side of the CTC

$$C_{\mathrm{L}}\left(t_{i,k_i}^+ + \Delta t\right) = k_s \left\| \bar{s}_i\left(t_{i,k_i}^+ + \Delta t\right) \right\|^2 + k_p k_g \left\| \bar{g}_i\left(t_{i,k_i}^+ + \Delta t\right) \right\|^2 + \eta \tag{B.43}$$

becomes using results of Section B.2 and $\eta = (1 - \boldsymbol{b} + \boldsymbol{b})\eta$,

$$\begin{aligned}
C_{\mathrm{L}}\left(t_{i,k_i}^+ + \Delta t\right) &\geqslant (1 - \boldsymbol{b}) \left[k_s \left\| \bar{s}_i\left(t_{i,k_i}^+\right) \right\|^2 + k_p k_g \left\| \bar{g}_i\left(t_{i,k_i}^+\right) \right\|^2 + \eta \right] + \boldsymbol{b}\eta \\
&\quad + \left(1 - \frac{1}{\boldsymbol{b}} \right) (\Delta t)^2 \left[k_s \left[\alpha_i \left(K + \hat{K} + 2K^* \right) + 2K_d^* \right]^2 \right. \\
&\quad \left. + k_p k_g \left(\alpha_i \left(K + \hat{K} + 2K^* \right) \right)^2 \right] \\
&\geqslant (1 - \boldsymbol{b}) C_{\mathrm{L}}\left(t_{i,k_i}^+\right) + \left(1 - \frac{1}{\boldsymbol{b}} \right) (\Delta t)^2 K_1 + \boldsymbol{b}\eta
\end{aligned} \tag{B.44}$$

with

$$K_1 = k_s \left[\alpha_i \left(K + \hat{K} + 2K^* \right) + 2K_d^* \right]^2 + k_p k_g \left(\alpha_i \left(K + \hat{K} + 2K^* \right) \right)^2 \tag{B.45}$$

In the same way, the right side of the CTC can be expressing using $\mathbb{E}\left(\left\|e_i^j\left(t_{i,k_i}^+ + \Delta t\right)\right\|^2\right)$ or its upper-bound $\overline{\mathbb{E}}\left(\left\|e_i^j\left(t_{i,k_i}^+ + \Delta t\right)\right\|^2\right)$. Thus

$$\begin{aligned} C_{\mathrm{R}}\left(t_{i,k_i}^+ + \Delta t\right) \leqslant \alpha_{\mathrm{M}} \Bigg[& \sum_{j=1}^{N} m_{ij} \left(k_e \overline{\mathbb{E}}\left(\left\|e_i^j\left(t_{i,k_i}^+ + \Delta t\right)\right\|^2\right) + k_p k_{\mathrm{M}} \overline{\mathbb{E}}\left(\left\|\dot{e}_i^j\left(t_{i,k_i}^+ + \Delta t\right)\right\|^2\right)\right) \\ & + 2k_p k_C^2 \sum_{j=1}^{N} m_{ij} \left(\overline{\mathbb{E}}\left(\left\|e_i^j\left(t_{i,k_i}^+ + \Delta t\right)\right\|^2\right) \left\|\dot{\hat{q}}_j^i\left(t_{i,k_i}^+ + \Delta t\right)\right\|^2 \right. \\ & \left. + \frac{1}{2}\overline{\mathbb{E}}\left(\left\|e_i^j\left(t_{i,k_i}^+ + \Delta t\right)\right\|^4\right) + \frac{1}{2}\overline{\mathbb{E}}\left(\left\|\dot{e}_i^j\left(t_{i,k_i}^+ + \Delta t\right)\right\|^4\right)\right)\Bigg] \end{aligned} \tag{B.46}$$

which becomes using results of Appendix B.2

$$\begin{aligned} C_{\mathrm{R}}\left(t_{i,k_i}^+ + \Delta t\right) \leqslant \alpha_{\mathrm{M}} \Bigg[& \sum_{j=1}^{N} m_{ij} \left(k_e \overline{\mathbb{E}}\left(\left\|e_i^j\left(t_{i,k_i}^+\right)\right\|^2\right) + k_p k_{\mathrm{M}} \overline{\mathbb{E}}\left(\left\|\dot{e}_i^j\left(t_{i,k_i}^+\right)\right\|^2\right)\right) \\ & + 2k_p k_C^2 \sum_{j=1}^{N} m_{ij} \left(\overline{\mathbb{E}}\left(\left\|e_i^j\left(t_{i,k_i}^+\right)\right\|^2\right) \left\|\dot{\hat{q}}_j^i\left(t_{i,k_i}^+\right)\right\|^2 \right. \\ & \left. + \frac{1}{2}\mathbb{E}\left(\left\|e_i^j\left(t_{i,k_i}^+\right)\right\|^4\right) + \frac{1}{2}\overline{\mathbb{E}}\left(\left\|\dot{e}_i^j\left(t_{i,k_i}^+\right)\right\|^4\right)\right)\Bigg] \\ & + \alpha_{\mathrm{M}} \left[\alpha_i \left(k_e 4\dot{q}_{\max} e_{\max} + 8\dot{q}_{\max}\ddot{q}_{\max}\right)\Delta t \right. \\ & + 2k_p k_C^2 \sum_{j=1}^{N} m_{ij} \left(\left(\ddot{q}_{\max} + 4\dot{q}_{\max} e_{\max}\right)\Delta t + 4\ddot{q}_{\max}\dot{q}_{\max} e_{\max}\left(\Delta t\right)^2 \right. \\ & \left. \left. + \left[4\dot{q}_{\max} e_{\max}\Delta t + 16\dot{q}_{\max}\ddot{q}_{\max}\right]\Delta t\right)\right] \\ & \leqslant C_{\mathrm{R}}\left(t_{i,k_i}^+\right) + B\Delta t + A\left(\Delta t\right)^2 \end{aligned} \tag{B.47}$$

with

$$A = 8\alpha_{\mathrm{M}} k_p k_C^2 \ddot{q}_{\max}\dot{q}_{\max} e_{\max} \sum_{j=1}^{N} m_{ij} \tag{B.48}$$

$$B = \alpha_{\mathrm{M}}\alpha_i \left(k_e 4\dot{q}_{\max} e_{\max} + 8\dot{q}_{\max}\ddot{q}_{\max}\right) + 2\alpha_{\mathrm{M}} k_p k_C^2 \left(\ddot{q}_{\max} + 8\dot{q}_{\max} e_{\max} + 16\dot{q}_{\max}\ddot{q}_{\max}\right) \sum_{j=1}^{N} m_{ij} \tag{B.49}$$

and remark $A > 0$ and $B > 0$.

At $t = t_{i,k_i}^+ + \Delta t$, the CTC is not triggering if

$$C_{\mathrm{R}}\left(t_{i,k_i}^+ + \Delta t\right) < C_{\mathrm{L}}\left(t_{i,k_i}^+ + \Delta t\right) \tag{B.50}$$

where (B.50) is satisfied if

$$\begin{aligned} C_{\mathrm{R}}\left(t_{i,k_i}^+\right) + B\Delta t + A\left(\Delta t\right)^2 &< \left(1 - \boldsymbol{b}\right) C_{\mathrm{L}}\left(t_{i,k_i}^+\right) + \left(1 - \frac{1}{\boldsymbol{b}}\right) K_1 \left(\Delta t\right)^2 + \boldsymbol{b}\eta \\ -\boldsymbol{b}\eta + B\Delta t + \left(A - \left(1 - \frac{1}{\boldsymbol{b}}\right)\left(\Delta t\right)^2 K_1\right)\left(\Delta t\right)^2 &< \left(1 - \boldsymbol{b}\right) C_{\mathrm{L}}\left(t_{i,k_i}^+\right) - C_{\mathrm{R}}\left(t_{i,k_i}^+\right). \end{aligned} \tag{B.51}$$

From (B.5), one has $C_{\mathrm{R}}\left(t_{i,k_i}^+\right) = \pi C_{\mathrm{L}}\left(t_{i,k_i}^+\right) + \left(1 - \pi\right) k_g b_i \left\|\dot{q}_i\left(t_{i,k_i}^+\right) - \dot{q}_i^*\left(t_{i,k_i}^+\right)\right\|^2$, thus one chooses $\boldsymbol{b} = 1 - \pi$ and (B.51) becomes

$$\begin{aligned} -\left(1 - \pi\right)\eta + B\Delta t + \left(A - \left(1 - \frac{1}{1 - \pi}\right)\left(\Delta t\right)^2 K_1\right)\left(\Delta t\right)^2 &< -\left(1 - \pi\right) k_g b_i \left\|\dot{q}_i\left(t_{i,k_i}^+\right) - \dot{q}_i^*\left(t_{i,k_i}^+\right)\right\|^2 \\ -\left(1 - \pi\right)\left(\eta - k_g b_i \left\|\dot{q}_i\left(t_{i,k_i}^+\right) - \dot{q}_i^*\left(t_{i,k_i}^+\right)\right\|^2\right) + B\Delta t + \left(A + \frac{\pi}{1 - \pi} K_1\right)\left(\Delta t\right)^2 &\leq 0 \end{aligned} \tag{B.52}$$

By solving (B.52) for Δt as unknown, one gets

$$\Delta t < \tau_{\min} \tag{B.53}$$

with

$$\tau_{\min} = \frac{-\boldsymbol{B} + \sqrt{\boldsymbol{B}^2 + 4\boldsymbol{A}\boldsymbol{C}}}{2\boldsymbol{A}} \tag{B.54}$$

$$\boldsymbol{A} = A + \frac{\pi}{1-\pi} K_1 \tag{B.55}$$

$$\boldsymbol{B} = B \tag{B.56}$$

$$\boldsymbol{C} = (1-\pi)\left(\eta - k_g b_i \left\|\dot{q}_i\left(t_{i,k_i}^+\right) - \dot{q}_i^*\left(t_{i,k_i}^+\right)\right\|^2\right) \tag{B.57}$$

Since $0 < \pi < 1$, one has $\boldsymbol{A} > 0$, $\boldsymbol{B} > 0$ and $\boldsymbol{C} > 0$ iff

$$\begin{aligned} \eta &> k_g b_i \left\|\dot{q}_i\left(t_{i,k_i}^+\right) - \dot{q}_i^*\left(t_{i,k_i}^+\right)\right\|^2 \\ &> 4 k_g b_i \dot{q}_{\max}^2. \end{aligned} \tag{B.58}$$

So $\tau_{\min} > 0$. From (B.53), we deduce than for $t > t_{i,k_i}$,the CTC is not triggering since $t < t_{i,k_i} + \Delta t$. Thus, we can conclude that

$$t_{i,k_i+1} - t_{i,k_i} \geqslant \tau_{\min}. \tag{B.59}$$