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ABSTRACT

In this paper, we consider a vibrating system of Timoshenko-type in a bounded one-dimensional domain under Dirichlet–Dirichlet or Dirichlet–Neumann boundary conditions with one or two discrete time delays and one or two internal frictional dampings. First, we show that the system is well posed in the sens of semigroup theory. Second, we prove the exponential stability regardless to the speeds of wave propagation of the system if the weights of the time delays are smaller than the ones of the corresponding dampings, respectively. However, when the weight of one time delay is not smaller than the one of the corresponding damping, we prove the exponential stability in case of equal-speed wave propagation, and the polynomial stability in the opposite case.

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1. Introduction

In this paper, we consider the following Timoshenko system:

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - k_1(\varphi_x+\psi)_x(x,t) + \lambda_1\varphi_t(x,t) + \mu_1\varphi_t(x,t-\tau_1) = 0, \\ \rho_2\psi_{tt}(x,t) - k_2\psi_{xx}(x,t) + k_1(\varphi_x+\psi)(x,t) + \lambda_2\psi_t(x,t) + \mu_2\psi_t(x,t-\tau_2) = 0, \\ \varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x), \\ \psi(x,0) = \psi_0(x),\ \psi_t(x,0) = \psi_1(x), \\ \varphi_t(x,-\tau_1\rho) = f_1(x,-\tau_1\rho),\ \ \psi_t(x,-\tau_2\rho) = f_2(x,-\tau_2\rho) \end{cases} \tag{1.1}$$

under the Dirichlet–Dirichlet boundary conditions

$$\varphi(0,t) = \varphi(L,t) = \psi(0,t) = \psi(L,t) = 0 \tag{1.2}$$

or the Dirichlet–Neumann boundary conditions

$$\varphi(0,t) = \varphi(L,t) = \psi_x(0,t) = \psi_x(L,t) = 0, \tag{1.3}$$

for $x \in]0,L[$, $t>0$, $\rho \in]0,1[$, $\mu_j \in \mathbb{R}$, $L,\ \rho_j,\ k_j,\ \tau_j > 0$, $\lambda_j \geq 0$ $(j=1,2)$,

$$(\varphi,\psi)\ :]0,L[\times]0,+\infty[\to\mathbb{R}^2$$

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is the state of (1.1) with (1.2) or (1.3),

$$\varphi_0,\ \varphi_1,\ \psi_0,\ \psi_1\ :]0,L[\to \mathbb{R} \quad \text{and} \quad f_j\ :]0,L[\times]-\tau_j,0[\to \mathbb{R}$$

($j = 1,2$) are given initial data. A subscript y as well as the notation ∂_y denote the derivative with respect to y. When a function has only one variable, its derivative is noted by $'$.

Our aim is the study of the well posedness and asymptotic behavior of the solutions of (1.1) with (1.2) or (1.3) in case of the equal-speed wave propagation

$$\frac{k_1}{\rho_1} = \frac{k_2}{\rho_2} \tag{1.4}$$

as well as in the opposite case.

The Timoshenko-type systems were introduced in [50] to describe the transverse vibration of a beam. Since then, the well posedness and stability of this model has attracted the attention of many researchers using diverse types of dissipative mechanisms. Let us mention here some of these results.

When no time delay is considered (i.e. $\mu_1 = \mu_2 = 0$), investigations showed that the presence of controls (linear or nonlinear feedbacks and/or finite or infinite memories) on both the rotation angle φ and the transverse displacement ψ guarantees the stability without any restriction on the constants ρ_j and k_j. However, in the case of only one control, the rate of decay depends heavily on the relation (1.4) and the regularity of the initial data. We quote in this regard [1–3,7,10–16,20,24–30,33,40–42,44,45,48,49].

When only one time delay is present (i.e. $\mu_1\mu_2 = 0$ and $(\mu_1, \mu_2) \neq (0,0)$), the questions related to well posedness and stability/instability of Timoshenko-type systems have attracted considerable attention in recent years and many researchers have shown that the time delay can destabilize a system that was asymptotically stable in the absence of time delay. Under smallness conditions on the weight of the time delay, it was proved that, when the time delay and control are considered on the same equation, the stability still holds; see [4,8,9,18,19,21,23,43,46].

In [47], the stability of Timoshenko systems with two internal time delays and two boundary linear feedbacks was proved under some smallness conditions on L and the weights of the delays.

For the stability of another kind of systems with delay, we refer the reader to [5,6,17,34–38] and the references therein.

As far as we know, the problem of stability of Timoshenko system with two discrete time delays and one or two frictional dampings, as well as the case of one time delay and one frictional damping not considered on the same equation, has never been treated in the literature. Our goal in this paper is to study the well posedness of (1.1) with (1.2) or (1.3) and investigate the effect of presence of one or two frictional dampings on the asymptotic behavior of their solutions. When the weight of one time delay is not smaller than the one of the corresponding damping, the system (1.1) with (1.2) or (1.3) is not necessarily dissipative with respect to its classical energy, so some new difficulties are generated.

The proof of the well posedness is based on the maximal monotone operators and semigroup approach. However, the proof of stability estimates is based on the multiplier method. The paper is organized as follows. Section 2 deals with the well posedness of (1.1) with (1.2) or (1.3). In section 3, we present our exponential and polynomial stability results. In section 4, we prove the exponential stability in both cases (1.2) and (1.3), when the weights of the time delays are smaller than the ones of the corresponding dampings, respectively. After, we consider the case where the weight of one time delay is not smaller than the one of the corresponding damping and we prove in sections 5 and 6, respectively, the exponential stability in case (1.4), and the polynomial stability in the opposite case. Finally, we conclude in Section 7 by some remarks and open questions.

2. Well posedness

In this section, we prove the existence, uniqueness, and smoothness of the solution of (1.1) with (1.2) or (1.3). For this purpose, we adopt the technique of [34], (see also [35–38]) to reformulate (1.1) with (1.2) or (1.3) in the first-order system (2.3) below and prove that the operator $\mathscr{A}+\mathscr{B}$ defined, respectively, in (2.4) and (2.6) generates a contraction semigroup on the Hilbert space $\mathscr{H}$ given in (2.7).

Let us consider the following new variables:

$$\begin{cases} z_1\left(x,\rho,t\right) := \varphi_t\left(x,t-\tau_1\rho\right), \text{ in}]0,L[\times]0,1[\times]0,+\infty[, \\ z_2\left(x,\rho,t\right) := \psi_t\left(x,t-\tau_2\rho\right), \text{in}]0,L[\times]0,1[\times]0,+\infty[. \end{cases} \tag{2.1}$$

Then it is easy to check that

$$\begin{cases} \tau_j z_{jt}\left(x,\rho,t\right) + z_{j\rho}\left(x,\rho,t\right) = 0, & \text{in}]0,L[\times]0,1[\times]0,+\infty[, \\ z_1\left(x,0,t\right) = \varphi_t(x,t),\ z_2\left(x,0,t\right) = \psi_t(x,t), & \text{in}]0,L[\times]0,+\infty[. \end{cases} \tag{2.2}$$

Now, we present a short discussion of the formulation of (1.1) with (1.2) or (1.3) in a first-order system. For this purpose, let

$$\mathscr{U} := \left(\varphi,\varphi_t,\psi,\psi_t,z_1,z_2\right)^T \quad \text{and} \quad \mathscr{U}_0 := \left(\varphi_0,\varphi_1,\psi_0,\psi_1,f_1(.,-\tau_1.),f_2(.,-\tau_2.)\right)^T,$$

then $\mathscr{U}$ satisfies the problem

$$\begin{cases} \mathscr{U}'(t) = (\mathscr{A}+\mathscr{B})\mathscr{U}(t), & t>0, \\ \mathscr{U}\left(0\right) = \mathscr{U}_0, \end{cases} \tag{2.3}$$

where the operators $\mathscr{B}$ and $\mathscr{A}$ are defined by

$$\mathscr{B}(u_1,u_2,v_1,v_2,z_1,z_2)^T = \left(0,\frac{\xi_1^0}{\rho_1}u_2,0,\frac{\xi_2^0}{\rho_2}v_2,0,0\right)^T, \tag{2.4}$$

$$\xi_j^0 = \begin{cases} 0 & \text{if } |\mu_j| \le \lambda_j, \\ |\mu_j| & \text{if } |\mu_j| > \lambda_j \end{cases} \tag{2.5}$$

and

$$\mathscr{A}\begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \\ w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} u_2 \\ \frac{k_1}{\rho_1}\left(u_{1xx}+v_{1x}\right) - \frac{\lambda_1+\xi_1^0}{\rho_1}u_2 - \frac{\mu_1}{\rho_1}w_1(1) \\ v_2 \\ \frac{k_2}{\rho_2}v_{1xx} - \frac{k_1}{\rho_2}\left(u_{1x}+v_1\right) - \frac{\lambda_2+\xi_2^0}{\rho_2}v_2 - \frac{\mu_2}{\rho_2}w_2(1) \\ \frac{-1}{\tau_1}w_{1\rho} \\ \frac{-1}{\tau_2}w_{2\rho} \end{pmatrix} \tag{2.6}$$

with domains $D\left(\mathscr{B}\right) = \mathscr{H}$ and

$$D\left(\mathscr{A}\right) = \Big\{\left(u_1,u_2,v_1,v_2,w_1,w_2\right)^T \in H : \\ (w_{1\rho},w_{2\rho}) \in L^2\left(]0,1[,L^2(]0,L[)\right) \times L^2\left(]0,1[,V_0\right),\ (w_1(0),w_2(0)) = (u_2,v_2)\Big\},$$

where

$$H := \left(H^2(]0,L[) \cap H_0^1(]0,L[)\right) \times H_0^1(]0,L[) \times \left(H_*^2(]0,L[) \cap V_1\right) \\ \times V_1 \times L^2\left(]0,1[, L^2(]0,L[)\right) \times L^2\left(]0,1[, V_0\right),$$

$\mathscr{H}$ is the energy space defined by

$$\mathscr{H} := H_0^1(]0,L[) \times L^2(]0,L[) \times V_1 \times V_0 \times L^2\left(]0,1[, L^2(]0,L[)\right) \times L^2\left(]0,1[, V_0\right), \tag{2.7}$$

$$H_*^2(]0,L[) = \begin{cases} H^2(]0,L[), & \text{in case (1.2)}, \\ \left\{v \in H^2(]0,L[),\ \partial_x v(0) = \partial_x v(L) = 0\right\}, & \text{in case (1.3)}, \end{cases} \tag{2.8}$$

$$V_1 = \begin{cases} H_0^1(]0,L[), & \text{in case (1.2)}, \\ \left\{v \in H^1(]0,L[),\ \int_0^L v\,\mathrm{d}x = 0\right\}, & \text{in case (1.3)} \end{cases} \tag{2.9}$$

and

$$V_0 = \begin{cases} L^2(]0,L[), & \text{in case (1.2)}, \\ \left\{v \in L^2(]0,L[),\ \int_0^L v\,\mathrm{d}x = 0\right\}, & \text{in case (1.3)}. \end{cases} \tag{2.10}$$

For $U = \left(u_1, u_2, v_1, v_2, w_1, w_2\right)^T$, $\overline{U} = \left(\overline{u}_1, \overline{u}_2, \overline{v}_1, \overline{v}_2, \overline{w}_1, \overline{w}_2\right)^T$ and

$$\xi_j = \begin{cases} \tau_j \lambda_j & \text{if } 0 < |\mu_j| \le \lambda_j, \\ \tau_j |\mu_j| & \text{if } |\mu_j| > \lambda_j \text{ or } \mu_j = 0, \end{cases} \tag{2.11}$$

the space $L^2\left(]0,1[, L^2(]0,L[)\right) \times L^2\left(]0,1[, V_0\right)$ endowed with the inner product

$$\langle (w_1, w_2), (\overline{w}_1, \overline{w}_2) \rangle_{L^2\left(]0,1[,L^2(]0,L[)\right) \times L^2\left(]0,1[,V_0\right)} \\ = \int_0^L \int_0^1 \left(\xi_1 w_1(x,\rho)\overline{w}_1(x,\rho) + \xi_2 w_2(x,\rho)\overline{w}_2(x,\rho)\right) \mathrm{d}\rho\,\mathrm{d}x$$

is a Hilbert space, and we define the inner product in $\mathscr{H}$ as follows:

$$\left\langle U, \overline{U} \right\rangle_{\mathscr{H}} = \int_0^L \left(\rho_1 u_2 \overline{u}_2 + \rho_2 v_2 \overline{v}_2 + k_1\left(u_{1x} + v_1\right)\left(\overline{u}_{1x} + \overline{v}_1\right) + k_2 v_{1x} \overline{v}_{1x}\right) \mathrm{d}x \\ + \langle (w_1, w_2), (\overline{w}_1, \overline{w}_2) \rangle_{L^2\left(]0,1[,L^2(]0,L[)\right) \times L^2\left(]0,1[,V_0\right)}.$$

Remark 2.1:

(1) If $\mu_j = 0$, the variable z_j is not considered, and therefore, the corresponding components in the definition of $\mathscr{U}$, $\mathscr{U}_0$, $\mathscr{B}$, $\mathscr{A}$, $D(\mathscr{A})$, H, and $\mathscr{H}$ will not appear.

(2) Let c_0 be the smallest positive constant depending only on L and satisfying (Poincaré's inequality)

$$\int_0^L v^2\,\mathrm{d}x \le c_0 \int_0^L v_x^2\,\mathrm{d}x, \quad \forall v \in H_*^1(]0,L[), \tag{2.12}$$

where

$$H_*^1(]0,L[) = \left\{ v \in H^1(]0,L[),\ v(0) = v(L) = 0 \text{ or } \int_0^L v\,\mathrm{d}x = 0 \right\}.$$

By applying (2.12) on v_1, we see that

$$\begin{aligned}
\int_0^L (u_{1x}^2 + v_{1x}^2)\,\mathrm{d}x &\le \int_0^L \left(2(u_{1x}+v_1)^2 + 2v_1^2 + v_{1x}^2\right)\,\mathrm{d}x \\
&\le \int_0^L \left(2(u_{1x}+v_1)^2 + (2c_0+1)v_{1x}^2\right)\,\mathrm{d}x \\
&\le \max\left\{\frac{2}{k_1}, \frac{2c_0+1}{k_2}\right\} \int_0^L \left(k_1(u_{1x}+v_1)^2 + k_2 v_{1x}^2\right)\,\mathrm{d}x. \qquad (2.13)
\end{aligned}$$

Because $\int_0^L u_{1x}^2\,\mathrm{d}x$ and $\int_0^L v_{1x}^2\,\mathrm{d}x$ define norms, for u_1 and v_1 on $H_0^1(]0,L[)$ and V_1, respectively, then

$$\int_0^L \left(k_1\left(u_{1x}+v_1\right)\left(\overline{u}_{1x}+\overline{v}_1\right) + k_2 v_{1x}\overline{v}_{1x}\right)\,\mathrm{d}x$$

generates a norm on $H_0^1(]0,L[) \times V_1$, for (u_1, v_1), equivalent to the one induced by $\left(H^1(]0,L[)\right)^2$. Consequently, $\mathscr{H}$ is a Hilbert space.

(3) In case of Dirichlet–Neumann boundary conditions (1.3), Poincaré's inequality (2.12) is not necessarily applicable for ψ. To overcome this problem, let us consider

$$g(t) = \int_0^L \psi(x,t)\,\mathrm{d}x \ \ (\text{for } t > 0), \quad g_{-1}(t) = \int_0^L f_2(x, t-\tau_2)\,\mathrm{d}x \ \ (\text{for } t \in]0,\tau_2[),$$

$$g_{-1}^0 = \int_0^L \psi_0(x)\,\mathrm{d}x \quad \text{and} \quad g_{-1}^1 = \int_0^L \psi_1(x)\,\mathrm{d}x.$$

Using the second equation and initial data (ψ_0, ψ_1, f_2) in (1.1) and the boundary conditions (1.3), we easily verify that

$$\begin{cases}
\rho_2 g''(t) + \lambda_2 g'(t) + k_1 g(t) + \mu_2 g'(t-\tau_2) = 0,\ t > 0, & \\
g'(t) = \int_0^L f_2(x,t)\,\mathrm{d}x, & t \in]-\tau_2, 0[, \\
g(0) = g_{-1}^0, \quad g'(0) = g_{-1}^1. &
\end{cases} \qquad (2.14)$$

In particular,

$$\begin{cases}
\rho_2 g''(t) + \lambda_2 g'(t) + k_1 g(t) + \mu_2 g_{-1}(t) = 0,\ t \in]0,\tau_2], \\
g(0) = g_{-1}^0, \quad g'(0) = g_{-1}^1.
\end{cases} \qquad (2.15)$$

The characteristic equation $\rho_2 s^2 + \lambda_2 s + k_1 = 0$ of the homogeneous equation associated to (2.15) has the solutions

$$\begin{cases}
s_0 = \frac{-\lambda_2}{2\rho_2} & \text{if } \lambda_2 = 2\sqrt{k_1\rho_2}, \\
s_1 = \frac{-\lambda_2 - \sqrt{|\lambda_2^2 - 4k_1\rho_2|}}{2\rho_2}, \ s_2 = \frac{-\lambda_2 + \sqrt{|\lambda_2^2 - 4k_1\rho_2|}}{2\rho_2} & \text{if } \lambda_2 > 2\sqrt{k_1\rho_2}, \\
s_0 - (s_2 - s_0)i, \ s_0 + (s_2 - s_0)i & \text{if } \lambda_2 \in [0, 2\sqrt{k_1\rho_2}[
\end{cases}$$

(here i is the imaginary number satisfying $i^2 = -1$). Then, using classical arguments, we find that the unique solution g_0 of (2.15) is given by $g_0 = g_0^0 + g_0^1$, where

$$g_0^0(t) = \begin{cases} \left(\left(g_{-1}^1 - s_0 g_{-1}^0\right) t + g_{-1}^0\right) e^{s_0 t} & \text{if } \lambda_2 = 2\sqrt{k_1\rho_2}, \\ \frac{1}{s_2-s_1}\left(s_2 g_{-1}^0 - g_{-1}^1\right) e^{s_1 t} + \frac{1}{s_2-s_1}\left(g_{-1}^1 - s_1 g_{-1}^0\right) e^{s_2 t} & \text{if } \lambda_2 > 2\sqrt{k_1\rho_2}, \\ \left(g_{-1}^0 \cos\left((s_2-s_0)t\right) + \frac{1}{s_2-s_0}\left(g_{-1}^1 - s_0 g_{-1}^0\right)\sin\left((s_2-s_0)t\right)\right) e^{s_0 t} & \text{if } \lambda_2 \in [0, 2\sqrt{k_1\rho_2}[\end{cases}$$

and

$$g_0^1(t) = \begin{cases} -\frac{\mu_2 e^{s_0 t}}{\rho_2} \int_0^t \int_0^s g_{-1}(\tau) e^{-s_0\tau}\, d\tau\, ds & \text{if } \lambda_2 = 2\sqrt{k_1\rho_2}, \\ -\frac{\mu_2 e^{s_1 t}}{\rho_2} \int_0^t e^{2(s_0-s_1)s} \int_0^s g_{-1}(\tau) e^{-(2s_0-s_1)\tau}\, d\tau\, ds & \text{if } \lambda_2 > 2\sqrt{k_1\rho_2}, \\ -\frac{\mu_2}{\rho_2} Re\left(e^{(s_0 - i(s_2-s_0))t} \int_0^t e^{2i(s_2-s_0)s} \int_0^s g_{-1}(\tau) e^{-(s_0+i(s_2-s_0))\tau}\, d\tau\, ds\right) & \text{if } \lambda_2 \in [0, 2\sqrt{k_1\rho_2}[, \end{cases}$$

where Re denotes the real part. Therefore, for any $n \in \mathbb{N}^*$, the unique solution g_n of

$$\begin{cases} \rho_2 g''(t) + \lambda_2 g'(t) + k_1 g(t) + \mu_2 g_{n-1}(t) = 0, & t \in]n\tau_2, (n+1)\tau_2], \\ g(n\tau_2) = g_{n-1}(n\tau_2), \quad g'(n\tau_2) = g'_{n-1}(n\tau_2) \end{cases}$$

is defined as g_0 with $g_{-1}, g_{-1}^0, g_{-1}^1, \int_0^t$ and $\int_0^s$ are replaced by $g_{n-1}, g_{n-1}(n\tau_2), g'_{n-1}(n\tau_2), \int_{n\tau_2}^t$ and $\int_{n\tau_2}^s$, respectively. Consequently, the unique two times differential solution g of (2.14) is given by

$$\begin{cases} g(t) = g_n(t), & t \in]n\tau_2, (n+1)\tau_2], \ n \in \mathbb{N}, \\ g(n\tau_2) = g_{n-1}(n\tau_2), \quad g'(n\tau_2) = g'_{n-1}(n\tau_2), \ n \in \mathbb{N}^*, \\ g(0) = g_0^0, \quad g'(0) = g_0^1. \end{cases}$$

Now, we put

$$\tilde{\psi} = \psi - \frac{1}{L} g. \tag{2.16}$$

Then, one can easily check that

$$\int_0^L \tilde{\psi}\, dx = 0, \tag{2.17}$$

and, hence, Poincaré's inequality (2.12) is applicable for $\tilde{\psi}$ provided that $\tilde{\psi} \in H^1(]0, L[)$. In addition, $(\varphi, \tilde{\psi})$ satisfies (1.1) with (1.3) and initial data

$$\tilde{\psi}_0 = \psi_0 - \frac{1}{L}\int_0^L \psi_0\, dx, \quad \tilde{\psi}_1 = \psi_1 - \frac{1}{L}\int_0^L \psi_1\, dx \quad \text{and} \quad \tilde{f}_2 = f_2 - \frac{1}{L}\int_0^L f_2\, dx$$

instead of ψ_0, ψ_1, and f_2, respectively. In the sequel, we work with $\tilde{\psi}$ instead of ψ when (1.3) holds, but, for simplicity of notation, we use always ψ instead of $\tilde{\psi}$.

Theorem 2.2: *For any $\mathscr{U}_0 \in \mathscr{H}$, there exists a unique solution*

$$\mathscr{U} \in C\left(\mathbb{R}^+, \mathscr{H}\right)$$

of problem (2.3). Moreover, if $\mathscr{U}_0 \in D\left(\mathscr{A}\right)$, then

$$\mathscr{U} \in C\left(\mathbb{R}^+, D\left(\mathscr{A}\right)\right) \cap C^1\left(\mathbb{R}^+, \mathscr{H}\right).$$

Proof: In order to prove the result stated in Theorem 2.2, we will use the semigroup approach; that is, we will show that the operator $\mathscr{A}+\mathscr{B}$ generates a C_0-semigroup in $\mathscr{H}$. In this step, we concern ourselves to prove that the operator $\mathscr{A}$ is dissipative. Indeed, exploiting (2.2), (2.6) and the definition of $\langle,\rangle_{\mathscr{H}}$, integrating by parts with respect to x and using the boundary conditions (1.2) or (1.3), we have, for $U=\left(u_1,u_2,v_1,v_2,w_1,w_2\right)^T\in D\left(\mathscr{A}\right)$,

$$\begin{aligned}\langle\mathscr{A}U,U\rangle_{\mathscr{H}}=&-\left(\lambda_1+\xi_1^0\right)\int_0^L u_2^2\,\mathrm{d}x-\left(\lambda_2+\xi_2^0\right)\int_0^L v_2^2\,\mathrm{d}x-\mu_1\int_0^L w_1\left(x,1\right)u_2\,\mathrm{d}x\\&-\mu_2\int_0^L w_2\left(x,1\right)v_2\,\mathrm{d}x-\frac{\xi_1}{\tau_1}\int_0^L\int_0^1 w_1\left(x,\rho\right)w_{1\rho}\left(x,\rho\right)\,\mathrm{d}\rho\,\mathrm{d}x\\&-\frac{\xi_2}{\tau_2}\int_0^L\int_0^1 w_2\left(x,\rho\right)w_{2\rho}\left(x,\rho\right)\,\mathrm{d}\rho\,\mathrm{d}x.\end{aligned}\tag{2.18}$$

Looking now at the last four integrals of the right-hand side of (2.18), we have

$$\begin{aligned}&-\frac{\xi_1}{\tau_1}\int_0^L\int_0^1 w_1\left(x,\rho\right)w_{1\rho}\left(x,\rho\right)\,\mathrm{d}\rho\,\mathrm{d}x-\frac{\xi_2}{\tau_2}\int_0^L\int_0^1 w_2\left(x,\rho\right)w_{2\rho}\left(x,\rho\right)\,\mathrm{d}\rho\,\mathrm{d}x\\&=-\frac{\xi_1}{\tau_1}\int_0^L\int_0^1\frac{1}{2}\frac{\partial}{\partial\rho}w_1^2\left(x,\rho\right)\,\mathrm{d}\rho\,\mathrm{d}x-\frac{\xi_2}{\tau_2}\int_0^L\int_0^1\frac{1}{2}\frac{\partial}{\partial\rho}w_2^2\left(x,\rho\right)\,\mathrm{d}\rho\,\mathrm{d}x\\&=\frac{\xi_1}{2\tau_1}\int_0^L\left(w_1^2\left(x,0\right)-w_1^2\left(x,1\right)\right)\,\mathrm{d}x+\frac{\xi_2}{2\tau_2}\int_0^L\left(w_2^2\left(x,0\right)-w_2^2\left(x,1\right)\right)\,\mathrm{d}x\end{aligned}$$

and, using Young's inequality,

$$\begin{aligned}&-\mu_1\int_0^L w_1\left(x,1\right)u_2\,\mathrm{d}x-\mu_2\int_0^L w_2\left(x,1\right)v_2\,\mathrm{d}x\le\frac{|\mu_1|}{2}\int_0^L\left(u_2^2+w_1^2(x,1)\right)\,\mathrm{d}x+\frac{|\mu_2|}{2}\\&\int_0^L\left(v_2^2+w_2^2(x,1)\right)\,\mathrm{d}x.\end{aligned}$$

Consequently, because $w_1^2\left(x,0\right)=u_2^2(x)$ and $w_2^2\left(x,0\right)=v_2^2(x)$, (2.18) becomes

$$\begin{aligned}\langle\mathscr{A}U,U\rangle_{\mathscr{H}}\le&\left(\frac{\xi_1}{2\tau_1}+\frac{|\mu_1|}{2}-\lambda_1-\xi_1^0\right)\int_0^L u_2^2\,\mathrm{d}x+\left(\frac{\xi_2}{2\tau_2}+\frac{|\mu_2|}{2}-\lambda_2-\xi_2^0\right)\int_0^L v_2^2\,\mathrm{d}x\\&+\left(\frac{|\mu_1|}{2}-\frac{\xi_1}{2\tau_1}\right)\int_0^L w_1^2\left(x,1\right)\,\mathrm{d}x+\left(\frac{|\mu_2|}{2}-\frac{\xi_2}{2\tau_2}\right)\int_0^L w_2^2\left(x,1\right)\,\mathrm{d}x.\end{aligned}\tag{2.19}$$

Using (2.5) and (2.11), we get

$$\frac{\xi_j}{2\tau_j}+\frac{|\mu_j|}{2}-\lambda_j-\xi_j^0\le 0\quad\text{and}\quad\frac{|\mu_j|}{2}-\frac{\xi_j}{2\tau_j}\le 0.$$

Hence, we deduce from (2.19) that $\langle\mathscr{A}U,U\rangle_{\mathscr{H}}\le 0$, and consequently, the operator $\mathscr{A}$ is dissipative.

Now, we show that the operator $Id-\mathscr{A}$ is surjective. Indeed, given

$$U_0=\left(\phi_1,\phi_2,g_1,g_2,h_1,h_2\right)^T\in\mathscr{H},$$

we seek $U=\left(u_1,u_2,v_1,v_2,w_1,w_2\right)^T\in D\left(\mathscr{A}\right)$ solution of $(Id-\mathscr{A})U=U_0$; that is,

$$\begin{pmatrix} u_1-u_2 \\ u_2-\frac{k_1}{\rho_1}\left(u_{1xx}+v_{1x}\right)+\frac{\lambda_1+\xi_1^0}{\rho_1}u_2+\frac{\mu_1}{\rho_1}w_1(1) \\ v_1-v_2 \\ v_2-\frac{k_2}{\rho_2}v_{1xx}+\frac{k_1}{\rho_2}\left(u_{1x}+v_1\right)+\frac{\lambda_2+\xi_2^0}{\rho_2}v_2+\frac{\mu_2}{\rho_2}w_2(1) \\ w_1+\frac{1}{\tau_1}w_{1\rho} \\ w_2+\frac{1}{\tau_2}w_{2\rho} \end{pmatrix}=\begin{pmatrix}\phi_1\\ \phi_2\\ g_1\\ g_2\\ h_1\\ h_2\end{pmatrix}. \tag{2.20}$$

Suppose that we have found

$$(u_1,v_1)\in\left(H^2(]0,L[)\cap H_0^1(]0,L[)\right)\times\left(H_*^2(]0,L[)\cap V_1\right). \tag{2.21}$$

Then the first and third equations in (2.20) give

$$\begin{cases} u_2=u_1-\phi_1, \\ v_2=v_1-g_1. \end{cases} \tag{2.22}$$

It is clear that $(u_2,v_2)\in H_0^1(]0,L[)\times V_1$. Furthermore,

$$\begin{cases} w_1(x,\rho)=(u_1(x)-\phi_1(x))e^{-\tau_1\rho}+\tau_1e^{-\tau_1\rho}\displaystyle\int_0^\rho h_1(x,\sigma)e^{\tau_1\sigma}\,\mathrm{d}\sigma, \\ w_2(x,\rho)=(v_1(x)-g_1(x))e^{-\tau_2\rho}+\tau_2e^{-\tau_2\rho}\displaystyle\int_0^\rho h_2(x,\sigma)e^{\tau_2\sigma}\,\mathrm{d}\sigma \end{cases} \tag{2.23}$$

satisfy the last two equations in (2.20),

$$(w_1,w_2),(w_{1\rho},w_{2\rho})\in L^2\left(]0,1[,L^2(]0,L[)\right)\times L^2\left(]0,1[,V_0\right)$$

and, according to (2.22), $(w_1(0),w_2(0))=(u_2,v_2)$. Puting

$$\begin{cases} w_1^0(x):=-\phi_1(x)e^{-\tau_1}+\tau_1e^{-\tau_1}\displaystyle\int_0^1 h_1(x,\sigma)e^{\tau_1\sigma}\,\mathrm{d}\sigma, \\ w_2^0(x):=-g_1(x)e^{-\tau_2}+\tau_2e^{-\tau_2}\displaystyle\int_0^1 h_2(x,\sigma)e^{\tau_2\sigma}\,\mathrm{d}\sigma. \end{cases} \tag{2.24}$$

We deduce from (2.23) that

$$\begin{cases} w_1(x,1)=e^{-\tau_1}u_1(x)+w_1^0(x), \\ w_2(x,1)=e^{-\tau_2}v_1(x)+w_2^0(x). \end{cases} \tag{2.25}$$

By using (2.22) and (2.25), the second and fourth equations in (2.20) are equivalent to

$$\begin{cases} \left(1+\frac{\lambda_1+\mu_1e^{-\tau_1}+\xi_1^0}{\rho_1}\right)u_1-\frac{k_1}{\rho_1}\left(u_{1xx}+v_{1x}\right)=\phi_2+\left(1+\frac{\lambda_1+\xi_1^0}{\rho_1}\right)\phi_1-\frac{\mu_1}{\rho_1}w_1^0, \\ \left(1+\frac{\lambda_2+\mu_2e^{-\tau_2}+\xi_2^0}{\rho_2}\right)v_1-\frac{k_2}{\rho_2}v_{1xx}+\frac{k_1}{\rho_2}\left(u_{1x}+v_1\right)=g_2+\left(1+\frac{\lambda_2+\xi_2^0}{\rho_2}\right)g_1-\frac{\mu_2}{\rho_2}w_2^0. \end{cases} \tag{2.26}$$

So, proving that $Id-\mathscr{A}$ is surjective is reduced to prove that (2.26) has at least one solution satisfying (2.21). To do so, multiplying the first and second equations in (2.26) by $\rho_1 l_1$ and $\rho_2 l_2$, respectively, where $(l_1, l_2) \in H_0^1(]0,L[) \times V_1$, and integrating by parts with respect to x, we see that any solution (2.21) of (2.26) satisfies the variational formulation

$$\mathscr{R}((u_1, v_1), (l_1, l_2)) = \mathscr{L}(l_1, l_2), \quad \forall (l_1, l_2) \in H_0^1(]0,L[) \times V_1, \tag{2.27}$$

where the bilinear form $\mathscr{R} : \left(H_0^1\left(]0,L[\right) \times V_1\right)^2 \to \mathbb{R}$ and the linear form $\mathscr{L} : H_0^1\left(]0,L[\right) \times V_1 \to \mathbb{R}$ are defined by

$$\begin{aligned}\mathscr{R}((\varphi,\psi),(w_1,w_2)) = &\int_0^L \left(\left(\rho_1 + \lambda_1 + \mu_1 e^{-\tau_1} + \xi_1^0\right) u_1 l_1 + \left(\rho_2 + \lambda_2 + \mu_2 e^{-\tau_2} + \xi_2^0\right) v_1 l_2\right) \mathrm{d}x \\ &+ \int_0^L \left(k_1(u_{1x} + v_1)(l_{1x} + l_2) + k_2 v_{1x} l_{2x}\right) \mathrm{d}x\end{aligned}$$

and

$$\mathscr{L}(l_1,l_2) = \int_0^L \left(\left(\rho_1\phi_2 + (\rho_1 + \lambda_1 + \xi_1^0)\phi_1 - \mu_1 w_1^0\right) l_1 + \left(\rho_2 g_2 + (\rho_2 + \lambda_2 + \xi_2^0) g_1 - \mu_2 w_2^0\right) l_2\right) \mathrm{d}x.$$

It is easy to verify that $\mathscr{R}$ is continuous and coercive, and $\mathscr{L}$ is continuous. So applying the Lax-Milgram theorem, we deduce that problem (2.27) admits a unique solution

$$(u_1, v_1) \in H_0^1\left(]0,L[\right) \times V_1.$$

Applying the classical elliptic regularity arguments, it follows that (u_1, v_1) satisfies (2.21). Therefore, the operator $Id - \mathscr{A}$ is surjective.

Since $\mathscr{A}$ is dissipative and $Id - \mathscr{A}$ is surjective, $\mathscr{A}$ is maximal monotone. Therefore, using Lummer-Phillips theorem (see [39]), we deduce that $\mathscr{A}$ is an infinitesimal generator of a linear contraction C_0-semigroup on $\mathscr{H}$. On the other hand, we see that the linear operator $\mathscr{B}$ is Lipschitz continuous. So, finally, also $\mathscr{A} + \mathscr{B}$ is an infinitesimal generator of a linear contraction C_0-semigroup on $\mathscr{H}$ (see [39]: Ch. 3 - Theorem 1.1). Consequently, the well-posedness results of Theorem 2.2 follow from the Hille-Yosida theorem (see [22] and [39]). □

3. Stability

To announce our stability results, we consider the energy functional associated with (1.1) and the boundary conditions (1.2) or (1.3) defined by

$$\begin{aligned}E(t) :&= \frac{1}{2}\|\mathscr{U}(t)\|_{\mathscr{H}}^2 \\ &= \frac{1}{2}\int_0^L \left(\rho_1\varphi_t^2 + \rho_2\psi_t^2 + k_1(\varphi_x + \psi)^2 + k_2\psi_x^2\right) \mathrm{d}x \\ &\quad + \frac{\xi_1}{2}\int_0^L\int_0^1 \varphi_t^2(x, t - \tau_1\rho)\,\mathrm{d}\rho\,\mathrm{d}x + \frac{\xi_2}{2}\int_0^L\int_0^1 \psi_t^2(x, t - \tau_2\rho)\,\mathrm{d}\rho\,\mathrm{d}x.\end{aligned} \tag{3.1}$$

Now, independently of (1.4) and in both cases (1.2) and (1.3), we give our first stability result which concerns the case

$$|\mu_1| < \lambda_1 \quad \text{and} \quad |\mu_2| < \lambda_2. \tag{3.2}$$

Theorem 3.1: *Assume that (3.2) is satisfied and let $\mathscr{U}_0 \in \mathscr{H}$. Then there exist positive constants c_1 and c_2 for which E satisfies*

$$E(t) \leq c_2 e^{-c_1 t}, \quad \forall t \geq 0. \tag{3.3}$$

Our second stability result concerns the case when (1.4) and one of the following situations hold:

$$|\mu_1| = \lambda_1 \quad \text{and} \quad |\mu_2| < \lambda_2, \tag{3.4}$$
$$|\mu_1| < \lambda_1 \quad \text{and} \quad |\mu_2| = \lambda_2, \tag{3.5}$$
$$|\mu_1| > \lambda_1 \quad \text{and} \quad |\mu_2| < \lambda_2 \tag{3.6}$$

and

$$|\mu_1| < \lambda_1 \quad \text{and} \quad |\mu_2| > \lambda_2. \tag{3.7}$$

Theorem 3.2: *Assume that (1.4), and (3.4) or (3.6) or [(1.3) and (3.5)] or [(1.3) and (3.7)] are satisfied, and let $\mathscr{U}_0 \in \mathscr{H}$. Then there exists a positive constant μ_j^0 independent of μ_j, where $j = 1$ in cases (3.4) and (3.6), and $j = 2$ in cases (3.5) and (3.7), such that, if*

$$\mu_j^2 + |\mu_j| < \mu_j^0, \tag{3.8}$$

the energy E satisfies (3.3).

Our last stability result concerns the case when (1.4) does not hold but (3.4) or (3.5) holds.

Theorem 3.3: *Assume that (1.4) does not hold, and (3.4) or [(1.3) and (3.5)] hold, and let $\mathscr{U}_0 \in D(\mathscr{A})$. Then there exists a positive constant μ_j^0 independent of μ_j, where $j = 1$ in case (3.4), and $j = 2$ in case (3.5), such that, if (3.8) holds, then there exists a positive constant c_1 for which E satisfies*

$$E(t) \leq \frac{c_1}{t}, \quad \forall t > 0. \tag{3.9}$$

We will use c (sometimes c_δ which depends on some parameter δ), throughout the proof of our stability results, to denote a generic positive constant which depends continuously on the initial data and can be different from step to step, but it does not depend neither on λ_j nor on μ_j.

By considering suitable multipliers in the next lemmas, we will construct a Lyapunov functional F satisfying some differential inequalities, for all $\mathscr{U}_0 \in D(\mathscr{A})$; so all the calculations are justified. By integrating these differential inequalities, we get the desired decay estimates (3.3) and (3.9). In case of Theorem 3.1 and Theorem 3.2, by a simple density arguments ($D(\mathscr{A})$ is dense in $\mathscr{H}$), (3.3) remains valid for any $\mathscr{U}_0 \in \mathscr{H}$.

Before starting the proof of our stability results, we give the following identity:

Lemma 3.4: *The energy functional satisfies*

$$E'(t) \leq \int_0^L \left(d_1 \varphi_t^2 + d_2 \psi_t^2\right) \, dx, \tag{3.10}$$

where

$$d_j = \begin{cases} \frac{-\lambda_j + |\mu_j|}{2} & \text{if } 0 < |\mu_j| \leq \lambda_j, \\ -\lambda_j + |\mu_j| & \text{if } |\mu_j| > \lambda_j \text{ or } \mu_j = 0. \end{cases} \tag{3.11}$$

Proof: By exploiting (2.3), (2.4), and (2.19), we obtain

$$E'(t) \leq \left(\frac{\xi_1}{2\tau_1} + \frac{|\mu_1|}{2} - \lambda_1\right) \int_0^L \varphi_t^2 \, dx + \left(\frac{\xi_2}{2\tau_2} + \frac{|\mu_2|}{2} - \lambda_2\right) \int_0^L \psi_t^2 \, dx + \left(\frac{|\mu_1|}{2} - \frac{\xi_1}{2\tau_1}\right) \int_0^L \varphi_t^2(x, t-\tau_1) \, dx + \left(\frac{|\mu_2|}{2} - \frac{\xi_2}{2\tau_2}\right) \int_0^L \psi_t^2(x, t-\tau_2) \, dx. \tag{3.12}$$

Then, using (2.11), we see that, if $0 < |\mu_j| \leq \lambda_j$,

$$\frac{\xi_j}{2\tau_j} + \frac{|\mu_j|}{2} - \lambda_j = \frac{|\mu_j|}{2} - \frac{\xi_j}{2\tau_j} = -\frac{\lambda_j - |\mu_j|}{2} \leq 0.$$

However, if $|\mu_j| > \lambda_j$ or $\mu_j = 0$, we have

$$\frac{\xi_j}{2\tau_j} + \frac{|\mu_j|}{2} - \lambda_j = |\mu_j| - \lambda_j \quad \text{and} \quad \frac{|\mu_j|}{2} - \frac{\xi_j}{2\tau_j} = 0.$$

Hence, (3.12) yields (3.10). □

Remark 3.5: When (3.2) or (3.4) or [(1.3) and (3.5)] hold, $E' \leq 0$, and then (1.1) is dissipative. However, when (3.6) or [(1.3) and (3.7)] hold, the sign of E' is not determined from (3.10), and therefore, (1.1) is not necessarily dissipative with respect to E at this stage.

4. Proof of Theorem 3.1

Assume that (3.2) holds and let $\mathscr{U}_0 \in D(\mathscr{A})$.

Lemma 4.1: *The functional*

$$J(t) := \int_0^L \left(\rho_1 \varphi \varphi_t + \rho_2 \psi \psi_t + \frac{\lambda_1}{2} \varphi^2 + \frac{\lambda_2}{2} \psi^2\right) dx \tag{4.1}$$

satisfies, for any $\delta > 0$,

$$J'(t) \leq -k_1 \int_0^L (\varphi_x + \psi)^2 \, dx - k_2 \int_0^L \psi_x^2 \, dx + \int_0^L \left(\rho_1 \varphi_t^2 + \rho_2 \psi_t^2\right) dx + \delta \int_0^L \left(\varphi_x^2 + \psi_x^2\right) dx + c_\delta \int_0^L \left(\mu_1^2 \varphi_t^2(x, t-\tau_1) + \mu_2^2 \psi_t^2(x, t-\tau_2)\right) dx. \tag{4.2}$$

Proof: By differentiating J, and using the first two equations in (1.1) and boundary conditions (1.2) or (1.3), we have

$$J'(t) = \int_0^L \left(\rho_1 \varphi_t^2 + \rho_2 \psi_t^2\right) dx - \int_0^L \left(k_1(\varphi_x + \psi)^2 + k_2 \psi_x^2\right) dx - \int_0^L \left(\mu_1 \varphi \varphi_t(x, t-\tau_1) + \mu_2 \psi \psi_t(x, t-\tau_2)\right) dx.$$

Consequently, applying Young's inequality, for the terms of the last integral of the above equality, and using Poincaré's inequality (2.12), for φ and ψ, we find (4.2). □

Lemma 4.2: *The functionals*

$$\begin{cases} I_1(t) = \xi_1 \int_0^L \int_0^1 e^{-2\tau_1 \rho} \varphi_t^2(x, t - \tau_1 \rho)\, \mathrm{d}\rho\, \mathrm{d}x, \\ I_2(t) = \xi_2 \int_0^L \int_0^1 e^{-2\tau_2 \rho} \psi_t^2(x, t - \tau_2 \rho)\, \mathrm{d}\rho\, \mathrm{d}x \end{cases} \tag{4.3}$$

satisfy

$$I_1'(t) \leq -2\xi_1 e^{-2\tau_1} \int_0^L \int_0^1 \varphi_t^2(x, t - \tau_1 \rho)\, \mathrm{d}\rho\, \mathrm{d}x + \frac{\xi_1}{\tau_1} \int_0^L \varphi_t^2\, \mathrm{d}x \tag{4.4}$$

$$- \frac{\xi_1 e^{-2\tau_1}}{\tau_1} \int_0^L \varphi_t^2(x, t - \tau_1)\, \mathrm{d}x \tag{4.5}$$

and

$$\begin{aligned} I_2'(t) \leq &-2\xi_2 e^{-2\tau_2} \int_0^L \int_0^1 \psi_t^2(x, t - \tau_2 \rho)\, \mathrm{d}\rho\, \mathrm{d}x + \frac{\xi_2}{\tau_2} \int_0^L \psi_t^2\, \mathrm{d}x \\ &- \frac{\xi_2 e^{-2\tau_2}}{\tau_2} \int_0^L \psi_t^2(x, t - \tau_2)\, \mathrm{d}x. \end{aligned} \tag{4.6}$$

Proof: Using (2.1) and (2.2), the derivative of I_1 entails

$$\begin{aligned} I_1'(t) &= 2\xi_1 \int_0^L \int_0^1 e^{-2\tau_1 \rho} \varphi_{tt}(x, t - \tau_1 \rho) \varphi_t(x, t - \tau_1 \rho)\, \mathrm{d}\rho\, \mathrm{d}x \\ &= -\frac{2\xi_1}{\tau_1} \int_0^L \int_0^1 e^{-2\tau_1 \rho} \varphi_{t\rho}(x, t - \tau_1 \rho) \varphi_t(x, t - \tau_1 \rho)\, \mathrm{d}\rho\, \mathrm{d}x \\ &= -\frac{\xi_1}{\tau_1} \int_0^L \int_0^1 e^{-2\tau_1 \rho} \partial_\rho \varphi_t^2(x, t - \tau_1 \rho)\, \mathrm{d}\rho\, \mathrm{d}x. \end{aligned}$$

Then, using an integrating by parts, the above formula can be rewritten as

$$I_1'(t) = -2\xi_1 \int_0^L \int_0^1 e^{-2\tau_1 \rho} \varphi_t^2(x, t - \tau_1 \rho)\, \mathrm{d}\rho\, \mathrm{d}x + \frac{\xi_1}{\tau_1} \int_0^L \varphi_t^2\, \mathrm{d}x - \frac{\xi_1 e^{-2\tau_1}}{\tau_1} \int_0^L \varphi_t^2(x, t - \tau_1)\, \mathrm{d}x,$$

which gives (4.4), since $-e^{-2\tau_1 \rho} \leq -e^{-2\tau_1}$, for any $\rho \in]0, 1[$. Similarily, (4.6) can be proved. □

Now, let $N_1,\ N_2 > 0$ and

$$F = N_1 E + N_2 (I_1 + I_2) + J. \tag{4.7}$$

By combining (4.2), (4.4), and (4.6), we obtain

$$
\begin{aligned}
F'(t) \leq{}& N_1 E'(t) - k_1 \int_0^L (\varphi_x + \psi)^2 \,\mathrm{d}x - k_2 \int_0^L \psi_x^2 \,\mathrm{d}x + \delta \int_0^L (\varphi_x^2 + \psi_x^2) \,\mathrm{d}x \\
&+ \int_0^L \left(\left(\rho_1 + \frac{\xi_1 N_2}{\tau_1} \right) \varphi_t^2 + \left(\rho_2 + \frac{\xi_2 N_2}{\tau_2} \right) \psi_t^2 \right) \mathrm{d}x \\
&+ \left(c_\delta \mu_1^2 - \frac{\xi_1 N_2 e^{-2\tau_1}}{\tau_1} \right) \int_0^L \varphi_t^2(x, t - \tau_1) \,\mathrm{d}x \\
&+ \left(c_\delta \mu_2^2 - \frac{\xi_2 N_2 e^{-2\tau_2}}{\tau_2} \right) \int_0^L \psi_t^2(x, t - \tau_2) \,\mathrm{d}x \\
&- 2N_2 \int_0^L \int_0^1 \left(\xi_1 e^{-2\tau_1} \varphi_t^2(x, t - \tau_1 \rho) + \xi_2 e^{-2\tau_2} \psi_t^2(x, t - \tau_2 \rho) \right) \mathrm{d}\rho \,\mathrm{d}x.
\end{aligned}
\tag{4.8}
$$

At this point, we choose $\delta > 0$ small enough and $N_2 > 0$ large enough so that (c_0 is defined in (2.12))

$$
\delta \max \left\{ \frac{2}{k_1}, \frac{2c_0 + 1}{k_2} \right\} \leq \frac{1}{2} \quad \text{and} \quad \min\{\xi_1, \xi_2\} N_2 \geq c_\delta \max \left\{ \tau_1 e^{2\tau_1} \mu_1^2, \tau_2 e^{2\tau_2} \mu_2^2 \right\}
$$

(notice that, thanks to (2.11), $\xi_j = 0$ implies $\mu_j = 0$; so N_2 exists). Therefore, using (2.13), for (φ, ψ) instead of (u_1, v_1), and the definition of E, we find that (4.8) implies that

$$
F'(t) \leq N_1 E'(t) - cE(t) + \tilde{c} \int_0^L (\varphi_t^2 + \psi_t^2) \,\mathrm{d}x,
$$

for some positive constant $\tilde{c}$ depending on λ_j and μ_j. Hence, according to (3.2) and (3.10), we have (notice that $\max\{d_1, d_2\} < 0$)

$$
\int_0^L \left(\varphi_t^2 + \psi_t^2 \right) \,\mathrm{d}x \leq \frac{1}{\max\{d_1, d_2\}} E'(t).
$$

By combining the above two inequalities, we get

$$
F'(t) \leq \left(N_1 + \frac{\tilde{c}}{\max\{d_1, d_2\}} \right) E'(t) - cE(t).
\tag{4.9}
$$

On the other hand, using again (2.13) (for (φ, ψ) instead of (u_1, v_1)) and the definition of J, I_1 and I_2, we can find that there exists a positive constant β (not depending on N_1) such that

$$
|N_2(I_1 + I_2) + J| \leq \beta E,
$$

which implies that

$$
(N_1 - \beta)E \leq F \leq (N_1 + \beta)E.
\tag{4.10}
$$

Thus, choosing N_1 large enough so that

$$
N_1 + \frac{\tilde{c}}{\max\{d_1, d_2\}} \geq 0 \quad \text{and} \quad N_1 > \beta,
$$

and using the fact that $E' \leq 0$ (see Remark (3.5)), we deduce from (4.9) and (4.10) that, for some positive constant c_1,

$$F' \leq -c_1 F. \tag{4.11}$$

Then, by integrating (4.11) over $[0, t]$ and using again (4.10), we get (3.3).

5. Proof of Theorem 3.2

Assume that (1.4) is satisfied, (3.4) or (3.5) or (3.6) or (3.7) holds and let $\mathscr{U}_0 \in D(\mathscr{A})$. We distinguish two cases.

Case 1: the boundary conditions (1.2) or (1.3) hold,

$$|\mu_1| \geq \lambda_1 \quad \text{and} \quad |\mu_2| < \lambda_2. \tag{5.1}$$

Lemma 5.1: *The functional*

$$I(t) = -\int_0^L \left(\rho_1 \varphi\varphi_t + \rho_2 \psi\psi_t + \frac{\lambda_1}{2}\varphi^2 + \frac{\lambda_2}{2}\psi^2\right) \mathrm{d}x$$

satisfies

$$\begin{aligned} I'(t) \leq & -\int_0^L \left(\rho_1 \varphi_t^2 + \rho_2 \psi_t^2\right) \mathrm{d}x + \int_0^L \left(k_1(\varphi_x + \psi)^2 + k_2 \psi_x^2\right) \mathrm{d}x \\ & + \delta \int_0^L \left(\varphi_x^2 + \psi_x^2\right) \mathrm{d}x + c_\delta \int_0^L \left(\mu_1^2 \varphi_t^2(x, t-\tau_1) + \mu_2^2 \psi_t^2(x, t-\tau_2)\right) \mathrm{d}x. \end{aligned} \tag{5.2}$$

Proof: The proof of (5.2) is identical to the one of (4.2). □

Similarly to [3], we consider the following lemma:

Lemma 5.2: *The functional*

$$I_3(t) = \rho_2 \int_0^L \psi_t(\varphi_x + \psi)\, \mathrm{d}x + \frac{k_2 \rho_1}{k_1} \int_0^L \psi_x \varphi_t \, \mathrm{d}x$$

satisfies, for any $\epsilon, \delta > 0$,

$$\begin{aligned} I_3'(t) \leq & \frac{k_2^2}{2\epsilon}\left(\psi_x^2(L,t) + \psi_x^2(0,t)\right) + \frac{\epsilon}{2}\left(\varphi_x^2(L,t) + \varphi_x^2(0,t)\right) + \rho_2 \int_0^L \psi_t^2 \, \mathrm{d}x \\ & - (k_1 - \delta) \int_0^L (\varphi_x + \psi)^2 \, \mathrm{d}x + \delta \int_0^L \psi_x^2 \, \mathrm{d}x + \left(\frac{k_2 \rho_1}{k_1} - \rho_2\right) \int_0^L \varphi_t \psi_{xt} \, \mathrm{d}x \\ & + c_\delta \int_0^L \left(\lambda_1^2 \varphi_t^2 + \lambda_2^2 \psi_t^2 + \mu_1^2 \varphi_t^2(x, t-\tau_1) + \mu_2^2 \psi_t^2(x, t-\tau_2)\right) \mathrm{d}x \end{aligned} \tag{5.3}$$

in case (1.2), *and*

$$I_3'(t) \le \rho_2 \int_0^L \psi_t^2 \,\mathrm{d}x - (k_1 - \delta) \int_0^L (\varphi_x + \psi)^2 \,\mathrm{d}x + \delta \int_0^L \psi_x^2 \,\mathrm{d}x + \left(\frac{k_2\rho_1}{k_1} - \rho_2\right) \int_0^L \varphi_t \psi_{xt} \,\mathrm{d}x + c_\delta \int_0^L \left(\lambda_1^2 \varphi_t^2 + \lambda_2^2 \psi_t^2 + \mu_1^2 \varphi_t^2(x, t - \tau_1) + \mu_2^2 \psi_t^2(x, t - \tau_2)\right) \mathrm{d}x \tag{5.4}$$

in case (1.3).

Proof: Using equations in (1.1) and the boundary conditions (1.2), and arguing as before, we have

$$I_3'(t) = -k_1 \int_0^L (\varphi_x + \psi)^2 \,\mathrm{d}x + \rho_2 \int_0^L \psi_t^2 \,\mathrm{d}x + \left(\frac{k_2\rho_1}{k_1} - \rho_2\right) \int_0^L \varphi_t \psi_{xt} \,\mathrm{d}x + k_2 \left(\varphi_x(L,t)\psi_x(L,t) - \varphi_x(0,t)\psi_x(0,t)\right) - \int_0^L (\varphi_x + \psi) \left(\lambda_2 \psi_t + \mu_2 \psi_t(x, t - \tau_2)\right) \mathrm{d}x - \frac{k_2}{k_1} \int_0^L \psi_x \left(\lambda_1 \varphi_t + \mu_1 \varphi_t(x, t - \tau_1)\right) \mathrm{d}x.$$

Using Young's inequality (for the last three terms of this equality), (5.3) is established. Similarly, in case (1.3), we get

$$I_3'(t) = -k_1 \int_0^L (\varphi_x + \psi)^2 \,\mathrm{d}x + \rho_2 \int_0^L \psi_t^2 \,\mathrm{d}x + \left(\frac{k_2\rho_1}{k_1} - \rho_2\right) \int_0^L \varphi_t \psi_{xt} \,\mathrm{d}x - \int_0^L (\varphi_x + \psi) \left(\lambda_2 \psi_t + \mu_2 \psi_t(x, t - \tau_2)\right) \mathrm{d}x - \frac{k_2}{k_1} \int_0^L \psi_x \left(\lambda_1 \varphi_t + \mu_1 \varphi_t(x, t - \tau_1)\right) \mathrm{d}x,$$

which, using Young's inequality (for the last two terms of this equality), implies (5.27). □

To estimate the boundary terms in (5.3), we proceed as in [3].

Lemma 5.3: *Let $m(x) = 2 - \frac{4}{L}x$. Then, for any $\epsilon > 0$, the functionals*

$$I_4 = \rho_2 k_2 \int_0^L m(x) \psi_t \psi_x \,\mathrm{d}x \quad \text{and} \quad I_5 = \rho_1 \int_0^L m(x) \varphi_t \varphi_x \,\mathrm{d}x$$

satisfy

$$I_4'(t) \le -k_2^2 \left(\psi_x^2(L,t) + \psi_x^2(0,t)\right) + \epsilon k_1 \int_0^L (\varphi_x + \psi)^2 \,\mathrm{d}x + c \int_0^L \psi_t^2 \,\mathrm{d}x + c\left(1 + \frac{1}{\varepsilon}\right) \int_0^L \psi_x^2 \,\mathrm{d}x + c \int_0^L \left(\lambda_2^2 \psi_t^2 + \mu_2^2 \psi_t^2(x, t - \tau_2)\right) \mathrm{d}x \tag{5.5}$$

and

$$I_5'(t) \le -k_1 \left(\varphi_x^2(L,t) + \varphi_x^2(0,t)\right) + c \int_0^L \left(\varphi_t^2 + \varphi_x^2 + \psi_x^2\right) \mathrm{d}x + c \int_0^L \left(\lambda_1^2 \varphi_t^2 + \mu_1^2 \varphi_t^2(x, t - \tau_1)\right) \mathrm{d}x. \tag{5.6}$$

Proof: Exploiting the second equation in (1.1) and using the boundary conditions (1.2), we get

$$I_4'(t) = \frac{2\rho_2 k_2}{L}\int_0^L \psi_t^2\,dx + \frac{2k_2^2}{L}\int_0^L \psi_x^2\,dx - k_2^2\left(\psi_x^2(L,t)+\psi_x^2(0,t)\right)$$
$$- k_2\int_0^L m\psi_x\left(k_1(\varphi_x+\psi)+\lambda_2\psi_t+\mu_2\psi_t(x,t-\tau_2)\right)\,dx.$$

Using Young's inequality, for the terms of the last integral of the above equality, (5.5) is established. Similarly, exploiting the first equation in (1.1) and using the boundary conditions (1.2), we find

$$I_5'(t) = \frac{2\rho_1}{L}\int_0^L \varphi_t^2\,dx + \frac{2k_1}{L}\int_0^L \varphi_x^2\,dx - k_1\left(\varphi_x^2(L,t)+\varphi_x^2(0,t)\right)$$
$$+ k_1\int_0^L m\varphi_x\psi_x\,dx - \int_0^L m\varphi_x\left(\lambda_1\varphi_t+\mu_1\varphi_t(x,t-\tau_1)\right)\,dx.$$

Then (5.6) can be proved by applying Young's inequality, for the terms of the last two integrals of the above equality. □

Lemma 5.4: *For any $\epsilon \in]0,1[$ and $\delta > 0$, the functional*

$$I_6 = \begin{cases} I_3 + \frac{1}{2\epsilon}I_4 + \frac{\epsilon}{2k_1}I_5, & \text{in case (1.2)},\\ I_3, & \text{in case (1.3)}\end{cases} \tag{5.7}$$

satisfies

$$I_6'(t) \le -\left(\frac{k_1}{2}-\delta-c\epsilon\right)\int_0^L(\varphi_x+\psi)^2\,dx + \frac{c}{\epsilon}\int_0^L \psi_t^2\,dx$$
$$+\left(\delta+\frac{c}{\epsilon^2}\right)\int_0^L\psi_x^2\,dx + c\epsilon\int_0^L\varphi_t^2\,dx + \left(\frac{\rho_1 k_2}{k_1}-\rho_2\right)\int_0^L\varphi_t\psi_{xt}\,dx$$
$$+(c_\epsilon+c_\delta)\int_0^L\left(\lambda_1^2\varphi_t^2+\lambda_2^2\psi_t^2+\mu_1^2\varphi_t^2(x,t-\tau_1)+\mu_2^2\psi_t^2(x,t-\tau_2)\right)\,dx. \tag{5.8}$$

Proof: Using Poincaré's inequality (2.12), for ψ, we obtain

$$\int_0^L\varphi_x^2\,dx \le 2\int_0^L(\varphi_x+\psi)^2\,dx + 2\int_0^L\psi^2\,dx$$
$$\le 2\int_0^L(\varphi_x+\psi)^2\,dx + c\int_0^L\psi_x^2\,dx. \tag{5.9}$$

Then (5.3)–(5.6) imply (5.8). □

Lemma 5.5: *The functional $I_7 = I_6 + \frac{1}{8}I$ satisfies*

$$I_7'(t) \le -\frac{k_1}{4}\int_0^L(\varphi_x+\psi)^2\,dx - \frac{\rho_1}{16}\int_0^L\varphi_t^2\,dx + c\int_0^L\left(\psi_t^2+\psi_x^2\right)\,dx$$
$$+\delta c\int_0^L\left((\varphi_x+\psi)^2+\psi_x^2\right)\,dx + \left(\frac{\rho_1 k_2}{k_1}-\rho_2\right)\int_0^L\varphi_t\psi_{xt}\,dx$$
$$+c_\delta\int_0^L\left(\lambda_1^2\varphi_t^2+\lambda_2^2\psi_t^2+\mu_1^2\varphi_t^2(x,t-\tau_1)+\mu_2^2\psi_t^2(x,t-\tau_2)\right)\,dx. \tag{5.10}$$

Proof: Using (5.9), inequalities (5.2) and (5.8) (with $\epsilon \in]0,1[$ small enough) imply (5.10). □

Now, as in [3], we use a function w to get a crucial estimate.

Lemma 5.6: *The function*

$$w(x,t) = -\int_0^x \psi(y,t)\,\mathrm{d}y + \frac{1}{L}\left(\int_0^L \psi(y,t)\,\mathrm{d}y\right)x \tag{5.11}$$

satisfies the estimates

$$\int_0^L w_x^2\,\mathrm{d}x \le c\int_0^L \psi^2\,\mathrm{d}x, \quad \forall t \ge 0 \tag{5.12}$$

and

$$\int_0^L w_t^2\,\mathrm{d}x \le c\int_0^L \psi_t^2\,\mathrm{d}x, \quad \forall t \ge 0. \tag{5.13}$$

Proof: We just have to calculate w_x and use Hölder's inequality to get (5.12). Applying (5.12) to w_t, we get

$$\int_0^L w_{xt}^2\,\mathrm{d}x \le c\int_0^L \psi_t^2\,\mathrm{d}x, \quad \forall t \ge 0.$$

Then, using Poincaré's inequality (2.12), for w_t (note that $w_t(0,t) = w_t(L,t) = 0$), we arrive at (5.13). $\square$

Lemma 5.7: *For any $\epsilon \in]0,1[$ and $\delta > 0$, the functional*

$$I_8(t) = \int_0^L \left(\rho_2\psi\psi_t + \rho_1 w\varphi_t + \frac{\lambda_2}{2}\psi^2\right)\mathrm{d}x$$

satisfies

$$\begin{aligned} I_8'(t) \le & -k_2\int_0^L \psi_x^2\,\mathrm{d}x + \frac{c}{\epsilon}\int_0^L \psi_t^2\,\mathrm{d}x + \epsilon\int_0^L \varphi_t^2\,\mathrm{d}x + \delta\int_0^L \psi_x^2\,\mathrm{d}x \\ & + c_\delta\int_0^L \left(\lambda_1^2\varphi_t^2 + \mu_1^2\varphi_t^2(x,t-\tau_1) + \mu_2^2\psi_t^2(x,t-\tau_2)\right)\mathrm{d}x. \end{aligned} \tag{5.14}$$

Proof: Exploiting the first two equations in (1.1), integrating by parts and using the boundary conditions (1.2) or (1.3), we get

$$\begin{aligned} I_8'(t) = & \int_0^L \left(\rho_2\psi_t^2 - k_2\psi_x^2\right)\mathrm{d}x - k_1\int_0^L (\varphi_x + \psi)(\psi + w_x)\,\mathrm{d}x \\ & + \rho_1\int_0^L w_t\varphi_t\,\mathrm{d}x - \mu_2\int_0^L \psi\psi_t(x,t-\tau_2)\,\mathrm{d}x \\ & - \int_0^L w\left(\lambda_1\varphi_t + \mu_1\varphi_t(x,t-\tau_1)\right)\mathrm{d}x. \end{aligned} \tag{5.15}$$

Notice that

$$\begin{aligned} -k_1 \int_0^L (\varphi_x + \psi)(\psi + w_x)\,\mathrm{d}x &= -k_1 \int_0^L (\varphi_x + \psi) \left(\frac{1}{L} \int_0^L \psi(y,t)\,\mathrm{d}y \right) \mathrm{d}x \\ &= -\frac{k_1}{L} \left(\int_0^L \psi(y,t)\,\mathrm{d}y \right)^2 \\ &\leq 0. \end{aligned}$$

Then, by applying Young's inequality, for the terms of the last three integrals of (5.29), and using (5.12) and (5.13), (5.14) is established. □

Now, for $N_1,\ N_2,\ N_3 > 0$, let

$$F = N_1 E + N_2(I_1 + I_2) + N_3 I_8 + I_7. \tag{5.16}$$

By combining (4.4), (4.6), (5.10) and (5.14), we obtain

$$\begin{aligned} F'(t) \leq &- \big(k_2 N_3 - c - \delta(N_3 + c)\big) \int_0^L \psi_x^2\,\mathrm{d}x - \left(\frac{\rho_1}{16} - \epsilon N_3 \right) \int_0^L \varphi_t^2\,\mathrm{d}x \\ &+ N_1 E'(t) + \left(\frac{c N_3}{\epsilon} + c + \frac{\xi_2 N_2}{\tau_2} \right) \int_0^L \psi_t^2\,\mathrm{d}x - \left(\frac{k_1}{4} - \delta c \right) \int_0^L (\varphi_x + \psi)^2\,\mathrm{d}x \\ &- 2N_2 \int_0^L \int_0^1 \big(\xi_1 e^{-2\tau_1} \varphi_t^2(x, t - \tau_1 p) + \xi_2 e^{-2\tau_2} \psi_t^2(x, t - \tau_2 p) \big)\,\mathrm{d}p\,\mathrm{d}x \\ &+ \frac{\xi_1 N_2}{\tau_1} \int_0^L \varphi_t^2\,\mathrm{d}x + \left(c_\delta \mu_1^2 (N_3 + 1) - \frac{\xi_1 e^{-2\tau_1}}{\tau_1} N_2 \right) \int_0^L \varphi_t^2(x, t - \tau_1)\,\mathrm{d}x \\ &+ \left(c_\delta \mu_2^2 (N_3 + 1) - \frac{\xi_2 e^{-2\tau_2}}{\tau_2} N_2 \right) \int_0^L \psi_t^2(x, t - \tau_2)\,\mathrm{d}x \\ &+ c_\delta \int_0^L \big(\lambda_1^2 (N_3 + 1) \varphi_t^2 + \lambda_2^2 \psi_t^2 \big)\,\mathrm{d}x + \left(\frac{\rho_1 k_2}{k_1} - \rho_2 \right) \int_0^L \varphi_t \psi_{xt}\,\mathrm{d}x. \end{aligned} \tag{5.17}$$

At this point, we choose $N_3 > 0$ large enough so that

$$k_2 N_3 - c > 0,$$

then $\epsilon \in]0,1[$ and $\delta > 0$ small enough so that

$$\frac{\rho_1}{16} - \epsilon N_3 > 0, \quad \delta(N_3 + c) < k_2 N_3 - c \quad \text{and} \quad \delta c < \frac{k_1}{4}.$$

Next, we pick $N_2 > 0$ large enough so that

$$c_\delta \mu_1^2 (N_3 + 1) - \frac{\xi_1 e^{-2\tau_1}}{\tau_1} N_2 \leq 0 \quad \text{and} \quad c_\delta \mu_2^2 (N_3 + 1) - \frac{\xi_2 e^{-2\tau_2}}{\tau_2} N_2 \leq 0.$$

Notice that, according to (2.11), if $\mu_j = 0$, then $\xi_j = 0$. Otherwise, in virtue of (2.11) and (5.1), $\xi_1 = \tau_1 |\mu_1|$ and $\xi_2 = \tau_2 \lambda_2$. So N_2 exists and can be taken in the form

$$N_2 = c(|\mu_1| + \lambda_2). \tag{5.18}$$

From the choice (5.18), the definition of E and the fact that $\lambda_1 \leq |\mu_1|$, we deduce from (5.17) that

$$F'(t) \leq -c\min\{1, |\mu_1| + \lambda_2\}E(t) + N_1E'(t) + c\left(\mu_1^2 + \lambda_2|\mu_1|\right)\int_0^L \varphi_t^2\,\mathrm{d}x$$
$$+ c\left(\lambda_2^2 + \lambda_2|\mu_1| + 1\right)\int_0^L \psi_t^2\,\mathrm{d}x + \left(\frac{\rho_1 k_2}{k_1} - \rho_2\right)\int_0^L \varphi_t\psi_{xt}\,\mathrm{d}x. \tag{5.19}$$

Hence, according to (3.10) and (5.1), we have

$$E'(t) \leq \int_0^L \left((|\mu_1| - \lambda_1)\varphi_t^2 + \frac{|\mu_2| - \lambda_2}{2}\psi_t^2\right)\mathrm{d}x. \tag{5.20}$$

By combining (5.31) and (5.35), we get ($\min\{1, |\mu_1| + \lambda_2\} \geq \min\{1, \lambda_2\}$)

$$F'(t) \leq -c\min\{1, \lambda_2\}E(t) + \left(\frac{\rho_1 k_2}{k_1} - \rho_2\right)\int_0^L \varphi_t\psi_{xt}\,\mathrm{d}x$$
$$+ \left(c(\mu_1^2 + \lambda_2|\mu_1|) + N_1(|\mu_1| - \lambda_1)\right)\int_0^L \varphi_t^2\,\mathrm{d}x$$
$$+ \left(c(\lambda_2^2 + \lambda_2|\mu_1| + 1) + \frac{N_1}{2}(|\mu_2| - \lambda_2)\right)\int_0^L \psi_t^2\,\mathrm{d}x. \tag{5.21}$$

On the other hand, by definition of the functionals I, $I_1 - I_8$ and E (notice that $\lambda_1 \leq |\mu_1|$), we have

$$|N_2(I_1 + I_2) + N_3 I_8 + I_7| \leq c(|\mu_1| + \lambda_2 + 1)E,$$

which implies that

$$(N_1 - c(|\mu_1| + \lambda_2 + 1))E \leq F \leq (N_1 + c(|\mu_1| + \lambda_2 + 1))E.$$

Then we choose N_1 large enough such that

$$c(\lambda_2^2 + \lambda_2|\mu_1| + 1) + \frac{N_1}{2}(|\mu_2| - \lambda_2) \leq 0 \quad \text{and} \quad N_1 > c(|\mu_1| + \lambda_2 + 1).$$

Because $|\mu_2| < \lambda_2$, then N_1 exists and can be taken in the form

$$N_1 = c\left(\frac{\lambda_2^2 + \lambda_2|\mu_1| + 1}{\lambda_2 - |\mu_2|} + |\mu_1| + \lambda_2 + 1\right), \tag{5.22}$$

so the last term in (5.31) is non-positive and $F \sim E$. In addition, using the definition of E, we deduce from (5.31) and (5.22) that

$$F'(t) \leq \qquad -c\min\{1, \lambda_2\}E(t) + \tilde{c}(\mu_1^2 + |\mu_1|)E(t) + \left(\frac{\rho_1 k_2}{k_1} - \rho_2\right)\int_0^L \varphi_t\psi_{xt}\,\mathrm{d}x, \tag{5.23}$$

where $\tilde{c}$ is a positive constant which depends on λ_2 and μ_2 but it does not depend neither on λ_1 nor on μ_1. Therefore, we assume that $|\mu_1|$ is small enough so that

$$\tilde{c}(\mu_1^2 + |\mu_1|) < c\min\{1, \lambda_2\}. \tag{5.24}$$

Because $\lambda_2 > 0$ (according to (5.1)), the set of μ_1 satisfying (5.37) is not empty and it is reduced to the one defined by a smallness condition of the form (3.8), for $j = 1$ and $\mu_1^0 = \frac{c}{\tilde{c}}\min\{1, \lambda_2\}$. Then

(5.34) and $F \sim E$ imply that, for some positive constant c_1,

$$F'(t) \leq -c_1 F(t) + \left(\frac{\rho_1 k_2}{k_1} - \rho_2\right) \int_0^L \varphi_t \psi_{xt} \, dx. \tag{5.25}$$

Because the last term of (5.25) vanishes (thanks to (1.4)), than (5.25) leads to (4.11), and then (3.3) is deduced as in the previous section.

Case 2: the boundary conditions (1.3) hold,

$$|\mu_1| < \lambda_1 \quad \text{and} \quad |\mu_2| \geq \lambda_2. \tag{5.26}$$

Similarly to [3], we consider the following lemma:

Lemma 5.8: *The functional*

$$J_1(t) = -\rho_2 \int_0^L \psi_t(\varphi_x + \psi) \, dx - \frac{k_2 \rho_1}{k_1} \int_0^L \psi_x \varphi_t \, dx$$

satisfies, for any $\delta > 0$,

$$J_1'(t) \leq (k_1 + \delta) \int_0^L (\varphi_x + \psi)^2 \, dx - \rho_2 \int_0^L \psi_t^2 \, dx + \delta \int_0^L \psi_x^2 \, dx + \left(\rho_2 - \frac{k_2 \rho_1}{k_1}\right) \int_0^L \varphi_t \psi_{xt} \, dx$$
$$+ c_\delta \int_0^L \left(\lambda_1^2 \varphi_t^2 + \lambda_2^2 \psi_t^2 + \mu_1^2 \varphi_t^2(x, t - \tau_1) + \mu_2^2 \psi_t^2(x, t - \tau_2)\right) dx. \tag{5.27}$$

Proof: Using equations in (1.1) and the boundary conditions (1.3), we have

$$J_1'(t) = k_1 \int_0^L (\varphi_x + \psi)^2 \, dx - \rho_2 \int_0^L \psi_t^2 \, dx + \left(\rho_2 - \frac{k_2 \rho_1}{k_1}\right) \int_0^L \varphi_t \psi_{xt} \, dx$$
$$+ \int_0^L (\varphi_x + \psi) \left(\lambda_2 \psi_t + \mu_2 \psi_t(x, t - \tau_2)\right) dx + \frac{k_2}{k_1} \int_0^L \psi_x \left(\lambda_1 \varphi_t + \mu_1 \varphi_t(x, t - \tau_1)\right) dx.$$

Arguing as for (5.27), we get (5.27). □

Lemma 5.9: *Let consider the functionals*

$$w(x, t) = \int_0^x \psi(y, t) \, dy$$

and

$$J_2(t) = \int_0^L \left(\rho_1 \varphi \varphi_t + \rho_1 w \varphi_t + \frac{\lambda_1}{2} \varphi^2\right) dx. \tag{5.28}$$

Then, for any ϵ, $\delta > 0$,

$$J_2'(t) \leq -k_1 \int_0^L (\varphi_x + \psi)^2 \, dx + \left(\rho_1 + \frac{c}{\epsilon}\right) \int_0^L \varphi_t^2 \, dx + \epsilon \int_0^L \psi_t^2 \, dx$$
$$+ \delta \int_0^L (\varphi_x^2 + \psi_x^2) \, dx + c_\delta \int_0^L \left(\lambda_1^2 \varphi_t^2 + \mu_1^2 \varphi_t^2(x, t - \tau_1)\right) dx. \tag{5.29}$$

Proof: First, we have $w(0, t) = w(L, t) = 0$ thanks to (2.17) (remeber that, in case (1.3), ψ plays the role of $\tilde{\psi}$). Then (5.12) and (5.13) hold (as for (5.11)). Exploiting the first equation in (1.1), integrating by parts, using (5.12), (5.13), and the boundary conditions (1.3), and arguing as for (5.14), we find (5.29). □

Now, for $N_1, N_2, N_3, N_4 > 0$, let

$$F = N_1 E + N_2(I_1 + I_2) + N_3 J + N_4 J_2 + J_1. \tag{5.30}$$

By combining (4.2), (4.4), (4.6), (5.27), and (5.29), and using (5.9), we obtain

$$\begin{aligned}
F'(t) \leq & -\left(k_2 N_3 - \delta(cN_3 + cN_4 + 1)\right) \int_0^L \psi_x^2 \,\mathrm{d}x - \left((1-N_3)\rho_2 - \epsilon N_4\right) \int_0^L \psi_t^2 \,\mathrm{d}x \\
& + N_1 E'(t) + \left(N_3 \rho_1 + N_4 \left(\rho_1 + \frac{c}{\epsilon}\right) + \frac{\xi_1 N_2}{\tau_1}\right) \int_0^L \varphi_t^2 \,\mathrm{d}x \\
& - \left((N_3 + N_4 - 1)k_1 - \delta(2N_3 + 2N_4 + 1)\right) \int_0^L (\varphi_x + \psi)^2 \,\mathrm{d}x \\
& - 2N_2 \int_0^L \int_0^1 \left(\xi_1 e^{-2\tau_1} \varphi_t^2(x, t - \tau_1 p) + \xi_2 e^{-2\tau_2} \psi_t^2(x, t - \tau_2 p)\right) dp \,\mathrm{d}x \\
& + \frac{\xi_2 N_2}{\tau_2} \int_0^L \psi_t^2 \,\mathrm{d}x + \left(c_\delta \mu_1^2 (N_3 + N_4 + 1) - \frac{\xi_1 e^{-2\tau_1}}{\tau_1} N_2\right) \int_0^L \varphi_t^2(x, t - \tau_1) \,\mathrm{d}x \\
& + \left(c_\delta \mu_2^2 (N_3 + 1) - \frac{\xi_2 e^{-2\tau_2}}{\tau_2} N_2\right) \int_0^L \psi_t^2(x, t - \tau_2) \,\mathrm{d}x \\
& + c_\delta \int_0^L \left(\lambda_1^2 (N_4 + 1)\varphi_t^2 + \lambda_2^2 \psi_t^2\right) \,\mathrm{d}x + \left(\rho_2 - \frac{\rho_1 k_2}{k_1}\right) \int_0^L \varphi_t \psi_{xt} \,\mathrm{d}x.
\end{aligned} \tag{5.31}$$

At this point, we choose

$$N_3 = \frac{1}{2}, \quad N_4 = 1 + \frac{1}{\rho_1} \quad \text{and} \quad \epsilon = \frac{1}{N_4^2},$$

then $\delta > 0$ small enough so that

$$\delta < \min\left\{\frac{k_2 N_3}{cN_3 + cN_4 + 1}, \frac{(N_3 + N_4 - 1)k_1}{2N_3 + 2N_4 + 1}\right\}.$$

Next, we pick $N_2 > 0$ large enough so that

$$c_\delta \mu_1^2 (N_3 + N_4 + 1) - \frac{\xi_1 e^{-2\tau_1}}{\tau_1} N_2 \leq 0 \quad \text{and} \quad c_\delta \mu_2^2 (N_3 + 1) - \frac{\xi_2 e^{-2\tau_2}}{\tau_2} N_2 \leq 0.$$

Notice that, according to (2.11), if $\mu_j = 0$, then $\xi_j = 0$. Otherwise, in virtue of (2.11) and (5.26), $\xi_1 = \tau_1 \lambda_1$ and $\xi_2 = \tau_2 |\mu_2|$. So N_2 exists and can be taken in the form

$$N_2 = c(|\mu_2| + \lambda_1). \tag{5.32}$$

From the choice (5.32), the definition of E and the fact that $\lambda_2 \leq |\mu_2|$, we deduce from (5.31) that

$$\begin{aligned}
F'(t) \leq & -c \min\{1, |\mu_2| + \lambda_1\} E(t) + N_1 E'(t) + c\left(\mu_2^2 + \lambda_1 |\mu_2|\right) \int_0^L \psi_t^2 \,\mathrm{d}x \\
& + c\left(\lambda_1^2 + \lambda_1 |\mu_2| + 1\right) \int_0^L \varphi_t^2 \,\mathrm{d}x + \left(\rho_2 - \frac{\rho_1 k_2}{k_1}\right) \int_0^L \varphi_t \psi_{xt} \,\mathrm{d}x.
\end{aligned} \tag{5.33}$$

Hence, according to (3.10) and (5.26), we have

$$E'(t) \le \int_0^L \left(\frac{|\mu_1| - \lambda_1}{2} \varphi_t^2 + (|\mu_2| - \lambda_2)\psi_t^2 \right) \mathrm{d}x. \tag{5.34}$$

By combining (5.35) and (5.34), we get ($\min\{1, \lambda_1\} \le \min\{1, |\mu_2| + \lambda_1\}$)

$$\begin{aligned} F'(t) \le & -c \min\{1, \lambda_1\} E(t) + \left(\rho_2 - \frac{\rho_1 k_2}{k_1} \right) \int_0^L \varphi_t \psi_{xt} \, \mathrm{d}x \\ & + \left(c(\mu_2^2 + \lambda_1 |\mu_2|) + N_1(|\mu_2| - \lambda_2) \right) \int_0^L \psi_t^2 \, \mathrm{d}x \\ & + \left(c(\lambda_1^2 + \lambda_1 |\mu_2| + 1) + \frac{N_1}{2}(|\mu_1| - \lambda_1) \right) \int_0^L \varphi_t^2 \, \mathrm{d}x. \end{aligned} \tag{5.35}$$

On the other hand, by definition of the functionals J, I_1, I_2, J_1, J_2 and E (notice that $\lambda_2 \le |\mu_2|$), we have

$$|N_2(I_1 + I_2) + N_3 J + N_4 J_2 + J_1| \le c(|\mu_2| + \lambda_1 + 1)E,$$

which implies that

$$(N_1 - c(|\mu_2| + \lambda_1 + 1))E \le F \le (N_1 + c(|\mu_2| + \lambda_1 + 1))E.$$

Then we choose N_1 large enough such that

$$c(\lambda_1^2 + \lambda_1 |\mu_2| + 1) + \frac{N_1}{2}(|\mu_1| - \lambda_1) \le 0 \quad \text{and} \quad N_1 > c(|\mu_2| + \lambda_1 + 1).$$

Because $|\mu_1| < \lambda_1$, then N_1 exists and can be taken in the form

$$N_1 = c \left(\frac{\lambda_1^2 + \lambda_1 |\mu_2| + 1}{\lambda_1 - |\mu_1|} + |\mu_2| + \lambda_1 + 1 \right), \tag{5.36}$$

so the last term in (5.35) is non-positive and $F \sim E$. In addition, using the definition of E, we deduce from (5.35) and (5.36) that

$$F'(t) \le -c \min\{1, \lambda_1\} E(t) + \tilde{c}(\mu_2^2 + |\mu_2|) E(t) + \left(\rho_2 - \frac{\rho_1 k_2}{k_1} \right) \int_0^L \varphi_t \psi_{xt} \, \mathrm{d}x, \tag{5.37}$$

where $\tilde{c}$ is a positive constant which depends on λ_1 and μ_1 but it does not depend neither on λ_2 nor on μ_2. Therefore, we assume that $|\mu_2|$ is small enough so that

$$\tilde{c}(\mu_2^2 + |\mu_2|) < c \min\{1, \lambda_1\}. \tag{5.38}$$

Because $\lambda_1 > 0$ (according to (5.26)), the set of μ_2 satisfying (5.39) is not empty and it is reduced to the one defined by a smallness condition of the form (3.8), for $j = 2$ and $\mu_2^0 = \frac{c}{\tilde{c}} \min\{1, \lambda_1\}$. Then (5.37) and $F \sim E$ imply that, for some positive constant c_1,

$$F'(t) \le -c_1 F(t) + \left(\rho_2 - \frac{\rho_1 k_2}{k_1} \right) \int_0^L \varphi_t \psi_{xt} \, \mathrm{d}x. \tag{5.39}$$

The end of the proof is the same as in the Case 1.

6. Proof of Theorem 3.3

Assume that (1.4) does not hold, (3.4) or (3.5) holds and let $\mathscr{U}_0 \in D(\mathscr{A})$. As in Theorem 3.2, we distinguish two cases.

Case 1: the boundary conditions (1.2) or (1.3) hold,

$$|\mu_1| = \lambda_1 \quad \text{and} \quad |\mu_2| < \lambda_2.$$

We will estimate the last term in (5.25) using the system (6) resulting from differentiating (1.1), (1.2), and (1.3) with respect to time; that is

$$\begin{cases} \rho_1\varphi_{ttt}(x,t) - k_1(\varphi_{xt} + \psi_t)_x(x,t) + \lambda_1\varphi_{tt}(x,t) + \mu_1\varphi_{tt}(x,t-\tau_1) = 0, \\ \rho_2\psi_{ttt}(x,t) - k_2\psi_{xxt}(x,t) + k_1(\varphi_{xt} + \psi_t)(x,t) + \lambda_2\psi_{tt}(x,t) + \mu_2\psi_{tt}(x,t-\tau_2) = 0 \end{cases}$$

with Dirichlet–Dirichlet boundary conditions

$$\varphi_t(0,t) = \varphi_t(L,t) = \psi_t(0,t) = \psi_t(L,t) = 0$$

or Dirichlet–Neumann boundary conditions

$$\varphi_t(0,t) = \varphi_t(L,t) = \psi_{xt}(0,t) = \psi_{xt}(L,t) = 0.$$

System (6) with (6) or (6) is well posed for initial data $\mathscr{U}_0 \in D(\mathscr{A})$ (see Theorem 2.2). Let E_2 be the second-order energy (the energy of (6)) defined by $E_2(t) = E(\mathscr{U}_t(t))$, where $E(\mathscr{U}(t)) = E(t)$ and E is defined by (3.1)). As for (3.10) and according to (6), a simple calculation implies that

$$E_2'(t) \leq d_2 \int_0^L \psi_{tt}^2 \, dx,$$

so, as E, also E_2 is non-increasing.

Lemma 6.1: *For any $\epsilon > 0$, there exists a positive constant α_ϵ such that*

$$\left(\frac{\rho_1 k_2}{k_1} - \rho_2\right) \int_S^T \int_0^L \varphi_{xt}\psi_t \, dx \, dt \leq \epsilon \int_S^T E(t) \, dt + \alpha_\epsilon \left(E(S) + E_2(S)\right), \quad \forall T \geq S \geq 0.$$

Proof: By integration with respect to t, we get

$$\int_S^T \int_0^L \varphi_{xt}\psi_t \, dx \, dt = \left[\int_0^L \varphi_x\psi_t \, dx\right]_S^T - \int_S^T \int_0^L \varphi_x\psi_{tt} \, dx \, dt.$$

Moreover, using the definition of E and its non-increasingness, we find

$$\left|\left(\frac{\rho_1 k_2}{k_1} - \rho_2\right) \int_0^L \varphi_x\psi_t \, dx\right| \leq cE(t) \leq cE(S), \quad \forall 0 \leq S \leq t.$$

Thus, from (6) we have

$$\left(\frac{\rho_1 k_2}{k_1} - \rho_2\right) \int_S^T \int_0^L \varphi_{xt}\psi_t \, dx \, dt \leq cE(S) + c\int_S^T \int_0^L |\varphi_x||\psi_{tt}| \, dx \, dt, \quad \forall T \geq S \geq 0.$$

On the other hand, because $d_2 < 0$, (6) leads to

$$\int_0^L \psi_{tt}^2 \, \mathrm{d}x \leq \frac{1}{d_2} E_2'(t).$$

Then, using Young's inequality, we estimate the last integral in (6) as follows:

$$\begin{aligned} c \int_S^T \int_0^L |\varphi_x||\psi_{tt}| \, \mathrm{d}x \, \mathrm{d}t &\leq \epsilon \int_S^T E(t) \, \mathrm{d}t - \frac{c_\epsilon}{d_2} \int_S^T E_2'(t) \, \mathrm{d}t \\ &\leq \epsilon \int_S^T E(t) \, \mathrm{d}t + \frac{c_\epsilon}{d_2} E_2(S), \quad \forall T \geq S \geq 0. \end{aligned}$$

Inserting this inequality into (6), we get (6.1) with $\alpha_\epsilon = \max\{c, \frac{c_\epsilon}{d_2}\}$. □

Now, exploiting (5.25) and (6.1), using the property $F \sim E$, and choosing $\epsilon > 0$ small enough, we get, for some positive constants α_1 and α_2,

$$\int_S^T F'(t) \, \mathrm{d}t \leq -\alpha_1 \int_S^T E(t) \, \mathrm{d}t + \alpha_2 \left(E(S) + E_2(S) \right), \quad \forall T \geq S \geq 0.$$

By combining (6) and the property $F \sim E$, we deduce that, for some positive constant α_3,

$$\int_S^T E(t) \, \mathrm{d}t \leq \alpha_3 \left(E(S) + E_2(S) \right), \quad \forall T \geq S \geq 0.$$

Choosing $S = 0$ in (6) and using the fact that E is non-increasing, we get

$$E(T)T \leq \int_0^T E(t) \, \mathrm{d}t \leq \alpha_3 \left(E(0) + E_2(0) \right) := c_1, \quad \forall T \geq 0,$$

which gives (3.9).

Case 2: the boundary conditions (1.3) hold,

$$|\mu_1| < \lambda_1 \quad \text{and} \quad |\mu_2| = \lambda_2.$$

According to (6), a simple calculation implies that, as for (6),

$$E_2'(t) \leq d_1 \int_0^L \varphi_{tt}^2 \, \mathrm{d}x.$$

Lemma 6.2: *For any $\epsilon > 0$, there exists a positive constant α_ϵ such that, for all $T \geq S \geq 0$,*

$$\left(\rho_2 - \frac{\rho_1 k_2}{k_1} \right) \int_S^T \int_0^L \varphi_{xt} \psi_t \, \mathrm{d}x \, \mathrm{d}t \leq \epsilon \int_S^T E(t) \, \mathrm{d}t + \alpha_\epsilon \left(E(S) + E_2(S) \right).$$

Proof: By integration with respect to x and t, and using the boundary condition (1.3), we get

$$\int_S^T \int_0^L \varphi_{xt} \psi_t \, \mathrm{d}x \, \mathrm{d}t = - \int_S^T \int_0^L \varphi_t \psi_{xt} \, \mathrm{d}x \, \mathrm{d}t = - \left[\int_0^L \varphi_t \psi_x \, \mathrm{d}x \right]_S^T + \int_S^T \int_0^L \varphi_{tt} \psi_x \, \mathrm{d}x \, \mathrm{d}t.$$

Using (6), the proof of (6.2) can be finished as for (6.1). □

Exploiting (5.39) and (6.2), the proof of (3.9) can be ended as in the Case 1.

7. Concluding remarks

In this section, we conclude with some remarks and list some open questions for the interested reader.

Remark 7.1: When (3.6) or [(1.3) and (3.7)] hold, we can take [$\lambda_1 = 0$ and $\mu_1\mu_2 \neq 0$] or [$\lambda_2 = 0$ and $\mu_1\mu_2 \neq 0$] or [$\lambda_1 = \mu_2 = 0$ and $\lambda_2\mu_1 \neq 0$] or [$\lambda_2 = \mu_1 = 0$ and $\lambda_1\mu_2 \neq 0$]. These cases show that, provided that (1.4) is satisfied, the exponential stability (3.3) of (1.1) holds also under one internal frictional damping and two discrete time delays or under one internal frictional damping and one discrete time delay not considered on the same equation.

Remark 7.2: Our results remain true if we consider Timoshenko-type systems with variables coeffiecients $\lambda_j(x)$ and $\mu_j(x)$ satisfying some smoothness and smallness conditions by modifying the operators and Lyapunov functionals considered in our proof (see [9] in case $\lambda_2 = \mu_2 = 0$).

Remark 7.3: We can consider distributed time delays

$$\int_O^{+\infty} h_1(s)\varphi_t(x,t-s)\,\mathrm{d}s \quad \text{and} \quad \int_O^{+\infty} h_2(s)\psi_t(x,t-s)\,\mathrm{d}s$$

instead of the discrete ones $\mu_1\varphi_t(x,t-\tau_1)$ and $\mu_2\psi_t(x,t-\tau_2)$, respectively, in both first two equations in (1.1) or in one of them, where h_1, $h_2 : \mathbb{R}^+ \to \mathbb{R}$ are some given functions (see [8] in case $\lambda_1 = \mu_1 = 0$).

Remark 7.4: The estimate (3.9) can be generalized by proving that, for any $n \in \mathbb{N}^*$ and $\mathscr{U}_0 \in D(\mathscr{A}^n)$, there exists a positive constant $c_n > 0$ such that

$$E(t) \leq \frac{c_n}{t^n}, \quad \forall t > 0$$

(see [9] in case $\lambda_2 = \mu_2 = 0$).

Remark 7.5: Our stability results can be generalized to the case where, for some given functions $\phi_1, \phi_2 : \mathbb{R} \to \mathbb{R}$, the linear frictional dampings $\lambda_1\varphi_t$ and $\lambda_2\psi_t$ are replaced by the nonlinear ones $\phi_1(\varphi_t)$ and $\phi_2(\psi_t)$, respectively.

Remark 7.6: We do not know if (1.1) is stable when (1.4) does not hold, and (3.6) or (3.7) holds. Similarly, we do not know if (1.1) is stable when [(1.2) and (3.5)] or [(1.2) and (3.7)] hold. On the other hand, the stability of (1.1) is an open question when (3.4) or (3.5) or (3.6) or (3.7) holds but $|\mu_j|$ is not small enough.

Remark 7.7: The stability of (1.1) with (1.2) or (1.3) when

$$|\mu_1| \geq \lambda_1 \quad \text{and} \quad |\mu_2| \geq \lambda_2$$

seems not being satisfied. It was proved in [34] that the stability of the wave equation with internal frictional damping and discrete time delay does not hold (even for small time delay) when the weight of the delay is bigger than the one of the damping.

Remark 7.8: It will be interesting to extend our results to the following system:

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - k_1(\varphi_x+\psi)_x(x,t) + \mu_1\varphi_t(x,t-\tau_1) = 0, \\ \rho_2\psi_{tt}(x,t) - k_2\psi_{xx}(x,t) + k_1(\varphi_x+\psi)(x,t) + \mu_2\psi_t(x,t-\tau_2) = 0, \\ \varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x), \\ \psi(x,0) = \psi_0(x),\ \psi_t(x,0) = \psi_1(x), \\ \varphi_t(x,-\tau_1\rho) = f_1(x,-\tau_1\rho),\ \ \psi_t(x,-\tau_2\rho) = f_2(x,-\tau_2\rho) \end{cases}$$

under the following boundary conditions:

$$\begin{cases} (\varphi_x + \psi)(L,t) + \lambda_1 \varphi_t(L,t) = 0, \\ \psi_x(L,t) + \lambda_2 \psi_t(L,t) = 0, \\ \varphi(0,t) = \psi(0,t) = 0. \end{cases}$$

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