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► **To cite this version:**

Tamas Balogh, Tamás Insperger, Islam Boussaada, Silviu-Iulian Niculescu. Towards an MID-based Delayed Design for Arbitrary-order Dynamical Systems with a Mechanical Application. IFAC 2020 - 21st IFAC World Congress, Jul 2020, Berlin / Virtual, Germany. 10.1016/j.ifacol.2020.12.065 . hal-02889938

**HAL Id: hal-02889938**

**<https://hal.science/hal-02889938>**

Submitted on 6 Jul 2020

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# Towards an MID-based Delayed Design for Arbitrary-order Dynamical Systems with a Mechanical Application

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**Abstract:** In this study we consider the stabilization of  $n^{\text{th}}$ -order linear time-invariant (LTI) dynamical systems using *Multiplicity-Induced-Dominancy* (MID)-based controller design in the presence of delays in the input/output channels. A *sufficient condition* is given for the dominance of a real root with multiplicity  $n + 1$  using an integral representation of the corresponding characteristic function. Furthermore, this sufficient condition is analyzed in the case when the characteristic function of the open-loop system is real-rooted, and delay intervals are derived for the set of parameters satisfying stabilizability and dominance conditions. The efficiency of the proposed controller design is shown in the case of a multi-link inverted pendulum.

*Keywords:* dynamical system, delay, spectral properties, inverted pendulum.

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## 1. INTRODUCTION

Stabilization of unstable equilibria and orbits in the presence of communication delay is an important and challenging task in engineering applications (see, for instance, Michiels et al. (2002); Olgac and Sipahi (2002); Gu et al. (2003); Landry et al. (2005)). In this context, as pointed out by Ma and Chen (2018), computing the control parameters allowing stable operation for large (feedback) delays is not a trivial task. Among the existing approaches, the so-called *parameter-based approach* is largely used in the case of low-order and/or low-complexity controllers (see, e.g., Bhattacharyya et al. (1995); Ackermann (2002); Michiels and Niculescu (2014) and the references therein). In particular, the stability diagrams can be used to visualize stability properties of the closed-loop system in the space of control parameters Bellman and Cooke (1963); Stépán (1989); Hassard (1997); Breda (2012).

It is commonly accepted that the delay in the feedback loop is generally seen as a source of unstable behavior. Furthermore, as the feedback delay gets larger, the stable region in the stability diagrams (charts) gets smaller. In this frame, a challenging task is to find the so-called *critical delay* or *delay margin*<sup>1</sup>. To the best of the au-

<sup>1</sup> i.e. the *maximum delay*, for which the system can still be stabilized by some control law, but for larger delay, the corresponding closed-loop system is unstable.

thors' knowledge, the idea of exploiting the delay effects in controllers' design was first introduced in Suh and Bien (1979) where it is shown that the conventional proportional controller equipped with an appropriate time-delay performs an averaged derivative action and thus it can replace the proportional-derivative controller. Furthermore, it was pointed out by Niculescu et al. (2010) that, under appropriate assumptions, the presence of some delay in the control law may induce stability in closed-loop. Finally, in the context of mechanical engineering problems, the effect of time-delay was emphasized in Stépán (1989) where some concrete applications are studied, such as the machine tool vibrations and robotic systems.

It is worth noting that, in the case of stable linear time-delay system, the rightmost root (or spectral abscissa) of the characteristic function is actually the exponential decay rate of its time-domain solution (see, for instance, Mori et al. (1982) for an estimate of the decay rate). Furthermore, to the best of the authors' knowledge, the first time an analytical proof of the dominance of a spectral value for the scalar equation with a single delay was presented in Hayes (1950). This *Multiplicity-Induced-Dominancy* (MID) property is further explored and analytically derived for scalar delay equations in Boussaada et al. (2016). Next, for second-order systems controlled by a delayed proportional is proposed in Boussaada et al. (2018b, 2017) where its applicability in damping active

vibrations for a piezo-actuated beam is proved. Finally, an extension to the delayed proportional-derivative controller case is studied in Boussaada and Niculescu (2018); Boussaada et al. (2018a) where the dominance property is parametrically characterized and proven by using the argument principle.

In this paper, we analyze a general  $n^{\text{th}}$ -order linear time-invariant dynamical system with a single delay and show that, under some appropriate conditions, the MID property can be used to assess the delay margin (see, e.g. Boussaada et al. (2015); Boussaada and Niculescu (2018); Boussaada et al. (2018b)). The novelty of the paper lies on the way we are exploiting the root location of the open-loop characteristic polynomial in order to have the multiplicity-induced-dominancy of the overall system, as done in Boussaada et al. (2020) in the particular case of second-order systems.

The remaining paper is organized as follows. The problem statement is presented in Section 2. Section 3 contains some motivating example. Section 4 provides the main ingredients of the dominance proof, which consists in writing the characteristic function with multiple real roots as an integral operator. The main results are presented in Section 5. An illustrative example is presented in Section 6, where the 5-link inverted pendulum is studied.

## 2. PROBLEM STATEMENT

Consider the quasipolynomial

$$D(s) = P(s) + e^{-s\tau}Q(s), \quad (1)$$

where the polynomials  $P(s)$  and  $Q(s)$  are real and have degrees  $n$  and  $n - 1$ , respectively:

$$\begin{aligned} P(s) &= a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0, \\ Q(s) &= b_{n-1} s^{n-1} + b_{n-2} s^{n-2} + \dots + b_1 s + b_0. \end{aligned} \quad (2)$$

Assume that the coefficients  $a_i$  of  $P$  are fixed and known, and that  $a_n > 0$ .

The problem we are addressing can be resumed as follows: *computing the values of the delay parameter  $\tau$  such that the system (1) is stabilizable<sup>2</sup> if the control parameters  $b_i$  are tuned such that the characteristic function  $D(s)$  has a real root  $s_0$  with multiplicity  $n + 1$ ?*

## 3. MOTIVATING EXAMPLE: THE INVERTED PENDULUM

Balancing an inverted pendulum in the presence of feedback delay is a frequently cited example in dynamics and control theory (Atay, 1999; Sieber and Krauskopf, 2004; Li et al., 2017). Different control methods are often implemented in simple inverted pendulum systems (Habib et al., 2017; Xu et al., 2017; Qin et al., 2014; Cieřkowski, 2016). The inverted pendulum is also a basic concept in human balancing models (Maurer and Peterka, 2005; Milton et al., 2016; Milton and Insperger, 2019; Morasso et al., 2019). The equation of motion of an inverted pendulum controlled by a proportional-derivative (PD) controller reads as:

$$\ddot{\varphi}(t) + a_0 \varphi(t) = -b_0 \varphi(t - \tau) - b_1 \dot{\varphi}(t - \tau), \quad (3)$$

<sup>2</sup> i.e. there exist some real parameters  $b_i$  ( $i = 0, 1, \dots, n - 1$ ) depending explicitly on  $(a_0, a_1, \dots, a_n, \tau)$  such that the characteristic roots of (1) have negative real part.

with a feedback delay  $\tau > 0$  and a system parameter  $a_0 < 0$ . The characteristic function corresponding to (3) is

$$D(s) = s^2 + a_0 + e^{-s\tau}(b_0 + b_1 s). \quad (4)$$

The open-loop characteristic function  $P(s) = s^2 + a_0$  has real roots  $\pm\sqrt{-a_0}$  since  $a_0 < 0$ . This property proves to be useful in Section 5.

The critical delay (delay margin) of the system (3) is well-known from the literature (Schurer, 1948; Stépán, 2009):

$$\tau_{\text{crit}} = \sqrt{-\frac{2}{a_0}}, \quad (5)$$

that is, the trivial solution of system (3) can be asymptotically stable if and only if  $\tau < \tau_{\text{crit}}$ . In the sequel, we will show that the critical delay (5) can be obtained by studying the multiple roots of the characteristic function.

Assume that  $D(s)$  has a real root  $s_0$  with algebraic multiplicity at least  $\deg P(s) + 1 = 3$ . Then  $D(s_0) = 0$ ,  $D'(s_0) = 0$  and  $D''(s_0) = 0$ :

$$\left. \begin{aligned} s_0^2 + a_0 + e^{-s_0\tau}(b_0 + b_1 s_0) &= 0, \\ 2s_0 + e^{-s_0\tau}(-\tau(b_0 + b_1 s_0) + b_1) &= 0, \\ 2 + e^{-s_0\tau}(\tau^2(b_0 + b_1 s_0) - 2\tau b_1) &= 0. \end{aligned} \right\} \quad (6)$$

From (6), one gets:

$$\left. \begin{aligned} b_0 &= e^{s_0\tau} \left( \tau s_0^3 + s_0^2 + a_0 \tau s_0 - a_0 \right), \\ b_1 &= -e^{s_0\tau} \left( \tau s_0^2 + 2s_0 + a_0 \tau \right), \\ s_0 &= \frac{-2 \pm \sqrt{2 - a_0 \tau^2}}{\tau} =: s_{0,\pm}. \end{aligned} \right\} \quad (7)$$

It can be shown that the triple root  $s_{0,+}$  is negative and dominant for every  $0 < \tau < \tau_{\text{crit}}$ , and therefore the system (3) is asymptotically stable. In particular, at the upper bound  $\tau = \tau_{\text{crit}}$  the triple root is  $s_{0,+} = 0$  and it is the dominant (rightmost) root of (4) with the control parameters  $b_0 = -a_0$  and  $b_1 = -a_0 \tau_{\text{crit}}$ . The dominance of  $s_{0,+}$  may be shown by using the argument principle (see, for instance, Boussaada and Niculescu, 2018; Boussaada et al., 2018a). In the next section, to show the dominance, we use a method first introduced in Boussaada et al. (2016) and based on an integral representation of the characteristic function (see also Boussaada et al. (2018b)).

## 4. INTEGRAL REPRESENTATION AND A SUFFICIENT CONDITION FOR DOMINANCY

We have the following result:

*Proposition 1.* If the quasipolynomial (1) has a real root  $s_0$  with multiplicity at least  $n$ , then it can be written as

$$D(s) = (s - s_0)^n \left( a_n + \int_0^1 e^{-(s-s_0)\tau t} \frac{\tau R_{n-1}(s_0; \tau t)}{(n-1)!} dt \right), \quad (8)$$

where the family of polynomials  $R_k(s; \tau)$  is defined as

$$R_k(s; \tau) = \sum_{i=0}^k \binom{k}{i} P^{(i)}(s) \tau^{k-i}, \quad k \in \mathbb{Z}_0^+. \quad (9)$$

**Proof.** The quasipolynomial  $D(s)$  has a root  $s_0$  with algebraic multiplicity at least  $n$  if and only if  $D^{(k)}(s_0) = 0$ ,  $k = 0, 1, \dots, n - 1$ :

## 5. OPEN-LOOP SYSTEMS WITH ONLY REAL ROOTS

$$\left. \begin{aligned} P(s_0) + e^{-s_0\tau} Q(s_0) &= 0, \\ P'(s_0) + e^{-s_0\tau} ((-\tau)Q(s_0) + Q'(s_0)) &= 0, \\ &\vdots \\ P^{(k)}(s_0) + e^{-s_0\tau} \sum_{i=0}^k \binom{k}{i} Q^{(i)}(s_0)(-\tau)^{k-i} &= 0, \\ &\vdots \\ P^{(n-1)}(s_0) + e^{-s_0\tau} \sum_{i=0}^{n-1} \binom{n-1}{i} Q^{(i)}(s_0)(-\tau)^{n-1-i} &= 0. \end{aligned} \right\} \quad (10)$$

Equation (10) gives a linear system of equations for the control parameters. Solving (10) for  $b_i$  enables the integral representation of the form (8).

If  $D(s)$  has a real root  $s_0$  with multiplicity at least  $n+1$  then (8) holds and, in addition,  $D^{(n)}(s_0) = 0$ :

$$\begin{aligned} D^{(n)}(s_0) &= n! \left( a_n + \int_0^1 \frac{\tau R_{n-1}(s_0; \tau t)}{(n-1)!} dt \right) \\ &= R_n(s_0; \tau) = 0. \end{aligned} \quad (11)$$

*Proposition 2.* Let  $s_0$  be a real root of  $R_n(s_0; \tau) = 0$ . Assume that the parameters  $b_i$  satisfy the conditions (10). If  $R_{n-1}(s_0; \tau t) \leq 0, \forall t, 0 < t \leq 1$  then the root  $s_0$  with multiplicity  $n+1$  is the dominant root of the characteristic function (8).

**Proof.** To prove that there exists no root  $s_1 = \gamma_1 + i\omega_1$  of (8) such that  $\gamma_1 > s_0$ , substitute  $s_1$  into (8). Since  $a_n > 0$  one can obtain that

$$\begin{aligned} a_n &= \left| \int_0^1 e^{-(s_1-s_0)\tau t} \frac{\tau}{(n-1)!} R_{n-1}(s_0; \tau t) dt \right| \\ &\leq \int_0^1 e^{-(\gamma_1-s_0)\tau t} \frac{\tau}{(n-1)!} |R_{n-1}(s_0; \tau t)| dt. \end{aligned} \quad (12)$$

Using the condition  $R_{n-1}(s_0; \tau t) \leq 0, \forall t, 0 < t \leq 1$  (12) can be written as

$$a_n \leq - \int_0^1 e^{-(\gamma_1-s_0)\tau t} \frac{\tau}{(n-1)!} R_{n-1}(s_0; \tau t) dt =: f(\gamma_1). \quad (13)$$

For  $\gamma_1 = s_0$  the function  $f$  takes the value

$$\begin{aligned} f(s_0) &= - \int_0^1 \frac{\tau}{(n-1)!} R_{n-1}(s_0; \tau t) dt \\ &= - \int_0^1 \frac{1}{n!} \frac{dR_n(s_0; \tau t)}{dt} dt \\ &= - \frac{1}{n!} \left( \underbrace{R_n(s_0; \tau)}_{=0} - \underbrace{R_n(s_0; 0)}_{=a_n n!} \right) = a_n. \end{aligned} \quad (14)$$

For  $\gamma_1 > s_0$  the value of the integral in (13) is  $f(\gamma_1) < a_n$  since  $0 < e^{-(\gamma_1-s_0)\tau t} < 1$  for  $\gamma_1 > s_0, \tau > 0, 0 < t \leq 1$ . Therefore, from (13), we obtain that  $a_n < a_n$  which proves the inconsistency of the hypothesis that the characteristic function (8) has a root  $s_1 = \gamma_1 + i\omega_1$  with  $\gamma_1 > s_0$ .

In this section, we assume that the polynomial  $P(s)$  corresponding to the open-loop system has only *real roots*. In this case,  $P(s)$  has the form  $P(s) = a_n \prod_{i=1}^n (s - s_i)$ ,  $s_i \in \mathbb{R}, s_n \leq s_{n-1} \leq \dots \leq s_1$ . To apply the sufficient condition in Proposition 2, first, we need to characterize the algebraic properties of the polynomials  $R_k(s; \tau)$ . These properties are outlined and discussed in the sequel.

### 5.1 Interlacing property of polynomials $R_k(s; \tau)$

The two-variable polynomials  $R_k(s; \tau), k \in \mathbb{Z}^+$  have the following properties:

$$R_k(s; \tau) = \tau R_{k-1}(s; \tau) + \frac{\partial R_{k-1}(s; \tau)}{\partial s}, \quad (15)$$

$$\frac{\partial R_k(s; \tau)}{\partial \tau} = k R_{k-1}(s; \tau). \quad (16)$$

The property (15) allows saying that, for a fixed  $\tau$ , the polynomials  $R_k(s; \tau)$  and  $R_{k-1}(s; \tau)$  interlace and  $R_k(s; \tau)$  has only real roots for  $s$  since  $R_0(s; \tau) = P(s)$  has only real roots (Fadeev and Sominski, 1965). Polynomials  $R_n(s; \tau)$  and  $R_{n-1}(s; \tau)$  have  $n$  distinct real roots for  $s$  if  $\tau \neq 0$ . Let  $s_{0,k}, k = 1, 2, \dots, n$  denote the roots of  $R_n(s; \tau), \tau \neq 0$  with  $s_{0,n} < s_{0,n-1} < \dots < s_{0,1}$ .

It can also be shown that, for a fixed  $s$ ,  $R_n(s; \tau)$  has only real roots for  $\tau$  (Pólya and Szegő, 1997). Moreover,  $R_k(s; \tau), k = 1, 2, \dots, n-1$  has only real roots for  $\tau$ , and  $R_k(s; \tau)$  and  $R_{k-1}(s; \tau)$  interlace which are direct consequences of the property (16) above and Rolle's theorem.

It should be mentioned that, in general, the solutions of (15) and (16) can be investigated by using computer algebra techniques (see, e.g., Chyzak et al. (2005)).

### 5.2 Monotonicity

In the  $(\tau, s)$ -plane, the algebraic curve  $R_n(s; \tau) = 0$  has distinct branches, and every branch is strictly increasing since the derivative of the implicit function  $R_n(s; \tau) = 0$  in a point  $(s, \tau)$  writes as (using (15) and (16)):

$$\frac{ds}{d\tau} = - \frac{\frac{\partial R_n(s; \tau)}{\partial \tau}}{\frac{\partial R_n(s; \tau)}{\partial s}} = - \frac{n R_{n-1}(s; \tau)}{R_{n+1}(s; \tau)} > 0, \quad (17)$$

where the fraction  $\frac{R_{n-1}(s; \tau)}{R_{n+1}(s; \tau)}$  is negative since for a fixed  $\tau \neq 0$  at a root  $s$  of the polynomial  $R_n(s; \tau)$  the function values  $R_{n-1}(s; \tau)$  and  $R_{n+1}(s; \tau)$  are nonzero and have different signs because of the interlacing property. A similar analysis can be done for the algebraic curve  $R_{n-1}(s; \tau) = 0$ .

### 5.3 Asymptotic properties

If  $\tau \rightarrow \infty$  (or  $\tau \rightarrow -\infty$ ) then the roots of  $R_k(s; \tau)$  for  $s$  approach the roots of  $P(s)$  (i.e.  $s_n \leq s_{n-1} \leq \dots \leq s_1$ ). Similarly, if  $s \rightarrow \infty$  or  $s \rightarrow -\infty$  then the roots of  $R_k(s; \tau)$  for  $\tau$  approach the roots of  $\tau^k = 0$  (i.e. 0 with multiplicity  $k$ ).

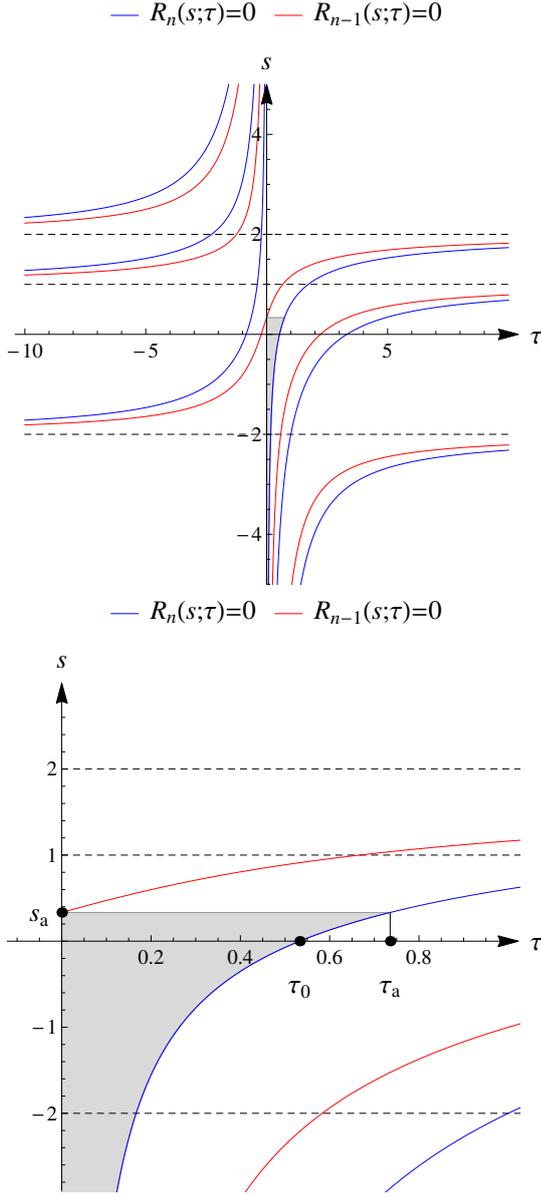


Fig. 1. Roots location of the polynomials  $R_n(s; \tau)$  and  $R_{n-1}(s; \tau)$  on the  $(\tau, s)$  plane for the open-loop characteristic function  $P(s) = (s - 2)(s - 1)(s + 2)$ . Top: Entire  $(\tau, s)$  plane. Bottom: Right half-plane near the origin.

#### 5.4 Roots of $R_k(s; 0)$

If  $\tau = 0$  then  $R_k(s; \tau) = R_k(s; 0) = P^{(k)}(s)$ . Therefore, if  $k = n$ , then  $R_n(s; 0) = n!a_n$  has no roots for  $s$ . If  $k = n - 1$ , then  $R_{n-1}(s; 0) = \frac{n!}{1!}a_n s + \frac{(n-1)!}{0!}a_{n-1}$  has one root for  $s$ :

$$s_a = -\frac{1}{n} \frac{a_{n-1}}{a_n} = \frac{1}{n} \sum_{i=1}^n s_i, \quad (18)$$

which is the average of the roots of  $P(s)$ .

#### 5.5 Sufficient conditions for dominancy and stabilizability if $P(s)$ is real-rooted

Fig. 1 shows the branches of the algebraic curves  $R_n(s; \tau) = 0$  and  $R_{n-1}(s; \tau) = 0$  corresponding to the interlacing and asymptotic properties on the  $(\tau, s)$  plane (top), as well as the right half-plane in the neighborhood of the origin. The horizontal asymptotes corresponding to the roots of  $P(s)$  are indicated with dashed lines.

Let  $\tau_0$  denote the smallest positive root of  $R_n(0; \tau) = 0$  for  $\tau$ . For  $\tau > 0$  the first branch of the algebraic curve  $R_n(s; \tau) = 0$  corresponds to the greatest  $s$  values, and it takes values in the interval  $]-\infty, s_1[$ . Therefore, if  $s_1 > 0$ , then  $\tau_0$  corresponds to the first branch of  $R_n(s; \tau) = 0$ . If  $s_1 \leq 0$ , then  $R_n(0; \tau) = 0$  has no positive roots. In this case, set  $\tau_0 = \infty$ .

Moreover, let  $\tau_a$  denote the smallest positive root of  $R_n(s_a; \tau) = 0$  for  $\tau$ . If  $P(s) \neq a_n(s - s_1)^n$  then  $\tau_a$  corresponds to the first branch of  $R_n(s; \tau) = 0$  since  $s_n < s_a < s_1$ . If  $P(s) = a_n(s - s_1)^n$ , then  $s_a = s_1$  and  $R_n(s_a; \tau) = 0$  has no roots. In this case, set  $\tau_a = \infty$ .

The curve  $R_n(s; \tau) = 0$  gives a connection between the delay  $\tau$  and the possible values of the real root  $s_0$  with multiplicity  $n + 1$ , while the curve  $R_{n-1}(s; \tau) = 0$  is needed to analyze the sufficient condition given in Proposition 2. It is clear that the condition  $R_{n-1}(s_{0,k}; \tau t) \leq 0, \forall t, 0 < t \leq 1$  can be satisfied if and only if  $k = 1$  and  $0 < \tau \leq \tau_a$  (i.e. for the greatest  $s_0$  and in a certain delay interval).

These observations are summarized as follows:

**Proposition 3.** Consider the case  $s_a \geq 0$ . Then

1.  $\tau_0 \leq \tau_a$
2.  $s_{0,1}$  is the dominant root of system (8) if  $0 < \tau \leq \tau_a$ , and system (8) is stabilizable if  $0 < \tau < \tau_0$ .

**Proof.** If  $s_a > 0$  then there is at least one positive root  $s_1$  of  $P(s)$ , therefore there is a finite  $\tau_0$  corresponding to the first branch of  $R_n(s; \tau) = 0$ . Then the inequality  $\tau_0 \leq \tau_a$  follows from the monotonicity of the curve  $R_n(s; \tau) = 0$ . Furthermore, if  $s_a = 0$  then  $\tau_0 = \tau_a$ . Next, the dominancy and stabilizability properties follow from the sufficient condition in Proposition 2 and the fact that  $s_{0,1} \geq 0$  if  $\tau_0 \leq \tau \leq \tau_a$ .

**Proposition 4.** Consider the case  $s_a < 0$ . Then

1.  $\tau_0 \geq \tau_a$
2.  $s_{0,1}$  is the dominant root of system (8) and system (8) is stabilizable if  $0 < \tau \leq \tau_a$ .

**Proof.** The proof follows the same lines as the proof of Proposition 3.

## 6. MULTI-DEGREE-OF-FREEDOM MECHANICAL EXAMPLE: 5-LINK INVERTED PENDULUM

Consider a 5-link inverted pendulum with rods of equal mass  $m$  and length  $l$ . The control torque is applied at the first (lowest) rod:

$$M = -\sum_{i=1}^5 p_i \varphi_i(t - \tau) - \sum_{i=1}^5 d_i \varphi_i'(t - \tau). \quad (19)$$

### 6.1 Derivation of the equation of motion

The equation of motion can be derived by using the Euler–Lagrange equations. The generalized coordinates are chosen to be the pendulum angles  $\varphi_i$  (i.e. angular displacement of the rods from the vertically upward position). The equation of motion linearized around the unstable equilibrium has the form:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{Q}, \quad (20)$$

where the mass matrix  $\mathbf{M}$  and the stiffness matrix  $\mathbf{K}$  can be written as

$$\mathbf{M} = \frac{1}{6}l^2m \begin{bmatrix} 26 & 21 & 15 & 9 & 3 \\ 21 & 20 & 15 & 9 & 3 \\ 15 & 15 & 14 & 9 & 3 \\ 9 & 9 & 9 & 8 & 3 \\ 3 & 3 & 3 & 3 & 2 \end{bmatrix}, \quad \mathbf{K} = -\frac{1}{2}glm \begin{bmatrix} 9 & 0 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The generalized force  $\mathbf{Q}$  reads

$$\mathbf{Q} = - \begin{bmatrix} p_1 & p_2 & p_3 & p_4 & p_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{q}(t-\tau) - \begin{bmatrix} d_1 & d_2 & d_3 & d_4 & d_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \dot{\mathbf{q}}(t-\tau).$$

Therefore, the characteristic function has the form (1) where the open-loop characteristic function  $P(s)$  reads

$$P(s) = \det \left( s^2 \mathbf{M} + \mathbf{K} \right). \quad (21)$$

### 6.2 Stabilizable delay interval

Since the mass matrix  $\mathbf{M}$  is positive-definite and the stiffness matrix  $\mathbf{K}$  is negative-definite, then  $P(s)$  has only real roots. The average of the roots is  $s_a = 0$  since the roots occur in real pairs  $\pm s_i$ . Therefore, we can apply the results of Proposition 3. More precisely, the system (20) is stabilizable if  $0 < \tau < \tau_0$ , where  $\tau_0$  can be calculated if the system parameters are known. For example, if the parameter  $a_0 := \frac{3g}{l} = 1$  then  $\tau_0 = 0.3816$ . Some numerical simulations are shown in Fig. 2 for  $\tau = 0.22$ .

### ACKNOWLEDGEMENTS

The research reported in this paper was supported by the Hungarian-French Bilateral Scientific and Technological Cooperation Fund under Grant no. 2017-2.2.5-TT-FR-2017-00007, by the Higher Education Excellence Program of the Ministry of Human Capacities in the frame of Artificial Intelligence research area of Budapest University of Technology and Economics (BME FIKP-MI), by the National Research, Development and Innovation Fund (TUDFO/51757/2019-ITM, Thematic Excellence Program). Finally, IB & SIN are partially financially supported by a public grant overseen by the French National Research Agency (ANR) as part of the "Investissement d'Avenir" program, through the "iCODE Institute project" funded by the IDEX Paris-Saclay.

### REFERENCES

Ackermann, J. (2002). *Robust control. The parameter space approach*. Springer-Verlag, London, UK.

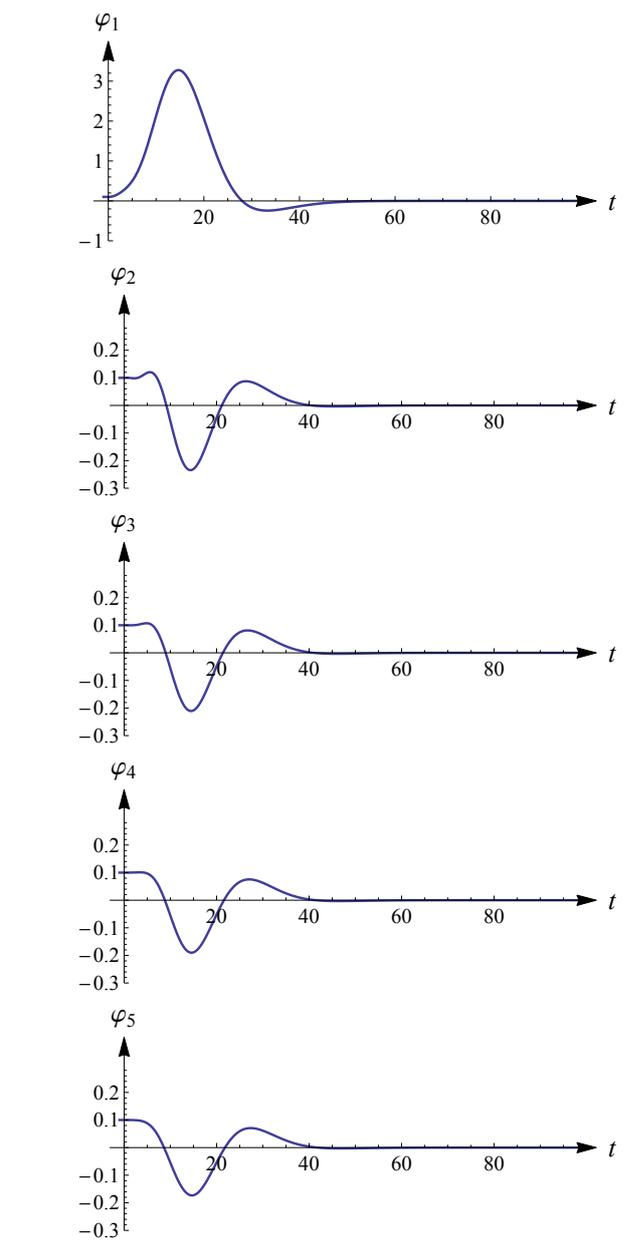


Fig. 2. Time-domain simulation for system (20) with  $3g/l = 1$  and  $\tau = 0.22 < \tau_0$ .

- Atay, F.M. (1999). Balancing the inverted pendulum using position feedback. *Appl. Math. Lett.*, 12(5), 51–56.
- Bellman, R.E. and Cooke, K.L. (1963). *Differential-difference equations*.
- Bhattacharyya, S., Chapellat, H., and Keel, L. (1995). *Robust control: the parametric approach*. Prentice-Hall PTR, Englewood Cliffs, NJ, USA.
- Boussaada, I., Morărescu, I.C., and Niculescu, S.I. (2015). Inverted pendulum stabilization: Characterization of codimension-three triple zero bifurcation via multiple delayed proportional gains. *Systems & Control Letters*, 82, 1–9.
- Boussaada, I. and Niculescu, S.I. (2018). On the dominance of multiple spectral values for time-delay systems with applications. *IFAC-PapersOnLine*, 51(14), 55–60.

- Boussaada, I., Niculescu, S.I., Tliba, S., and Vyhlídal, T. (2017). On the coalescence of spectral values and its effect on the stability of time-delay systems: Application to active vibration control. *Procedia IUTAM*, 22(Supplement C), 75–82.
- Boussaada, I., Niculescu, S.I., and Trabelsi, K. (2018a). Toward a decay rate assignment based design for time-delay systems with multiple spectral values. In *Proceeding of the 23rd International Symposium on Mathematical Theory of Networks and Systems*, 864–871.
- Boussaada, I., Tliba, S., Niculescu, S.I., Ünal, H., and Vyhlídal, T. (2018b). Further remarks on the effect of multiple spectral values on the dynamics of time-delay systems. application to the control of a mechanical system. *Linear Algebra and its Applications*, 542, 589–604.
- Boussaada, I., Unal, H., and Niculescu, S.I. (2016). Multiplicity and stable varieties of time-delay systems: A missing link. In *Proceeding of the 22nd International Symposium on Mathematical Theory of Networks and Systems*, 1–6.
- Boussaada, I., Niculescu, S.I., El Ati, A., Perez-Ramos, R., and Trabelsi, K.L. (2020). Multiplicity-induced-dominancy in parametric second-order delay differential equations: Analysis and application in control design. *ESAIM Control Optim. Calc. Var.*
- Breda, D. (2012). On characteristic roots and stability charts of delay differential equations. *International journal of robust and nonlinear control*, 22(8), 892–917.
- Chyzak, F., Quadrat, A., and Robertz, D. (2005). Effective algorithms for parametrizing linear control systems over Ore algebras. *Applicable Algebra in Engineering, Communication and Computing*, 16(5), 319–376.
- Ciezkowski, M. (2016). Method for determination of interaction between a two-wheeled self-balancing vehicle and its rider. *Mechanics*, 22(5), 416–424.
- Fadeev, D. and Sominski, I. (1965). *Problems in Higher Algebra*. W.H. Freeman.
- Gu, K., Chen, J., and Kharitonov, V.L. (2003). *Stability of time-delay systems*. Springer Science & Business Media.
- Habib, G., Miklós, A., Enikov, E.T., Stépán, G., and Rega, G. (2017). Nonlinear model-based parameter estimation and stability analysis of an aero-pendulum subject to digital delayed control. *International Journal of Dynamics and Control*, 5(3), 629–643.
- Hassard, B. (1997). Counting roots of the characteristic equation for linear delay-differential systems. *Journal of Differential Equations*, 136(2), 222–235.
- Hayes, N.D. (1950). Roots of the transcendental equation associated with a certain difference-differential equation. *Journal of the London Mathematical Society*, s1-25(3), 226–232.
- Landry, M., Campbell, S.A., Morris, K., and Aguilar, C.O. (2005). Dynamics of an inverted pendulum with delayed feedback control. *SIAM Journal on Applied Dynamical Systems*, 4(2), 333–351.
- Li, X., Niculescu, S., Çela, A., Zhang, L., and Li, X. (2017). A frequency-sweeping framework for stability analysis of time-delay systems. *IEEE Transactions on Automatic Control*, 62(8), 3701–3716.
- Ma, D. and Chen, J. (2018). Delay margin of low-order systems achievable by PID controllers. *IEEE Transactions on Automatic Control*, 64(5), 1958–1973.
- Maurer, C. and Peterka, R.J. (2005). A new interpretation of spontaneous sway measures based on a simple model of human postural control. *Journal of neurophysiology*, 93(1), 189–200.
- Michiels, W. and Niculescu, S.I. (2014). *Stability, Control, and Computation for Time-Delay Systems. An Eigenvalue-Based Approach*. Advances in design and control. Society for Industrial and Applied Mathematics, Philadelphia, 2 edition.
- Michiels, W., Engelborghs, K., Vansevenant, P., and Roose, D. (2002). Continuous pole placement for delay equations. *Automatica*, 38(5), 747–761.
- Milton, J. and Insperger, T. (2019). Acting together, destabilizing influences can stabilize human balance. *Philosophical Transactions of the Royal Society A*, 377(2153), 20180126.
- Milton, J., Meyer, R., Zhvanetsky, M., Ridge, S., and Insperger, T. (2016). Control at stability’s edge minimizes energetic costs: expert stick balancing. *Journal of the Royal Society Interface*, 13(119), 20160212.
- Morasso, P., Cherif, A., and Zenzeri, J. (2019). Quiet standing: The single inverted pendulum model is not so bad after all. *PloS one*, 14(3), e0213870.
- Mori, T., Fukuma, N., and Kuwahara, M. (1982). On an estimate of the decay rate for stable linear delay systems. *International Journal of Control*, 36(1), 95–97.
- Niculescu, S.I., Michiels, W., Gu, K., and Abdallah, C. (2010). *Delay Effects on Output Feedback Control of Dynamical Systems*, 63–84. Springer Berlin Heidelberg, Berlin, Heidelberg.
- Olgac, N. and Sipahi, R. (2002). An exact method for the stability analysis of time-delayed linear time-invariant (LTI) systems. *IEEE Transactions on Automatic Control*, 47(5), 793–797.
- Pólya, G. and Szegő, G. (1997). *Problems and Theorems in Analysis II: Theory of Functions. Zeros. Polynomials. Determinants. Number Theory. Geometry*. Springer Science & Business Media.
- Qin, Z.C., Li, X., Zhong, S., and Sun, J.Q. (2014). Control experiments on time-delayed dynamical systems. *Journal of Vibration and Control*, 20(6), 827–837.
- Schurer, F. (1948). Zur Theorie des Balancierens. *Mathematische Nachrichten*, 1, 295–331.
- Sieber, J. and Krauskopf, B. (2004). Bifurcation analysis of an inverted pendulum with delayed feedback control near a triple-zero eigenvalue singularity. *Nonlinearity*, 17, 85–103.
- Stépán, G. (1989). *Retarded dynamical systems: stability and characteristic functions*. Longman Scientific & Technical.
- Stépán, G. (2009). Delay effects in the human sensory system during balancing. *Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 367, 1195–1212.
- Suh, I. and Bien, Z. (1979). Proportional minus delay controller. *IEEE Trans. on Aut. Cont.*, 24, 370–372.
- Xu, Q., Stépán, G., and Wang, Z. (2017). Balancing a wheeled inverted pendulum with a single accelerometer in the presence of time delay. *Journal of Vibration and Control*, 23(4), 604–614.