A Hole in the Ladder: Interleaved Variables in Iterative Conditional Branching
Yoann Marquer, Tania Richmond

To cite this version:

HAL Id: hal-02889212
https://hal.archives-ouvertes.fr/hal-02889212
Submitted on 3 Jul 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
A Hole in the Ladder: 
Interleaved Variables in Iterative Conditional Branching

Yoann Marquer  
TAMIS team, TeamPlay project  
Inria, Univ Rennes, CNRS, IRISA  
Rennes, France  
yoann.marquer@inria.fr

Tania Richmond  
TAMIS team, TeamPlay project  
Inria, Univ Rennes, CNRS, IRISA  
Rennes, France  
tania.richmond@inria.fr

Abstract—The modular exponentiation is crucial to the RSA 
cryptographic protocol, and variants inspired by the Montgomery 
ladder have been studied to provide more secure algorithms. In 
this paper, we abstract away the iterative conditional branching 
used in the Montgomery ladder, and formalize systems of 
equations necessary to obtain what we call the semi-interleaved 
and fully-interleaved cases. The considered attacks cannot be used against the fully-interleaved cases, but not against fully-interleaved ladders that are more secure. We also apply these equations to extend the Montgomery ladder for both the semi- and fully-interleaved cases, thus proposing novel and more secure algorithms to compute the modular exponentiation.

Index Terms—Cryptography, Countermeasures (computer), 
Fault detection, Iterative algorithms, Public-key cryptography, 
Security, Side-channel attacks

I. INTRODUCTION

A. Contribution

We present common algorithms used to compute the modu-
lar exponentiation, based on an iterative conditional bran-
ching, where a conditional branching depending on the se-
cret updates the value of a variable \( x \) on every iteration of 
one/ several loop(s). Amongst these algorithms the Mont-
gomery ladder, that uses a fresh variable \( y \), satisfies desirable 
properties against side-channel or fault-injection attacks.

In this paper, we formalize these properties with equations 
corresponding to two families of cases: the semi-interleaved 
ladders where the value of \( x \) or \( y \) may (depending on the 
secret) depend only on its previous value, and the fully-
interleaved ladders where the value of \( x \) and \( y \) both depend 
on both previous values.

We use our formalization to design novel algorithms for the 
modular exponentiation, including a semi-interleaved ladder 
with a mask updated at every iteration, and a fully-interleaved 
ladder. We also propose an attacker model using fault-injection 
able to obtain some bits of the secret in the semi-interleaved 
cases (including the Montgomery ladder) but none in the fully-
interleaved cases, and a stronger variant of the attacker model 
able to obtain all bits of the secret in the semi-interleaved 
cases. The considered attacks cannot be used against the fully-
interleaved cases, but other attacks might exist.

This work was supported by the EU Horizon 2020 project TeamPlay (https://www.teamplay-h2020.eu), grant number 779882.

B. Organization of the Paper

As introduction, we present in Section II some related 
works on the modular exponentiation and the Montgomery 
ladder. In Section III we formalize the iterative conditional 
branching, and we deduce the equations satisfied by the 
semi-interleaved and fully-interleaved ladders, and thus the 
requirements for ladderizable programs. In Section IV we 
introduce two attacker models using fault injection techni-
uques, and compare the vulnerability of the non-, semi- and fully-
interleaved ladders. Finally, in Section V we detail how to 
produce examples of the semi- and fully-interleaved ladders, 
including novel variants of the Montgomery ladder.

II. RELATED WORKS

In this section, we introduce known algorithms for the 
modular exponentiation, their relevance regarding security, and 
the desirable properties of the usual Montgomery ladder.

A. Modular Exponentiation

Let \( k \) be a secret key, and \( k = \sum_{0 \leq i < d} k[i] 2^i \) be its 
binary expansion of size \( d + 1 \), i.e. \( k[i] \) is the bit \( i \) of \( k \).
The square-and-multiply algorithm described in Table I com-
putes the (left-to-right) modular exponentiation \( a^k \mod n \), by 
using \( a^\sum_{0 \leq i < d} k[i] 2^i = \prod_{0 \leq i < d} (a^{2^i})^{k[i]} \). This exponentiation is 
commonly used in crypto-systems like RSA [1].

For every iteration, the multiplication \( ax \) is computed only if 
\( k[i] = 1 \), which can be detected by observing execution 
time\(^1\) [2] or power profiles\(^2\) by means of e.g. SPA (Simple

\(^1\)Even observing the duration of the whole execution can leak the Hamming 
weight of the secret key, and thus can narrow down the exploration space.

\(^2\)A multiplication can be distinguished from a squaring, hence variants with 
squaring only have been proposed in [3] and improved in [4].
Power Analysis\(^3\) [8], and thus leads to information leakage from both time and power side-channel attacks.

To prevent SPA, regularity of the modular exponentiation algorithms is required, which means that both branches of the 

4

Table III

<table>
<thead>
<tr>
<th>Iterative Conditional Branching</th>
</tr>
</thead>
<tbody>
<tr>
<td>input public a, n; secret k</td>
</tr>
<tr>
<td>1: x ← init</td>
</tr>
<tr>
<td>2: for i = 1 to n do</td>
</tr>
<tr>
<td>3: if k[i] = 1 then</td>
</tr>
<tr>
<td>4: x ← x * y mod n</td>
</tr>
<tr>
<td>5: y ← y * x mod n</td>
</tr>
<tr>
<td>6: else</td>
</tr>
<tr>
<td>7: x ← x * (\varphi(x))</td>
</tr>
<tr>
<td>8: end if</td>
</tr>
<tr>
<td>9: end for</td>
</tr>
<tr>
<td>output (x \equiv a^k \mod n)</td>
</tr>
</tbody>
</table>

Montgomery Ladder

This is not the algorithm in the context of the algorithm proposed by Montgomery [11] and described in Table III, where a fault injected in a register will eventually propagate to the other one, and thus will alter the final result, preventing the attacker to obtain information. But, as described in Section IV, some information on the last digits may still leak, weakening the protection obtained from the ladder.

The Montgomery ladder is algorithmically equivalent [12] to the square-and-multiply(-always) algorithm(s), in the sense that \(x\) has the same value for every iteration. Actually, some variants [13] of the square-and-multiply-always algorithm may be as resistant as the Montgomery ladder [17], both by checking invariants [18] violated if a fault is injected. In the case of the Montgomery ladder, the invariant \(y = ax\) is satisfied for every iteration. These invariants are important for the self-secure exponentiation countermeasures [19].

Note that the else branch in Table III is identical to the then branch, except that \(x\) and \(y\) are swapped, which

5

provides also (partial)\(^5\) protection against timing and power leakage. Moreover, the variable dependency makes these variables interleaved, so this exponentiation is algorithmically (but partially) protected against safe-error attacks. Finally, as opposed to square-and-multiply-always in Table II, the code in the else branch is not dead, so will not be removed by compiler optimisations.

III. Ladder Equations

In this section, we formalize the iterative conditional branching occurring in algorithms like the Montgomery ladder, and deduce the requirements to optimize them with semi- or fully-interleaved ladders.

A. Iterative Conditional Branching

In this paper, we focus on programs as in Table IV called iterative conditional branching. It appears in programs like the modular exponentiation, its counterpart the double-and-add used for elliptic curve point multiplication [20], [21], or the secure bit permutation in the McEliece cryptosystem [22] attacked in [23]. It also appears naturally (Table V) when trying to turn a loop on the secret into a conditional branching depending on the secret, that can itself be balanced to remove or reduce the dependency on the secret.

But our approach does not depend on the number/depth of the considered loops, hence the dots in Table IV. We assume only that the conditional branching uses only one variable \(x\), the multivariate case being future work (see Section VI).

Definition 1 (Iterative Conditional Branching). A program as in Table IV is said with an (univariate) iterative conditional branching with two (unary) functions \(\varphi\) and \(\psi\).

B. Semi-Interleaved Ladders

To prevent information leakage from side-channels or fault injections, we use another variable \(y\) in the algorithm described in Table VI. As in the Montgomery ladder in Table III, we need to find two functions \(\ell\) and \(f\) such that for every iteration:

6

\[\psi(y) = y \times \varphi(x)\]

\[\ell(y) = y \times x\]

\[f(y) = y \times x\]

\[\varphi(x) = x \times y\]

\[\psi(x) = x \times y\]

\[\ell(x) = x \times y\]

\[f(x) = x \times y\]

\[\varphi(y) = y \times x\]

\[\psi(y) = y \times x\]

\[\ell(y) = y \times x\]

\[f(y) = y \times x\]

\[\varphi(x) = x \times y\]

\[\psi(x) = x \times y\]

\[\ell(x) = x \times y\]

\[f(x) = x \times y\]

\[\varphi(y) = y \times x\]

\[\psi(y) = y \times x\]

\[\ell(y) = y \times x\]

\[f(y) = y \times x\]

\[\varphi(x) = x \times y\]

\[\psi(x) = x \times y\]

\[\ell(x) = x \times y\]

\[f(x) = x \times y\]

\[\varphi(y) = y \times x\]

\[\psi(y) = y \times x\]

\[\ell(y) = y \times x\]

\[f(y) = y \times x\]

\[\varphi(x) = x \times y\]

\[\psi(x) = x \times y\]

\[\ell(x) = x \times y\]

\[f(x) = x \times y\]

\[\varphi(y) = y \times x\]

\[\psi(y) = y \times x\]

\[\ell(y) = y \times x\]

\[f(y) = y \times x\]

\[\varphi(x) = x \times y\]

\[\psi(x) = x \times y\]

\[\ell(x) = x \times y\]

\[f(x) = x \times y\]

\[\varphi(y) = y \times x\]

\[\psi(y) = y \times x\]

\[\ell(y) = y \times x\]

\[f(y) = y \times x\]

\[\varphi(x) = x \times y\]

\[\psi(x) = x \times y\]

\[\ell(x) = x \times y\]

\[f(x) = x \times y\]

\[\varphi(y) = y \times x\]

\[\psi(y) = y \times x\]

\[\ell(y) = y \times x\]

\[f(y) = y \times x\]

\[\varphi(x) = x \times y\]

\[\psi(x) = x \times y\]

\[\ell(x) = x \times y\]

\[f(x) = x \times y\]

\[\varphi(y) = y \times x\]

\[\psi(y) = y \times x\]

\[\ell(y) = y \times x\]

\[f(y) = y \times x\]
\[ y = \ell(x), \text{ and} \]
\[ x \text{ has the same value for every iteration as in Table IV.} \]
\[ y = \ell(x) \text{ is satisfied at the initialization. By induction, let’s} \]
\[ assume y = \ell(x) \text{ at the beginning of an iteration. In the then} \]
\[ branch we have } x \leftarrow f(x,y) \text{ then } y \leftarrow \psi(y), \text{ thus in order to} \]
\[ have y = \ell(x) \text{ satisfied at the end of an iteration the following} \]
\[ equation must hold:} \]
\[ \forall x, \psi(\ell(x)) = \ell(f(x,\ell(x))) \]
\[ and to have } x \text{ updated to } \varphi(x) \text{ during the iteration the following} \]
\[ equation must hold:} \]
\[ \forall x, f(x,\ell(x)) = \varphi(x) \]

In the else branch we have \( y \leftarrow f(y,x) \text{ then } x \leftarrow \psi(x), \) thus in order to have \( y = \ell(x) \) satisfied at the end of an iteration the following equation must hold:
\[ \forall x, f(\ell(x),x) = \psi(x) \]
and \( x \) is already updated to \( \psi(x) \) during the iteration.

**Definition 2 (Semi-Ladderizable).** Let \( P \) be a program with a univariate iterative conditional branching with two unary functions denoted \( \varphi \) and \( \psi \). \( P \) is semi-ladderizable if there exists a unary function \( \ell \) and a binary function \( f \) such that, for every considered value \( x \):

\[
\begin{align*}
\psi(\ell(x)) &= \ell(\varphi(x)) \\
f(x,\ell(x)) &= \varphi(x) \\
f(\ell(x),x) &= \ell(\psi(x))
\end{align*}
\]

For the square-and-multiply algorithm we have \( \varphi(x) = ax^2 \) and \( \psi(x) = x^2 \), and we know that it can be semi-ladderized by using the Montgomery ladder with \( \ell(x) = ax \) and \( f(x,y) = xy \), but we show in Section V that there are other solutions. Note that to respect the form of the semi-interleaved ladder, we should have written \( y \leftarrow yx \) and not \( y \leftarrow xy \) in the else branch\(^6\) of the Montgomery ladder.

**C. Fully-Interleaved Ladders**

Unfortunately, the semi-interleaved ladder is vulnerable to fault injection techniques, because in every branch at least one variable depends only on its previous value and not the previous value of both variables (see Section IV). Moreover, an attacker able to determine whether the output of one operation is used as the input to another one can [25] apply collision attacks \(^3\) to deduce whether two following bits are the same. To solve these issues, we propose in Table VII a fully-interleaved ladder using three functions \( \ell, f \) and \( g \).

As for the semi-interleaved ladders, \( y = \ell(x) \) is satisfied at the initialization. We assume again by induction that \( y = \ell(x) \) at the beginning of an iteration. In the then branch we have \( x \leftarrow f(x,y) \text{ then } y \leftarrow g(x,y), \) thus in order to have \( y = \ell(x) \) satisfied at the end of an iteration the following equation must hold:
\[
\forall x, g(f(x,\ell(x)),\ell(x)) = \ell(f(x,\ell(x)))
\]

\(^6\)The former is actually better regarding vulnerability to the M safe-error [17] or collision [24] attacks, showing that this method is good practice.

\(^3\)A countermeasure proposed in [26] is to randomly blend variants of the ladder, or compute the exponentiation by taking a random (bounded) walk.

and to have \( x \) updated to \( \varphi(x) \) during the iteration the following equation must hold:
\[
\forall x, f(x,\ell(x)) = \varphi(x)
\]

In the else branch we have \( y \leftarrow f(y,x) \text{ then } x \leftarrow g(y,x), \) thus, in order to have \( y = \ell(x) \) satisfied at the end of an iteration, the following equation must hold:
\[
\forall x, f(\ell(x),x) = \ell(g(f(\ell(x),x),x))
\]
and to have \( x \) updated to \( \psi(x) \) during the iteration the following equation must hold:
\[
\forall x, g(f(\ell(x),x),x) = \psi(x)
\]

**Definition 3 (Fully-Ladderizable).** Let \( P \) be a program with a univariate iterative conditional branching with two unary functions denoted \( \varphi \) and \( \psi \). \( P \) is fully-ladderizable if there exists a unary function \( \ell \) and two binary functions \( f \) and \( g \) such that, for every considered value \( x \):

\[
\begin{align*}
g(\varphi(x),\ell(x)) &= \ell(\varphi(x)) \\
f(x,\ell(x)) &= \varphi(x) \\
f(\ell(x),x) &= \ell(\psi(x)) \\
g(f(\ell(x),x),x) &= \psi(x)
\end{align*}
\]

Note that Equation (2) is Equation (5) and Equation (3) is Equation (6). Without surprise, if \( g \) is chosen such that \( g(x,y) = \psi(y) \) then Equation (1) is a special case of Equation (4), and Equation (7) is satisfied. Thus, semi-interleaved ladders are subcases of fully-interleaved ladders.

**D. Ladderizable Programs**

**Theorem 4.** Let \( P \) be a program with an iterative conditional branching with two unary functions denoted \( \varphi \) and \( \psi \). If \( P \) is semi-ladderizable with \( \ell \) and \( f \), or fully-ladderizable with \( \ell, f \) and \( g \), then for every iteration of the ladder variant:

- \( y = \ell(x) \)
- \( x \) is updated as in \( P \):
  \[
  x \leftarrow \begin{cases} \varphi(x) & \text{if secret} \\ \psi(x) & \text{otherwise} \end{cases}
  \]

Thus, for every program with an iterative conditional branching, if there exists \( \ell \) and \( f \) satisfying the equations in Definition 2 then the conditional branching can be rewritten as a semi-interleaved ladder, or even better if there exists \( \ell, f \) and \( g \) satisfying the equations in Definition 3 then it can be rewritten as a fully-interleaved ladder.

Because in a semi- or fully-interleaved ladder the operations performed are the same for both the then and the else branch, this transformation is an algorithmic countermeasure against side-channel attacks [2], [8]. We detail in Section IV the impact of the semi- and fully-interleaved ladderization against fault injection techniques. Then, to demonstrate the concept, we construct examples of semi- and fully-interleaved ladders in Section V.

**IV. Fault Injection**

In this section, we introduce two attacker models using fault injection techniques, and compare the vulnerability of the non-, semi- and fully-interleaved ladders.
According to [10] a fault is a physical defect, imperfection or flaw that occurs within some hardware or software component, while an error is a deviation from accuracy or correctness, and is the manifestation of a fault. Hardware/physical faults can be permanent, transient or intermittent, while software faults are the consequence of incorrect design, specification or at coding time. Fault injection is defined [27] as the validation technique of the dependability of fault tolerant systems, which consists in performing controlled experiments where the observation of the system’s behavior in presence of faults is induced explicitly by the writing introduction (called injection) of faults in the system.

A. Attacker Model

For the iterative conditional branching (Table IV) and the semi- and fully-interleaved ladders (Tables VI and VII), we assume as in the Montgomery ladder (Table III) that “secret” is the condition \( k[i] = 1 \), where \( k[i] \) is the \( i \)-th bit of the secret key.

**Definition 5** (Attacker Model). We assume that:

- The attacker wants to obtain the secret key stored in the chip and copied in the register \( k \).
- The attacker can run the program any number of times:
  - inputting \( x_{\text{init}} \) and \( y_{\text{init}}, \) the initial values in the register \( x \) and \( y \),
  - obtaining \( x_{\text{final}} \) and/or \( y_{\text{final}}, \) the final value(s) returned by the program.
- A run consists of iterations \( i \) over \( 1, \ldots, n \), where:
  - \( k[i] \) is the \( i \)-th bit of \( k \),
  - \( x_i \) and \( y_i \) denote the values in the register \( x \) and \( y \) between the iterations \( i-1 \) and \( i \).
- The attacker can \( \ell(x_i) \) (resp. \( \ell(y_i) \)):
  - inject a random fault\(^6\) (the affected variable is set to a random value) in the register of \( x \) (resp. \( y \)),
  - between the iterations \( i-1 \) and \( i \).

The attacker can run a program for the square-and-multiply-(always) (Tables I and II) algorithm(s) for a given input \( x_{\text{init}} \) obtain a value \( x_{\text{final}} \), then run the program again with the same input while injecting a fault \( y_i \) in the register \( y = ax \) between the iterations \( i-1 \) and \( i \), and obtain a value \( x_{\text{final}} \). If \( x_{\text{fault}} = x_{\text{final}} \) then \( k[i] = 0 \), otherwise \( k[i] = 1 \). This can be done for every iteration (in any order), thus the attacker can obtain that way all the bits of the secret key.

In this algorithm, because the current value in \( x \) determines the next value in \( x \), a faulted value \( \ell(x_i) \) for an iteration \( i \) always propagates to the next iteration \( x_{i+1} \), which we denote by \( \ell(x_i) \Rightarrow \ell(x_{i+1}) \). To obtain the bit \( k[i] \), the attacker has exploited

\[ \ell(x_i) \Rightarrow \ell(x_{i+1}) \]

\[ \ell(y_i) \Rightarrow \ell(y_{i+1}) \]

\[ \ell(x_i) \Rightarrow \ell(x_{i+1}) \]

\[ \ell(y_i) \Rightarrow \ell(y_{i+1}) \]

that a fault in \( y = ax \) propagates to \( x \), denoted \( \ell(y_i) \Rightarrow \ell(x_{i+1}) \), only if \( k[i] = 1 \). As opposed to this non-ladderized variant, in Tables VI and VII the values of \( x \) and \( y \) are interleaved, and the fault propagation patterns are the following:

1) For the semi-interleaved ladder:

\[ \ell(x_i) \Rightarrow \ell(x_{i+1}) \]
\[ \ell(y_i) \Rightarrow \ell(y_{i+1}) \]
\[ \ell(x_i) \Rightarrow \ell(y_{i+1}) \] only if \( k[i] = 0 \)
\[ \ell(y_i) \Rightarrow \ell(x_{i+1}) \] only if \( k[i] = 1 \)

which means that a fault always propagates to the same register, but propagates to the other depending on the current bit of the secret key.

2) For the fully-interleaved ladder:

\[ \ell(x_i) \Rightarrow \ell(x_{i+1}) \]
\[ \ell(y_i) \Rightarrow \ell(y_{i+1}) \]

which means that any fault in one register propagates in every case to both.

The fully-interleaved ladder has a lower fault tolerance, i.e. it is easier to disrupt the computation. This is not convenient for properties like functionality, availability or redundancy, but prevents an attacker from obtaining the secret key, thus increases security. Thus, we reduced the information leakage from fault injection by reducing also the fault tolerance.

The semi-interleaved ladder is more robust, but this comes at the price of a fault propagation pattern depending on the secret key, which can be exploited. Indeed, the attacker can \( \ell(y_i) \) as in Figure 1 and compare the \( x \) output. If \( x_{\text{fault}} = x_{\text{final}} \) then \( k[n] = 0 \), otherwise \( k[n] = 1 \). Thus, the attacker can always obtain \( k[n] \), the last bit of the secret key.

\[ k \cdots k[n-1] k[n] \]

\[ x \]
\[ \ell(x_{\text{final}}) \]
\[ y \]
\[ \ell(y_i) \Rightarrow \ell(y_{\text{final}}) \]

Fig. 1. Attack the Last Bit

If the obtained bit was 0 as in the left part of Figure 2, the attacker can \( \ell(y_{n-1}) \). In that case if \( x_{\text{fault}} = x_{\text{final}} \) then \( k[n-1] = 0 \), otherwise \( k[n-1] = 1 \). This process can be repeated until a 1 is found. If the obtained bit was 1 as in the right part of Figure 2, the attacker can \( \ell(x_{n-1}) \). In that case if \( y_{\text{fault}} = y_{\text{final}} \) then \( k[n-1] = 1 \), otherwise \( k[n-1] = 0 \). This process can be repeated until a 0 is found.

\[ k \cdots k[n-1] 0 \]
\[ k \cdots k[n-1] 1 \]

\[ x \]
\[ \ell(x_{n-1}) \Rightarrow \ell(x_n) \Rightarrow \ell(x_{\text{final}}) \]
\[ y \]
\[ \ell(y_{n-1}) \Rightarrow \ell(y_n) \Rightarrow \ell(y_{\text{final}}) \]

Fig. 2. Attack the Penultimate Bit

One may think that these processes can be alternated in order to recover all the bits of the secret key, but if there is a bit alternation, i.e. \( (k[i], k[i+1]) = (0, 1) \) or \( (1, 0) \), then the bits before \( i \) cannot be obtained, as illustrated in Figures 3 and 4. So, the attacker could obtain that way all the bits only if the key is trivial, and it seems improbable that the attacker can obtain that way more than a small number of bits.
Therefore, the vulnerability to fault injection can be summarized in Table VIII. Against the attacker model described in Definition 5, fully-interleaved ladders are more secure than semi-interleaved ladders, which are more secure than without interleaving at all. But we show in the next subsection that a stronger attacker is able to obtain all the bits of the key from the semi-interleaved ladders.

**B. Stronger Attacker Model**

**Definition 6** (Stronger Attacker Model). We assume that:

- The stronger attacker has the same goal and means that the attacker in Definition 5.
- The stronger attacker can also \( f_{k>1} = 0 \) (resp. \( f_{k>1} = 1 \)):
  - stuck-at \([10] \) 0 (resp. 1) all the bits, or at least those \( i > 0 \), of the register \( k \)
  - between iterations \( i \) and \( i + 1 \).

The stronger attacker is able to break the semi-interleaved ladders by using the attack protocol described in Table IX in order to obtain all the bits of the secret key\(^{10}\), where \( \text{EXE}(f_{k>1} = 1) \) at Line 4 means that the studied program is executed with inputs \( x_{\text{init}}, y_{\text{init}} \) and a stuck-at \( f_{k>1} = 1 \), and \( \text{EXE}(f_{k>1} = 1, f_{x_i}) \) at Line 5 is the same but with a fault \( f_{x_i} \). In particular, if both \( x_{\text{final}} \) and \( y_{\text{final}} \) can be read, then the Montgomery Ladder (Table III) can be broken (by iterating over 0 to 0). This attack does not work against fully-interleaved ladders, but other attacks might exist, and a fully-interleaved ladder is more difficult to obtain (when possible), as discussed in Subsection V-C.

V. EXTEND THE MONTGOMERY LADDER

The purpose of this section is to provide concrete examples of the ladder equations for cryptography, and to generalize the idea behind the Montgomery ladder to improve its protection against side-channel and fault injection attacks. To do that, we assume that \( \varphi(x), \psi(x), f(x, y), g(x, y) \) and \( \ell(x) \) are quadratic polynomials with the following coefficients:

\[
\begin{align*}
\varphi(x) &= \varphi_2 x^2 \\
\psi(x) &= x^2 \\
f(x, y) &= f_{20} x^2 + f_{11} x y + f_{02} y^2 + f_{10} x + f_{01} y + f_{00} \\
g(x, y) &= g_{20} x^2 + g_{11} x y + g_{02} y^2 + g_{10} x + g_{01} y + g_{00} \\
\ell(x) &= \ell_1 x + \ell_0
\end{align*}
\]

\(^{10}\)Note that the iterations are reversed in the attack protocol.

---

**Table VIII**

**Vulnerability against the Attack Described for the Attacker Model from Definition 5**

1: \( F_{\text{FX}} \leftarrow 0 \)
2: for \( i = n \) to 1 do
3: if \( F_{\text{FX}} = 1 \) then
4: \( (x_{\text{final}}, y_{\text{final}}) \leftarrow \text{EXE}(f_{k>1} = 1) \)
5: \( (x_{\text{fault}}, y_{\text{fault}}) \leftarrow \text{EXE}(f_{k>1} = 1, f_{x_i}) \)
6: if \( \text{EXE}(i = \text{final}) \) then
7: \( k[i] \leftarrow 1 \)
8: else
9: \( k[i] \leftarrow 0 \)
10: \( F_{\text{FX}} \leftarrow 0 \)
11: end if
12: end else
13: \( (x_{\text{final}}, y_{\text{final}}) \leftarrow \text{EXE}(f_{k>1} = 0) \)
14: \( (x_{\text{final}}, y_{\text{init}}) \leftarrow \text{EXE}(f_{k>1} = 0, f_{x_i}) \)
15: if \( x_{\text{final}} = x_{\text{final}} \) then
16: \( k[i] \leftarrow 1 \)
17: else
18: \( k[i] \leftarrow 0 \)
19: \( F_{\text{FX}} \leftarrow 1 \)
20: end if
21: end if
22: end for
23: return \( k \)

**Table IX**

**Protocol to Attack the Semi-Interleaved Ladders**

More general quadratic polynomials, e.g. \( \ell_2 \neq 0 \) or more complex \( \varphi(x) \) and \( \psi(x) \), can be investigated but the general systems of equations tend to be complicated, and in this paper we want to focus on the exponentiation algorithms. To ensure that \( \varphi(x) = \varphi_2 x^2 \) and \( \ell(x) = \ell_1 x + \ell_0 \) depends on \( x \), we will assume that \( \varphi_2 \neq 0 \) and \( \ell_1 \neq 0 \).

**A. Both Semi- and Fully-Interleaved Ladders**

Equation (2) (resp. Equation (5)) \( f(x, \ell(x)) = \varphi(x) \) for the semi- (resp. fully-) interleaved cases is equivalent to:

\[
\begin{align*}
\{ f_{20} + f_{11} \ell_1 + f_{02} \ell_1 \ell_0 = \varphi_2 \\
f_{11} \ell_0 + 2 f_{02} \ell_1 \ell_0 + f_{10} + f_{01} \ell_1 = 0 \\
f_{02} \ell_1^2 + f_{01} \ell_1 + f_{00} = 0
\end{align*}
\]

Equation (3) (resp. Equation (6)) \( f(\ell(x), x) = \ell(\psi(x)) \) for the semi- (resp. fully-) interleaved cases is equivalent to:

\[
\begin{align*}
\{ 2 f_{20} \ell_1 + f_{11} \ell_0 + f_{10} \ell_1 + f_{01} \ell_0 = 0 \\
f_{20} \ell_1^2 + f_{10} \ell_0 + f_{00} = \ell_0
\end{align*}
\]

**B. Semi-Interleaved Ladders**

The remaining Equation (1) \( \psi(\ell(x)) = \ell(\varphi(x)) \) for the semi-interleaved cases is equivalent to:

\[
\begin{align*}
\{ \ell_1^2 = \ell_1 \varphi_2 \\
2 \ell_1 \ell_0 = 0 \\
\ell_0^2 = \ell_0
\end{align*}
\]

So, because \( \ell_1 \neq 0 \), we have \( \ell_1 = \varphi_2 \) and \( \ell_0 = 0 \). Therefore Equation (2) is equivalent to:

\[
\begin{align*}
\{ f_{20} + f_{11} \varphi_2 + f_{02} \varphi_2^2 = \varphi_2 \\
f_{10} + f_{01} \varphi_2 = 0 \\
f_{00} = 0
\end{align*}
\]
and Equation (3) is equivalent to:

\[
\begin{align*}
20\varphi_2^2 + f_{11}\varphi_2 + f_{10} &= \varphi_2 \\
10\varphi_2 + f_{10} &= 0 \\
f_{00} &= 0
\end{align*}
\]

From \( f_{10} + f_{10}\varphi_2 = f_{10}\varphi_2 + f_{10} \) we deduce \( f_{10}(\varphi_2 - 1) = f_{10}(\varphi_2 - 1) \), which is always true without constraint on \( \varphi_2 \) if \( f_{10} = 0 \). Moreover, from \( f_{10} + f_{10}\varphi_2 = 0 \) we obtain \( f_{10} = -f_{10}\varphi_2 \), thus from \( f_{10}\varphi_2 + f_{10} = 0 \) we obtain \( f_{00}(\varphi_2 - 1) = 0 \), which is always true without constraint on \( \varphi_2 \) if \( f_{10} = 0 \). Therefore we assume \( f_{10} = 0 \).

From \( 20\varphi_2^2 + f_{11}\varphi_2 + f_{10}/2 = 20\varphi_2^2 + f_{11}\varphi_2 + f_{10} \) we deduce \( 20\varphi_2^2 - 1 = 0 \), which is always true without constraint on \( \varphi_2 \) if \( f_{20} = 0 \). The remaining constraint is \( 20\varphi_2^2 + 1 + f_{11}\varphi_2 = \varphi_2 \), which is equivalent to \( 20\varphi_2^2 = 1 - f_{11} \). By assuming as in RSA that the coefficients are integers there exists \( m \) such that \( 1 - f_{11} = m(\varphi_2^2 + 1) \). Thus, we have \( f_{20} = m\varphi_2^2 \) and \( f_{11} = 1 - m(\varphi_2^2 + 1) \).

Therefore, from \( f(x) = \varphi_2 x^2 \) and \( \psi(x) = x^2 \) the equations require (for integer coefficients):

\[ f(x, y) = m\varphi_2(x^2 + y^2) + (1 - m(\varphi_2^2 + 1)x)
\]

\( \ell(x) = \varphi_2 x \)

Note that if \( m = 0 \) then the solution in Table X is the common Montgomery ladder. But \( m \) can be chosen randomly for every iteration, providing a mask for the intermediate values and thus reducing the opportunity of information leakage.

C. Fully-Interleaved Ladders

Equation (4) \( g(\varphi(x), \ell(x)) = \ell(\varphi(x)) \) for the fully-interleaved cases is equivalent to:

\[
\begin{align*}
g_{20}\varphi_2^2 &= 0 \\
g_{11}\ell_1\varphi_2 &= 0 \\
g_{11}\ell_0\varphi_2 + g_{10}\varphi_2^2 + g_{10}\varphi_2 &= \ell_1\varphi_2 \\
2g_{02}\ell_1\ell_0 + g_{01}\ell_0 &= 0 \\
g_{02}\ell_0^2 + g_{01}\ell_0 + g_{00} &= \ell_0
\end{align*}
\]

So, because \( \varphi_2 \neq 0 \) and \( \ell_1 \neq 0 \), we have \( g_{20} = 0 \) from the first equation, \( g_{11} = 0 \) from the second, and the remaining constraints are:

\[
\begin{align*}
g_{20}\ell_2^2 + g_{10}\varphi_2^2 &= \ell_1\varphi_2 \\
2g_{02}\ell_1\ell_0 + g_{01}\ell_0 &= 0 \\
g_{02}\ell_0^2 + g_{01}\ell_0 + g_{00} &= \ell_0
\end{align*}
\]

According to Equation (6) \( f(\ell(x), x) = \ell(\psi(x)) = \ell_1 x^2 + \ell_0 \), so by using \( g_{20} = g_{11} = 0 \) we have:

\[
g(f(\ell(x), x), x) = (g_{20} + g_{10}\ell_1)x^2 + (g_{10}\ell_0 + g_{00})x
\]

Thus, Equation (7) \( g(f(\ell(x), x), x) = \psi(x) \) for the fully-interleaved cases is equivalent to:

\[
\begin{align*}
g_{20} + g_{10}\ell_1 &= 1 \\
g_01 &= 0 \\
g_{10}\ell_0 + g_{00} &= 0
\end{align*}
\]

Because \( g_{20} = g_{11} = g_{01} = 0 \) and we require \( g(x, y) = g_{20} y^2 + g_{10} x + g_{00} \) to depend on \( x \) and \( y \), we have to assume \( g_{20} \neq 0 \) and \( g_{10} \neq 0 \). By using \( g_{01} = 0 \) the remaining constraints from Equation (4) are:

\[
\begin{align*}
g_{20}\ell_2^2 + g_{10}\varphi_2 &= \ell_1\varphi_2 \\
g_{02}\ell_0 + g_{00} &= \ell_0
\end{align*}
\]

Because \( g_{20} \neq 0 \) and \( \ell_1 \neq 0 \), the second equation implies that \( \ell_0 = 0 \). Thus, the remaining constraints from Equation (4) and Equation (7) are:

\[
\begin{align*}
g_{20}\ell_2^2 + g_{10}\varphi_2 &= \ell_1\varphi_2 \\
g_{02}\ell_0^2 + g_{00} &= \ell_0
\end{align*}
\]

By using the second equation \( g_{20} = 1 - g_{10}\ell_1 \), the first one is equivalent to:

\[
g_{10}(\varphi_2 - \ell_1^2) = \ell_1(\varphi_2 - \ell_1)
\]

Because \( g_{10} \neq 0 \) and \( \ell_1 \neq 0 \), note that \( \varphi_2 = \ell_1^2 \iff \varphi_2 = \ell_1 \), in which case \( \ell_1^2 = \ell_1 \), thus \( \varphi_2 = \ell_1 = \pm 1 \). To remove the constraint on \( \varphi_2 \), we assume that \( \varphi_2 \neq \ell_1^2 \), and thus \( g_{10} \) and \( g_{02} \) depends only on \( \varphi_2 \) and \( \ell_1 \):

\[
g_{10} = \ell_1 \varphi_2 - \ell_1 \\
g_{02} = 1 - y_0\ell_1 = \ell_2 - \ell_1^2
\]

\[
g(x, y) = g_{10} y^2 + g_{10} x
\]

\[
= \frac{1}{\varphi_2 - \ell_1^2} (\varphi_2(1 - \ell_1^2)\ell_2^2 + \ell_1(\varphi_2 - \ell_1)\ell_1)
\]

Note that because \( \ell_1 \neq 0 \) and \( g_{10} \neq 0 \) we have to assume that, as opposed to the semi-interleaved cases, \( \ell_1 \neq \varphi_2 \). Moreover, because \( \ell_0 = 0 \), the constraints from the common Equations (5) and (6) can be simplified:

\[
\begin{align*}
f_{20} + f_{11}\ell_1 + f_{02}\ell_1^2 &= \varphi_2 \\
f_{20}\ell_1^2 + f_{11}\ell_1 + f_{02} &= \ell_1 \\
f_{10} + f_{01}\ell_1 &= 0 \\
f_{10}\ell_1 + f_{01} &= 0 \\
f_{00} &= 0
\end{align*}
\]

In order to have a non-trivial ladder \( \ell(x) = \ell_1 x \), we will assume that \( \ell_1 \neq \pm 1 \). With \( (L_4) - (L_3) \) we obtain \( (f_{10} - f_{01})(\ell_1 - 1) = 0 \) thus \( f_{10} = f_{01} \), and with \( (L_3) \) we have \( f_{10}(\ell_1 + 1) = 0 \), so \( f_{10} = f_{01} = 0 \). With \( (L_1) - (L_2) \) we
obtain $(f_{02} - f_{20})(\ell_1^2 - 1) = \varphi_2 - \ell_1$, so $f_{02} = f_{20} + \frac{\varphi_2 - \ell_1}{\ell_1^2 - 1}$.

Therefore, remain the following constraints:

$$
\begin{aligned}
    f_{20}(\ell_1^2 + 1) + f_{11}\ell_1 &= \varphi_2 - \ell_1 - \varphi_2 - \ell_1 \\
    &= \ell_1 - \varphi_2 - \ell_1 \\
    f_{02} &= f_{20} + \frac{\varphi_2 - \ell_1}{\ell_1^2 - 1} \\
    f_{10} &= f_{01} = f_{00} = 0
\end{aligned}
$$

The first line being obtained from $(L_1)$ and the second from $(L_2)$. Note that $\varphi_2 - \ell^2 \frac{\varphi_2 - \ell_1}{\ell_1^2 - 1} = \ell_1 - \frac{\varphi_2 - \ell_1}{\ell_1^2 - 1}$ is satisfied and thus is not a constraint. Moreover, because $\ell_1 \neq 0$, by using the second line, $f_{11}$ can be written as:

$$f_{11} = \frac{1}{\ell_1} \left( \frac{\ell_1^2 - \varphi_2 - \ell_1}{\ell_1^2 - 1} - f_{20}(\ell_1^2 + 1) \right) = \frac{1}{\ell_1} \left( \frac{\ell_1^2 - \varphi_2 - \ell_1}{\ell_1^2 - 1} - f_{20}(\ell_1^2 + 1) \right) t_1 \text{ mod } n \neq 1 \Rightarrow \text{invertible}$$

Thus, we have:

$$f(x, y) = f_{20}x^2 + f_{11}xy + f_{02}y^2$$

$$= f_{20}x^2 + \frac{1}{\ell_1} \left( \frac{\ell_1^2 - \varphi_2 - \ell_1}{\ell_1^2 - 1} - f_{20}(\ell_1^2 + 1) \right) xy $$

$$+ \left( f_{20} + \frac{\varphi_2 - \ell_1}{\ell_1^2 - 1} \right) y^2$$

$$= f_{20} \left( x^2 - \frac{\ell_1^2 + 1}{\ell_1} xy + y^2 \right) $$

$$+ \frac{1}{\ell_1^2 - 1} \left( \frac{\ell_1^2 - \varphi_2 - \ell_1}{\ell_1^2 - 1} xy + (\varphi_2 - \ell_1)y^2 \right)$$

Finally, $f_{20}$ has no constraint so, for sake of simplicity, we assume $f_{20} = 0$ and obtain the following functions:

$$f(x, y) = \frac{1}{\ell_1^2 - 1} \left( \frac{\ell_1^2 - \varphi_2 - \ell_1}{\ell_1^2 - 1} xy - (\ell_1 - \varphi_2)y^2 \right)$$

$$g(x, y) = \frac{1}{\ell_1^2 - \varphi_2} (\varphi_2(\ell_1^2 - 1)y^2 + \ell_1(\ell_1 - \varphi_2)x)$$

where $\ell_1 \neq \varphi_2$ and where $\ell_1, \ell_1^2 - 1$ and $\ell_1^3 - \varphi_2$ should be invertible. The former can be satisfied by checking whether $(\ell_1 - \varphi_2) \mod n = 0$. The latter can be satisfied by using the Extended Euclidean Algorithm $(d, u) \leftarrow \text{EEA}(v, n)$, where $u \equiv d \mod n$ such that if $d = 1$ then $u$ is invertible modulo $n$ and $v^{-1} = u \mod n$. Note that when $\ell_1$ is chosen, these coefficients are constant during the iterative conditional branching, thus they can be pre-computed at the beginning of the algorithm in Table XI. These computations cause an overhead that does not depend on the secret, thus trading execution time and energy consumption for more security. If no such $\ell_1$ exists then the algorithm returns an error.

We provide in Table XII a comparison of complexities for the Montgomery, semi- and fully-interleaved ladders. The complexities are given in terms of cost per key bit, where $M$ stands for multiplication, $S$ for squaring and $A$ for addition/subtraction, all modulo $n$. We did not include the cost of the pre-computations, which is far from negligible for the fully-interleaved ladder in Table XI.

While investigating the fully-interleaved cases, we found another solution shown but not detailed in Table XIII as additional material for this paper. It is a modular exponentiation for bases of the form $a = b^2(b^2 + b - 1)$ where $b$ is invertible, for instance, $b = n - 1$. Actually, if $b$ is not invertible then the initialization could be $x \leftarrow b$ and $\ell \leftarrow 1 \mod n$ instead, in which case the result would be $x = b^{d+1}(b^2(b^2 - b - 1))^{k \mod n}$ where $b^{d+1}$ is public. Note also that this time the condition is $k[i] = 0$.

VI. CONCLUSION AND FUTURE WORK

In this paper, we abstract away the algorithmic strength of the Montgomery ladder against side-channel and fault-injection attacks, by defining semi- and fully-ladderizable programs. We designed also fault-injection attacks able to obtain some/all bits of the secret key from the semi-interleaved ladders, like the Montgomery ladder, but none from the fully-interleaved ladders. As examples, we also provided for the modular exponentiation a better semi-interleaved ladder using a random mask updated at every iteration, a fully-interleaved ladder depending on an appropriate ladder constant, and a simpler fully-interleaved ladder for some bases.

The algorithm in Table XI is a general fully-interleaved ladder for the exponentiation, and depends on the existence of a ladder constant $\ell$. Its existence is a mathematical problem
input public b, n; secret k
1: x ← 1
2: y ← b⁻¹ mod n
3: for i = d to 0 do
4: if k[i] = 0 then
5: x ← b²(b + 1)y² - b²xy mod n
6: y ← (b + 1)x - b(b² + b - 1)y² mod n
7: else
8: y ← b²(b + 1)x² - b²gx mod n
9: x ← (b + 1)y - b(b² + b - 1)x² mod n
10: end if
11: end for
12: return x
output x = (b²(b² + b - 1))d mod n

TABLE XIII
Simpler Fully-Interleaved Ladder for Some Bases

that should be investigated to ensure that the proposed solution
is relevant. Moreover, obtaining a formula for ℓ depending only on a and n would reduce the overhead of the algorithm.
Because in RSA we have n = pq with p, q primes, verifying whether a solution exists or not might be easier, but a formula
for ℓ should not depend on p, q because they are secret.
We have only investigated the univariate case for con-
ditional branching, but the multivariate case may be of
interest. For instance, a multi-variable polynomial
\[ \sum_{0 \leq i \leq k} \sum_{n_1 + \ldots + n_k = n} c_{n_1, \ldots, n_k} \prod_{1 \leq i \leq k} x_i^{n_i} \]
can be represented by a multidimensional array of coefficients, and thus the manipulation of equations in Section V could be handled by using matrix operations.
Finally, the double-and-add and the secure bit permutation
algorithms mentioned in Subsection III-A could be investi-
gated as well, providing more examples to demonstrate the
generality of the method described in this paper.

ACKNOWLEDGEMENTS
The authors want to thank the reviewers for their relevant
and very useful comments.

REFERENCES
digital signatures and public-key cryptosystems,” Communications of the
RSA, DSS, and Other Systems,” in Advances in Cryptology — CRYPTO
’96, N. Koblitz, Ed. Berlin, Heidelberg: Springer Berlin Heidelberg, 1996,
pp. 104–113.
“Square Always Exponentiation,” in Progress in Cryptology – INDOCRYPT
2011, D. J. Bernstein and S. Chatterjee, Eds. Berlin, Heidelberg: Springer
the Right-to-Left Square-Always Exponentiation Algorithm,” J. Internet
leakage model,” in International workshop on cryptographic hardware
Curve Cryptosystems,” in Cryptographic Hardware and Embedded Systems.
C. K. Koç and C. Paar, Eds. Berlin, Heidelberg: Springer Berlin Heidelberg,
Attacks of Modular Exponentiation in Smartcards,” in Cryptographic
Hardware and Embedded Systems, C. K. Koç and C. Paar, Eds. Berlin,
against One Physical Cryptanalysis May Benefit Another Attack,” in
hit://hal.archives-ouvertes.fr/hal-00105562
Curve Methods of Factorization. Math. Comp. 48, 243–264,” Mathemat-
Languages,” Fundamenta Informaticae, vol. 168, no. 1, pp. 51–77, July
2019.
Smart Cards, Mobile, and Wireless Computing Systems, D. Sauveron,
K. Markantonakis, A. Bilas, and J.-J. Quisquater, Eds. Berlin, Heidel-
[14] M. Joyce, “Highly Regular Right-to-Left Algorithms for Scalar Multi-
P. Paillier and I. Verbauwhede, Eds. Berlin, Heidelberg: Springer
[15] ———, “Highly Regular m-Ary Powering Ladders,” in Selected Areas in
[16] C. D. Waller, “The Montgomery and Joye Powering Ladders are Dual,”
IACR ePrint archive, vol. 1081, pp. 1–6, 2017. [Online]. Available:
Cryptographic Hardware and Embedded Systems - CHES 2002, B. S.
Kaliski, c. K. Koç, and C. Paar, Eds. Berlin, Heidelberg: Springer
[18] A. Kiss, J. Krämer, P. Rauzy, and J.-P. Seifert, “Algorithmic Counter-
measures Against Fault Attacks and Power Analysis for RSA-CRT,” in
Constructive Side-Channel Analysis and Secure Design. F-X. Standaert
and E. Oswald, Eds. Cham: Springer International Publishing, 2016,
pp. 111–129.
to Simple Power Analysis,” IEEE Transactions on Computers, vol. 55,
[20] V. S. Miller, “Use of Elliptic Curves in Cryptography,” in Advances in
Cryptology — CRYPTO ’85 Proceedings, H. C. Williams, Ed. Berlin,
“Side Channels in the McEliece PKC,” in Post-Quantum Cryptography,
J. Buchmann and J. Ding, Eds. Berlin, Heidelberg: Springer Berlin
[23] M. Petrvalsky, T. Richmond, M. Drutarovsky, P. Cayrel, and V. Fischer,
“Differential power analysis attack on the secure bit permutation in
the McEliece cryptosystem,” in 2016 26th International Conference
Radioelektronika (RADIOELEKTRONIKA), April 2016, pp. 132–137.
Correlation Power Analysis on the Message Blinding Method and Its
Chain-Based Exponentiation Algorithms Using a Single Trace,” in
Topics in Cryptology — CT-RSA 2015, K. Nyberg, Ed. Cham: Springer
169–184.
[27] J. Arlat, “Validation de la sûreté de fonctionnement par injection de
fautes : méthode, mise en oeuvre, application,” Ph.D. dissertation,
Institut national polytechnique (Toulouse, France), 1990.