NON-SIMPLICIAL QUANTUM TORIC VARIETIES

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Abstract. In this paper, we define quantum toric varieties associated to an arbitrary fan in a finitely generated subgroup of some $\mathbb{R}^d$ generalizing the article [KLMV20] of Katzarkov, Lupercio, Meersseman and Verjovsky.

Contents

1. Introduction

A toric variety is a complex algebraic variety with an action of an algebraic torus $(\mathbb{C}^*)^n$ with a Zariski open orbit isomorphic to this torus. Such a variety can be described by a fan of rational strongly convex polyhedral cones on a lattice Γ (see, for example, [CLS11] or [Ful93]).

More precisely, there is an equivalence of categories between the category of toric varieties and the category of fans. This central result gives us a dictionary between geometric properties of toric varieties and combinatorial properties of fans making of toric varieties one of the most studied class of complex algebraic varieties.

However, in the classical theory, this fan has to be rational. Therefore the toric varieties are rigid i.e. we can not deform them. Indeed, if we deform a lattice, it can become not discrete (for instance, the group $\mathbb{Z} + \alpha\mathbb{Z}$ is discrete if α is rational but is dense in $\mathbb{R}$ if α is irrational)

The authors of [KLMV20] gave a construction of "quantum toric variety" (which is a stack) described by a simplicial fan (i.e. the 1-cones of

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each cones of the fan are $\mathbb{R}$-linearly independent) on a finitely generated (possibly irrational) subgroup of some $\mathbb{R}^d$ (named "quantum fan"). When the fan is rational, one recovers the classical toric variety but the case of irrational simplicial fans is also covered. This construction is functorial and defines an equivalence of categories between the category of quantum toric varieties and the category of quantum simplicial fans (see theorems 5.18 and 6.24 of [KLMV20]).

Now, the simplicial fans form only a small part of all the fans. In the classical theory, they correspond to the toric varieties which are orbifold (i.e. with cyclic singularities). Hence, the restriction to simplicial fans is strong.

In this paper, we extend the construction of [KLMV20] to the general case i.e. we define the quantum toric variety associated to an arbitrary fan.

The non-simplicial case brings new problems. Indeed, in the simplicial case, the family of 1-cones of a cone is $\mathbb{R}$-linearly free and can be completed in a basis of $\mathbb{R}^d$. Hence, up to isomorphism, a simplicial cone is a standard cone $\mathrm{Cone}(e_1, \ldots, e_k) \subset \mathbb{R}^d$ (where $\{e_1, \ldots, e_d\}$ is the canonical basis of $\mathbb{R}^d$). Moreover, we can easily find the faces of these cones: it is the cones generated by a subfamily of $\{e_1, \ldots, e_k\}$. This basic fact is repeatedly used in construction of [KLMV20]. It is not the case for non-simplicial case anymore. Indeed, as the 1-cones of non-simplicial cone is linearly dependant, we do not have a notion of standard cone and the cones generated by a subfamily of the 1-cones is not necessarily a face of the cone since the cone can have an arbitrarily big number of 1-cones.

One of the key technical points is to replace the calibration h of the group Γ used in [KLMV20] by a calibration φ of a group isomorphic to Γ but which is a subgroup of an higher dimensional $\mathbb{R}^p$ for each maximal cone of the fan (p is the number of 1-cones of the maximal cone).

In section 3, we give a suitable definition of quantum tori in order to deal with the non-simplicial case and more precisely, these quantum tori encode the change of calibration of the last paragraph.

In section 4, we define the affine quantum toric variety (which will be calibrated as we have to keep track of the relations between the 1-cones) associated to a (non-simplicial) cone and more generally, the quantum toric variety with the descent data of the affine pieces associated to a quantum fan. In particular, this construction works for simplicial fans and coincide with that of [KLMV20].

In the end of this section, we prove the main theorem of this paper (extending the similar theorem of [KLMV20]) :

Theorem 1.1. *The category of calibrated quantum fans and the category of (calibrated) quantum toric varieties are equivalent.*

Finally, we realize these quantum toric varieties as a global quotient in section 5 and as a gerbe over a "non-calibrated quantum toric varieties" associated to it (i.e. where we replace the action of $\mathbb{Z}^{N-d}$ through the epimorphism φ by the action of $\varphi(\mathbb{Z}^{N-d})$ and hence removing the ineffectivity of the action) in section 6.

2. Notations and conventions

2.1. **Stacks.** We will take the same conventions on stacks as [KLMV20] :

Let $\mathfrak{A}$ be the category of affine toric varieties with toric morphisms. We endow this category a structure of site with the coverings $\{U_i \hookrightarrow T\}_{i\in\{1,\dots,n\}}$, where U_i are open subsets of T and $T = \bigcup_{i=1}^{n} U_i$.

Let $\mathfrak{G}$ be the category of complex analytic spaces with an action of an abelian complex Lie group G with an open orbit isomorphic to G.

Definition 2.1. Let H be an abelian Lie group and X be an object of $\mathfrak{G}$ with an action of H commute with the action of G.
The stack $[X/H]$ is the stack over $\mathfrak{A}$ whose objects over T are H-principal bundle $\widetilde{T} \to T$ (in $\mathfrak{G}$) with an H-equivariant morphism $\widetilde{T} \to X$ and morphisms over $S \to T$ are a bundle morphism $\widetilde{S} \to \widetilde{T}$ compatible with the equivariant maps.

All the stacks in this paper will be of this form or given by the descent data of stacks of this form.

2.2. **Fans.** We will recall some definitions on (calibrated) quantum fans :

Definition 2.2. Let Γ be a finitely generated subgroup of $\mathbb{R}^d$ such that $\mathrm{Vect}_{\mathbb{R}}(\Gamma) = \mathbb{R}^d$. A calibration of Γ is given by :

- A group epimorphism $h\colon \mathbb{Z}^N \to \Gamma$
- A subset $\mathcal{I} \subset \{1,\dots,N\}$ such that $\mathrm{Vect}_{\mathbb{C}}(h(e_j), j \notin \mathcal{I}) = \mathbb{C}^d$ (this is the set of virtual generators)

This is a standard calibration if $\mathbb{Z}^d \subset \Gamma$, $h(e_i) = e_i$ for $i = 1,\dots,d$ and $\mathcal{I}$ is of the form $\{n - |\mathcal{I}| + 1,\dots,n\}$

Definition 2.3. A calibrated quantum fan $(\Delta, h, \mathcal{I})$ in Γ is the data of

- a collection Δ of strongly convex polyhedral cones generated by elements of Γ such that every intersection of cones of Δ is a cone of Δ, every face of a cone of Δ is a cone and $\{0\}$ is a cone of Δ.
- a standard calibration h with $\mathcal{I}$ its set of virtual generators
- A set of generators A i.e. a subset of $\{1,\dots,N\} \setminus \mathcal{I}$ such that the 1-cone generated by the $h(e_i)$ for $i \in A$ are exactly the 1-cones of Δ

Let $h : \mathbb{Z}^N \to \Gamma$ a calibration (with $\mathcal{I}$ as set of virtual generators) and $(\Delta, h, \mathcal{I})$ a calibrated quantum fan in Γ. Note Δ_h the fan in $\mathbb{Z}^N$ such that

$$\mathrm{Cone}(e_i, i \in I) \in \Delta_h \text{ if, and only if, } \mathrm{Cone}(h(e_i), i \in I) \in \Delta \tag{2.1}$$

Definition 2.4. A morphism of calibrated quantum fans between $(\Delta, h, \mathcal{I})$ in Γ and $(\Delta', h', \mathcal{I}')$ is a pair of linear morphisms $(L : \mathbb{R}^d \to \mathbb{R}^{d'}, H : \mathbb{R}^N \to \mathbb{R}^{N'})$ where

- The diagram

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{h_{\mathbb{R}}} & \mathbb{R}^d \\ \downarrow H & & \downarrow L \\ \mathbb{R}^{n'} & \xrightarrow{h'_{\mathbb{R}}} & \mathbb{R}^{d'} \end{array} \tag{2.2}$$

  commutes (where $h_{\mathbb{R}}$ (resp. $h'_{\mathbb{R}}$) is the $\mathbb{R}$-linear map associated to h (resp. to h'))
- $L(\Gamma) \subset \Gamma'$
- If $\sigma \in \Delta$ then it exists $\sigma' \in \Delta'$ such that $L(\sigma) \subset \sigma'$ (thanks to the first point, the same statement is true for H)
- If $H(e_i) \in \mathrm{Cone}(e_j, j \in \mathcal{I}) \in \Delta_h$ then $H(e_i) \in \bigoplus_{j \in \mathcal{I}'} \mathbb{N}e_j$ (same statement is true for L and the vectors $h(e_i)$, thanks to the first point)
- For all $i \notin \mathcal{I}$, $H(e_i) \in \bigoplus_{j \notin \mathcal{I}'} \mathbb{Z}e_j$
- There exists a map $s : \mathcal{I} \to \mathcal{I}'$ such that for all $i \in \mathcal{I}, H(e_i) = e_{s(i)}$

Remark 2.5. We can define non-calibrated counterpart of these definitions by replacing the data of the calibration by the data of the 1-cones $h(e_i)$, $i \in \{1, \ldots, N\}$

3. Quantum tori

Let Γ a finitely generated subgroup of $\mathbb{R}^d$ and $h : \mathbb{Z}^N \to \Gamma$ a calibration of Γ (note $\mathcal{I}$ the set of virtual generators).
The standard calibrated quantum torus associated to these data is the quotient stack $\mathcal{T}^{cal}_{h,\mathcal{I}} = [\mathbb{C}^d/\mathbb{Z}^N]$ where $\mathbb{Z}^N$ acts on $\mathbb{C}^d$ through the morphism h. However, since a non-simplicial cone can have more than d generators, we have to consider quantum tori as a quotient stack of a higher-dimensional space $\mathbb{C}^p$. In order to do this, we will replace the group Γ and its calibration h by a subgroup G of $\mathbb{R}^p$ isomorphic to it with a calibration φ of it. Then, we describe these quantum tori by the data of the underlying standard quantum torus (in order to keep track of the combinatorial data of the group Γ) with the data of a stack isomorphism, respecting the virtual generators, with such quotient stack. More precisely,

Definition 3.1. A presented calibrated quantum torus is a 6-uple $(\mathcal{T}^{cal}_{h,\mathcal{I}}, \varphi : \mathbb{Z}^N \to G \subset \mathbb{C}^p, \mathcal{I}', L, H, s)$ where φ is a calibration of the group G (with $\mathcal{I}'$ its set of virtual generators), $L : \mathbb{R}^p \to \mathbb{R}^d$ is a linear epimorphism, $H : \mathbb{Z}^N \to \mathbb{Z}^N$ is a group isomorphism and $s : \mathcal{I} \to \mathcal{I}'$ is a bijection such that :

- $L(G) = \Gamma$ and $L_{|G} : G \to \Gamma$ is a group isomorphism.
- The diagram $\begin{array}{ccc} \mathbb{Z}^N & \xrightarrow{H} & \mathbb{Z}^N \\ \downarrow \varphi & & \downarrow h \\ G & \xrightarrow{L_{|G}} & \Gamma \end{array}$ is commutative.
- For all $i \in \mathcal{I}, H(e_i) = e_{s(i)}$ and for all $i \notin \mathcal{I}$, $H(e_i) \in \bigoplus_{j \notin \mathcal{I}'} \mathbb{Z}e_j$

The morphism φ is the calibration of this presented calibrated quantum torus.

With these data, we can define the quotient stack $[\mathbb{C}^p/\mathbb{Z}^N \times \ker(L \otimes_{\mathbb{R}} id_{\mathbb{C}})]$, where $\mathbb{Z}^N \times \ker(L \otimes_{\mathbb{R}} id_{\mathbb{C}})$ acts on $\mathbb{C}^p$ through

$$(m, w) \cdot z = z + \varphi(m) + w,$$

Moreover, these data encode a stack isomorphism

$$[\mathbb{C}^p/\mathbb{Z}^N \times \ker(L \otimes_{\mathbb{R}} id_{\mathbb{C}})] \simeq \mathcal{T}^{cal}_{h,\mathcal{I}}.$$

If L is a linear isomorphism then this stack isomorphism is a calibrated torus isomorphism as defined in [KLMV20].

We can define in the same way non-calibrated quantum tori :

Definition 3.2. A presented non-calibrated quantum torus is a couple $(\mathcal{T}_{d,\Gamma}, L)$ where L is a linear epimorphism $\mathbb{R}^p \to \mathbb{R}^d$

In particular, the linear morphism descends to a stack isomorphism

$$[\mathbb{C}^p/L_{\mathbb{C}}^{-1}(\Gamma)] \simeq \mathcal{T}_{d,\Gamma}$$

(where $L_{\mathbb{C}} = L \otimes_{\mathbb{R}} id_{\mathbb{C}}$ and $L_{\mathbb{C}}^{-1}(\Gamma)$ acts on $\mathbb{C}^p$ by translations) which is a quantum torus isomorphism (in the sense of [KLMV20]) if L is a linear isomorphism.

In both cases, we can define a multiplicative form of the torus thanks to the exponential map

$$E : (z_1, \dots, z_p) \in \mathbb{C}^p \mapsto (e^{2i\pi z_1}, \dots, e^{2i\pi z_p}) \in \mathbb{T}^p := (\mathbb{C}^*)^p$$

(1) In the non-calibrated case, we have an isomorphism (see diagram 3.12 of [KLMV20]) :

$$[\mathbb{C}^p/L_{\mathbb{C}}^{-1}(\Gamma)] \simeq [\mathbb{T}^p/E(L_{\mathbb{C}}^{-1}(\Gamma))]$$

where $E(L_{\mathbb{C}}^{-1}(\Gamma))$ acts multiplicatively on $\mathbb{T}^p$.

(2) In the calibrated case, we have to suppose h standard i.e. we suppose that there is a subset $\widetilde{I} = \{i_1, \dots, i_d\} \subset \{1, \dots, p\}$ such that $\mathbb{Z}^{\widetilde{I}} \subset G$ and $h(e_k) = e_{i_k}$ for $k = 1..d$. Then, the exponential map gives us :

$$[\mathbb{C}^p/\mathbb{Z}^N \times \ker(L_{\mathbb{C}})] \simeq [\mathbb{T}^p/\mathbb{Z}^{N-d} \times E(\ker(L_{\mathbb{C}}))]$$

where $\mathbb{Z}^{N-d} \times E(\ker(L_{\mathbb{C}}))$ acts on $\mathbb{T}^p$ through

$$(m, E(w)) \cdot z = E(\varphi(0 \oplus m) + w)z$$

Definition 3.3. A morphism of presented non-calibrated quantum tori

$$(\mathcal{T}_{d,\Gamma}, L) \to (\mathcal{T}_{d',\Gamma'}, L')$$

or presented torus morphism is a couple $(\mathcal{L}, \mathcal{L}')$ of linear morphisms such that

$$\begin{array}{ccc}
\mathbb{C}^d & \xleftarrow{[L_\mathbb{C}]} & \mathbb{C}^p / \ker(L_\mathbb{C}) \\
\downarrow \mathcal{L} & & \downarrow \mathcal{L}' \\
\mathbb{C}^{d'} & \xleftarrow[{[L'_\mathbb{C}]}]{} & \mathbb{C}^{p'} / \ker(L'_\mathbb{C})
\end{array}$$

where $[L]$ is the morphism induced by L on the quotient.

Definition 3.4. A morphism of presented calibrated quantum tori

$(\mathcal{T}^{cal}_{h,\mathcal{I}}, \varphi : \mathbb{Z}^N \to G \subset \mathbb{C}^p, \widetilde{\mathcal{I}}, L, H, s) \to (\mathcal{T}^{cal}_{h',\mathcal{I}'}, \varphi' : \mathbb{Z}^{N'} \to G' \subset \mathbb{C}^{p'}, \widetilde{\mathcal{I}}', L', \varphi, s')$

or presented calibrated torus morphism is a 6-uple $(\mathcal{L}, \mathcal{H}, \mathcal{S}, \mathcal{L}', \mathcal{H}', \mathcal{S}')$ of morphisms such that

- $(\mathcal{L}, \mathcal{H}, \mathcal{S})$ induces a torus morphism $\mathcal{T}^{cal}_{h,\mathcal{J}} \to \mathcal{T}^{cal}_{h',\mathcal{J}'}$ as defined in [KLMV20] (definition 3.11),
- for all $j \in \widetilde{\mathcal{I}}, \mathcal{H}'(e_j) = e_{\mathcal{S}'(j)}$ and for all $j \notin \widetilde{\mathcal{I}}$,
$$\mathcal{H}'(e_j) \in \bigoplus_{k \notin \widetilde{\mathcal{I}}'} \mathbb{Z} e_k$$
- The following diagrams commute :

$$\begin{array}{ccccccc}
\mathbb{C}^N & \xleftarrow{H} & \mathbb{C}^N & \xrightarrow{h} & \mathbb{C}^d & \xleftarrow{[L_\mathbb{C}]} & \mathbb{C}^p / \ker(L_\mathbb{C}) \\
\downarrow \mathcal{H}' & & \downarrow \mathcal{H} & & \downarrow \mathcal{L} & & \downarrow \mathcal{L}' \\
\mathbb{C}^{N'} & \xleftarrow[H']{} & \mathbb{C}^{N'} & \xrightarrow[h']{} & \mathbb{C}^{d'} & \xleftarrow[{[L'_\mathbb{C}]}]{} & \mathbb{C}^{p'} / \ker(L'_\mathbb{C})
\end{array}$$

$$\begin{array}{ccc}
\mathcal{I} & \xrightarrow{s} & \widetilde{\mathcal{I}} \\
\mathcal{S} \downarrow & & \downarrow \mathcal{S}' \\
\mathcal{I}' & \xrightarrow{s'} & \widetilde{\mathcal{I}}'
\end{array}$$

In particular, we have the following commutative diagram showing the relation between the different calibrations :

$$\begin{array}{ccccccc}
\mathbb{Z}^N & \xrightarrow{H} & \mathbb{Z}^N & \xrightarrow{\mathcal{H}} & \mathbb{Z}^{N'} & \xrightarrow{H'^{-1}} & \mathbb{Z}^{N'} \\
\downarrow \varphi & & \downarrow h & & \downarrow h' & & \downarrow \varphi' \\
G & \xrightarrow[L]{} & \Gamma & \xrightarrow[\mathcal{L}]{} & \Gamma' & \xrightarrow[(L'_{|G'})^{-1}]{} & G'
\end{array}$$

We can reformulate this in term of quotient stack :

Lemma 3.5. *We use the same notations as definition 3.4.*
Let $\ell' : [\mathbb{C}^p / \mathbb{Z}^N \times \ker(L_\mathbb{C})] \to [\mathbb{C}^{p'} / \mathbb{Z}^{N'} \times \ker(L'_\mathbb{C})]$ *be the stack morphism described by the linear morphisms* $(\mathcal{L}', \mathcal{H}')$ *and* $\ell^{cal} : \mathcal{T}^{cal}_{d,\Gamma} \to \mathcal{T}^{cal}_{d',\Gamma'}$ *be the stack morphisms described by the linear morphisms* $(\mathcal{L}, \mathcal{H})$*. Then, the diagram*

$$\begin{array}{ccc} [\mathbb{C}^p/\mathbb{Z}^N \times \ker(L_{\mathbb{C}})] & \xrightarrow{\ell'} & [\mathbb{C}^{p'}/\mathbb{Z}^{N'} \times \ker(L'_{\mathbb{C}})] \\ \downarrow \simeq & & \downarrow \simeq \\ \mathcal{T}^{cal}_{d,\Gamma} & \xrightarrow{\ell^{cal}} & \mathcal{T}^{cal}_{d',\Gamma'} \end{array}$$

commutes (where the isomorphisms are the stack isomorphisms encoded by the presentation of the presented calibrated quantum tori).

The presented quantum tori are (essentially) not new quantum tori. In fact, all presented quantum tori with the same underlying quantum tori are isomorphic :

Lemma 3.6. $(\mathcal{T}^{cal}_{h,\mathcal{I}}, \varphi : \mathbb{Z}^N \to G \subset \mathbb{C}^p, \mathcal{I}', L, H, s)$ *and* $(\mathcal{T}^{cal}_{h,\mathcal{I}}, h : \mathbb{Z}^N \to \Gamma, \mathcal{I}, id, id, id)$ *(resp.* $(\mathcal{T}_{d,\Gamma}, L)$ *and* $(\mathcal{T}_{d,\Gamma}, id)$*) are isomorphic.*

Proof. An isomorphism is given by $(id, id, id, [L], H, s)$ (resp. $(id, [L])$). □

Proposition 3.7. *The forgetful functor*

$$(\mathcal{T}^{cal}_{h,\mathcal{I}}, \varphi : \mathbb{Z}^N \to G, \mathcal{I}', L, H, s) \to \mathcal{T}^{cal}_{h,\mathcal{I}},$$

resp. $(\mathcal{T}_{d,\Gamma}, L) \to \mathcal{T}_{d,\Gamma}$*, is an equivalence of categories between the category of presented calibrated quantum tori and the category of standard calibrated quantum tori, resp. between the category of non-calibrated quantum tori and the category of non-calibrated standard quantum tori.*

Proof. Use lemma 3.6 □

4. Definition of (non-simplicial) quantum toric varieties

Let Γ be a finitely generated subgroup of $\mathbb{R}^d$ such that $\mathrm{Vect}_{\mathbb{R}}(\Gamma) = \mathbb{R}^d$, $(\Delta, h^{cal} : \mathbb{Z}^N \to \Gamma, \mathcal{I})$ a standard calibrated quantum fan (with $\mathcal{I}$ its set of virtual generators). In order to define a quantum variety associated to this fan, we will describe the affine quantum variety associated to each cone of this fan. The crucial point is to replace the calibration h of Γ by an adapted calibration φ for each maximal cone σ of the fan Δ.

After that, we will define the quantum toric varieties with the descent data of affine pieces as [KLMV20].

At the end of this section, we will prove that the correspondence

$$(\Delta, h^{cal} : \mathbb{Z}^N \to \Gamma, \mathcal{I}) \mapsto \mathcal{X}^{cal}_{\Delta, h^{cal}, \mathcal{I}}$$

is an equivalence of categories.

4.1. Affine quantum toric varieties.

Let

$$\sigma = \sigma_I = \mathrm{Cone}(v_{i_1} := h^{cal}(e_{i_1}), \ldots, v_{i_p} := h^{cal}(e_{i_p}))$$

(with $I := \{i_1, \ldots, i_p\} \subset \{1, \ldots, N\}$) be a strongly convex cone of Δ. There are two possibilities : σ is of maximal dimension i.e. of dimension d or not. In the latter, we have to do a choice of a completion of $\mathrm{Vect}_{\mathbb{R}}(\sigma)$ in

$\mathbb{R}^d$. Fortunately, all obtained quantum varieties are isomorphic and this isomorphism respects a cocycle condition.

4.1.1. *Cones of dimension d.* Suppose σ is a cone of dimension d i.e. $\text{Vect}_{\mathbb{R}}(\sigma) = \mathbb{R}^d$.
Note $h_\sigma : \mathbb{Z}^I \to \Gamma$ the restriction of h^{cal} on $\mathbb{Z}^I$, $h_{\sigma\mathbb{C}} : \mathbb{C}^I \to \mathbb{C}^d$ the $\mathbb{C}$-linear map associated to it and $\widehat{\sigma}$ the cone of $\mathbb{R}^N$ defined by

$$\widehat{\sigma} = \text{Cone}(e_i, i \in I) \tag{4.1}$$

and $\widetilde{\sigma}$ its restriction on $\mathbb{R}^I$

Let $\mathcal{B} = (v_i, i \in \widetilde{I})$ a subfamily of $(v_1, \dots, v_p)$ which is a basis of $\mathbb{C}^d$. Then, we will consider the decomposition $\mathbb{C}^I = \mathbb{C}^{\widetilde{I}} \oplus \ker(h_{\sigma\mathbb{C}})$. The map $h_{\sigma\mathbb{C}}$ induces a linear isomorphism ψ between $\mathbb{C}^{\widetilde{I}}$ and $\mathbb{C}^d$.
Let $\chi \in \mathfrak{S}_N$ be a permutation such that $\chi(\{1, \dots, d\}) = \widetilde{I}$.

Definition 4.1. The linear morphism $\varphi : \mathbb{C}^N \to \mathbb{C}^{\widetilde{I}} \hookrightarrow \mathbb{C}^I$ defined by

$$e_k \mapsto \psi^{-1}(h_{\mathbb{C}}^{cal}(e_{\chi(k)}))$$

is called calibration associated to σ (and $\mathcal{B}$ and χ)

We can think this morphism as a calibration induced by the calibration h on $\mathbb{C}^I$. Indeed, the image of $\mathbb{Z}^d \oplus 0$ by the morphism φ is $\mathbb{Z}^{\widetilde{I}}$ (by construction) and the group $\varphi(\mathbb{Z}^N)$ is isomorphic to Γ.

We can define an action of $\mathbb{Z}^{N-d} \times E(\ker(h_{\sigma\mathbb{C}}))$ on $\mathbb{C}^I$ by setting for $(m, E(t)) \in \mathbb{Z}^{N-d} \times E(\ker(h_{\sigma\mathbb{C}}))$ and $z \in \mathbb{C}^I$,

$$(m, t) \cdot z = E(\varphi(0 \oplus m)) + t)z$$

Remark 4.2. The non-effectiveness of this action only comes from the calibration i.e. the subgroup of ineffectivity of this action is $\ker(h^{cal})$.

The action of $\mathbb{T}^I$ on $\mathbb{C}^I = U_{\widetilde{\sigma}}$ commutes with this action. Hence, we can form the quotient $[\mathbb{C}^I / \mathbb{Z}^{N-d} \times E(\ker(h_{\sigma\mathbb{C}}))]$.

Lemma 4.3. *The quotient stack $[\mathbb{C}^I / \mathbb{Z}^{N-d} \times E(\ker(h_{\sigma\mathbb{C}}))]$ depends neither on the choice of the basis $\mathcal{B}$ nor on the permutation χ. Hence, the action induced by the morphism φ depends only on σ.*

Proof. Let $(\mathcal{B} = (v_i, i \in \widetilde{I}), \chi)$ and $(\mathcal{B}' = (v_i, i \in \widetilde{I}'), \chi')$ be two pairs of basis and permutation. Then we can define two associated isomorphisms $\psi : \mathbb{C}^{\widetilde{I}} \to \mathbb{C}^d$ and $\psi' : \mathbb{C}^{\widetilde{I}'} \to \mathbb{C}^d$ and two morphisms $\varphi : \mathbb{C}^N \to \mathbb{C}^I$, $\varphi' : \mathbb{C}^N \to \mathbb{C}^{I'}$. Then, the identity is a α-equivariant morphism, where α is the morphism $\mathbb{Z}^{N-d} \times E(\ker(h_\sigma)) \to \mathbb{Z}^{N-d} \times E(\ker(h_\sigma))$ defined by

$$\alpha(m, E(y)) = (P_{\chi'}^{-1} P_\chi(m), E(\varphi(m) - \psi'^{-1} h(\varphi(m))) E(y))$$

where P_χ is the linear map associated to χ.

Indeed, the following diagram is commutative

$$\begin{array}{ccc} E(h_\sigma^{-1}(\Gamma)) & \xrightarrow{\quad = \quad} & E(h_\sigma^{-1}(\Gamma)) \\ \text{action through } \varphi \Big\uparrow & & \Big\uparrow \text{action through } \varphi' \\ \mathbb{Z}^{N-d} \times E(\ker(h_\sigma)) & \xrightarrow{\alpha} & \mathbb{Z}^{N-d} \times E(\ker(h_\sigma)) \end{array}$$

□

Definition 4.4. The stack $\mathcal{U}_\sigma^{cal} := [\mathbb{C}^I / \mathbb{Z}^{N-d} \times E(\ker(h_{\sigma\mathbb{C}}))]$ is the quantum toric variety associated to the cone σ and to the calibration $h^{cal} : \mathbb{Z}^N \to \Gamma$.

Remark 4.5. If σ is non-simplicial (i.e. $\ker(h_{\sigma\mathbb{C}})$ is non-empty and thus is not discrete) then the quantum toric variety associated to σ cannot be describe as a quasifold (i.e. locally the quotient of a space $\mathbb{R}^n$ by a discrete subgroup, see [Pra01]) in contrast to the simplicial case.

The associated "torus" to $\mathcal{U}_\sigma^{cal}$ is the stack $[\mathbb{T}^I / \mathbb{Z}^{N-d} \times E(\ker(h_{\sigma\mathbb{C}}))]$. We can endow this stack with a structure of presented calibrated quantum torus thanks to the morphisms used to define $\mathcal{U}_\sigma^{cal}$:

Proposition 4.6. *$(\mathcal{T}_{h,\mathcal{I}}^{cal}, \varphi : \mathbb{Z}^N \to \psi^{-1}(\Gamma), h_{\sigma\mathbb{C}}, \chi^{-1}(\mathcal{I}), P_\chi : e_i \mapsto e_{\chi(i)}, \chi)$ is a presented calibrated quantum torus which encodes the stack $[\mathbb{T}^I / \mathbb{Z}^{N-d} \times E(\ker(h_{\sigma\mathbb{C}}))]$*

Proof. This proposition comes from the fact that ψ is induced by $h_{\sigma\mathbb{C}}$ on $\mathbb{C}^I / \ker(h_{\sigma\mathbb{C}}) \simeq \mathbb{C}^{\tilde{I}}$ and the equality $\varphi = \psi^{-1} h P_\chi$. □

Notation 4.7. In what follows, we will omit the isomorphism and just write $[\mathbb{T}^p / \mathbb{Z}^{N-d} \times E(\ker(h_{\sigma\mathbb{C}}))]$ instead of this 6-uple.

Warning 4.8. *In the classical non-simplicial case (i.e. Γ is a lattice and the calibration is an isomorphism), the stack $\mathcal{U}_\sigma^{cal}$ is not a (stack representable by a) variety.*

However, if we replace the stack quotient by the GIT quotient, we get the classical toric variety associated to σ :

Proposition 4.9. *If Γ is discrete, h is an isomorphism and the set of virtual generators is empty, then we can define the (classical) toric variety U_σ associated to σ (and Γ) and*

$$U_\sigma = \mathbb{C}^I /\!\!/ \mathbb{Z}^{N-d} \times E(\ker(h_{\sigma\mathbb{C}}))$$

where the $/\!\!/$ denotes the GIT quotient. Moreover, all the points of $\mathbb{C}^I$ are semi-stable (see the definition in [MFK94]*) for the action of $\mathbb{Z}^{N-d} \times E(\ker(h_{\sigma\mathbb{C}}))$ for the trivial line bundle $\mathbb{C}^I \times \mathbb{C} \to \mathbb{C}^I$ with the linearization defined for all $(m, t) \in \mathbb{Z}^{N-d} \times E(\ker(h_{\sigma\mathbb{C}}))$ and for $(z, w) \in \mathbb{C}^I \times \mathbb{C}$,*

$$(m, t) \cdot (z, w) = ((m, t) \cdot z, E(\langle \varphi(0 \oplus m) + t, a \rangle) w)$$

for any $a \in \mathbb{Z}^I$.

Proof. Since h is an isomorphism, the action of the group $\mathbb{Z}^{n-d} \times E(\ker(h_{\sigma\mathbb{C}}))$ on $\mathbb{C}^I$ is the same action as the action of $E(h_{\sigma\mathbb{C}}^{-1}(\Gamma))$ on it thanks to the equality

$$\{E(\varphi(0 \oplus m) + t) \mid m \in \mathbb{Z}^{N-d}, E(t) \in E(\ker(h_{\sigma\mathbb{C}}))\} = E(h_{\sigma\mathbb{C}}^{-1}(\Gamma)). \quad (4.2)$$

If we suppose Γ discrete, we can suppose too that $\Gamma = \mathbb{Z}^d$ without loss of generality. In [CLS11] (theorem 5.1.11), the authors prove that $\mathbb{C}^I \to U_\sigma$ is an almost geometric quotient for the action of G defined by

$$G := \mathrm{Hom}_{\mathbb{Z}}(\mathrm{coker}(M = (v_i, i \in I)^T : \mathbb{Z}^I \to \mathbb{Z}^d), \mathbb{C}^*)$$

(The cokernel in this expression is also defined as the class group of the variety U_σ see [CLS11] p172/173).

Since $\mathbb{C}^*$ is a divisible group, we have a exact sequence :

$$0 \longrightarrow G \longrightarrow \mathbb{T}^I \xrightarrow{\varphi} \mathbb{T}^d \longrightarrow 0 \tag{4.3}$$

where the morphism φ is the induced morphism by $\mathrm{Hom}(M, \mathbb{C}^*)$ and the isomorphisms $\mathrm{Hom}(\mathbb{Z}^I, \mathbb{C}^*) \to \mathbb{T}^I$, $(u : \mathbb{Z}^I \to \mathbb{C}^*) \mapsto (u(e_i), i \in I)$ and $\mathrm{Hom}(\mathbb{Z}^d, \mathbb{C}^*) \to \mathbb{T}^d$, $(u : \mathbb{Z}^d \to \mathbb{C}^*) \mapsto (u(e_1), \ldots, u(e_d))$.

Then, we deduce from the equality $h_{\sigma\mathbb{C}} = M_{\mathbb{C}}^T$ that φ is the morphism $\mathbb{T}^I \to \mathbb{T}^d$ induced by h. Finally, thanks to the exact sequence (4.3), we get the isomorphism $G \simeq E(h_{\sigma\mathbb{C}}^{-1}(\mathbb{Z}^d))$ (since $\ker(E) = \mathbb{Z}^d$)
The semi-stability is proved in chapter 12 of [Dol03]. □

We cannot have the equality

$$\mathbb{C}^I /\!\!/ \mathbb{Z}^{N-d} \times E(\ker(h_{\sigma\mathbb{C}})) = [\mathbb{C}^I / \mathbb{Z}^{N-d} \times E(\ker(h_{\sigma\mathbb{C}}))].$$

Indeed, the categorical quotient (for the action of G) $\mathbb{C}^I \to U_\sigma$ is geometric if, and only if, σ is a simplicial cone because the set of semi-stable points is equal to the set of stable points if, and only if, the cone is simplicial (see [Dol03] proposition 12.1)

Example 4.10. Let

$$\Gamma = \mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3 + \mathbb{Z}(ae_1 - be_2 + ce_3) + \sum_{i=5}^{N} \mathbb{Z}v_i \subset \mathbb{R}^3$$

with $a, b, c \in \mathbb{R}_{>0}, v_i \in \mathbb{R}^3$ be a subgroup of $\mathbb{R}^3$, $h^{cal} : \mathbb{Z}^N \to \Gamma$ be the calibration of Γ defined by $h^{cal}(e_i) = e_i$ for $i = 1, 2, 3$, $h^{cal}(e_4) = ae_1 - be_2 + ce_3$ and $h^{cal}(e_i) = v_i$ for $i \geq 5$. Let $\sigma = \mathrm{Cone}(e_1, e_2, e_3, ae_1 - be_2 + ce_3)$ be a strongly convex cone of $\mathbb{R}^4$.

The morphism h_σ is the restricted map $h^{cal}_{|\mathbb{Z}^4 \oplus 0}$ and the kernel $\ker(h_{\sigma\mathbb{C}})$ is the line

$$\ker(h_{\sigma\mathbb{C}}) = \mathbb{C}(-a, b, -c, 1)$$

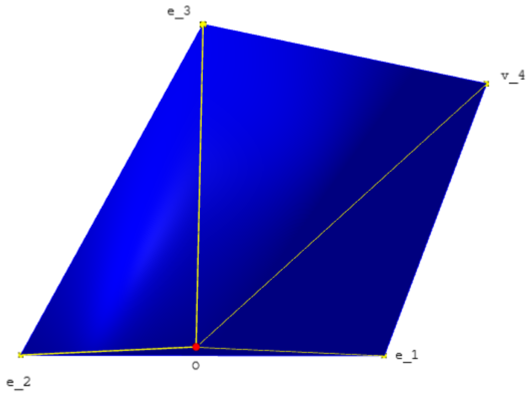
We have a decomposition $\mathbb{C}^4 = (\mathbb{C}^3 \oplus 0) \oplus \ker(h_{\sigma\mathbb{C}})$ and hence, an action of $\mathbb{Z}^{N-3} \times E(\ker(h_{\sigma\mathbb{C}}))$ on $\mathbb{C}^4$ defined by :

$$(m, E(t)) \cdot s = E(h(0 \oplus m) + t)s \tag{4.4}$$

Then, $\mathcal{U}_\sigma^{cal} = [\mathbb{C}^4 / \mathbb{Z}^{N-3} \times E(\mathbb{C}(-a, b, -c, 1))]$

For the case $N = 4$ and $a = b = c = 1$, thanks to the decomposition of $\mathcal{S}$, we find a stacky version of the toric variety $\mathbb{C}^4 /\!\!/ G = V(yt - xz) \subset \mathbb{C}^4$ (see [Ful93] p17)

FIGURE 1. The cone σ



4.1.2. *Cone of dimension $k < d$.* Suppose $\sigma = \sigma_I \subset \mathbb{R}^d$ is a cone of dimension $k < d$.
Let J a subset of $[\![1, N]\!]$ of cardinal $d - k$ such that

$$\mathbb{C}^d = \mathrm{Vect}(\sigma) \oplus \mathrm{Vect}(v_j, j \in J)$$

Hence, $\overline{h}_\sigma : \mathbb{C}^I \oplus \mathbb{C}^J \to \mathbb{C}^d, e_i \mapsto v_i$ is an epimorphism. Now, we can reuse the discussion of subsection 4.1.1 and define an action of $\mathbb{Z}^{N-d} \times E(\ker(\overline{h}_{\sigma\mathbb{C}}))$ on $\mathbb{C}^I \oplus \mathbb{C}^J$ (we will suppose that the permutation χ sends $\{k+1, \ldots d\}$ on J). We remark that the toric variety $\mathbb{C}^I \times \mathbb{T}^J = U_{\tilde{\sigma}\times 0}$ is preserved by this action and thus, we can define the affine quantum toric variety associated to σ :

$$\mathcal{U}_\sigma^{cal} := [\mathbb{C}^I \times \mathbb{T}^J / \mathbb{Z}^{N-d} \times \ker(\overline{h}_\sigma)]$$

Proposition 4.11. *The affine quantum toric variety $\mathcal{U}_\sigma^{cal}$ is well-defined i.e. if we change the family $(v_j, j \in J)$, we get an isomorphism which respects a cocycle condition.*

Proof. Let J and J' be two subsets of $[\![1, N]\!]$ of cardinal $d - k$ such that $\dim(\mathrm{Vect}(v_j, j \in J)) = \dim(\mathrm{Vect}(v_j, j' \in J')) = d - k$ and χ, χ' be the associated permutations (and P_χ, $P_{\chi'}$ the associated linear maps). The morphism $P_{\chi'} \circ P_\chi^{-1} : \mathbb{C}^J \to \mathbb{C}^{J'}$ is an isomorphism and $id \oplus P_{\chi'} \circ P_\chi^{-1}$: $\mathbb{C}^I \oplus \mathbb{C}^J \to \mathbb{C}^I \oplus \mathbb{C}^{J'}$ too.

We will prove that this morphism induces an isomorphism of stacks

$$[\mathbb{C}^I \times \mathbb{T}^J / \mathbb{Z}^{N-d} \times \ker(h_{\sigma\mathbb{C}} \oplus h_J)] \simeq [\mathbb{C}^I \times \mathbb{T}^{J'} / \mathbb{Z}^{N-d} \times \ker(h_{\sigma\mathbb{C}} \oplus h_{J'})]$$

where h_J (resp. $h_{J'}$) is the restriction of $h_\mathbb{C}^{cal}$ on $\mathbb{Z}^J$ (resp. $\mathbb{Z}^{J'}$)

Firstly, we can remark that $x \oplus y \in \ker(h_{\sigma\mathbb{C}} \oplus h_J) = \ker(h_{\sigma\mathbb{C}} \oplus h_{J'})$ if, and only if, $x \in \ker(h_{\sigma\mathbb{C}})$ and $y = 0$. Thus, we get the following commutative diagram (every arrow is an isomorphism) :

$$\begin{array}{ccc} \mathbb{C}^I \oplus \mathbb{C}^J / \ker(h_{\sigma\mathbb{C}} \oplus h_J) & \xrightarrow{id \oplus P_{\chi'} \circ P_\chi^{-1}} & \mathbb{C}^I \oplus \mathbb{C}^{J'} / \ker(h_{\sigma\mathbb{C}} \oplus h_{J'}) \\ \downarrow & & \downarrow \\ \mathbb{C}^I / \ker(h_{\sigma\mathbb{C}}) \oplus \mathbb{C}^J & \longrightarrow & \mathbb{C}^I / \ker(h_{\sigma\mathbb{C}}) \oplus \mathbb{C}^{J'} \end{array}$$

By construction, the isomorphism $id \oplus P_{\chi'}^{-1} \circ P_\chi$ descends to quotient. Hence, we get :

$$\begin{array}{ccccccc} \mathbb{C}^I \oplus \mathbb{C}^J & \xrightarrow{E} & \mathbb{T}^I \times \mathbb{T}^J & \hookrightarrow & \mathbb{C}^I \times \mathbb{T}^J & \longrightarrow & [\mathbb{C}^I \times \mathbb{T}^J / \mathbb{Z}^{N-d} \times \ker(h_{\sigma\mathbb{C}} \oplus h_J)] \\ \downarrow{\scriptstyle id \oplus P_{\chi'} \circ P_\chi^{-1}} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \hbar_{JJ'}} \\ \mathbb{C}^I \oplus \mathbb{C}^{J'} & \xrightarrow{E} & \mathbb{T}^I \times \mathbb{T}^{J'} & \hookrightarrow & \mathbb{C}^I \times \mathbb{T}^J & \longrightarrow & [\mathbb{C}^I \times \mathbb{T}^J / \mathbb{Z}^{N-d} \times \ker(h_{\sigma\mathbb{C}} \oplus h_{J'})] \end{array}$$

The rightmost morphism is the desired morphism, which is an isomorphism.

If we take a third subset J'' (and a permutation χ'') , we get two other isomorphisms $\hbar_{JJ''}$ and $\hbar_{J'J''}$. The equality

$$(id \oplus P_{\chi''} \circ P_{\chi'}^{-1}) \circ (id \oplus P_{\chi'} \circ P_\chi^{-1}) = id \oplus P_{\chi''} \circ P_\chi^{-1}$$

induces an equality between the stack (iso)morphisms

$$\hbar_{J''J'} \circ \hbar_{J'J} = \hbar_{J''J}$$

□

The following proposition is proved by the same manner as proposition 4.9.

Proposition 4.12. *If Γ is discrete, h is an isomorphism and the set of virtual generators is empty then*

$$U_\sigma = \mathbb{C}^I \times \mathbb{T}^J /\!\!/ \mathbb{Z}^{N-d} \times E\left(\ker\left(\overline{h}_\sigma\right)\right)$$

Example 4.13. We will describe a variant of example 4.10 :

Let

$$\Gamma = \mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3 + \mathbb{Z}(ae_1 - be_2 + ce_3) + \sum_{i=5}^{N} v_i\mathbb{Z} \subset \mathbb{R}^4$$

be a subgroup of $\mathbb{R}^4$, $h\colon \mathbb{Z}^N \to \Gamma$ be the calibration of Γ defined by $h(e_i) = e_i$ for $i = 1,2,3$, $h(e_4) = ae_1 - be_2 + ce_3$ and $h(e_i) = v_i$ for $i \geq 5$, and $\sigma = \mathrm{Cone}(e_1, e_2, e_3, ae_1 - be_2 + ce_3)$.

Note $k \in \{5, \dots, N\}$ an integer such that (e_1, e_2, e_3, v_k) is a basis of $\mathbb{C}^4$. The kernel of the morphism $\overline{h_\sigma}\colon \mathbb{C}^4 \oplus \mathbb{C}^{\{k\}} \to \mathbb{C}^4$ is $\mathbb{C}(-a, b, -c, 1, 0)$ i.e. $\ker(h_{\sigma\mathbb{C}}) \oplus 0$. Then, $\mathcal{U}_\sigma^{cal} = [\mathbb{C}^4 \times \mathbb{C}^* / \mathbb{Z}^{N-3} \times E(\mathbb{C}(-a, b, -c, 1, 0))]$

We will conclude this subsection with the compatibility of the construction with the restriction to a face of a cone.

Proposition 4.14. *Let $\sigma = \sigma_I$ be a cone and let $\tau = \sigma_{I'}$ be a face of σ. Then we have an isomorphism*

$$\mathcal{U}_\tau^{cal} \simeq [\mathbb{C}^{I'} \times \mathbb{T}^{I\setminus I'} \times \mathbb{T}^J/\mathbb{Z}^{N-d} \times E(\ker(h_{\sigma\mathbb{C}}))] \hookrightarrow \mathcal{U}_\sigma^{cal}$$

which restricts to a torus isomorphism

$$[\mathbb{T}^{I'} \times \mathbb{T}^{J'}/\mathbb{Z}^{N-d} \times E(\ker(h_{\tau\mathbb{C}}))] \simeq [\mathbb{T}^{I'} \times \mathbb{T}^{I\setminus I'} \times \mathbb{T}^J/\mathbb{Z}^{N-d} \times E(\ker(h_{\sigma\mathbb{C}}))]$$

induced by the identity of $\mathcal{T}_{h,\mathcal{I}}^{cal}$.

Proof. Without loss of generality, we can suppose that $J' = J \coprod K$ and $\mathbb{C}^{I\cup J} = \mathbb{C}^{\widetilde{I}} \oplus \ker(h_{\sigma\mathbb{C}}) \oplus \mathbb{C}^K$.
Then, the map $f\colon \mathbb{C}^{I'} \times \mathbb{T}^{J'} = (\mathbb{C}^{\widetilde{I}} \oplus \ker(h_{\tau\mathbb{C}})) \times \mathbb{T}^{J'} \to (\mathbb{C}^{\widetilde{I}} \oplus \overline{E}_{I\setminus I}(\ker(h_{\sigma\mathbb{C}})) \times \mathbb{T}^{J'} = \mathbb{C}^{I'} \times \mathbb{T}^{I\setminus I'} \times \mathbb{T}^J$ (where $\overline{E}_{I\setminus I} : (z,w) \in \mathbb{C}^{I'} \times \mathbb{C}^{I\setminus I'} \mapsto (z, E(w)) \in \mathbb{C}^{I'} \times \mathbb{T}^{I\setminus I'}$) defined by

$$f(x \oplus t, y) = (x, \overline{E}_{I\setminus I}(t,0), z)$$

descends to the desired stack isomorphism □

4.1.3. *Toric morphisms of affine quantum toric varieties.* We will use the same definition of toric morphism as [KLMV20] :

Definition 4.15. A toric morphism between the two affine quantum toric varieties $\mathcal{U}_{\sigma_1}^{cal} = [\mathbb{C}^I \times \mathbb{T}^J/G_1]$ and $\mathcal{U}_{\sigma_2}^{cal} = [\mathbb{C}^{I'} \times \mathbb{T}^{J'}/G_2]$ is a stack morphism $[\mathbb{C}^I \times \mathbb{T}^J/G_1] \to [\mathbb{C}^{I'} \times \mathbb{T}^{J'}/G_2]$ which restricts to a presented calibrated torus morphism $[\mathbb{T}^I \times \mathbb{T}^J/G_1] \to [\mathbb{T}^{I'} \times \mathbb{T}^{J'}/G_2]$

By definition of torus morphism, such torus morphism induces a torus morphism between standard quantum tori $\mathcal{T}_{h,\mathcal{I}}^{cal} \to \mathcal{T}_{h',\mathcal{I}'}^{cal}$ and thus two linear maps ($L : \mathbb{R}^d \to \mathbb{R}^{d'}, H : \mathbb{R}^N \to \mathbb{R}^{N'}$). Moreover, $L(\sigma) \subset \sigma'$ and $H(\widehat{\sigma}) \subset \widehat{\sigma}'$. These morphisms form a calibrated quantum fan morphism. Conversely, we will discuss how we can associate to a morphism of calibrated quantum fans a toric morphism :

Let σ be a cone of dimension k of $\mathbb{R}^d$, σ' be a cone of dimension k' of $\mathbb{R}^{d'}$. Let $\widetilde{\sigma}$ and $\widetilde{\sigma}'$ be the associated cone in $\mathbb{R}^N$ and $\mathbb{R}^{N'}$.
Let $(L, H) : (\sigma, h) \to (\sigma', h')$ be a calibrated quantum fan morphism.
Let $\sigma = \sigma_I$ be a cone of dimension k of Δ and $\widetilde{\sigma}$ the associated cone. By definition of calibrated quantum fan morphism, there exists a cone $\widetilde{\sigma}' = \widetilde{\sigma_{I'}}$ of $\Delta'_{h'}$ such that $H(\widetilde{\sigma}) \subset \widetilde{\sigma}'$.
We will adapt the construction of [KLMV20] (section 5.1). Firstly, we begin by replacing the calibration h by the morphism φ (used in definition of $\mathcal{U}_\sigma^{cal}$) in diagram (2.2) :
Let J be a subset of cardinal $d-k$ of $\{1,\dots,N\}$ such that

$$\mathbb{C}^d = \mathrm{Vect}_\mathbb{C}(\sigma) \oplus \mathrm{Vect}_\mathbb{C}(v_j, j \in J).$$

Let $\widetilde{J}$ be a subset of J such that for all $j \in \widetilde{J}$, $L(v_j) \notin L(\mathrm{Vect}_\mathbb{C}(\sigma))$ and such that the family $(L(v_j), j \in \widetilde{J})$ is free. Let J' be a subset of cardinal

$d' - \dim(\sigma')$ of $\{1, \ldots, N\}$ containing $\widetilde{J}$ such that

$$\mathbb{C}^{d'} = \mathrm{Vect}_{\mathbb{C}}(\sigma') \oplus \mathrm{Vect}_{\mathbb{C}}(v'_{j'}, j \in J').$$

Note $h_J : \mathbb{C}^J \to \mathbb{C}^d$ (resp. $h_{J'} : \mathbb{C}^{J'} \to \mathbb{C}^{d'}$) the linear map $e_j \mapsto v_j$ for all $j \in J$ (resp. $e_j \mapsto v'_j$ for all $j \in J'$) To sum up, we have the following commutative diagram :

$$\begin{array}{ccc} \mathbb{C}^I \oplus \mathbb{C}^J & \xrightarrow{h_{\sigma\mathbb{C}}+h_J} & \mathbb{C}^d \\ \downarrow{\scriptstyle \widetilde{L}} & & \downarrow{\scriptstyle L} \\ \mathbb{C}^{I'} \oplus \mathbb{C}^{J'} & \xrightarrow{h_{\sigma'\mathbb{C}}+h_{J'}} & \mathbb{C}^{d'} \end{array} \tag{4.5}$$

where $\widetilde{L}$ is the map $\mathbb{C}^I \oplus \mathbb{C}^J = \ker(h_\sigma) \oplus (\mathbb{C}^{\widetilde{I}} \oplus \mathbb{C}^J) \to \ker(h'_{\sigma'}) \oplus (\mathbb{C}^{\widetilde{I'}} \oplus \mathbb{C}^{J'})$ defined by :

$$\widetilde{L}(w, z) = (H(w), \psi'^{-1}(L(h_{\sigma\mathbb{C}}(z))))$$

where ψ' is the map induced by $h'_{\sigma'}$ used in the definition of $\mathcal{U}^{cal}_{\sigma}$. This map is well defined i.e. $H(\ker(h_{\sigma\mathbb{C}})) \subset \ker(h'_{\sigma'\mathbb{C}})$ since $H(\widehat{\sigma}) \subset \widehat{\sigma}'$ and (L, H) is a morphism of calibrated quantum fans.

Let χ and χ' be the permutations used to defined the toric varieties $\mathcal{U}^{cal}_{\sigma}$ and $\mathcal{U}^{cal}_{\sigma'}$ i.e. permutations such that

$$\chi(\{1, \ldots, k\}) = \widetilde{I}, \chi(\{k+1, \ldots, d\}) = J$$

and

$$\chi'(\{1, \ldots, k'\}) = \widetilde{I}', \chi'(\{k'+1, \ldots, d'\}) = J'.$$

Let $P_\chi \in \mathrm{GL}_N(\mathbb{R})$ and $P_{\chi'} \in \mathrm{GL}_{N'}(\mathbb{R})$ be the associated linear maps.

Moreover, we can extend this diagram with the morphisms $\varphi = \psi^{-1} \circ h_{\mathbb{C}} \circ P_\chi \colon \mathbb{C}^N \to \mathbb{C}^I \oplus \mathbb{C}^J$ and $\varphi' = \psi'^{-1} \circ h'_{\mathbb{C}} \circ P_{\chi'} \colon \mathbb{C}^{N'} \to \mathbb{C}^{I'} \oplus \mathbb{C}^{J'}$ used to define the quantum toric varieties $\mathcal{U}^{cal}_{\sigma}$ and $\mathcal{U}^{cal}_{\sigma'}$ (and more precisely, used to define an action of $\mathbb{Z}^{N-d}$ (resp. $\mathbb{Z}^{n'-d'}$) on $\mathbb{C}^I \oplus \mathbb{C}^J$ (resp. $\mathbb{C}^{I'} \oplus \mathbb{C}^{J'}$) :

$$\begin{array}{ccccc} \mathbb{C}^N & \xrightarrow{\varphi} & \mathbb{C}^I \oplus \mathbb{C}^J & \xrightarrow{h_{\sigma\mathbb{C}}+h_J} & \mathbb{C}^d \\ \downarrow{\scriptstyle P_{\chi'}^{-1}HP_\chi} & & \downarrow{\scriptstyle \widetilde{L}} & & \downarrow{\scriptstyle L} \\ \mathbb{C}^{N'} & \xrightarrow{\varphi} & \mathbb{C}^{I'} \oplus \mathbb{C}^{J'} & \xrightarrow{h_{\sigma'\mathbb{C}}+h_{J'}} & \mathbb{C}^{d'} \end{array} \tag{4.6}$$

After this replacement, we can follow the construction of corollary 5.6 of [KLMV20] (i.e. the section 5.1) in order to associate to the morphisms $(H_{\chi\chi'} := P_{\chi'}^{-1}HP_\chi, \widetilde{L})$ a toric morphism $\mathcal{U}^{cal}_{\sigma} \to \mathcal{U}^{cal}_{\sigma'}$:

Note $E_I : \mathbb{C}^I \times \mathbb{C}^J \to \mathbb{T}^I \times \mathbb{C}^J$ the map defined by :

$$((z_i)_{i \in I}, (w_j)_{j \in J}) \mapsto ((E(z_i))_{i \in I}, (w_j)_{j \in J})$$

and $\overline{E}_J : \mathbb{C}^I \times \mathbb{C}^J \to \mathbb{C}^I \times \mathbb{T}^J$ the map defined by :

$$((z_i)_{i \in I}, (w_j)_{j \in J}) \mapsto ((z_i)_{i \in I}, (E(w_j))_{j \in J})$$

In the same way, we define $E'_{I'}, \overline{E}'_{J'}$ and $E' = E'_{I'} \circ \overline{E}'_{J'}$. Then, we have the commutative diagram :

$$\begin{array}{ccccccc} \mathbb{C}^I \oplus \mathbb{C}^J & \xrightarrow{E_I} & \mathbb{T}^I \times \mathbb{C}^J & \hookrightarrow & \mathbb{C}^I \times \mathbb{C}^J & \xrightarrow{\overline{E}_J} & \mathbb{C}^I \times \mathbb{T}^J \\ \downarrow \widetilde{L} & & \downarrow \overline{L} & & \downarrow \overline{L} & & \vdots \\ \mathbb{C}^{I'} \oplus \mathbb{C}^{J'} & \xrightarrow{E'_{I'}} & \mathbb{T}^{I'} \times \mathbb{C}^{J'} & \hookrightarrow & \mathbb{C}^{I'} \times \mathbb{C}^{J'} & \xrightarrow{\overline{E}'_{J'}} & \mathbb{C}^{I'} \times \mathbb{T}^{J} \end{array} \tag{4.7}$$

The morphism $\widetilde{L}$ descends to $\mathbb{T}^I \times \mathbb{C}^J$ because $\widetilde{L}(\widetilde{\sigma}) \subset \widetilde{\sigma}'$ but since we do not make enough restriction on J and J', the morphism $\overline{L}$ has no reason to descend to a morphism $\mathbb{C}^I \times \mathbb{T}^J \to \mathbb{C}^{I'} \times \mathbb{T}^{J'}$ (like the simplicial case).

Let $T \in \mathfrak{A}$ and

$$\begin{array}{ccc} \widetilde{T}^{cal} & \xrightarrow{m^{cal}} & \mathbb{C}^I \times \mathbb{T}^J \\ \downarrow & & \\ T & & \end{array}$$

be an object of $\mathcal{U}_\sigma^{cal}$ over T.

Let $\widehat{T}^{cal}$ be the fibre product

$$\widetilde{T}_m \times_{\widetilde{E}_J} (\mathbb{C}^I \oplus \mathbb{C}^J) = \left\{ (\widetilde{t}, z) \mid m^{cal}\left(\widetilde{t}\right) = \overline{E}_J(z) \right\}.$$

The group $\mathbb{Z}^{N-k} \times E_I(\ker(h_\sigma))$ acts on $\widehat{T}^{cal}$ by

$$(p, E_I(w_1, w_2)) \cdot (\widetilde{t}, z_1, z_2) = ((pr_k(p), E(w_1, w_2)) \cdot \widetilde{t}, z_1 + w_1, E(w_2) z_2)$$

where $pr_k : \mathbb{Z}^{N-k} = \mathbb{Z}^{N-d} \oplus \mathbb{Z}^{d-k} \to \mathbb{Z}^{N-d}$ is the projection (same definition for $pr_{k'} : \mathbb{Z}^{N'-k'} \to \mathbb{Z}^{N'-d'}$).

The linear map $H_{\chi\chi'}$ satisfies, for $i \in \{1, \ldots, k\}$,

$$H_{\chi\chi'}(e_i) \in \mathbb{Z}^{k'} \oplus 0$$

Then, the map $H_{\chi\chi'}$ is of the form $\begin{pmatrix} * & * \\ 0 & M \end{pmatrix} \in \mathcal{M}_{N,N'}(\mathbb{Z})$ (we do not care of the upper part since we apply the map E_I).

In consequence, we can define the $\mathbb{Z}^{N'-d'} \times E'(\ker(h_{\sigma'\mathbb{C}}))$-principal bundle $\widehat{T}^{cal}$ as

$$\widetilde{T'}^{cal} := \widehat{T}^{cal} \times_{(pr_{k'} \circ M) \times \overline{E}'_J \circ \overline{L}} (\mathbb{Z}^{N'-d'} \times E'(\ker(h_{\sigma'\mathbb{C}})))$$

with associated equivariant map

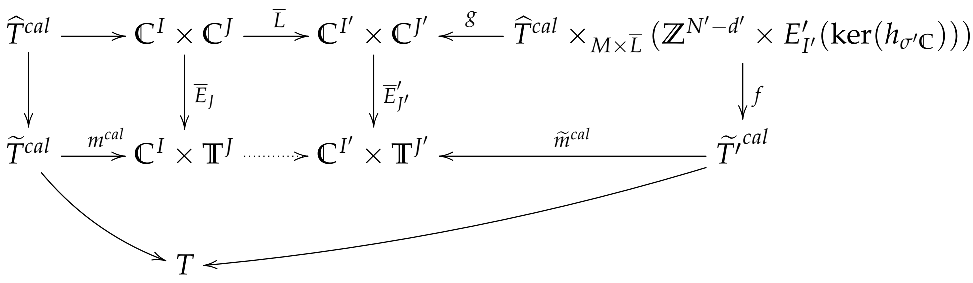
$$\widetilde{m}^{cal} : (\widetilde{t}, z, (q, E'(w))) \mapsto \overline{E}'_J\left(\overline{L}(z)\right) \cdot E'(\varphi'(0 \oplus q) + w)$$

Hence,

$$\begin{array}{ccc} \widetilde{T'}^{cal} & \xrightarrow{m^{cal}} & \mathbb{C}^{I'} \times \mathbb{T}^{J'} \\ \downarrow & & \\ T & & \end{array}$$

is an object of $\mathcal{U}_{\sigma'}^{cal}$ over T.

Lemma 4.16. *The left square and the right square of the following diagram are cartesian :*



where

$$f(\widetilde{t}, z, q, E'_{I'}(t)) = (\widetilde{t}, z, pr_{K'}(q), E'(t))$$

and

$$g(\widetilde{t}, z, q, E'_I(t)) = E'_I(\varphi'(0 \oplus q) + t) \cdot \overline{L}(z)$$

Proof. See proof of lemma 5.3 of [KLMV20] □

We define the image of a morphism between objects of $\mathcal{U}_\sigma^{cal}$ by the same way as [KLMV20] in diagram 6.10.

We get a stack morphism $\ell^{cal} : \mathcal{U}_\sigma^{cal} \to \mathcal{U}_{\sigma'}^{cal}$.

Theorem 4.17.

- *The stack morphism ℓ^{cal} is a toric morphism.*
- *Let $\sigma \subset \mathbb{R}^d$, $\sigma' \subset \mathbb{R}^{d'}$, $\sigma'' \subset \mathbb{R}^{d''}$ be cones and (L, H), (L', H') be calibrated quantum fans morphisms between the calibrated quantum fans associated to it. Note ℓ, ℓ' the toric morphisms associated to it and ℓ'' the toric morphism associated to $(L' \circ L, H' \circ H)$. Then,*

$$\ell'' = \ell' \circ \ell \tag{4.8}$$

- *Let ℓ^{cal} be a torus morphism between $\mathcal{T}_{h,\mathcal{I}}^{cal}$ and $\mathcal{T}_{h',\mathcal{I}'}^{cal}$ and let (L, H) be the induced linear morphisms. Then, ℓ^{cal} extends as a toric morphism $\mathcal{U}_\sigma^{cal} \to \mathcal{U}_{\sigma'}^{cal}$ if, and only if, (L, H) is a morphism of calibrated quantum fans.*

Proof. In [KLMV20], see the proof of lemma 5.4 for the second point and the proof of theorem 5.5 and theorem 6.2 for the first and third points. □

The two first points tell us that we defined a functor between the full subcategory of the quantum fans given by a cone and the category of affine quantum toric varieties. This functor is an equivalence of category thanks to the third one.

Now, we can give the link between the construction of affine quantum variety of this paper and the construction in [KLMV20] :

Proposition 4.18. *Suppose σ simplicial. We have a toric isomorphism :*

$$\mathcal{U}_\sigma^{cal} \simeq Q_{k,d,P_\chi^{-1} \circ \varphi}^{cal}$$

(in the same manner as paragraph 6.3 in [KLMV20]*).*

Proof. If σ is simplicial then $h_{\sigma\mathbb{C}}$ is a monomorphism and $\overline{h}_\sigma$ is an isomorphism. The morphism $(\overline{h}_\sigma P_\chi, P_\chi)$ is an isomorphism of calibrated quantum fans between the fan induced by the cone $C_{k,d} = \mathrm{Cone}(e_1, \ldots, e_k)$ and the calibration $P_\chi^{-1} \circ \varphi : \mathbb{Z}^N \to P_{\chi^{-1}}(\overline{h}_\sigma^{-1}(\Gamma))$ and the fan induced by σ and the calibration h :
Firstly, the following diagram commute :

$$\begin{array}{ccc} \mathbb{C}^N & \xrightarrow{P_\chi^{-1}\varphi} & \mathbb{C}^I \\ {\scriptstyle P_\chi}\downarrow & & \downarrow{\scriptstyle \overline{h}_{\sigma\mathbb{C}} P_\chi} \\ \mathbb{C}^N & \xrightarrow{h} & \mathbb{C}^d \end{array}$$

since $\varphi = \overline{h}_{\sigma\mathbb{C}}^{-1} \circ h \circ P_\chi$. The map $\overline{h}_{\sigma\mathbb{C}} P_\chi$ sends $C_{k,d}$ onto σ since

$$\overline{h}_{\sigma\mathbb{C}} P_\chi(e_i) = \overline{h}_{\sigma\mathbb{C}}(e_{\chi(i)}) = v_{\chi(i)} \in \sigma$$

The other points are obvious.

Hence, it induces a toric isomorphism (thanks to theorem 4.17)

$$\mathcal{U}_\sigma^{cal} \simeq Q^{cal}_{k,d,P_\chi^{-1}\circ\varphi}$$

□

4.2. **Quantum toric varieties.**

4.2.1. *Definition.* Let $\sigma = \sigma_I = \mathrm{Cone}(v_i, i \in I)$ and $\tau = \sigma_J = \mathrm{Cone}(v_j, j \in J)$ be two cones of Δ with a non-empty intersection. Note $\widetilde{\sigma}$(resp. $\widetilde{\tau}$) the associated cone (see (4.1)) of $\mathbb{R}^I$ (resp. $\mathbb{R}^J$), h_σ, h_τ the associated group homomorphisms. In this subsection, we will see how to glue the quantum toric varieties associated to them.

We will suppose that σ and τ are dimension d but the following construction works for cones of any dimension.

Let $H_{\sigma\tau} : \mathbb{C}^{I\cup J} \to \mathbb{C}^d$ be the linear map such that $H_{\sigma\tau|\mathbb{C}^I} = h_{\sigma\mathbb{C}}$ and $H_{\sigma\tau|\mathbb{C}^J} = h_{\tau\mathbb{C}}$. The identity morphism $\mathcal{T}^{cal}_{d,\mathcal{I}} \to \mathcal{T}^{cal}_{d,\mathcal{I}}$ induces torus isomorphisms

$$[\mathbb{T}^I/\mathbb{Z}^{N-d} \times E(\ker(h_{\sigma\mathbb{C}}))] \simeq [\mathbb{T}^{I\cup J}/\mathbb{Z}^{N-d} \times E(\ker(H_{\sigma\tau\mathbb{C}}))] \tag{4.9}$$

$$[\mathbb{T}^I/\mathbb{Z}^{N-d} \times \ker(h_{\tau\mathbb{C}})] \simeq [\mathbb{T}^{I\cup J}/\mathbb{Z}^{N-d} \times E(\ker(H_{\sigma\tau\mathbb{C}}))] \tag{4.10}$$

Moreover, these morphisms extend to toric isomorphisms between $\mathcal{U}_\sigma^{cal} = [\mathbb{C}^I/\mathbb{Z}^{N-d} \times E(\ker(h_{\sigma\mathbb{C}})]$ and $[\mathbb{C}^I \times \mathbb{T}^J/\mathbb{Z}^{N-d} \times E(\ker(H_{\sigma\tau}))]$ and between $\mathcal{U}_\tau^{cal} = [\mathbb{C}^I/\mathbb{Z}^{N-d} \times E(\ker(h_{\tau\mathbb{C}}))]$ and $[\mathbb{T}^I \times \mathbb{C}^J/\mathbb{Z}^{N-d} \times E(\ker(H_{\sigma\tau}))]$ in the same way as proposition 4.14.

Note $K_I := (I \cup J) \setminus I$ and $K_J := (I \cup J) \setminus J$.
As the intersection $\sigma \cap \tau$ is non-empty, the intersection $\{0\} \times \widetilde{\tau} \cap \widetilde{\sigma} \times \{0\}$ in $\mathbb{R}^{I\cup J}$ is non-empty too. Hence, the classical theory gives us an open toric subvariety $\mathcal{S}_{\sigma\tau}$ of $U_{\{0\}\times\widetilde{\tau}}$, $\mathcal{S}_{\tau\sigma}$ an open toric subvariety of $U_{\widetilde{\sigma}\times\{0\}}$ and

an toric isomorphism $\varphi : \mathcal{S}_{\sigma\tau} \to \mathcal{S}_{\tau\sigma}$. With some computations, we see that (with the decomposition $\mathbb{C}^{I\cup J} = \mathbb{C}^{K_J} \oplus \mathbb{C}^{I\cap J} \oplus \mathbb{C}^{K_I}$) :

$$U_{\widetilde{\sigma}\times\{0\}} = \mathbb{C}^I \times \mathbb{T}^{K_I}, U_{\{0\}\times\widetilde{\tau}} = \mathbb{T}^{K_J} \times \mathbb{C}^J \subset \mathbb{C}^{I\cup J} \tag{4.11}$$

and

$$\varphi = id \colon \mathcal{S}_{\tau\sigma} = \mathbb{T}^{K_J} \times \mathbb{C}^{I\cap J} \times \mathbb{T}^{K_I} \to \mathcal{S}_{\sigma\tau} = \mathbb{T}^{K_J} \times \mathbb{C}^{I\cap J} \times \mathbb{T}^{K_I} \tag{4.12}$$

The identity descends to a stack isomorphism (thanks to the linear isomorphism induced by the permutation $\chi'^{-1}\chi$ and the theorem 4.17)

$$[\mathcal{S}_{\tau\sigma}/\mathbb{Z}^{N-d} \times E(\ker(H_{\sigma\tau}))] \simeq [\mathcal{S}_{\sigma\tau}/\mathbb{Z}^{N-d} \times E(\ker(H_{\sigma\tau}))]$$

(since the toric varieties $\mathcal{S}_{\sigma\tau}$, $\mathcal{S}_{\tau\sigma}$ are preserved by the actions).

Now, thanks to the equality in (4.12) and the isomorphisms (4.9) and (4.10), we get a toric isomorphism g_{IJ} between $\mathcal{U}^{cal}_{\tau\sigma} := [\mathbb{T}^{K_J} \times \mathbb{C}^{I\cap J}/\mathbb{Z}^{N-d} \times E(\ker(h_{\sigma\mathbb{C}}))] \hookrightarrow [\mathbb{C}^I/\mathbb{Z}^{N-d} \times E(\ker(h_{\sigma\mathbb{C}})]$ and $\mathcal{U}^{cal}_{\sigma\tau} := [\mathbb{C}^{I\cap J} \times \mathbb{T}^{K_I}/\mathbb{Z}^{N-d} \times E(\ker(h_{\tau\mathbb{C}})] \hookrightarrow [\mathbb{C}^J/\mathbb{Z}^{N-d} \times E(\ker(h_{\tau\mathbb{C}})]$.

Remark 4.19. This transitions maps verify a cocycle condition since the identity map does.

With the previous discussion, we can define quantum toric varieties :

Definition 4.20. Let $T \in \mathfrak{A}$. An object of $\mathcal{X}^{cal}_{\Delta,h^{cal},\mathcal{I}}$ over T is a covering $(T_I := T_{\sigma_I})$ of T indexed by the set of maximal cones $\mathcal{I}_{max}$ together with an object

$$\begin{array}{ccc} \widetilde{T}_I & \xrightarrow{m_I} & \mathbb{C}^I \times \mathbb{T}^K \\ \downarrow & & \\ T_I & & \end{array}$$

of $[\mathbb{C}^I \times \mathbb{T}^K/\mathbb{Z}^{N-d} \times E(\ker(\overline{h}_{\sigma_I\mathbb{C}}))](T_I)$ for every $\sigma_I \in \mathcal{I}_{max}$, satisfying for any couple (I, I') with non-empty intersection J

$$g_{II'}\left(\begin{array}{ccc} \widetilde{T}_{I\supset J} & \xrightarrow{m_I} & \mathcal{S}_{\sigma_I\sigma_{I'}} \\ \downarrow & & \\ T_I & & \end{array}\right) = \begin{array}{ccc} \widetilde{T}_{I'\supset J} & \xrightarrow{m_{I'}} & \mathcal{S}_{\sigma_{I'}\sigma_I} \\ \downarrow & & \\ T_{I'} & & \end{array}$$

where $\widetilde{T}_{I'\supset J} := m_I^{-1}(\mathcal{S}_{\sigma_I\sigma_{I'}})$ and $\widetilde{T}_{I'\supset J} := m_{I'}^{-1}(\mathcal{S}_{\sigma_{I'}\sigma_I})$

A morphism of $\mathcal{X}_{\Delta,h^{cal},\mathcal{I}}$ over a toric morphism $T \to S$ is defined as in [KLMV20] with the necessary modifications.

4.2.2. *Toric morphisms.* A morphism of quantum toric varieties $\mathcal{X}^{cal}_{\Delta,h^{cal},\mathcal{I}} \to \mathcal{X}^{cal}_{\Delta',(h^{cal})',\mathcal{I}'}$ is a collection of compatible toric morphisms between their affine pieces. In other words,

Definition 4.21. A morphism $\mathcal{X}^{cal}_{\Delta,h^{cal},\mathcal{I}} \to \mathcal{X}^{cal}_{\Delta',(h^{cal})',\mathcal{I}'}$ of calibrated quantum toric varieties is a collection of toric morphisms $\ell_\sigma \colon \mathcal{U}^{cal}_\sigma \to \mathcal{U}^{cal}_{\sigma'}$ for all maximal cones of Δ compatible with the glueing i.e. if the intersection of σ and τ is not empty then the morphism ℓ_σ (resp. ℓ_τ) restricts to a morphism $\mathcal{U}^{cal}_{\tau\sigma} \to \mathcal{U}^{cal}_{\tau'\sigma'}$ (resp. $\mathcal{U}^{cal}_{\sigma\tau} \to \mathcal{U}^{cal}_{\sigma'\tau'}$) and such that the following equality holds on $\mathcal{U}^{cal}_{\tau\sigma}$

$$g'_{\sigma'\tau'}\ell_\sigma = \ell_\tau g_{\sigma\tau} \tag{4.13}$$

for each cones σ, τ with non-empty intersection.

In paragraph 4.1.3, we proved a correspondence between the affine quantum toric varieties morphisms and the calibrated quantum morphisms. We will complete it :

Let $\mathcal{X}^{cal}_{\Delta,h^{cal},\mathcal{I}} \to \mathcal{X}^{cal}_{\Delta',(h^{cal})',\mathcal{I}'}$ be a morphism of calibrated quantum toric varieties i.e. a collection of toric morphisms $\ell_\sigma \colon \mathcal{U}^{cal}_\sigma \to \mathcal{U}^{cal}_{\sigma'}$ respecting the equalities (4.13) Let (L_σ, H_σ) be the linear maps associated to ℓ_σ, $(g_{\sigma\tau}, k_{\sigma\tau})$ the linear isomorphisms associated to $g_{\sigma\tau}$ and $(g_{\sigma'\tau'}, k'_{\sigma'\tau'})$ the linear isomorphisms associated to $g'_{\sigma'\tau'}$. Then, thanks to the second point of theorem 4.17, the equations (4.13) becomes :

$$g'_{\sigma'\tau'} \circ H_\sigma = H_\tau \circ g_{\sigma\tau} \text{ and } k'_{\sigma'\tau'} \circ L_\sigma = L_\tau \circ k_{\sigma\tau} \tag{4.14}$$

Hence, we can glue these calibrated quantum fans morphisms into one between $(\Delta, h^{cal}, \mathcal{I}) \to (\Delta', (h^{cal})', \mathcal{I}')$.

Conversely, a calibrated quantum fans morphism $(\Delta, h^{cal}, \mathcal{I}) \to (\Delta', (h^{cal})', \mathcal{I}')$ defines toric morphism between affine calibrated quantum toric varieties verifying (4.13) i.e. defines a toric morphism $\mathcal{X}^{cal}_{\Delta,h^{cal},\mathcal{I}} \to \mathcal{X}^{cal}_{\Delta',(h^{cal})',\mathcal{I}'}$

We proved our main theorem :

Theorem 4.22. *The correspondence $(\Delta, h^{cal}, \mathcal{I}) \to \mathcal{X}^{cal}_{\Delta,h^{cal},\mathcal{I}}$ is functorial and defines an equivalence of categories between the category of calibrated quantum fan and the category of calibrated quantum toric varieties.*

5. GIT-like construction

In this section, we will discuss the realization of a quantum toric variety as a global quotient stack.

We can build the glueing of quantum toric varieties in an another way than the previous section :

Let $\mathcal{S}$ be the glueing of the toric varieties $\mathbb{C}^I$ and $\mathbb{C}^J$ along the intersection $\{0\} \times \widetilde{\tau} \cap \widetilde{\sigma} \times \{0\}$. Then, we define the glueing of $\mathcal{U}^{cal}_\sigma$ and $\mathcal{U}^{cal}_\tau$ is the quotient stack $[\mathcal{S}/\mathbb{Z}^{N-d} \times E(\ker(H_{\sigma\tau}))]$. Then, we deduce

Theorem 5.1. *Let A be the set $A := \bigcup_{\sigma_I \in \Delta} I$. Let $\widetilde{A}$ a subset of $\{1, \ldots, N\} \setminus \mathcal{I}$ such that $(v_i, i \in \widetilde{A})$ is free and such that $\mathbb{C}^d = \mathrm{Vect}(v_i, i \in A) \oplus \mathrm{Vect}(v_i, i \in \widetilde{A})$ Let $h_A : \mathbb{Z}^{A \cup \widetilde{A}} \to \Gamma$ be the group homomorphism defined by $h(e_i) = v_i$ for $i \in A \cup \widetilde{A}$. Note $\mathcal{S}_A$ the toric variety given by the associated fan (see* (4.1)*) of Δ in $\mathbb{R}^A$ given by*

$$\Delta_h \cap \mathbb{R}^A = \{\tau \mid \exists \sigma \in \Delta, \tau \preceq \widehat{\sigma} \cap \mathbb{R}^A\}.$$

We define an action of $\mathbb{Z}^{N-d} \times E(\ker(h_A))$ *on* $\mathcal{S}_A \times \mathbb{T}^{\widetilde{A}}$ *in the same way as 4.1.2. Then,*

$$\mathfrak{X}^{cal}_{\Delta,h^{cal},\mathcal{I}} \simeq [\mathcal{S}_A \times \mathbb{T}^{\widetilde{A}} / \mathbb{Z}^{N-d} \times E(\ker(h_A))]$$

as stacks

Moreover, in the same way as in the beginning of subsection 4.2, we have a toric isomorphism

$$[\mathbb{C}^I \times \mathbb{T}^J / \mathbb{Z}^{N-d} \times \ker(h_{\sigma\mathbb{C}})] \simeq [\mathbb{C}^I \times \mathbb{T}^{I^c} / \mathbb{Z}^{N-d} \times \ker(h^{cal}_{\mathbb{C}})]$$

Hence, we get a GIT-like realization of $\mathfrak{X}^{cal}_{\Delta,h^{cal},\mathcal{I}}$:

Theorem 5.2. *Let* $\mathcal{S}$ *be the toric variety associated to the associated fan* Δ_h*. Then, we have a stack isomorphism*

$$\mathfrak{X}^{cal}_{\Delta,h^{cal},\mathcal{I}} \simeq [\mathcal{S} / \mathbb{Z}^{N-d} \times E(\ker(h^{cal}_{\mathbb{C}}))] \tag{5.1}$$

which restricts to a torus isomorphism between the associated quantum torus on each affine chart.

By contrast to the simplicial case, we cannot realize $\mathfrak{X}^{cal}_{\Delta,h,\mathcal{I}}$ as a quotient of a toric variety by an action of $\mathbb{C}^{N-d}$ via Gale transform (the "quantum GIT" of [KLMV20]).

Indeed, let $k : \mathbb{C}^{N-d} \to \mathbb{C}^N$ be the map induced by a Gale transform (see [GKKZ03] for the details) of the family $(h^{cal}(e_i), i \in [\![1, N]\!])$ i.e. a map such that the sequence

$$0 \longrightarrow \mathbb{C}^{N-d} \xrightarrow{k} \mathbb{C}^N \xrightarrow{h^{cal}_{\mathbb{C}}} \mathbb{C}^d \longrightarrow 0$$

is exact.

We would like to prove that $\mathfrak{X}^{cal}_{\Delta,h^{cal},\mathcal{I}}$ and $[\mathcal{S}/\mathbb{C}^{N-d}]$ are isomorphic (where $\mathbb{C}^{N-d}$ acts on $\mathcal{S}$ through $E \circ k$) in order to generalize the theorem 7.6 of [KLMV20]).

For each $\sigma \in \Delta$, the corresponding open substack of $\mathfrak{X}^{cal}_{\Delta,h^{cal},\mathcal{I}}$ is $\mathcal{U}^{cal}_\sigma = [\mathbb{C}^I \times \mathbb{T}^J / \mathbb{Z}^{N-d} \times E(\ker(h_{\sigma\mathbb{C}}))]$ and the substack of $[\mathcal{S}/\mathbb{C}^{N-d}]$ is $\mathcal{U}'_\sigma := [\mathbb{C}^I \times \mathbb{T}^{I^c} / \mathbb{C}^{N-d}]$ where $I^c = \{1, \dots, N\} \setminus I$.

Proposition 5.3. *The stacks* $\mathcal{U}^{cal}_\sigma$ *and* $\mathcal{U}'_\sigma$ *are not isomorphic if* σ *is not simplicial.*

Proof. The groupoid associated to $\mathcal{U}^{cal}_\sigma$ and the groupoid associated to $\mathcal{U}'_\sigma$ cannot be Morita-equivalent since their isotropy groups (i.e. the stabilizer on a point by the action) are not isomorphic (see theorem 4.4 of [Xu04]).

More precisely, the stabilizer of the action of $\mathbb{Z}^{N-d} \times E(\ker(h_{\sigma\mathbb{C}}))$ at each point of $(\mathbb{C}^{\widetilde{I}} \oplus 0) \times \mathbb{T}^J$ is $E(\ker(h_\sigma)) \times \ker(h^{cal})$ and there is no point of $\mathbb{C}^I \times \mathbb{T}^{I^c}$ with an isomorphic stabilizer group for the $\mathbb{C}^{N-d}$-action. Indeed, the different stabilizer groups for the $\mathbb{C}^{N-d}$-action are $k^{-1}(\ker(h_{\sigma\mathbb{C}}) \cap \mathbb{C}^K + \ker(h^{cal}))$ at $\mathbb{C}^{\widetilde{I}} \oplus (\ker(h_\sigma) \cap \{0\}^K) \times \mathbb{T}^{I^c}$ for $\emptyset \neq K \subsetneq I$ and $k^{-1}(\ker(h_{\sigma\mathbb{C}}) + \ker(h^{cal}))$ at $(\mathbb{C}^{\widetilde{I}} \oplus \{0\}) \times \mathbb{T}^{I^c}$. These groups cannot be isomorphic to $E(\ker(h_\sigma)) \times \ker(h^{cal})$(think of the lack of isomorphism between $\mathbb{C}^n$ and $(\mathbb{C}^*)^n \times \mathbb{Z}^n$). ☐

Conversely, if we replace the construction of quantum toric varieties by the quotient stack of $\mathcal{S}$ by the $\mathbb{C}^{N-d}$-action, the construction would not be fonctorial. Indeed, its "torus" is the quotient stack $[\mathbb{T}^N/\mathbb{C}^{N-d}]$ which is not isomorphic to the calibrated quantum torus $\mathcal{T}^{cal}_{h,\mathcal{I}}$ (for the same reason as the proof of 5.3) and hence, this torus is not entirely characterized by the calibration (up to isomorphism). Therefore, we cannot associate a stack morphism to a calibrated quantum fan morphism $(L\colon \mathbb{R}^d \to \mathbb{R}^{d'}, H\colon \mathbb{R}^N \to \mathbb{R}^{N'})$.

6. Forgetting calibration and gerbe structure

We can associate to each affine quantum toric variety $\mathcal{U}^{cal}_\sigma = [\mathbb{C}^I \times \mathbb{T}^J/\mathbb{Z}^{N-d} \times E(\ker(h_{\sigma\mathbb{C}}))]$ a "non-calibrated quantum toric variety" $\mathcal{U}_\sigma := [\mathbb{C}^I \times \mathbb{T}^J/E(h^{-1}_{\sigma\mathbb{C}}(\Gamma))]$.
More precisely, note Ξ be the kernel of h^{cal} and let T be an object of $\mathfrak{A}$ and $(\widetilde{T}^{cal}, m^{cal})$ be an object of $\mathcal{U}^{cal}_\sigma(T)$.
Since

$$h \times id\colon \mathbb{Z}^{N-d} \times E(\ker(h_{\sigma\mathbb{C}})) \to \frac{\mathbb{Z}^{N-d} \times E(\ker(h_{\sigma\mathbb{C}}))}{\Xi \times 0} = E(\Gamma) \times E(\ker(h_{\sigma\mathbb{C}})) = E(h^{-1}_{\sigma\mathbb{C}}(\Gamma))$$

is a Ξ-covering then $\widetilde{T} = \widetilde{T}^{cal}/\Xi \to T$ is a $E(h^{-1}_{\sigma\mathbb{C}}(\Gamma))$-principal bundle.
Note $m : [\widetilde{t}^{cal}] \in \widetilde{T} \mapsto m^{cal}(\widetilde{t}^{cal}) \in \mathbb{C}^I \times \mathbb{T}^J$. This map is well-defined :

$$m^{cal}((\xi, 0) \cdot \widetilde{t}^{cal}) = E(\varphi(\xi, 0)) m^{cal}(t^{cal}) = m^{cal}(t^{cal})$$

We can define morphisms of non-calibrated affine quantum toric varieties in the same way as 4.15 :

Definition 6.1. A toric morphism between two non-calibrated affine quantum toric varieties $\mathcal{U}_{\sigma_1} = [\mathbb{C}^I \times \mathbb{T}^J/G_1]$ and $\mathcal{U}_{\sigma_2} = [\mathbb{C}^{I'} \times \mathbb{T}^{J'}/G_2]$ is a stack morphism $[\mathbb{C}^I \times \mathbb{T}^J/G_1] \to [\mathbb{C}^{I'} \times \mathbb{T}^{J'}/G_2]$ which restricts to a presented non-calibrated torus morphism $[\mathbb{T}^I \times \mathbb{T}^J/G_1] \to [\mathbb{T}^{I'} \times \mathbb{T}^{J'}/G_2]$

We can see that morphism of affine quantum toric varieties descends to quotient and induce a morphism of non-calibrated affine quantum toric varieties. Hence, we have defined a functor f from the category of affine quantum toric varieties to the category of non-calibrated affine quantum toric varieties.

We can follow the proof of 4.17 in order to see that a morphism (L, H) of calibrated quantum fan morphism (we need the two morphisms due to the presence of the kernel of $h_{\sigma\mathbb{C}}$) leads to a morphism ℓ of non-calibrated affine quantum toric varieties.

Lemma 6.2. *This functor coincide with the functor* f *of* [KLMV20] *(section 6.2) on simplicial quantum toric variety.*

Moreover, the functor f induces (with lemma 3.5) a functor $\widetilde{\mathrm{f}}$ on the category of presented quantum tori defined by

$$\widetilde{\mathrm{f}}(\mathcal{T}^{cal}_{h,\mathcal{I}}, h' : \mathbb{Z}^N \to G, \mathcal{I}', L, H, s) = (\mathcal{T}_{d,\Gamma}, L)$$

and

$$\widetilde{\mathrm{f}}(\mathcal{L},\mathcal{H},\mathcal{S},\mathcal{L}',\mathcal{H}',\mathcal{S}') = (\mathcal{L},\mathcal{L}')$$

Lemma 6.3. *We have a commutative diagram*

$$\begin{array}{ccc} \mathcal{U}_\sigma^{cal} & \xrightarrow{\ell^{cal}} & \mathcal{U}_{\sigma'}^{cal} \\ \mathrm{f}\big\downarrow & & \big\downarrow \mathrm{f} \\ \mathcal{U}_\sigma & \xrightarrow{\ell} & \mathcal{U}_{\sigma'} \end{array}$$

We can adapt the definition of quantum toric varieties to the non-calibrated case :

Definition 6.4. Let $T \in \mathfrak{A}$. An object of $\mathcal{X}_{\Delta,\Gamma}$ over T is a covering $(T_I := T_{\sigma_I})$ of T indexed by the set of maximal cones $\mathcal{I}_{max}$ together with the image by f of an object

$$\begin{array}{ccc} \widetilde{T}_I & \xrightarrow{m_I} & \mathbb{C}^I \times \mathbb{T}^K \\ \big\downarrow & & \\ T_I & & \end{array}$$

of $[\mathbb{C}^I \times \mathbb{T}^K/\mathbb{Z}^{N-d} \times E(\ker(\overline{h}_{\sigma_I\mathbb{C}})](T_I)$ for every $\sigma_I \in \mathcal{I}_{max}$, satisfying for any couple (I, I') with non-empty intersection J

$$\mathrm{f}\left(g_{II'}\left(\begin{array}{ccc} \widetilde{T}_{I\supset J} & \xrightarrow{m_I} & \mathcal{S}_{\sigma_I\sigma_{I'}} \\ \big\downarrow & & \\ T_I & & \end{array}\right)\right) = \mathrm{f}\left(\begin{array}{ccc} \widetilde{T}_{I'\supset J} & \xrightarrow{m_{I'}} & \mathcal{S}_{\sigma_{I'}\sigma_I} \\ \big\downarrow & & \\ T_{I'} & & \end{array}\right)$$

where $\widetilde{T}_{I'\supset J} := m_I^{-1}(\mathcal{S}_{\sigma_I\sigma_{I'}})$ and $\widetilde{T}_{I'\supset J} := m_{I'}^{-1}(\mathcal{S}_{\sigma_{I'}\sigma_I})$

A morphism of $\mathcal{X}_{\Delta,\Gamma}$ over a toric morphism $T \to S$ is defined by applying f to a morphism of $\mathcal{X}^{cal}_{\Delta,h^{cal},\mathcal{I}}$ over this morphism.

Theorem 6.5. *The quantum toric variety $\mathcal{X}^{cal}_{\Delta,h,\mathcal{I}}$ is a gerbe over $\mathcal{X}_{\Delta,\Gamma}$ with band $\mathbb{Z}^{\mathrm{rk}(\Xi)}$. In particular, if $\Xi = 0$ then these two stacks are isomorphic.*

This structure of gerbe induces a $\mathbb{T}^{\mathrm{rk}(\Xi)}$-bundle over $\mathcal{X}_{\Gamma,\Delta}$ up to homotopy. The end of this section will be devoted to describe it :

Let $\sigma = \sigma_I \in \Delta$ be a maximal cone, $h_{\sigma\mathbb{C}}$ be the $\mathbb{C}$-linear morphism associated to it, $\psi : \mathbb{C}^I/\ker(h_{\sigma\mathbb{C}}) \to \mathbb{C}^d$ induced by $h_{\sigma\mathbb{C}}$.
Let $\widetilde{I}$ be a subset of I such that $(h(e_i), i \in \widetilde{I})$ is a basis of $\mathrm{Vect}(\sigma)$, J be a set of cardinal $d - \dim(\sigma)$ such that

$$\mathrm{Vect}(h^{cal}(e_i), i \in I \cup J\} = \mathbb{C}^d,$$

let $\chi \in \mathfrak{S}_N$ be a permutation such that

$$\chi(\{1,\ldots,\dim(\sigma)\}) = \widetilde{I} \text{ and } \chi(\{\dim(\sigma)+1,\ldots,d\}) = J \tag{6.1}$$

and $P_\chi \in \mathrm{GL}_N(\mathbb{R})$ be the map associated to χ.

Let $\varphi = (\varphi_1, \varphi_2)$ be the linear map $\psi^{-1}hP_\chi : \mathbb{C}^N \to \mathbb{C}^I/\ker(h_\sigma) \times \mathbb{C}^J$. Thanks to the conditions (6.1), we get the following diagram

$$\begin{array}{ccc}
\mathbb{C}^N & \xrightarrow{\quad\varphi\quad} & \mathbb{C}^I \times \mathbb{C}^J/\ker(h_\sigma) \\
\Big\downarrow{\scriptstyle (\psi^{-1}, id_{\mathbb{C}^{N-d}})\circ E_k} & & \Big\downarrow{\scriptstyle E_I} \\
\mathbb{T}^I/E(\ker(h_{\sigma\mathbb{C}}) \times \mathbb{C}^J \times \mathbb{C}^{N-d} & \xrightarrow{\overline{\varphi}} & \mathbb{T}^I \times \mathbb{C}^J/E(\ker(h_{\sigma\mathbb{C}})) \\
\Big\downarrow & & \Big\downarrow \\
\left[\mathbb{C}^I/E(\ker(h_{\sigma\mathbb{C}})\right] \times \mathbb{C}^J \times \mathbb{C}^{N-d} & \xrightarrow{\overline{\varphi}} & \left[\mathbb{C}^I \times \mathbb{C}^J/E(\ker(h_\sigma))\right] \\
\Big\downarrow{\scriptstyle (id,E,id)} & & \Big\downarrow{\scriptstyle \overline{E_J}} \\
\left[\mathbb{C}^I/E(\ker(h_{\sigma\mathbb{C}})\right] \times \mathbb{T}^J \times \mathbb{C}^{N-d} & \xrightarrow{\widehat{\varphi}} & \left[\mathbb{C}^I \times \mathbb{T}^J/E(\ker(h_\sigma))\right] \\
\Big\downarrow{\scriptstyle \overline{E}_d} & & \Big\downarrow \\
\left[\mathbb{C}^I/E(\ker(h_{\sigma\mathbb{C}})\right] \times \mathbb{T}^J \times \mathbb{T}^{N-d} & \xrightarrow{\quad\varphi^{cal}\quad} & \mathcal{U}_\sigma^{cal}
\end{array}$$

where for all $(z_1, z_2) \in \mathbb{T}^I \times \mathbb{C}^J$, $w \in \mathbb{C}^{N-d}$

$$\overline{\varphi}([z_1], z_2, w) = ([E(\varphi_1(0 \oplus w))z_1], \varphi_2(0 \oplus w) + z_2)$$

and $z \in \mathbb{C}^I \times \mathbb{T}^J$, $w \in \mathbb{C}^{N-d}$,

$$\widehat{\varphi}([z_1], z_2, w) = ([E(\varphi(0 \oplus w))z]$$

(we can translate this equality in terms of principal bundles)

The group $\mathbb{Z}^{N-d}$ acts on $\left[\mathbb{C}^I/E(\ker(h_{\sigma\mathbb{C}}))\right] \times \mathbb{T}^J \times \mathbb{C}^{N-d}$ by translation on the last factor and on $\left[\mathbb{C}^I/E(\ker(h_{\sigma\mathbb{C}}))\right] \times \mathbb{T}^J \times \mathbb{C}^{N-d}$ through the morphism φ. The morphism $\widehat{\varphi}$ is equivariant for these two actions. Hence, $\widehat{\varphi}$ descends to a stack morphism $\left[\mathbb{C}^I/E(\ker(h_{\sigma\mathbb{C}})\right] \times \mathbb{T}^J \times \mathbb{T}^{N-d} \to \mathcal{U}_\sigma^{cal}$

In the same manner, the projection on the first factor

$$\left([\mathbb{C}^I/E(\ker(h_{\sigma\mathbb{C}}))] \times \mathbb{T}^J\right) \times \mathbb{C}^{N-d} \to [\mathbb{C}^I/E(\ker(h_{\sigma\mathbb{C}}))] \times \mathbb{T}^J$$

is equivariant for the action (on the source) of $\mathbb{Z}^{N-d}$ defined by, for $p \in \mathbb{Z}^{N-d}$, $(z, w) \in \mathbb{C}^I \times \mathbb{T}^J$

$$p \cdot (z, w) = (E(\varphi(0 \oplus p))z, w + p) \tag{6.2}$$

(it is well-defined since the actions are multiplicative) and the action of $\mathbb{Z}^{N-d}$ on the target defined by, for $p \in \mathbb{Z}^{N-d}$ and for $(z, w) \in (\mathbb{C}^I \times \mathbb{T}^J)$

$$p \cdot z = E(\varphi(0 \oplus p))z \tag{6.3}$$

Hence, the projection descends to a stack morphism :

(6.4)

$$\begin{array}{ccc}
\left([\mathbb{C}^I/E(\ker(h_{\sigma\mathbb{C}}))] \times \mathbb{T}^J\right) \times \mathbb{C}^{N-d} & \xrightarrow{\quad\pi_1\quad} & [\mathbb{C}^I/E(\ker(h_{\sigma\mathbb{C}}))] \times \mathbb{T}^J \\
\Big\downarrow & & \Big\downarrow \\
\left[(\mathbb{C}^I \times \mathbb{T}^J) \times \mathbb{C}^{N-d}/E(\ker(h_\sigma)) \times \mathbb{Z}^{N-d}\right] & \xrightarrow{\quad p\quad} & \mathcal{U}_\sigma^{cal}
\end{array}$$

Moreover, we have the following commutative diagram

$$\begin{array}{ccc} \big([\mathbb{C}^I/E(\ker(h_{\sigma\mathbb{C}}))]\times\mathbb{T}^J\big)\times\mathbb{C}^{N-d} & \xrightarrow{\pi_1} & [\mathbb{C}^I\times\mathbb{T}^J/E(\ker(h_\sigma))] \\ \big\downarrow {\scriptstyle(\widehat{\varphi},id)^{-1}} & \overset{\widehat{\varphi}}{\nearrow} & \\ [\mathbb{C}^I/E(\ker(h_{\sigma\mathbb{C}}))]\times\mathbb{T}^J\times\mathbb{C}^{N-d} & & \end{array}$$

Since the morphism $(\widehat{\varphi},id)^{-1}$ is equivariant for the action (6.2) and for the translation in the last factor. Hence, we get the following commutative diagram

$$\begin{array}{ccc} \big[(\mathbb{C}^I\times\mathbb{T}^J)\times\mathbb{C}^{N-d}/E(\ker(h_\sigma))\times\mathbb{Z}^{N-d}\big] & \xrightarrow{p} & \mathcal{U}^{cal}_\sigma \\ \big\downarrow {\scriptstyle(\widehat{\varphi},id)^{-1}} & \overset{\varphi^{cal}}{\nearrow} & \\ [\mathbb{C}^I/E(\ker(h_{\sigma\mathbb{C}}))]\times\mathbb{T}^J\times\mathbb{T}^{N-d} & & \end{array} \tag{6.5}$$

The diagrams (6.4) and (6.5) can be seen as trivialization of the morphism φ^{cal}. Thus, we can see φ^{cal} as a $\mathbb{C}^{N-d}$-fibre bundle.

Note $\mathcal{X}_\Delta$ the stack over $\mathfrak{A}$ given by the descent data of the family of stacks $[\mathbb{C}^I/E(\ker(h_{\sigma\mathbb{C}}))]\times\mathbb{T}^J\times\mathbb{T}^{N-d}$ indexed by every maximal cones of Δ (in the same manner as 4.2). By functoriality, we get

Theorem 6.6. *The map $\varphi^{cal}\colon\mathcal{X}_\Delta\to\mathcal{X}^{cal}_{\Delta,h,\mathcal{I}}$ is a $\mathbb{C}^{N-d}$-principal bundle (in the sense that this morphism can be locally trivialized on the affine pieces, the trivialization map induced the identity on the factors $\mathbb{C}^{N-d}$ and the transition maps are given by the action of $\mathbb{C}^{N-d}$)*

This statement gives further details on the theorem 6.21 of [KLMV20].

Proof. It remains us to explain to the action induced by the transition maps. By the definitions of $\mathcal{X}_\Delta$ and $\mathcal{X}^{cal}_{\Delta,h,\mathcal{I}}$, we can restrict us to the simplicial fans i.e. $\mathcal{X}_\Delta=\mathcal{S}$.

Each cone σ_I of Δ induces an action of the group $\mathbb{C}^{N-d}$ on each variety $\mathbb{C}^I\times\mathbb{T}^J\times\mathbb{C}^{(I\cup J)^c}\subset\mathcal{S}$. Namely, it is the induced $\mathbb{C}^{N-d}$-action by the action by translation on the last factor of $\mathbb{C}^I\times\mathbb{T}^J\times\mathbb{C}^{N-d}$ and the isomorphism $(\widehat{\varphi}_I,id)\circ(id,P_{\chi_I|\mathbb{C}^{N-d}})^{-1}$ i.e.

$$\lambda\cdot(z_1,z_2,w)=(E(-\widetilde{\varphi}^1_I(\lambda))z_1,E(-\widetilde{\varphi}^2_I(\lambda))z_2,w+P_{\chi_I}\lambda) \tag{6.6}$$

where $\widetilde{\varphi}_I=(\widetilde{\varphi}^1_I,\widetilde{\varphi}^2_I):\mathbb{C}^{N-d}\to\mathbb{C}^I\times\mathbb{C}^J$ is the restriction of φ_I on $\mathbb{C}^{N-d}$.

Thus, the morphism $\widehat{\varphi_I}\circ(id,P_{\chi_I|\mathbb{C}^{N-d}})^{-1}\colon\mathbb{C}^I\times\mathbb{T}^J\times\mathbb{C}^{(I\cup J)^c}\to\mathbb{C}^I\times\mathbb{T}^J$ is a $\mathbb{C}^{N-d}$ trivializable principal bundle. Hence, it has a global cross section $s_\sigma:\mathbb{C}^I\times\mathbb{T}^J\to\mathbb{C}^I\times\mathbb{T}^J\times\mathbb{C}^{(I\cup J)^c}$ which is defined by

$$s_\sigma(x,y)=(x,y,0)$$

This morphism descends as a morphism $\mathcal{U}^{cal}_\sigma\to\mathbb{C}^I\times\mathbb{T}^J\times\mathbb{T}^{N-d}$.

In order to conclude, we have to find, for each cones $\sigma=\sigma_I$, $\tau=\sigma_{I'}$ with a non-empty intersection, a morphism $t_{\sigma\tau}\colon\mathcal{U}^{cal}_{\sigma\tau}\to\mathbb{C}^{n-d}$ such that

$$s_\sigma=t_{\sigma\tau}\cdot s_\tau \tag{6.7}$$

Note $\varphi_I = \psi_I^{-1} h P_{\chi_I}$ and $\varphi_{I'} = \psi_{I'}^{-1} h P_{\chi_{I'}}$ the calibration associated, respectively, to σ and τ. Recall the commutative diagram used in the definition of quantum toric varieties :

$$\begin{array}{ccccc}
\mathbb{C}^{I\cup J} & \xrightarrow{\psi_I} & \mathbb{C}^d & \xrightarrow{\psi_{I'}^{-1}} & \mathbb{C}^{I'\cup J'} \\
\uparrow & & \uparrow & & \uparrow \\
\psi_I^{-1}(\Gamma) & \xrightarrow{\psi_I} & \Gamma & \xrightarrow{\psi_{I'}^{-1}} & \psi_{I'}^{-1}(\Gamma) \\
{\scriptstyle \varphi_I}\uparrow & & {\scriptstyle h}\uparrow & & \uparrow{\scriptstyle \varphi_{I'}} \\
\mathbb{Z}^n & \xrightarrow{P_{\chi_I}} & \mathbb{Z}^n & \xrightarrow{P_{\chi_{I'}}^{-1}} & \mathbb{Z}^n
\end{array}$$

Since $\psi_{I'}^{-1}\psi_I\varphi_I = \varphi_{I'}(P_{\chi_I'}^{-1}P_{\chi_I})$ then for all point m of $\mathbb{Z}^d$,

$$\psi_{I'}^{-1}\psi_I(P_\chi m) = \varphi_{I'}(P_{\chi_{I'}}^{-1}P_{\chi_I})(m) = (Id + \widetilde{\varphi}_{I'}) \circ (P_{\chi_{I'}}^{-1}P_{\chi_I})(m) \tag{6.8}$$

Since we consider the simplicial case, we can use the quasifold formalism (like the equation (6.33) of [KLMV20]) :

The transition map between $\mathcal{U}_{\tau\sigma}^{cal}$ and $\mathcal{U}_{\sigma\tau}^{cal}$ is $[z^{P_\chi}] \in \mathcal{U}_{\tau\sigma}^{cal} \mapsto \left[z^{\psi_{I'}^{-1}\psi_I P_\chi}\right] \in \mathcal{U}_{\sigma\tau}^{cal}$

Note $K_{II'}$ the set $P_{\chi_{I'}}^{-1}(I) \setminus \{1, \dots, d\}$. Then, the maps $t_{\sigma\tau} \colon z \mapsto \left[z^{K_{II'}}\right]$ verify the equality (6.7) thanks to the equalities (6.6) and (6.8). ☐

As $\mathcal{X}_{\Delta,h,\mathcal{I}}^{cal}$ is a gerbe over $\mathcal{X}_{\Delta,\Gamma}$ with band $\mathbb{Z}^{\mathrm{rk}(\Xi)}$ then

Theorem 6.7. *The map* $\mathrm{f} \circ \varphi^{cal} \colon \mathcal{X}_\Delta \to \mathcal{X}_{\Delta,\Gamma}$ *is a* $\mathbb{C}^{N-d-\mathrm{rk}(\Xi)} \times \mathbb{T}^{\mathrm{rk}(\Xi)}$*-principal bundle and hence, a* $\mathbb{T}^{\mathrm{rk}(\Xi)}$*-principal bundle up to homotopy.*

References

[CLS11] D.A. Cox, J.B. Little, and H.K. Schenck. *Toric Varieties*. Graduate studies in mathematics. American Mathematical Soc., 2011.

[Dol03] Igor Dolgachev. *Lectures on Invariant Theory*. London Mathematical Society Lecture Note Series. Cambridge University Press, 2003.

[Ful93] William Fulton. *Introduction to toric varieties*. Annals of mathematics studies. Princeton Univ. Press, Princeton, NJ, 1993.

[GKKZ03] B. Grünbaum, V. Kaibel, V. Klee, and G.M. Ziegler. *Convex Polytopes*. Graduate Texts in Mathematics. Springer, 2003.

[KLMV20] Ludmil Katzarkov, Ernesto Lupercio, Laurent Meersseman, and Alberto Verjovsky. Quantum (non-commutative) toric geometry: Foundations. *arXiv preprint math/2002.03876*, 2020.

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric Invariant Theory*. Number vol. 34 in Ergebnisse der Mathematik und Ihrer Grenzgebiete, 3 Folge/A Series of Modern Surveys in Mathematics Series. Springer Berlin Heidelberg, 1994.

[Pra01] Elisa Prato. Simple non-rational convex polytopes via symplectic geometry. *Topology*, 40:961–975, 2001.

[Xu04] Ping Xu. Momentum maps and morita equivalence. *Journal of Differential Geometry*, 67:289–333, 2004.

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