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Solution to a problem of Diestel

R. M. Causey, E. M. Galego, and C. Samuel

Dedicated to the memory of Professor Joe Diestel.

ABSTRACT. In the present paper, we prove that the 3-fold projective tensor product $c_0 \widehat{\otimes}_\pi c_0 \widehat{\otimes}_\pi c_0$ of c_0 is not isomorphic to any subspace of $c_0 \widehat{\otimes}_\pi c_0$. In particular, this settles the long-standing open problem, originally raised by Joe Diestel in a private communication, of whether $c_0 \widehat{\otimes}_\pi c_0$ is isomorphic to $c_0 \widehat{\otimes}_\pi c_0 \widehat{\otimes}_\pi c_0$.

1. Introduction

Since Grothendieck established the theory of tensor products [7], it has been realized that projective tensor products would have a great impact on the geometry of Banach spaces. Many surprising results are known which show that the geometric structure of a projective tensor product can be quite complicated, even for the simplest component spaces. For example, Stehle showed that there exists a subspace of $c_0 \widehat{\otimes}_\pi c_0$ which fails to have the Dunford-Pettis property [13], despite the well known fact that every subspace of c_0 has this property [8], [6]. For further results concerning the geometric structure of projective tensor product of Banach spaces, see for instance [2], [4] and [11].

On the other hand, due to the somewhat intractable nature of the definition of the projective tensor norm, various elementary questions on the spaces $X \widehat{\otimes}_\pi Y$ still remain unanswered. This is the case for the following problem attributed to Aleksander Pełczyński [2, p.517]. Does $c_0 \widehat{\otimes}_\pi c_0$ have the uniform approximation property (UAP)? Here, for Banach spaces X, Y , $X \widehat{\otimes}_\pi Y$ denotes the projective tensor product of X and Y .

Recall that a Banach space X has the UAP if there exist a constant K and a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that, given $E \subset X$ with $\dim E = k$, there exists continuous, linear operator $T : X \rightarrow X$ with $\|T\| \leq K$, $\dim T(X) \leq f(k)$, and $T(x) = x$ for every $x \in E$ [1].

However, it was proved in [2, Corollary 1.7] that the quadruple projective tensor product of c_0 , $c_0 \widehat{\otimes}_\pi c_0 \widehat{\otimes}_\pi c_0 \widehat{\otimes}_\pi c_0$, does not have the UAP. Thus, in view of Pełczyński's problem, this last result also raised the following problem.

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PROBLEM 1.1. *Is $c_0 \widehat{\otimes}_\pi c_0$ isomorphic to $c_0 \widehat{\otimes}_\pi c_0 \widehat{\otimes}_\pi c_0 \widehat{\otimes}_\pi c_0$?*

As already noted by the authors of [2], by the associativity of the projective tensor product, a positive solution to the next problem involving the triple projective tensor product of c_0 would imply a positive solution to Problem 1.1.

PROBLEM 1.2. *Is $c_0 \widehat{\otimes}_\pi c_0$ isomorphic to $c_0 \widehat{\otimes}_\pi c_0 \widehat{\otimes}_\pi c_0$?*

Of course, the geometric structure of 3-fold projective tensor products is even more complicated than the 2-fold products, and so far very little is known about the geometric properties of these spaces. In particular, Problem 1.2 is another longstanding open question, attributed to Joe Diestel [2, p.517].

The initial motivation for studying the theme of this paper was to look for the solution of Problem 1.2. Although we have solved Problem 1.2 negatively, our main result also resolves Problem 1.1 negatively. In fact, in Theorem 1.3 we establish something stronger about the family of subspaces of $c_0 \widehat{\otimes}_\pi c_0$. This result is a contribution to better understand the fruitful work on projective tensor products started by Grothendieck in 1953, with which many open problems are related (see, e.g., [5, Introduction]).

THEOREM 1.3. *$c_0 \widehat{\otimes}_\pi c_0 \widehat{\otimes}_\pi c_0$ is not isomorphic to a subspace of $c_0 \widehat{\otimes}_\pi c_0$.*

In the next section, we provide some preliminary definitions needed to prove Theorem 1.3, as well as indicate the strategy of the proof.

Finally, observe that Theorem 1.3 suggests some new questions. We only highlight one that is closely related to the subject of this work. As usual, we denote by $\widehat{\otimes}_\pi^n c_0$ the n -fold projective tensor product of c_0 .

PROBLEM 1.4. *Suppose that for some $m, n \in \mathbb{N}$ with $m, n \geq 3$, $\widehat{\otimes}_\pi^m c_0$ is isomorphic to $\widehat{\otimes}_\pi^n c_0$. Is it true that $m = n$?*

2. Preliminaries

Our notation is standard as may be found in [12]. We just remember that if X and Y are Banach spaces and $\mathcal{B}(X, Y)$ is the space of bounded bilinear functionals on $X \times Y$, then the projective tensor norm of $u = \sum_{i=1}^n a_i \otimes b_i \in X \otimes Y$ is defined by

$$\|u\| = \sup \left\{ \left| \sum_{i=1}^n \varphi(a_i, b_i) \right| : \varphi \in \mathcal{B}(X, Y), \|\varphi\| \leq 1 \right\}.$$

Thus, $X \widehat{\otimes}_\pi Y$ is the completion of $X \otimes Y$ with respect to this norm [12]. We denote by $\|\cdot\|_{X \widehat{\otimes}_\pi Y}$ the projective norm on $X \widehat{\otimes}_\pi Y$.

The idea behind our proof of Theorem 1.3 is to argue that $c_0 \widehat{\otimes}_\pi c_0$ is 2-asymptotically uniformly smoothable (shown by Dilworth and Kutzarova in [3, Theorem 9]), while $c_0 \widehat{\otimes}_\pi c_0 \widehat{\otimes}_\pi c_0$ is not. This amounts to exhibiting normalized, weakly null trees in $c_0 \widehat{\otimes}_\pi c_0 \widehat{\otimes}_\pi c_0$ which do not admit uniform upper ℓ_2 estimates on their branches, for which we will use the Hilbert matrices in a manner similar to Kwapien and Pełczyński's use of the Hilbert matrices in [9]. One consequence of the results of [9] is that

$$\left\| \sum_{i=1}^n e_i \otimes s_i \right\|_{c_0 \widehat{\otimes}_\pi c_0} \geq \Omega \log n,$$

where Ω is a constant > 0 , $(e_i)_{i=1}^\infty$ is the unit vector basis of c_0 and $s_i = \sum_{j=1}^i e_j$ denotes the summing basis of c_0 . The proof proceeded by using the Hilbert matrix h_n as a member of $(c_0 \hat{\otimes}_\pi c_0)^*$ to norm $\sum_{i=1}^n e_i \otimes s_i$. More precisely, the Hilbert matrix h_n was used to norm a tensor whose rows are a permutation of the rows of $\sum_{i=1}^n e_i \otimes s_i$. A crucial portion of that argument is the use of an appropriate upper estimate on the operator norm of h_n , when viewed as an operator from c_0 to ℓ_1 . We will use a similar upper estimate on the operator norm of h_n from c_0 to ℓ_2 and then use the Hilbert matrices (actually, row permutations of the Hilbert matrices) to norm $\sum_{i=1}^n e_i \otimes s_i \otimes f_i^n$ and provide the lower estimate

$$\left\| \sum_{i=1}^n e_i \otimes s_i \otimes f_i^n \right\|_{c_0 \hat{\otimes}_\pi c_0 \hat{\otimes}_\pi c_0} \geq \Omega n^{1/2} \log(n)$$

for a Rademacher system $(f_i^n)_{i=1}^n$ and a constant Ω . Note that for each n , we are using a different Rademacher system $(f_i^n)_{i=1}^n$. We then use this estimate to prove that $c_0 \hat{\otimes}_\pi c_0 \hat{\otimes}_\pi c_0$ is not 2-asymptotically uniformly smoothable, and is therefore not isomorphic to a subspace of $c_0 \hat{\otimes}_\pi c_0$. However, we will deal with 2-asymptotic uniform smoothness only implicitly, choosing to deal with weakly null trees instead. We started defining this notion.

For $n \in \mathbb{N}$, let

$$\mathcal{A}_n = \{(m_i)_{i=1}^l : 1 \leq l \leq n, m_1 < \dots < m_l, m_i \in \mathbb{N}\}.$$

Given $t \in \{\emptyset\} \cup \bigcup_{l=1}^\infty \mathcal{A}_l$ and $m \in \mathbb{N}$, we let $t < m$ denote the relation that either $t = \emptyset$ or $t = (m_1, \dots, m_l)$ and $m_l < m$. We let $\frown$ denote concatenation, so that if $t = \{\emptyset\} \cup \mathcal{A}_{n-1}$ and $t < m$, $m \in \mathbb{N}$, it follows that $t \frown (m) \in \mathcal{A}_n$.

Given a Banach space X , a family $(u_t)_{t \in \mathcal{A}_n}$ of X is said to be *weakly null* if for any $t \in \{\emptyset\} \cup \mathcal{A}_{n-1}$, $(u_{t \frown (m)})_{t < m}$ is a weakly null sequence in X .

For each $k \in \mathbb{N}$, let

$$E_k = \text{span}\{e_i \otimes e_j : \max\{i, j\} = k\} \subset c_0 \hat{\otimes}_\pi c_0.$$

Then the sequence $(E_k)_{k=1}^\infty$ is a Schauder finite dimensional decomposition (FDD) for $c_0 \hat{\otimes}_\pi c_0$. Moreover, since $(c_0 \hat{\otimes}_\pi c_0)^* = \mathfrak{L}(c_0, \ell_1) = \mathfrak{K}(c_0, \ell_1)$, the space of compact operators from c_0 to ℓ_1 , it follows that the sequence $(E_k^*)_{k=1}^\infty$ given by

$$E_k^* = \text{span}\{e_i^* \otimes e_j^* : \max\{i, j\} = k\}$$

is a FDD of $(c_0 \hat{\otimes}_\pi c_0)^*$. It was shown in [3] that this FDD $(E_k^*)_{k=1}^\infty$ satisfies a uniform ℓ_2 lower estimate. That is, there exists C_1 such that for any $n \in \mathbb{N}$, any integers $0 = r_0 < r_1 < \dots < r_n$, and any $u_i \in \text{span}\{E_j^* : r_{i-1} < j \leq r_i\}$,

$$C_1 \left\| \sum_{i=1}^n u_i \right\|_{(c_0 \hat{\otimes}_\pi c_0)^*}^2 \geq \sum_{i=1}^n \|u_i\|_{(c_0 \hat{\otimes}_\pi c_0)^*}^2.$$

By standard duality arguments, there exists a constant $C_2 > 0$ such that for any $k \in \mathbb{N}$, any integers $0 = r_0 < r_1 < \dots < r_n$, and any $u_i \in \text{span}\{E_j : r_{i-1} < j \leq r_i\}$,

$$\left\| \sum_{i=1}^n u_i \right\|_{c_0 \hat{\otimes}_\pi c_0}^2 \leq C_2 \sum_{i=1}^n \|u_i\|_{c_0 \hat{\otimes}_\pi c_0}^2.$$

Therefore for any $n \in \mathbb{N}$, any $C_3 > C_2$, and any weakly null family $(u_t)_{t \in \mathcal{A}_n}$ of $B_{c_0 \hat{\otimes}_\pi c_0}$, there exists $(m_1, \dots, m_n) \in \mathcal{A}_n$ such that

$$\left\| \sum_{i=1}^n u_{(m_1, \dots, m_i)} \right\|_{c_0 \hat{\otimes}_\pi c_0} \leq C_3 n^{1/2}.$$

We isolate this result in the following proposition.

PROPOSITION 2.1. *There exists a constant C such that for any $n \in \mathbb{N}$ and any weakly null family $(u_t)_{t \in \mathcal{A}_n}$ of $B_{c_0 \hat{\otimes}_\pi c_0}$, there exists $(m_1, \dots, m_n) \in \mathcal{A}_n$ such that*

$$\left\| \sum_{i=1}^n u_{(m_1, \dots, m_i)} \right\|_{c_0 \hat{\otimes}_\pi c_0} \leq C n^{1/2}.$$

We also note that the isolated property states in Proposition 2.1 is strictly weaker than 2-asymptotic uniform smoothability. Moreover, we will ultimately show that $c_0 \hat{\otimes}_\pi c_0 \hat{\otimes}_\pi c_0$ lacks this weaker property (Proposition 4.2), which means we will prove something stronger than the fact that $c_0 \hat{\otimes}_\pi c_0 \hat{\otimes}_\pi c_0$ is not 2-asymptotically uniformly smoothable. In particular, joining Propositions 2.1 and 4.2 it immediately follows that $c_0 \hat{\otimes}_\pi c_0 \hat{\otimes}_\pi c_0$ is not isomorphic to a subspace of $c_0 \hat{\otimes}_\pi c_0$, so we will have proved Theorem 1.3.

3. The Hilbert matrices h_n and the Rademacher system $(f_i^n)_{i=1}^n$

In this section first we define the *Hilbert matrices* h_n and some permuted versions thereof, which we denote by p_n . Throughout, our matrices will be identified with the operators they induce via matrix multiplication. We will denote the row i , column j entry of a matrix M by $M(i, j)$. We define

$$h_n(i, j) = \begin{cases} \frac{1}{n+1-i-j} & : i, j \leq n \text{ and } i+j \neq n+1 \\ 0 & : \text{otherwise} \end{cases}$$

and

$$p_n(i, j) = \begin{cases} \frac{1}{i-j} & : i, j \leq n \text{ and } i \neq j \\ 0 & : \text{otherwise.} \end{cases}$$

REMARK 3.1. Notice that for any $j \in \mathbb{N}$, $p_n(i, j) = h_n(n+1-i, j)$ for $1 \leq i \leq n$ and $p_n(i, j) = h_n(i, j)$ for all $i > n$. Therefore $\|h_n : \ell_2 \rightarrow \ell_2\| = \|p_n : \ell_2 \rightarrow \ell_2\|$ for all $n \in \mathbb{N}$. As noted in [9, Inequality 1.7], and there attributed to Titchmarsh [14], there exists a constant $\tau = \tau(2)$ such that for all $n \in \mathbb{N}$, $\|h_n : \ell_2 \rightarrow \ell_2\| \leq \tau$. Since the rows of p_n are simply the rows of h_n permuted, $\|p_n : \ell_2 \rightarrow \ell_2\| \leq \tau$ for all $n \in \mathbb{N}$. Since the map $I_n : c_0 \rightarrow \ell_2$ given by $I_n \sum_{i=1}^\infty a_i e_i = \sum_{i=1}^n a_i e_i$ has norm $n^{1/2}$, and since $p_n : c_0 \rightarrow \ell_2$ is equal to the composition $p_n I_n : c_0 \rightarrow \ell_2$, we have

$$(3.1) \quad \|p_n : c_0 \rightarrow \ell_2\| \leq \tau n^{1/2} \text{ for all } n \in \mathbb{N}.$$

Next we need to remember the definitions of the Haar and the Rademacher systems introduced by A. Pełczyński and Singer [10] in a 2^n -dimensional space with respect to a symmetric basis $(x_i)_{1 \leq i \leq 2^n}$. The Haar system $(y_i)_{1 \leq i \leq 2^n}$ is the sequence defined by

$$y_1 = \sum_{i=1}^{2^n} x_i, \quad y_{2^k+l} = \sum_{i=1}^{2^n} \beta_i^{k,l} x_i, \quad (l = 1, \dots, 2^k; k = 0, \dots, n-1)$$

where

$$\beta_i^{k,l} = \begin{cases} 1 & \text{for } (2l-2)2^{n-k-1} + 1 \leq i \leq (2l-1)2^{n-k-1} \\ -1 & \text{for } (2l-1)2^{n-k-1} + 1 \leq i \leq 2l \cdot 2^{n-k-1} \\ 0 & \text{for } 1 \leq i \leq (2l-2)2^{n-k-1} \text{ and } 2l \cdot 2^{n-k-1} + 1 \leq i \leq 2^n \end{cases}$$

We shall call Rademacher system the sequence $(r_k)_{1 \leq k \leq n}$ defined by

$$r_k = \sum_{l=1}^{2^{k-1}} y_{2^{k-1}+l}$$

We denote $(f_i^n)_{1 \leq i \leq n}$ the Rademacher system associated to the unit basis $\ell_\infty^{2^n}$ and $(g_i^n)_{1 \leq i \leq n}$ the normalized Rademacher system associated to the unit basis $\ell_1^{2^n}$. In the duality $\langle \ell_1^{2^n}, \ell_\infty^{2^n} \rangle$ we have $g_i^n(f_i^n) = 1$.

LEMMA 3.2. *For any scalars $(a_i)_{1 \leq i \leq n}$ we have*

$$\left\| \sum_{i=1}^n a_i g_i^n \right\| \leq \left(\sum_{i=1}^n |a_i|^2 \right)^{\frac{1}{2}}$$

PROOF. Let $(r_k)_{1 \leq k}$ be the sequence of the usual Rademacher system. It follows from the claim (10) of [10] and Hölder's inequality that

$$\left\| \sum_{i=1}^n a_i g_i^n \right\| = \int_{[0,1]} \left| \sum_{i=1}^n a_i r_i(t) \right| dt \leq \left(\int_{[0,1]} \left| \sum_{i=1}^n a_i r_i(t) \right|^2 dt \right)^{\frac{1}{2}}.$$

The usual Rademacher system is orthonormal so the right hand inequality follows. $\square$

The following lemma will play a key role in section 4.

LEMMA 3.3. *For every integer n there exists a unique bounded linear operator $P_n : c_0 \widehat{\otimes}_\pi c_0 \rightarrow \ell_1$ such that $P_n(e_i \otimes e_j) = p_n(i, j) g_i^n$. Moreover $\|P_n\| \leq \tau n^{\frac{1}{2}}$.*

PROOF. It is obvious that there exists a bilinear map b_n from $c_0 \times c_0 \rightarrow \ell_1$ such that $b_n(e_i, e_j) = p_n(i, j) g_i^n$. We shall show that b_n is bounded. Let $x = \sum_{i=1}^\infty a_i e_i$ and $y = \sum_{j=1}^\infty b_j e_j$ be two elements of B_{c_0} . Then

$$b_n(x, y) = \sum_{i=1}^n \sum_{j=1}^n a_i b_j p_n(i, j) g_i^n = \sum_{i=1}^n a_i \left(\sum_{j=1}^n b_j p_n(i, j) \right) g_i^n.$$

By Lemma 3.2 and (3.1) it follows that

$$\|b_n(x, y)\| \leq \left(\sum_{i=1}^n \left| a_i \sum_{j=1}^n b_j p_n(i, j) \right|^2 \right)^{\frac{1}{2}} \leq \left(\sum_{i=1}^n \left| \sum_{j=1}^n b_j p_n(i, j) \right|^2 \right)^{\frac{1}{2}} \leq \tau n^{\frac{1}{2}}.$$

$\square$

4. On the geometric structure of $c_0 \hat{\otimes}_\pi c_0 \hat{\otimes}_\pi c_0$

The objective of this last section is to prove Proposition 4.2. It contains the fact that $c_0 \hat{\otimes}_\pi c_0 \hat{\otimes}_\pi c_0$ lacks the previously isolated property states in Proposition 2.1.

LEMMA 4.1. *There exists a constant $\Delta > 0$ such that for any $n \in \mathbb{N}$,*

$$\left\| \sum_{i=1}^n e_i \otimes s_i \otimes f_i^n \right\|_{\hat{\otimes}_\pi^3 c_0} \geq \Delta n^{1/2} \log(n).$$

PROOF. We recall that the spaces $(c_0 \hat{\otimes}_\pi c_0 \hat{\otimes}_\pi c_0)^*$ and $\mathcal{L}(c_0 \hat{\otimes}_\pi c_0, \ell_1)$ are isometrically isomorphic so, for every integer n ,

$$\tau n^{\frac{1}{2}} \left\| \sum_{i=1}^n e_i \otimes s_i \otimes f_i^n \right\|_{\hat{\otimes}_\pi^3 c_0} \geq \left| \sum_{i=1}^n P_n(e_i \otimes s_i)(f_i^n) \right| = \left| \sum_{i=1}^n \sum_{j=1}^i p_n(i, j) g_i^n(f_i^n) \right|$$

and letting $k = i - 1, l = i - j$, by an elementary computation we have

$$\geq \sum_{k=1}^{n-1} \sum_{l=1}^k \frac{1}{l} \geq \frac{1}{2} n \log n.$$

We conclude by letting $\Delta = 2\tau$. $\square$

PROPOSITION 4.2. *There exists a constant $\Delta > 0$ such that for all $n \in \mathbb{N}$, there exists a weakly null tree $(u_t)_{t \in \mathcal{A}_n}$ of $B_{\hat{\otimes}_\pi^3 c_0}$ such that, for every $(m_1, \dots, m_n) \in \mathcal{A}_n$,*

$$\left\| \sum_{i=1}^n u_{(m_1, \dots, m_i)} \right\|_{\hat{\otimes}_\pi^3 c_0} \geq \Delta n^{1/2} \log(n).$$

PROOF. Let Δ be the constant from Lemma 4.1. Fix $n \in \mathbb{N}$. For $(m_1, \dots, m_i) \in \mathcal{A}_n$, let $u_t = e_{m_i} \otimes s_i \otimes f_i^n \in S_{\hat{\otimes}_\pi^3 c_0}$. By 1-symmetry of the unit basis of c_0 , for any $(m_1, \dots, m_n) \in \mathcal{A}_n$,

$$\begin{aligned} \left\| \sum_{i=1}^n u_{(m_1, \dots, m_i)} \right\|_{\hat{\otimes}_\pi^3 c_0} &= \left\| \sum_{i=1}^n e_{m_i} \otimes s_i \otimes f_i^n \right\|_{\hat{\otimes}_\pi^3 c_0} = \left\| \sum_{i=1}^n e_i \otimes s_i \otimes f_i^n \right\|_{\hat{\otimes}_\pi^3 c_0} \\ &\geq \Delta n^{1/2} \log(n). \end{aligned}$$

It remains to show that $(u_t)_{t \in \mathcal{A}_n}$ is weakly null. For this, fix $t \in \{\emptyset\} \cup \mathcal{A}_{n-1}$ and let $0 \leq l$ be the length of t . Then for each $t < m$, $u_{t \wedge (m)} = e_m \otimes s_{l+1} \otimes f_{l+1}^n$. Since l does not depend on m , and since $\|s_{l+1}\|_{c_0} = \|f_{l+1}^n\|_{c_0} = 1$, it follows that

$$(u_{t \wedge (m)})_{t < m} = (e_m \otimes s_{l+1} \otimes f_{l+1}^n)_{t < m}^\infty$$

is isometrically equivalent to the canonical c_0 basis in $\hat{\otimes}_\pi^3 c_0$, and therefore a weakly null sequence. $\square$

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