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▶ To cite this version:

Li-Ping Wang, Yi-Ming Chen, Da-Yan Liu, Driss Boutat. Numerical algorithm to solve generalized fractional pantograph equations with variable coefficients based on shifted Chebyshev polynomials. International Journal of Computer Mathematics, 2019, 96 (12), pp.2487-2510. 10.1080/00207160.2019.1573992 . hal-02884254

HAL Id: hal-02884254
https://hal.science/hal-02884254

Submitted on 4 May 2021

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Numerical solutions of generalized fractional pantograph equations with variable coefficients using shifted Chebyshev polynomials

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Abstract

In this paper, an efficient numerical technique based on the shifted Chebyshev polynomials (SCPs) is established to obtain numerical solutions of generalized fractional pantograph equations with variable coefficients. These polynomials are orthogonal and have compact support on $[0, L]$. We use these polynomials to approximate the unknown function. Using the properties of the SCPs, we derive the generalized pantograph operational matrix of SCPs and the one of fractional-order differentiation. Then the original problems can be transformed to a system of algebraic equations based on these matrices. By solving these algebraic equations, we can obtain numerical solutions. In addition, we investigate the error analysis and introduce the process of error correction for improving the precision of numerical solutions. Lastly, by giving some examples and comparing with other existing methods, the validity and efficiency of our method is demonstrated.

Keywords: Shifted Chebyshev polynomials, Operational matrix, Generalized fractional pantograph equations, Numerical solution, Error analysis

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1. Introduction

Fractional calculus is a branch of calculus theory, which makes calculus theory more perfect. In recent decades, fractional calculus have been widely used in various areas, such as viscoelasticity [1, 2], economics [3], control theory [4, 5], and fractals dynamics [6]. One of the interesting research topics is the design of fractional differentiators [7] to compute fractional differentials of unknown signals in a noisy environment. With the application of fractional differential equations in more and more scientific fields, the study of numerical calculations of the differential equations of fractional order is particularly important. At present, the majority of scholars have studied different kind of vigorous numerical methods to obtained an approximate solution of fractional differential equations. These methods include Chebyshev collocation method [8], Laplace transform method [9], differential transform method [10, 11], Adomian decomposition method [12], Legendre operational matrix [13], and CAS wavelet method [14], etc.

Delay differential equations have many applications in different fields, such as biological, industrial, electronic, chemical and transportation systems [15, 16, 17]. To obtain the numerical solutions of delay differential equations, Many researchers have studied different kind of vigorous techniques [18]. The functional differential equations with proportional delay are generally called to pantograph equations or generalized pantograph equations. As the one of the most important types of delay differential equations, the pantograph equation or the generalized pantograph equation can explaining various physical phenomena. And they are used in many fields. In recent years, there have been many numerical methods for solving pantograph differential equations or generalized pantograph equations of integer order, such as Chebyshev polynomials [19], Bernoulli polynomials [20], variational iteration method [21], etc. Further, [22] introduces the stability properties of many numerical techniques for nonlinear generalized pantograph equations.

The fractional delay differential equation is a generalization of the delay differential equation to arbitrary non-integer order. Fractional delay differential equations are adapted to many fields, such as hydraulic networks, automatic control, long transmission lines, economy and biology [23]. The numerical calculation of fractional delay differential equations has also attracted the attention of many scholars. Because the fractional delay differential equations cannot be analytically solved, different numerical methods [24, 25, 26] have been devoted to obtain the approximate solutions. Sherif et al. [27] considered the Spline functions to solve fractional delay differential equation. Authors of [28] investigated modified Laguerre wavelets method. Modified Chebyshev wavelet methods and a operational matrix based on Bernoulli wavelets are utilized in [29, 30]. However, there are few scholars that pay attention to study the numerical methods of fractional pantograph delay differential equations. From these works we can mention, Y Yang and Y Huang [32] have studied the existence of solutions of nonlinear fractional pantograph equations with the order of the derivative is in $[0, 1]$; Using spectral-collocation methods, Yang and Huang [31] obtained the approximate solution for fractional pantograph delay-integro-differential equations; the approximate solution of fractional pantograph differential equations can be obtained by using the explicit formula of the generalized fractional-order Bernoulli wavelet in [33].

The polynomial approximation theory is an important branch of the function approximation theory. As the name suggests, polynomials are used to approximate a function whose analytical form is more complex or whose analytical form is unknown. In general, the polynomial has many advantages, such as its structure is clear, its calculation is simple and it is relatively easy to integral and derivative. For some complex problems, applying polynomials to approximate function , and then studying the laws of actual problems, the problems can be simplified. At present, polynomial approximation theory has been widely used in different fields, such as numerical approximation theory, engineering calculation, and practical life. In this paper, based on the properties of the shifted Chebyshev polynomials, we derive SCPs generalized pantograph operational matrix. And with the aid of the operational matrix of fractional differentiation of SCPs, generalized pantograph operational matrix of fractional-order differentiation is obtained. We combine polynomial approximation theory and operational matrix to solve the following generalized fractional pantograph equation with variable coefficients

$$ {}^{c}D^{\beta}u\left(t\right)=b\left(t\right)u\left(t\right)+\sum_{j=0}^{J}\sum_{n=0}^{r-1}v_{j,n}\left(t\right){}^{c}D^{\alpha_n}u\left(q_{j,n}t-r_{j,n}\right)+g\left(t\right), \quad 0\le t\le L, \tag{1} $$

subject to the initial conditions

$$ u^{(n)}\left(0\right)=d_n, \quad n=0,1,\cdots,r-1, \tag{2} $$

where d_n, $q_{j,n}$ and $r_{j,n}$ are real or complex coefficients, $r-1<\alpha\le r$, $0<\alpha_0<\alpha_1<\cdots<\alpha_{r-1}<\beta$, while $b\left(t\right)$, $v_{j,n}\left(t\right)$ and $g\left(t\right)$ are continuous functions in the interval $\left[0,L\right]$, ${}^{c}D^{\beta}$ and ${}^{c}D^{\alpha_n}$ denote fractional derivatives in the Caputo's sense.

The rest of the paper is organized as follows: Section 2 introduces some mathematical preliminaries of fractional calculus. In Section 3, we review the basic definitions of shifted Chebyshev polynomials and discuss the polynomial approximation theory. In Section 4, we derive the SCPs generalized pantograph operational matrix and the one of fractional-order differentiation. In Section 5, we apply the proposed method to solve the generalized fractional pantograph equations. The error correction and error analysis are given in Section 6. In Section 7, the proposed approach is tested through several numerical examples. Finally, a conclusion is given in Section 8.

2. Basic definitions of fractional calculus

In this section, we review the necessary definitions and preliminaries of fractional calculus theory that will be used in this article.

Definition 1. *The Riemann-Liouville fractional integral operator of $f\left(t\right)$ is defined as*

$$
{}^{RL}I^{\beta} f(t) = \begin{cases} \dfrac{1}{\Gamma(\beta)} \displaystyle\int_0^t (t-T)^{\beta-1} f(T)\, dT, & \beta > 0,\ t > 0, \\ f(t), & \beta = 0. \end{cases} \tag{3}
$$

The Riemann-Liouville fractional differential operator of order β is derived by the definition of the Riemann-Liouville fractional integral operator

$$
{}^{RL}D^{\beta} f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\beta)} \dfrac{\mathrm{d}^n}{\mathrm{d}t^n} \displaystyle\int_0^t \dfrac{f(T)}{(t-T)^{\beta-n+1}} \mathrm{d}T, & \beta > 0,\ n-1 \le \beta < n, \\ \dfrac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}, & \beta = n,\ t > 0. \end{cases} \tag{4}
$$

Definition 2. *The fractional differential operator of order β in the Caputo sense is defined as*

$$
{}^{c}D^{\beta} f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\beta)} \displaystyle\int_0^t \dfrac{f^{(n)}(T)}{(t-T)^{\beta-n+1}} dT, & n-1 \le \beta < n, \\ \dfrac{d^n f(t)}{dt^n}, & \beta = n,\ t > 0. \end{cases} \tag{5}
$$

For the Caputo differential operator, we have

$$
{}^{c}D_t^{\beta} t^m = \begin{cases} 0, & for\ m \in \mathrm{N}_0\ and\ m < \lceil \beta \rceil, \\ \dfrac{\Gamma(m+1)}{\Gamma(m+1-\beta)} t^{m-\beta}, & for\ m \in \mathrm{N}_0\ and\ m \ge \lceil \beta \rceil\ \ or\ m \notin \mathrm{N}_0\ and\ m > \lfloor \beta \rfloor. \end{cases} \tag{6}
$$

where $\mathrm{N}_0 = \{0, 1, 2, \cdots\}$.

3. Shifted Chebyshev polynomials

3.1. Properties of the shifted Chebyshev polynomials

The well-known Chebyshev polynomials are defined on the interval $[-1, 1]$, and are derived by orthogonalizing the sequence $\{1, t, \cdots t^n \cdots\}$.

The specific form of the Chebyshev polynomials can be obtained by the following recurrence relation

$$
T_{i+1}(z) = 2(2z-1) T_i(z) - T_{i-1}(z), \quad i = 1, 2 \cdots
$$

where $T_0(z) = 1$ and $T_1(z) = z$. To use the Chebyshev polynomials on the extended interval $[0, L]$, it is necessary to shift the defining domain $[-1, 1]$. The shifted Chebyshev polynomials on $[0, L]$ can easily be derived by introducing the change of variables $z = \frac{2t}{L} - 1$. We can denote the shifted Chebyshev polynomials $T_i\left(\frac{2t}{L} - 1\right)$ by $H_{L,i}(t)$, then $H_{L,i}(t)$ can be determined with the aid of the following recurrence formula

$$H_{L,i+1}(t) = 2\left(\frac{2t}{L} - 1\right) H_{L,i}(t) - H_{L,i-1}(t), \quad i = 1, 2, \cdots$$

where $H_{L,0}(t) = 1$ and $H_{L,1}(t) = \frac{2t}{L} - 1$. The analytical form of the shifted Chebyshev polynomials $H_{L,i}(t)$ of degree i is given by

$$H_{L,i}(t) = T_i\left(\frac{2t}{L} - 1\right) = i \sum_{k=0}^{i} (-1)^{i-k} \frac{(i+k-1)!}{(2k)!\,(i-k)!} \frac{(2)^{2k}}{L^k} (t)^k, \quad i = 1, 2, \cdots \tag{7}$$

where $H_{L,i}(0) = (-1)^i$ and $H_{L,i}(L) = 1$.

The shifted Chebyshev polynomials satisfy the following orthogonality relation

$$\int_0^L H_{L,j}(t)\, H_{L,k}(t)\, \omega_L(t)\, dt = h_\kappa, \tag{8}$$

where the weight function $\omega_L(t) = \frac{1}{\sqrt{Lt - t^2}}$ and $h_k = \begin{cases} \frac{b_k}{2}\pi, & k = j, \\ 0, & k \neq j, \end{cases}$ $b_0 = 2$, $b_k = 1$, $k \geq 1$.

3.2. Function approximation

A function $u(t) \in L^2([0, L])$ can be expanded in terms of the shifted Chebyshev polynomials as follows

$$u(t) = \sum_{i=0}^{\infty} c_i H_{L,i}(t), \tag{9}$$

where the coefficients c_i are obtained by

$$c_i = \frac{1}{h_i} \int_0^L u(t) H_{L,i}(t)\, \omega_L(t)\, dt, \quad i = 0, 1, 2, \cdots$$

If we consider truncated series in Eq. (9), we can get

$$u(t) \approx \sum_{i=0}^{m} c_i H_{L,i}(t) = C^{\mathrm{T}} \Phi_m(t), \tag{10}$$

where

$$C = \left[c_0, c_1, \cdots, c_m\right]^{\mathrm{T}}, \Phi_m\left(t\right) = \left[H_{L,0}\left(t\right), H_{L,1}\left(t\right), \cdots, H_{L,m}\left(t\right)\right]^{\mathrm{T}}. \tag{11}$$

4. SCPs operational matrix for solving the pantograph equations

In this part, we derive the necessary SCPs generalized pantograph operational matrix and the fractional generalized pantograph operational matrix .

From Eq. (7), $\Phi_m\left(t\right)$ can be denoted by the product of two matrices

$$\Phi_m\left(t\right) = AZ_m\left(t\right), \tag{12}$$

where

$$Z_m\left(t\right) = \left[1, t, \cdots, t^m\right]^{\mathrm{T}}.$$

The matrix A is SCPs coefficient matrix and we assume each item of the matrix A can be write as follows

$$A = \begin{bmatrix} P_{0,0} & 0 & ... & 0 \\ P_{1,0} & P_{1,1} & ... & 0 \\ ... & ... & \ddots & \vdots \\ P_{m,0} & P_{m,1} & ... & P_{m,m} \end{bmatrix}, \tag{13}$$

where

$$\begin{cases} P_{0,0} = 1, \\ P_{i,j} = 2\left(\dfrac{2}{L}P_{i-1,j-1} - P_{i-1,j}\right) - P_{i-2,j}, \\ P_{i,j} = 0, \quad for\ i < j\ or\ i < 0\ or\ j < 0. \end{cases}$$

According to [34], inverse matrix of the coefficient matrix A can be expressed as follows

$$A^{-1} = \begin{bmatrix} P^{-1}_{0,0} & 0 & 0 & \cdots & 0 & 0 \\ P^{-1}_{0,0}a_{1,2} & P^{-1}_{1,1} & 0 & \cdots & 0 & 0 \\ P^{-1}_{0,0}a_{1,3} & P^{-1}_{1,1}a_{2,3} & P^{-1}_{2,2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ P^{-1}_{0,0}a_{1,m} & P^{-1}_{1,1}a_{2,m} & P^{-1}_{2,2}a_{3,m} & \cdots & P^{-1}_{m-1,m-1} & 0 \\ P^{-1}_{0,0}a_{1,m+1} & P^{-1}_{1,1}a_{2,m+1} & P^{-1}_{2,2}a_{3,m+1} & \cdots & P^{-1}_{m-1,m-1}a_{m,m+1} & P^{-1}_{m,m} \end{bmatrix}, \tag{14}$$

where

$$\begin{cases} a_{i,i+1} = -P_{i,i}^{-1} P_{i,i-1}, \quad i = 1, 2, \cdots m, \\ a_{i,j} = -P_{j-1,j-1}^{-1} \left(P_{j-1,i-1} + \sum\limits_{i<k<j} a_{i,k} P_{j-1,k-1} \right), \quad i = 1, 2, \cdots m-1, \ j = 3, 4, \cdots m+1. \end{cases}$$

4.1. SCPs generalized pantograph operational matrix

The shifted Chebyshev vector with delay parameter $r\,(0 < r < 1)$ and pantograph coefficient $q\,(0 < q < 1)$ is given as follows

$$\Phi_m (qt - r) = \left[H_{L,0} (qt - r), H_{L,1} (qt - r), \cdots, H_{L,i} (qt - r), \cdots, H_{L,m} (qt - r) \right]^{\mathrm{T}}. \tag{15}$$

Theorem 1. *Let $\Phi_m (qt)$ be the special case of $r = 0$ in Eq. (15) and suppose $0 < q < 1$, then*

$$\Phi_m (qt) = F \Phi_m (t), \tag{16}$$

where the matrix F is called the pantograph operational matrix of SCPs, and it is defined as follows

$$F = \begin{bmatrix} f_{0,0} & 0 & \cdots & 0 & 0 \\ f_{1,0} & f_{1,1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{m-1,0} & f_{m-1,1} & \cdots & f_{m-1,m-1} & 0 \\ f_{m,0} & f_{m,1} & \cdots & f_{m,m-1} & f_{m,m} \end{bmatrix}$$

where

$$f_{i,j} = \begin{cases} P_{j,j}^{-1} \left(P_{i,j} q^j + \sum\limits_{l=j+1}^{i} P_{i,l} q^l a_{j+1,l+1} \right), & i \neq j \\ q^i, & i = j \end{cases}$$

$$i = 0, 1, \cdots, m.$$

Proof. The $H_{L,i} (qt), i = 0, 1, \cdots m$ must be expanded in terms of $\left(H_{L,j} (t) \right)_{j=0,1,\cdots i}$. Let

$$H_{L,i} (qt) = \sum_{j=0}^{i} f_{i,j} H_{L,j} (t).$$

And then we can get

$$\Phi_m(qt) = \begin{bmatrix} f_{0,0} & 0 & \cdots & 0 & 0 \\ f_{1,0} & f_{1,1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{m-1,0} & f_{m-1,1} & \cdots & f_{m-1,m-1} & 0 \\ f_{m,0} & f_{m,1} & \cdots & f_{m,m-1} & f_{m,m} \end{bmatrix} \Phi_m(t) = F\Phi_m(t), \quad (17)$$

where

$$F = \begin{bmatrix} f_{0,0} & 0 & \cdots & 0 & 0 \\ f_{1,0} & f_{1,1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{m-1,0} & f_{m-1,1} & \cdots & f_{m-1,m-1} & 0 \\ f_{m,0} & f_{m,1} & \cdots & f_{m,m-1} & f_{m,m} \end{bmatrix}.$$

145 According to Eq. (12), we get

$$\begin{aligned} \Phi_m(qt) = AZ_m(qt) = A \begin{bmatrix} 1 \\ qt \\ q^2t^2 \\ \vdots \\ q^mt^m \end{bmatrix} \\ = A \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & q & 0 & \cdots & 0 \\ 0 & 0 & q^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & q^m \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^m \end{bmatrix} \\ = A \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & q & 0 & \cdots & 0 \\ 0 & 0 & q^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & q^m \end{bmatrix} A^{-1}\Phi_m(t). \end{aligned}$$

By substituting the formula of matrix A^{-1} into the above equation, we obtain

$$\Phi_m(qt) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ P_{0,0}^{-1}\left(P_{1,0} + P_{1,1}qa_{1,2}\right) & q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ P_{0,0}^{-1}\left(P_{m,0} + \sum\limits_{i=1}^{m} P_{m,i}q^i a_{1,i+1}\right) & P_{1,1}^{-1}\left(P_{m,1}q + \sum\limits_{i=2}^{m} P_{m,i}q^i a_{2,i+1}\right) & \cdots & q^m \end{bmatrix} \Phi_m(t),$$

Combining the above equation with Eq. (17), we can get the recurrence formula of $f_{i,j}$

$$f_{i,j} = \begin{cases} P_{j,j}^{-1}\left(P_{i,j}q^j + \sum\limits_{l=j+1}^{i} P_{i,l}q^l a_{j+1,l+1}\right), & i \neq j \\ q^i, & i = j \end{cases}$$

Theorem. 1 is proved . ■

Theorem 2. *Consider $\Phi_m(qt - r)$ is the shifted Chebyshev vector defined in Eq. (15) and suppose $0 < q < 1$, $0 < r < 1$, then*

$$\Phi_m(qt - r) = W\Phi_m(t), \tag{18}$$

where the matrix W is called the generalized pantograph operational matrix of SCPs, and it is defined as follows

$$W = \left[w_0, w_1, \cdots, w_i, \cdots, w_m\right]^{\mathrm{T}}, \quad i = 0, 1, \cdots m,$$

where

$$w_i = \sum_{j=0}^{i} P_{i,j}N_j,$$

$$N_j = \sum_{k=0}^{j} \binom{j}{k} (-r)^{j-k} \left[\alpha_0, \alpha_1, \cdots, \alpha_n, \cdots \alpha_k, 0, \cdots, 0\right],$$

where

$$\alpha_n = \sum_{l=n}^{k} P_{l,l}^{-1} a_{l+1,k+1} f_{l,n},$$

where

$$f_{l,n} = \begin{cases} P_{n,n}^{-1}\left(P_{l,n}q^n + \sum\limits_{s=n+1}^{l} P_{l,s}q^s a_{n+1,s+1}\right), & l \neq n, \\ q^l, & l = n. \end{cases}$$

Proof. By Eq. (12), we have

$$\Phi_m\left(qt-r\right)=AZ_m\left(qt-r\right)=A\begin{bmatrix}1\\ qt-r\\ \left(qt-r\right)^2\\ \vdots\\ \left(qt-r\right)^i\\ \vdots\\ \left(qt-r\right)^m\end{bmatrix}. \tag{19}$$

We expand the formula $\left(qt-r\right)^i$ in Eq. (19)

$$\left(qt-r\right)^i=\sum_{k=0}^{i}\begin{pmatrix}i\\ k\end{pmatrix}\left(-r\right)^{i-k}(qt)^k. \tag{20}$$

And by Eq. (12), we get

$$Z_m\left(qt\right)=\begin{bmatrix}1\\ qt\\ \left(qt\right)^2\\ \vdots\\ \left(qt\right)^i\\ \vdots\\ \left(qt\right)^m\end{bmatrix}=A^{-1}\Phi_m\left(qt\right),$$

therefore

$$\left(qt\right)^i=A^{-1}_{[i+1]}\Phi_m\left(qt\right), \tag{21}$$

where $A^{-1}_{[i+1]}$ is the $\left(i+1\right)^{th}$ row of A^{-1}, $i=0,1,\cdots,m$.

By substituting Eqs. (16) and (21) into Eq. (20), we obtain

$$\left(qt-r\right)^i=\sum_{k=0}^{i}\begin{pmatrix}i\\ k\end{pmatrix}\left(-r\right)^{i-k}A^{-1}_{[k+1]}F\Phi_m\left(t\right).$$

Using the formula of A^{-1} and Eq. (16), $A^{-1}_{[k+1]}F$ can be written as follows

$$A^{-1}_{[k+1]}F=\left[\alpha_0,\alpha_1,\cdots,\alpha_n,\cdots\alpha_k,0,\cdots,0\right],$$

where

$$\alpha_n = \sum_{l=n}^{k} P_{l,l}^{-1} a_{l+1,k+1} f_{l,n}.$$

In conclude

$$\Phi_m\left(qt-r\right) = A \begin{bmatrix} N_0 \\ N_1 \\ \vdots \\ N_i \\ \vdots \\ N_m \end{bmatrix} \Phi_m\left(t\right) = W\Phi_m\left(t\right),$$

where

$$N_i = \sum_{k=0}^{i} \binom{i}{k} \left(-r\right)^{i-k} \left[\alpha_0, \alpha_1, \cdots, \alpha_n, \cdots \alpha_k, 0, \cdots, 0\right]. \tag{22}$$

$$W = A \begin{bmatrix} N_0 \\ N_1 \\ \vdots \\ N_i \\ \vdots \\ N_m \end{bmatrix} = \begin{bmatrix} P_{0,0}N_0 \\ P_{1,0}N_0 + P_{1,1}N_1 \\ \vdots \\ P_{i,0}N_0 + P_{i,1}N_1 + \cdots + P_{i,i}N_i \\ \vdots \\ P_{m,0}N_0 + P_{m,1}N_1 + \cdots + P_{m,m}N_m \end{bmatrix}.$$

Let $W = \left[w_0, w_1, \cdots, w_i, \cdots w_m\right]^{\mathrm{T}}$, hence the precise expression of w_i can be concluded as follows

$$w_i = \sum_{j=0}^{i} P_{i,j} N_j. \tag{23}$$

By substituting Eq. (22) into Eq. (23), Theorem. 2 is proved . ■

4.2. SCPs operational matrix of derivative

In order to build the SCPs operational matrix of derivative, the differentiation of vector $\Phi_m\left(t\right)$ can be expressed by

$$\Phi_m^{(1)}\left(t\right) = P^{(1)}\Phi_m\left(t\right), \tag{24}$$

where $P^{(1)}$ is called the $(m+1)\times(m+1)$ SCPs operational matrix of derivative. According to Eq. (12), we can get

$$\Phi_m^{(1)}(t) = A \begin{bmatrix} 0 \\ 1 \\ \vdots \\ mt^{m-1} \end{bmatrix} = AV_{(m+1)\times m} Z_m^*(t), \tag{25}$$

where

$$V_{(m+1)\times m} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m \end{bmatrix},$$

$$Z_{_m}^*(t) = \begin{bmatrix} 1 \\ t \\ \vdots \\ t^{m-2} \\ t^{m-1} \end{bmatrix}.$$

We now expand vector $Z_{_m}^*(t)$ in terms of $\Phi_m(t)$. From Eq. (12), we have

$$Z_{_m}^*(t) = B^* \Phi_m(t), \tag{26}$$

where

$$B^* = \begin{bmatrix} A_{[1]}^{-1} \\ A_{[2]}^{-1} \\ \vdots \\ A_{[m]}^{-1} \end{bmatrix},$$

$A_{[k]}^{-1}$ is the k^{th} row of A^{-1}, $k = 1, 2, \cdots, m$.

Then Eq. (25) can be rewritten as

$$\Phi_{_m}^{(1)}(t) = AV_{(m+1)\times m} B^* \Phi_m(t). \tag{27}$$

Therefore we have the operational matrix of derivative as

$$P^{(1)} = AV_{(m+1)\times m}B^*.$$

Further, we can get

$$\Phi_m^{(n)}(t) = \left(P^{(1)}\right)^n \Phi_m(t), \quad n = 1, 2, \cdots \tag{28}$$

When $n = 1$, from Eq. (24), we get

$$\Phi_m^{(1)}(t) = P^{(1)}\Phi_m(t).$$

Suppose Eq. (28) is correct, when $n = s$. Then we obtain

$$\Phi_m^{(s)}(t) = \left(P^{(1)}\right)^s \Phi_m(t).$$

Thus, when $n = s + 1$, we have

$$\begin{aligned} \Phi_m^{(s+1)}(t) &= \frac{\partial^s}{\partial t^s}\left(\frac{\partial \Phi_m(t)}{\partial t}\right) = P^{(1)}\frac{\partial^s}{\partial t^s}\Phi_m(t) \\ &= P^{(1)}\left(P^{(1)}\right)^s \Phi_m(t) = \left(P^{(1)}\right)^{s+1}\Phi_m(t) \end{aligned}.$$

For any integer s, the Eq. (28) holds.

Therefore, Eq. (28) can be proved.

4.3. SCPs generalized pantograph operational matrix of fractional-order differentiation

In order to build the operational matrix of fractional-order differentiation of SCPs. Let

$$^cD^\beta \Phi_m(t) = P^\beta(t)\Phi_m(t), \quad \beta > 0, \tag{29}$$

where $\Phi_m(t)$ is the shifted Chebyshev vector defined in Eq. (10) and the matrix P^β is called the SCPs operational matrix of fractional derivatives.

Theorem 3. *Suppose P^β is the SCPs operational matrix of Caputo fractional-*

order differentiation of order $\beta > 0$, then the elements of P^{β} are given as follows

$$P^{\beta}(t) = \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ S_{\beta}(\lceil\beta\rceil, 0) & \cdots & S_{\beta}(\lceil\beta\rceil, \lceil\beta\rceil) & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ S_{\beta}(i, 0) & \cdots & S_{\beta}(i, \lceil\beta\rceil) & \cdots & S_{\beta}(i, i) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ S_{\beta}(m, 0) & \cdots & S_{\beta}(m, \lceil\beta\rceil) & \cdots & S_{\beta}(m, i) & \cdots & S_{\beta}(m, m) \end{bmatrix},$$

where

$$S_{\beta}(i, j) = \sum_{k=\lceil\beta\rceil}^{i} t^{-\beta} P^{-1}_{j,j} a_{j+1,k+1} P_{i,k} \frac{\Gamma(k+1)}{\Gamma(k+1-\beta)}, \quad i = \lceil\beta\rceil, \lceil\beta\rceil + 1, \cdots, m.$$

Proof. From Eq. (12) , we get

$$^{c}D^{\beta}\Phi_m(t) = A\,^{c}D^{\beta}Z_m(t) = A\,^{c}D^{\beta} \begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^m \end{bmatrix}. \tag{30}$$

Using Eq. (6), we can derive $^{c}D^{\beta}Z_m(t)$ in Eq. (30) as

$$^{c}D^{\beta}Z_m(t) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \frac{\Gamma(\lceil\beta\rceil+1)}{\Gamma(\lceil\beta\rceil+1-\beta)} t^{\lceil\beta\rceil-\beta} \\ \vdots \\ \frac{\Gamma(i+1)}{\Gamma(i+1-\beta)} t^{i-\beta} \\ \vdots \\ \frac{\Gamma(m+1)}{\Gamma(m+1-\beta)} t^{m-\beta} \end{bmatrix}, i = \lceil\beta\rceil, \lceil\beta\rceil + 1, \cdots, m. \tag{31}$$

Define the $(m+1)\times(m+1)$ matrix $V^*_{(m+1)\times(m+1)}(t)$ as

$$V^*_{(m+1)\times(m+1)}(t)=\begin{bmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \frac{\Gamma(\lceil\beta\rceil+1)}{\Gamma(\lceil\beta\rceil+1-\beta)}t^{-\beta} & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & \frac{\Gamma(i+1)}{\Gamma(i+1-\beta)}t^{-\beta} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & \frac{\Gamma(m+1)}{\Gamma(m+1-\beta)}t^{-\beta} \end{bmatrix}.$$

Eq. (31) may be restated as

$$^cD^\beta Z_m(t)=V^*_{(m+1)\times(m+1)}(t)\,Z_m(t). \tag{32}$$

Using Eq. (12), Eq. (28) can be rewritten as

$$^cD^\beta Z_m(t)=V^*_{(m+1)\times(m+1)}(t)\,A^{-1}\Phi_m(t).$$

Therefore, we have

$$^cD^\beta \Phi_m(t)=AV^*_{(m+1)\times(m+1)}(t)\,A^{-1}\Phi_m(t)=P^\beta(t)\,\Phi_m(t) \tag{33}$$

Substituting the formulas of A and A^{-1} into Eq. (33), we get

$$P^\beta(t)=\begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ S_\beta(\lceil\beta\rceil,0) & \cdots & S_\beta(\lceil\beta\rceil,\lceil\beta\rceil) & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ S_\beta(i,0) & \cdots & S_\beta(i,\lceil\beta\rceil) & \cdots & S_\beta(i,i) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ S_\beta(m,0) & \cdots & S_\beta(m,\lceil\beta\rceil) & \cdots & S_\beta(m,i) & \cdots & S_\beta(m,m) \end{bmatrix},$$

where

$$S_\beta(i,j)=\sum_{k=\lceil\beta\rceil}^{i} t^{-\beta}P^{-1}_{j,j}a_{j+1,k+1}P_{i,k}\frac{\Gamma(k+1)}{\Gamma(k+1-\beta)}.$$

Theorem. 3 is proved. ■

Theorem 4. *Let $\Phi_m(qt-r)$ be the shifted Chebyshev vector defined in Eq. (15) and suppose $0<q<1$, $0<r<1$, then*

$$ {}^{c}D^{\beta}\Phi_m(qt-r) = K^{\beta}\Phi_m(t), \tag{34}$$

where the matrix K^{β} is called the generalized pantograph operational matrix of fractional-order differentiation, and the elements are given

$$K^{\beta} = \left[0,0,\cdots,0,\tau_{\lceil\beta\rceil},\cdots,\tau_i,\cdots,\tau_m\right]^{\mathrm{T}}, \tag{35}$$

where

$$\tau_i = \sum_{j=0}^{i} x_{\beta}(i,j)\,\omega_j, \quad i=\lceil\beta\rceil,\lceil\beta\rceil+1,\cdots,m,$$

where

$$x_{\beta}(i,j) = \sum_{k=\lceil\beta\rceil}^{i} (qt-r)^{-\beta} P_{j,j}^{-1} a_{j+1,k+1} P_{i,k} \frac{\Gamma(k+1)}{\Gamma(k+1-\beta)},$$

The formula of ω_j can be represented as Eq. (18).

Proof. By Eqs. (18) and (29), we have

$$ {}^{c}D^{\beta}\Phi_m(qt-r) = P^{\beta}(qt-r)\,\Phi_m(qt-r) = P^{\beta}(qt-r)\,W\Phi_m(t) = K^{\beta}\Phi_m(t), \tag{36}$$

where $K^{\beta} = P^{\beta}(qt-r)\,W$.

According to the formula of $P^{\beta}(t)$, we can derive the expression of $P^{\beta}(qt-r)$

$$P^{\beta}(qt-r) = \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ x_{\beta}(\lceil\beta\rceil,0) & \cdots & x_{\beta}(\lceil\beta\rceil,\lceil\beta\rceil) & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ x_{\beta}(i,0) & \cdots & x_{\beta}(i,\lceil\beta\rceil) & \cdots & x_{\beta}(i,i) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{\beta}(m,0) & \cdots & x_{\beta}(m,\lceil\beta\rceil) & \cdots & x_{\beta}(m,i) & \cdots & x_{\beta}(m,m) \end{bmatrix}, \tag{37}$$

where

$$x_\beta\left(i,j\right) = \sum_{k=\lceil\beta\rceil}^{i} \left(qt-r\right)^{-\beta} P_{j,j}^{-1} a_{j+1,k+1} P_{i,k} \frac{\Gamma\left(k+1\right)}{\Gamma\left(k+1-\beta\right)}, \quad i=\lceil\beta\rceil, \lceil\beta\rceil+1, \cdots, m. \tag{38}$$

Substuting the concrete formulas of W and $P^\beta\left(qt-r\right)$ into Eq. (36), we get

$$K^\beta = P^\beta\left(qt-r\right) W = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \tau_{\lceil\beta\rceil} \\ \vdots \\ \tau_i \\ \vdots \\ \tau_m \end{bmatrix},$$

where

$$\tau_i = \sum_{j=0}^{i} x_\beta\left(i,j\right)\omega_j, \quad i=\lceil\beta\rceil, \lceil\beta\rceil+1, \cdots, m.$$

Theorem. 4 is proved by substituting Eqs. (18) and (38) into the above equation . ■

4.4. SCPs operational matrix of product

Let C be the vector with the parameters c_i given in Eq. (11). By multiplying C with the outer product of two shifted orthonormal Chebyshev polynomial vectors, we can get the row vector. And this row vector can be approximated based on the shifted Chebyshev polynomial vector.

Let

$$C^{\mathrm{T}}\Phi_m\left(t\right)\Phi_m^{\mathrm{T}}\left(t\right) \approx \Phi_m^{\mathrm{T}}\left(t\right)\overline{C}. \tag{39}$$

$\overline{C}$ which satisfies in the above relation is called the operational matrix of product of two shifted Chebyshev polynomial vectors.

To derive the operational matrix of product, inserting Eq. (12) into Eq. (39), we can get

$$\begin{aligned} C^{\mathrm{T}}\Phi_m(t)\,\Phi_m^{\mathrm{T}}(t) &= C^{\mathrm{T}}\Phi_m(t)\,Z_m(t)^{\mathrm{T}}A^{\mathrm{T}} \\ &= \left[C^{\mathrm{T}}\Phi_m(t),\, t\left(C^{\mathrm{T}}\Phi_m(t)\right),\, t^2\left(C^{\mathrm{T}}\Phi_m(t)\right),\cdots,\, t^m\left(C^{\mathrm{T}}\Phi_m(t)\right)\right]A^{\mathrm{T}} \\ &= \left[\sum_{i=0}^{m} c_i H_{L,i}(t), \sum_{i=0}^{m} c_i t H_{L,i}(t), \sum_{i=0}^{m} c_i t^2 H_{L,i}(t), \cdots, \sum_{i=0}^{m} c_i t^m H_{L,i}(t)\right]A^{\mathrm{T}}. \end{aligned} \tag{40}$$

For each of $t^y H_{L,i}(t), y = 0, 1, \cdots, m$, it can be approximated by the shifted Chebyshev polynomials , then

$$t^y H_{L,i}(t) \approx \sum_{k=0}^{m} e_k^{y,i} H_{L,k}(t) = E_{y,i}^{\mathrm{T}}\Phi_m(t),$$

where $E_{y,i}^{\mathrm{T}} = \left[e_0^{y,i}, e_1^{y,i}, \cdots, e_m^{y,i}\right]$, $y = 0, 1, \cdots, m$, $i = 0, 1, \cdots, m$, $e_k^{y,i} = \frac{1}{h_k}\int_0^L t^y H_{L,i}(t)\,H_{L,k}(t)\,\omega_L(t)\,dt$.

Thus, we obtain

$$\begin{aligned} \sum_{i=0}^{m} c_i t^y H_{L,i}(t) &\simeq \sum_{i=0}^{m} c_i \left(\sum_{k=0}^{m} e_k^{y,i} H_{L,k}(t)\right) \\ &= \sum_{k=0}^{m} H_{L,k}(t)\left(\sum_{i=0}^{m} c_i e_k^{y,i}\right) \\ &= \Phi_m^{\mathrm{T}}(t)\,[E_{y,0}, E_{y,1}, \cdots, E_{y,m}]\,C \\ &= \Phi_m^{\mathrm{T}}(t)\,U_{y,c}, \end{aligned} \tag{41}$$

where $U_{y,c} = [E_{y,0}, E_{y,1}, \cdots, E_{y,m}]\,C$.

By defining matrix $U_c = [U_{0,c}, U_{1,c}, \cdots, U_{m,c}]$, and substituting Eq. (41) into Eq. (40), we can get

$$C^{\mathrm{T}}\Phi_m(t)\,\Phi_m^{\mathrm{T}}(t) = \Phi_m^{\mathrm{T}}(t)\,U_c A^{\mathrm{T}}, \tag{42}$$

therefore

$$\overline{C} \approx U_c A^{\mathrm{T}}.$$

5. Numerical algorithms

For the generalized fractional pantograph equation Eq. (1) that satisfies the initial condition Eq. (2), we first approximate

$$u\left(t\right)\approx C^{\mathrm{T}}\Phi_{m}\left(t\right),\tag{43}$$

$$b\left(t\right)\approx B^{\mathrm{T}}\Phi_{m}\left(t\right),\tag{44}$$

$$g\left(t\right)\approx G^{\mathrm{T}}\Phi_{m}\left(t\right),\tag{45}$$

$$v_{j,n}\left(t\right)\approx C_{vjn}^{\mathrm{T}}\Phi_{m}\left(t\right),\tag{46}$$

where $G^{\mathrm{T}}=\{g_i\}_{i=0}^{m}$, ${C_{vjn}}^{\mathrm{T}}=\{c_{i,vjn}\}_{i=0}^{m}$, $B^{\mathrm{T}}=\{b_i\}_{i=0}^{m}$.

Now, using Eqs. (29) and (43), we have

$${}^{c}D^{\beta}u\left(t\right)\approx {}^{c}D^{\beta}C^{\mathrm{T}}\Phi_{m}\left(t\right)=C^{\mathrm{T}}P^{\beta}\left(t\right)\Phi_{m}\left(t\right).\tag{47}$$

For solving ${}^{c}D^{\alpha_n}u\left(q_{j,n}t-r_{j,n}\right)$ in Eq. (1), by using Eqs. (34) and (43) , we obtain

$${}^{c}D^{\alpha_n}u\left(q_{j,n}t-r_{j,n}\right)\approx C^{\mathrm{T}}K_{j,n}^{\alpha_n}\Phi_{m}\left(t\right).\tag{48}$$

Moreover, by the product operational matrix of SCPs and Eq. (46), we have

$$\begin{aligned}
v_{j,n}\left(t\right){}^{c}D^{\alpha_n}u\left(q_{j,n}t-r_{j,n}\right)&\approx C_{vjn}^{\mathrm{T}}\Phi_{m}\left(t\right)C^{\mathrm{T}}K_{j,n}^{\alpha_n}\Phi_{m}\left(t\right)\\
&=C_{vjn}^{\mathrm{T}}\Phi_{m}\left(t\right)\Phi_{m}^{\mathrm{T}}\left(t\right)\left(C^{\mathrm{T}}K_{j,n}^{\alpha_n}\right)^{\mathrm{T}}\\
&\approx\Phi_{m}^{\mathrm{T}}\left(t\right)\overline{C_{vjn}}\left(C^{\mathrm{T}}K_{j,n}^{\alpha_n}\right)^{\mathrm{T}}\\
&=C^{\mathrm{T}}K_{j,n}^{\alpha_n}\left(\overline{C_{vjn}}\right)^{\mathrm{T}}\Phi_{m}\left(t\right)\\
&=D^{(jn)^{\mathrm{T}}}\Phi_{m}\left(t\right),
\end{aligned}\tag{49}$$

where $\overline{C_{vjn}}$ is product operational matrix for the vector C_{vjn}, $D^{(jn)^{\mathrm{T}}}=C^{\mathrm{T}}K_{j,n}^{\alpha_n}\left(\overline{C_{vjn}}\right)^{\mathrm{T}}$.

And we also have

$$\begin{aligned} b\left(t\right)u\left(t\right) &\approx B^{\mathrm{T}}\Phi_m\left(t\right)C^{\mathrm{T}}\Phi_m\left(t\right) \\ &= B^{\mathrm{T}}\Phi_m\left(t\right)\Phi_m^{\mathrm{T}}\left(t\right)C \\ &\approx \Phi_m^{\mathrm{T}}\left(t\right)\overline{B}C \\ &= C^{\mathrm{T}}\left(\overline{B}\right)^{\mathrm{T}}\Phi_m\left(t\right) \\ &= R^{\mathrm{T}}\Phi_m\left(t\right), \end{aligned} \tag{50}$$

where $\overline{B}$ is product operational matrix for the vector B, $R^{\mathrm{T}} = C^{\mathrm{T}}\left(\overline{B}\right)^{\mathrm{T}}$.

Substituting Eqs. (45), (47), (49) and (50) into Eq. (1), we obtain

$$C^{\mathrm{T}}P^{\beta}\left(t\right)\Phi_m\left(t\right) = R^{\mathrm{T}}\Phi_m\left(t\right) + \sum_{j=0}^{J}\sum_{n=0}^{r-1} D^{(jn)^{\mathrm{T}}}\Phi_m\left(t\right) + G^{\mathrm{T}}\Phi_m\left(t\right). \tag{51}$$

For the initial conditions, we can write

$$f = C^{\mathrm{T}}\left(P^{(1)}\right)^{n}\Phi_m\left(0\right). \tag{52}$$

We collocate this system at the following points

$$t_i = \frac{2i-1}{2\left(m+1\right)}, \quad i = 1, 2, \cdots, m+1.$$

These equations can be transferred to algebraic equations. Combining Matlab soft-ware and least square method, the unknown vector C can be solved.

6. Error analysis and error correction

6.1. Error analysis

Lemma 1. *We assume that $u \in C^{m+1}\left[0, L\right]$ with $m \in N^{*}$, and $\beta < m$ with $\beta \in R_{+}\backslash N$. Let $\xi = -r + qt$, then we have*

$${}^{c}D^{\beta}u\left(\xi\right) = \sum_{i=n}^{m}\frac{\left(\xi\right)^{i-\beta}}{\Gamma\left(i-\beta+1\right)}u^{(i)}\left(0\right) + \frac{1}{\Gamma\left(m-\beta+1\right)}\int_0^{\xi}\left(\xi-T\right)^{m-\beta}u^{(m+1)}\left(T\right)dT$$

where $n-1 < \beta < n \leq m$ with $n \in N^{*}$.

Proof. Let $\xi = -r + qt$. Using Eq. (5), we obtain

$${}^{c}D^{\beta}u\left(\xi\right) = \frac{1}{\Gamma\left(n-\beta\right)}\int_0^{\xi}\left(\xi-T\right)^{n-1-\beta}u^{(n)}\left(T\right)dT. \tag{53}$$

By applying $m-n+1$ times integration by parts in Eq. (53), we have

$$^{c}D^{\beta}u\left(\xi\right)=\sum_{i=n}^{m}\frac{\left(\xi\right)^{i-\beta}}{\Gamma\left(i-\beta+1\right)}u^{(i)}\left(0\right)+\frac{1}{\Gamma\left(m-\beta+1\right)}\int_{0}^{\xi}\left(\xi-T\right)^{m-\beta}u^{(m+1)}\left(T\right)dT.$$

Lemma. 1 is proved . ■

Lemma 2. *Assume that $u\in C^{m+1}\left[0,L\right]$ with $m\in N^{*}$, and $\beta<m$ with $\beta\in R_{+}\backslash N$. Let $Y=span\left\{H_{L,o},H_{L,1},\cdots,H_{L,m}\right\}$, $u_m=C^{\mathrm{T}}\Phi_m$ is the approximate function to u from Y. Let $\xi=-r+qt$, then we have*

$$^{c}D^{\beta}u_m\left(\xi\right)=\int_{o}^{L}{}^{c}D^{\beta}Q\left(t,\xi\right)u\left(t\right)dt, \tag{54}$$

where

$$Q\left(t,\xi\right)=\sum_{i=0}^{m}\frac{1}{h_i}H_{L,i}\left(t\right)\omega_L\left(t\right)H_{L,i}\left(\xi\right).$$

Proof. Let $\xi=-r+qt$. From Eqs. (9) and (10), we get

$$\begin{aligned}
u_m\left(\xi\right)&=\sum_{i=0}^{m}c_iH_{L,i}\left(\xi\right)\\
&=\sum_{i=0}^{m}\frac{1}{h_i}\int_{o}^{L}u\left(t\right)H_{L,i}\left(t\right)\omega_L\left(t\right)dtH_{L,i}\left(\xi\right)\\
&=\int_{o}^{L}\left(\sum_{i=0}^{m}\frac{1}{h_i}H_{L,i}\left(t\right)\omega_L\left(t\right)H_{L,i}\left(\xi\right)\right)u\left(t\right)dt\\
&=\int_{o}^{L}Q\left(t,\xi\right)u\left(t\right)dt,
\end{aligned}$$

where

$$Q\left(t,\xi\right)=\sum_{i=0}^{m}\frac{1}{h_i}H_{L,i}\left(t\right)\omega_L\left(t\right)H_{L,i}\left(\xi\right).$$

And we can obtain

$$^{c}D^{\beta}u_m\left(\xi\right)=\int_{o}^{L}{}^{c}D^{\beta}Q\left(t,\xi\right)u\left(t\right)dt.$$

Lemma. 2 is proved . ■

Theorem 5. *Suppose that $u\in C^{m+1}\left[0,L\right]$ with $m\in N^{*}$, and $\beta<m$ with $\beta\in R_{+}\backslash N$. Let $Y=span\left\{H_{L,o},H_{L,1},\cdots,H_{L,m}\right\}$, $u_m=C^{\mathrm{T}}\Phi_m$ is the approximate*

function to u from Y. $u^{(i)}\,(i=0,1,2,\cdots,m+1)$ *are continuous functions. Let* $\xi=-r+qt$*, the error of* ${}^{c}D^{\beta}u(\xi)$ *and* ${}^{c}D^{\beta}u_m(\xi)$ *is represented as follows*

$$e(t)=\int_{o}^{L}{}^{c}D^{\beta}Q(t,\xi)\,I_1(t)\,dt+I_2(\xi), \tag{55}$$

where $Q(t,\xi)$ *is given by Lemma. 2, and*

$$I_1(t)=t^{m+1}\int_0^1\frac{(1-T)^m}{m!}u^{(m+1)}(tT)\,dT,$$

$$I_2(\xi)=\frac{\xi^{m-\beta+1}}{\Gamma(m-\beta+1)}\int_0^1(1-T)^{m-\beta}u^{(m+1)}(\xi T)\,dT.$$

If $M_m=\left\|u^{(m+1)}\right\|_\infty=\sup\left\{\left|u^{(m+1)}(t)\right|,t\in R\right\}$ *exists, then* $e(t)$ *can be bounded as follows*

$$|e(t)|\le M_{m+1}\left(\left|\int_{o}^{L}{}^{c}D^{\beta}Q(t,\xi)\,\frac{t^{m+1}}{(m+1)!}dt\right|+\frac{\xi^{m-\beta+1}}{\Gamma(m-\beta+2)}\right).$$

Proof. $u(t)$ can be expanded into Taylor formula as

$$u(t)=\sum_{i=0}^{m}\frac{(t)^i}{i!}u^{(i)}(0)+\int_0^t\frac{(t-T)^m}{m!}u^{(m+1)}(T)\,dT. \tag{56}$$

Then, we define the following truncated Taylor series expansion

$$\widehat{u_m}(t)=\sum_{i=0}^{m}\frac{(t)^i}{i!}u^{(i)}(0). \tag{57}$$

Let $\xi=-r+qt$, from Eqs. (56) and (57), we get

$$u(\xi)=\sum_{i=0}^{m}\frac{(\xi)^i}{i!}u^{(i)}(0)+\int_0^{\xi}\frac{(\xi-T)^m}{m!}u^{(m+1)}(T)\,dT, \tag{58}$$

$$\widehat{u_m}(\xi)=\sum_{i=0}^{m}\frac{(\xi)^i}{i!}u^{(i)}(0). \tag{59}$$

Hence, the β^{th} order derivative of Eq. (59) can be calculated as follows

$${}^{c}D^{\beta}\widehat{u_m}(\xi)=\sum_{i=n}^{m}\frac{(\xi)^{i-\beta}}{\Gamma(i-\beta+1)}u^{(i)}(0). \tag{60}$$

Let us consider the following equality

$$e(\xi) = \left({}^cD^\beta u_m(\xi) - {}^cD^\beta \widehat{u_m}(\xi)\right) + \left({}^cD^\beta \widehat{u_m}(\xi) - {}^cD^\beta u(\xi)\right).$$

Similar to the proof process of Lemma. 2, we get

$$\begin{aligned}
\widehat{u_m}(\xi) &= \sum_{i=0}^{m} g_i H_{L,i}(\xi) \\
&= \sum_{i=0}^{m} \frac{1}{h_i} \int_o^L \widehat{u_m(t)} H_{L,i}(t)\,\omega_L(t)\,dt H_{L,i}(\xi) \\
&= \int_o^L \left(\sum_{i=0}^{m} \frac{1}{h_i} H_{L,i}(t)\,\omega_L(t)\,H_{L,i}(\xi) \right) \widehat{u_m(t)} dt \\
&= \int_o^L Q(t,\xi)\,\widehat{u_m(t)} dt.
\end{aligned}$$

The β^{th} order derivative of $\widehat{u_m}(\xi)$ can be calculated as follows

$${}^cD^\beta \widehat{u_m}(\xi) = \int_o^L {}^cD^\beta Q(t,\xi)\,\widehat{u_m(t)} dt.$$

Therefore

$$\begin{aligned}
{}^cD^\beta u_m(\xi) - {}^cD^\beta \widehat{u_m}(\xi) &= \int_o^L {}^cD^\beta Q(t,\xi)\,u(t)\,dt - \int_o^L {}^cD^\beta Q(t,\xi)\,\widehat{u_m(t)} dt \\
&= \int_o^L {}^cD^\beta Q(t,\xi) \left(u(t) - \widehat{u_m(t)}\right) dt.
\end{aligned} \tag{61}$$

According to Eqs. (56) and (57) , we obtain

$$I_1(t) = u(t) - \widehat{u_m(t)} = \int_0^t \frac{(t-T)^m}{m!} u^{(m+1)}(T)\,dT. \tag{62}$$

Applying the following change of variables $T \to tT$ in Eq. (62)

$$\begin{aligned}
I_1(t) &= \int_0^t \frac{(t-T)^m}{m!} u^{(m+1)}(T)\,dT \\
&= t^{m+1} \int_0^1 \frac{(1-T)^m}{m!} u^{(m+1)}(tT)\,dT.
\end{aligned}$$

The β^{th} order derivative of Eq. (58) can be represented as follows

$${}^cD^\beta \widehat{u_m}(\xi) - {}^cD^\beta u(\xi) = I_2.$$

According to Eq. (60) and Lemma. 1, we get

$$I_2(\xi) = \frac{1}{\Gamma(m-\beta+1)} \int_0^{\xi} (\xi - T)^{m-\beta} u^{(m+1)}(T)\, dT \tag{63}$$

Applying the following change of variables $T \to \xi T$ in Eq. (63)

$$I_2(\xi) = \frac{\xi^{m-\beta+1}}{\Gamma(m-\beta+1)} \int_0^{1} (1 - T)^{m-\beta} u^{(m+1)}(\xi T)\, dT.$$

Using Eqs. (61), (62) and (63), we get

$$e(t) = \int_o^L {}^cD^{\beta} Q(t,\xi)\, I_1(t)\, dt + I_2(\xi).$$

Finally, this proof can be completed by taking the absolute value of $e(t)$ and the following inequalities

$$|I_1(t)| \le M_{m+1} \frac{t^{m+1}}{(m+1)!}$$

$$|I_2(t)| \le M_{m+1} \frac{\xi^{m-\beta+1}}{\Gamma(m-\beta+2)}.$$

.

Theorem. 5 is proved . ■

6.2. Error correction

For Eq. (1), we consider the following residual function

$$R_m(t) = L[u_m(t)] - g(t), \tag{64}$$

where

$$L[u_m(t)] = {}^cD^{\beta} u_m(t) - b(t)\, u_m(t) - \sum_{j=0}^{J} \sum_{n=0}^{r} v_{j,n}(t)\, {}^cD^{\alpha_n} u_m(q_{j,n} t - r_{j,n}).$$

Eq. (64) satisfies the following form

$$L[u_m(t)] = R_m(t) + g(t).$$

It needs to be pointed out, $u_m(t)$ is the approximate solution for $u(t)$, and $u(t)$ is the exact solution of Eq. (1).

Defining the error function, as follows

$$e_m(t) = u(t) - u_m(t),$$

then we can get the differential equation about the error function

$$\begin{aligned} L[e_m(t)] &= L[u(t)] - L[u_m(t)] \\ &= \mathrm{g}(t) - R_m(t) \text{-} \mathrm{g}(t) \\ &= \text{-}R_m(t). \end{aligned}$$

The formula of the error function is giving

$$\begin{aligned} L[e_m(t)] = {}^{c}D^{\beta} e_m(t) - b(t) e_m(t) - \sum_{j=0}^{J} \sum_{n=0}^{r} v_{j,n}(t) {}^{c}D^{\alpha_n} e_m(q_{j,n}t - r_{j,n}) \\ = \text{-}R_m(t). \end{aligned} \tag{65}$$

In order to construct the approximate $e_m^*(x,t)$ to $e_m(t)$, only Eq. (65) needs to be recalculated in the same way as we did before for the solution of Eq. (1). And we define the $e_m^*(t)$ as the approximate error function.

According to the numerical solution $u_m(t)$ of Eq. (1) and the numerical solution $e_m^*(t)$ of Eq. (65), corrective solution $u^*(t)$ is obtained

$$u^*(t) = u_m(t) + e_m^*(t). \tag{66}$$

From Eq.(66), we can get the corrective error function

$$e_r(t) = e_m(t) - e_m^*(t) = u(t) - u_m(t) - e_m^*(t).$$

7. Numerical experiments

In this section, to demonstrate the applicability and accuracy of our method, we shows some numerical examples in the form Eq. (1) with intial conditions. All the numerical computations have been done using Matlab.

Example 1. *Consider the fractional pantograph differential equation with variable coefficients*

$$D^{\beta}u\left(t\right)=\frac{1}{2}e^{\frac{t}{2}}u\left(\frac{t}{2}\right)+\frac{1}{2}u\left(t\right),\quad 0<\beta\leq 1,\ 0\leq t\leq 1,$$
$$u\left(0\right)=1.$$

In the case, when $\beta=1$, the exact solution is $u\left(t\right)=e^{t}$.

Table 1 shows the comparison of the absolute errors of the proposed method for $m=9,10,11$ with that of the Taylor method [35] for $N=9$. Also we do correction for the numerical solutions with $m=9$, and obtain the absolute corrective errors for $m=9$, $me=11$. We see that the approximation solutions obtained by the present method have good agreement with the exact solution, and the absolute errors of corrective solutions are smaller than the absolute errors of numerical solutions.

Table 1: Absolute errors at some points for Example 1.

t	Taylor method	Present method with			
		$m=9$	$m=9$,$me=11$	$m=10$	$m=11$
0.2	0.70×10^{-14}	3.03×10^{-11}	3.18×10^{-14}	1.03×10^{-12}	2.82×10^{-14}
0.4	0.10×10^{-10}	3.76×10^{-11}	3.86×10^{-14}	1.26×10^{-12}	3.42×10^{-14}
0.6	0.29×10^{-9}	4.69×10^{-11}	4.69×10^{-14}	1.53×10^{-12}	4.29×10^{-14}
0.8	0.38×10^{-8}	5.75×10^{-11}	5.73×10^{-14}	1.88×10^{-12}	5.20×10^{-14}
1	0.29×10^{-7}	7.10×10^{-11}	7.11×10^{-14}	2.26×10^{-12}	6.35×10^{-14}

Example 2. *Consider the generalized fractional pantograph differential equation*

$$D^{\beta}u\left(t\right)=-u\left(t\right)-u\left(\frac{t}{2}-0.3\right)+g\left(t\right),\quad 0<\beta\leq 3,$$
$$u\left(0\right)=1,u^{(1)}\left(0\right)=-1,u^{(2)}\left(0\right)=1.$$

In the case, when $\beta=3$, $g\left(t\right)=e^{-\frac{1}{2}t+0.3}$, the exact solution is $u\left(t\right)=e^{-t}$.

Table 2 shows the absolute errors between the exact solution and approximate solutions of our method for different values of m. From Table 2, we can

Table 2: Absolute errors for different values of m for Example 2.

t	$m = 6$	$m = 8$	$m = 10$	$m = 12$
0.2	5.18×10^{-7}	1.45×10^{-9}	2.86×10^{-12}	5.22×10^{-15}
0.4	1.93×10^{-6}	5.24×10^{-9}	1.21×10^{-11}	2.39×10^{-14}
0.6	3.27×10^{-6}	1.21×10^{-8}	2.77×10^{-11}	5.67×10^{-14}
0.8	5.34×10^{-6}	2.13×10^{-8}	4.99×10^{-11}	1.03×10^{-13}
1	8.62×10^{-6}	3.29×10^{-8}	7.80×10^{-11}	1.62×10^{-13}

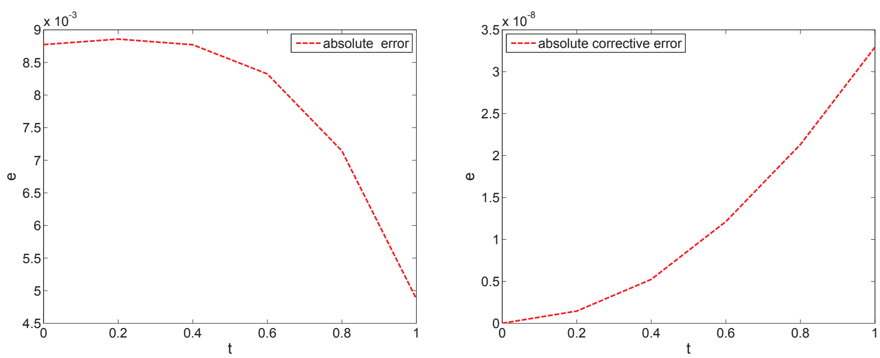

Figure 1: The absolute errors for $m = 4$, and the absolute corrective errors for $m = 4$, $me = 8$ for Example 2.

say that the numerical solutions come close to the exact solution with the increasing value of m. In Figure 1, the absolute errors of our method are given for $m = 4$. And the errors are not perfect enough. By doing correction for the numerical solutions for $m = 4$, $me = 8$, the absolute corrective errors achieve about 10^{-8}. Figure 2 displays the computational results for $m = 7$ on $[0, 2]$ when β takes different values. By comparing these computational results with exact solution, it is evident from Figure 2 that as β approaches 3, the numerical solutions converge to those of integer order differential equations.

Example 3. *Consider the fractional multi-pantograph differential equation [33]*

$$D^{\beta} u\,(t) = -\frac{5}{6} u\,(t) + 4u\left(\frac{1}{2}t\right) + 9u\left(\frac{1}{3}t\right) + t^2 - 1, \quad 0 < \beta \leq 1,$$

$$u\,(0) = 1.$$

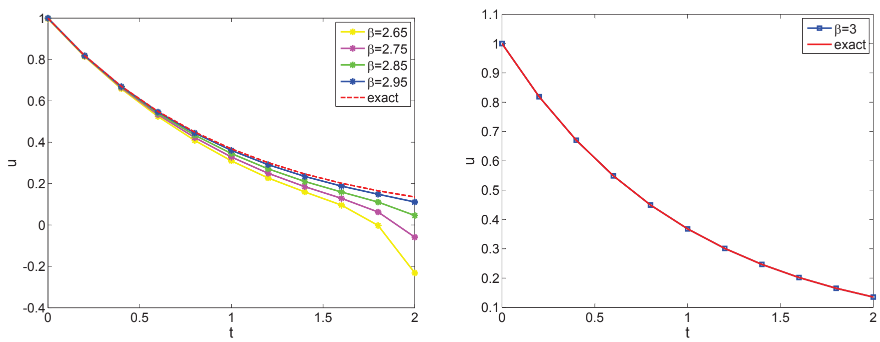

Figure 2: The comparison of $u(t)$ for $m = 7$, with $\beta = 2.65, 2.75, 2.85, 2.95, 1$, and the exact solution for Example 2.

In the case, when $\beta = 1$, *the exact solution is* $u(t) = 1 + \frac{67}{6}t + \frac{1675}{72}t^2 + \frac{12157}{1296}t^3$.

Table 3: The absolute errors for various intervals for Example 3.

t	[0, 5]		[0, 10]	
	GFBWFs method	Present method	GFBWFs method	Present method
0	1.60×10^{-10}	5.34×10^{-12}	1.31×10^{-9}	7.28×10^{-12}
1	5.49×10^{-9}	4.31×10^{-10}	5.31×10^{-8}	4.13×10^{-10}
2	2.33×10^{-8}	1.93×10^{-9}	2.26×10^{-7}	1.48×10^{-9}
3	3.31×10^{-6}	5.36×10^{-9}	5.85×10^{-7}	3.05×10^{-9}
4	1.20×10^{-5}	1.16×10^{-8}	1.19×10^{-6}	4.98×10^{-9}
5	5.18×10^{-7}	2.14×10^{-8}	2.13×10^{-6}	7.12×10^{-9}

In Table 3, we compare the absolute errors of our method for $m = 3$ with those of the GFBWFs method of [33] for $k = 2$, $M = 4$ on various intervals. Figure 3 gives the numerical results for different choices of β with $m = 3$ on the interval $[0, 10]$. The Figure 4 shows the absolute errors between the exact solution and approximate solutions for $m = 3$, $\beta = 1$ on $[0, 10]$, and the absolute corrective errors between the exact solution and corrective solutions for $m = 3$, $me = 3$. These results explained that as β approaches 1, the numerical solutions converge to the exact solution. And it is evident from Figure 4 that the corrective solutions obtained by doing correction have better convergence

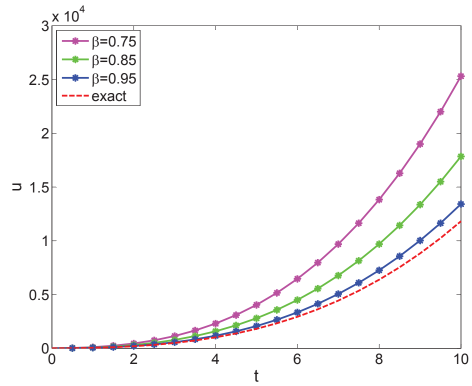

Figure 3: The comparison of $u(t)$ for $m = 3$, with $\beta = 0.75, 0.85, 0.95$, and the exact solution for Example 3.

to exact solution than numerical solutions.

Example 4. *Consider the fractional neutral pantograph differential equation*

$$D^{\beta}u(t) = -u(t) + 0.1u(qt) + 0.5D^{\beta}u(qt) + (0.32t - 0.5)\,e^{-0.8t} + e^{-t}, 0 < \beta \leq 1,$$

$$u(0) = 0.$$

In the case, when $\beta = 1$, $q = 0.8$, the exact solution is $u(t) = te^{-t}$.

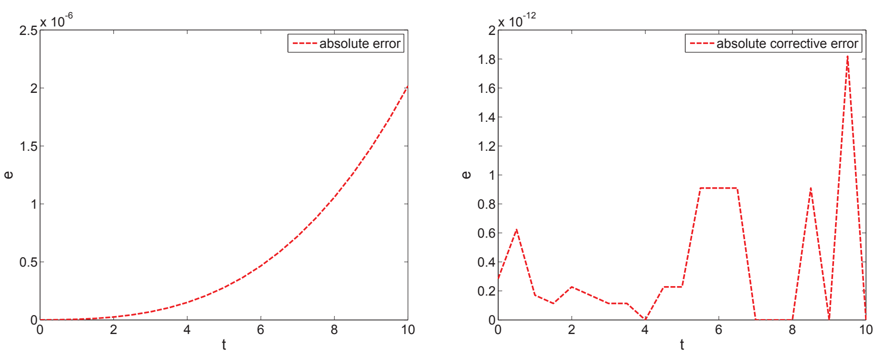

Figure 4: The absolute errors for $m = 3$, and the absolute corrective errors for $m = 3$, $me = 3$ for Example 3.

In Table 4, the compassion, the absolute errors of the proposed method for $m = 6, 8, 10$ with those of the one-Leg θ [36] with $\theta = 0.8$, $h = 0.01$, the variational iteration method (V-I method) [37] for $\widehat{m} = 6$, and the GFBWFs method (G method) [33] for $k = 2$, $M = 6$ on the interval $[0, 1]$, is given.In Figure 5, we do correction for the numerical solutions with $m = 6$, and obtain the absolute corrective errors for $m = 6$, $me = 10$. Also, the Figure 6 displays the numerical results for $m = 10$, $\beta = 1$ on $[0, 2]$, when q takes different values. And by comparing these results and exact solution, we can see that, as q approaches 0.8, the numerical solutions converge to the exact solution.

Table 4: The comparison of the absolute errors with other methods for Example 4.

t	θ method	V-I method	G method	Present method with		
				$m = 6$	$m = 8$	$m = 10$
0.1	4.65×10^{-3}	1.30×10^{-3}	1.98×10^{-8}	2.13×10^{-6}	5.08×10^{-9}	7.33×10^{-12}
0.3	2.57×10^{-2}	2.63×10^{-3}	7.78×10^{-9}	8.65×10^{-7}	2.70×10^{-9}	4.31×10^{-12}
0.5	4.43×10^{-2}	2.83×10^{-3}	6.34×10^{-5}	7.01×10^{-7}	1.68×10^{-9}	2.82×10^{-12}
0.7	5.37×10^{-2}	2.39×10^{-3}	4.36×10^{-5}	3.14×10^{-7}	1.10×10^{-9}	1.79×10^{-12}
0.9	5.35×10^{-2}	1.64×10^{-3}	2.80×10^{-5}	2.94×10^{-7}	7.32×10^{-10}	1.09×10^{-12}

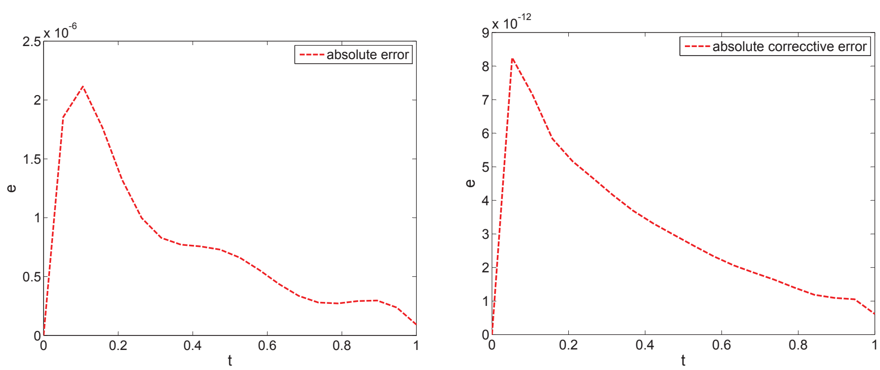

Figure 5: The absolute errors for $m = 6$, and the absolute corrective errors for $m = 6$, $me = 10$ for Example 4.

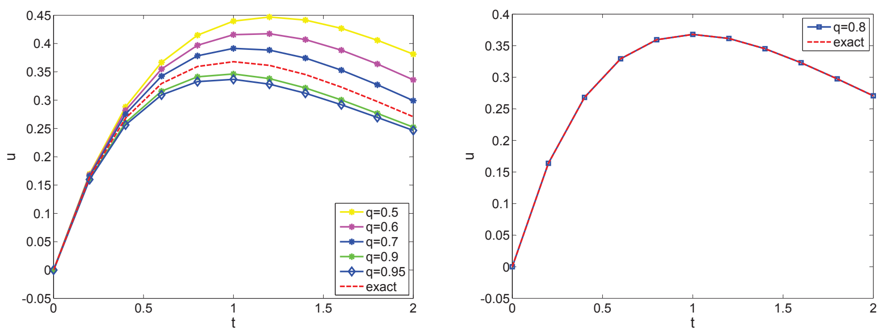

Figure 6: The comparison of $u(t)$ for $m = 10$, $\beta = 1$ with $q = 0.5, 0.6, 0.7, 0.8, 0.9, 0.95$, and the exact solution for Example 4.

8. Conclusion

In this article, applying the properties of the shifted Chebyshev polynomials, we have derived the generalized pantograph operational matrix. Also, according to the SCPs fractional differential operational matrix, the generalized pantograph operational matrix of fractional-order differentiation is introduced. These matrices combined with collocation method are used to simplify and effectively calculate the numerical solutions of the generalized fractional pantograph delay equations. By constructing the generalized fractional pantograph delay equations of error function, we obtain the approximate error function to correct numerical solutions . Numerical examples show our method is effective. From examples, it is seen that with the increasing value of m, the absolute error is smaller and the convergence effect between the numerical solutions and the exact solution is better. The corrective solutions have better convergence to exact solutions than the numerical solutions. In addition, we find that the present method is an excellent mathematical method, when the function defined on the interval $[0, L]$ and various order $\beta > 0$.

Acknowledgements

This work is supported by the Natural Science Foundation of Hebei Province (A2017203100) in China and the LE STUDIUM RESEARCH PROFESSOR-

SHIP award of Centre-Val de Loire region in France.

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