Linear demand systems for differentiated goods: Overview and user's guide

Philippe Choné, Laurent Linnemer

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Abstract

Linear demand systems and quasi-linear quadratic utility models are widely used in industrial economics. We clarify the link between the two settings and explain their exact origin as it seems to be little known by practitioners. We offer practical recommendations to achieve consistency, tractability and a reasonable degree of generality when using the linear demand framework. We show that all tractable versions of the model used in practice are (almost) identical and have a mean-variance structure. We provide concise, ready-to-use formulae for the symmetric model. Finally, we revisit and extend the asymmetric model of Shubik and Levitan.

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1 Introduction

The usage of a Linear Demand System for differentiated goods (henceforth LDS) is widespread in oligopoly theory, especially when closed-form solutions are needed. Historically, the micro-foundation of an LDS has been the Quasilinear Quadratic Utility Model (hereafter QQUM).

As Amir, Erickson, and Jin (2017) point out, "this framework has become so widely invoked that virtually no author nowadays cites any of the(se) early works when adopting this convenient setting." This lack of reference is not new, however. Some researchers find it so natural to use linear (direct or inverse) demands (and to derive them from a QQUM) that they do not try to give a source.[1] This lack of reference, however, can confuse other economists who try to cite a source but have difficulties coördinating on the correct one. Whereas users can also be disoriented by the various existing ways to write an LDS and/or a QQUM, we show that models presented as different are in fact isomorphic.

The goal of this paper is to help IO economists find their way into the intricacies of these models. For that purpose, we structured the paper as follows. In section 2, we start by briefly recounting how LDS and QQUM were introduced and extensively used by Richard E. Levitan and Martin Shubik[2] in the 1960s and we show these pioneers deserve credit for having paved the way. In fact, by analogy with the Cobb-Douglas utility function, it would not be farfetched to name QQUM after Levitan and Shubik. Next, in section 3, we introduce a general LDS and explain its relation to QQUM. We then survey the main properties of an oligopoly game based on such linear demands. In section 4 we explain the ins and outs of a simple yet rich enough symmetric model. We provide concise ready-to-use formulae and emphasize the mean-variance structure of the model. Finally, in section 5, we extend the symmetric model to a richer asymmetric set-up (revisiting the asymmetric model of chapter 9 of Shubik and Levitan (1980)). Closed-forms formulae can still be obtained and could be of more use in future research to emphasize results absent in the symmetric context.

2 Genesis of LDS and QQUM

Martin (2002) states that part of QQUM originates in Bowley (1924) (see section 3.6 of Martin's book).[3] However, a close look at page 56 of Bowley's book shows that it is farfetched as, actually, this author did not make the quasi-linearity assumption. Bowley, indeed, considered, for two commodities, a consumer with a general quadratic utility (as later would Dixit (1979) and Singh and Vives (1984)) but the derived demands are very different from QQUM, in particular, they are not linear in prices.

After an extensive search, we traced back the usage of LDS to the 1960s and to the collaboration between Richard E. Levitan and Martin Shubik.

Fact 1 (Dawn of LDS)**.** *Martin Shubik introduced a linear demand system to model a differentiated good oligopoly game in the early 1960s. In 1961, he joined forces with Richard E. Levitan at I.B.M..*

The following quote is of particular interest:

> "I went to IBM in October 1961 and started to work with Dick Levitan to move my game from an IBM 650 to a bigger, better machine-also to add any new features to it. (Our first game used a template as we had no way to print out the format. We could only print numbers.) Levitan did his thesis with Dorfmann and me on a quadratic programming method for allocating demand among oligopolistic firms with product differentiation."
>
> Martin Shubik (see Smith (1992), page 252)

[1] By analogy, no sane economist would look for a reference when using a linear demand like $D(p) = a - bp$. Although, Cournot (1838) might be a decent try.

[2] March 24, 1926 – August 22, 2018. He had a long and productive career. See the Special Issue in his Honor published by *Games and Economic Behavior* (Volume 65, Issue 1, Pages 1-288, January 2009).

[3] In the 1993 first edition there is no such reference to Bowley and only a slight reference to Levitan and Shubik.

From Vernon Smith's perspective the work of Levitan and Shubik is one of the first laboratory experiment in oligopoly theory. Shubik (1961),[4] describes a more general demand function which is then simplified[5] into a linear one *à la* LDS. That is (again changing the notations):

$$q_i = \frac{1}{n}\left[a - b\left(1 + \sigma - \frac{\sigma}{n}\right)p_i + b\frac{\sigma}{n}\left(\sum_{j\neq i} p_j\right)\right] = \frac{1}{n}\left[a - bp_i + b\sigma\left(\overline{\mathbf{p}} - p_i\right)\right]$$

where $\overline{\mathbf{p}} = \frac{1}{n}\sum p_i$ is the average price. Note that throughout our paper, whenever there is no ambiguity the subscript is skipped in the sum terms. That is the lengthy $\sum_{i=1}^{n} x_i$ is simply written $\sum x_i$. Shubik placed a factor $1/n$ in front of the demand expression so that total demand $\sum q_i = a - b\,\overline{\mathbf{p}}$, is independent of n when all prices are equal. This point is made in Martin's book and we further discuss it in section 4.2.

Levitan's name soon appeared. First, in two IBM research reports describing the same business/experimental game: Levitan and Shubik (1962a) and Levitan and Shubik (1962b). Next, in Shubik (1964) the collaboration with Levitan is also made clear: "This paper is part of a continuing study done by the author in coöperation with Richard Levitan of the IBM corporation." The content of the missing IBM reports is probably used in the Cowles Foundation research papers published later, in particular: Levitan and Shubik (1967a) (part of which is published as Levitan and Shubik (1971b)) and Levitan and Shubik (1967b).

Fact 2 (QQUM)**.** *From the mid 1960s, Levitan and Shubik founded their LDS on a quasilinear quadratic utility. Moreover, they started to study both price and quantity competition.*

Indeed, the understanding of QQUM by Levitan and Shubik had evolved from the astute but *ad hoc* linear demands to a more structural model with a representative consumer. "We assume that consumer preferences can be represented by a general quadratic utility function. Our somewhat strong special assumption is that to a first approximation there is no income effect between this class of goods and the remainder of the consumer's purchases." (Levitan and Shubik (1967b), page 2). They also refer to the PhD thesis Levitan (1966) for a detailed analysis. They write:[6]

$$U = a\sum q_i - \frac{1}{2\beta}\left[2\sigma\sum_i\sum_{j>i} q_i q_j + \sum\left(\sigma + \frac{1-\sigma}{w_i}\right)q_i^2\right] - \sum p_i q_i \tag{1}$$

where they call w_i a weight reflecting the size of firm i, for all i, $0 < w_i < 1$ and $\sum w_i = 1$. Notice that in both Levitan and Shubik (1967a) and Levitan and Shubik (1967b) matrix notations are used to solve for the Nash equilibrium of both the price and quantity games.[7]

Fact 3 (Book and slow diffusion)**.** *In 1980, Levitan and Shubik gathered their previous work on LDS and QQUM in Shubik and Levitan (1980). The reception of the book in the academic arena was cold and it was not mentioned in the main IO surveys which flourished at the end of the 1980s.*

The first five chapters of the book Shubik and Levitan (1980) can be seen as an update of the book Shubik (1959). Chapter 6 introduces QQUM for a symmetric duopoly, chapter 7 extends it to a

[4] In the book Shubik (1959) no LDS is mentioned.

[5] The whole framework is fairly complicated and business oriented as it aims to incorporate the effect of advertising on demand, inventory constraints, and financial variables like loans, and dividends.

[6] Again, adjusting the notations to keep formulae homogeneous throughout this paper. From their formula page 2, the following notational changes have been made: $V \to a$ and $\gamma \to \frac{\sigma}{1-\sigma}$.

[7] After Levitan and Shubik have been less involved with QQUM. They have another working paper together (duopoly model) Levitan and Shubik (1969) (published in the *Journal of Economic Theory*: Levitan and Shubik (1971a)). A duopoly variant of (1) is used in Shapley and Shubik (1969) where they do not cite Levitan and Shubik (1967a) nor Levitan and Shubik (1967b). Levitan and Shubik published several articles on related topics (always in a duopoly setting): Levitan and Shubik (1971b), Levitan and Shubik (1972), and Levitan and Shubik (1978).

n firm symmetric oligopoly, and chapter 9 to an asymmetric oligopoly where their chosen quadratic utility (9.5) page 132 is a slight modification of (1).

Despite their thorough work, Levitan and Shubik's approach was not immediately popular. In particular, their 1980 book *Market structure and behavior*, which from today's perspective has certainly been a success (most academic libraries hold the book and it is still in print), received mixed reviews to say the least. It is almost painful to read some reviews. Rothschild (1982) in the *Journal of Economic Literature* is unmerciful: "Much of the book is devoted to computing the solutions of different variants of a single model. Although it is interesting and important to know that this can be done, it is difficult to stay awake while watching the process. I found it hard to make anything of the many numerical results that are presented and so, I suspect, did the authors. The results of an attempt to apply the oligopoly model to a real problem can charitably be described as eccentric." In the *Journal of Political Economy*, Telser (1982) is not enthusiastic either: "A mere catalog of some models of oligopoly does not constitute a useful contribution to economics. Readers deserve a coherent set of principles that can relate the theories, a demonstration of their explanatory power, if any, and a statement of which survives these tests." Telser is also quite harsh on the empirical part: "The book also contains a section purporting to apply the theories to the study of the automobile industry, but it does not pass even the loosest standards of econometric rigor." In the *Economic Journal*, Reid (1982) is more positive and spends more space than the previous two reviews on praising the book, concluding "On balance, however, the reading of the book is a tonic. It stimulates, fascinates and informs, and will repay frequent re-reading." Chapter 9 is praised (both Reid and Telser find chapter 9 the most ambitious) but also criticized: "...but the great weight of attention is still given to pure theory and occasionally, as in chapter 9, to one of its least attractive varieties, namely the intricate manipulation of specialized functional forms." Similarly, Pagoulatos (1983) in *Southern Economic Journal* writes "Finally, the mathematical manipulations of different linear functions presented in Chapter 9 leave the reader in strong doubt about the usefulness of following every step of the various exercises. Relegating the nonessential manipulations to an appendix would have added considerably to the enjoyment of the book."

Later the IO Bible, Tirole (1988), put forward the address models, Hotelling and Salop, (see chap. 7 of Tirole's book) to deal with product differentiation. Similarly, in the "Product Differentiation" chapter of the Handbook of Industrial Organization (vol. 1, chap. 12) Eaton and Lipsey (1989) focus on address models and when briefly discussing the representative consumer approach they do not mention Levitan and Shubik (see also their Figure 12.4. "Historical perspective", page 762). They only refer to the seminal papers on monopolistic competition: Spence (1976a), Spence (1976b) and Dixit and Stiglitz (1977). Finally, in Anderson, De Palma, and Thisse (1992) there is no reference to QQUM and (therefore) no reference to Levitan and Shubik.

How to explain the relative lack of success of Levitan and Shubik's work on QQUM? First, as shown by the surveys by Tirole and Eaton and Lipsey, by the end of the 1980s the representative consumer approach was not seen as having appropriate microeconomic foundations. Consumers have different tastes and each individual buys only a tiny subset of all available varieties (see Section 3.1 below for more on this point). Second, as the monopolistic competition literature (Dixit and Stiglitz (1977)) finally made the representative consumer approach popular, most researchers (e.g. in trade) adopted the constant elasticity of substitution (CES) utility function rather than the QQUM.

Regarding specifically the use of LDS in industrial economics, some confusion might come from the fact that QQUM was independently introduced by Spence (1976a) and Dixit (1979). In Spence (1976a) there is no representative consumer. A n goods, completely symmetric LDS is assumed (see (2) page 411) and Spence derives (footnote 6, page 412) consumers' surplus (which takes a QQUM form). In Dixit (1979), a general (two-good) quasi-linear utility function is introduced (see (1) page 21) and used to derive inverse demands (see (2) page 22). In order to derive comparative statics results, Dixit assumes a quadratic form (see (4) page 26). He gives the precise conditions under which the utility is concave. It is Spence (1976a) which is referred to in the seminal work of Singh and Vives (1984) (they also cite Shubik and Levitan (1980), but they do not present it as a predecessor).

Fact 4 (QQUM usage in IO)**.** *From the 1980s up until today, QQUM has been used in IO, typically when closed-form formulae are needed.*

Despite the dominant view that address models were sounder, and the competing references of Dixit and Spence, the spirit of QQUM endured and proved itself useful in IO. Some authors started to cite the book Shubik and Levitan (1980) and also, but to a lesser extend, the chapter Levitan and Shubik (1971b). Prominent examples are Deneckere and Davidson (1985),[8] Vives (1985),[9] Shaked and Sutton (1990), Bagwell and Ramey (1991), Shaffer (1991), Dobson and Waterson (1997), and Sutton (1997). In, Motta (2004), influential book "Competition Policy: Theory and Practice", the Levitan and Shubik's model is used (in particular in chapter 5 on horizontal mergers) to illustrate some properties with a closed-form model.

Among the articles relying on QQUM, there is a literature on comparing prices, quantities, profits, welfare, between Bertrand and Cournot competition. Levitan and Shubik themselves have compared prices when all goods are substitutes, see Levitan and Shubik (1967b) page 7, but this strand of literature really started with Singh and Vives (1984) and Vives (1985), the main reference remaining Amir and Jin (2001).[10]

Table 1 lists a sample of articles, on various topics, that (by and large) adopt a similar modeling strategy. A general product differentiation oligopoly framework is used at the start of the paper, some results are derived, and at the end the QQUM is introduced in order to derive more specific results which are unclear in the general framework. In almost all these examples only a symmetric QQUM is used.

QQUM is not particularly popular among econometricians probably because in its general form it involves too many coefficients to estimate. Pinkse, Slade, and Brett (2002) is an exception. There QQUM is presented as a second order approximation of a general demand model. This is a clever remark which could very well explain the success of QQUM in practice when a result cannot be shown with a general (nonlinear) demand function.

[8]They give Shubik as the sole author of the book because on the book cover, of the first editions, the author is "Martin Shubik with Richard Levitan". Many authors give credit to both authors and we follow this tradition here.

[9]There Levitan and Shubik's book is cited although the publication year is wrong: 1971 instead of 1980. The same mistake is made in Vives (2001). Maybe a confusion between the 1980 book and the 1971 chapter. In Vives (2008) the year is correct but the QQUM origin is attributed to Shapley and Shubik (1969). Theilen (2012) also cites this article.

[10]Häckner (2000) presents new results, in particular for $\sigma < 0$. See Chang and Peng (2012) for a survey.

Table 1: Sample of articles using QQUM

Article	Journal	year	Nb firms	Type of **B**	Hetero. **a** or **c**	L&S
Spence	AER	1976	n	Symmetric	No	No
Dixit	BJE	1979	2	General	Yes	No
Friedman	BJE	1983	n	Symmetric	Yes	Yes
Singh and Vives	RJE	1984	2	General	Yes	Yes
Deneckere and Davidson	RJE	1985	n	Symmetric	No	Yes
Vives	JET	1985	n	Symmetric	No	Yes
Shaked and Sutton	RJE	1990	n	Symmetric	No	Yes
Bagwell and Ramey	RJE	1991	n	Symmetric	No	Yes
Shaffer	RJE	1991	2	Symmetric	No	Yes
Besanko and Perry	RJE	1993	3	Symmetric	No	Yes
Röller and Tombak	MS	1993	n	Symmetric	No	Yes
Raju, Sethuraman, and Dhar	MS	1995	n	Symmetric	No	Yes
Raith	JET	1996	n	Symmetric	Yes	No
Dobson and Waterson	JINDEC	1997	n	Symmetric	Yes	No
Sutton	RJE	1997	n	Symmetric	No	Yes
Sayman, Hoch, and Raju	MkS	2002	3	Asymmetric	Yes	Yes
Pinkse and Slade	EER	2004	n	Symmetric	Yes	No
Marx and Shaffer	IJIO	2004	2	Symmetric	Yes	No
Motta	Book	2004	n	Symmetric	No	Yes
Chen and Gayle	IJIO	2007	2	Symmetric	No	No
Daughety and Reinganum	RJE	2008	n	Symmetric	Yes	No
Fumagalli and Motta	EJ	2008	n	Symmetric	No	Yes
Vives	JINDEC	2008	n	Symmetric	Yes	Yes
Abito and Wright	IJIO	2008	2	Symmetric	Yes	Yes
Foros, Hagen, and Kind	MS	2009	n	Symmetric	No	Yes
Kind, Nilssen, and Sørgard	MkS	2009	n	Symmetric	No	Yes
Lu and Wright	IJIO	2010	2	Symmetric	No	No
Rey and Vergé	JINDEC	2010	4	General	Yes	No
Subramanian, Raju, Dhar, and Wang	MS	2010	2	Symmetric	Yes	Yes
Bourreau, Hombert, Pouyet, and Schutz	JINDEC	2011	3	Symmetric	Yes	No
Inderst and Valletti	EER	2011	n	Symmetric	No	Yes
Calzolari and Denicolo	AER	2015	2	Symmetric	No	Yes
Edelman and Wright	QJE	2015	n	Symmetric	No	Yes
Abhishek, Jerath, and Zhang	MS	2016	2	Symmetric	No	Yes
Allain, Henry, and Kyle	MS	2016	n	Symmetric	No	Yes
Cho and Wang	MS	2016	n	Symmetric	No	Yes
Herweg and Müller	IJIO	2016	2	Symmetric	Yes	No
Ulsaker	IJIO	2020	2	Symmetric	Yes	No

3 Foundation of LDS and QQUM: an interlocking relationship?

In this section, we first discuss necessary properties of LDS, their micro-foundations from individual preferences with no income effect, and the conditions for integrability. Next, focusing on the main case where integrability into a QQUM is guaranteed, we show how various oligopoly games can be solved with arbitrary many products and various market structures.

Notations Mostly for compactness in presentation, it is convenient to use the following notations. Let $\mathbf{x}$ (bold font) denote a vector of size n: $\mathbf{x} = (x_1, \cdots, x_n)'$ where the $'$ stands for transposition. A capital bold letter, as $\mathbf{X}$, denotes a $n \times n$ matrix which elements are x_{ij}. Levitan and Shubik already resorted to matrix notations but mostly for their proofs. As IO economists are not always at ease with matrix notations, standard expressions are (most of the time) also given throughout this paper. Let q_i denote the quantity of good i, $i = 1$ to n, and let $\mathbf{q} = (q_1, \cdots, q_n)'$ denote the column vector of such quantities. Let p_i denote the price of good i, $i = 1$ to n, and let $\mathbf{p} = (p_1, \cdots, p_n)'$ denote the column vector of prices.[11]

3.1 Linear Demand Systems

Ideally, an LDS would be micro-founded, i.e. consistent with a detailed description of the market under study: a population of individual consumers with possibly heterogeneous preferences over the set of differentiated goods, including possible search or transportation costs, as well as any property of the underlying environment. Then aggregation of individual demands would lead to an LDS, that is n relationships of the form

$$\text{for each product } i, \; q_i = d_i - \sum_j h_{ij} p_j \,.$$

Throughout the paper we assume that prices are such that all q_i are positive.[12] These expressions can be gathered in matrix form:

$$\mathbf{q}\,(\mathbf{p}) = \mathbf{d} - \mathbf{H}\mathbf{p}\,. \tag{2}$$

First and foremost the domains of $\mathbf{d}$ and $\mathbf{H}$ have to be restricted to make it a meaningful demand system. Assuming $\mathbf{d} > 0$ and $h_{ii} > 0$ is a first step but it is not enough. It also has to satisfy the law of demand, which extends to the multiproduct case the idea that demand decreases with price:[13]

$$\text{for any pair of prices } \mathbf{p^1},\, \mathbf{p^2},\; \left(\mathbf{p^1} - \mathbf{p^2}\right)' \left(\mathbf{q}\left(\mathbf{p^1}\right) - \mathbf{q}\left(\mathbf{p^2}\right)\right) \leqslant 0\,.$$

In the linear case, the above condition boils down to the matrix $\mathbf{H}$ being positive semi-definite. That is, symmetric and for any $\mathbf{p} > 0$, the scalar $\mathbf{p}'\mathbf{H}\mathbf{p}$ should be nonnegative. If $\mathbf{p}'\mathbf{H}\mathbf{p}$ is never zero, $\mathbf{H}$ is positive definite.

Even for a micro-founded LDS, a natural question is whether or not it can be derived from the utility function of a representative consumer. Formally, following definition 3 of Nocke and Schutz (2017), an LDS is quasi-linearly integrable if there exists a function $U(\mathbf{q})$ such that the LDS is the unique solution to $\max_{q_0,\mathbf{q}} U(\mathbf{q}) + q_0$ s.t. $q_0 + \mathbf{p}'\mathbf{q} = m$, where m denotes the wealth of the consumer.

Fact 5 (Law of demand and integrability)**.** *An LDS, $\mathbf{q} = \mathbf{d} - \mathbf{H}\mathbf{p}$, satisfies the law of demand if and only if matrix $\mathbf{H}$ is positive semi-definite, and in that case it is quasi-linearly integrable. Moreover, it is quasi-linearly integrable into a QQUM if and only if matrix $\mathbf{H}$ is positive definite.*[14]

[11] It goes without saying that throughout prices and quantities are nonnegative.

[12] The logic behind this usual approach is to neglect these positivity constraints, solve for a unique Nash equilibrium and verify that the constraints are satisfied for the equilibrium values.

[13] See Lemma 1 of Amir, Erickson, and Jin (2017) and the discussion above, and Theorem 1 of Nocke and Schutz (2017).

[14] The proof can be found (under slightly different forms) in LaFrance (1985), Amir, Erickson, and Jin (2017), and Nocke and Schutz (2017).

The Hotelling line model illustrates that an LDS can be micro-founded, satisfy the low of demand, be quasi-linearly integrable but not into a QQUM. Slightly generalizing the textbook two-good example, assume that consumers are described by their location $x \in [0,1]$. If consumer x buys good 1 (located a zero) at price p_1, s/he has a surplus $u_1 - p_1 - tx$ where t is a transportation cost. Buying good 2 (located at one) at price p_2 leads to a surplus $u_2 - p_2 - t(1-x)$. Not buying (i.e. the outside option) gives a surplus normalized at zero. For prices such that the demand of both goods are positive, the corresponding LDS is:

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \frac{1}{2t}\begin{pmatrix} t + u_1 - u_2 \\ t + u_2 - u_1 \end{pmatrix} - \frac{1}{2t}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}.$$

The matrix $((1,-1),(-1,1))$ is positive semi-definite but not positive definite. Therefore it is a well defined, quasi-linearly integrable LDS but it is not quasi-linearly integrable into a QQUM. The price competition duopoly game has a unique Nash equilibrium and consumers' surplus can be easily computed. One could not, however, define a quantity competition duopoly.

The above Hotelling example brings us naturally to a negative result provided by Jaffe and Weyl (2010). They show that a linear demand system cannot be generated from continuous discrete choice foundations when there are at least two products and buyers can consume an outside option. This seems at odds with the basic Hotelling and Salop models, however. Armstrong and Vickers (2015) generalize Jaffe and Weyl and show, in particular, that linear demand can be consistent with a discrete choice model in which the support of valuations does not have full dimension. And, indeed, let $\theta_1 = u_1 - tx$ and $\theta_2 = u_2 - t(1-x)$ then $(\theta_1, \theta_2) \in \Theta = [u_1 - t, u_1] \times [u_2 - t, u_2]$. A pair (θ_1, θ_2) characterizes a consumer but consumers are not continuously distributed over Θ as they are all on the line $\theta_2 + \theta_1 = u_1 + u_2 - t$.

Fact 6 (Micro foundation of an LDS)**.** *An LDS* $\mathbf{q} = \mathbf{d} - \mathbf{Hp}$ *with for all* i, $\sum_j h_{ij} > 0$ *can be micro founded by heterogeneous consumers distributed along the lines and at the nodes of a* n *node complete graph with Hotelling-like utility functions for each pair of goods.*

In the spirit of the spokes model of Chen and Riordan (2007) (see also section 4.3 of Amir, Jin, Pech, and Tröge (2016)) and also building on the duopoly example given in the working paper Bos and Vermeulen (2019),[15] one can proceed as follows. Let μ_{ii} be the mass of consumers interested uniquely in brand i, $i = 1$ to n and μ_{ij} the mass of consumers potentially interested in both brand i and $j \neq i$ with $\mu_{ij} = \mu_{ji}$.[16] Notice that good i and $j \neq i$ could be substitute or complement. Local demands are derived as in the Hotelling line example and take the form:

$$q_{ii} = x_{ii} - \tau_{ii}\, p_i \text{ and } q_{ij} = \begin{cases} x_{ij} + \tau_{ij}\,(p_j - p_i) & \text{if } h_{ij} < 0 \\ x_{ij} + \tau_{ij}\,(p_j + p_i) & \text{if } h_{ij} > 0 \end{cases}$$

where $\tau_{ij} = \tau_{ji} = 1/t_{ij}$ is the inverse of the transportation cost on the line between i and j. When $h_{ij} < 0$ then $\tau_{ij} > 0$ and $0 < x_{ij} < 1$ is the demand for good i when $p_i = p_j$, with $x_{ji} = 1 - x_{ij}$. Whereas when $h_{ij} > 0$ then $\tau_{ij} < 0$ and $0 < x_{ij} = x_{ji}$ is the demand for both good i and j when $p_i = p_j = 0$.[17] In Chen and Riordan, two goods are always substitute, $q_{ii} = \frac{N-n}{N-1}\frac{2}{N}(v - p_i)$ if $0 \leqslant v - p_i \leqslant 1$ and $q_{ii} = \frac{N-n}{N-1}\frac{2}{N}$ otherwise, and $q_{ij} = \frac{1}{N(N-1)}(1 + p_j - p_i)$.

[15] We are grateful to a referee for pointing out this contribution.

[16] Products are located at n different nodes and there is complete graph linking these nodes. Consumers are located along the graph lines and at the nodes.

[17] In the case $n = 2$, the Hotelling graph model can be seen as a search model à la Rosenthal (1980) Varian (1980) with a fraction of consumers loyal to good 1 (say, coffee), another loyal to good 2 (say, tea), and a fraction of shoppers who could buy one or the other. In these two seminal models, shoppers would all buy from the cheapest brand leading to a mixed strategy equilibrium. Here shoppers are heterogenous.

Then total demand for brand i (assuming that all local demands are positive) writes:

$$q_i = \sum_j \mu_{ij} q_{ij} = \left(\sum_j \mu_{ij} x_{ij} \right) - \left(\sum_j \mu_{ij} \tau_{ij} \right) p_i + \sum_{j \neq i} \mu_{ij} \tau_{ij} p_j$$

which could be matched with

$$q_i = d_i - h_{ii} p_i - \sum_{j \neq i} h_{ij} p_j$$

Therefore, for $i \neq j$, one should have $\mu_{ij}\tau_{ij} = -h_{ij}$. Then the equation $\sum_j \mu_{ij}\tau_{ij} = h_{ii}$ writes $\mu_{ii}\tau_{ii} - \sum_{j \neq i} h_{ij} = h_{ii}$ and here the only issue is the constraint $\mu_{ii}\tau_{ii} > 0$. That is, the matrix $\mathbf{H}$ should be such that for all i, $h_{ii} + \sum_{j \neq i} h_{ij} > 0$ which is true by assumption. This condition is fairly natural as it means that if the prices of all goods are increased by the same amount then the demands of all goods decrease. It is, however, more restrictive than the Law of demand.

3.2 Quasilinear Quadratic Utility Model

As explained at the end of Section 2, when studying an economic question in a general product differentiation oligopoly, the need of a tractable demand system arises in order to have closed-form solutions. The natural choice is an LDS. In order to have a foundation, most papers choose to derive this LDS from a QQUM (see Table 1 for articles on various topics where this modeling strategy is followed). Formally, the assumption is made to restrict the attention to the family of positive definite H matrices in (2). Thus excluding from the analysis the singular positive semi-definite matrices. As generically a positive semi-definite matrix is positive definite, this choice makes sense. Another advantage of this choice is that both Bertrand and Cournot competition can be studied and that their equilibria can be expressed in a symmetric way.

If QQUM started with Levitan and Shubik, several academics have, since then, generalized it. In this section, we use a general framework to introduce formally QQUM and derive its main properties. This section (and the next where oligopoly games are solved) builds on previous articles which can be divided into two groups. First, Economics oriented articles: Jin (1997), Amir and Jin (2001), Bernstein and Federgruen (2004), Choné and Linnemer (2008), Chang and Peng (2012), and Amir, Erickson, and Jin (2017). Second, Operation Research oriented ones: Farahat and Perakis (2009), Farahat and Perakis (2011a), Farahat and Perakis (2011b), Kluberg and Perakis (2012). Cross-citations between the two groups tend to be rare.

Quasi-linear Quadratic utility The quasi-linear quadratic utility model (QQUM) first assumes quasi-linearity. That is, there is a numéraire good q_0 which price is normalized to 1 and the utility function (of the representative consumer) writes $U(\mathbf{q}) + q_0$. The maximization problem of the consumer writes: $\max_{q_0,\mathbf{q}} U(\mathbf{q}) + q_0$ s.t. $q_0 + \mathbf{p}'\mathbf{q} = m$ where m denotes the wealth of the consumer. Eliminating[18] q_0 and dropping the constant term m, leads to $\max_{\mathbf{q}} U(\mathbf{q}) - \mathbf{p}'\mathbf{q}$. Assuming a quadratic form for $U(.)$ allows to write the maximization problem as

$$\max_{\mathbf{q}} U(\mathbf{q}) - \mathbf{p}'\mathbf{q} = \max_{\mathbf{q}} (\mathbf{a} - \mathbf{p})'\,\mathbf{q} - \frac{1}{2}\mathbf{q}'\mathbf{B}\mathbf{q} \tag{3}$$

where $\mathbf{a}$ is the column vector of the (marginal) quality (or utility) indexes, a_i, one for each variety i, and $\mathbf{B}$ is a $n \times n$ positive definite matrix a necessary condition for U to be strictly concave. The $\mathbf{B}$ matrix captures the complementarity/substitution patterns. The diagonal terms, $b_{ii} = b_i$, correspond to $-\partial^2 U/\partial^2 q_i$ and capture the concavity of U with respect to q_i (or how quickly is the marginal utility

[18] This cannot be done for all values of m. If m is too small the constraint $q_0 \geqslant 0$ could be binding and the optimal quantities would depend on m. See Varian (1992) (chapter 10 section 3) and Amir, Erickson, and Jin (2017).

of good i decreases). The off-diagonal terms, b_{ij} $i \neq j$, correspond to $-\partial^2 U/\partial q_i \partial q_j$ and capture the (possibly rich) pattern of complementarity and substitutability among the goods.[19]

Without matrix notations, the objective function of (3) writes (noting $b_{ii} = b_i$):

$$U(\mathbf{q}) - \mathbf{p}'\mathbf{q} = \sum (a_i - p_i)\, q_i - \sum_i \sum_{j>i} b_{ij} q_i q_j - \frac{1}{2} \sum b_i q_i^2 \,. \qquad (3 \text{ bis})$$

Some authors normalize the b_i to one. This is possible (by changing the units with which each quantity is measured, i.e. using $x_i = \sqrt{b_i} q_i$). Very often it would be useless because the b_{ij} terms would not be made simpler. In practice, one chooses a specific model (i.e. $\mathbf{B}$) for tractability. So it is more a question of choice than one of normalization. Sometimes, however, a (re)normalization is useful to show that two models are isomorphic (see section 4.1 and section 5).

Fact 7 (Limiting cases where $\mathbf{B}$ becomes singular)**.** *The utility function $U(\mathbf{q})$ can be rewritten in order to emphasize two limiting cases: perfect substitutes and perfect complements. In the neighborhood of perfect complements, the no-income-effect assumption (i.e. $q_0 > 0$) cannot hold.*

Indeed, (3 bis) can be rewritten as

$$U(\mathbf{q}) = \sum a_i q_i - \sum_i \sum_{j>i} \left(b_{ij} - \sqrt{b_i b_j} \right) q_i q_j - \frac{1}{2} \left(\sum \sqrt{b_i}\, q_i \right)^2 .$$

Now, if for all i, j, $b_i = b_{ij} = b > 0$, then $U - \sum p_i q_i = \sum (a_i - p_i) q_i - \frac{b}{2} \left(\sum q_i \right)^2$. This utility is maximized by buying all units from the seller offering the largest surplus $a_i - p_i$ (Bertrand competition for vertically differentiated goods). The case $a_i = a > 0$ being Bertrand competition for an homogeneous good where only the $\sum q_i$ is relevant and it corresponds to the perfect substitutes case.

The polar case is when all goods are perfect complements. Formally, this case corresponds to a Leontief utility function (which is concave but not strictly concave), for example, $q_0 + a \min \{q_1, \cdots, q_n\}$ with $a > 1$ then maximizing under a budget constraint $q_0 + \sum p_i q_i \leqslant m$ leads to $q_0 = 0$ and, for all i, $q_i = m / \sum p_i$. Quantities cannot be independent of income. It means that this model does not capture fully the economic environment of the consumer. Indeed, the choice of m should be modelled. QQUM can "in spirit" replicate this case (with the same drawback). Indeed, rewriting (3 bis) as

$$U(\mathbf{q}) = \sum a_i q_i - \sum_i \sum_{j>i} \left(b_{ij} + \frac{\sqrt{b_i b_j}}{n-1} \right) q_i q_j - \frac{1}{2(n-1)} \sum_i \sum_{j>i} \left(\sqrt{b_i}\, q_i - \sqrt{b_j}\, q_j \right)^2 .$$

If for all i, $a_i = a > 1$, $b_i = b > 0$ and for $j \neq i$, $b_{ij} = \frac{-b}{n-1}$, then $U = a \sum q_i - \frac{b}{2(n-1)} \sum_i \sum_{j>i} (q_i - q_j)^2$. Maximizing $U(\mathbf{q}) + q_0$ under a budget constraint $q_0 + \sum p_i q_i \leqslant m$ also leads to $q_0 = 0$ and, for all i, $q_i = m / \sum p_i$.

More generally, as discussed by Varian (1992) (chapter 10 section 3), for a quasi-linear utility function, the available income should be large enough in order to have demand functions which are independent of income. This assumption can be mild for substitutes but, as pointed out by Amir, Erickson, and Jin (2017) (see their Proposition 14), it is, indeed, incompatible with perfect complements.

[19] Some authors, while working within the general framework, assume that all elements of $\mathbf{B}$ are positive and that $\mathbf{B}$ is diagonal dominant: Bernstein and Federgruen (2004), Farahat and Perakis (2009), Farahat and Perakis (2010), Farahat and Perakis (2011b), and Kluberg and Perakis (2012). Diagonal dominance implis that $\mathbf{B}$ is nonsingular. Also some authors start with $U = (\mathbf{a} - \mathbf{p})' \mathbf{q} - \frac{1}{2} \mathbf{q}' \mathbf{B}^{-1} \mathbf{q}$ inverting (in the notation only) the role played by $\mathbf{B}$ and $\mathbf{B}^{-1}$. This is a question of taste but this can be slightly confusing as the off-diagonal elements of $\mathbf{B}^{-1}$ are negative when those of $\mathbf{B}$ are positive.

A similar problem could arise for independent goods. That is, assuming for all i, j, $j \neq i$, $b_{ij} = 0$. If each variant i is sold by a monopoly (it would not really change the problem if goods were sold at marginal cost) and denoting c_i the constant marginal cost, demand is $q_i = (a_i - p_i)/b_i$, price $p_i = (a_i + c_i)/2$, and the expenditure of the consumer would be $\sum p_i q_i = \sum \left(a_i^2 - c_i^2\right)/(4b_i)$. Clearly, without restricting further the values of a_i, c_i, and b_i this sum would diverge when the number of good, n, tends to infinity, and the budget constraint cannot be satisfied.

Demand and inverse demand functions The first-order condition of the maximization of U with respect to $\mathbf{q}$ provides immediately the expression of inverse demand:

$$\mathbf{Bq} = \mathbf{a} - \mathbf{p} \;\text{ i.e. }\; \mathbf{p}\left(\mathbf{q}\right) = \mathbf{a} - \mathbf{Bq}\,. \tag{4}$$

Without matrix notations, it writes:

$$\text{for all } i,\; p_i = a_i - b_i q_i - \sum_{j \neq i} b_{ij} q_j \tag{4 bis}$$

which shows that, up to this point, matrix notations do not particularly simplify the writing of the model. However, to characterize direct demand, one needs $\mathbf{B}^{-1}$ the inverse of $\mathbf{B}$, and here matrix notations are useful.[20] Inverting (4) gives direct demand:

$$\mathbf{q}(\mathbf{p}) = \mathbf{B}^{-1}\left(\mathbf{a} - \mathbf{p}\right) \tag{5}$$

which can be linked to (2) with $\mathbf{H} = \mathbf{B}^{-1}$ and $\mathbf{d} = \mathbf{B}^{-1}\mathbf{a}$. To avoid matrix notations, let β_{ij} denote the elements[21] of $\mathbf{B}^{-1}$, and using $\beta_i = \beta_{ii}$, we can write

$$\text{for all } i,\; q_i = \sum_j \beta_{ij}(a_j - p_j) = \beta_i(a_i - p_i) + \sum_{j \neq i} \beta_{ij}(a_j - p_j)\,. \tag{5 bis}$$

Three observations are in order here. First, if $p_i = 0$ and all $p_j = a_j$, then $q_i = \beta_i a_i$. Second, if all prices are set to zero, then $q_i = \sum_j \beta_{ij} a_j < \beta_i a_i$. Third, a comparative static on b_{ij} is different from a comparative static on β_{ij} as a variation of b_{ij} typically implies a variation of several $\beta_{k\ell}$.[22]

In Appendix , the expressions of consumers' surplus, V, aggregate profits Π, and welfare W are derived (repectively (A.1), (A.2), and (A.3)) as functions of prices, that is, independently of the type of competition, or the ownership structure of firms, for any positive definite matrix $\mathbf{B}$.

3.3 Oligopoly games

Firms To complete the description of the oligopoly two elements are needed. First, production costs have to be introduced and, most of the time, they are assumed to be linear. Let c_i denote the marginal cost of production of firm i and let $\mathbf{c} = (c_1, \ldots, c_n)'$ denote the column vector of these marginal costs. Assuming convex costs is usually problematic when looking for closed-form expressions. See, however, Bernstein and Federgruen (2004) for existence and uniqueness of equilibrium with convex costs, and various comparative statics results. Second, an ownership structure has to be specified.

Fact 8 (Ownership structure)**.** *While most existing studies consider oligopolies with single-product firms, the model naturally extends to multi-product competition, delivering simple expressions for first-order conditions at Cournot-Nash and Bertrand-Nash equilibria.*

[20] Recall here that as $\mathbf{B}$ is positive definite it has an inverse and $\mathbf{B}^{-1}$ is also positive definite.

[21] Formally, let $\mathbf{B}^{ij}$ be the matrix where lign j and column i are deleted from $\mathbf{B}$ then $\beta_{ij} = (-1)^{i+j}\det\left(\mathbf{B}^{ij}\right)/\det\left(\mathbf{B}\right)$.

[22] See Kopel, Ressi, and Lambertini (2017) for this point being made in the case of a duopoly.

This question appears naturally in the context of mergers, see Choné and Linnemer (2008) for example. It is also considered in Farahat and Perakis (2009), Farahat and Perakis (2010), and Farahat and Perakis (2011b). Let $N = \{1, \ldots, n\}$ denote the set of all brands. The structure of the industry is described by a partition of N into r subsets: $\{I_1, \cdots, I_r\}$ where I_k denotes the set of brands owned by firm k.

More generally, one could assume cross-ownership. In that case, let $\alpha_{kj} \in [0,1]$ denote the share of profit generated by the sales of good j owned by firm k, with for all j, $\sum_k \alpha_{kj} = 1$. One would also need to specify how the price (or quantity) of a multi-owned good is chosen. For example, it could be chosen by the firm with the largest share but when this share is less than 0.5 there might be no obvious choice.

Parameter space For the model to have economic sense, its parameters have to be further constrained and two assumptions are often made: $\mathbf{a} - \mathbf{c} > \mathbf{0}$ (Positive primary markups) and $\mathbf{q}(\mathbf{c}) \geqslant \mathbf{0}$ (Positive primary outputs, i.e. all varieties are produced when all prices are set at marginal costs). Both terms were introduced by Amir and Jin (2001), see Chang and Peng (2012) for a discussion. Although these assumptions are quite natural, in equilibrium prices are larger than marginal costs, and a variety can be profitable even if it would not be sold at the first best.[23] In both types of competition, price margins $\mathbf{p} - \mathbf{c}$ and quantities $\mathbf{q}$ depend only on the marginal surpluses $\mathbf{a} - \mathbf{c}$. It is, therefore, useful to introduce a new notation for these marginal surpluses:

$$\text{let } v_i = a_i - c_i\,, \text{ or in matrix form } \mathbf{v} = \mathbf{a} - \mathbf{c}\,.$$

Equilibrium The purpose of establishing direct and inverse demand functions is to use them to find the Nash equilibrium of an oligopoly game where each product is produced by one firm. As firms either compete in prices or quantities there are two games: a Bertrand-like one (i.e. competition in prices) and a Cournot-like one (i.e. competition in quantities).

Although in this section matrix $\mathbf{B}$ is assumed to be invertible, most of the computations carry out to an LDS given by (2) with $\mathbf{H}$ being singular. What would be lost is the possibility to replace prices by quantities in the f.o.c. to maintain a symmetry between the Cournot and Bertrand formulae. Before turning to oligopoly games, we present briefly two benchmarks: first-best and monopoly.

First-best Obviously, to maximize welfare, each good should be priced at marginal cost and the first-best quantities, using (4), are given by

$$\mathbf{B}\mathbf{q}^* = \mathbf{a} - \mathbf{p} = \mathbf{a} - \mathbf{c} = \mathbf{v} \tag{6}$$

using (A.3), we have $W^* = \frac{1}{2}\mathbf{v}'\mathbf{B}^{-1}\mathbf{v}$.

Monopoly It can be useful (e.g. to study collusion,[24] or merger to monopoly) to characterize the quantities that a monopoly controlling all goods would choose.[25] Its profit is

$$\Pi = \sum \pi_i = (\mathbf{p} - \mathbf{c})'\,\mathbf{q} = (\mathbf{a} - \mathbf{c})'\,\mathbf{q} - \mathbf{q}'\mathbf{B}\,\mathbf{q}$$

which is maximized here with respect to $\mathbf{q}$ and the f.o.c. writes in matrix notations

$$2\mathbf{B}\mathbf{q}^{\mathbf{m}} = \mathbf{a} - \mathbf{c} = \mathbf{v}\,.$$

[23] See Zanchettin (2006) for a detailed comparison of Betrand and Cournot competition in a duopoly setting.

[24] See Deneckere (1983) for $n = 2$, Deneckere (1984) for a key correction, and Majerus (1988) for n firms.

[25] One such application is undertook by Amir, Jin, Pech, and Tröge (2016) who study prices and deadweight loss in multiproduct monopoly.

Fact 9 (Monopoly)**.** *Monopoly quantities are half first-best quantities. Monopoly prices are invariant with* $\mathbf{B}$*, they are* $\mathbf{p^m} = \frac{1}{2}(\mathbf{a}+\mathbf{c})$*. Or equivalently,* $\mathbf{p^m} - \mathbf{c} = \frac{1}{2}\mathbf{v}$*. Moreover*

$$\Pi^m = \frac{1}{4}\mathbf{v}'\mathbf{B}^{-1}\mathbf{v} \ , \ \textit{and} \ \ W^m = \frac{3}{8}\mathbf{v}'\mathbf{B}^{-1}\mathbf{v} = \frac{3}{4}W^* \, . \tag{7}$$

Cournot competition: f.o.c. Following Jin (1997) and Amir and Jin (2001),[26] one can elegantly write the f.o.c. of the maximization of the profit of firm i quite generally (i.e. for an arbitrary positive definite $\mathbf{B}$ matrix). Indeed, the profit of firm i being

$$\pi_i = (p_i - c_i)q_i = \left(a_i - c_i - \sum_j b_{ij}q_j\right) q_i$$

which is maximized with respect to q_i. The f.o.c. (the s.o.c. is satisfied as b_i is positive) writes

$$p_i - c_i + \frac{\partial(p_i - c_i)}{\partial q_i} q_i = p_i - c_i - b_i q_i = 0$$

or in matrix form

$$\mathbf{p} - \mathbf{c} = \mathbf{diag}(\mathbf{b})\mathbf{q}$$

where $\mathbf{diag}(\mathbf{b})$ is a diagonal matrix which elements are the diagonal elements of $\mathbf{B}$ (i.e. the b_i). Now, using $\mathbf{p} - \mathbf{c} = \mathbf{a} - \mathbf{c} - \mathbf{B}\mathbf{q}$ the f.o.c. collected in matrix form are

$$\left(\mathbf{B} + \mathbf{diag}(\mathbf{b})\right)\mathbf{q}^C = \mathbf{a} - \mathbf{c} = \mathbf{v} \, . \tag{8}$$

Equation (8) is helpful to study the existence and uniqueness of Cournot equilibria. The answer is simple. As the matrices $\mathbf{B}$ and $\mathbf{diag}(\mathbf{b})$ are both positive definite so is the sum $\mathbf{B} + \mathbf{diag}(\mathbf{b})$, hence the existence and uniqueness of a Cournot equilibrium. In Appendix B, f.o.c. are similarly computed for the case where firms are multi-products.

Bertrand competition: f.o.c. Again following Jin (1997) and Amir and Jin (2001), one can write the f.o.c. of the maximization of the profit of firm i with respect to price quite generally, i.e. for an arbitrary positive definite $\mathbf{B}$ matrix. Indeed, the profit of firm i being

$$\pi_i = (p_i - c_i)q_i = (p_i - c_i)\left(\beta_i(a_i - p_i) + \sum_{j\neq i} \beta_{ij}(a_j - p_j)\right)$$

is maximized here with respect to p_i. The f.o.c. (the s.o.c. is satisfied as β_i is positive) writes

$$q_i + \frac{\partial q_i}{\partial p_i}(p_i - c_i) = q_i - \beta_i(p_i - c_i) = 0$$

or in matrix form

$$\mathbf{q} = \mathbf{diag}(\boldsymbol{\beta})(\mathbf{p} - \mathbf{c})$$

where $\mathbf{diag}(\boldsymbol{\beta})$ is a diagonal matrix which elements are the diagonal elements of $\mathbf{B^{-1}}$ (i.e. the β_i). Now, using $\mathbf{p} - \mathbf{c} = \mathbf{a} - \mathbf{c} - \mathbf{B}\mathbf{q}$ the f.o.c. collected in matrix form are[27]

$$\left(\mathbf{B} + \mathbf{diag}(\boldsymbol{\beta})^{-1}\right)\mathbf{q}^B = \mathbf{a} - \mathbf{c} = \mathbf{v} \, . \tag{9}$$

[26] In Levitan and Shubik (1967b), the same derivation is done, for a particular, $\mathbf{B}$ matrix. The writing is slightly messy but the idea is sound.

[27] If one can only use an LDS given by (2) with a singular matrix, then a similar computation would lead to the characterization of equilibrium prices as $\mathbf{p} - \mathbf{c} = \left(\mathbf{H} + \mathbf{diag}(\mathbf{h})\right)^{-1}(\mathbf{d} - \mathbf{H}\mathbf{c})$, under the assumption that $\mathbf{H} + \mathbf{diag}(\mathbf{h})$ is non singular.

Prices have been eliminated in order to show the similarity of the Bertrand and Cournot characterization. Here also as $\mathbf{B}$ and $\mathbf{diag}(\boldsymbol{\beta})^{-1}$ are both positive definite (as the β_i are positive), also is their sum $\mathbf{B}+\mathbf{diag}(\boldsymbol{\beta})^{-1}$ and thus invertible. Hence the existence of a unique equilibrium. In Appendix B, f.o.c. are similarly computed for multi-product firms. To move from the Cournot characterization to the Bertrand one, the $\mathbf{diag}(\mathbf{b})$ matrix has to be replaced by $\mathbf{diag}(\boldsymbol{\beta})^{-1}$. In Jin (1997) (see Appendix A), it is shown that $b_i > 1/\beta_i$, that is the elements of the $\mathbf{diag}(\mathbf{b})$ matrix are larger than those of $\mathbf{diag}(\boldsymbol{\beta})^{-1}$. In that sense, the intuition is that the Bertrand quantities should be larger than the Cournot ones. However, this is not mechanical and counterexamples exist.

To summarize:

Fact 10 (Equilibrium quantities)**.** *Let*

$$\mathbf{q}^* (\mathbf{X}) \equiv \mathbf{q}^* (\mathbf{X}; \mathbf{B}, \mathbf{v}) = (\mathbf{B} + \mathbf{X})^{-1} \mathbf{v} \tag{10}$$

where $\mathbf{X}$ *is a positive definite matrix. Then the First-best, Monopoly, Cournot, and Bertrand equilibrium quantities are respectively given by*

$$\mathbf{q}^* = \mathbf{q}^* (\mathbf{0}), \ \mathbf{q}^{\mathbf{m}} = \mathbf{q}^* (\mathbf{B}), \ \mathbf{q}^C = \mathbf{q}^* (\mathbf{diag}(\mathbf{b})), \ \ \textit{and} \ \ \mathbf{q}^B = \mathbf{q}^* \left(\mathbf{diag}(\boldsymbol{\beta})^{-1}\right).$$

For example, in the case of Cournot (resp. Bertrand), as the matrix $\mathbf{X}$ is diagonal, it is particularly easy to move from the expression of the first-best quantity q_i^* to the expression of q_i^C. One has to change all the b_j in $2b_j$, $j = 1$ to n. In Appendix A, we show how consumers' surplus, total profit, and welfare can be written in this general framework. For example, the equilibrium welfare is

$$2W^* (\mathbf{X}) = \mathbf{v}' \left[(\mathbf{B} + \mathbf{X})^{-1} (\mathbf{B} + 2\,\mathbf{X}) (\mathbf{B} + \mathbf{X})^{-1}\right] \mathbf{v}.$$

4 A user's guide

To the puzzled researcher considering the many variants of LDS, we would suggest to start with the symmetric QQUM introduced by Spence (1976a). This simple version, indeed, encompasses all other symmetric variants encountered in the literature. If the researcher needs to vary the number of product n, she may check the Levitan and Shubik formulation (see our discussion in section 4.2), and should, in any case, cite Shubik and Levitan (1980). We would like to emphasize two points. First, there is no need to restrict to a duopoly framework as dealing with n firms rather than two hardly increases the computational complexity. Second, there is no difficulty allowing for heterogeneous (inverse) demand intercepts $\mathbf{a}$ and marginal costs $\mathbf{c}$. The formulation is

$$U - \sum p_i q_i = \sum (a_i - p_i)\, q_i - \sigma \sum_i \sum_{j>i} q_i q_j - \frac{b}{2} \sum q_i^2. \tag{11}$$

Moreover, this model seems to be the most popular. For example, Majerus (1988) and later Häckner (2000) (see also Hsu and Wang (2005), and many others) who all "simplify" further by assuming $b = 1$. However, we do not recommend this as it simplifies very little the expressions and makes the results harder to read. For the corresponding matrix $\mathbf{B}$ to be definite positive, a sufficient and necessary condition is $\sigma \in \left]\frac{-b}{n-1}, b\right[$. The set of all these matrices can be represented in the (σ, b) plan as in Figure 1. Each pair (σ_0, b_0) such that $\sigma_0 \in \left]\frac{-b_0}{n-1}, b_0\right[$ (in the graph they are above the two darkblue lines) correspond to a positive definite matrix. Figure 1 also highlights how natural is the comparative statics on σ (or b) in this model. The usual one is the horizontal change of σ for a given b. If a matrix is above the line $b = (n-1)\sigma$, then it is diagonally dominant. Matrix (σ_0, b_0) satisfies this condition.

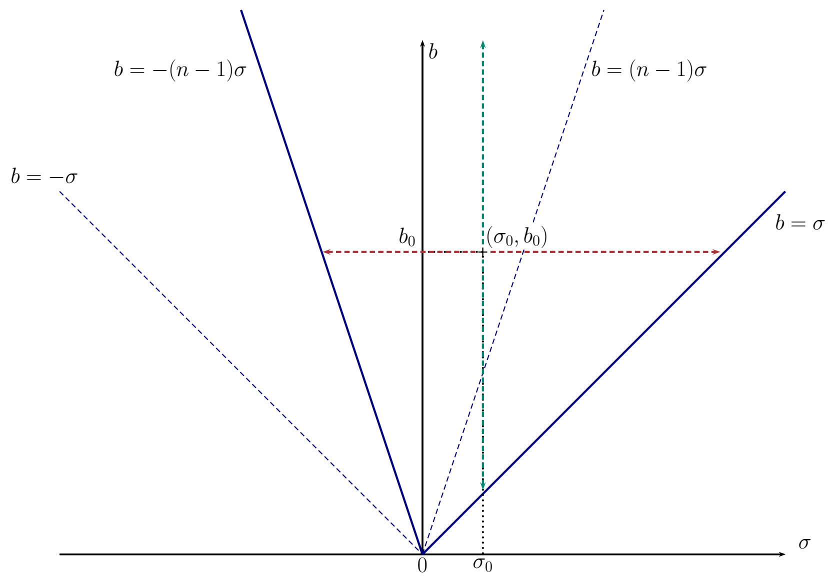

Figure 1: Matrix space, and comparative statics

As the first step is always to establish the relevant equilibrium entities (quantities, prices, profits, surplus, welfare) and as this could be cumbersome (even exhausting), we have collected all the needed results in Table 2 (proofs are in Appendix C). The Table also gives these results for the symmetric Levitan-Shubik's formulation. The aggregate values exhibit a mean-variance structure. Using, for any vectors $\mathbf{x}$, $\mathbf{y}$ the notation $\overline{\mathbf{x}} = \frac{1}{n}\sum x_i$ for the mean and $\mathrm{Var}(\mathbf{x}) = \frac{1}{n}\sum(x_i - \overline{\mathbf{x}})^2$ for the variance and $\mathrm{Cov}(\mathbf{x}, \mathbf{y}) = \frac{1}{n}\sum(x_i - \overline{\mathbf{x}})(y_i - \overline{\mathbf{y}})$ for the covariance.

Fact 11 (Main property of the Spence's formulation)**.** *When* $\mathbf{B}$ *is the Spence's matrix, for any vectors* $\mathbf{x}$ *and* $\mathbf{y}$*:*

$$\mathbf{x}'\mathbf{B}^{-1}\mathbf{y} = \frac{n}{(b-\sigma)}\mathrm{Cov}(\mathbf{x}, \mathbf{y}) + \frac{n}{(b+(n-1)\sigma)}\overline{\mathbf{x}}\,\overline{\mathbf{y}} \tag{12}$$

using (A.1) *and* (A.2)*, for any* $\mathbf{p}$ *such that* $\mathbf{q}(\mathbf{p}) > 0$*, consumers' surplus and total profit are*

- $2V(\mathbf{p}) = \frac{n}{(b-\sigma)}\mathrm{Var}(\mathbf{a} - \mathbf{p}) + \frac{n}{(b+(n-1)\sigma)}\overline{\mathbf{a} - \mathbf{p}}^2$
- $\Pi(\mathbf{p}) = \frac{n}{(b-\sigma)}\mathrm{Cov}(\mathbf{p} - \mathbf{c}, \mathbf{a} - \mathbf{p}) + \frac{n}{(b+(n-1)\sigma)}\left(\overline{\mathbf{p} - \mathbf{c}}\right)\left(\overline{\mathbf{a} - \mathbf{p}}\right).$

For all type T *of competition,* T = *First-best, Monopoly, Cournot, or Bertrand, equilibrium quantities take the form*

$$q_i^{\mathrm{T}} = \lambda^{\mathrm{T}}\left(v_i - \overline{\mathbf{v}}\right) + \mu^{\mathrm{T}}\,\overline{\mathbf{v}}$$

and all aggregate quantities $Y = S$*(urplus),* Π*(rofit), and* W*(elfare) take the form*

$$Y^{\mathrm{T}} = \Gamma_Y^{\mathrm{T}}\mathrm{Var}(\mathbf{v}) + \Upsilon_Y^{\mathrm{T}}\,\overline{\mathbf{v}}^2$$

See Table 2 for Cournot and Bertrand. A direct inspection of the coefficients of the mean and variance terms shows that $V^B > V^C$ *and* $W^B > W^C$*. For the First-best,* $\lambda^* = \frac{1}{b-\sigma}$*,* $\mu^* = \frac{1}{b+(n-1)\sigma}$*, and* $S^* = W^* = \frac{n}{2(b-\sigma)}\mathrm{Var}(\mathbf{v}) + \frac{n}{2(b+(n-1)\sigma)}\overline{\mathbf{v}}^2$*. For the monopoly,* $\lambda^{\mathrm{m}} = \lambda^*/2$*,* $\mu^{\mathrm{m}} = \mu^*/2$*, and* $S^{\mathrm{m}} = W^*/4$*,* $\Pi^{\mathrm{m}} = W^*/2$*, and* $W^{\mathrm{m}} = 3W^*/4$*.*

Because of this property, the formulae are more intuitive with heterogenous rather than homogenous v_i. This mean-variance property was first noticed for the Cournot model, where it holds even for a general demand, by Linnemer (2003) and Valletti (2003). Valletti also gives the result for QQUM, in a variant close to symmetric formulation (using $b - \sigma$ instead of b). It can be used to study the effects of shocks on v_i in the spirit of Zhao (2001) and Février and Linnemer (2004).

4.1 Levitan and Shubik's and Sutton's forms

As a symmetric QQUM has only two parameters: b and σ, (11) covers all cases. Yet, the imagination of economists is such that at least two models have been proposed for particular values of these two terms. Figure 3 summarizes the links between the various QQUM encountered in practice. First, there is the Levitan and Shubik's formulation.[28] The general case given by (1) can be written, using $\sigma^{\mathrm{LS}} = \sigma/\beta$ to allow a better comparison with (11):

$$U^{\mathrm{LS}} - \sum p_i q_i = \sum (a - p_i)\, q_i - \sigma^{\mathrm{LS}} \sum_i \sum_{j>i} q_i q_j - \frac{1}{2} \sum \left(\sigma^{\mathrm{LS}} + \frac{1/\beta - \sigma^{\mathrm{LS}}}{w_i} \right) q_i^2$$

with, for all i, $0 < w_i < 1$ and $\sum w_i = 1$, in addition marginal costs are heterogeneous (chapter 9 of their book). The symmetric case being $w_i = 1/n$ for all i (see chapter 7 of their book where marginal costs are homogeneous). The symmetric Levitan and Shubik's formulation is used, for example, by Motta (2004) and Wang and Zhao (2007). For the symmetric case, $b^{\mathrm{LS}} = n/\beta - (n-1)\sigma^{\mathrm{LS}}$ a peculiar option. Nevertheless, as shown in Figure 2, varying σ^{LS} and β allows to describe the same space as (11). Of course, n is given. Then for a given β, when σ^{LS} varies, b^{LS} varies along the maroon

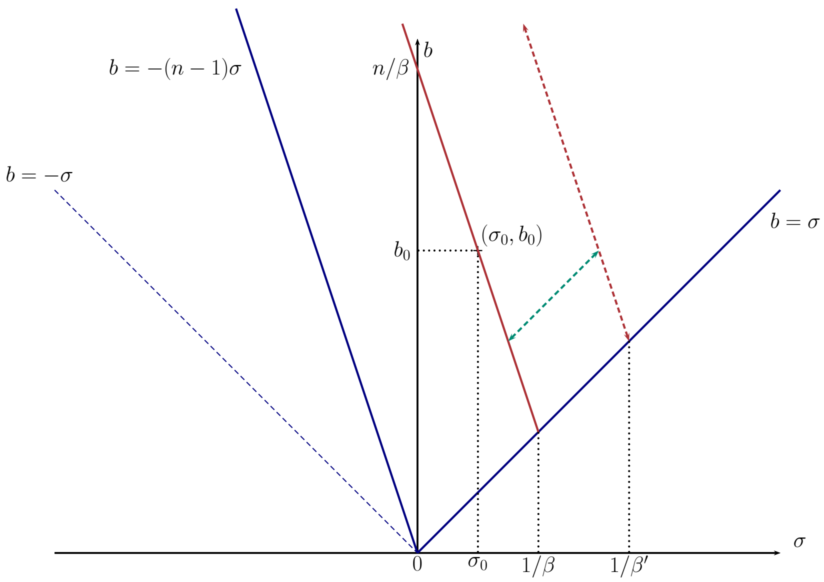

Figure 2: Matrix space, and comparative statics in Levitan and Shubik

line. Changing β to β' allows to move to the (parallel) line $b^{\mathrm{LS}} = n/\beta' - (n-1)\sigma^{\mathrm{LS}}$. Usually the parameter β is fixed (to one) which means that only one line is described. Yet β is crucial in order to describe the space of all definite positive matrices (of our symmetric type). The comparative static

[28] It dates back to Levitan and Shubik (1967a) (part of which is published as Levitan and Shubik (1971b)) and Levitan and Shubik (1967b) and the chapter 9 of their 1980 book.

in σ^{LS} (along the maroon line of Figure 2) is not as natural as in Figure 1.[29] Misunderstanding the relationship between σ and b can lead to confusing statements about the effects of σ.

Second, and visually more different, there is the formulation of Sutton (1997) (see also Sutton (1996) but the working paper of the 1997 RAND article pre-dates the 1996 EER article):

$$U^{\mathrm{Su}} - \sum \tilde{p}_i x_i = \sum (1 - \tilde{p}_i)\, x_i - \sigma \sum_i \sum_{j>i} \frac{x_i}{u_i} \frac{x_j}{u_j} - \frac{1}{2} \sum_i \frac{x_i^2}{u_i^2}$$

with $u_i > 0$ and where (for convenience) the factor $1/2$ is introduced before the squared terms, an innocuous modification of Sutton's formula. See also Symeonidis (1999), Symeonidis (2003b) (duopoly), and Symeonidis (2003a). To see that (11) encompasses Sutton's case, just write $q_i = \frac{x_i}{u_i}$ and $p_i = u_i \tilde{p}_i$ (the marginal cost should also be normalized)[30] it comes that Sutton's expression writes:

$$U^{\mathrm{Su}} = \sum (u_i - p_i)\, q_i - \sigma \sum_i \sum_{j>i} q_i q_j - \frac{1}{2} \sum_i q_i^2$$

which is exactly the Spence's formulation with $a_i = u_i$ and $b = 1$.

A last model is presented in Amir, Erickson, and Jin (2017) (see their section 7) where it is attributed to Bresnahan (1987). They assume that the matrix $\mathbf{B}$ is such that $b_{ij} = \sigma^{|i-j|}$. They call it "a linear demand with local interaction" or the KMS model.[31] They show that, in addition to its own price, the demand of firm i, $2 \leqslant i \leqslant n-1$ only depends on the prices of $i-1$ and $i+1$, whereas the demand for firm 1 (resp. n) depends only on its price and the price of firm 2 (resp. $n-1$). Formally,

$$U^{\mathrm{KMS}} = \sum (a_i - p_i)\, q_i - \sum_i \sum_{j>i} \sigma^{|i-j|} q_i q_j - \frac{1}{2} \sum_i q_i^2 \text{ , and}$$

$$\begin{aligned}
(1-\sigma^2) q_1 &= a_1 - p_1 - \sigma (a_2 - p_2) \\
(1-\sigma^2) q_i &= -\sigma (a_{i-1} - p_{i-1}) + (1+\sigma^2)(a_i - p_i) - \sigma (a_{i+1} - p_{i+1}) \quad \text{for } 2 \leqslant i \leqslant n-1 \\
(1-\sigma^2) q_n &= -\sigma (a_{n-1} - p_{n-1}) + a_n - p_n
\end{aligned}$$

The local interaction property would disappear if $b_{ii} = b \neq 1$, however.

4.2 Dealing with a variation of n

This is an important research question and we touch only lightly on the subject here. When one product variety is added, there are two main effects. First, there is a variety effect: the utility tends to increase with one more product. Second, there is a competition effect: prices tend to decrease with the arrival of a new competitor. A priori, it is not easy to disentangle them. One can start to look at total demand for a given list of prices. Using the expression of the direct demands given in Table 2, total demand $Q = \sum_i q_i$ is:

$$Q^{LS} = \beta\, (\overline{\mathbf{a} - \mathbf{p}}) \quad \text{and} \quad Q = \frac{n}{b + (n-1)\sigma} (\overline{\mathbf{a} - \mathbf{p}})$$

[29] Levitan and Shubik always considered $\sigma^{\mathrm{LS}} > 0$ but with their symmetric specification, the matrix $\mathbf{B}$ is positive definite for any $\sigma^{\mathrm{LS}} \in]-\infty, 1/\beta[$. In fact, they use a parameter γ which correspond to $\sigma^{\mathrm{LS}}/(1-\sigma^{\mathrm{LS}})$ and they (implicitly) assume γ between 0 and $+\infty$. The model allows γ between -1 and and $+\infty$.

[30] The normalization is innocuous. For example, if x_i is measured in grams, and $u_i = 1000$, then q_i is measured in kilograms. For the price, if $\tilde{p}_i$ is the price for one gram, then $1000\tilde{p}_i$ is indeed the price for one kilogram.

[31] Kac–Murdock–Szegö matrices are asymmetric n-Toeplitz matrices.

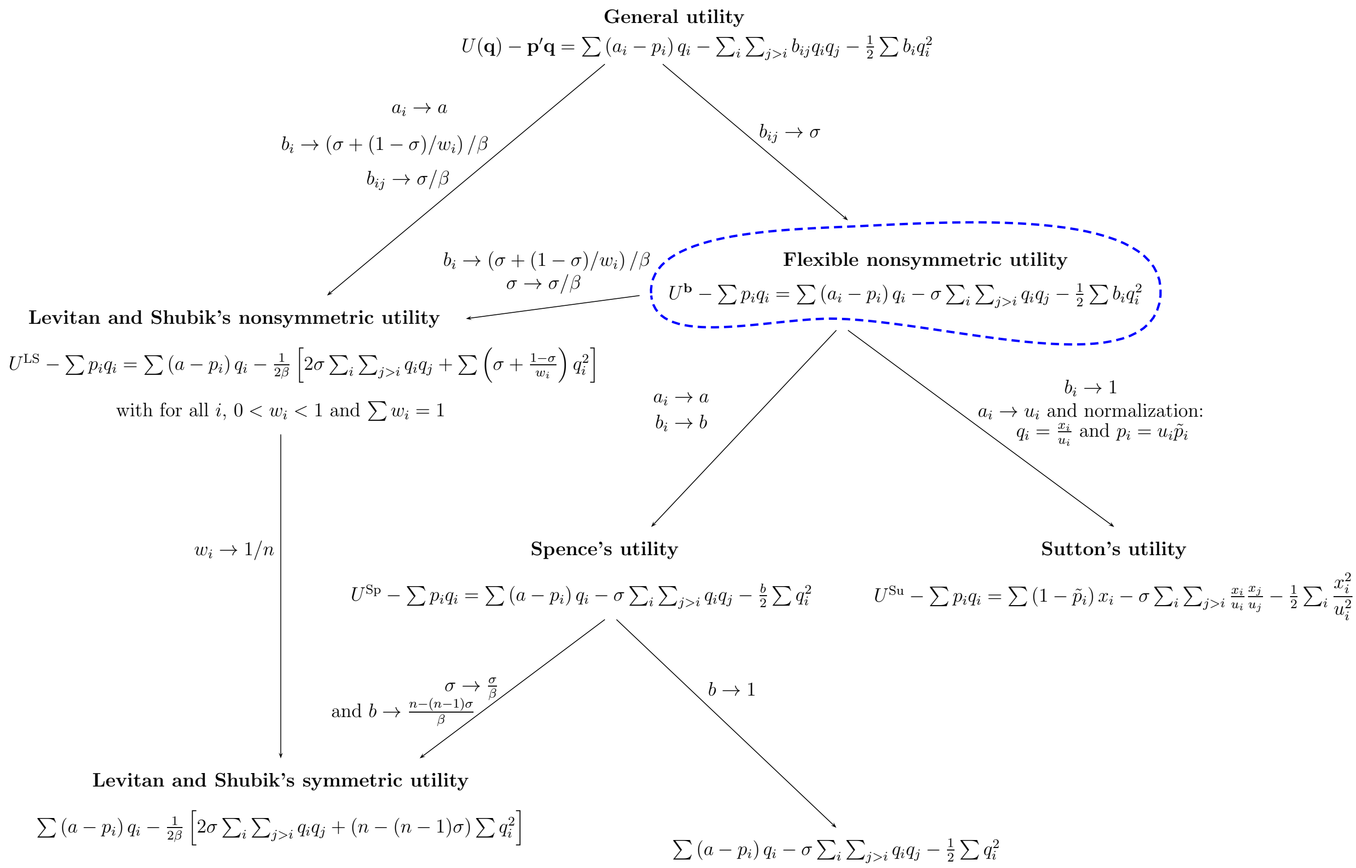

Figure 3: Links between the various formulations

respectively for Levitan-Shubik's and Spence's formulation. As long as β is independent of n, adding a new variety such that $a_{n+1} - p_{n+1} = (\overline{\mathbf{a} - \mathbf{p}})$ would not change Q^{LS} but would increase Q by $\frac{b-\sigma}{(b+(n-1)\sigma)(b+n\sigma)} (\overline{\mathbf{a} - \mathbf{p}})$, admittedly a small amount when n is large (it goes to zero at a rate of $1/n^2$). In Levitan-Shubik's formulation the parameter b increases with n, which decreases the utility, and offsets the positive effect of an additional variety. Also notice that Q^{LS} does not vary with σ.

In Levitan-Shubik's formulation consumers' surplus is $\frac{\beta}{(1-\sigma)} \mathrm{Var}(\mathbf{a} - \mathbf{p}) + \beta \, \overline{\mathbf{a} - \mathbf{p}}^2$, the coefficients do not vary with n (but the variance coefficient does vary with σ). Yet, the introduction of a $(n+1)$-th variety would not leave in general both the variance and the mean constant. So even if total quantity is unchanged, the surplus could change. The only case where the surplus is unaffected happens when for all i, $v_i = \overline{\mathbf{v}}$ as then there is no variance term. Therefore in this no-variance case, the entry of an additional variety such that $a_{n+1} - p_{n+1} = (\overline{\mathbf{a} - \mathbf{p}})$ would not change consumers' surplus unless prices are changed by the entry. Then if an increase of n decreases prices, the surplus would increase and this increase would come exclusively from a competition effect. Is it a desirable property? Even in the no-variance case, the introduction of a new variety could (or even should) create a market expansion effect. This is captured by Spence's formulation where total quantity and surplus would grow at a rate $\frac{n}{b+(n-1)\sigma}$. Whereas this effect is shutdown by Levitan and Shubik. Outside the no-variance case, an additional variety (or a change of σ) modifies consumers' surplus even in the Levitan and Shubik case.

Table 2: The two commonly used models with heterogenous **a** and **c**

Levitan and Shubik	**Spence**
Inverse demands:	
$\begin{aligned} p_i &= a_i - \frac{1}{\beta}\left((\sigma + n(1-\sigma))q_i + \sigma \sum_{j\neq i} q_j\right) \\ &= a_i - \frac{n}{\beta}\left((1-\sigma)q_i + \sigma\overline{\mathbf{q}}\right) \end{aligned}$	$\begin{aligned} p_i &= a_i - bq_i - \sigma \sum_{j\neq i} q_j \\ &= a_i - (b-\sigma)q_i - n\sigma\overline{\mathbf{q}} \end{aligned}$
Direct demands:	
$\begin{aligned} q_i &= \frac{\beta}{n^2(1-\sigma)}\left[(n-\sigma)(a_i-p_i) - \sigma\sum_{j\neq i}(a_j-p_j)\right] \\ &= \frac{\beta}{n(1-\sigma)}\left[(a_i-p_i) - \sigma\left(\overline{\mathbf{a}-\mathbf{p}}\right)\right] \end{aligned}$	$\begin{aligned} q_i &= \frac{(b+(n-2)\sigma)(a_i-p_i)-\sigma\sum_{j\neq i}(a_j-p_j)}{(b-\sigma)(b+(n-1)\sigma)} \\ &= \frac{1}{b-\sigma}\left[(a_i-p_i) - \frac{n\sigma}{b+(n-1)\sigma}\left(\overline{\mathbf{a}-\mathbf{p}}\right)\right] \end{aligned}$
Equilibrium quantities and prices under Cournot competition:	
$\begin{aligned} q_i^C &= \frac{\beta}{2n-(2n-1)\sigma}\left[v_i - \frac{n\sigma}{2n-(n-1)\sigma}\overline{\mathbf{v}}\right] \\ &= \frac{\beta}{2n-(2n-1)\sigma}\left[(v_i-\overline{\mathbf{v}}) + \frac{2n-(2n-1)\sigma}{2n-(n-1)\sigma}\overline{\mathbf{v}}\right] \\ p_i^C - c_i &= \frac{n-(n-1)\sigma}{\beta}q_i^C \end{aligned}$	$\begin{aligned} q_i^C &= \frac{1}{2b-\sigma}\left[v_i - \frac{n\sigma}{2b+(n-1)\sigma}\overline{\mathbf{v}}\right] \\ &= \frac{1}{2b-\sigma}\left[(v_i-\overline{\mathbf{v}}) + \frac{2b-\sigma}{2b+(n-1)\sigma}\overline{\mathbf{v}}\right] \\ p_i^C - c_i &= bq_i^C \end{aligned}$
Equilibrium quantities and prices under Bertrand competition:	
Let $\Psi = \frac{(n-1)\sigma^2}{n-\sigma}$	**Let** $\Phi = \frac{(n-1)\sigma^2}{b+(n-2)\sigma}$
$\begin{aligned} q_i^B &= \frac{\beta}{2n-(2n-1)\sigma-\Psi}\left[v_i - \frac{n\sigma}{2n-(n-1)\sigma-\Psi}\overline{\mathbf{v}}\right] \\ &= \frac{\beta}{2n-(2n-1)\sigma-\Psi}\left[(v_i-\overline{\mathbf{v}}) + \frac{2n-(2n-1)\sigma-\Psi}{2n-(n-1)\sigma-\Psi}\overline{\mathbf{v}}\right] \\ p_i^B - c_i &= \frac{n-(n-1)\sigma-\Psi}{\beta}q_i^B \end{aligned}$	$\begin{aligned} q_i^B &= \frac{1}{2b-\sigma-\Phi}\left[v_i - \frac{n\sigma}{2b+(n-1)\sigma-\Phi}\overline{\mathbf{v}}\right] \\ &= \frac{1}{2b-\sigma-\Phi}\left[(v_i-\overline{\mathbf{v}}) + \frac{2b-\sigma-\Phi}{2b+(n-1)\sigma-\Phi}\overline{\mathbf{v}}\right] \\ p_i^B - c_i &= (b-\Phi)q_i^b \end{aligned}$
Equilibrium surplus under Cournot and Bertrand competition:	
$2V_C = \frac{n^2(1-\sigma)\beta}{(2n-(2n-1)\sigma)^2}\mathrm{Var}(\mathbf{v}) + \frac{n^2}{(2n-(n-1)\sigma)^2}\overline{\mathbf{v}}^2$	$2V_C = \frac{n(b-\sigma)}{(2b-\sigma)^2}\mathrm{Var}(\mathbf{v}) + \frac{n(b+(n-1)\sigma)}{(2b+(n-1)\sigma)^2}\overline{\mathbf{v}}^2$
$2V_B = \frac{n^2(1-\sigma)\beta}{(2n-(2n-1)\sigma-\Psi)^2}\mathrm{Var}(\mathbf{v}) + \frac{n^2}{(2n-(n-1)\sigma-\Psi)^2}\overline{\mathbf{v}}^2$	$2V_B = \frac{n(b-\sigma)}{(2b-\sigma-\Phi)^2}\mathrm{Var}(\mathbf{v}) + \frac{n(b+(n-1)\sigma)}{(2b+(n-1)\sigma-\Phi)^2}\overline{\mathbf{v}}^2$
Equilibrium aggregate profit under Cournot and Bertrand competition:	
$\Pi_C = \frac{n(n-(n-1)\sigma)\beta}{(2n-(2n-1)\sigma)^2}\left[\mathrm{Var}(\mathbf{v}) + \frac{(2n-(2n-1)\sigma)^2\overline{\mathbf{v}}^2}{(2n-(n-1)\sigma)^2}\right]$	$\Pi_C = \frac{nb}{(2b-\sigma)^2}\left[\mathrm{Var}(\mathbf{v}) + \frac{(2b-\sigma)^2\overline{\mathbf{v}}^2}{(2b+(n-1)\sigma)^2}\right]$
$\Pi_B = \frac{n(n-(n-1)\sigma-\Psi)\beta}{(2n-(2n-1)\sigma-\Psi)^2}\left[\mathrm{Var}(\mathbf{v}) + \frac{(2n-(2n-1)\sigma-\Psi)^2\overline{\mathbf{v}}^2}{(2n-(n-1)\sigma-\Psi)^2}\right]$	$\Pi_B = \frac{n(b-\Phi)}{(2b-\sigma-\Phi)^2}\left[\mathrm{Var}(\mathbf{v}) + \frac{(2b-\sigma-\Phi)^2\overline{\mathbf{v}}^2}{(2b+(n-1)\sigma-\Phi)^2}\right]$
Equilibrium welfare under Cournot and Bertrand competition:	
$2\mathrm{W}_C = \frac{n\beta(3n-(3n-2)\sigma)}{(2n-(2n-1)\sigma)^2}\mathrm{Var}(\mathbf{v}) + \frac{n\beta(3n-2(n-1)\sigma)}{(2n-(n-1)\sigma)^2}\overline{\mathbf{v}}^2$	$2\mathrm{W}_C = \frac{n(3b-\sigma)}{(2b-\sigma)^2}\mathrm{Var}(\mathbf{v}) + \frac{n(3b+(n-1)\sigma)}{(2b+(n-1)\sigma)^2}\overline{\mathbf{v}}^2$
$2\mathrm{W}_B = \frac{n\beta(3n-(3n-2)\sigma-2\Psi)}{(2n-(2n-1)\sigma-\Psi)^2}\mathrm{Var}(\mathbf{v}) + \frac{n\beta(3n-2(n-1)\sigma-2\Psi)}{(2n-(n-1)\sigma-\Psi)^2}\overline{\mathbf{v}}^2$	$2\mathrm{W}_B = \frac{n(3b-\sigma-2\Phi)}{(2b-\sigma-\Phi)^2}\mathrm{Var}(\mathbf{v}) + \frac{n(3b+(n-1)\sigma-2\Phi)}{(2b+(n-1)\sigma-\Phi)^2}\overline{\mathbf{v}}^2$

5 Asymmetric formulation: more flexible substitution patterns

In this section, we compute the Bertrand and Cournot Nash equilibria of the "Flexible nonsymmetric utility" (see Figure 3). We extend, and hopefully clarify, the seminal work of Levitan and Shubik: Levitan and Shubik (1967a) (part of which is published as Levitan and Shubik (1971b)) and Levitan and Shubik (1967b) and the chapter 9 of their 1980 book. The maximization program (3) is modified to:

$$U(\mathbf{q}) - \mathbf{p}'\mathbf{q} = \sum (a_i - p_i)\, q_i - \sigma \sum_i \sum_{j>i} q_i q_j - \frac{1}{2} \sum b_i q_i^2 \tag{3 ter}$$

A larger b_i implies that additional units of good i are less valued which limits consumption. The most popular products are those with large $v_i = a_i - c_i$ and low b_i. Another interpretation of the asymmetric model is available. Normalizing the b_i to one:

$$\begin{aligned}
U(\mathbf{x}) - \hat{\mathbf{p}}'\mathbf{x} &= \sum (\hat{a}_i - \hat{p}_i)\, x_i - \sigma \sum_i \sum_{j>i} x_i x_j - \frac{1}{2} \sum b_i x_i^2 \\
&= \sum \left(\frac{\hat{a}_i}{\sqrt{b_i}} - \frac{\hat{p}_i}{\sqrt{b_i}} \right) (\sqrt{b_i} x_i) - \sum_i \sum_{j>i} \frac{\sqrt{\sigma}}{\sqrt{b_i}} \frac{\sqrt{\sigma}}{\sqrt{b_i}} (\sqrt{b_i} x_i)(\sqrt{b_j} x_j) - \frac{1}{2} \sum (\sqrt{b_i} x_i)^2 \\
&= \sum (a_i - p_i)\, q_i - \sum_i \sum_{j>i} \sigma_i \sigma_j q_i q_j - \frac{1}{2} \sum q_i^2
\end{aligned}$$

where $\sigma_i = \sqrt{\sigma / b_i}$, $q_i = \sqrt{\sigma} x_i / \sigma_i$, $a_i = \sigma_i \hat{a}_i / \sqrt{\sigma}$, and $p_i = \sigma_i \hat{p}_i / \sqrt{\sigma}$. This interpretation highlights the differentiation parameters $b_{ij} = \sigma_i \sigma_j$, an environment richer than the symmetric case where for all i and j, $b_{ij} = \sigma$. See Appendix D for how to move from one model to the other. The model has not been used in IO applications because computations are more involved. Yet, closed-form expressions can be derived and their interpretation remains intuitive.

Notation We introduce $w_i = 1/(b_i - \sigma)$, $i = 1$ to n. We denote $\mathbf{W}$ the diagonal matrix where the diagonal terms are these w_i.

Inverse demands When $\mathbf{B} = \mathbf{W}^{-1} + \sigma \mathbf{J}$, the inverse demand functions are

$$\mathbf{p} = \mathbf{a} - \mathbf{B}\mathbf{q} = \mathbf{a} - \mathbf{diag}(\mathbf{b} - \sigma)\mathbf{q} - n\sigma \overline{\mathbf{q}}\, \mathbf{e} \tag{13}$$

where $\overline{\mathbf{q}} = \frac{1}{n} \sum q_i$ is the arithmetic mean and $\mathbf{e}$ is the unitary vector, $\mathbf{e} = (1, \cdots, 1)'$. In extended form, for all i,

$$p_i = a_i - (b_i - \sigma) q_i - n\sigma \overline{\mathbf{q}}$$

Direct demands When $\mathbf{B} = \mathbf{W}^{-1} + \sigma \mathbf{J}$, the direct demand functions are

$$\mathbf{q} = \mathbf{B}^{-1}(\mathbf{a} - \mathbf{p}) = \mathbf{W} \left(\mathbf{a} - \mathbf{p} - \frac{\sigma \sum w_j (a_j - p_j)}{1 + \sigma \sum w_j} \mathbf{e} \right) \tag{14}$$

In extended form:

$$q_i = w_i \left[(a_i - p_i) - \frac{\sigma \sum w_j (a_j - p_j)}{1 + \sigma \sum w_j} \right]$$

Weighted mean and variance For any vector $\mathbf{x}$, and positive diagonal matrix $\widetilde{\mathbf{W}} = \mathbf{diag}(\tilde{\mathbf{w}})$, with diagonal elements $\widetilde{w}_i > 0$, let $\widetilde{n} = \sum \widetilde{w}_i$ the weighted average be

$$\tilde{\mathbf{x}} = \frac{\sum \widetilde{w}_i x_i}{\widetilde{n}} = \frac{1}{\widetilde{n}} \mathbf{e}' \widetilde{\mathbf{W}} \mathbf{x} \,. \tag{15}$$

Similarly, let the weighted variance be

$$\widetilde{\mathrm{Var}}\,(\mathbf{x}) = \frac{1}{\widetilde{n}} \sum \widetilde{w}_i \,(x_i - \tilde{\mathbf{x}})^2 = \frac{1}{\widetilde{n}} \,(\mathbf{x} - \tilde{\mathbf{x}}\,\mathbf{e})' \,\widetilde{\mathbf{W}}\,(\mathbf{x} - \tilde{\mathbf{x}}\,\mathbf{e}) \,. \tag{16}$$

If, for all i, $\widetilde{w}_i = w$, then $\tilde{\mathbf{x}}$ and $\widetilde{\mathrm{Var}}\,(\mathbf{x})$ are the usual mean and variance.

Fact 12 (**Equilibrium quantities**). *When $\mathbf{B} = \mathbf{W}^{-1} + \sigma \mathbf{J}$, there exist weights $\widetilde{\mathbf{W}}$ specific to each type of competition such that equilibrium quantities take the form*

$$\mathbf{q} = \widetilde{\mathbf{W}} \left[\mathbf{v} - \frac{\widetilde{n}\sigma}{1 + \widetilde{n}\sigma}\, \tilde{\mathbf{v}}\,\mathbf{e} \right] = \widetilde{\mathbf{W}} \left[(\mathbf{v} - \tilde{\mathbf{v}}\,\mathbf{e}) + \frac{\tilde{\mathbf{v}}\,\mathbf{e}}{1 + \widetilde{n}\sigma} \right] \tag{17}$$

In extended form:

$$q_i = \widetilde{w}_i \left[v_i - \frac{\widetilde{n}\sigma}{1 + \widetilde{n}\sigma} \tilde{\mathbf{v}} \right] = \widetilde{w}_i \left[(v_i - \tilde{\mathbf{v}}) + \frac{\tilde{\mathbf{v}}}{1 + \widetilde{n}\sigma} \right] \tag{17 bis}$$

One can move from one expression to another by changing the weights. More precisely: First-best weights are $\widetilde{w}_i = w_i = 1/(b_i - \sigma)$. Monopoly quantities are $q_i^m = q_i^*/2$. Cournot weights are $\widetilde{w}_i = 1/(2b_i - \sigma)$. Bertrand weights are $\widetilde{w}_i = 1/(2b_i - \Phi_i - \sigma)$ where $\Phi_i = \left(\sigma^2 \sum_{j \neq i} w_j\right) / \left(1 + \sigma \sum_{j \neq i} w_j\right)$.

A detailed proof is given in Appendix D.1. In terms of normalization, a different one would be suitable for each type of competition. Indeed, one would like to have $\sum \widetilde{w}_j = 1$ but these weights depend on the type of competition. If, however, one is interested in only one type of competition, then the normalization would certainly be useful. The formulae of Fact 12 are easily comparable with their counterparts in Table 2. The standard averages present in Table 2 have been replaced by weighted averages and the (somehow) mysterious coefficients have been replaced by symmetric expressions.

To complete the characterization of the equilibrium we now give the equilibrium prices. They are directly inferred from the equilibrium quantities and from the f.o.c.

Fact 13 (**Equilibrium prices and profits**). *When $\mathbf{B} = \mathbf{W}^{-1} + \sigma \mathbf{J}$, First-best prices are $p_i^* = c_i$ and profits are zero. Monopoly prices are $p_i^m = (a_i + c_i)/2$ and profits are $\pi_i^m = v_i q_i^m/2$. Cournot prices are such that $p_i^C - c_i = b_i q_i^C$. Hence Cournot profits are $\pi_i^C = b_i \left(q_i^C\right)^2$. Bertrand prices are such that $p_i^B - c_i = \frac{1}{\beta_i} q_i^B$. Hence Bertrand profits are $\pi_i^B = \frac{1}{\beta_i} \left(q_i^B\right)^2$ where β_i is the ith diagonal term of the $\mathbf{B}^{-1}$ matrix.*

The computation of consumers' surplus follows. The surplus keeps a mean-variance structure but it is a weighted variance and also two different weighted means are involved.

Fact 14 (**Equilibrium surplus**). *When $\mathbf{B} = \mathbf{W}^{-1} + \sigma \mathbf{J}$, there exist weights $\widetilde{\mathbf{W}}$ specific to each type of competition and weights $\widehat{\mathbf{W}} = \widetilde{\mathbf{W}} \mathbf{W}^{-1} \widetilde{\mathbf{W}}$ such that equilibrium quantities take the following form:*

$$2V = \widehat{n} \left[\widehat{\mathrm{Var}}\,(\mathbf{v}) + \left(\hat{\mathbf{v}} - \frac{\widetilde{n}\sigma}{1 + \widetilde{n}\sigma}\, \tilde{\mathbf{v}} \right)^2 \right] + \sigma \left(\frac{\widetilde{n}\tilde{\mathbf{v}}}{1 + \widetilde{n}\sigma} \right)^2 \tag{18}$$

where $\widehat{n} = \sum \widehat{w}_i$, $\hat{\mathbf{v}}$ and $\widehat{Var}\,(\mathbf{v})$ are respectively the weighted average and variance associated to the weights $\widehat{w}_j$, and $\tilde{\mathbf{v}}$ is the weighted average associated to the weights $\widetilde{w}_j$.

- *First-best weights are* $\widetilde{w}_i = w_i = 1/(b_i - \sigma)$*. Hence* $\widehat{w}_i = w_i$*, and* (18) *takes a simpler form*

$$2V^* = \widetilde{n}\left[\widetilde{\mathrm{Var}}\left(\mathbf{v}\right) + \frac{\widetilde{\mathbf{v}}^2}{1+\widetilde{n}\sigma}\right]$$

- *Monopoly:* $V^m = V^*/4$*.*
- *Cournot weights are* $\widetilde{w}_i = 1/(2b_i - \sigma)$*. Hence* $\widehat{w}_i = (b_i - \sigma)/(2b_i - \sigma)^2$*.*
- *Bertrand weights are* $\widetilde{w}_i = 1/(2b_i - \Phi_i - \sigma)$ *where* $\Phi_i = \left(\sigma^2 \sum_{j\neq i} w_j\right) / \left(1 + \sigma \sum_{j\neq i} w_j\right)$*. Hence* $\widehat{w}_i = (b_i - \sigma)/(2b_i - \Phi_i - \sigma)^2$*.*

The proofs are in Appendix D.2. Similar computations lead to the expression of total profit.

Fact 15 (Aggregate equilibrium profit). *When* $\mathbf{B} = \mathbf{W}^{-1} + \sigma\mathbf{J}$*,*

- *First-best profits are null.*
- *Monopoly profits are* $\Pi^m = V^*/2$*.*

For Cournot and Bertrand, there exist weights $\widetilde{\mathbf{W}}$*, a matrix* $\boldsymbol{X}$*, and weights* $\breve{\mathbf{W}} = \widetilde{\mathbf{W}}\mathbf{X}\widetilde{\mathbf{W}}$ *all three specific to each type of competition such that equilibrium quantities take the following form:*

$$\Pi = \breve{n}\left[\breve{\mathrm{Var}}\left(\mathbf{v}\right) + \left(\breve{\mathbf{v}} - \frac{\widetilde{n}\sigma}{1+\widetilde{n}\sigma}\,\widetilde{\mathbf{v}}\right)^2\right] \tag{19}$$

where $\breve{n} = \sum \breve{w}_i$*,* $\breve{\mathbf{v}}$ *and* $\breve{Var}(\mathbf{v})$ *are respectively the weighted average and variance associated to the weights* $\widehat{w}_j$*, andwhere* $\widetilde{\mathbf{v}}$ *is the weighted average associated to the weights* $\widetilde{w}_j$*.*

- *Cournot weights are* $\widetilde{w}_i = 1/(2b_i - \sigma)$ *and* $\mathbf{X} = \mathbf{diag}(\mathbf{b})$*. Hence* $\breve{w}_i = b_i/(2b_i - \sigma)^2$*.*
- *Bertrand weights are* $\widetilde{w}_i = 1/(2b_i - \Phi_i - \sigma)$ *where* $\Phi_i = \left(\sigma^2 \sum_{j\neq i} w_j\right) / \left(1 + \sigma \sum_{j\neq i} w_j\right)$ *and* $\mathbf{X} = \mathbf{diag}(\boldsymbol{\beta})^{-1}$*. Hence* $\breve{w}_i = (b_i - \Phi_i)/(2b_i - \Phi_i - \sigma)^2$*.*

The proof is in Appendix D.3.

Welfare The expression of welfare in equilibrium is obtained by summation from the expression given in Facts 14 and 15. The sum takes a simple form in the case of the first-best and of the monopoly but, unfortunately, there is no obvious simplification in the case of Cournot and Bertrand. This is because the weights which are useful for the computation of the surplus are different from the weights used in the total profit expression.

6 Conclusion

In this paper, we hope to have clarified recurring questions that IO economists face when dealing with linear demand systems. Our analysis delivers the following takeaway points. For a simple application (i.e. symmetric model in terms of second derivatives of U) the burden of keeping n firms, heterogenous valuations $\mathbf{a} = (a_1, \cdots, a_n)$, and heterogenous marginal costs $\mathbf{c} = (c_1 \cdots, c_n)$ is minimal. It has the advantage of highlighting the Mean-Variance structure of the symmetric model. Although we have emphasized the seminal work of Levitan and Shubik, we would not recommend the use of their astute symmetric model where the parameter b depends on σ and n (and also their parameter β). Formally, their modelling is not restrictive and can be used to describe the same set of models. We tend to prefer the Spence formulation (see Figure 3), which leads to more intuitive comparative static exercises. The gain of keeping consumers' surplus constant after the introduction of a new variety (before prices adjust to a new equilibrium) only exists for marginal costs and inverse demand intercepts that are uniform across products, that is, when the model is the least attractive to describe differentiated products. We have provided concise, closed-form expressions for the First-best, the monopoly, Cournot and Bertrand outcomes, so future users do not bother with cumbersome equations. We have also explained how to compute the equilibria for general ownership structures. Finally, we have provided closed-form expressions for a tractable asymmetric model which should allow future users to venture into richer environments.

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Appendix

A Surplus, profit, welfare

Consumers' surplus, Aggregate profits, and Welfare Using (5), the indirect utility function as a function of prices.

$$V(\mathbf{p}) = \max_{\mathbf{q}} U(\mathbf{q}) - \mathbf{p}'\mathbf{q} = \frac{1}{2}(\mathbf{a}-\mathbf{p})'\mathbf{B}^{-1}(\mathbf{a}-\mathbf{p}) = \frac{1}{2}(\mathbf{q}(\mathbf{p}))'\mathbf{B}(\mathbf{q}(\mathbf{p})) \tag{A.1}$$

As $\mathbf{B}^{-1}$ is also a $n \times n$ positive definite matrix, $V(\mathbf{p})$ is a quadratic form in $\mathbf{a}-\mathbf{p}$.

Aggregate profit (i.e. the sum of all profits) is:

$$\Pi(\mathbf{p}) = (\mathbf{p}-\mathbf{c})'\mathbf{B}^{-1}(\mathbf{a}-\mathbf{p}) = (\mathbf{a}-\mathbf{c})'\mathbf{q}(\mathbf{p}) - \mathbf{q}(\mathbf{p})'\mathbf{B}\,\mathbf{q}(\mathbf{p}) \tag{A.2}$$

and therefore welfare writes:

$$W(\mathbf{p}) = \frac{1}{2}(\mathbf{a}-\mathbf{c}+\mathbf{p}-\mathbf{c})'\mathbf{B}^{-1}(\mathbf{a}-\mathbf{p}) = (\mathbf{a}-\mathbf{c})'\mathbf{q}(\mathbf{p}) - \frac{1}{2}\mathbf{q}(\mathbf{p})'\mathbf{B}\,\mathbf{q}(\mathbf{p}) \tag{A.3}$$

These expressions (A.1), (A.2), and (A.3) hold independently of the type of competition, or the ownership structure of firms, for any positive definite matrix $\mathbf{B}$.

Consumers' surplus, Aggregate profits, and Welfare in equilibrium First, combining (A.1) with (8) and (9) leads to (using the fact that for a symmetric matrix $\mathbf{M}' = \mathbf{M}$)

$$2V^{\mathrm{C}} = \mathbf{v}'(\mathbf{B}+\mathbf{diag}(\mathbf{b}))^{-1}\mathbf{B}(\mathbf{B}+\mathbf{diag}(\mathbf{b}))^{-1}\mathbf{v} \tag{A.4}$$

$$2V^{\mathrm{B}} = \mathbf{v}'\left(\mathbf{B}+\mathbf{diag}(\beta)^{-1}\right)^{-1}\mathbf{B}\left(\mathbf{B}+\mathbf{diag}(\beta)^{-1}\right)^{-1}\mathbf{v} \tag{A.5}$$

Next, combining (A.1) with (8) and (9) leads to

$$\begin{aligned}\Pi^{\mathrm{C}} &= \mathbf{v}'\left[(\mathbf{B}+\mathbf{diag}(\mathbf{b}))^{-1} - (\mathbf{B}+\mathbf{diag}(\mathbf{b}))^{-1}\mathbf{B}(\mathbf{B}+\mathbf{diag}(\mathbf{b}))^{-1}\right]\mathbf{v}\\ &= \mathbf{v}'\left[(\mathbf{B}+\mathbf{diag}(\mathbf{b}))^{-1}\mathbf{diag}(\mathbf{b})(\mathbf{B}+\mathbf{diag}(\mathbf{b}))^{-1}\right]\mathbf{v}\end{aligned} \tag{A.6}$$

$$\begin{aligned}\Pi^{\mathrm{B}} &= \mathbf{v}'\left[\left(\mathbf{B}+\mathbf{diag}(\beta)^{-1}\right)^{-1} - \left(\mathbf{B}+\mathbf{diag}(\beta)^{-1}\right)^{-1}\mathbf{B}\left(\mathbf{B}+\mathbf{diag}(\beta)^{-1}\right)^{-1}\right]\mathbf{v}\\ &= \mathbf{v}'\left[\left(\mathbf{B}+\mathbf{diag}(\beta)^{-1}\right)^{-1}\mathbf{diag}(\beta)^{-1}\left(\mathbf{B}+\mathbf{diag}(\beta)^{-1}\right)^{-1}\right]\mathbf{v}\end{aligned} \tag{A.7}$$

Finally, combining (A.1) with (8) and (9) leads to

$$\begin{aligned}W^{\mathrm{C}} &= \mathbf{v}'\left[(\mathbf{B}+\mathbf{diag}(\mathbf{b}))^{-1} - \frac{1}{2}(\mathbf{B}+\mathbf{diag}(\mathbf{b}))^{-1}\mathbf{B}(\mathbf{B}+\mathbf{diag}(\mathbf{b}))^{-1}\right]\mathbf{v}\\ 2W^{\mathrm{C}} &= \mathbf{v}'\left[(\mathbf{B}+\mathbf{diag}(\mathbf{b}))^{-1}(\mathbf{B}+2\,\mathbf{diag}(\mathbf{b}))(\mathbf{B}+\mathbf{diag}(\mathbf{b}))^{-1}\right]\mathbf{v}\end{aligned} \tag{A.8}$$

$$\begin{aligned}W^{\mathrm{B}} &= \mathbf{v}'\left[\left(\mathbf{B}+\mathbf{diag}(\beta)^{-1}\right)^{-1} - \frac{1}{2}\left(\mathbf{B}+\mathbf{diag}(\beta)^{-1}\right)^{-1}\mathbf{B}\left(\mathbf{B}+\mathbf{diag}(\beta)^{-1}\right)^{-1}\right]\mathbf{v}\\ 2W^{\mathrm{B}} &= \mathbf{v}'\left[\left(\mathbf{B}+\mathbf{diag}(\beta)^{-1}\right)^{-1}\left(\mathbf{B}+2\,\mathbf{diag}(\beta)^{-1}\right)\left(\mathbf{B}+\mathbf{diag}(\beta)^{-1}\right)^{-1}\right]\mathbf{v}\end{aligned} \tag{A.9}$$

B F.o.c. with multi-product firms

Cournot competition Recall that I_k is the set of product under the control of firm k. We introduce the following r matrices. For all k, $1 \leqslant k \leqslant r$, let $\mathbf{B_k} = (b_{ij})_{i,j \in I_k}$. That is, $\mathbf{B_k}$ is a square matrix of size $\#I_k$ the number of varieties under the control of firm k. If each firm owns only one good, $\mathbf{B_k}$ is simply $b_{kk} = b_k$. Next, let $\mathbf{diag}(\mathbf{B_1}, \ldots, \mathbf{B_r})$ denote the block diagonal matrix, whose blocks are $\mathbf{B_k}$, $k = 1, \cdots, r$. The matrix $\mathbf{diag}(\mathbf{B_1}, \ldots, \mathbf{B_r})$ is positive definite. Computations similar to the ones leading to (8) now gives:

$$\mathbf{p} - \mathbf{c} = \mathbf{diag}(\mathbf{B_1}, \ldots, \mathbf{B_r})\mathbf{q}$$

and therefore, using $\mathbf{p} - \mathbf{c} = \mathbf{a} - \mathbf{c} - \mathbf{Bq}$,

$$(\mathbf{B} + \mathbf{diag}(\mathbf{B_1}, \ldots, \mathbf{B_r}))\, \mathbf{q} = \mathbf{a} - \mathbf{c} = \mathbf{v} \tag{B.1}$$

Bertrand competition Recall that I_k is the set of product under the control of firm k. We introduce the following r matrices. For all k, $1 \leqslant k \leqslant r$, let $\mathcal{B}_\mathbf{k} = (\beta_{ij})_{i,j \in I_k}$ where the β_{ij} are the elements of matrix $\mathbf{B^{-1}}$. That is, $\mathcal{B}_\mathbf{k}$ is a square matrix of size $\#I_k$ the number of varieties under the control of firm k. If each firm owns only one good, $\mathcal{B}_\mathbf{k}$ is simply $\beta_{kk} = \beta_k$. Next, let $\mathbf{diag}(\mathcal{B}_\mathbf{1}, \ldots, \mathcal{B}_\mathbf{r})$ denote the block diagonal matrix, whose blocks are $\mathcal{B}_\mathbf{k}$, $k = 1, \cdots, r$. The matrix $\mathbf{diag}(\mathcal{B}_\mathbf{1}, \ldots, \mathcal{B}_\mathbf{r})$ is positive definite. Computations similar to the ones leading to (8) now gives:

$$\mathbf{q} = \mathbf{diag}(\mathcal{B}_\mathbf{1}, \ldots, \mathcal{B}_\mathbf{r})(\mathbf{p} - \mathbf{c})$$

and therefore, using $\mathbf{p} - \mathbf{c} = \mathbf{a} - \mathbf{c} - \mathbf{Bq}$,

$$\left(\mathbf{B} + \mathbf{diag}(\mathcal{B}_\mathbf{1}, \ldots, \mathcal{B}_\mathbf{r})^{-1}\right) \mathbf{q} = \mathbf{a} - \mathbf{c} = \mathbf{v} \tag{B.2}$$

where, as in the text, prices have been eliminated in order to show the similarity of the Bertrand and Cournot characterization. The only difference from the Cournot f.o.c. is that $\mathbf{diag}(\mathbf{B_1}, \ldots, \mathbf{B_r})$ as been replaced by $\mathbf{diag}(\mathcal{B}_\mathbf{1}, \ldots, \mathcal{B}_\mathbf{r})^{-1}$.

C Common symmetric model

In section 3, we have shown how to find the Cournot-Nash or Bertrand-Nash equilibrium of QQUM. So here we only have to compute the relevant matrices. First, notice that

$$\mathbf{B} = \begin{pmatrix} b & \sigma & \cdots & \cdots & \sigma \\ \sigma & \ddots & \ddots & & \vdots \\ \vdots & \ddots & b & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \sigma \\ \sigma & \cdots & \cdots & \sigma & b \end{pmatrix} \text{ using } \mathbf{I} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \text{ and } \mathbf{J} = \begin{pmatrix} 1 & 1 & \cdots & \cdots & 1 \\ 1 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 1 \\ 1 & \cdots & \cdots & 1 & 1 \end{pmatrix}$$

one can write

$$\mathbf{B} = (b - \sigma)\mathbf{I} + \sigma\mathbf{J}$$

It is important to emphasize, here, a convenient property of the matrix $\mathbf{J}$, namely that it is a sum or, if one divides by n, a mean operator. This property plays a key role in the analysis. Indeed, let $\mathbf{e}' = (1, \cdots, 1)$, then for any vector $\mathbf{x}$,

$$\frac{1}{n}\mathbf{Jx} = \overline{\mathbf{x}}\,\mathbf{e} \;\text{ and }\; \frac{1}{n}\mathbf{e}'\,\mathbf{Jx} = \overline{\mathbf{x}} \;\text{ and }\; \mathbf{x}'\mathbf{Jy} = n^2\overline{\mathbf{x}}\,\overline{\mathbf{y}}$$

Direct and inverse demands The expression of the inverse demand functions given at the top of Table 2:

$$\mathbf{p} = \mathbf{a} - \mathbf{B}\mathbf{q} = \mathbf{a} - ((b-\sigma)\mathbf{I} + \sigma\mathbf{J})\,\mathbf{q} = \mathbf{a} - (b-\sigma)\mathbf{q} + n\sigma\overline{\mathbf{x}}\,\mathbf{e}$$

It is readily confirmed (one can simply verify it by computing $\mathbf{B}\mathbf{B}^{-1}$, using $\mathbf{J}^2 = n\mathbf{J}$) that

$$\mathbf{B}^{-1} = \frac{1}{b-\sigma}\left(\mathbf{I} - \frac{\sigma}{b+(n-1)\sigma}\mathbf{J}\right)$$

this gives the demand functions given at the top of Table 2:

$$\mathbf{q} = \mathbf{B}^{-1}(\mathbf{a}-\mathbf{p}) = \frac{1}{b-\sigma}\left(\mathbf{I} - \frac{\sigma}{b+(n-1)\sigma}\mathbf{J}\right)(\mathbf{a}-\mathbf{p}) = \frac{1}{b-\sigma}\left((\mathbf{a}-\mathbf{p}) - \frac{n\sigma}{b+(n-1)\sigma}(\overline{\mathbf{a}}-\overline{\mathbf{p}})\right)$$

We can also compute $\mathbf{x}'\mathbf{B}^{-1}\mathbf{y}$ to prove Fact 11. Using $\mathbf{x}'\mathbf{y} = n\mathrm{Cov}\,(\mathbf{x},\mathbf{y}) + n\overline{\mathbf{x}}\,\overline{\mathbf{y}}$.

$$\begin{aligned}
\mathbf{x}'\mathbf{B}^{-1}\mathbf{y} &= \frac{1}{b-\sigma}\left(\mathbf{x}'\mathbf{y} - \frac{\sigma}{b+(n-1)\sigma}\mathbf{x}'\mathbf{J}\mathbf{y}\right)\\
&= \frac{1}{b-\sigma}\left(n\mathrm{Cov}\,(\mathbf{x},\mathbf{y}) + n\overline{\mathbf{x}}\,\overline{\mathbf{y}} - \frac{n^2\sigma}{b+(n-1)\sigma}\overline{\mathbf{x}}\,\overline{\mathbf{y}}\right)\\
&= \frac{1}{b-\sigma}\left(n\mathrm{Cov}\,(\mathbf{x},\mathbf{y}) + \left(1 - \frac{n\sigma}{b+(n-1)\sigma}\right)n\overline{\mathbf{x}}\,\overline{\mathbf{y}}\right)\\
&= \frac{n}{b-\sigma}\mathrm{Cov}\,(\mathbf{x},\mathbf{y}) + \frac{n}{b+(n-1)\sigma}\overline{\mathbf{x}}\,\overline{\mathbf{y}}
\end{aligned}$$

Cournot equilibrium prices and quantities As shown by (8), one only needs to compute the inverse of $\mathbf{B} + \mathbf{diag}(\mathbf{b})$ in order to compute the equilibrium quantities in the Cournot game. This is straightforward,

$$(\mathbf{B} + \mathbf{diag}(\mathbf{b}))^{-1} = ((2b-\sigma)\mathbf{I} + \sigma\mathbf{J})^{-1} = \frac{1}{2b-\sigma}\left(\mathbf{I} - \frac{\sigma}{2b+(n-1)\sigma}\mathbf{J}\right)$$

hence

$$\mathbf{q}^C = \frac{1}{2b-\sigma}\left(\mathbf{I} - \frac{\sigma}{2b+(n-1)\sigma}\mathbf{J}\right)\mathbf{v} = \frac{1}{2b-\sigma}\left(\mathbf{v} - \frac{n\sigma}{2b+(n-1)\sigma}\overline{\mathbf{v}}\mathbf{e}\right)$$

and

$$\mathbf{p}^C - \mathbf{c} = \mathbf{diag}(\mathbf{b})\mathbf{q}^C = \frac{b}{2b-\sigma}\left(\mathbf{v} - \frac{n\sigma}{2b+(n-1)\sigma}\overline{\mathbf{v}}\mathbf{e}\right)$$

Bertrand equilibrium prices and quantities As shown by (9), one only needs to compute the inverse of $\mathbf{B} + \mathbf{diag}(\boldsymbol{\beta})^{-1}$ in order to compute the equilibrium quantities in the Bertrand game. This is again straightforward (although a little bit cumbersome),

$$\begin{aligned}
\left(\mathbf{B} + \mathbf{diag}(\boldsymbol{\beta})^{-1}\right)^{-1} &= \left(\frac{(b-\sigma)(2b+(2n-3)\sigma)}{b+(n-2)\sigma}\mathbf{I} + \sigma\mathbf{J}\right)^{-1}\\
&= ((2b-\sigma-\Phi)\,\mathbf{I} + \sigma\mathbf{J})^{-1}\\
&= \frac{1}{2b-\sigma-\Phi}\left(\mathbf{I} - \frac{\sigma}{2b-(n-1)\sigma-\Phi}\mathbf{J}\right)
\end{aligned}$$

where

$$\Phi = \frac{(n-1)\sigma^2}{b+(n-2)\sigma}$$

hence

$$\mathbf{q}^B = \frac{1}{2b-\sigma-\Phi}\left(\mathbf{I} - \frac{\sigma}{2b+(n-1)\sigma-\Phi}\mathbf{J}\right)\mathbf{v} = \frac{1}{2b-\sigma-\Phi}\left(\mathbf{v} - \frac{n\sigma}{2b+(n-1)\sigma-\Phi}\overline{\mathbf{v}}\mathbf{e}\right)$$

and

$$\mathbf{p}^B - \mathbf{c} = \mathbf{diag}(\mathbf{B^{-1}})^{-1}\mathbf{q}^B = \frac{b-\Phi}{2b-\sigma-\Phi}\left(\mathbf{v} - \frac{n\sigma}{2b+(n-1)\sigma-\Phi}\overline{\mathbf{v}}\mathbf{e}\right)$$

Cournot and Bertrand equilibrium consumers' surplus One can either use (A.1) and replace q by $\mathbf{q}^C$ and $\mathbf{q}^B$ respectively, using the expressions

$$\begin{aligned}
\mathbf{q}^C &= \frac{1}{2b-\sigma}\left[(\mathbf{v}-\overline{\mathbf{v}}\mathbf{e}) - \frac{2b-\sigma}{2b+(n-1)\sigma}\overline{\mathbf{v}}\mathbf{e}\right] \\
\mathbf{q}^B &= \frac{1}{2b-\sigma-\Phi}\left[(\mathbf{v}-\overline{\mathbf{v}}\mathbf{e}) - \frac{2b-\sigma-\Phi}{2b+(n-1)\sigma-\Phi}\overline{\mathbf{v}}\mathbf{e}\right]
\end{aligned}$$

which are the most convenient to see the variance terms. Or, more directly (but with slightly more matrix computations), one can use (A.4). For example, in the case of Cournot (the computations for Bertrand are almost exactly the same), we start from

$$2V^{\mathrm{C}} = \mathbf{v}'\left(\mathbf{B}+\mathbf{diag}(\mathbf{b})\right)^{-1}\mathbf{B}\left(\mathbf{B}+\mathbf{diag}(\mathbf{b})\right)^{-1}\mathbf{v}$$

so we have to compute the matrix which is between $\mathbf{v}'$ and $\mathbf{v}$, we do it in two steps. First (using again $\mathbf{J}^2 = n\mathbf{J}$),

$$\begin{aligned}
\left(\mathbf{B}+\mathbf{diag}(\mathbf{b})\right)^{-1}\mathbf{B} &= \frac{1}{2b-\sigma}\left[\mathbf{I} - \frac{\sigma}{2b+(n-1)\sigma}\mathbf{J}\right]\left[(b-\sigma)\mathbf{I}+\sigma\mathbf{J}\right] \\
&= \frac{b-\sigma}{2b-\sigma}\left[\mathbf{I} - \frac{\sigma}{2b+(n-1)\sigma}\mathbf{J}\right]\left[\mathbf{I} + \frac{\sigma}{b-\sigma}\mathbf{J}\right] \\
&= \frac{b-\sigma}{2b-\sigma}\left[\mathbf{I} + \left(\frac{\sigma}{b-\sigma} - \frac{\sigma}{2b+(n-1)\sigma}\right)\mathbf{J} - \frac{\sigma^2}{(2b+(n-1)\sigma)(b-\sigma)}\mathbf{J}^2\right] \\
&= \frac{b-\sigma}{2b-\sigma}\left[\mathbf{I} + \frac{\sigma b}{(2b+(n-1)\sigma)(b-\sigma)}\mathbf{J}\right]
\end{aligned}$$

Now, we compute $\mathbf{M} = \left(\mathbf{B}+\mathbf{diag}(\mathbf{b})\right)^{-1}\mathbf{B}\left(\mathbf{B}+\mathbf{diag}(\mathbf{b})\right)^{-1}$

$$\begin{aligned}
\mathbf{M} &= \frac{(b-\sigma)}{(2b-\sigma)^2}\left[\mathbf{I} + \frac{\sigma b}{(2b+(n-1)\sigma)(b-\sigma)}\mathbf{J}\right]\left[\mathbf{I} - \frac{\sigma}{2b+(n-1)\sigma}\mathbf{J}\right] \\
&= \frac{(b-\sigma)}{(2b-\sigma)^2}\left[\mathbf{I} + \left(\frac{b(2b+(n-1)\sigma)-(2b+(n-1)\sigma)(b-\sigma)-n\sigma b)}{(2b+(n-1)\sigma)(b-\sigma)}\right)\frac{\sigma}{2b+(n-1)\sigma}\mathbf{J}\right] \\
&= \frac{(b-\sigma)}{(2b-\sigma)^2}\left[\mathbf{I} - \left(\frac{\sigma^2\left((n-2)b-(n-1)\sigma\right)}{(2b+(n-1)\sigma)^2(b-\sigma)}\right)\mathbf{J}\right]
\end{aligned}$$

Finally, using

$$\mathrm{Var}\left(\mathbf{v}\right) = \frac{1}{n}\mathbf{v}'\mathbf{v} - \overline{\mathbf{v}}^2 \quad \text{and } \mathbf{v}'\,\mathbf{J}\mathbf{v} = n^2\overline{\mathbf{v}}^2$$

$$
\begin{aligned}
2V^{\mathrm{C}} &= \mathbf{v}'\mathbf{M}\mathbf{v} \\
&= \frac{(b-\sigma)}{(2b-\sigma)^2}\mathbf{v}'\left[\mathbf{I}-\left(\frac{\sigma^2\left((n-2)b-(n-1)\sigma\right)}{(2b+(n-1)\sigma)^2(b-\sigma)}\right)\mathbf{J}\right]\mathbf{v} \\
&= \frac{n(b-\sigma)}{(2b-\sigma)^2}\left[\mathrm{Var}\left(\mathbf{v}\right)+\left(1-\frac{n\sigma^2\left((n-2)b-(n-1)\sigma\right)}{(2b+(n-1)\sigma)^2(b-\sigma)}\right)\overline{\mathbf{v}}^2\right] \\
&= \frac{n(b-\sigma)}{(2b-\sigma)^2}\mathrm{Var}\left(\mathbf{v}\right)+\frac{n}{(2b-\sigma)^2}\left((b-\sigma)-\frac{n\sigma^2\left((n-2)b-(n-1)\sigma\right)}{(2b+(n-1)\sigma)^2}\right)\overline{\mathbf{v}}^2 \\
&= \frac{n(b-\sigma)}{(2b-\sigma)^2}\mathrm{Var}\left(\mathbf{v}\right)+\frac{n(b+(n-1)\sigma)}{(2b+(n-1)\sigma)^2}\overline{\mathbf{v}}^2
\end{aligned}
$$

which is the formula given in Table 2. One can check that if $n=1$ it is the consumers' surplus of the linear demand monopoly, i.e. $2V=\frac{(a-c)^2}{4b}$. If $\sigma=0$ the formula becomes $2V=\frac{n}{4b}\left(\mathrm{Var}\left(\mathbf{v}\right)+\overline{\mathbf{v}}^2\right)$. More generally, it increases with the variance of $\mathbf{v}$ because consumers enjoy diversity and it increases with the average marginal surplus $\overline{\mathbf{v}}$.

Cournot and Bertrand firms' profits Individual profits (again we show the computation for Cournot as the Bertrand ones are similar) are immediately given deduced from the expressions of q_i^C and $p_i^C-c_i$. One can either sum these individual profits or directly use the equilibrium expression for total profit given by (A.6). That is,

$$
\Pi^C=\mathbf{v}'\left[\left(\mathbf{B}+\mathbf{diag}(\mathbf{b})\right)^{-1}-\left(\mathbf{B}+\mathbf{diag}(\mathbf{b})\right)^{-1}\mathbf{B}\left(\mathbf{B}+\mathbf{diag}(\mathbf{b})\right)^{-1}\right]\mathbf{v}
$$

we need to compute the matrix between $\mathbf{v}'$ and $\mathbf{v}$ which is

$$
\frac{1}{2b-\sigma}\left[\mathbf{I}-\frac{\sigma}{2b+(n-1)\sigma}\mathbf{J}\right]-\frac{(b-\sigma)}{(2b-\sigma)^2}\left[\mathbf{I}-\left(\frac{\sigma^2\left((n-2)b-(n-1)\sigma\right)}{(2b+(n-1)\sigma)^2(b-\sigma)}\right)\mathbf{J}\right]
$$

and simplifies into

$$
\frac{b}{(2b-\sigma)^2}\left[\mathbf{I}-\frac{\sigma(4b+(n-2)\sigma)}{(2b+(n-1)\sigma)^2}\mathbf{J}\right]
$$

and finally

$$
\Pi^C=\frac{nb}{(2b-\sigma)^2}\left[\mathrm{Var}\left(\mathbf{v}\right)+\frac{(2b-\sigma)^2}{(2b+(n-1)\sigma)^2}\overline{\mathbf{v}}^2\right]
$$

D Proofs of section 5

Thus, the matrix $\mathbf{B}$ of this asymmetric model is:

$$
\mathbf{B}=\begin{pmatrix}
b_1 & \sigma & \cdots & \cdots & \sigma \\
\sigma & \ddots & \ddots & & \vdots \\
\vdots & \ddots & b_i & \ddots & \vdots \\
\vdots & & \ddots & \ddots & \sigma \\
\sigma & \cdots & \cdots & \sigma & b_n
\end{pmatrix}
\equiv
\begin{pmatrix}
1 & \sigma_1\sigma_2 & \cdots & \cdots & \sigma_1\sigma_n \\
\sigma_2\sigma_1 & \ddots & \ddots & & \vdots \\
\vdots & \ddots & 1 & \ddots & \vdots \\
\vdots & & \ddots & \ddots & \sigma_{n-1}\sigma_n \\
\sigma_n\sigma_1 & \cdots & \cdots & \sigma_n\sigma_{n-1} & 1
\end{pmatrix}
$$

When $n=2$, all definite positive matrix are covered by this description. For $n\geqslant 3$, the fact that all the cross-derivative terms b_{ij}, $i\neq j$, are all equal to σ is obviously restrictive.

The change of variables to move from the b_i and σ model to the 1, $\sigma_i\sigma_j$ model is

$$\begin{array}{ccc} q_i & \rightarrow & \dfrac{\sigma_i q_i}{\sqrt{\sigma}} \\ v_i \text{ (resp. } a_i,\, c_i,\, p_i) & \rightarrow & \dfrac{\sqrt{\sigma} v_i}{\sigma_i} \text{ (resp. } \cdots) \end{array}$$

the other terms should adjust automatically. For example direct demand in the b_i and σ model writes

$$q_i = w_i \left[(a_i - p_i) - \frac{\sigma \sum w_j (a_j - p_j)}{1 + \sigma \sum w_j} \right]$$

with $w_i = 1/(b_i - \sigma)$. Therefore the demand in the 1, $\sigma_i\sigma_j$ model writes (after some elementary rearranging)

$$\sigma_i q_i = w_i \left[\frac{a_i - p_i}{\sigma_i} - \frac{\sum w_j \frac{a_j - p_j}{\sigma_j}}{1 + \sum w_j} \right]$$

with $w_i = \sigma_i^2/(1 - \sigma_i^2)$.

Determinant of the B matrix

$$\det(\mathbf{B}) = \sigma \sum_i \prod_{j \neq i} (b_j - \sigma) + \prod (b_i - \sigma)$$

See Bernstein (2009) page 141-142 Fact 2.13.12.

This determinant is strictly positive as we have assumed that $\mathbf{B}$ is positive definite. So the values of the elements of $\mathbf{b}$ are restricted. As $\mathbf{B}$ is symmetric if it were diagonal dominant (i.e. assuming $b_1 > (n-1)\sigma$), it would automatically be positive definite. We cannot compute the eigenvalues of $\mathbf{B}$ but for our computations, it is enough to assume that $\mathbf{B}$ is positive definite and that $b_1 > \sigma$.[32] Under this assumption (recall that $b_1 \leqslant b_2 \leqslant \cdots \leqslant b_n$)

$$\det(\mathbf{B}) = \prod (b_i - \sigma) \left(1 + \sigma \sum \frac{1}{b_i - \sigma} \right) \tag{D.1}$$

Decomposition of the B matrix It is useful to introduce the following matrix:

$$\mathbf{W} = \mathbf{diag}(\mathbf{b} - \boldsymbol{\sigma})^{-1} = \begin{pmatrix} w_1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & w_i & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & w_n \end{pmatrix} = \begin{pmatrix} \frac{1}{b_1-\sigma} & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \frac{1}{b_i-\sigma} & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \frac{1}{b_n-\sigma} \end{pmatrix}$$

We can then rewrite matrix $\mathbf{B}$ as,

$$\mathbf{B} = \mathbf{diag}(\mathbf{b} - \boldsymbol{\sigma}) + \sigma \mathbf{J} = \mathbf{W}^{-1} + \sigma \mathbf{J} \tag{D.2}$$

and (D.1) can be rewritten

$$\det(\mathbf{B}) = \det\left(\mathbf{W}^{-1}\right) \left(1 + \sigma \sum w_i\right)$$

[32] The $\mathbf{B}$ matrix could still be positive definite in the case $b_1 = \sigma$ but the expression of the $\mathbf{B}^{-1}$ would be different.

Inverse of B When $\mathbf{B} = \mathbf{W}^{-1} + \sigma\mathbf{J}$, then

$$\mathbf{B}^{-1} = \mathbf{W} - \frac{\sigma}{1+\sigma\sum w_i}\mathbf{W}\mathbf{J}\,\mathbf{W} \tag{D.3}$$

That is, in extended form,

$$\mathbf{B}^{-1} = \begin{pmatrix} w_1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & w_i & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & w_n \end{pmatrix} - \frac{\sigma}{1+\sigma\sum w_i}\begin{pmatrix} w_1^2 & w_1 w_2 & \cdots & \cdots & w_1 w_n \\ w_1 w_2 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & w_i^2 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & w_{n-1}w_n \\ w_n w_1 & \cdots & \cdots & w_n w_{n-1} & w_n^2 \end{pmatrix}$$

notice that $1+\sigma\sum w_i = \det(\mathbf{W})\det(\mathbf{B})$. One can directly verify that $\mathbf{B}\mathbf{B}^{-1} = \mathbf{I}$. Indeed,

$$\begin{aligned} &\left(\mathbf{W}^{-1} + \sigma\mathbf{J}\right)\left(\mathbf{W} - \sigma(1+\sigma\delta)^{-1}\mathbf{W}\mathbf{J}\,\mathbf{W}\right) \\ =&\,\mathbf{I} - \sigma(1+\sigma\delta)^{-1}\mathbf{J}\,\mathbf{W} + \sigma\mathbf{J}\,\mathbf{W} - \sigma^2(1+\sigma\delta)^{-1}\mathbf{J}\,\mathbf{W}\mathbf{J}\,\mathbf{W} \end{aligned}$$

now it is easy to check that $\mathbf{J}\,\mathbf{W}\mathbf{J}\,\mathbf{W} = \delta\,\mathbf{J}\,\mathbf{W}$ therefore

$$\mathbf{B}\mathbf{B}^{-1} = \mathbf{I} - \sigma\left[\frac{1-1-\sigma\delta+\sigma\delta}{1+\sigma\delta}\right]\mathbf{J}\,\mathbf{W} = \mathbf{I}$$

D.1 Quantities

Using Fact 10, the First-best, Monopoly, Cournot, and Bertrand equilibrium quantities are respectively given by

$$\mathbf{q}^* = \mathbf{q}^*(\mathbf{0}),\ \mathbf{q}^{\mathbf{m}} = \mathbf{q}^*(\mathbf{B}),\ \mathbf{q}^C = \mathbf{q}^*(\mathbf{diag}(\mathbf{b})),\ \text{ and }\ \mathbf{q}^B = \mathbf{q}^*\left(\mathbf{diag}(\beta)^{-1}\right).$$

Proof of Fact 12 In the first-best case, $\mathbf{X} = 0$ and $\mathbf{q}^* = \mathbf{B}^{-1}\mathbf{v}$ that is the same expression as for demand (14) except that $\mathbf{p}$ has been replaced by $\mathbf{c}$. Hence the result. For the monopoly, we have $\mathbf{q}^{\mathbf{m}} = \mathbf{2B}^{-1}\mathbf{v}$, hence $\mathbf{q}^{\mathbf{m}} = \mathbf{q}^*/2$.

In the cases of Cournot, and Bertrand, $\mathbf{X}$ is a diagonal matrix, therefore $\mathbf{B}+\mathbf{X} = \left(\mathbf{W}^{-1} + \mathbf{X}\right) + \sigma\mathbf{J}$ which can be written $\mathbf{B}+\mathbf{X} = \widetilde{\mathbf{W}}^{-1} + \sigma\mathbf{J}$ with

$$\widetilde{\mathbf{W}} = \left(\mathbf{W}^{-1} + \mathbf{X}\right)^{-1}$$

Using that $\mathbf{W}^{-1} = \mathbf{diag}\left((b_i - \sigma)\right)$, and that in the Cournot case $\mathbf{X} = \mathbf{diag}(\mathbf{b})$

$$\widetilde{\mathbf{W}} = \left(\mathbf{diag}\left((b_i-\sigma)\right) + \mathbf{diag}(\mathbf{b})\right)^{-1} = \mathbf{diag}\left(\left(\frac{1}{2b_i - \sigma}\right)\right)$$

In the Bertrand case $\mathbf{X} = \mathbf{diag}(\beta)^{-1} = \mathbf{diag}\left(\left((b_i-\sigma)\frac{1+\sigma\sum w_j}{1+\sigma\sum_{j\neq i} w_j}\right)\right)$, so

$$\begin{aligned} (\beta_i)^{-1} &= (b_i - \sigma)\left[\frac{1+\sigma\sum w_j}{1+\sigma\sum_{j\neq i} w_j}\right] = (b_i-\sigma)\left[1 + \frac{\sigma w_i}{1+\sigma\sum_{j\neq i} w_j}\right] \\ &= b_i - \sigma + \frac{\sigma}{1+\sigma\sum_{j\neq i} w_j} = b_i - \frac{\sigma^2\sum_{j\neq i} w_j}{1+\sigma_{j\neq i}\sum w_j} = b_i - \Phi_i \end{aligned}$$

where $\Phi_i = \dfrac{\sigma^2\sum_{j\neq i} w_j}{1+\sigma_{j\neq i}\sum w_j}$ and therefore

$$(\widetilde{w}_i)^{-1} = (b_i - \sigma) + 1/\beta_i = 2b_i - \Phi_i - \sigma \text{ or } \widetilde{w}_i = \frac{1}{2b_i - \Phi_i - \sigma}$$

D.2 Equilibrium Surplus

Proof of Fact 14 Consumers' surplus is given by $2V = (\mathbf{q})' \mathbf{B} (\mathbf{q})$ where $\mathbf{q}$ are the equilibrium quantities given in Fact 12. Therefore using $\mathbf{B} = \mathbf{W}^{-1} + \sigma \mathbf{J}$, one can write $\mathbf{q} = \widetilde{\mathbf{W}} \mathbf{z}$ with $\mathbf{z} = \mathbf{v} - \left(\frac{\sigma \sum \widetilde{w}_j}{1+\sigma \sum \widetilde{w}_j} \right) \tilde{\mathbf{v}} \, \mathbf{e}$

$$2V = (\mathbf{q})' \left(\mathbf{W}^{-1} + \sigma \mathbf{J} \right) (\mathbf{q}) = \mathbf{z}' \left(\widetilde{\mathbf{W}} \mathbf{W}^{-1} \widetilde{\mathbf{W}} \right) \mathbf{z} + \sigma \, \mathbf{z}' \widetilde{\mathbf{W}} \mathbf{J} \widetilde{\mathbf{W}} \mathbf{z}$$

Now, by definition of the weighted variance the first term is

$$\mathbf{z}' \left(\widetilde{\mathbf{W}} \mathbf{W}^{-1} \widetilde{\mathbf{W}} \right) \mathbf{z} = \hat{n} \left[\widehat{Var} \left(\mathbf{z} \right) + \left(\hat{\mathbf{z}} \right)^2 \right]$$

then using the value of $\mathbf{z}$

$$\mathbf{z}' \left(\widetilde{\mathbf{W}} \mathbf{W}^{-1} \widetilde{\mathbf{W}} \right) \mathbf{z} = \hat{n} \left[\widehat{Var} \left(\mathbf{v} \right) + \left(\hat{\mathbf{v}} - \frac{\tilde{n} \sigma}{1 + \tilde{n} \sigma} \tilde{\mathbf{v}} \right)^2 \right]$$

and the second term is

$$\begin{aligned} \mathbf{z}' \widetilde{\mathbf{W}} \mathbf{J} \widetilde{\mathbf{W}} \mathbf{z} &= \left(\sum \widetilde{w}_i \right)^2 \left(\tilde{\mathbf{z}} \right)^2 = \tilde{n}^2 \left(\tilde{\mathbf{v}} - \frac{\tilde{n} \sigma}{1 + \tilde{n} \sigma} \tilde{\mathbf{v}} \right)^2 \\ &= \left(\frac{\tilde{n} \tilde{\mathbf{v}}}{1 + \tilde{n} \sigma} \right)^2 \end{aligned}$$

thus finally (18).

D.3 Total profit

Proof of Fact 15 The result for the first-best and the monopoly are straightforward (they are true in general, not only for the specific $\mathbf{B}$ matrix considered in this paper). In any case, total profits are

$$\Pi = \mathbf{q}' (\mathbf{p} - \mathbf{c}) \quad \text{where } \mathbf{q} \text{ and } \mathbf{p} \text{ are the equilibrium quantities and prices}$$

For Cournot and Bertrand, the relationship between $\mathbf{q}$ and $\mathbf{p} - \mathbf{c}$ is given by the f.o.c. which have the form $\mathbf{p} - \mathbf{c} = \mathbf{X} \mathbf{q}$ where $\mathbf{X} = \mathbf{diag}(\mathbf{b})$ for Cournot and $\mathbf{X} = \mathbf{diag}(\boldsymbol{\beta})^{-1}$ for Bertrand. Hence, using Fact 12 one can write $\mathbf{q} = \widetilde{\mathbf{W}} \mathbf{z}$ with $\mathbf{z} = \mathbf{v} - \left(\frac{\sigma \sum \widetilde{w}_j}{1+\sigma \sum \widetilde{w}_j} \right) \tilde{\mathbf{v}} \, \mathbf{e}$, we have

$$\Pi = \mathbf{z}' \widetilde{\mathbf{W}} \mathbf{X} \widetilde{\mathbf{W}} \mathbf{z} = \mathbf{z}' \breve{\mathbf{W}} \mathbf{z}$$

and as $\breve{\mathbf{W}}$ is a positive diagonal matrix, we can use the variance formula:

$$\Pi = \breve{n} \left[\breve{Var} \left(\mathbf{z} \right) + \left(\breve{\mathbf{z}} \right)^2 \right]$$

hence (19).