A braid-like presentation of the integral Steinberg group of type C_2

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A BRAID-LIKE PRESENTATION OF THE INTEGRAL STEINBERG GROUP OF TYPE C_2

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Abstract. We show that the Steinberg group $\mathrm{St}(C_2, \mathbb{Z})$ associated with the Lie type C_2 and with integer coefficients can be realized as a quotient of the braid group B_6 by one relation. As an application we give a new braid-like presentation of the symplectic modular group $\mathrm{Sp}_4(\mathbb{Z})$.

In memoriam amici
Patrick Dehornoy
(1952–2019)

1. Introduction

Let B_6 be the braid group on six strands. It has a standard presentation with five generators σ_1, σ_2, σ_3, σ_4, σ_5, and the following ten relations ($1 \leq i, j \leq 5$):

$$\sigma_i \sigma_j = \sigma_j \sigma_i \qquad \text{if } |i-j| > 1 \tag{1.1}$$

and

$$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \quad \text{if } |i-j| = 1. \tag{1.2}$$

The purpose of this note is to show that if we add the single relation

$$(\sigma_1\sigma_2\sigma_1)^2(\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_1\sigma_2\sigma_1)^{-2}(\sigma_1\sigma_3^{-1}\sigma_5) = 1, \tag{1.3}$$

to Relations (1.1) and (1.2), we obtain a presentation of the Steinberg group $\mathrm{St}(C_2, \mathbb{Z})$ associated with the Lie type C_2 over the ring of integers (see Theorem 4.1).

If we further add the relation

$$(\sigma_1\sigma_2\sigma_1)^4 = 1, \tag{1.4}$$

we obtain a presentation of the symplectic modular group $\mathrm{Sp}_4(\mathbb{Z})$ (see Corollary 4.2). It is known that the Steinberg group $\mathrm{St}(C_2, \mathbb{Z})$ is a group extension of $\mathrm{Sp}_4(\mathbb{Z})$ with infinite cyclic kernel.

In order to prove these results we construct a surjective group homomorphism $f : B_6 \to \mathrm{St}(C_2, \mathbb{Z})$, show that f vanishes on the normal subgroup N of B_6 generated by the element represented by the braid word in (1.3), and construct a group homomorphism $\varphi : \mathrm{St}(C_2, \mathbb{Z}) \to B_6/N$ such that $\varphi \circ f = \mathrm{id}$. The homomorphism f is a lifting of a map $B_6 \to \mathrm{Sp}_4(\mathbb{Z})$, which is a special case of the homomorphism $\overline{f} : B_{2g+2} \to \mathrm{Sp}_{2g}(\mathbb{Z})$ constructed in [11, Sect. 4] for any $g \geq 1$.

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The note is organized as follows. In Section 2 we give a presentation of the Steinberg group $\mathrm{St}(C_2, \mathbb{Z})$ and we prove a number of relations between special elements of $\mathrm{St}(C_2, \mathbb{Z})$. In Section 3 we construct the homomorphism f from the braid group to the Steinberg group. In Section 4 we state and prove our results. In Appendix A we give a presentation of the Steinberg group $\mathrm{St}(C_2, R)$ with coefficients in an arbitrary commutative ring R.

Notation. All the groups we consider are noted multiplicatively. We denote their identity elements by 1 and we use brackets for the commutators:

$$[x, y] = xyx^{-1}y^{-1} .$$

Recall that $[y, x] = [x, y]^{-1}$.

2. The Steinberg group $\mathrm{St}(C_2, \mathbb{Z})$

The positive roots of the root system C_2 consist of four vectors α, β, $\alpha + \beta$ and $2\alpha + \beta$ of the Euclidean plane $\mathbb{R}^2$, where the roots α and $\alpha + \beta$ are of length $\sqrt{2}$, the roots β and $2\alpha+\beta$ are of length 2, and α is orthogonal to $\alpha + \beta$. Together with $-\alpha$, $-\beta$, $-(\alpha + \beta)$ and $-(2\alpha + \beta)$, they form the root system Φ of type C_2 (see e.g. [6, 7]).

2.1. **Defining relations for the Steinberg group.** We have the following presentation for the Steinberg group $\mathrm{St}(C_2, \mathbb{Z})$ of type C_2 over the ring $\mathbb{Z}$ of integers.

Proposition 2.1. *The Steinberg group* $\mathrm{St}(C_2, \mathbb{Z})$ *has a presentation with eight generators*

$$x_\alpha,\ x_\beta,\ x_{\alpha+\beta},\ x_{2\alpha+\beta},\ x_{-\alpha},\ x_{-\beta},\ x_{-(\alpha+\beta)},\ x_{-(2\alpha+\beta)}$$

and the following 24 relations:

$$[x_\alpha, x_{2\alpha+\beta}] = [x_\beta, x_{\alpha+\beta}] = [x_\beta, x_{2\alpha+\beta}] = [x_{\alpha+\beta}, x_{2\alpha+\beta}] = 1, \tag{2.1}$$

$$[x_\alpha, x_{-\beta}] = [x_\beta, x_{-\alpha}] = [x_\beta, x_{-(2\alpha+\beta)}] = [x_{-\beta}, x_{2\alpha+\beta}] = 1, \tag{2.2}$$

$$\begin{aligned}[x_{-\alpha}, x_{-(2\alpha+\beta)}] &= [x_{-\beta}, x_{-(\alpha+\beta)}] \\ &= [x_{-\beta}, x_{-(2\alpha+\beta)}] = [x_{-(\alpha+\beta)}, x_{-(2\alpha+\beta)}] = 1,\end{aligned} \tag{2.3}$$

$$[x_\alpha, x_\beta] = x_{\alpha+\beta}\, x_{2\alpha+\beta} = x_{2\alpha+\beta}\, x_{\alpha+\beta} , \tag{2.4}$$

$$[x_\alpha, x_{\alpha+\beta}] = x_{2\alpha+\beta}^2 , \tag{2.5}$$

$$[x_\alpha, x_{-(\alpha+\beta)}] = x_{-\beta}^{-2} , \tag{2.6}$$

$$[x_\alpha, x_{-(2\alpha+\beta)}] = x_{-\beta}\, x_{-(\alpha+\beta)}^{-1} = x_{-(\alpha+\beta)}^{-1}\, x_{-\beta} , \tag{2.7}$$

$$[x_\beta, x_{-(\alpha+\beta)}] = x_{-\alpha}\, x_{-(2\alpha+\beta)} = x_{-(2\alpha+\beta)}\, x_{-\alpha} , \tag{2.8}$$

$$[x_{\alpha+\beta}, x_{-\alpha}] = x_\beta^{-2}, \tag{2.9}$$

$$[x_{\alpha+\beta}, x_{-\beta}] = x_\alpha\, x_{2\alpha+\beta}^{-1} = x_{2\alpha+\beta}^{-1}\, x_\alpha , \tag{2.10}$$

$$[x_{\alpha+\beta}, x_{-(2\alpha+\beta)}] = x_{-\alpha} x_\beta^{-1} = x_\beta^{-1} x_{-\alpha} , \tag{2.11}$$

$$[x_{2\alpha+\beta}, x_{-\alpha}] = x_{\alpha+\beta}^{-1}\, x_{\beta}^{-1} = x_{\beta}^{-1}\, x_{\alpha+\beta}^{-1}\,, \tag{2.12}$$

$$[x_{2\alpha+\beta}, x_{-(\alpha+\beta)}] = x_{\alpha} x_{-\beta} = x_{-\beta} x_{\alpha}\,, \tag{2.13}$$

$$[x_{-\alpha}, x_{-\beta}] = x_{-(\alpha+\beta)}^{-1}\, x_{-(2\alpha+\beta)} = x_{-(2\alpha+\beta)}\, x_{-(\alpha+\beta)}^{-1}\,, \tag{2.14}$$

$$[x_{-\alpha}, x_{-(\alpha+\beta)}] = x_{-(2\alpha+\beta)}^{-2}\,. \tag{2.15}$$

Note that in view of Relations (2.10), (2.11) and (2.14) the generators x_{α}, $x_{-\alpha}$ and $x_{-(\alpha+\beta)}$ can be expressed in terms of the five remaining generators x_{β}, $x_{\alpha+\beta}$, $x_{2\alpha+\beta}$, $x_{-\beta}$, $x_{-(2\alpha+\beta)}$.

Proposition 2.1 will be proved in Section 2.3 below.

2.2. **The surjection $\pi : \mathrm{St}(C_2, \mathbb{Z}) \to \mathrm{Sp}_4(\mathbb{Z})$.** Recall that the symplectic modular group $\mathrm{Sp}_4(\mathbb{Z})$ is the group of automorphisms of the free abelian rank 4 group $\mathbb{Z}^4$ preserving the standard alternating form, namely the group of 4×4-matrices M with integer entries such that

$$M^T \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} M = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix},$$

where M^T is the transpose of M and I_2 is the identity 2×2-matrix.

The group $\mathrm{Sp}_4(\mathbb{Z})$ is generated by the following eight matrices:

$$X_{\alpha} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \quad X_{-\alpha} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.16}$$

$$X_{\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad X_{-\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \tag{2.17}$$

$$X_{\alpha+\beta} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad X_{-(\alpha+\beta)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \tag{2.18}$$

$$X_{2\alpha+\beta} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad X_{-(2\alpha+\beta)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.19}$$

Observe that each matrix $X_{-\gamma}$ ($\gamma \in \Phi$) is the transpose of X_{γ}.

It is easy to check that the matrices X_{γ} ($\gamma \in \Phi$) satisfy the relations (2.1)–(2.15) defining the Steinberg group $\mathrm{St}(C_2, \mathbb{Z})$. Therefore there exists a unique homomorphism

$$\pi : \mathrm{St}(C_2, \mathbb{Z}) \to \mathrm{Sp}_4(\mathbb{Z}) \tag{2.20}$$

such that $\pi(x_{\gamma}) = X_{\gamma}$ for all $\gamma \in \Phi$. Since the matrices X_{γ} ($\gamma \in \Phi$) generate $\mathrm{Sp}_4(\mathbb{Z})$, the homomorphism π is surjective.

2.3. **Proof of Proposition 2.1.** By [2, Sect. 3] the Steinberg group $\mathrm{St}(C_2, \mathbb{Z})$ has a presentation with the set of generators $\{x_\gamma : \gamma \in \Phi\}$ subject to the following relations: if $\gamma, \delta \in \Phi$ such that $\gamma + \delta \neq 0$, then

$$[x_\gamma, x_\delta] = \prod x_{i\gamma+j\delta}^{c_{i,j}^{\gamma,\delta}}\,, \tag{2.21}$$

where i and j are positive integers such that $i\gamma + j\delta$ belongs to Φ and the exponents $c_{i,j}^{\gamma,\delta}$ are integers depending only on the structure of $\mathrm{Sp}_4(\mathbb{Z})$. In order to find the structure constants $c_{i,j}^{\gamma,\delta}$ it is enough to apply the homomorphism π of (2.20) and to compute the integers $c_{i,j}^{\gamma,\delta}$ appearing in the relations

$$[X_\gamma, X_\delta] = \prod X_{i\gamma+j\delta}^{c_{i,j}^{\gamma,\delta}}\,,$$

where $X_\gamma = \pi(x_\gamma)$ are the elements of $\mathrm{Sp}_4(\mathbb{Z})$ defined by (2.16)–(2.19).

In particular, when $X_\gamma X_\delta = X_\delta X_\gamma$ ($\gamma, \delta \in \Phi$), then $x_\gamma x_\delta = x_\delta x_\gamma$. Relations (2.1)–(2.3) follow from this remark. We next have the following equalities in $\mathrm{Sp}_4(\mathbb{Z})$:

$$[X_\alpha, X_\beta] = X_{\alpha+\beta}\, X_{2\alpha+\beta}\,, \quad [X_\alpha, X_{\alpha+\beta}] = X_{2\alpha+\beta}^2\,,$$

$$[X_\alpha, X_{-(\alpha+\beta)}] = X_{-\beta}^{-2}\,, \quad [X_\alpha, X_{-(2\alpha+\beta)}] = X_{-\beta}\, X_{-(\alpha+\beta)}^{-1}\,,$$

$$[X_\beta, X_{-(\alpha+\beta)}] = X_{-\alpha}\, X_{-(2\alpha+\beta)}\,, \quad [X_{\alpha+\beta}, X_{-\alpha}] = X_\beta^{-2},$$

$$[X_{\alpha+\beta}, X_{-\beta}] = X_\alpha\, X_{2\alpha+\beta}^{-1}\,, \quad [X_{\alpha+\beta}, X_{-(2\alpha+\beta)}] = X_{-\alpha} X_\beta^{-1}\,,$$

$$[X_{2\alpha+\beta}, X_{-\alpha}] = X_{\alpha+\beta}^{-1}\, X_\beta^{-1}\,, \quad [X_{2\alpha+\beta}, X_{-(\alpha+\beta)}] = X_\alpha X_{-\beta}\,,$$

$$[X_{-\alpha}, X_{-\beta}] = X_{-(\alpha+\beta)}^{-1}\, X_{-(2\alpha+\beta)}\,, \quad [X_{-\alpha}, X_{-(\alpha+\beta)}] = X_{-(2\alpha+\beta)}^{-2}\,.$$

From these relations the remaining relations of Proposition 2.1 follow.

2.4. **The elements w_γ.** For any root $\gamma \in \Phi$, set

$$w_\gamma = x_\gamma\, x_{-\gamma}^{-1}\, x_\gamma\,. \tag{2.22}$$

In Steinberg's notation (see [15, 16]), we have $x_\gamma = x_\gamma(1)$ and $w_\gamma = w_\gamma(1)$.

In the sequel we need to know that the kernel of $\pi : \mathrm{St}(C_2, \mathbb{Z}) \to \mathrm{Sp}_4(\mathbb{Z})$ is generated by w_γ^4 for any long root γ, for instance by w_β^4 or by $w_{2\alpha+\beta}^4$; this follows from [2, Kor. 3.2]. The fact that such a generator of the kernel is of infinite order was established in [13, Th. 6.3].

We now list a few properties of the elements w_γ.

Lemma 2.2. *We have $w_\gamma = w_{-\gamma}^{-1}$ for all $\gamma \in \Phi$.*

Proof. By [15] (see also [13, Lemme 5.2 (g)]) we have

$$\begin{aligned} w_\gamma &= w_\gamma(1) = w_{-\gamma}(-1) \\ &= x_{-\gamma}(-1) x_\gamma(1) x_{-\gamma}(-1) = x_{-\gamma}^{-1} x_\gamma x_{-\gamma}^{-1} \\ &= \left(x_{-\gamma} x_\gamma^{-1} x_{-\gamma}\right)^{-1} = w_{-\gamma}^{-1}\,, \end{aligned}$$

which was to be proved. □

Lemma 2.3. *We have*

$$
\begin{aligned}
w_\beta\, x_\alpha\, w_\beta^{-1} &= x_{\alpha+\beta}^{-1}\,, & w_\beta\, x_{-\alpha}\, w_\beta^{-1} &= x_{-(\alpha+\beta)}^{-1}\,,\\
w_\beta\, x_{\alpha+\beta}\, w_\beta^{-1} &= x_\alpha\,, & w_\beta\, x_{-(\alpha+\beta)}\, w_\beta^{-1} &= x_{-\alpha}\,,\\
w_\beta\, x_{2\alpha+\beta}\, w_\beta^{-1} &= x_{2\alpha+\beta}\,, & w_\beta\, x_{-(2\alpha+\beta)}\, w_\beta^{-1} &= x_{-(2\alpha+\beta)}\,,\\
w_{2\alpha+\beta}\, x_\beta\, w_{2\alpha+\beta}^{-1} &= x_\beta\,, & w_{2\alpha+\beta}\, x_{-\beta}\, w_{2\alpha+\beta}^{-1} &= x_{-\beta}\,,\\
w_{2\alpha+\beta}\, x_\alpha\, w_{2\alpha+\beta}^{-1} &= x_{-(\alpha+\beta)}^{-1}\,, & w_{2\alpha+\beta}\, x_{-(\alpha+\beta)}\, w_{2\alpha+\beta}^{-1} &= x_\alpha\,,\\
w_{2\alpha+\beta}\, x_{\alpha+\beta}\, w_{2\alpha+\beta}^{-1} &= x_{-\alpha}\,, & w_{2\alpha+\beta}\, x_{-\alpha}\, w_{2\alpha+\beta}^{-1} &= x_{\alpha+\beta}^{-1}\,,\\
w_\alpha\, x_\beta\, w_\alpha^{-1} &= x_{2\alpha+\beta}\,, & w_\alpha\, x_{-\beta}\, w_\alpha^{-1} &= x_{-(2\alpha+\beta)}\,,\\
w_\alpha\, x_{\alpha+\beta}\, w_\alpha^{-1} &= x_{\alpha+\beta}^{-1}\,, & w_\alpha\, x_{-(\alpha+\beta)}\, w_\alpha^{-1} &= x_{-(\alpha+\beta)}^{-1}\,,\\
w_\alpha\, x_{2\alpha+\beta}\, w_\alpha^{-1} &= x_\beta\,, & w_\alpha\, x_{-(2\alpha+\beta)}\, w_\alpha^{-1} &= x_{-\beta}\,,\\
w_{\alpha+\beta}\, x_\alpha\, w_{\alpha+\beta}^{-1} &= x_\alpha^{-1}\,, & w_{\alpha+\beta}\, x_{-\alpha}\, w_{\alpha+\beta}^{-1} &= x_{-\alpha}^{-1}\,,\\
w_{\alpha+\beta}\, x_\beta\, w_{\alpha+\beta}^{-1} &= x_{-(2\alpha+\beta)}^{-1}\,, & w_{\alpha+\beta}\, x_{-(2\alpha+\beta)}\, w_{\alpha+\beta}^{-1} &= x_\beta^{-1}\,,\\
w_{\alpha+\beta}\, x_{2\alpha+\beta}\, w_{\alpha+\beta}^{-1} &= x_{-\beta}^{-1}\,, & w_{\alpha+\beta}\, x_{-\beta}\, w_{\alpha+\beta}^{-1} &= x_{2\alpha+\beta}^{-1}\,,
\end{aligned}
$$

Proof. By Relation (R7) in [16, Chap. 3, p. 23], for any two roots γ and δ such that $\gamma+\delta \neq 0$ we have $w_\gamma\, x_\delta\, w_\gamma^{-1} = x_{\delta'}^{\varepsilon}$, where δ' is the image of δ under the reflection in the line orthogonal to γ and $\varepsilon = \pm 1$. To determine the root δ' and the sign ε it is enough to compute the image $\pi(w_\gamma\, x_\delta\, w_\gamma^{-1})$ in $\mathrm{Sp}_4(\mathbb{Z})$. □

3. From the braid group to the Steinberg group

We now construct a homomorphism from the braid group B_6 to the Steinberg group $\mathrm{St}(C_2,\mathbb{Z})$.

Proposition 3.1. *There is exists a unique homomorphism $f : B_6 \to \mathrm{St}(C_2,\mathbb{Z})$ such that*

$$
f(\sigma_1) = x_{2\alpha+\beta}\,, \qquad f(\sigma_2) = x_{-(2\alpha+\beta)}^{-1}\,,
$$
$$
f(\sigma_3) = x_\beta\, x_{\alpha+\beta}^{-1}\, x_{2\alpha+\beta}\,,
$$
$$
f(\sigma_4) = x_{-\beta}^{-1}\,, \qquad f(\sigma_5) = x_\beta\,.
$$

The homomorphism f is surjective.

The homomorphism f lifts the homomorphism $\overline{f} : B_6 \to \mathrm{Sp}_4(\mathbb{Z})$ constructed in [11, Sect. 4.2] in the sense that $\overline{f} = \pi \circ f$, where π is the natural surjection $\mathrm{St}(C_2,\mathbb{Z}) \to \mathrm{Sp}_4(\mathbb{Z})$ defined in (2.20).

Proof. For the existence and the uniqueness of f it suffices to check that the five elements $f(\sigma_i)$ $(1 \leq i \leq 5)$ of the Steinberg group $\mathrm{St}(C_2,\mathbb{Z})$ satisfy the braid relations (1.1) and (1.2).

(i) Let us first check the commutation relations (1.1).

- *Commutation of $f(\sigma_1)$ with $f(\sigma_3)$, $f(\sigma_4)$ and $f(\sigma_5)$.* This follows from the fact that $x_{2\alpha+\beta}$ commutes with x_β and $x_{\alpha+\beta}$ by (2.1), and with $x_{-\beta}$ by (2.2).
- *Commutation of $f(\sigma_2)$ with $f(\sigma_4)$ and $f(\sigma_5)$.* It follows from (2.2) and (2.3).

- *Commutation of $f(\sigma_3)$ with $f(\sigma_5)$.* It follows from (2.1).

(ii) The relation $f(\sigma_1)f(\sigma_2)f(\sigma_1) = f(\sigma_2)f(\sigma_1)f(\sigma_2)$ reads as

$$x_{2\alpha+\beta}\, x^{-1}_{-(2\alpha+\beta)}\, x_{2\alpha+\beta} = x^{-1}_{-(2\alpha+\beta)}\, x_{2\alpha+\beta}\, x^{-1}_{-(2\alpha+\beta)}\,,$$

which is equivalent to $w_{2\alpha+\beta} = w^{-1}_{-(2\alpha+\beta)}$. The latter holds by Lemma 2.2.

(iii) The relation $f(\sigma_2)f(\sigma_3)f(\sigma_2) = f(\sigma_3)f(\sigma_2)f(\sigma_3)$ reads as

$$x^{-1}_{-(2\alpha+\beta)}\, x_\beta\, x^{-1}_{\alpha+\beta}\, x_{2\alpha+\beta}\, x^{-1}_{-(2\alpha+\beta)} = x_\beta\, x^{-1}_{\alpha+\beta}\, x_{2\alpha+\beta}\, x^{-1}_{-(2\alpha+\beta)}\, x_\beta\, x^{-1}_{\alpha+\beta}\, x_{2\alpha+\beta}\,.$$

Let LHS (resp. RHS) be the element of $\mathrm{St}(C_2,\mathbb{Z})$ represented by the left-hand (resp. right-hand) side of the previous equation. By (2.1), (2.2), (2.3), (2.11), (2.22) we obtain

$$\begin{aligned}
\mathrm{LHS} &= x_\beta\, \underbrace{x^{-1}_{-(2\alpha+\beta)}\, x^{-1}_{\alpha+\beta}}\, x_{-(2\alpha+\beta)}\, w^{-1}_{-(2\alpha+\beta)} \\
&= x_\beta\, x^{-1}_{\alpha+\beta}\, \underbrace{x^{-1}_{-(2\alpha+\beta)}\, x^{-1}_{-\alpha}\, x_\beta\, x_{-(2\alpha+\beta)}}\, w^{-1}_{-(2\alpha+\beta)} \\
&= \underbrace{x_\beta\, x^{-1}_{\alpha+\beta}\, x^{-1}_{-\alpha}\, x_\beta}\, w^{-1}_{-(2\alpha+\beta)} \\
&= x^2_\beta\, x^{-1}_{\alpha+\beta}\, x^{-1}_{-\alpha}\, w^{-1}_{-(2\alpha+\beta)}\,.
\end{aligned}$$

(The above underbraces $\underbrace{\qquad}$ mark the places to which we apply the relations we refer to.)

Let us now deal with RHS. Since by (2.1) and (2.2) x_β commutes with $x_{\alpha+\beta}$ and with $x_{\pm(2\alpha+\beta)}$, and $x_{\alpha+\beta}$ commutes with $x_{2\alpha+\beta}$, we obtain

$$\mathrm{RHS} = x^2_\beta\, x^{-1}_{\alpha+\beta}\, w_{2\alpha+\beta}\, x^{-1}_{\alpha+\beta}\,.$$

Now by Lemma 2.3, we have $w_{2\alpha+\beta}\, x^{-1}_{\alpha+\beta} = x^{-1}_{-\alpha}\, w_{2\alpha+\beta}$. Therefore,

$$\mathrm{RHS} = x^2_\beta\, x^{-1}_{\alpha+\beta}\, x^{-1}_{-\alpha}\, w_{2\alpha+\beta}\,.$$

It follows that LHS = RHS is equivalent to $w^{-1}_{-(2\alpha+\beta)} = w_{2\alpha+\beta}$, which again holds by Lemma 2.2.

(iv) The relation $f(\sigma_3)f(\sigma_4)f(\sigma_3) = f(\sigma_4)f(\sigma_3)f(\sigma_4)$ reads as

$$x_\beta\, x^{-1}_{\alpha+\beta}\, x_{2\alpha+\beta}\, x^{-1}_{-\beta}\, x_\beta\, x^{-1}_{\alpha+\beta}\, x_{2\alpha+\beta} = x^{-1}_{-\beta}\, x_\beta\, x^{-1}_{\alpha+\beta}\, x_{2\alpha+\beta}\, x^{-1}_{-\beta}\,.$$

Let LHS′ (resp. RHS′) be the element of $\mathrm{St}(C_2,\mathbb{Z})$ represented by the left-hand (resp. right-hand) side of the previous equation. Since by (2.1) and (2.2) $x_{2\alpha+\beta}$ commutes with x_α, with $x_{\alpha+\beta}$ and with $x_{\pm\beta}$, hence with w_β, and x_β commutes with $x_{\alpha+\beta}$, we obtain

$$\mathrm{LHS}' = x^{-1}_{\alpha+\beta}\, x^2_{2\alpha+\beta}\, w_\beta\, x^{-1}_{\alpha+\beta} = x^{-1}_{\alpha+\beta}\, x^2_{2\alpha+\beta}\, x^{-1}_{\alpha}\, w_\beta = x^{-1}_{\alpha+\beta}\, x^{-1}_{\alpha}\, x^2_{2\alpha+\beta}\, w_\beta\,,$$

the second equality holding by Lemma 2.3. For RHS′, by Lemma 2.2 we have

$$\mathrm{RHS}' = w_\beta\, x_{-\beta}\, x^{-1}_{\alpha+\beta}\, x^{-1}_{-\beta}\, x_{2\alpha+\beta}\,.$$

Now by (2.10), we have $x_{-\beta}\, x^{-1}_{\alpha+\beta}\, x^{-1}_{-\beta} = x^{-1}_{\alpha+\beta}\, x_\alpha\, x^{-1}_{2\alpha+\beta}$. Therefore,

$$\mathrm{RHS}' = w_\beta\, x^{-1}_{\alpha+\beta}\, x_\alpha\,.$$

It follows from Lemma 2.3 that $w_\beta\, x_{\alpha+\beta}^{-1} = x_\alpha^{-1}\, w_\beta$ and $w_\beta\, x_\alpha = x_{\alpha+\beta}^{-1}\, w_\beta$. Moreover by (2.5) we have $x_\alpha^{-1}\, x_{\alpha+\beta}^{-1} = x_{\alpha+\beta}^{-1}\, x_\alpha^{-1}\, x_{2\alpha+\beta}^2$. Therefore,

$$\mathrm{RHS}' = x_\alpha^{-1}\, w_\beta\, x_\alpha = x_\alpha^{-1}\, x_{\alpha+\beta}^{-1}\, w_\beta = x_{\alpha+\beta}^{-1}\, x_\alpha^{-1}\, x_{2\alpha+\beta}^2\, w_\beta = \mathrm{LHS}'.$$

(v) The relation $f(\sigma_4)f(\sigma_5)f(\sigma_4) = f(\sigma_4)f(\sigma_5)f(\sigma_4)$ reads as

$$x_{-\beta}^{-1}\, x_\beta\, x_{-\beta}^{-1} = x_\beta\, x_{-\beta}^{-1}\, x_\beta\,,$$

which is equivalent to $w_{-\beta}^{-1} = w_\beta$. The last equality holds by Lemma 2.2.

As we noted after stating Proposition 2.1, the five elements x_β, $x_{\alpha+\beta}$, $x_{2\alpha+\beta}$, $x_{-\beta}$, $x_{-(2\alpha+\beta)}$ generate $\mathrm{St}(C_2,\mathbb{Z})$. They clearly are in the image of f, which implies the surjectivity of the latter. □

4. Results

Let $f : B_6 \to \mathrm{St}(C_2,\mathbb{Z})$ be the surjective homomorphism defined in Proposition 3.1. We now state our main result.

Theorem 4.1. *The kernel of the homomorphism $f : B_6 \to \mathrm{St}(C_2,\mathbb{Z})$ is the normal subgroup of B_6 generated by*

$$(\sigma_1\sigma_2\sigma_1)^2(\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_1\sigma_2\sigma_1)^{-2}(\sigma_1\sigma_3^{-1}\sigma_5).$$

This means that the Steinberg group $\mathrm{St}(C_2,\mathbb{Z})$ has a presentation with five generators σ_1, σ_2, σ_3, σ_4, σ_5, and eleven relations consisting of the ten braid relations (1.1)–(1.2) and the additional relation

$$(\sigma_1\sigma_2\sigma_1)^2(\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_1\sigma_2\sigma_1)^{-2} = (\sigma_1\sigma_3^{-1}\sigma_5)^{-1}. \tag{4.1}$$

This relation is clearly equivalent to (1.3).

As mentioned at the beginning of Section 2.4, the kernel of the projection $\pi : \mathrm{St}(C_2,\mathbb{Z}) \to \mathrm{Sp}_4(\mathbb{Z})$ is generated by $w_{2\alpha+\beta}^4$. Since

$$w_{2\alpha+\beta} = x_{2\alpha+\beta}x_{-(2\alpha+\beta)}^{-1}x_{2\alpha+\beta} = f(\sigma_1\sigma_2\sigma_1), \tag{4.2}$$

we deduce the following presentation[1] of $\mathrm{Sp}_4(\mathbb{Z})$.

Corollary 4.2. *The symplectic modular group $\mathrm{Sp}_4(\mathbb{Z})$ has a presentation with five generators σ_1, σ_2, σ_3, σ_4, σ_5, and twelve relations consisting of Relations (1.1)–(1.2) and the two relations*

$$(\sigma_1\sigma_2\sigma_1)^2(\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_1\sigma_2\sigma_1)^{-2} = (\sigma_1\sigma_3^{-1}\sigma_5)^{-1}$$

and

$$(\sigma_1\sigma_2\sigma_1)^4 = 1.$$

Let N be the normal subgroup of B_6 generated by

$$\beta = (\sigma_1\sigma_2\sigma_1)^2(\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_1\sigma_2\sigma_1)^{-2}(\sigma_1\sigma_3^{-1}\sigma_5).$$

Theorem 4.1 is a consequence of the following two propositions.

Proposition 4.3. *We have $f(N) = 1$.*

[1] Group presentations of $\mathrm{Sp}_4(\mathbb{Z})$ of a different kind have been given in [1, 3].

Proof. It suffices to check that $f(\beta) = 1$. By (4.2) we have $f(\sigma_1\sigma_2\sigma_1) = w_{2\alpha+\beta}$. We also have

$$f(\sigma_1\sigma_3^{-1}\sigma_5) = x_{2\alpha+\beta}(x_{2\alpha+\beta}^{-1}x_{\alpha+\beta}x_\beta^{-1})x_\beta = x_{\alpha+\beta}\,.$$

Therefore, using Lemma 2.3, we obtain

$$f(\beta) = w_{2\alpha+\beta}^2 x_{\alpha+\beta} w_{2\alpha+\beta}^{-2} x_{\alpha+\beta} = w_{2\alpha+\beta} x_{-\alpha} w_{2\alpha+\beta}^{-1} x_{\alpha+\beta} = x_{\alpha+\beta}^{-1} x_{\alpha+\beta} = 1.$$

□

It follows from the previous proposition that $f : B_6 \to \mathrm{St}(C_2,\mathbb{Z})$ factors through a homomorphism $B_6/N \to \mathrm{St}(C_2,\mathbb{Z})$, which we still denote by f.

Proposition 4.4. *There exists a homomorphism $\varphi : \mathrm{St}(C_2,\mathbb{Z}) \to B_6/N$ such that*

$$\begin{aligned}
\varphi(x_\alpha) &= (\sigma_5\sigma_4)(\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_5\sigma_4)^{-1},\\
\varphi(x_\beta) &= \sigma_5\,,\\
\varphi(x_{\alpha+\beta}) &= \sigma_1\sigma_3^{-1}\sigma_5\,,\\
\varphi(x_{2\alpha+\beta}) &= \sigma_1\,,\\
\varphi(x_{-\alpha}) &= (\sigma_1\sigma_2)(\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_1\sigma_2)^{-1},\\
\varphi(x_{-\beta}) &= \sigma_4^{-1},\\
\varphi(x_{-(\alpha+\beta)}) &= (\sigma_1\sigma_2\sigma_5\sigma_4)(\sigma_1^{-1}\sigma_3\sigma_5^{-1})(\sigma_1\sigma_2\sigma_5\sigma_4)^{-1},\\
\varphi(x_{-(2\alpha+\beta)}) &= \sigma_2^{-1}
\end{aligned}$$

modulo N. Moreover, $\varphi \circ f = \mathrm{id}$.

Before we prove Proposition 4.4, let us record the following four equalities in the braid group B_6. One may check them using one's favorite algorithm for solving the word problem in braid groups (see for instance the monographs [5, 8, 9, 10, 12]).

Lemma 4.5. *The following equalities hold in B_6:*

$$[\varphi(x_\alpha),\varphi(x_\beta)]^{-1}\varphi(x_{\alpha+\beta})\,\varphi(x_{2\alpha+\beta}) = (\sigma_4\sigma_5\sigma_4)^2(\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_4\sigma_5\sigma_4)^{-2}(\sigma_1\sigma_3^{-1}\sigma_5),$$

$$[\varphi(x_\alpha),\varphi(x_{\alpha+\beta})]^{-1}\varphi(x_{2\alpha+\beta}^2) = (\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_4\sigma_5\sigma_4)^2(\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_4\sigma_5\sigma_4)^{-2},$$

$$[\varphi(x_\alpha),\varphi(x_{-(\alpha+\beta)})]\,\varphi(x_{-\beta}^2) = (\sigma_5\sigma_4\sigma_1\sigma_2)(\sigma_1\sigma_2\sigma_3)\,\gamma^{-1}\,(\sigma_1\sigma_2\sigma_3)^{-1}(\sigma_5\sigma_4\sigma_1\sigma_2)^{-1},$$

where $\gamma = (\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_1\sigma_2\sigma_1)^{-2}(\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_1\sigma_2\sigma_1)^2$, and

$$[\varphi(x_{\alpha+\beta}),\varphi(x_{-\alpha})]\,\varphi(x_\beta^2) = (\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_1\sigma_2\sigma_1)^2(\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_1\sigma_2\sigma_1)^{-2}.$$

We make also the following observation. Let

$$\Delta = (\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5)(\sigma_1\sigma_2\sigma_3\sigma_4)(\sigma_1\sigma_2\sigma_3)(\sigma_1\sigma_2)\sigma_1$$

be the 'half-twist' in the braid group B_6. It has the following important property (see for instance [5] or [12, Sect. 1.3.3]):

$$\Delta\sigma_i\Delta^{-1} = \sigma_{6-i}\,, \qquad (i = 1,\ldots,5) \tag{4.3}$$

which implies that its square Δ^2 is central (actually Δ^2 generates the center of the braid group). Note that the braids $\sigma_1\sigma_5$, $\sigma_2\sigma_4$ and σ_3 are invariant under conjugation by Δ so that

$$\Delta(\sigma_1\sigma_3^{-1}\sigma_5)\Delta^{-1} = \sigma_5\sigma_3^{-1}\sigma_1 = \sigma_1\sigma_3^{-1}\sigma_5\,,$$

whereas $\sigma_1\sigma_2\sigma_1$ and $\sigma_4\sigma_5\sigma_4$ are interchanged:

$$\Delta(\sigma_1\sigma_2\sigma_1)\Delta^{-1} = \sigma_5\sigma_4\sigma_5 = \sigma_4\sigma_5\sigma_4\,. \tag{4.4}$$

In view of (4.3) and of the definition of φ we have the following "symmetries":

$$\Delta\,\varphi(x_\beta)\,\Delta^{-1} = \varphi(x_{2\alpha+\beta})\,, \tag{4.5}$$

$$\Delta\,\varphi(x_{-\beta})\,\Delta^{-1} = \varphi(x_{-(2\alpha+\beta)})\,, \tag{4.6}$$

$$\Delta\,\varphi(x_\alpha)\,\Delta^{-1} = \varphi(x_{-\alpha})\,, \tag{4.7}$$

$$\Delta\,\varphi(x_{\pm(\alpha+\beta)})\,\Delta^{-1} = \varphi(x_{\pm(\alpha+\beta)})\,. \tag{4.8}$$

Geometrically speaking, conjugating the elements $\varphi(x_\gamma)$ ($\gamma \in \Phi$) by Δ corresponds in the root system C_2 to the reflection in the one-dimensional vector space spanned by $\alpha+\beta$.

Remark 4.6. It follows from [4, Th. 0.2] that the centralizer of Δ is an Artin group of type C_3. Actually it is generated by the above-mentioned elements $\beta_1 = \sigma_1\sigma_5$, $\beta_2 = \sigma_2\sigma_4$ and $\beta_3 = \sigma_3$, which satisfy the relations

$$\beta_1\beta_2\beta_1 = \beta_2\beta_1\beta_2\,, \quad \beta_1\beta_3 = \beta_3\beta_1\,, \quad \beta_2\beta_3\beta_2\beta_3 = \beta_3\beta_2\beta_3\beta_2\,.$$

Proof of Proposition 4.4. To prove the existence of $\varphi : \mathrm{St}(C_2,\mathbb{Z}) \to B_6/N$ it suffices to check that the elements $\varphi(x_\gamma)$ ($\gamma \in \Phi$) satisfy the 24 relations (2.1)–(2.15) in the quotient group B_6/N.

(a) Relations (2.1): in view of the braid relations (1.1), we have

$$\begin{aligned}
[\varphi(x_\alpha), \varphi(x_{2\alpha+\beta})] &= [(\sigma_5\sigma_4)(\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_5\sigma_4)^{-1}, \sigma_1] = 1\,,\\
[\varphi(x_\beta), \varphi(x_{\alpha+\beta})] &= [\sigma_5, \sigma_1\sigma_3^{-1}\sigma_5] = 1\,,\\
[\varphi(x_\beta), \varphi(x_{2\alpha+\beta})] &= [\sigma_5, \sigma_1] = 1\,,\\
[\varphi(x_{\alpha+\beta}), \varphi(x_{2\alpha+\beta})] &= [\sigma_1\sigma_3^{-1}\sigma_5, \sigma_1] = 1\,.
\end{aligned}$$

(b) Relations (2.2): again in view of the commutation relations (1.1), but also of the braid relations (1.2), we have

$$\begin{aligned}
[\varphi(x_\alpha), \varphi(x_{-\beta})] &= [(\sigma_5\sigma_4)(\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_5\sigma_4)^{-1}, \sigma_4^{-1}]\\
&= \sigma_5\sigma_4(\sigma_1\sigma_3^{-1}\sigma_5)\,\underbrace{\sigma_4^{-1}\sigma_5^{-1}\sigma_4^{-1}}\,\sigma_5\sigma_4(\sigma_1\sigma_3^{-1}\sigma_5)^{-1}\,\underbrace{\sigma_4^{-1}\sigma_5^{-1}\sigma_4}\\
&= \sigma_5\sigma_4(\sigma_1\sigma_3^{-1}\sigma_5)\,\underbrace{\sigma_5^{-1}\,\underbrace{\sigma_4^{-1}\sigma_5^{-1}\sigma_5\sigma_4}(\sigma_1\sigma_3^{-1}\sigma_5)^{-1}\sigma_5}\,\sigma_4^{-1}\sigma_5^{-1}\\
&= \sigma_5\sigma_4(\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_1\sigma_3^{-1}\sigma_5)^{-1}\sigma_4^{-1}\sigma_5^{-1} = 1.
\end{aligned}$$

(Here and below the underbraces indicate the places where we apply the braid relations and the trivial relations $\sigma_i\sigma_i^{-1} = \sigma_i^{-1}\sigma_i = 1$.)

Similarly, by (1.1) we have

$$[\varphi(x_\beta), \varphi(x_{-\alpha})] = [\sigma_5, (\sigma_1\sigma_2)(\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_1\sigma_2)^{-1}] = 1\,,$$

$$[\varphi(x_\beta), \varphi(x_{-(2\alpha+\beta)})] = [\sigma_5, \sigma_2^{-1}] = 1\,,$$
$$[\varphi(x_{-\beta}), \varphi(x_{2\alpha+\beta})] = [\sigma_4^{-1}, \sigma_1] = 1\,.$$

Note that only $[\varphi(x_\alpha), \varphi(x_{-\beta})] = 1$ requires one of the braid relations (1.2).

(c) Relations (2.3): for the first one, using (4.6) and (4.7), we have

$$\begin{aligned}[\varphi(x_{-\alpha}), \varphi(x_{-(2\alpha+\beta)})] &= [\Delta\,\varphi(x_\alpha)\,\Delta^{-1}, \Delta\,\varphi(x_{-\beta})\,\Delta^{-1}]\\ &= \Delta\,[\varphi(x_\alpha), \varphi(x_{-\beta})]\,\Delta^{-1} = 1\end{aligned}$$

since $[\varphi(x_\alpha), \varphi(x_{-\beta})] = 1$, as we have just proved in Item (b).

For the second one we use the equality $\varphi(x_{-(\alpha+\beta)}) = (\sigma_1\sigma_2)\,\varphi(x_\alpha)^{-1}(\sigma_1\sigma_2)^{-1}$, which we derive from the definition of φ. Then

$$\begin{aligned}[\varphi(x_{-\beta}), \varphi(x_{-(\alpha+\beta)}] &= [\varphi(x_{-\beta}), (\sigma_1\sigma_2)\,\varphi(x_\alpha)^{-1}(\sigma_1\sigma_2)^{-1}]\\ &= [\sigma_4^{-1}, (\sigma_1\sigma_2)\,\varphi(x_\alpha)^{-1}(\sigma_1\sigma_2)^{-1}]\\ &= [(\sigma_1\sigma_2)\sigma_4^{-1}(\sigma_1\sigma_2)^{-1}, (\sigma_1\sigma_2)\,\varphi(x_\alpha)^{-1}(\sigma_1\sigma_2)^{-1}]\\ &= (\sigma_1\sigma_2)\,[\varphi(x_{-\beta}), \varphi(x_\alpha)^{-1}]\,(\sigma_1\sigma_2)^{-1}.\end{aligned}$$

Since $\varphi(x_{-\beta})$ commutes with $\varphi(x_\alpha)$ by Item (b), it commutes with its inverse $\varphi(x_\alpha)^{-1}$. Hence, $[\varphi(x_{-\beta}), \varphi(x_{-(\alpha+\beta)}] = (\sigma_1\sigma_2)(\sigma_1\sigma_2)^{-1} = 1$.

For the third one, we have $[\varphi(x_{-\beta}), \varphi(x_{-(2\alpha+\beta)})] = [\sigma_4^{-1}, \sigma_2^{-1}] = 1$ by (1.1).

For the fourth one, conjugating by Δ, and using (4.6) and (4.8), we reduce the desired equality $[\varphi(x_{-(\alpha+\beta)}), \varphi(x_{-(2\alpha+\beta)})] = 1$ to the previous equality $[\varphi(x_{-\beta}), \varphi(x_{-(\alpha+\beta)}] = 1$.

(d) Relation (2.4): using the first equality in Lemma 4.5 and Equation (4.4), we have

$$\begin{aligned}&[\varphi(x_\alpha), \varphi(x_\beta)]^{-1}\varphi(x_{\alpha+\beta})\,\varphi(x_{2\alpha+\beta})\\ &\quad= (\sigma_4\sigma_5\sigma_4)^2(\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_4\sigma_5\sigma_4)^{-2}(\sigma_1\sigma_3^{-1}\sigma_5)\\ &\quad= \Delta\,(\sigma_1\sigma_2\sigma_1)^2(\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_1\sigma_2\sigma_1)^{-2}(\sigma_1\sigma_3^{-1}\sigma_5)\,\Delta^{-1}.\end{aligned}$$

It follows that $[\varphi(x_\alpha), \varphi(x_\beta)]^{-1}\varphi(x_{\alpha+\beta})\,\varphi(x_{2\alpha+\beta})$ belongs to the normal subgroup N, hence is trivial in B_6/N.

(e) Relation (2.5): we use the second equality in Lemma 4.5 and argue as in Item (d). It follows that $[\varphi(x_\alpha), \varphi(x_{\alpha+\beta})]^{-1}\varphi(x_{2\alpha+\beta}^2)$ is trivial in B_6/N.

(f) Relation (2.6): consider the third equality in Lemma 4.5; the braid γ is trivial in B_6/N; hence, so is $[\varphi(x_\alpha), \varphi(x_{-(\alpha+\beta)})]\,\varphi(x_{-\beta}^2)$.

(g) Relation (2.7): we have

$$\begin{aligned}&[\varphi(x_\alpha), \varphi(x_{-(2\alpha+\beta)})]^{-1}\varphi(x_{-(\alpha+\beta)})^{-1}\varphi(x_{-\beta})\\ &\quad= \underbrace{\sigma_2^{-1}\sigma_5\sigma_4\sigma_1\sigma_3^{-1}\sigma_5}\,\underbrace{\sigma_4^{-1}\sigma_5^{-1}\sigma_2\sigma_5\sigma_4}\,\sigma_1^{-1}\sigma_3\sigma_5^{-1}\\ &\qquad\times\,\underbrace{\sigma_4^{-1}\sigma_5^{-1}\sigma_1\sigma_2\sigma_5\sigma_4}\,\underbrace{\sigma_1\sigma_3^{-1}\sigma_5}\,\underbrace{\sigma_4^{-1}\sigma_5^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_4^{-1}}\\ &\quad= \sigma_5\sigma_4\sigma_5\sigma_2^{-1}\sigma_1\sigma_3^{-1}\sigma_2\sigma_1^{-1}\sigma_3\,\underbrace{\sigma_5^{-1}\sigma_1\sigma_2\sigma_5}\,\sigma_3^{-1}\,\underbrace{\sigma_1\sigma_2^{-1}\sigma_1^{-1}}\,\underbrace{\sigma_4^{-1}\sigma_5^{-1}\sigma_4^{-1}}\\ &\quad= \sigma_5\sigma_4\sigma_5\sigma_2^{-1}\sigma_1\sigma_3^{-1}\sigma_2\,\underbrace{\sigma_1^{-1}\sigma_3\sigma_1}\,\underbrace{\sigma_2\sigma_3^{-1}\sigma_2^{-1}}\,\sigma_1^{-1}\sigma_2\sigma_5^{-1}\sigma_4^{-1}\sigma_5^{-1}\\ &\quad= \sigma_5\sigma_4\sigma_5\sigma_2^{-1}\sigma_1\sigma_3^{-1}\sigma_2\sigma_3\sigma_3^{-1}\sigma_2^{-1}\sigma_3\sigma_1^{-1}\sigma_2\sigma_5^{-1}\sigma_4^{-1}\sigma_5^{-1}\\ &\quad= (\sigma_5\sigma_4\sigma_5\sigma_2^{-1}\sigma_1\sigma_3^{-1}\sigma_2\sigma_3)(\sigma_5\sigma_4\sigma_5\sigma_2^{-1}\sigma_1\sigma_3^{-1}\sigma_2\sigma_3)^{-1} = 1.\end{aligned}$$

(h) Relation (2.8): we have

$$\begin{aligned}
&[\varphi(x_\beta), \varphi(x_{-(\alpha+\beta)})]^{-1}\varphi(x_{-\alpha})\,\varphi(x_{-(2\alpha+\beta)}) \\
&\quad = \sigma_1\sigma_2\sigma_5\sigma_4\sigma_1^{-1}\sigma_3\sigma_5^{-1}\sigma_4^{-1}\,\underbrace{\sigma_5^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_5}_{}\,\sigma_1\sigma_2\sigma_5\sigma_4\sigma_1\sigma_3^{-1}\sigma_5 \\
&\quad\qquad \times \sigma_4^{-1}\sigma_5^{-1}\,\underbrace{\sigma_2^{-1}\sigma_1^{-1}\sigma_5^{-1}\sigma_1\sigma_2}_{}\,\sigma_1\sigma_3^{-1}\sigma_5\,\underbrace{\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}}_{} \\
&\quad = \sigma_1\sigma_2\sigma_5\sigma_4\sigma_1^{-1}\sigma_3\sigma_5^{-1}\sigma_4^{-1}\,\underbrace{\sigma_2^{-1}\sigma_1^{-1}\sigma_1\sigma_2}_{}\,\sigma_5\sigma_4\sigma_1\sigma_3^{-1}\sigma_5 \\
&\quad\qquad \times \sigma_4^{-1}\sigma_5^{-1}\,\underbrace{\sigma_5^{-1}\sigma_1\sigma_3^{-1}\sigma_5\sigma_1^{-1}}_{}\,\sigma_2^{-1}\sigma_1^{-1} \\
&\quad = \sigma_1\sigma_2\sigma_5\sigma_4\,\underbrace{\sigma_1^{-1}\sigma_3\sigma_5^{-1}\,\underbrace{\sigma_4^{-1}\sigma_5\sigma_4}_{}\,\sigma_1}_{}\,\sigma_3^{-1}\sigma_5\sigma_4^{-1}\,\underbrace{\sigma_5^{-1}\sigma_3^{-1}}_{}\,\sigma_2^{-1}\sigma_1^{-1} \\
&\quad = \sigma_1\sigma_2\sigma_5\sigma_4\sigma_3\,\underbrace{\sigma_5^{-1}\sigma_5}_{}\,\sigma_4\,\underbrace{\sigma_5^{-1}\sigma_3^{-1}\sigma_5}_{}\,\sigma_4^{-1}\sigma_3^{-1}\sigma_5^{-1}\sigma_2^{-1}\sigma_1^{-1} \\
&\quad = \sigma_1\sigma_2\sigma_5\sigma_4\sigma_3\sigma_4\,\underbrace{\sigma_3^{-1}\sigma_4^{-1}\sigma_3^{-1}}_{}\,\sigma_5^{-1}\sigma_2^{-1}\sigma_1^{-1} \\
&\quad = \sigma_1\sigma_2\sigma_5\sigma_4\sigma_3\sigma_4\sigma_4^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_5^{-1}\sigma_2^{-1}\sigma_1^{-1} \\
&\quad = (\sigma_1\sigma_2\sigma_5\sigma_4\sigma_3\sigma_4)(\sigma_1\sigma_2\sigma_5\sigma_4\sigma_3\sigma_4)^{-1} = 1.
\end{aligned}$$

(i) Relation (2.9): the fourth equality in Lemma 4.5 implies that the braid $[\varphi(x_{\alpha+\beta}), \varphi(x_{-\alpha})]\,\varphi(x_\beta^2)$ belongs to N, hence is trivial in B_6/N.

(j) Relation (2.10): we have

$$\begin{aligned}
&[\varphi(x_{\alpha+\beta}), \varphi(x_{-\beta})]^{-1}\varphi(x_\alpha)\,\varphi(x_{2\alpha+\beta}^{-1}) \\
&\quad = \sigma_4^{-1}\,\underbrace{\sigma_1\sigma_3^{-1}\sigma_5\sigma_4\sigma_1^{-1}}_{}\,\sigma_3\,\underbrace{\sigma_5^{-1}\sigma_5}_{}\,\sigma_4\,\underbrace{\sigma_1\sigma_3^{-1}\sigma_5\sigma_4^{-1}\sigma_5^{-1}\sigma_1^{-1}}_{} \\
&\quad = \sigma_4^{-1}\sigma_3^{-1}\sigma_5\,\underbrace{\sigma_4\sigma_3\sigma_4}_{}\,\sigma_3^{-1}\,\underbrace{\sigma_5\sigma_4^{-1}\sigma_5^{-1}}_{} \\
&\quad = \sigma_4^{-1}\,\underbrace{\sigma_3^{-1}\sigma_5\sigma_3}_{}\,\underbrace{\sigma_4\sigma_3\sigma_3^{-1}\sigma_4^{-1}}_{}\,\sigma_5^{-1}\sigma_4 \\
&\quad = \sigma_4^{-1}\sigma_5\sigma_5^{-1}\sigma_4 = 1.
\end{aligned}$$

(k) Relation (2.11): using the computation in Item (j) and conjugating by Δ, we obtain

$$\begin{aligned}
&[\varphi(x_{\alpha+\beta}), \varphi(x_{-(2\alpha+\beta)})]^{-1}\varphi(x_{-\alpha})\,\varphi(x_\beta^{-1}) \\
&\quad = \Delta\,[\varphi(x_{\alpha+\beta}), \varphi(x_{-\beta})]^{-1}\varphi(x_\alpha)\,\varphi(x_{2\alpha+\beta}^{-1})\,\Delta^{-1} = 1.
\end{aligned}$$

(l) Relation (2.12): by (4.5), (4.7) and (4.8) we have

$$\begin{aligned}
&[\varphi(x_{2\alpha+\beta}), \varphi(x_{-\alpha})]\,\varphi(x_{\alpha+\beta})\,\varphi(x_\beta) \\
&\qquad\qquad = \Delta\,[\varphi(x_\beta), \varphi(x_\alpha)]\,\varphi(x_{\alpha+\beta})\,\varphi(x_{2\alpha+\beta})\,\Delta^{-1},
\end{aligned}$$

which in view of Item (d) above implies that $[\varphi(x_{2\alpha+\beta}), \varphi(x_{-\alpha})]\,\varphi(x_{\alpha+\beta})\,\varphi(x_\beta)$ is trivial in B_6/N.

(m) Relation (2.13) reduces to Relation (2.8) after conjugating by Δ. In the same way Relations (2.14) and (2.15) reduce to Relations (2.7) and (2.6) respectively.

To complete the proof it remains to check that $\varphi \circ f = \mathrm{id}$. This is a consequence of the equalities $(\varphi \circ f)(\sigma_i) = \sigma_i$ $(1 \le i \le 5)$, which follow immediately from the definitions of f and φ. □

As can be seen in the previous proof, 18 out of the 24 images under φ of the relations defining $\mathrm{St}(C_2, \mathbb{Z})$ hold in the braid group B_6 itself. Only six of them hold modulo the normal subgroup N.

Remark 4.7. In [11, Cor. 7] we gave a braid-like presentation of the symplectic modular group $\mathrm{Sp}_4(\mathbb{Z})$ with four additional relations rather than the two additional ones in Corollary 4.2 above. In [11] the first relation in Equation (27) is equivalent (modulo the braid relations) to the relation $(\sigma_1\sigma_2\sigma_1)^4 = 1$ above. Relation (28) in *loc. cit.* is equivalent to the relation $(\sigma_1\sigma_2\sigma_1)^2(\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_1\sigma_2\sigma_1)^{-2} = (\sigma_1\sigma_3^{-1}\sigma_5)^{-1}$. Our corollary 4.2 shows that the relation $\Delta^2 = 1$ of [11, Eq. (26)] is not needed. Actually, one can prove

$$f(\Delta^2) = w_\beta^{12} = w_{2\alpha+\beta}^{12} = f\left((\sigma_4\sigma_5\sigma_4)^{12}\right) = f\left((\sigma_1\sigma_2\sigma_1)^{12}\right).$$

Appendix A. The Steinberg group of type C_2 over a commutative ring

For completeness we give a presentation of the Steinberg group $\mathrm{St}(C_2, R)$ over an arbitrary commutative ring R.

By [14, Sect. 3] (see also [15] or [16, Chap. 6]) the group $\mathrm{St}(C_2, R)$ has a presentation with the set of generators $\{x_\gamma(u) : \gamma \in \Phi,\ u \in R\}$ subject to the following relations:

- $x_\gamma(u+v) = x_\gamma(u)x_\gamma(v)$ for all $u, v \in R$;
- if $\gamma, \delta \in \Phi$ such that $\gamma + \delta \neq 0$, then
$$[x_\gamma(u), x_\delta(v)] = \prod x_{i\gamma+j\delta}(c_{i,j}^{\gamma,\delta} u^i v^j),$$
where i and j are positive integers such that $i\gamma + j\delta$ belongs to Φ.

The coefficients $c_{i,j}^{\gamma,\delta}$ are the integers which have already come up in Relations (2.21) in the proof of Proposition 2.1.

In this way we obtain the following 24 defining relations for $\mathrm{St}(C_2, R)$ (where $u, v \in R$):

$$\text{(A.1)}\quad [x_\alpha(u), x_{2\alpha+\beta}(v)] = [x_\beta(u), x_{\alpha+\beta}(v)] = \\ = [x_\beta(u), x_{2\alpha+\beta}(v)] = [x_{\alpha+\beta}(u), x_{2\alpha+\beta}(v)] = 1,$$

$$\text{(A.2)}\quad [x_\alpha(u), x_{-\beta}(v)] = [x_\beta(u), x_{-\alpha}(v)] = \\ = [x_\beta(u), x_{-(2\alpha+\beta)}(v)] = [x_{-\beta}(u), x_{2\alpha+\beta}(v)] = 1,$$

$$\text{(A.3)}\quad [x_{-\alpha}(u), x_{-(2\alpha+\beta)}(v)] = [x_{-\beta}(u), x_{-(\alpha+\beta)}(v)] = \\ = [x_{-\beta}(u), x_{-(2\alpha+\beta)}(v)] = [x_{-(\alpha+\beta)}(u), x_{-(2\alpha+\beta)}(v)] = 1,$$

$$\text{(A.4)}\quad [x_\alpha(u), x_\beta(v)] = x_{\alpha+\beta}(uv)\, x_{2\alpha+\beta}(u^2v),$$

$$\text{(A.5)}\quad [x_\alpha(u), x_{\alpha+\beta}(v)] = x_{2\alpha+\beta}(2uv),$$

$$\text{(A.6)}\quad [x_\alpha(u), x_{-(\alpha+\beta)}(v)] = x_{-\beta}(-2uv),$$

$$\text{(A.7)}\quad [x_\alpha(u), x_{-(2\alpha+\beta)}(v)] = x_{-\beta}(u^2v)\, x_{-(\alpha+\beta)}(-uv),$$

$$[x_\beta(u), x_{-(\alpha+\beta)}(v)] = x_{-\alpha}(uv)\, x_{-(2\alpha+\beta)}(uv^2), \tag{A.8}$$

$$[x_{\alpha+\beta}(u), x_{-\alpha}(v)] = x_\beta(-2uv), \tag{A.9}$$

$$[x_{\alpha+\beta}(u), x_{-\beta}(v)] = x_\alpha(uv)\, x_{2\alpha+\beta}(-u^2v), \tag{A.10}$$

$$[x_{\alpha+\beta}(u), x_{-(2\alpha+\beta)}(v)] = x_{-\alpha}(uv) x_\beta(-u^2v), \tag{A.11}$$

$$[x_{2\alpha+\beta}(u), x_{-\alpha}(v)] = x_{\alpha+\beta}(-uv)\, x_\beta(-uv^2), \tag{A.12}$$

$$[x_{2\alpha+\beta}(u), x_{-(\alpha+\beta)}(v)] = x_\alpha(uv) x_{-\beta}(uv^2), \tag{A.13}$$

$$[x_{-\alpha}(u), x_{-\beta}(v)] = x_{-(\alpha+\beta)}(-uv)\, x_{-(2\alpha+\beta)}(u^2v), \tag{A.14}$$

$$[x_{-\alpha}(u), x_{-(\alpha+\beta)}(v)] = x_{-(2\alpha+\beta)}(-2uv). \tag{A.15}$$

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