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HIGH ORDER HOMOGENIZATION OF THE STOKES SYSTEM IN A PERIODIC POROUS MEDIUM*

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Abstract. We derive high order homogenized models for the incompressible Stokes system in a cubic domain filled with periodic obstacles. These models have the potential to unify the three classical limit problems (namely the "unchanged" Stokes system, the Brinkman model, and the Darcy's law) corresponding to various asymptotic regimes of the ratio $\eta \equiv a_\varepsilon/\varepsilon$ between the radius a_ε of the holes and the size ε of the periodic cell. What is more, a novel, rather surprising feature of our higher order effective equations is the occurrence of odd order differential operators when the obstacles are not symmetric. Our derivation relies on the method of two-scale power series expansions and on the existence of a "criminal" ansatz, which allows to reconstruct the oscillating velocity and pressure $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$ as a linear combination of the derivatives of their formal average $(\boldsymbol{u}_\varepsilon^*, p_\varepsilon^*)$ weighted by suitable corrector tensors. The formal average $(\boldsymbol{u}_\varepsilon^*, p_\varepsilon^*)$ is itself the solution to a formal, infinite order homogenized equation, whose truncation at any finite order is in general ill-posed. Inspired by the variational truncation method of [53, 27], we derive, for any $K \in \mathbb{N}$, a well-posed model of order $2K+2$ which yields approximations of the original solutions with an error of order $O(\varepsilon^{K+3})$ in the L^2 norm. Furthermore, the error improves up to the order $O(\varepsilon^{2K+4})$ if a slight modification of this model remains well-posed. Finally, we find asymptotics of all homogenized tensors in the low volume fraction limit $\eta \to 0$ and in dimension $d \geq 3$. This allows us to obtain that our effective equations converge coefficient-wise to either of the Brinkman or Darcy regimes which arise when η is respectively equivalent, or greater than the critical scaling $\eta_{\text{crit}} \sim \varepsilon^{2/(d-2)}$.

Key words. Homogenization, higher order models, porous media, Stokes system, strange term.

AMS subject classifications. 35B27, 76M50, 35330

1. Introduction.

This article is concerned with the high order homogenization of the Stokes system in a periodic porous medium. Let $D := (0, L)^d$ be a d-dimensional box filled with periodic obstacles $\omega_\varepsilon := \varepsilon(\mathbb{Z}^d + \eta T) \cap D$ (the setting is illustrated on Figure 1). The parameter ε denotes the size of the periodic cell, it is equal to $\varepsilon := L/N$ where $N \in \mathbb{N}$ is a large integer and L is the length of the box. The parameter η is the scaling ratio between the radius $a_\varepsilon := \eta\varepsilon$ of the obstacles and the length ε of the cells. The total fluid domain is denoted by $D_\varepsilon := D \backslash \overline{\omega_\varepsilon}$ and it is assumed to be connected. $P = (0, 1)^d$ is the unit cell and $Y = P \backslash \eta T$ denotes its fluid component.

We consider $(\boldsymbol{u}_\varepsilon, p_\varepsilon) \in H^1(D_\varepsilon, \mathbb{R}^d) \times L^2(D_\varepsilon)/\mathbb{R}$ the solution to the Stokes system

$$
\begin{cases}
-\Delta \boldsymbol{u}_\varepsilon + \nabla p_\varepsilon = \boldsymbol{f} \text{ in } D_\varepsilon \\
\operatorname{div}(\boldsymbol{u}_\varepsilon) = 0 \\
\boldsymbol{u}_\varepsilon = 0 \text{ on } \partial\omega_\varepsilon \\
\boldsymbol{u}_\varepsilon \text{ is } D\text{–periodic,}
\end{cases}
\tag{1.1}
$$

where $\boldsymbol{f} \in \mathcal{C}^\infty_{per}(D, \mathbb{R}^d)$ (and all its derivatives) is a smooth, D–periodic right hand-side. The goal of this paper is to derive high order effective models for (1.1); i.e. a family of well-posed partial differential equations posed in the homogeneous domain D (without the holes) and whose solutions approximate the macroscopic behavior of $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$ at any desired order of accuracy in ε as $\varepsilon \to 0$.

*Submitted to the editors DATE.

Funding: This work was supported by the Association Nationale de la Recherche et de la Technologie (ANRT) [grant number CIFRE 2017/0024] and by the project ANR-18-CE40-0013 SHAPO financed by the French Agence Nationale de la Recherche (ANR).

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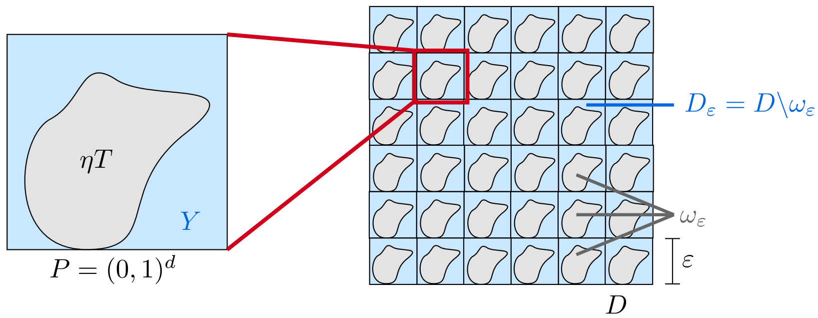

FIG. 1. *The perforated domain $D_\varepsilon = D\backslash\omega_\varepsilon$ and the unit cell $Y = P\backslash(\eta T)$.*

The literature [52, 47, 29, 5, 7, 4, 8] describes the occurrence of different asymptotic regimes depending on how the size $a_\varepsilon = \eta\varepsilon$ of the holes compares to the critical size $\sigma_\varepsilon := \varepsilon^{d/(d-2)}$ in dimension $d \geq 3$ (if $d = 2$, then these regimes depend on how $\log(a_\varepsilon)$ compares to $-\varepsilon^{-2}$, see [7]). In loose mathematical terms, these can be summarized as follows (see e.g. [5, 7] for the precise statements):

- if $a_\varepsilon = o(\sigma_\varepsilon)$, then the holes have no effect and $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$ converges as $\varepsilon \to 0$ to the solution $(\boldsymbol{u}, p)$ of the Stokes equation in the homogeneous domain D:

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \text{ in } D \\ \operatorname{div}(\boldsymbol{u}) = 0 \\ \boldsymbol{u} \text{ is } D\text{–periodic.} \end{cases} \tag{1.2}$$

- if $a_\varepsilon = c\sigma_\varepsilon$ for a constant $c > 0$, then $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$ converges as $\varepsilon \to 0$ to the solution $(\boldsymbol{u}, p)$ of the Brinkman equation

$$\begin{cases} -\Delta \boldsymbol{u} + cF\boldsymbol{u} + \nabla p = \boldsymbol{f} \text{ in } D \\ \operatorname{div}(\boldsymbol{u}) = 0 \\ \boldsymbol{u} \text{ is } D\text{–periodic,} \end{cases} \tag{1.3}$$

  where the so-called *strange term* $cF\boldsymbol{u}$ involves a symmetric positive definite $d \times d$ matrix F which can be computed by means of an exterior problem in $\mathbb{R}^d\backslash T$ (see [4] and section 5).
- if $\sigma_\varepsilon = o(a_\varepsilon)$ and $a_\varepsilon = \eta\varepsilon$ with $\eta \to 0$ as $\varepsilon \to 0$, then the holes are "large" and $(a_\varepsilon^{d-2}\varepsilon^{-d}\boldsymbol{u}_\varepsilon, p_\varepsilon)$ converges to the solution $(\boldsymbol{u}, p)$ of the Darcy problem

$$\begin{cases} F\boldsymbol{u} + \nabla p = \boldsymbol{f} \text{ in } D \\ \operatorname{div}(\boldsymbol{u}) = 0 \text{ in } D \\ \boldsymbol{u} \text{ is } D\text{–periodic,} \end{cases} \tag{1.4}$$

  where F is the same symmetric positive definite $d \times d$ matrix as in (1.3).
- if $a_\varepsilon = \eta\varepsilon$ with the ratio η *fixed*, then $(\varepsilon^{-2}\boldsymbol{u}_\varepsilon, p_\varepsilon)$ converges to the solution $(\boldsymbol{u}, p)$ of the Darcy problem

$$\begin{cases} M^0\boldsymbol{u} + \nabla p = \boldsymbol{f} \text{ in } D \\ \operatorname{div}(\boldsymbol{u}) = 0 \text{ in } D \\ \boldsymbol{u} \text{ is } D\text{–periodic,} \end{cases} \tag{1.5}$$

where M^0 is another positive symmetric $d \times d$ matrix (which depends on η). Furthermore $M^0/|\log(\eta)| \to F$ if $d = 2$, and $M^0/\eta^{d-2} \to F$ (if $d \geq 3$) when $\eta \to 0$, so that there is a continuous transition from (1.5) to (1.4), see [6].

One of the long-term motivations driving this work is the need to lay down theoretical material that would allow to optimize the design of fluid systems by homogenization methods similar to those available in the context of mechanical structures [21, 20, 10, 50, 14]. To date, the Brinkman [24, 25, 30] and the Darcy models [56, 51] are commonly used by topology optimization algorithms in order to conveniently interpolate the physics of the fluid at intermediate "gray" regions featuring locally a mixture of fluid and solid. However, the above conclusions imply that these models are consistent only in specific ranges of obstacle sizes a_ε: the Brinkman model (1.3) is relevant when there are none or tiny obstacles, while the Darcy models (1.4) and (1.5) should be used at locations where the obstacles are large enough. The arising of these different regimes (1.2)–(1.5) is consequently a major obstacle towards the development of 'de-homogenization' methods [14, 37, 50, 39, 40] for the optimal design of fluid systems, which would enable to interpret "gray" designs as locally periodic "black and white" microstructures (featuring for instance many small tubes or thin plates).

It turns out that there is a continuous transition between these regimes which can be captured by higher order homogenized equations, which is the object of the present article. These higher order models are obtained by adding corrective terms scaled by increasing powers of ε to the Darcy equation (1.5); they yield more accurate approximations of $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$ when ε is "not so small". For a desired order $K \in \mathbb{N}$, the homogenized model of order $2K+2$ reads

$$(1.6) \qquad \begin{cases} \displaystyle\sum_{k=0}^{2K+2} \varepsilon^{k-2}\mathbb{D}_K^k \cdot \nabla^k \boldsymbol{v}_{\varepsilon,K}^* + \nabla q_{\varepsilon,K}^* = \boldsymbol{f}, \\ \operatorname{div}(\boldsymbol{v}_{\varepsilon,K}^*) = 0, \\ \boldsymbol{v}_{\varepsilon,K}^* \text{ is } D\text{–periodic,} \end{cases}$$

where $(\boldsymbol{v}_{\varepsilon,K}^*, q_{\varepsilon,K}^*)$ is a high order homogenized approximation of $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$. The coefficient $\mathbb{D}_K^k$ is a k-th order *matrix valued* tensor which can be computed by a procedure involving the resolution of cell problems; it makes $\mathbb{D}_K^k \cdot \nabla^k$ a differential operator of order k (the notation is defined in section 2 below). Finally, the high order equation (1.6) encompasses at least the Brinkman and the Darcy regimes in the sense that it converges coefficient-wise to either of (1.3) and (1.4) for the corresponding asymptotic regime of the scaling η (see Remarks 5.6 and 5.7) (the analysis of the subcritical case leading to the Stokes regime (1.2) requires more sophisticated arguments which are to be investigated in future works).

A rather striking feature of (1.6) is the arising of *odd* order differential operators (these vanish, however, in case the obstacle ηT is symmetric with respect to the cell axes; see Corollary 3.16). This fact is closely related to the vectorial nature of the Stokes system (1.1): the tensors $\mathbb{D}_K^k$ are symmetric and antisymmetric valued matrices for respectively even and odd values of k. This property ensures that eventually, $\mathbb{D}_K^k \cdot \nabla^k$ is a symmetric operator for any $0 \leq k \leq 2K+2$ (see Remark 3.12). To our knowledge, such terms have so far not been proposed in the literature seeking similar higher order corrections for the Stokes system, although these have been observed in other vectorial contexts [27, 28, 53]. Most of the available works have focused on situations with low regularity for $\boldsymbol{f}$, T and D (see [52, 5]), where the homogenization process can be justified only for the approximation at the leading order in ε. Error bounds for higher order approximations of $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$ (namely for the truncation of the ansatz (1.7) below) have been obtained in [46, 26], without relating these to effective models. A few additional works have sought corrector terms from physical modelling considerations [35, 18, 17], without considering odd order operators.

Our derivation is inspired from the works [19, 53, 15]; it is based on (non standard) two-scale asymptotic expansions and formal operations on related power series which give rise to several families of tensors and homogenized equations at any order. We extend our previous works [34, 33] where we investigated the cases of the perforated Poisson problem and of the perforated elasticity system. Expectedly, the major difficulty in extending the analysis to (1.6) is the treatment of the pressure variable p_ε and of the incompressibility constraint $\operatorname{div}(\boldsymbol{u}_\varepsilon) = 0$. Note that the D–periodicity assumption on $\boldsymbol{f}$ and $\boldsymbol{u}_\varepsilon$ is made in order to eliminate additional difficulties related to the arising of boundary layers (see [43, 22, 23, 11]).

The starting point of the method of two-scale expansions is to postulate an ansatz for the velocity and pressure solution $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$:

$$(1.7)\quad \boldsymbol{u}_\varepsilon(x) = \sum_{i=0}^{+\infty} \varepsilon^{i+2} \boldsymbol{u}_i(x, x/\varepsilon), \quad p_\varepsilon(x) = \sum_{i=0}^{+\infty} \varepsilon^i (p_i^*(x) + \varepsilon p_i(x, x/\varepsilon)), \quad x \in D_\varepsilon,$$

where the functions $\boldsymbol{u}_i(x,y)$ and $p_i(x,y)$ are P–periodic with respect to $y \in P$, and D–periodic with respect to $x \in D$. In (1.7), the oscillating function $p_i(x,y)$ is required to be of zero average with respect to y:

$$\int_Y p_i(x,y)\mathrm{d}y = 0, \quad \forall i \geq 0.$$

The aim of the homogenization process is to obtain effective equations for the formal "infinite order" homogenized averages $\boldsymbol{u}_\varepsilon^*$ and p_ε^* defined by

$$(1.8)\quad \boldsymbol{u}_\varepsilon^*(x) := \sum_{i=0}^{+\infty} \varepsilon^{i+2} \int_Y \boldsymbol{u}_i(x,y)\mathrm{d}y, \quad p_\varepsilon^*(x) := \sum_{i=0}^{+\infty} \varepsilon^i p_i^*(x), \quad x \in D.$$

In Proposition 3.7 below, we obtain that $(\boldsymbol{u}_\varepsilon^*, p_\varepsilon^*)$ solves the following formal "infinite-order" homogenized equation,

$$(1.9)\quad \begin{cases} \displaystyle\sum_{k=0}^{+\infty} \varepsilon^{k-2} M^k \cdot \nabla^k \boldsymbol{u}_\varepsilon^* + \nabla p_\varepsilon^* = \boldsymbol{f}, \\ \operatorname{div}(\boldsymbol{u}^*) = 0, \\ \boldsymbol{u}_\varepsilon^* \text{ is } D\text{–periodic}, \end{cases}$$

which involves a family of constant matrix-valued tensors $(M^k)_{k\in\mathbb{N}}$. Classically, truncating directly (1.9) yields, in general, an ill-posed model [12]. Several methods have been proposed to address this issue in order to obtain nonetheless well-posed higher order equations [16, 13, 1, 2, 15]. In our case, we adapt an idea from [53], whereby the coefficients $\mathbb{D}_K^k$ are obtained thanks to a minimization principle (described in section 4) which makes indeed (1.6) well-posed. It is based on the existence of remarkable identities which relate the oscillating solution $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$ to its formal average $(\boldsymbol{u}^*_\varepsilon,\, p^*_\varepsilon)$:

$$(1.10)\qquad \begin{cases} \boldsymbol{u}_\varepsilon(x) = \displaystyle\sum_{i=0}^{+\infty} \varepsilon^i N^i(x/\varepsilon)\cdot \nabla^i \boldsymbol{u}^*_\varepsilon(x) \\ p_\varepsilon(x) = p^*_\varepsilon(x) + \displaystyle\sum_{i=0}^{+\infty} \varepsilon^{i-1} \boldsymbol{\beta}^i(x/\varepsilon)\cdot \nabla^i \boldsymbol{u}^*_\varepsilon(x), \end{cases} \qquad \forall x \in D_\varepsilon,$$

where $(N^i(y))_{i\in\mathbb{N}}$ and $(\boldsymbol{\beta}^i(y))_{i\in\mathbb{N}}$ are different families of respectively matrix valued and vector valued P–periodic tensors (of order i). The ansatz (1.10) is substantially different from (1.7); following [15], we call it "criminal" because the expansions of (1.10) depend on $\boldsymbol{u}^*_\varepsilon$ which is itself a formal power series in ε (eqn. (1.8)).

The order of accuracy at which the solution $(\boldsymbol{v}^*_{\varepsilon,K}, p^*_{\varepsilon,K})$ yields an approximation of the original solution $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$ is determined by how many leading coefficients of (1.6) and (1.9) coincide (Proposition 4.5). In Proposition 4.10, we show that $\mathbb{D}^k_K = M^k$ for $0 \le k \le K$, which allows to infer error estimates of order $O(\varepsilon^{K+3})$ in the $L^2(D)$ norm. It may seem disappointing that one needs to solve an equation of order $2K+2$ in order to obtain approximations of order $O(\varepsilon^{K+3})$ "only". This shortcoming is related to the zero-divergence constraint: in the scalar and elasticity cases considered in [34, 33], it turns out that $K+1$ extra coefficients coincide, namely $\mathbb{D}^k_K = M^k$ for $0 \le k \le 2K+1$, which yields error estimates of order $O(\varepsilon^{2K+4})$. In the present context devoted to the Stokes system (1.1), the equation obtained by substituting $\mathbb{D}^k_K$ with M^k in (1.6) for $K+1 \le k \le 2K+1$,

$$(1.11)\qquad \begin{cases} \varepsilon^{2K}\mathbb{D}^{2K+2}_K \cdot \nabla^{2K+2}\widehat{\boldsymbol{v}}^*_{\varepsilon,K} + \displaystyle\sum_{k=0}^{2K+1} \varepsilon^{k-2} M^k \cdot \nabla^k \boldsymbol{v}^*_{\varepsilon,K} + \nabla \widehat{q}^*_{\varepsilon,K} = \boldsymbol{f} \\ \operatorname{div}(\widehat{\boldsymbol{v}}^*_{\varepsilon,K}) = 0 \\ \widehat{\boldsymbol{v}}^*_{\varepsilon,K} \text{ is } D\text{–periodic,} \end{cases}$$

corresponds to applying the truncation method of [53] to the mixed variational formulation rather than to the minimization problem associated with (1.1) (see Remark 4.11). While the minimization principle ensures that (1.6) is well-posed, we do not know whether this is the case for (1.11). However if it is, then Proposition 4.5 implies that (1.11) improves the approximation accuracy up to the order $O(\varepsilon^{2K+4})$.

The article outlines as follows. Notation conventions related to tensors and technical assumptions are exposed in section 2.

In section 3, we introduce cell problems and their solution tensors $(\mathcal{X}^k, \boldsymbol{\alpha}^k)$ which allow to identify the functions $\boldsymbol{u}_i$, p^*_i and p_i in the ansatz (1.7). We show that the formal average $(\boldsymbol{u}^*_\varepsilon, p^*_\varepsilon)$ solves the infinite order homogenized equation (1.9) involving the tensors M^k. After defining the tensors $N^k(y)$ and $\beta^k(y)$, we derive the "criminal" ansatz (1.10) expressing $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$ in terms of p^*_ε and of the derivatives of $\boldsymbol{u}^*_\varepsilon$. Throughout this section, a number of algebraic properties are stated for the various tensors coming at play, such as the symmetry and the antisymmetry of the matrix valued tensors M^k for respectively even and odd values of k, and the simplifications taking place in case the obstacle ηT is symmetric with respect to the cell axes.

Section 4 details the truncation process of the infinite order equation (1.9) leading to the well-posed model (1.6). We then provide an error analysis of the homogenized approximations of $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$ generated by our procedure: our main result is stated in

Corollary 4.15 where we show that the solution $(\boldsymbol{v}^*_{\varepsilon,K}, q^*_{\varepsilon,K})$ of (1.6) yield approximations of $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$ in the $L^2(D_\varepsilon)$ norm of order $K+3$ and $K+1$ for the velocity and the pressure respectively. We establish explicit formulas relating the coefficients $\mathbb{D}^k_K$ to the coefficients M^k and we briefly discuss the improvement provided by (1.11) in case it is well-posed.

The last section 5 investigates asymptotics of the tensors M^k in the low volume fraction limit where the scaling of the obstacle η converges to zero. Our main result is Corollary 5.5 where we obtain the "coefficient-wise" convergence of the infinite order homogenized equation as well as the one of (1.6) towards either of the Brinkman or Darcy regimes (1.4) and (1.5) when η is respectively equivalent or greater than the critical size $\eta_{\text{crit}} \sim \varepsilon^{2/(d-2)}$, and towards the Stokes regime (1.3) for $\eta = o(\varepsilon^{2/(d-2)})$ in the case $K=0$. Although our error estimates for (1.6), are a priori not uniform in η, this suggests that our higher order model (1.6) has the potential to yield valid approximations in any regime of size of holes (at least for $K=0$ or above the critical scale). Note that our analysis is unfortunately unsufficient to establish the convergence of the high order coefficients $\varepsilon^{k-2}M^k$ with $k>2$ towards 0 as $\eta \to 0$. Future works will investigate higher order asymptotics of the tensors M^k in the subcritical regime $\eta = o(\varepsilon^{2/(d-2)})$ which are required to establish or invalidate such a claim.

2. Setting and notation conventions related to tensors.

In the sequel, we consider the following two classical assumptions for the distributions of the holes ω_ε (we recall the schematic of Figure 1), following [5]:

(H1) $Y = P \backslash (\eta T) \subset P$, as a subset of the unit torus (opposite matching faces of $(0,1)^d$ are identified) is a smooth connected set with non-empty interior.

(H2) The fluid component $D_\varepsilon = D \backslash \omega_\varepsilon$ is a smooth connected set.

Remark 2.1. Assumption (H1) does not necessarily imply (H2), see [3] for a counterexample. Assumption (H1) is not very restrictive and can easily be generalized to the case where the subset Y has m connected components with $m \in \mathbb{N}$ (see Appendix 7.5.6 in [33]). Assumption (H2) is stronger, but is also more connected to physical applications. It forbids the existence of isolated fluid inclusions. Most of our derivations only assume (H1). However, we rely on both assumptions (H1) and (H2) in order to obtain error bounds section 4, because we use some technical results of [5].

Below and further on, we consider scalar and vectorial functions such as

$$u \;:\; \begin{array}{rcl} D\times P & \to & \mathbb{R} \\ (x,y) & \mapsto & u(x,y) \end{array}, \qquad \boldsymbol{u} \;:\; \begin{array}{rcl} D\times P & \to & \mathbb{R}^d \\ (x,y) & \mapsto & \boldsymbol{u}(x,y) \end{array} \tag{2.1}$$

which are both D and P–periodic with respect to respectively the first and the second variable, and which vanish on the hole $D\times(\eta T)$. The arguments x and y of $u(x,y)$ are respectively called the "slow" and the "fast" or "oscillating" variable. With a small abuse of notation, the partial derivative with respect to the variable y_j (respectively x_j) is simply written ∂_j instead of ∂_{y_j} (respectively ∂_{x_j}) when the context is clear.

The star–"$*$"– symbol is used to indicate that a quantity is "macroscopic" in the sense that it does not depend on the fast variable x/ε; e.g. $(\boldsymbol{v}^*_{\varepsilon,K}, q^*_{\varepsilon,K})$ or $(\boldsymbol{u}^*_\varepsilon, p^*_\varepsilon)$ in (1.6) and (1.9). In the particular case where a two-variable quantity $u(x,y)$ is given such as (2.1), $u^*(x)$ always denotes the average of $y \mapsto u(x,y)$ with respect to the y variable:

$$u^*(x) := \int_P u(x,y)\mathrm{d}y = \int_Y u(x,y)\mathrm{d}y, \qquad x \in D,$$

where the last equality is a consequence of u vanishing on $P\backslash Y = \overline{\eta T}$. When a function $\mathcal{X} : P \to \mathbb{R}$ depends only on the y variable, we find occasionally more convenient to write its cell average with the usual angle bracket symbols:

$$\langle \mathcal{X} \rangle := \int_P \mathcal{X}(y) \mathrm{d}y.$$

In all what follows, unless otherwise specified, the Einstein summation convention over repeated *subscript* indices is assumed (but never on *superscript* indices). Vectors $\boldsymbol{b} \in \mathbb{R}^d$ are written in bold face notation.

The notation conventions used for tensor related operations are summarized in the nomenclature below. Some of them are not standard; they allow to avoid to systematically write partial derivative indices (e.g. $1 \leq i_1 \dots i_k \leq d$) and to distinguish them from spatial indices (e.g. $1 \leq l, m \leq d$) associated with vector or matrix components.

Scalar, vector, and matrix valued tensors and their coordinates

$\boldsymbol{b}$	Vector of $\mathbb{R}^d$
$(b_j)_{1\leq j\leq d}$	Coordinates of the vector $\boldsymbol{b}$
b^k	Scalar valued tensor of order k ($b^k_{i_1\dots i_k} \in \mathbb{R}$ for $1 \leq i_1, \dots, i_k \leq d$)
$\boldsymbol{b}^k$	Vector valued tensor of order k ($\boldsymbol{b}^k_{i_1\dots i_k} \in \mathbb{R}^d$ for $1 \leq i_1, \dots, i_k \leq d$)
B^k	Matrix valued tensor of order k ($B^k_{i_1\dots i_k} \in \mathbb{R}^{d\times d}$ for $1 \leq i_1, \dots, i_k \leq d$)
$(b^k_j)_{1\leq j\leq d}$	Coordinates of the vector valued tensor $\boldsymbol{b}^k$ (b^k_j is a *scalar* tensor of order k).
$(B^k_{lm})_{1\leq l,m\leq d}$	Coefficients of the matrix valued tensor B^k (B^k_{lm} is a *scalar* tensors of order k).
$b^k_{i_1\dots i_k,j}$	Coefficient of the vector valued tensor $\boldsymbol{b}^k$ ($1 \leq i_1, \dots i_k, j \leq d$)
$B^k_{i_1\dots i_k,lm}$	Coefficients of the matrix valued tensor B^k ($1 \leq i_1, \dots i_k, l, m \leq d$)

Tensor products

$b^p \otimes c^{k-p}$ Tensor product of scalar tensors b^p and c^{k-p}:

$$(b^p \otimes c^{k-p})_{i_1\dots i_k} := b^p_{i_1\dots i_p} c^{k-p}_{i_{p+1}\dots i_k}. \tag{2.2}$$

$a^p \otimes \boldsymbol{b}^{k-p}$ Tensor product of a scalar tensors a^p and a vector valued tensor $\boldsymbol{b}^{k-p}$:

$$(a^p \otimes \boldsymbol{b}^{k-p})_{i_1\dots i_k} := a^p_{i_1\dots i_p} \boldsymbol{b}^{k-p}_{i_{p+1}\dots i_k}. \tag{2.3}$$

$B^p \otimes C^{k-p}$ Tensor product of matrix valued tensors B^p and C^{k-p}:

$$(B^p \otimes C^{k-p})_{i_1\dots i_k,lm} := B^p_{i_1\dots i_p,lj} C^{k-p}_{i_{p+1}\dots i_k,jm}. \tag{2.4}$$

Hence a matrix product is implicitly assumed in the notation $B^p \otimes C^{k-p}$.

$B^p : C^{k-p}$ Tensor product and Frobenius product of matrix tensors B^p and C^{k-p}:

$$(B^p : C^{k-p})_{i_1\dots i_k} := B^p_{i_1\dots i_p,lm} C^{k-p}_{i_{p+1}\dots i_k,lm}. \tag{2.5}$$

$\boldsymbol{b}^p \cdot \boldsymbol{c}^{k-p}$ Tensor product and inner product of vector valued tensors $\boldsymbol{b}^p$ and $\boldsymbol{c}^{k-p}$:

$$(\boldsymbol{b}^p \cdot \boldsymbol{c}^{k-p})_{i_1\dots i_k} := b^p_{i_1\dots i_p,m} c^{k-p}_{i_{p+1}\dots i_k,m}. \tag{2.6}$$

$B^p \cdot \boldsymbol{c}^{k-p}$ — Tensor product of a matrix tensor B^p and a vector tensors $\boldsymbol{c}^{k-p}$:

$$(B^p \cdot \boldsymbol{c}^{k-p})_{i_1\dots i_k,l} := B^p_{i_1\dots i_p,lm} c^{k-p}_{i_{p+1}\dots i_k,m}. \tag{2.7}$$

Hence a matrix-vector product is implicitly assumed in $B^p \cdot \boldsymbol{c}^{k-p}$.

Contraction with partial derivatives

$b^k \cdot \nabla^k$ — Differential operator of order k associated with a scalar tensor b^k:

$$b^k \cdot \nabla^k := b^k_{i_1\dots i_k} \partial^k_{i_1\dots i_k}. \tag{2.8}$$

$\boldsymbol{b}^k \cdot \nabla^k$ — Differential operator of order k associated with a vector tensor $\boldsymbol{b}^k$: for any smooth vector field $\boldsymbol{v} \in \mathcal{C}^\infty_{per}(D, \mathbb{R}^d)$,

$$\boldsymbol{b}^k \cdot \nabla^k \boldsymbol{v} = b^k_{i_1\dots i_k,l} \partial^k_{i_1\dots i_k} v_l. \tag{2.9}$$

$B^k \cdot \nabla^k$ — Differential operator of order k associated with a matrix valued tensor B^k: for any smooth vector field $\boldsymbol{v} \in \mathcal{C}^\infty_{per}(D, \mathbb{R}^d)$,

$$(B^k \cdot \nabla^k \boldsymbol{v})_l = B^k_{i_1\dots i_k,lm} \partial^k_{i_1\dots i_k} v_m. \tag{2.10}$$

Special tensors

$(\boldsymbol{e}_j)_{1\le j\le d}$ — Vectors of the canonical basis of $\mathbb{R}^d$.

e_j — Scalar valued tensor of order 1 given by $e_{j,i_1} := \delta_{i_1 j}$ (with $1 \le j \le d$).

δ_{ij} — Kronecker symbol: $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \ne j$.

I — Identity tensor of order 2:

$$I_{i_1 i_2} = \delta_{i_1 i_2}.$$

The identity tensor is another notation for the Kronecker tensor and it holds $I = e_j \otimes e_j$ with summation on the index $1 \le j \le d$.

J^{2k} — Tensor of order $2k$ defined by:

$$J^{2k} := \overbrace{I \otimes I \otimes \cdots \otimes I}^{k \text{ times}}.$$

With a small abuse of notation, we consider zeroth order tensors b^0 to be constants (i.e. $b^0 \in \mathbb{R}$ if b^0 is scalar) and we still denote by $b^0 \otimes c^k := b^0 c^k$ the tensor product with a k-th order tensor c^k. The same convention also applies to vector valued and matrix valued tensors.

In all what follows, a k-th order tensor b^k (scalar, vector or matrix valued) truly makes sense when contracted with k partial derivatives, as in (2.8)–(2.10). Therefore all the tensors considered throughout this work are identified to their symmetrization:

$$b^k_{i_1\dots i_k} \equiv \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} b_{i_{\sigma(1)}\dots i_{\sigma(k)}},$$

where $\mathfrak{S}_k$ is the permutation group of order k. Consequently, the order in which the (derivative) indices $i_1, \dots, i_k$ are written in $b^k_{i_1\dots i_k}$ does not matter.

Finally, in the whole work, we write C, C_K or $C_K(\boldsymbol{f})$ to denote universal constants that do not depend on ε but whose values may change from lines to lines (and which may depend on η or on the obstacle T).

Remark 2.2. In a limited number of places, the superscript or subscript indices $p, q \in \mathbb{N}$ are used. Naturally, these are not to be confused with the pressure variables p_ε or q_ε introduced in (1.1).

3. Infinite order homogenized equation and criminal ansatz.

We start by identifying the two-scale structure of $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$ which arise in the form of the ansatz (1.7). Because it helps emphasizing the arising of Cauchy products, we assume, *in this section only*, that the right-hand side $\boldsymbol{f}$ can be formally decomposed into a power series in ε:

$$\forall x \in D,\ \boldsymbol{f}(x) = \sum_{i=0}^{+\infty} \varepsilon^i \boldsymbol{f}^i(x). \tag{3.1}$$

3.1. Identification of the "classical" ansatz: tensors $(\mathcal{X}^k, \boldsymbol{\alpha}^k)$.

Inserting (1.7) into the Stokes system (1.1) yields the following cascade of equations:

$$\begin{cases} -\Delta_{yy}\boldsymbol{u}_{i+2} + \nabla_y p_{i+2} = \boldsymbol{f}_{i+2} - \nabla_x p^*_{i+2} - \nabla_x p_{i+1} + \Delta_{xy}\boldsymbol{u}_{i+1} + \Delta_{xx}\boldsymbol{u}_i, \\ \operatorname{div}_y(\boldsymbol{u}_{i+2}) = -\operatorname{div}_x(\boldsymbol{u}_{i+1}), \\ \boldsymbol{u}_{-2} = \boldsymbol{u}_{-1} = 0,\ p_{-1} = 0, \\ \boldsymbol{u}_i(x,\cdot) = 0 \text{ on } \partial(\eta T) \\ \boldsymbol{u}_i(x,\cdot) \text{ is } P\text{–periodic for any } x \in D, \\ \boldsymbol{u}_i(\cdot,y) \text{ is } D\text{–periodic for any } y \in P, \end{cases} \tag{3.2}$$

for any $i \geq -2$, where the operators $-\Delta_{yy}$, $-\Delta_{xy}$, $-\Delta_{yy}$ are defined by

$$-\Delta_{xx} = -\operatorname{div}_x(\nabla_x \cdot), \quad -\Delta_{xy} = -\operatorname{div}_x(\nabla_y \cdot) - \operatorname{div}_y(\nabla_x \cdot), \quad -\Delta_{yy} := -\operatorname{div}_y(\nabla_y \cdot).$$

In order to solve (3.2), we introduce a family of respectively vector valued tensors $(\boldsymbol{\mathcal{X}}^k_j(y))_{1\leq j\leq d}$ and scalar valued tensors $(\alpha^k_j(y))_{1\leq j\leq d}$ defined by induction as the unique solutions in $H^1_{per}(Y,\mathbb{R}^d)\times L^2(Y)/\mathbb{R}$ to the following cell problems:

$$\begin{cases} -\Delta_{yy}\boldsymbol{\mathcal{X}}^0_j + \nabla_y \alpha^0_j = \boldsymbol{e}_j \text{ in } Y, \\ \operatorname{div}_y(\boldsymbol{\mathcal{X}}^0_j) = 0 \text{ in } Y \end{cases} \tag{3.3}$$

$$\begin{cases} -\Delta_{yy}\boldsymbol{\mathcal{X}}^1_j + \nabla_y \alpha^1_j = (2\partial_l \boldsymbol{\mathcal{X}}^0_j - \alpha^0_j \boldsymbol{e}_l) \otimes e_l \text{ in } Y \\ \operatorname{div}_y(\boldsymbol{\mathcal{X}}^1_j) = -(\boldsymbol{\mathcal{X}}^0_j - \langle \boldsymbol{\mathcal{X}}^0_j \rangle)\cdot \boldsymbol{e}_l \otimes e_l \text{ in } Y, \end{cases} \tag{3.4}$$

$$\begin{cases} -\Delta_{yy}\boldsymbol{\mathcal{X}}^{k+2}_j + \nabla_y \alpha^{k+2}_j = (2\partial_l \boldsymbol{\mathcal{X}}^{k+1}_j - \alpha^{k+1}_j \boldsymbol{e}_l) \otimes e_l + \boldsymbol{\mathcal{X}}^k_j \otimes I \text{ in } Y \\ \operatorname{div}_y(\boldsymbol{\mathcal{X}}^{k+2}_j) = -(\boldsymbol{\mathcal{X}}^{k+1}_j - \langle \boldsymbol{\mathcal{X}}^{k+1}_j \rangle)\cdot \boldsymbol{e}_l \otimes e_l \text{ in } Y \end{cases} \quad \forall k \geq 0. \tag{3.5}$$

Equations (3.3)–(3.5) are supplemented with the following boundary conditions:

$$\begin{cases} \displaystyle\int_Y \alpha^k_j \mathrm{d}y = 0 \\ \boldsymbol{\mathcal{X}}^k_j = 0 \text{ on } \partial(\eta T) \\ (\boldsymbol{\mathcal{X}}^k_j, \alpha^k_j) \text{ is } P\text{–periodic} \end{cases} \quad \forall k \geq 0. \tag{3.6}$$

Remark 3.1. In view of the notation conventions of section 2, the non bold symbols $\otimes e_l$ and $\otimes I$ indicate the arising of extra partial derivatives indices. For instance, the first line of (3.5) must be understood as

$$-\Delta_{yy}\boldsymbol{\mathcal{X}}^{k+2}_{j,i_1,\dots i_{k+2}} + \nabla \alpha^{k+2}_{j,i_1\dots i_{k+2}} = 2\partial_{i_{k+2}}\boldsymbol{\mathcal{X}}^{k+1}_{j,i_1\dots i_{k+1}} - \alpha^{k+1}_{j,i_1\dots i_{k+1}}\boldsymbol{e}_{i_{k+2}} + \boldsymbol{\mathcal{X}}^k_{j,i_1\dots i_k}\delta_{i_{k+1}i_{k+2}}.$$

We introduce the k-th order matrix valued tensors $\mathcal{X}^k$ whose columns are the vector valued tensors $(\boldsymbol{\mathcal{X}}_j^k)$:

$$(\mathcal{X}_{ij}^k(y))_{1\le i,j\le d} := \begin{bmatrix}\boldsymbol{\mathcal{X}}_1^k(y) & \dots & \boldsymbol{\mathcal{X}}_d^k(y)\end{bmatrix}, \forall y \in Y, \quad \forall k \ge 0.$$

We also denote by $\boldsymbol{\alpha}^k$ the k-th order vector valued tensor whose coordinates are the scalar tensors α_j^k:

$$\boldsymbol{\alpha}^k(y) := (\alpha_j^k(y))_{1\le j\le d}, \forall y \in Y, \quad \forall k \ge 0.$$

Following the conventions of section 2, we use a star notation to denote the average of respectively the tensor $\mathcal{X}^k$ and of the vector fields $\boldsymbol{u}_i$:

$$\mathcal{X}^{k*} := \int_Y \mathcal{X}^k(y)\mathrm{d}y, \forall k \ge 0, \qquad \boldsymbol{u}_i^*(x) := \int_Y \boldsymbol{u}_i(x,y)\mathrm{d}y, \forall x \in D, \forall i \ge 0. \tag{3.7}$$

The tensors $\mathcal{X}^k$ and $\boldsymbol{\alpha}^k$ enable to solve the cascade of equations (3.2):

PROPOSITION 3.2. *Assume (H1). The solutions $\boldsymbol{u}_i(x,y)$, $p_i(x,y)$ of the cascade of equations (3.2) are given by*

$$\begin{aligned} \boldsymbol{u}_i(x,y) &= \sum_{k=0}^{i} \mathcal{X}^k(y)\cdot\nabla^k(\boldsymbol{f}_{i-k}(x) - \nabla p_{i-k}^*(x)) \\ p_i(x,y) &= \sum_{k=0}^{i} \boldsymbol{\alpha}^k(y)\cdot\nabla^k(\boldsymbol{f}_{i-k}(x) - \nabla p_{i-k}^*(x)), \end{aligned} \tag{3.8}$$

where the functions p_i^* are uniquely determined recursively as the solutions to the following elliptic system: for any $i \ge 0$,

$$\begin{cases} -\mathrm{div}_x(\mathcal{X}^{0*}\nabla_x p_i^*) = -\mathrm{div}_x(\mathcal{X}^{0*}\boldsymbol{f}_i) \\ \qquad\qquad - \displaystyle\sum_{k=1}^{i} \mathrm{div}(\mathcal{X}^{k*}\cdot\nabla^k(\boldsymbol{f}_{i-k} - \nabla_x p_{i-k}^*)) \text{ in } D_\varepsilon, \\ \displaystyle\int_D p_i^* \mathrm{d}x = 0 \\ p_i^* \text{ is } D\text{–periodic.} \end{cases} \tag{3.9}$$

Recognizing Cauchy products, the identities (3.8) and (3.9) rewrite formally in terms of equality of formal power series:

$$\boldsymbol{u}_\varepsilon(x) = \sum_{i=0}^{+\infty} \varepsilon^{i+2}\mathcal{X}^i(x/\varepsilon)\cdot\nabla^i(\boldsymbol{f}(x) - \nabla p_\varepsilon^*(x)), \tag{3.10}$$

$$p_\varepsilon(x) = p_\varepsilon^*(x) + \sum_{i=0}^{+\infty} \varepsilon^{i+1}\boldsymbol{\alpha}^i(x/\varepsilon)\cdot\nabla^i(\boldsymbol{f}(x) - \nabla p_\varepsilon^*(x)), \tag{3.11}$$

$$\mathrm{div}(\boldsymbol{u}_\varepsilon^*(x)) = 0 \text{ where } \boldsymbol{u}_\varepsilon^*(x) = \sum_{i=0}^{+\infty} \varepsilon^{i+2}\mathcal{X}^{i*}\cdot\nabla^i(\boldsymbol{f}(x) - \nabla p_\varepsilon^*(x)). \tag{3.12}$$

Proof. The result is proved by induction. The case $i = -1$ is straightforward thanks to the convention $\boldsymbol{u}_{-1} = p_{-1} = 0$. In this proof we use the short-hand notation $\boldsymbol{h}_i(x) = \boldsymbol{f}_i(x) - \nabla p_i^*(x)$. Assuming (3.8) and (3.9) hold till rank $i+1$ with $i \geq -2$, we compute, substituting (3.8) into (3.2):

$$(3.13)\quad \begin{cases} (-\Delta_{yy}\boldsymbol{u}_{i+2} + \nabla_y p_{i+2})(x,y) \\ \quad = h_{i+2,j}(x)\boldsymbol{e}_j + (2\partial_l \boldsymbol{\mathcal{X}}_j^0(y) - \alpha_j^0(y)\boldsymbol{e}_l) \otimes e_l) \cdot \nabla h_{i+1,j}(x) \\ \quad\quad + \displaystyle\sum_{k=0}^{i}((2\partial_l \boldsymbol{\mathcal{X}}_j^{k+1}(y) - \alpha_j^{k+1}(y)\boldsymbol{e}_l) \otimes e_l + \boldsymbol{\mathcal{X}}_j^k(y) \otimes I) \cdot \nabla^{k+2} h_{i-k,j}(x) \\ \mathrm{div}_y(\boldsymbol{u}_{i+2})(x,y) = -\displaystyle\sum_{k=0}^{i+1}(\boldsymbol{\mathcal{X}}_j^k(y) \cdot \boldsymbol{e}_l \otimes e_l) \cdot \nabla^{k+1} h_{i+1-k,j}(x). \end{cases}$$

The system (3.13) admits a unique solution $(\boldsymbol{u}_{i+2}, p_{i+2})$ with $\int_Y p_{i+2}(x,y)\mathrm{d}y = 0$ if and only if the following compatibility condition (the so-called "Fredholm alternative") holds (for any $i \geq -1$):

$$\int_Y \mathrm{div}_y(\boldsymbol{u}_{i+2})(x,y)\mathrm{d}y = -\sum_{k=0}^{i+1}[\langle \boldsymbol{\mathcal{X}}_j^k \rangle \cdot \boldsymbol{e}_l \otimes e_l] \cdot \nabla^{k+1} h_{i+1-k,j}(x) = 0.$$

The above equation determines p_{i+1}^* given the values of p_k^* for $0 \leq k \leq i$:

$$(\langle \boldsymbol{\mathcal{X}}_j^0 \rangle \cdot \boldsymbol{e}_l)\partial_l(f_{i+1,j} - \partial_j p_{i+1}^*) = -\sum_{k=1}^{i+1}[\langle \boldsymbol{\mathcal{X}}_j^k \rangle \cdot \boldsymbol{e}_l \otimes e_l] \cdot \nabla^{k+1}(f_{i+1-k,j} - \partial_j p_{i+1-k}^*),$$

which is (3.9) at order $i+1$. This identity allows to rewrite $\mathrm{div}_y(\boldsymbol{u}_{i+2})$ as

$$(3.14)\qquad \mathrm{div}_y(\boldsymbol{u}_{i+2})(x,y) = -\sum_{k=0}^{i+1}[(\boldsymbol{\mathcal{X}}_j^k(y) - \langle \boldsymbol{\mathcal{X}}_j^k \rangle) \cdot \boldsymbol{e}_l \otimes e_l] \cdot \nabla^{k+1} h_{i+1-k,j}(x).$$

By linearity, (3.13) and (3.14) and the definitions of $(\boldsymbol{\mathcal{X}}_j^k, \alpha_j^k)$ through the cell problems (3.3)–(3.5) imply the result at rank $i+2$. ∎

Remark 3.3. The truncation of the series (3.12) at first order yields the well-known Darcy's law [52]. The next terms of the series have been obtained in [46, 26], at least up to the order $i = 1$.

Remark 3.4. The ansatz (3.10) is already non standard (when compared to (1.7)) because it features p_ε^* which is a formal power series in ε (recall (1.8)).

The next proposition establishes the symmetry and antisymmetry of the matrices $\mathcal{X}^{k*}$ (eqn. (3.7)) for respectively odd and even values of k. We note that similar identities have been found for the Poisson [34] or the wave equation [1].

Proposition 3.5. *For any $k \geq 0$ and $0 \leq p \leq k$, $1 \leq i,j \leq d$, the following identity holds for the matrix valued tensor $\mathcal{X}^{k*}$:*

$$(3.15)\quad \mathcal{X}_{ij}^{k*} = (-1)^p \int_Y ((-\Delta_{yy}\boldsymbol{\mathcal{X}}_i^p + \nabla\alpha_i^p) \cdot \boldsymbol{\mathcal{X}}_j^{k-p} + \nabla\alpha_j^{k-p} \cdot \boldsymbol{\mathcal{X}}_i^p - \boldsymbol{\mathcal{X}}_j^{k-p-1} \cdot \boldsymbol{\mathcal{X}}_i^{p-1} \otimes I)\mathrm{d}y$$

with $\boldsymbol{\mathcal{X}}_i^{-1} = 0$ by convention. In particular, for any $k \geq 0$, $\mathcal{X}^{2k*}$ and $\mathcal{X}^{2k+1*}$ take values respectively in the set of $d \times d$ symmetric and antisymmetric matrices:

$$(3.16)\quad \mathcal{X}_{ij}^{2k*} = (-1)^k \int_Y (\nabla \boldsymbol{\mathcal{X}}_i^k : \nabla \boldsymbol{\mathcal{X}}_j^k + \nabla\alpha_i^k \cdot \boldsymbol{\mathcal{X}}_j^k + \nabla\alpha_j^k \cdot \boldsymbol{\mathcal{X}}_i^k - \boldsymbol{\mathcal{X}}_i^{k-1} \cdot \boldsymbol{\mathcal{X}}_j^{k-1} \otimes I)\mathrm{d}y$$

$$\mathcal{X}_{ij}^{2k+1*} = (-1)^k \int_Y (\boldsymbol{\mathcal{X}}_i^k \cdot \nabla \boldsymbol{\mathcal{X}}_j^k - \boldsymbol{\mathcal{X}}_j^k \cdot \nabla \boldsymbol{\mathcal{X}}_i^k + \alpha_i^k \boldsymbol{\mathcal{X}}_j^k - \alpha_j^k \boldsymbol{\mathcal{X}}_i^k) \cdot \boldsymbol{e}_l \otimes e_l \mathrm{d}y + (-1)^k \int_Y (\boldsymbol{\mathcal{X}}_j^{k-1} \cdot \boldsymbol{\mathcal{X}}_j^k - \boldsymbol{\mathcal{X}}_i^{k-1} \cdot \boldsymbol{\mathcal{X}}_j^k) \mathrm{d}y. \tag{3.17}$$

Proof. The result holds for $p = 0$ because

$$\mathcal{X}_{ij}^{k*} = \int_Y \boldsymbol{\mathcal{X}}_j^k \cdot \boldsymbol{e}_i \mathrm{d}y = \int_Y \boldsymbol{\mathcal{X}}_j^k \cdot (-\Delta_{yy} \boldsymbol{\mathcal{X}}_i^0 + \nabla \alpha_i^0) \mathrm{d}y.$$

Assuming now that (3.15) holds till rank p with $k > p \geq 0$, we prove the result at rank $p+1$. We write, after an integration by parts and by using (3.3)–(3.5):

$$\begin{aligned}
\mathcal{X}_{ij}^{k*} = (-1)^p \int_Y &[-\boldsymbol{\mathcal{X}}_i^p \cdot \Delta \boldsymbol{\mathcal{X}}_j^{k-p} - \alpha_i^p \mathrm{div}(\boldsymbol{\mathcal{X}}_j^{k-p}) - \alpha_j^{k-p} \mathrm{div}(\boldsymbol{\mathcal{X}}_i^p) \\
&- \boldsymbol{\mathcal{X}}^{k-p-1} \cdot \boldsymbol{\mathcal{X}}_i^{p-1} \otimes I] \mathrm{d}y \\
= (-1)^p \int_Y &\big[(\boldsymbol{\mathcal{X}}_i^p \cdot (2\partial_l \boldsymbol{\mathcal{X}}_j^{k-p-1} - \alpha_j^{k-p-1} \boldsymbol{e}_l) \otimes e_l + \boldsymbol{\mathcal{X}}_j^{k-p-2} \otimes I - \nabla \alpha_j^{k-p}) \cdot \boldsymbol{\mathcal{X}}_i^p \\
&+ \alpha_i^p \boldsymbol{\mathcal{X}}_j^{k-p-1} \cdot \boldsymbol{e}_l \otimes e_l + \alpha_j^{k-p} \boldsymbol{\mathcal{X}}_i^{p-1} \cdot \boldsymbol{e}_l \otimes e_l - \boldsymbol{\mathcal{X}}_j^{k-p-1} \cdot \boldsymbol{\mathcal{X}}_i^{p-1} \otimes I\big] \mathrm{d}y \\
= (-1)^p \int_Y &\big[-\boldsymbol{\mathcal{X}}_j^{k-p-1} \cdot ((2\partial_l \boldsymbol{\mathcal{X}}_i^p - \alpha_i^p \boldsymbol{e}_l) \otimes e_l + \boldsymbol{\mathcal{X}}_i^{p-1} \otimes I) + \alpha_j^{k-p-1} \mathrm{div}(\boldsymbol{\mathcal{X}}_i^{p+1}) \\
&- \nabla \alpha_j^{k-p} \cdot \boldsymbol{\mathcal{X}}_i^p - \alpha_j^{k-p} \mathrm{div}(\boldsymbol{\mathcal{X}}_i^p) + \boldsymbol{\mathcal{X}}_j^{k-p-2} \cdot \boldsymbol{\mathcal{X}}_i^p \otimes I\big] \mathrm{d}y \\
= (-1)^p \int_Y &[-\boldsymbol{\mathcal{X}}_j^{k-p-1} \cdot (-\Delta_{yy} \boldsymbol{\mathcal{X}}_i^{p+1} + \nabla \alpha_i^{p+1}) \\
&- \nabla \alpha_j^{k-p-1} \cdot \boldsymbol{\mathcal{X}}_i^{p+1} + \boldsymbol{\mathcal{X}}_j^{k-p-2} \cdot \boldsymbol{\mathcal{X}}_i^p \otimes I] \mathrm{d}y,
\end{aligned}$$

whence (3.15) at rank $p+1$. Finally, the expression (3.16) for $\mathcal{X}_{ij}^{2k*}$ is obtained by setting $k \leftarrow 2k$ and $p \leftarrow k$ in (3.15). The expression for $\mathcal{X}_{ij}^{2k+1*}$ is obtained by setting $k \leftarrow 2k+1$ and $p \leftarrow k$ and performing an integration by parts. ∎

3.2. Derivation of the infinite order homogenized equation and of the criminal ansatz.

We now proceed on the derivation of the infinite order homogenized equation (1.9). Let us recall the classical positive definiteness of the Darcy tensor $\mathcal{X}^{0*}$.

Corollary 3.6. *Assume (H1). The matrix $\mathcal{X}^{0*} = (\mathcal{X}_{ij}^{0*})_{1 \leq i,j \leq d}$ (defined in (3.7)) is positive symmetric definite.*

Proof. See [52] or Corollary 7.8 in [33]. ∎

Hence, the following definition of the tensors $(M^k)_{k \in \mathbb{N}}$ makes sense.

Proposition 3.7. *Let M^k be the tensor of order k defined by induction as follows:*

$$\begin{cases} M^0 = (\mathcal{X}^{0*})^{-1} \\ M^k = -(\mathcal{X}^{0*})^{-1} \displaystyle\sum_{p=0}^{k-1} \mathcal{X}^{k-p*} \otimes M^p, \quad \forall k \geq 1. \end{cases} \tag{3.18}$$

Then the source terms $\boldsymbol{f}_i$ (eqn. (3.1)) can be expressed in terms of the averaged summands $\boldsymbol{u}_i^*(x)$ and $p_i^*(x)$ ((1.8) and (3.7)) through the following identity:

$$\forall i \geq 0,\ \boldsymbol{f}_i(x) - \nabla p_i^*(x) = \sum_{k=0}^{i} M^k \cdot \nabla^k \boldsymbol{u}_{i-k}^*(x). \tag{3.19}$$

Recognizing a Cauchy product, (3.19) and (3.12) rewrite formally as the "infinite order" homogenized system (1.9) for the formal average $(\boldsymbol{u}_\varepsilon^*, p_\varepsilon^*)$ defined in (1.8).

Proof. The proof is identical to the one of Proposition 5 in [33], it amounts to average the first line of (3.8) with respect to y and to solve the resulting triangular system determining $\boldsymbol{f}_{i-k} - \nabla p_{i-k}^*$ in terms of $\boldsymbol{u}_i^*$. ∎

The definition (3.18) essentially states that $\sum_{k=0}^{+\infty} \varepsilon^{k-2} M^k \cdot \nabla^k$ is the inverse of the formal power series $\sum_{k=0}^{+\infty} \varepsilon^{k+2} \mathcal{X}^{k*} \cdot \nabla^k$. In this spirit, it is even possible to write a fully explicit formula (see [34], Proposition 6 and Remark 2 for the proof):

Proposition 3.8. *For any $k \geq 1$, the tensor M^k is explicitly given by*

$$M^k = \sum_{p=1}^{k} (-1)^p \sum_{\substack{i_1+\cdots+i_p=k \\ 1\leq i_1,\ldots,i_p \leq k}} (\mathcal{X}^{0*})^{-1} \otimes \mathcal{X}^{i_1*} \otimes \cdots \otimes (\mathcal{X}^{0*})^{-1} \otimes \mathcal{X}^{i_p*} \otimes (\mathcal{X}^{0*})^{-1}. \tag{3.20}$$

We now introduce matrix valued tensors N^k and vector valued tensors $\boldsymbol{\beta}^k$ which allow to obtain the "criminal ansatz" (1.10) expressing the velocity and pressure $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$ in terms of their formal average $(\boldsymbol{u}_\varepsilon^*, p_\varepsilon^*)$.

Proposition 3.9. *Let N^k and $\boldsymbol{\beta}^k$ be respectively the $k-$th order matrix valued and vector valued tensors defined for any $k \in \mathbb{N}$ by*

$$N^k(y) := \sum_{p=0}^{k} \mathcal{X}^{k-p}(y) \otimes M^p, \qquad \boldsymbol{\beta}^k(y) := \sum_{p=0}^{k} (-1)^p M^p \cdot \boldsymbol{\alpha}^{k-p}(y), \qquad \forall y \in Y.$$

Then the summands $\boldsymbol{u}_i(x,y)$ and $p_i(x,y)$ of (3.10) and (3.11) are given for any $i \geq 0$ in terms of the averages $\boldsymbol{u}_i^*$ (eqn. (3.7)) and p_i^* as follows:

$$\boldsymbol{u}_i(x,y) = \sum_{k=0}^{i} N^k(y) \cdot \nabla^k \boldsymbol{u}_{i-k}^*(x), \qquad p_i(x,y) = \sum_{k=0}^{i} \boldsymbol{\beta}^k(y) \cdot \nabla^k \boldsymbol{u}_{i-k}^*(x). \tag{3.21}$$

Recognizing Cauchy products, the identities (3.21) can be rewritten formally as the "criminal ansatz" (1.10).

Proof. The result is obtained by substituting (3.19) into (3.8) which yields

$$\begin{aligned}\boldsymbol{u}_i(x,y) &= \sum_{p=0}^{i} \sum_{q=0}^{i-p} \mathcal{X}^p(y) \otimes M^q \cdot \nabla^{p+q} \boldsymbol{u}_{i-p-q}^*(x) \\ &= \sum_{k=0}^{i} \sum_{p=0}^{k} (\mathcal{X}^p(y) \otimes M^{p-k}) \cdot \nabla^k \boldsymbol{u}_{i-k}^*(x) \text{ (change of indices } k = p+q)\end{aligned}$$

from where the identity (3.21) for $\boldsymbol{u}_i(x,y)$ follows by inverting the summation. Similarly, we obtain

$$p_i(x,y) = \sum_{k=0}^{i} \sum_{p=0}^{k} ((M^{p-k})^T \cdot \boldsymbol{\alpha}^p(y)) \cdot \nabla^k \boldsymbol{u}_{i-k}^*(x),$$

hence (3.21) by using $(M^{p-k})^T=(-1)^{p-k}M^{p-k}$ (see Corollary 3.11 below). □

In what follows, we denote by $(\boldsymbol{N}_j^k)_{1\le j\le d}$ and by $(\beta_j^k)_{1\le j\le d}$ respectively the column vectors and the coefficients of $N^k(y)$ and $\boldsymbol{\beta}^k(y)$:

$$\forall 1\le i,j\le d,\ \boldsymbol{N}_j^k:=N^k\boldsymbol{e}_j \text{ and } \beta_j^k:=\boldsymbol{\beta}^k\cdot\boldsymbol{e}_j.$$

In addition, the convention $\boldsymbol{N}_j^{-1}=0$ is assumed. We shall in the sequel use several times the following properties of $(\boldsymbol{N}_j^k,\beta_j^k)$ which are dual to those of $(\boldsymbol{\mathcal{X}}_j^k,\alpha_j^k)$.

Proposition 3.10. *The k-th order tensors N^k, $(\boldsymbol{N}_j^k)_{1\le j\le d}$, $\boldsymbol{\beta}^k$ and $(\beta_j^k)_{1\le j\le d}$ satisfy:*

(i) $\int_Y N^0(y)\mathrm{d}y=I$ and $\int_Y N^k(y)\mathrm{d}y=0$ for any $k\ge 1$;

(ii) $\int_Y \boldsymbol{\beta}^k(y)\mathrm{d}y=0$ for any $k\ge 0$;

(iii) For any $k\ge -2$ and $1\le j\le d$,

$$\begin{cases}-\Delta_{yy}\boldsymbol{N}_j^{k+2}+\nabla\beta_j^{k+2}=(2\partial_l\boldsymbol{N}_j^{k+1}-\beta_j^{k+1}\boldsymbol{e}_l)\otimes e_l+\boldsymbol{N}_j^k\otimes I+M^{k+2}\boldsymbol{e}_j,\\ \operatorname{div}(\boldsymbol{N}_j^{k+2})=-(\boldsymbol{N}_j^{k+1}-\langle\boldsymbol{N}_j^{k+1}\rangle)\cdot\boldsymbol{e}_l\otimes e_l;\end{cases}\tag{3.22}$$

Proof. (i) and (ii) are straightforward consequences of (3.18).

(iii) is obtained by writing, for $k\ge 0$ (implicit summation on the repeated index j assumed):

$$\begin{aligned}-\Delta_{yy}\boldsymbol{N}_j^{k+2}+\nabla\beta_j^{k+2}&=-\Delta_{yy}\left(\sum_{p=0}^{k+2}\boldsymbol{\mathcal{X}}_i^{k+2-p}(y)\otimes M_{ij}^p\right)+\nabla\left(\sum_{p=0}^{k+2}\alpha_i^{k+2-p}(y)\otimes M_{ij}^p\right)\\ &=\sum_{p=0}^{k}\left[(2\partial_l\boldsymbol{\mathcal{X}}_i^{k+1-p}-\alpha_i^{k+1-p}\boldsymbol{e}_l)\otimes e_l+\boldsymbol{\mathcal{X}}_i^{k-p}\otimes I\right]M_{ij}^p\\ &\quad+(2\partial_l\boldsymbol{\mathcal{X}}_i^0-\alpha_i^0\boldsymbol{e}_l)M_{ij}^{k+1}+M_{ij}^{k+2}\boldsymbol{e}_i\\ &=(2\partial_l\boldsymbol{N}_j^{k+1}-\beta_i^{k+1}\boldsymbol{e}_l)\otimes e_l+\boldsymbol{N}_j^k\otimes I+M^{k+2}\boldsymbol{e}_j.\end{aligned}$$

$$\operatorname{div}(\boldsymbol{N}_j^{k+2})=\sum_{p=0}^{k+2}\operatorname{div}(\boldsymbol{\mathcal{X}}_i^{k+2-p})M_{ij}^p=-\sum_{p=0}^{k+1}M_{ij}^p(\boldsymbol{\mathcal{X}}_i^{k+1-p}-<\boldsymbol{\mathcal{X}}_i^{k+1-p}>)\cdot\boldsymbol{e}_l\otimes e_l.$$

The proof is identical for $k=-1$ and $k=-2$. □

The identity (3.22) allows to infer important properties characterizing the tensors M^k which are similar to those of Proposition 3.5.

Corollary 3.11. *For any $1\le p\le k-1$, it holds*

$$M_{ij}^k=(-1)^{p+1}\int_Y((-\Delta_{yy}\boldsymbol{N}_i^p+\nabla\beta_i^p)\cdot\boldsymbol{N}_j^{k-p}+\nabla\beta_j^{k-p}\cdot\boldsymbol{N}_i^p-\boldsymbol{N}_i^{p-1}\cdot\boldsymbol{N}_j^{k-p-1}\otimes I)\mathrm{d}y.$$

Consequently, for any $k\ge 0$,

- *M^{2k} is a symmetric matrix valued tensor, and the following identities hold:*

$$M_{ij}^0=\int_Y\nabla\boldsymbol{N}_i^0:\nabla\boldsymbol{N}_j^0\mathrm{d}y,$$

$$\forall k\ge 1,\ M_{ij}^{2k}=(-1)^{k+1}\int_Y(\nabla\boldsymbol{N}_i^k:\nabla\boldsymbol{N}_j^k+\nabla\beta_i^k\cdot\boldsymbol{N}_j^k+\nabla\beta_j^k\cdot\boldsymbol{N}_i^k-\boldsymbol{N}_i^{k-1}\cdot\boldsymbol{N}_j^{k-1}\otimes I)\mathrm{d}y.$$

- *M^{2k+1} is an antisymmetric matrix valued tensor and it holds:*

$$M_{ij}^{2k+1} = (-1)^{k+1} \int_Y (\boldsymbol{N}_i^k \cdot \nabla \boldsymbol{N}_j^k - \boldsymbol{N}_j^k \cdot \nabla \boldsymbol{N}_i^k + \beta_i^k \boldsymbol{N}_j^k - \beta_j^k \boldsymbol{N}_i^k) \cdot \boldsymbol{e}_l \otimes e_l \mathrm{d}y$$
$$+ (-1)^{k+1} \int_Y (\boldsymbol{N}_j^{k-1} \cdot \boldsymbol{N}_i^k - \boldsymbol{N}_i^{k-1} \cdot \boldsymbol{N}_j^k) \otimes I \mathrm{d}y.$$

Proof. The proof is very similar to the one of Proposition 3.5 and is omitted, see also Proposition 7.34 in [33]. ∎

Remark 3.12. The antisymmetry of odd order tensors M^{2k+1} ensures that the associated differential operators $\varepsilon^{2k-1} M^{2k+1} \cdot \nabla^{2k+1}$ arising in the "infinite order" homogenized equation (1.9) are symmetric. Indeed, the antisymmetry of M^{2k+1} "compensates" the one induced by odd order derivatives which makes $M^{2k+1} \cdot \nabla^{2k+1}$ be a symmetric operator: for two vector fields $\boldsymbol{u} := (u_i)_{1 \le i \le d}, \boldsymbol{v} = (v_i)_{1 \le i \le d}$, it holds

$$\int_Y \boldsymbol{v} \cdot M^{2k+1} \cdot \nabla^{2k+1} \boldsymbol{u} \mathrm{d}y = \int_Y (M_{ij}^{2k+1} \cdot \nabla^{2k+1} u_j) v_i \mathrm{d}y = -\int_Y (M_{ij}^{2k+1} \cdot \nabla^{2k+1} v_i) u_j \mathrm{d}y$$
$$= \int_Y (M_{ji}^{2k+1} \cdot \nabla^{2k+1} v_i) u_j \mathrm{d}y = \int_Y \boldsymbol{u} \cdot M^{2k+1} \cdot \nabla^{2k+1} \boldsymbol{v} \mathrm{d}y.$$

Remark 3.13. It is not completely straightforward to exhibit an instance of hole ∂T and $k \in \mathbb{N}$ for which we can actually prove that M^{2k+1} is not zero. However simple numerical evidences tend to confirm this conjecture, see section 7.4.5 in [33] for an example featuring $M^1 \neq 0$ in the case of the elasticity system .

3.3. Simplifications for the tensors $\mathcal{X}^{k*}$ and M^k in case of symmetries. In the final part of this section, we examine how the symmetries of the obstacle ηT with respect to the cell axes reflect into the coefficients of the matrix valued tensors $\mathcal{X}^{k*}$ and M^k. Our final result is stated in Corollary 3.16, which implies that odd order tensors $\mathcal{X}^{2k+1}$ and M^{2k+1} vanish in case ηT is symmetric with respect to the cell axes. It is based on the following elementary lemma:

LEMMA 3.14. *Let $S \in \mathbb{R}^{d \times d}$ an orthogonal symmetry, i.e. $S = S^T$ and $SS = I$. The following identities hold for any smooth vector field $\boldsymbol{\mathcal{X}}$ and scalar field α:*

$$-\Delta(S\boldsymbol{\mathcal{X}} \circ S) + \nabla(\alpha \circ S) = S(-\Delta \boldsymbol{\mathcal{X}} + \nabla \alpha) \circ S, \tag{3.23}$$
$$\mathrm{div}(S\boldsymbol{\mathcal{X}} \circ S) = \mathrm{div}(\boldsymbol{\mathcal{X}}) \circ S, \tag{3.24}$$
$$\partial_i (S\boldsymbol{\mathcal{X}} \circ S) = S_{ij} S(\partial_j \boldsymbol{\mathcal{X}}) \circ S. \tag{3.25}$$

Proof. The first two identities are obtained by writing

$$-\Delta(S\boldsymbol{\mathcal{X}} \circ S) + \nabla(\alpha \circ S) = -S\partial_{ij} \boldsymbol{\mathcal{X}} \circ S S_{il} S_{jl} + S(\nabla \alpha) \circ S$$
$$= -S(\Delta \boldsymbol{\mathcal{X}} + \nabla \alpha) \circ S,$$

$$\mathrm{div}(S\boldsymbol{\mathcal{X}} \circ S) = \mathrm{Tr}(\nabla(S\boldsymbol{\mathcal{X}} \circ S)) = \mathrm{Tr}(S(\nabla \boldsymbol{\mathcal{X}}) \circ SS) = \mathrm{Tr}((\nabla \boldsymbol{\mathcal{X}}) \circ S) = \mathrm{div}(\boldsymbol{\mathcal{X}}) \circ S.$$

Identity (3.25) is an elementary consequence of the chain rule. ∎

PROPOSITION 3.15. *If the cell $Y = P \backslash (\eta T)$ is invariant with respect to a symmetry S, i.e. $S(Y) = Y$, then the following identity holds for the tensors $(\boldsymbol{\mathcal{X}}_l^k, \alpha_l^k)$ (defined in (3.3)–(3.5)):*

$$S \boldsymbol{\mathcal{X}}_{i_1 \dots i_k, l}^k \circ S = S_{i_1 j_1} \dots S_{i_k j_k} S_{lm} \boldsymbol{\mathcal{X}}_{j_1 \dots j_k, m}^k, \tag{3.26}$$

$$\alpha^k_{i_1\ldots i_k,l}\circ S = S_{i_1j_1}\ldots S_{i_kj_k}S_{lm}\alpha^k_{j_1\ldots j_k,m}, \tag{3.27}$$

with implicit summation over the repeated indices $j_1,\ldots,j_k$ and m. As a consequence, the following identities hold for the constant matrix valued tensors $\mathcal{X}^{k*}$ and M^k:

$$\mathcal{X}^{k*}_{i_1\ldots i_k,lm} = S_{i_1j_1}\ldots S_{i_kj_k}S_{lp}S_{mq}\mathcal{X}^{k*}_{j_1\ldots j_k,pq} \tag{3.28}$$

$$M^k_{i_1\ldots i_k,lm} = S_{i_1j_1}\ldots S_{i_kj_k}S_{lp}S_{mq}M^k_{j_1\ldots j_k,pq}. \tag{3.29}$$

Proof. We prove (3.26) and (3.27) by induction. Applying Proposition 3.15 yields

$$\begin{cases} -\Delta_{yy}(S\boldsymbol{\mathcal{X}}^0_l\circ S)+\nabla_y(\alpha^0_l\circ S) = S\boldsymbol{e}_l\circ S = S\boldsymbol{e}_l = S_{mj}\boldsymbol{e}_m,\\ \qquad\qquad \operatorname{div}(S\boldsymbol{\mathcal{X}}^0_l\circ S)=0.\end{cases}$$

Since the cell is symmetric with respect to S, $(S\boldsymbol{\mathcal{X}}^0_l\circ S,\alpha^0_l\circ S)$ satisfies the same boundary conditions (3.6) than $S_{mj}(\boldsymbol{\mathcal{X}}^0_m,\alpha^0_m)$. Therefore these vector fields are equal and we infer (3.26) and (3.27) at rank $k=0$. We then write, for a given $1\le i_1\le d$:

$$\begin{cases} -\Delta_{yy}(S\boldsymbol{\mathcal{X}}^1_{i_1,l}\circ S)+\nabla_y(\alpha^1_{i_1,l}\circ S) = S(2\partial_{i_1}\boldsymbol{\mathcal{X}}^0_l-\alpha^0_l\boldsymbol{e}_{i_1})\circ S\\ \qquad = S_{i_1j_1}(2\partial_{j_1}(S\boldsymbol{\mathcal{X}}^0_l\circ S)-\alpha^0_l\circ S\boldsymbol{e}_{j_1}) = S_{i_1j_1}S_{lm}(2\partial_{j_1}\boldsymbol{\mathcal{X}}^0_m-\alpha^0_m\boldsymbol{e}_{j_1}),\\ \operatorname{div}_y(S\boldsymbol{\mathcal{X}}^1_{i_1,l}\circ S) = -(\boldsymbol{\mathcal{X}}^0_l\circ S-\langle\boldsymbol{\mathcal{X}}^0_l\rangle)\cdot\boldsymbol{e}_{i_1}\\ \qquad = -S_{lm}S(\boldsymbol{\mathcal{X}}^0_m-\langle\boldsymbol{\mathcal{X}}^0_m\rangle)\cdot\boldsymbol{e}_{i_1} = -S_{i_1j_1}S_{lm}(\boldsymbol{\mathcal{X}}^0_m-\langle\boldsymbol{\mathcal{X}}^0_m\rangle)\cdot\boldsymbol{e}_{j_1},\end{cases}$$

where we have used $\langle\boldsymbol{\mathcal{X}}^0_l\rangle=\langle\boldsymbol{\mathcal{X}}^0_l\circ S\rangle$. This implies similarly (3.26) and (3.27) at rank $k=1$. Assuming now the result holds till rank $k+1$ with $k\ge0$, it holds:

$$\begin{cases} -\Delta_{yy}(S\boldsymbol{\mathcal{X}}^{k+2}_{i_1\ldots i_{k+2},l}\circ S)+\nabla_y(\alpha^{k+2}_{i_1\ldots i_{k+2},l}\circ S)\\ \qquad = S(2\partial_{i_{k+2}}\boldsymbol{\mathcal{X}}^{k+1}_{i_1\ldots i_{k+1},l}-\alpha^{k+1}_{i_1\ldots i_{k+1},l}\boldsymbol{e}_{i_{k+2}})\circ S+S\boldsymbol{\mathcal{X}}^k_{i_1\ldots i_k,l}\circ S\delta_{i_{k+1}i_{k+2}}\\ \qquad = S_{i_{k+2}j_{k+2}}(2\partial_{j_{k+2}}(S\boldsymbol{\mathcal{X}}^{k+1}_{i_1\ldots i_{k+1},l}\circ S)-\alpha^{k+1}_{i_1\ldots i_{k+1},l}\circ S\boldsymbol{e}_{j_{k+2}})\\ \qquad\quad + S_{i_{k+1}j_{k+1}}S_{i_{k+2}j_{k+2}}\delta_{j_{k+1}j_{k+2}}S\boldsymbol{\mathcal{X}}^k_{i_1\ldots i_k,l}\circ S\\ \qquad = S_{i_1j_1}\ldots S_{i_{k+2}i_{k+2}}S_{lm}[(2\partial_{j_{k+2}}\boldsymbol{\mathcal{X}}^{k+1}_{j_1\ldots j_{k+1},m}-\alpha^{k+1}_{j_1\ldots j_{k+1},m}\boldsymbol{e}_{j_{k+2}})\\ \qquad\qquad\qquad\qquad + \delta_{j_{k+1}j_{k+2}}\boldsymbol{\mathcal{X}}^k_{j_1\ldots j_k,m}]\\ \operatorname{div}_y(S\boldsymbol{\mathcal{X}}^{k+2}\circ S) = -(\boldsymbol{\mathcal{X}}^{k+1}_{i_1\ldots i_{k+1},l}\circ S-<\boldsymbol{\mathcal{X}}^{k+1}_{i_1\ldots i_{k+1},l}>)\ldots\boldsymbol{e}_{i_{k+2}}\\ \qquad = -S_{i_1j_1}\ldots S_{i_{k+1}j_{k+1}}S_{lm}S(\boldsymbol{\mathcal{X}}^{k+1}_{j_1\ldots j_{k+1},m}-<\boldsymbol{\mathcal{X}}^{k+1}_{j_1\ldots j_{k+1},m}>)\cdot\boldsymbol{e}_{i_{k+2}}\\ \qquad = -S_{i_1j_1}\ldots S_{i_{k+2}j_{k+2}}S_{lm}(\boldsymbol{\mathcal{X}}^{k+1}_{j_1\ldots j_{k+1},m}-<\boldsymbol{\mathcal{X}}^{k+1}_{j_1\ldots j_{k+1},m}>)\cdot\boldsymbol{e}_{j_{k+2}},\end{cases}$$

hence (3.26) and (3.27) at rank $k+2$. A change of variable then yields:

$$\mathcal{X}^{k*}_{i_1\ldots i_k,lm} = \int_Y \boldsymbol{e}_l\cdot\boldsymbol{\mathcal{X}}^k_{i_1\ldots i_k,m}\mathrm{d}y = \int_Y (S\boldsymbol{e}_l)\cdot(S\boldsymbol{\mathcal{X}}^k_{i_1\ldots i_k,m}\circ S)\mathrm{d}y.$$

This implies (3.28), and then (3.29) by using (3.20). ∎

We apply the above result to two possible families of symmetries:

- for $1\le l\le d$, the symmetry S^l with respect to the cell axis $\boldsymbol{e}_l$:

$$S^l = 1-2\boldsymbol{e}_l\boldsymbol{e}_l^T;$$

- for $1 \leq m, l \leq d$ with $m \neq l$, the symmetry S^{lm} with respect to the diagonal axis $\boldsymbol{e}_l - \boldsymbol{e}_m$:

$$S^{lm} = I - \boldsymbol{e}_l \boldsymbol{e}_l^T - \boldsymbol{e}_m \boldsymbol{e}_m^T + \boldsymbol{e}_l \boldsymbol{e}_m^T + \boldsymbol{e}_l \boldsymbol{e}_m^T + \boldsymbol{e}_m \boldsymbol{e}_l^T.$$

COROLLARY 3.16. 1. *If the cell Y is symmetric with respect to all cell axes $(\boldsymbol{e}_l)_{1\leq l\leq d}$, i.e. $S^l(Y) = Y$ for any $1 \leq l \leq d$, then*

$$\mathcal{X}^{k*}_{i_1\dots i_k,pq} = 0 \text{ and } M^k_{i_1\dots i_k,pq} = 0$$

whenever any given integer $1 \leq l \leq d$ occurs an odd number of times in the indices $i_1 \dots i_k, p, q$. In particular, this implies $\mathcal{X}^{2k+1*} = 0$ and $M^{2k+1} = 0$.

2. *If the cell Y is symmetric with respect to all diagonal axes $\boldsymbol{e}_l - \boldsymbol{e}_m$, i.e. $S^{l,m}(Y) = Y$ for any $1 \leq l < m \leq d$, then for any permutation $\sigma \in \mathfrak{S}_d$,*

$$\mathcal{X}^{k*}_{\sigma(i_1)\dots\sigma(i_k),\sigma(p)\sigma(q)} = \mathcal{X}^{k*}_{i_1\dots i_k,pq} \text{ and } M^k_{\sigma(i_1)\dots\sigma(i_k),\sigma(p)\sigma(q)} = M^k_{i_1\dots i_k,pq}.$$

Proof. The result is obtained by applying (3.28) and (3.29) to the particular symmetries S^l and S^{lm}. See also Corollary 3 in [34]. ∎

Let us illustrate how the previous properties translate for the tensors M^0, M^2 and M^4:

- if the cell Y is symmetric with respect to all cell axes $(\boldsymbol{e}_l)_{1\leq l\leq d}$, only the coefficients of the form $M^0_{i,i}$ are non zero. For M^2, only

$$M^2_{ii,jj}, M^2_{ij,ij}, M^2_{ii,ii}$$

with $i \neq j$ are non zero. For M^4, only the coefficients of the form

$$M^4_{iijj,kk}, M^4_{iijk,jk}, M^4_{iiii,jj}, M^4_{iijj,ii}, M^4_{iiij,ij}, M^4_{iiii,ii}$$

are non zero with distinct integers i, j, k.

- If in addition the obstacle is symmetric with respect to all diagonal axes, then the values of the above coefficients do not depend on the choice of the distinct integers i, j, k. As a result, M^0 is proportional to the identity tensor, M^2 reduces to at most three coefficients (the material is said to be orthotropic), and M^4 reduces to at most 6 coefficients for $d \geq 3$, and to 4 coefficients for $d = 2$. For instance there are three constants α, β, γ such that $M^2 \cdot \nabla$ is the operator

$$M^2 \cdot \nabla \boldsymbol{v} = \alpha \Delta \boldsymbol{v} + \beta \nabla \operatorname{div}(\boldsymbol{v}) + \gamma \sum_{i=1}^{d} \partial_{ii} v_i \boldsymbol{e}_i.$$

4. Homogenized equations of order $2K + 2$: tensors $\mathbb{D}^k_K$.

In this section, we derive the well-posed high order homogenized system (1.6) and we justify the homogenization process by means of quantitative error estimates.

The formal identities (1.10) lead us to introduce, for any order $K \in \mathbb{N}$, the truncated ansatz $\boldsymbol{W}_{\varepsilon,K}(\boldsymbol{v})$ and $Q_{\varepsilon,K}(\boldsymbol{v}, \phi)$ for the reconstructed velocity and pressure:

$$\boldsymbol{W}_{\varepsilon,K}(\boldsymbol{v})(x,y) := \sum_{k=0}^{K} \varepsilon^k N^k(y) \cdot \nabla^k \boldsymbol{v}(x), \qquad x \in D,\ y \in Y \tag{4.1}$$

$$Q_{\varepsilon,K}(\boldsymbol{v},\phi)(x,y) := \phi(x) + \sum_{k=0}^{K} \varepsilon^{k-1} \boldsymbol{\beta}^k(y) \cdot \nabla^k \boldsymbol{v}(x), \qquad x \in D,\ y \in Y \tag{4.2}$$

for any $\boldsymbol{v} \in H^{K+1}(D,\mathbb{R}^d)$ and $\phi \in L^2(D)$ which are sought to approximate the homogenized averages $\boldsymbol{u}_\varepsilon^*$ and p_ε^* respectively. Similarly we denote by $\widetilde{\boldsymbol{W}}_{\varepsilon,K}(\boldsymbol{v})$ and $\widetilde{Q}_{\varepsilon,K}(\boldsymbol{v},\phi)$ the reconstructed *oscillating* functions defined for any $x \in D_\varepsilon$ by

$$\widetilde{\boldsymbol{W}}_{\varepsilon,K}(\boldsymbol{v})(x) := \boldsymbol{W}_{\varepsilon,K}(\boldsymbol{v})(x, x/\varepsilon), \quad \widetilde{Q}_{\varepsilon,K}(\boldsymbol{v},\phi)(x) := Q_{\varepsilon,K}(\boldsymbol{v},\phi)(x,x/\varepsilon). \tag{4.3}$$

Most of the results of this section are consequences of the following observation:

LEMMA 4.1. *For any $K' \in \mathbb{N}$, $(\boldsymbol{v},\phi) \in H^1(D,\mathbb{R}^d)\times L^2(D)$, the reconstructed velocity and pressure $(\widetilde{\boldsymbol{W}}_{\varepsilon,K'}(\boldsymbol{v}), \widetilde{Q}_{\varepsilon,K'}(\boldsymbol{v},\phi))$ of (4.3) satisfy*

$$\begin{aligned} -\Delta\widetilde{\boldsymbol{W}}_{\varepsilon,K'}(\boldsymbol{v}) &+ \nabla\widetilde{Q}_{\varepsilon,K'}(\boldsymbol{v},\phi) \\ &= \sum_{k=0}^{K'} \varepsilon^{k-2} M^k \nabla^k \boldsymbol{v} - \varepsilon^{K'-1}((2\partial_l \boldsymbol{N}_j^{K'} - \beta_j^{K'}\boldsymbol{e}_l)\otimes e_l)(\cdot/\varepsilon)\cdot\nabla^{K'+1}v_j. \\ &\quad + (\boldsymbol{N}_j^{K'-1}\otimes I)(\cdot/\varepsilon)\cdot\nabla^{K'+1}v_j - \varepsilon^{K'}\boldsymbol{N}_j^{K'}(\cdot/\varepsilon)\otimes I\cdot\nabla^{K'+2}v_j, \end{aligned} \tag{4.4}$$

$$\operatorname{div}(\widetilde{\boldsymbol{W}}_{\varepsilon,K'}(\boldsymbol{v})) = \operatorname{div}(\boldsymbol{v}) + \varepsilon^{K'}\boldsymbol{N}_j^{K'}(\cdot/\varepsilon)\cdot\boldsymbol{e}_l\otimes e_l\cdot\nabla^{K'+1}v_j. \tag{4.5}$$

Proof. (4.4) and (4.5) are obtained by applying the Laplace and gradient operators on (4.1) and (4.2) and by using the identity (3.22). ◻

4.1. Sufficient conditions leading to error estimates. The purpose of this part is to demonstrate that a sequence of functions $(\boldsymbol{v}_\varepsilon^*, q_\varepsilon^*)_{\varepsilon>0}$ yields an approximation of $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$ at the order $O(\varepsilon^{K'})$ provided it solves the infinite order homogenized equation (1.9) up to a remainder of order $O(\varepsilon^{K'+1})$. The derivation of a *finite-order* homogenized equation such as (1.6) reduces then to determine $2K+2-K'$ tensors $\mathbb{D}_K^k$ for $K'+1 \le k \le 2K+2$ such that the equation

$$\sum_{k=K'+1}^{2K+2} \varepsilon^{k-2}\mathbb{D}_K^k\cdot\nabla^k\boldsymbol{v}_\varepsilon^* + \sum_{k=0}^{K'}\varepsilon^{k-2}M^k\cdot\nabla^k\boldsymbol{v}_\varepsilon^* + \nabla q_\varepsilon^* = \boldsymbol{f} \tag{4.6}$$

is well-posed. The proof is based on the next three technical results.

LEMMA 4.2. *There exists a constant C independent of $\varepsilon > 0$ such that for any $\boldsymbol{v} \in H^1(D_\varepsilon,\mathbb{R}^d)$ with $\boldsymbol{v} = 0$ on $\partial\omega_\varepsilon$, the following Poincaré inequality holds:*

$$||\boldsymbol{v}||_{L^2(D_\varepsilon,\mathbb{R}^d)} \le C\varepsilon||\nabla\boldsymbol{v}||_{L^2(D,\mathbb{R}^{d\times d})}.$$

Proof. See e.g. [44] or the appendix of [49]. ◻

The next lemma states the existence of a continuous right inverse for the divergence B_ε—so-called a Bogovskii's operator—with a bound explicit in ε on the uniform continuity constant.

LEMMA 4.3. *Assume (H1) and (H2). Then there exists a linear operator $B_\varepsilon : L^2(D_\varepsilon) \to H^1(D_\varepsilon,\mathbb{R}^d)$ satisfying, for any $\phi \in L^2(D_\varepsilon)$ with $\int_{D_\varepsilon}\phi \mathrm{d}x = 0$:*

(i) $\operatorname{div}(B_\varepsilon\phi) = \phi$ *in* D_ε,
(ii) $B_\varepsilon\phi = 0$ *on* $\partial\omega_\varepsilon$ *and* $B_\varepsilon\phi$ *is* D*–periodic,*
(iii) $||\nabla(B_\varepsilon\phi)||_{L^2(D_\varepsilon,\mathbb{R}^{d\times d})} \le C\varepsilon^{-1}||\phi||_{L^2(D_\varepsilon)}$, *for a constant* $C > 0$ *independent of* ϕ *and* ε.

Proof. See [32], Lemma 2.1, or [33], Lemma 7.9. ∎

COROLLARY 4.4. *Assume (H1) and (H2). For any $\boldsymbol{h} \in L^2(D_\varepsilon, \mathbb{R}^d)$ and $g \in L^2(D_\varepsilon)$ satisfying $\int_{D_\varepsilon} g \mathrm{d}x = 0$, let $(\boldsymbol{v}, \phi) \in H^1(D_\varepsilon, \mathbb{R}^d) \times L^2(D_\varepsilon)$ be the unique solution to the Stokes problem*

$$\begin{cases} -\Delta \boldsymbol{v} + \nabla \phi = \boldsymbol{h} \text{ in } D_\varepsilon \\ \operatorname{div}(\boldsymbol{v}) = g \text{ in } D_\varepsilon \\ \displaystyle\int_{D_\varepsilon} \phi \mathrm{d}x = 0 \\ \boldsymbol{v} = 0 \text{ on } \partial\omega_\varepsilon \\ \boldsymbol{v} \text{ is } D\text{–periodic.} \end{cases} \tag{4.7}$$

There exists a constant C independent of ε, $\boldsymbol{h}$ and g such that

$$||\nabla \boldsymbol{v}||_{L^2(D_\varepsilon, \mathbb{R}^{d\times d})} + \varepsilon ||\phi||_{L^2(D_\varepsilon)} \leq C(\varepsilon ||\boldsymbol{h}||_{L^2(D_\varepsilon, \mathbb{R}^d)} + \varepsilon^{-1} ||g||_{L^2(D_\varepsilon)}), \tag{4.8}$$

Proof. We use the operator B_ε of Lemma 4.3 to lift the divergence of $\boldsymbol{v}$. Let us define the vector field $\boldsymbol{w} := \boldsymbol{v} - B_\varepsilon g \in H^1_{per}(D_\varepsilon, \mathbb{R}^d)$ which satisfies

$$\begin{cases} \operatorname{div}(\boldsymbol{w}) = 0 \text{ in } D_\varepsilon, \\ \boldsymbol{w} = 0 \text{ on } \partial\omega_\varepsilon. \end{cases}$$

After an integration by part, we obtain:

$$\begin{aligned} ||\nabla \boldsymbol{w}||^2_{L^2(D_\varepsilon, \mathbb{R}^{d\times d})} &= \int_{D_\varepsilon} \boldsymbol{h} \cdot \boldsymbol{w} \mathrm{d}x - \int_{D_\varepsilon} \nabla(B_\varepsilon g) : \nabla \boldsymbol{w} \mathrm{d}x \\ &\leq ||\boldsymbol{h}||_{L^2(D_\varepsilon, \mathbb{R}^d)} ||\boldsymbol{w}||_{L^2(D_\varepsilon, \mathbb{R}^d)} + ||\nabla(B_\varepsilon g)||_{L^2(D_\varepsilon, \mathbb{R}^{d\times d})} ||\nabla \boldsymbol{w}||_{L^2(D_\varepsilon, \mathbb{R}^{d\times d})} \\ &\leq C(\varepsilon ||\boldsymbol{h}||_{L^2(D_\varepsilon, \mathbb{R}^d)} + ||\nabla(B_\varepsilon g)||_{L^2(D_\varepsilon, \mathbb{R}^{d\times d})}) ||\nabla \boldsymbol{w}||_{L^2(D_\varepsilon, \mathbb{R}^{d\times d})}, \end{aligned}$$

where the last inequality is a consequence of Lemma 4.2. Therefore, simplifying by $||\nabla \boldsymbol{w}||_{L^2(D_\varepsilon, \mathbb{R}^{d\times d})}$ and using the point (iii) of Lemma 4.3 yields

$$\begin{aligned} ||\nabla \boldsymbol{v}||_{L^2(D, \mathbb{R}^{d\times d})} &\leq ||\nabla \boldsymbol{w}||_{L^2(D_\varepsilon, \mathbb{R}^{d\times d})} + ||\nabla(B_\varepsilon g)||_{L^2(D_\varepsilon, \mathbb{R}^{d\times d})} \\ &\leq C(\varepsilon ||\boldsymbol{h}||_{L^2(D_\varepsilon, \mathbb{R}^d)} + \varepsilon^{-1} ||g||_{L^2(D_\varepsilon)}), \end{aligned}$$

which proves the first part of the bound (4.8) on $\nabla \boldsymbol{v}$. The bound on the pressure is then obtained by using $B_\varepsilon \phi$ as a test function: we write

$$\begin{aligned} ||\phi||^2_{L^2(D_\varepsilon)} &= \int_{D_\varepsilon} \phi \operatorname{div}(B_\varepsilon \phi) \mathrm{d}x = -\int_{D_\varepsilon} \nabla \phi \cdot B_\varepsilon \phi \mathrm{d}x \\ &= \int_{D_\varepsilon} (-\Delta \boldsymbol{v} - \boldsymbol{h}) \cdot B_\varepsilon \phi \mathrm{d}x = \int_{D_\varepsilon} (\nabla \boldsymbol{v} \cdot \nabla(B_\varepsilon \phi) - \boldsymbol{h} \cdot B_\varepsilon \phi) \mathrm{d}x \\ &\leq ||\nabla \boldsymbol{v}||_{L^2(D_\varepsilon, \mathbb{R}^{d\times d})} ||\nabla(B_\varepsilon \phi)||_{L^2(D_\varepsilon, \mathbb{R}^{d\times d})} + ||\boldsymbol{h}||_{L^2(D_\varepsilon, \mathbb{R}^d)} ||B_\varepsilon \phi||_{L^2(D_\varepsilon, \mathbb{R}^d)} \\ &\leq C(\varepsilon ||\boldsymbol{h}||_{L^2(D_\varepsilon, \mathbb{R}^d)} + \varepsilon^{-1} ||g||_{L^2(D_\varepsilon)}) ||\nabla(B_\varepsilon \phi)||_{L^2(D_\varepsilon, \mathbb{R}^{d\times d})} \\ &\leq C\varepsilon^{-1}(\varepsilon ||\boldsymbol{h}||_{L^2(D_\varepsilon, \mathbb{R}^d)} + \varepsilon^{-1} ||g||_{L^2(D_\varepsilon)}) ||\phi||_{L^2(D_\varepsilon)}, \end{aligned}$$

which concludes the proof. ∎

We are now in position to state the main result of this section.

Proposition 4.5. *Let $(\boldsymbol{v}_\varepsilon^*, q_\varepsilon^*)_{\varepsilon>0}$ be a sequence of functions of $H^1(D_\varepsilon,\mathbb{R}^d)\times L^2(D_\varepsilon)$, D–periodic, depending on ε (and possibly on K' and $\boldsymbol{f}$) satisfying the following conditions:*

1. *$(\boldsymbol{v}_\varepsilon, q_\varepsilon)$ solves the infinite order homogenized equation (1.9) up to an error of order $O(\varepsilon^{K'+1})$:*

$$\left|\left|\sum_{k=0}^{K'}\varepsilon^{k-2}M^k\cdot\nabla^k\boldsymbol{v}_\varepsilon^*+\nabla q_\varepsilon^*-\boldsymbol{f}\right|\right|_{L^2(D,\mathbb{R}^d)}\leq C_{K'}(\boldsymbol{f})\varepsilon^{K'+1} \tag{4.9}$$

$$\operatorname{div}(\boldsymbol{v}_\varepsilon^*)=0 \text{ in } D \tag{4.10}$$

$$(\boldsymbol{v}_\varepsilon^*, p_\varepsilon^*) \text{ is } D\text{–periodic.} \tag{4.11}$$

2. *For any $m\in\mathbb{N}$, there exists a constant C_m independent of ε such that*

$$||\boldsymbol{v}_\varepsilon^*||_{H^m(D,\mathbb{R}^d)}\leq C_m\varepsilon^2. \tag{4.12}$$

Then the reconstructed functions $(\widetilde{\boldsymbol{W}}_{\varepsilon,K'}(\boldsymbol{v}_\varepsilon^*), \widetilde{Q}_{\varepsilon,K'-1}(q_\varepsilon^*,\boldsymbol{v}_\varepsilon^*))$ (eqn. (4.1) and (4.2)) yield approximations of $(u_\varepsilon, p_\varepsilon)$ of order $O(K'+2)$ in the $H^1(D_\varepsilon,\mathbb{R}^d)$ norm and $O(\varepsilon^{K'+3})$ in the $L^2(D_\varepsilon,\mathbb{R}^d)$ norm:

$$\left|\left|\nabla(\boldsymbol{u}_\varepsilon-\widetilde{\boldsymbol{W}}_{\varepsilon,K'}(\boldsymbol{v}_\varepsilon^*))\right|\right|_{L^2(D_\varepsilon,\mathbb{R}^{d\times d})}+\varepsilon\left|\left|p_\varepsilon-\widetilde{Q}_{\varepsilon,K'-1}(q_\varepsilon^*,\boldsymbol{v}_\varepsilon^*)\right|\right|_{L^2(D_\varepsilon,\mathbb{R}^d)}\leq C_{K'}(\boldsymbol{f})\varepsilon^{K'+2},$$

$$\left|\left|\boldsymbol{u}_\varepsilon-\widetilde{\boldsymbol{W}}_{\varepsilon,K'}(\boldsymbol{v}_\varepsilon^*)\right|\right|_{L^2(D_\varepsilon,\mathbb{R}^d)}\leq C_{K'}(\boldsymbol{f})\varepsilon^{K'+3}.$$

Proof. According to Lemma 4.1 and (4.10), it holds

$$-\Delta\widetilde{\boldsymbol{W}}_{\varepsilon,K'+1}(\boldsymbol{v}_\varepsilon^*)+\nabla\widetilde{Q}_{\varepsilon,K'+1}(q_\varepsilon^*,\boldsymbol{v}_\varepsilon^*)=\sum_{k=0}^{K'}\varepsilon^{k-2}M^k\cdot\nabla^k\boldsymbol{v}_\varepsilon^*+\nabla q_\varepsilon^*+O_{L^2(D_\varepsilon,\mathbb{R}^d)}(\varepsilon^{K'+1})$$

$$\operatorname{div}(\widetilde{\boldsymbol{W}}_{\varepsilon,K'+1})=O_{L^2(D_\varepsilon)}(\varepsilon^{K'+3}),$$

where we have used (4.12) to estimate the right-hand side terms. Applying now Corollary 4.4 to $(\boldsymbol{v},\phi)\equiv(\boldsymbol{u}_\varepsilon-\widetilde{\boldsymbol{W}}_{\varepsilon,K'+1}(\boldsymbol{v}_\varepsilon^*), p_\varepsilon-\widetilde{Q}_{\varepsilon,K'}(q_\varepsilon^*,\boldsymbol{v}_\varepsilon^*))$ yields the error estimate

$$\left|\left|\nabla(\boldsymbol{u}_\varepsilon-\widetilde{\boldsymbol{W}}_{\varepsilon,K'+1}(\boldsymbol{v}_\varepsilon^*))\right|\right|_{L^2(D_\varepsilon,\mathbb{R}^{d\times d})}+\varepsilon\left|\left|p_\varepsilon-\widetilde{Q}_{\varepsilon,K'+1}(q_\varepsilon^*,\boldsymbol{v}_\varepsilon^*)\right|\right|_{L^2(D_\varepsilon)}\leq C_{K'}(\boldsymbol{f})\varepsilon^{K'+2}.$$

Finally, remarking that the highest order terms are already of order $O(\varepsilon^{K'+2})$, i.e.

$$||\nabla(\widetilde{\boldsymbol{W}}_{\varepsilon,K'+1}(\boldsymbol{v}_\varepsilon^*)-\widetilde{\boldsymbol{W}}_{\varepsilon,K'}(\boldsymbol{v}_\varepsilon^*))||_{L^2(D_\varepsilon,\mathbb{R}^d)}\leq C_{K'}\varepsilon^{K'+2},$$

$$\varepsilon||\widetilde{Q}_{\varepsilon,K'+1}(q_\varepsilon^*,\boldsymbol{v}_\varepsilon^*)-\widetilde{Q}_{\varepsilon,K'-1}(q_\varepsilon^*,\boldsymbol{v}_\varepsilon^*)||_{L^2(D_\varepsilon)}\leq C_{K'}\varepsilon^{K'+2},$$

we obtain the result by using the triangle's inequality. ∎

Remark 4.6. We need only $K-1$ derivatives in the truncated criminal ansatz $Q_{\varepsilon,K-1}(\boldsymbol{v}_\varepsilon^*, q_\varepsilon^*)$ for the pressure (eqn. (4.2)), because the term of highest order has a norm of order ε^K while $\boldsymbol{v}_\varepsilon^*$ is of order ε^2 by the assumption (4.12).

Remark 4.7. As a result of the scaling ε^{-1} in Corollary 4.4, we pay a factor ε^{-1} in the error induced by the non zero divergence constraint. However we are able to obtain the right order of ε in the error estimates of Proposition 4.5 thanks to the use of higher order terms of the ansatz (3.21) which are removed at the end of the proof. This strategy is quite classical in the truncation analysis of two-scale expansions, see e.g. [26, 11].

4.2. Construction of a well-posed higher order effective models by a minimization principle. We now derive the well-posed homogenized equation (1.6) of (finite) order $2K+2$ by following the variational method introduced by Smyshlyaev and Cherednychenko in [53] and used in the further works [28, 27, 34]. In the present context, of the Stokes system (1.1), we shall see that (1.6) can be obtained as (4.6) with $K' = K$, which yields error estimates of order $O(\varepsilon^{K+3})$ in the $L^2(D_\varepsilon)$ norm.

Recall that the velocity $\boldsymbol{u}_\varepsilon$ solution to the Stokes system (1.1) is the unique minimizer of the constrained minimization problem

$$\begin{aligned} \boldsymbol{u}_\varepsilon = \arg \min_{\boldsymbol{w}\in H^1(D_\varepsilon,\mathbb{R}^d)} \quad & J(\boldsymbol{w},\boldsymbol{f}) := \int_D \left(\frac{1}{2}\nabla \boldsymbol{w} : \nabla \boldsymbol{w} - \boldsymbol{f}\cdot\boldsymbol{w}\right)\mathrm{d}y \\ \text{s.t.} & \begin{cases} \operatorname{div}(\boldsymbol{w}) = 0 \text{ in } D_\varepsilon \\ \boldsymbol{w} = 0 \text{ on } \partial\omega_\varepsilon \\ \boldsymbol{w} \text{ is } D\text{–periodic.} \end{cases} \end{aligned} \tag{4.13}$$

In the context of the homogenization of linearized elasticity, the main idea of the method of [53] is to restrict (4.13) to functions of the form $\boldsymbol{w} = \widetilde{\boldsymbol{W}}_{\varepsilon,K}(\boldsymbol{v})$ given by (4.1), where $\boldsymbol{v} \in H^{K+1}(D,\mathbb{R}^d)$ is an unknown function sought to approximate $\boldsymbol{u}^*_\varepsilon$. In the present setting, we consider the following approximation of (4.13):

$$\begin{aligned} \min_{\boldsymbol{v}\in H^{K+1}(D,\mathbb{R}^d)} \quad & J(\widetilde{\boldsymbol{W}}_{\varepsilon,K}(\boldsymbol{v}),\boldsymbol{f}) \\ s.t. \quad & \begin{cases} \operatorname{div}(\boldsymbol{v}) = 0 \text{ in } D, \\ \boldsymbol{v} \text{ is } D\text{–periodic.} \end{cases} \end{aligned} \tag{4.14}$$

Note that (4.14) is not exactly the restriction of (4.13) to such functions $\widetilde{\boldsymbol{W}}_{\varepsilon,K}(\boldsymbol{v})$ because $\operatorname{div}(\widetilde{\boldsymbol{W}}_{\varepsilon,K}(\boldsymbol{v})) \neq 0$ (it is of order ε^K, see (4.5)). The next step of the process is to eliminate the oscillating variable x/ε in $J(\widetilde{\boldsymbol{W}}_{\varepsilon,K}(\boldsymbol{v}),\boldsymbol{f})$ so as to obtain an effective energy $J^*_K(\boldsymbol{v},\boldsymbol{f},\varepsilon) \simeq J(\widetilde{\boldsymbol{W}}_{\varepsilon,K}(\boldsymbol{v}),\boldsymbol{f})$ which does not involve oscillating functions. Such is achieved thanks to the classical lemma of two-scale convergence [9].

Lemma 4.8. *Let ϕ be a $P = (0,1)^d$–periodic function and $f \in \mathcal{C}^\infty_{per}(D)$ be a smooth D–periodic function. Then for any $p \in \mathbb{N}$, there exists a constant $C_p(f,\phi)$ independent of ε such that:*

$$\left|\int_D f(x)\phi(x/\varepsilon)\mathrm{d}x - \int_D\int_P f(x)\phi(y)\mathrm{d}y\mathrm{d}x\right| \leq C_p(f,\phi)\varepsilon^p.$$

Proof. See Appendix C. of [53] or Lemma 7.3 in [33]. ☐

Applying Lemma 4.8 to (4.14) in order to pass to the limit in the terms of $J(\widetilde{\boldsymbol{W}}_{\varepsilon,K}(\boldsymbol{v}),\boldsymbol{f})$ which depends on the oscillating variable x/ε, we obtain the existence of a functional J^*_K such that for any $\boldsymbol{v} \in \mathcal{C}^\infty_{per}(D,\mathbb{R}^d)$, it holds

$$J(\widetilde{\boldsymbol{W}}_{\varepsilon,K}(\boldsymbol{v}),\boldsymbol{f}) = J^*_K(\boldsymbol{v},\boldsymbol{f},\varepsilon) + o(\varepsilon^p)$$

with $p \in \mathbb{N}$ arbitrarily large. The functional J^*_K is given explicitly by

$$J^*_K(\boldsymbol{v},\boldsymbol{f},\varepsilon) := \int_D\int_P \frac{1}{2}\left|\left|(\nabla_x + \varepsilon^{-1}\nabla_y)\Big(\boldsymbol{W}_{\varepsilon,K}(\boldsymbol{v})(x,y)\Big)\right|\right|^2 \mathrm{d}y\mathrm{d}x - \int_D \boldsymbol{f}\cdot\boldsymbol{v}\mathrm{d}x, \tag{4.15}$$

where we have used the point 1 of Proposition 3.10 to simplify the linear part of the energy. Replacing $J(\widetilde{\boldsymbol{W}}_{\varepsilon,K}(\boldsymbol{v}),\boldsymbol{f})$ by $J^*_K(\boldsymbol{v},\boldsymbol{f},\varepsilon)$ in (4.14) allows to obtain the homogenized equation (1.6) of order $2K+2$:

DEFINITION 4.9. *For any $K\in\mathbb{N}$, we call homogenized equation of order $2K+2$ associated with the Stokes system (1.1) the Euler-Lagrange equation of the minimization problem*

$$\begin{aligned} \min \quad & J_K^*(\boldsymbol{v},\boldsymbol{f},\varepsilon) \\ s.t. \quad & \begin{cases} \boldsymbol{v}\in H^{K+1}(D,\mathbb{R}^d), \\ \operatorname{div}(\boldsymbol{v})=0 \text{ in } D, \\ \boldsymbol{v} \text{ is } D\text{–periodic.} \end{cases} \end{aligned} \tag{4.16}$$

This Euler-Lagrange equation can be written as (1.6) where the constant (matrix valued) tensors $\mathbb{D}_K^k$ are inferred from (4.15) and where $(\boldsymbol{v}^*_{\varepsilon,K},q^*_{\varepsilon,K})\in H^{K+1}(D,\mathbb{R}^d)\times L^2(D)$ defines the higher order homogenized solution.

The next two propositions verify that (1.6) is indeed a "good" candidate effective model, by relating the coefficients $\mathbb{D}_K^k$ to the tensors M^k (in view of (4.6)), and by establishing the well-posedness of (1.6).

PROPOSITION 4.10. *The coefficients of the matrix valued tensor $\mathbb{D}_K^k$ are explicitly given for any $1\le i,j\le d$ by:*

$$\mathbb{D}_{K,ij}^k = \begin{cases} M^k \text{ if } 0\le k\le K \\ M^k+\mathbb{A}_K^k \text{ if } K+1\le k\le 2K+1 \\ (-1)^{K+1}\displaystyle\int_Y \boldsymbol{N}_i^K\cdot\boldsymbol{N}_j^K\otimes I\mathrm{d}y \text{ if } k=2K+2. \end{cases} \tag{4.17}$$

where the matrix valued tensor $\mathbb{A}_K^k$ is given for any $K+1\le k\le 2K+1$ by

$$\mathbb{A}_{K,ij}^k := (-1)^{K+1}\int_Y (\nabla\beta_j^{k-K-1}\cdot\boldsymbol{N}_i^{K+1}+(-1)^k\nabla\beta_i^{k-K-1}\cdot\boldsymbol{N}_j^{K+1})\mathrm{d}y. \tag{4.18}$$

Proof. Let us denote by V_K the space

$$V_K := \{\boldsymbol{v}\in H^{K+1}(D,\mathbb{R}^d)\,|\,\operatorname{div}(\boldsymbol{v})=0 \text{ and } \boldsymbol{v} \text{ is } D\text{–periodic}\}. \tag{4.19}$$

We identify the coefficients $\mathbb{D}_K^k$ by computing the Euler-Lagrange equation associated with (4.15). For any $(\boldsymbol{v},\boldsymbol{w})\in V_K$, it holds, in a distributional sense:

$$\begin{aligned} &\int_D\int_P (\nabla_x+\varepsilon^{-1}\nabla_y)\boldsymbol{W}_{\varepsilon,K}(\boldsymbol{v}) : (\nabla_x+\varepsilon^{-1}\nabla_y)\boldsymbol{W}_{\varepsilon,K}(\boldsymbol{w})\mathrm{d}x\mathrm{d}y \\ &= \int_D\int_P [(-\Delta_{xx}-\varepsilon^{-1}\Delta_{xy}-\varepsilon^{-2}\Delta_{yy})\boldsymbol{W}_{\varepsilon,K}(\boldsymbol{v}) \\ &\qquad + (\nabla_x+\varepsilon^{-1}\nabla_y)Q_{\varepsilon,K}(\boldsymbol{v},0)]\cdot\boldsymbol{W}_{\varepsilon,K}(\boldsymbol{w})\mathrm{d}x\mathrm{d}y \\ &\quad + \int_D\int_Y Q_{\varepsilon,K}(\boldsymbol{v},0)[(\operatorname{div}_x+\varepsilon^{-1}\operatorname{div}_y)\boldsymbol{W}_{\varepsilon,K}(\boldsymbol{w})]\mathrm{d}x\mathrm{d}y. \end{aligned}$$

By using (4.1) and the point (i) of Proposition 3.10, the above quantity is equal to

$$\begin{aligned} &\int_D\int_Y \left[\sum_{k=0}^K \varepsilon^{k-2}M^k\nabla^k\boldsymbol{v}(x)\right]\cdot\sum_{k=0}^K\varepsilon^k N^k(y)\cdot\nabla^k\boldsymbol{w}(x)\mathrm{d}x\mathrm{d}y \\ &-\int_D\int_Y \varepsilon^{K-1}\left[(2\partial_l\boldsymbol{N}_j^K(y)-\beta_j^K(y)\boldsymbol{e}_l)\otimes e_l)\cdot\nabla^{K+1}v_j(x)\right]\cdot\boldsymbol{W}_{\varepsilon,K}(\boldsymbol{w})\mathrm{d}x\mathrm{d}y \end{aligned}$$

$$
\begin{aligned}
&- \int_D \int_Y [\varepsilon^{K-1} (\boldsymbol{N}_j^{K-1}(y) \otimes I) \cdot \nabla^{K+1} \boldsymbol{v}(x)] \cdot \boldsymbol{W}_{\varepsilon,K}(\boldsymbol{w}) \mathrm{d}x \mathrm{d}y \\
&\quad - \int_D \int_Y [\varepsilon^{K} (\boldsymbol{N}_j^{K}(y) \otimes I) \cdot \nabla^{K+2} \boldsymbol{v}(x)] \cdot \boldsymbol{W}_{\varepsilon,K}(\boldsymbol{w}) \mathrm{d}x \mathrm{d}y \\
&+ \int_D \int_Y \sum_{k=0}^{K} \varepsilon^{k+K-1} (\beta_j^k(y) \cdot \nabla^k v_j(x)) (\boldsymbol{N}_i^K(y) \cdot \boldsymbol{e}_l \otimes e_l \cdot \nabla^{K+1} w_i(x)) \mathrm{d}y \mathrm{d}x \\
= &\int_D \left(\sum_{k=0}^{K} \varepsilon^{k-2} M^k \cdot \nabla^k \boldsymbol{v} + \sum_{k=K+1}^{2K+2} \varepsilon^{k-2} \mathbb{D}_K^k \cdot \nabla^k \boldsymbol{v} \right) \cdot \boldsymbol{w} \mathrm{d}x
\end{aligned}
$$

where we identify (by integration by parts) $\mathbb{D}_{K,ij}^{2K+2} := -(-1)^K \int_Y \boldsymbol{N}_i^K \cdot \boldsymbol{N}_j^K \otimes I \mathrm{d}y$ as claimed. The coefficients of the tensor $\mathbb{D}_K^k$ are given for $K+1 \le k \le 2K+1$ by

$$
\begin{aligned}
\mathbb{D}_{K,ij}^k &= -(-1)^{k-K-1} \int_Y ((2\partial_l \boldsymbol{N}_j^K - \beta_j^K \boldsymbol{e}_l) \otimes e_l + \boldsymbol{N}_j^{K-1} \otimes I) \cdot \boldsymbol{N}_i^{k-K-1} \mathrm{d}y \\
&\quad - (-1)^{k-K-2} \int_Y \boldsymbol{N}_j^K \cdot \boldsymbol{N}_i^{k-K-2} \otimes I \mathrm{d}y + (-1)^{K+1} \int_Y \beta_j^{k-K-1} \boldsymbol{N}_i^K \cdot \boldsymbol{e}_l \otimes e_l \mathrm{d}y \\
&= -(-1)^{K+1} \int_Y (-1)^k (-\Delta_{yy} \boldsymbol{N}_j^{K+1} + \nabla \beta_j^{K+1} - M^{K+1} \boldsymbol{e}_j) \cdot \boldsymbol{N}_i^{k-K-1} \mathrm{d}y \\
&\quad + (-1)^{K+1} \int_Y [(-1)^k \boldsymbol{N}_j^K \cdot \boldsymbol{N}_i^{k-K-2} \otimes I + \beta_j^{k-K-1} \boldsymbol{N}_i^K \cdot \boldsymbol{e}_l \otimes e_l] \mathrm{d}y,
\end{aligned}
$$

where we have used extensively Proposition 3.10. We now distinguish two cases:
1. if $k = K+1$, then the above expression reads

$$
\begin{aligned}
\mathbb{D}_{K,ij}^{K+1} = (M^{K+1} \boldsymbol{e}_j) \cdot \boldsymbol{e}_i - \int_Y \boldsymbol{N}_j^{K+1} \cdot (-\Delta_{yy} \boldsymbol{N}_i^0 + \nabla \beta_i^0) \mathrm{d}y + \int_Y \nabla \beta_i^0 \cdot \boldsymbol{N}_j^{K+1} \\
+ (-1)^{K+1} \int_Y \nabla \beta_j^{k-K-1} \cdot \boldsymbol{N}_i^{K+1} \mathrm{d}y = M_{ij}^{K+1} + \mathbb{A}_{ij}^{K+1},
\end{aligned}
$$

whence the result for $k = K+1$;
2. if $K+2 \le k \le 2K+1$, then we read instead

$$
\begin{aligned}
\mathbb{D}_{K,ij}^k &= (-1)^{k+K} \int_Y ((-\Delta_{yy} \boldsymbol{N}_j^{K+1} + \nabla \beta_j^{K+1}) \cdot \boldsymbol{N}_i^{k-K-1} + \nabla \beta_i^{k-K-1} \cdot \boldsymbol{N}_j^{K+1}) \mathrm{d}y \\
&- (-1)^{k+K} \int_Y \boldsymbol{N}_j^K \cdot \boldsymbol{N}_i^{k-K-2} \otimes I \mathrm{d}y + \mathbb{A}_{ij}^k.
\end{aligned}
$$

Applying finally Corollary 3.11 with $p = K+1$, we obtain that the first two term of the above equation are equal to $(-1)^k M_{ji}^k = M_{ij}^k$, which yields the final result. ∎

Remark 4.11. In view of the proof of Proposition 4.10, it is possible to show that (1.11) (with the same definition for the leading coefficient $\mathbb{D}_K^{2K+2}$) is the strong form of the following "mixed" variational formulation: find $(\widehat{\boldsymbol{v}}_{\varepsilon,K}^*, \widehat{q}_{\varepsilon,K}^*) \in V_K \times L^2(D)$ such that for any $(\boldsymbol{w}, \phi) \in V_K \times L^2(D)$,

$$
\begin{aligned}
(4.20) \quad &\int_D \int_Y \left[(\varepsilon^{-1} \nabla_y + \nabla_x) \boldsymbol{W}_{\varepsilon,K}(\widehat{\boldsymbol{v}}_{\varepsilon,K}^*) \right] : \left[(\varepsilon^{-1} \nabla_y + \nabla_x) \boldsymbol{W}_{\varepsilon,K}(\boldsymbol{w}) \right] \mathrm{d}y \mathrm{d}x \\
&\quad - \int_D \int_Y Q_{\varepsilon,K}(\widehat{\boldsymbol{v}}_{\varepsilon,K}^*, \widehat{q}_{\varepsilon,K}^*) \left[(\varepsilon^{-1} \mathrm{div}_y + \mathrm{div}_x) \boldsymbol{W}_{\varepsilon,K}(\boldsymbol{w}) \right] \mathrm{d}y \mathrm{d}x
\end{aligned}
$$

$$-\int_D\int_Y Q_{\varepsilon,K}(\boldsymbol{w},\phi)\left[(\varepsilon^{-1}\mathrm{div}_y+\mathrm{div}_x)\boldsymbol{W}_{\varepsilon,K}(\widehat{\boldsymbol{v}}^*_{\varepsilon,K})\right]\mathrm{d}y\mathrm{d}x=\int_D \boldsymbol{f}\cdot\boldsymbol{w}\mathrm{d}x.$$

This result is quite surprising. Indeed, (4.20) is built from the truncated ansatz $(\boldsymbol{W}_{\varepsilon,K}(\widehat{\boldsymbol{v}}^*_{\varepsilon,K}), Q_{\varepsilon,K}(\widehat{\boldsymbol{v}}^*_{\varepsilon,K},\widehat{q}^*_{\varepsilon,K}))$ which is expected to yield approximations of $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$ at order $O(\varepsilon^{K+3})$ only (eqn. (4.1) and (4.2)). However, the strong form (1.11) turns out to exhibit $2K+1$ "correct" coefficients M^k. As a result, if (1.11) is well-posed, the reconstructed oscillating functions $\widetilde{\boldsymbol{W}}_{\varepsilon,2K+1}(\widehat{\boldsymbol{v}}^*_{\varepsilon,K})$ and $\widetilde{Q}_{\varepsilon,2K}(\widehat{\boldsymbol{v}}^*_{\varepsilon,K},\widehat{q}^*_{\varepsilon,K})$ approximate $\boldsymbol{u}_\varepsilon$ and p_ε with an error rate as good as $O(\varepsilon^{2K+4})$ in the $L^2(D_\varepsilon)$ norm (Proposition 4.5). This improvement (which had not been noticed in the original paper [53]) actually holds in the context of the Poisson or elasticity equations for which there is no difference between (1.6) and (1.11) (see [34, 33]). Unfortunately in the case of the Stokes system, we do not know whether the mixed formulation (4.20) with the "velocity-dependent" pressure $Q_{\varepsilon,K}(\boldsymbol{w},\phi)$ yields a well-posed problem, hence our commitment to consider (1.6) instead of (1.11).

The leading tensor $\mathbb{D}^{2K+2}_K$ is nonnegative according to (4.17). Under a rather unrestrictive additional non-degeneracy assumption, we obtain that the minimization principle (4.16) makes (1.6) be a well posed problem.

Proposition 4.12. *Assume the dominant tensor $\mathbb{D}^{2K+2}_K=(-1)^{K+1}\mathbb{B}^{K+1,K+1}_K$ is non-degenerate, that is there exists a constant $\nu>0$ such that for any constant vector tensor $\boldsymbol{\xi}^{K+1}=\xi^{K+1}_{i_1\dots i_{K+1},p}\in\mathbb{R}^{d^{K+1}}\times\mathbb{R}^d$ of order $K+1$, it holds*

$$\int_Y [(N^K\otimes e_l)\cdot\boldsymbol{\xi}^{K+1}]\cdot[(N^K\otimes e_l)\cdot\boldsymbol{\xi}^{K+1}]\mathrm{d}y\geq \nu\,\boldsymbol{\xi}^{K+1}_{i_1\dots i_{K+1}}\cdot\boldsymbol{\xi}^{K+1}_{i_1\dots i_{K+1}}. \tag{4.21}$$

Then there exists a unique velocity and pressure couple $(\boldsymbol{v}^*_{\varepsilon,K},q^*_{\varepsilon,K})\in H^{K+1}(D,\mathbb{R}^d)\times L^2(D)/\mathbb{R}$ solving the higher order homogenized equation (1.6).

Proof. The proof relies on the positivity of the quadratic part of the energy $J^*_K(\boldsymbol{v},\boldsymbol{f},\varepsilon)$. By adapting the arguments of the proof of Proposition 12 in [34], we obtain indeed that the bilinear form associated with the energy (4.15) is coercive on the space V_K defined in (4.19). This is enough to apply standard theory for saddle point problems involving the zero divergence constraint (see e.g. the textbooks [55, 54, 38, 31]) which ensures the existence and uniqueness of a solution for (1.6). ∎

Remark 4.13. The assumption (4.21) could fail for $K\geq 1$ in case the obstacle ηT is invariant along some of the directions $\boldsymbol{e}_i$ of the cell P, however it is not restrictive. Indeed, since the leading order tensor $\mathbb{D}^{2K+2}_K$ has no influence on the error estimates of Proposition 4.5, it is always possible to add to $\mathbb{D}^{2K+2}_K$ a small non-negative tensor making it non-degenerate.

4.3. Error estimates: justification of the homogenization process. We conclude this section by stating error estimates holding for the solution $(\boldsymbol{v}^*_{\varepsilon,K},q^*_{\varepsilon,K})$ to the high order homogenized model (1.6). We know from Proposition 4.10 that $\mathbb{D}^k_K=M^k$ for $0\leq k\leq K$, therefore the assumptions of Proposition 4.5 are satisfied provided we verify the uniform regularity estimate (4.12).

Lemma 4.14. *The solution $(\boldsymbol{v}^*_{\varepsilon,K},q^*_{\varepsilon,K})$ of (1.6) is smooth and for any $m\in\mathbb{N}$, there exists a constant $C_m(\boldsymbol{f})$ depending only on m and $\boldsymbol{f}$ such that*

$$||\boldsymbol{v}^*_{\varepsilon,K}||_{H^m(D,\mathbb{R}^d)}\leq C_m(\boldsymbol{f})\varepsilon^2.$$

Proof. This result can be obtained by solving (1.6) explicitly with Fourier series in the periodic domain D and by adapting the proof of Lemma 5 in [34]. ∎

Since we have verified that all the assumptions of Proposition 4.5 hold with $K' = K$, we are finally in position to state the following error bounds.

COROLLARY 4.15. *Let $(\boldsymbol{v}^*_{\varepsilon,K}, q^*_{\varepsilon,K})$ be the unique solution to the high order homogenized equation (1.6). There exists a constant $C_K(\boldsymbol{f})$ independent of ε (but depending on K, $\boldsymbol{f}$, and a priori on the shape of the hole (ηT)) such that the following error estimates hold:*

$$\left|\left|\boldsymbol{u}_\varepsilon - \sum_{k=0}^{K} \varepsilon^k N^k(\cdot/\varepsilon)\cdot\nabla^k \boldsymbol{v}^*_{\varepsilon,K}\right|\right|_{L^2(D_\varepsilon,\mathbb{R}^d)} \leq C_K(\boldsymbol{f})\varepsilon^{K+3},$$

$$\left|\left|\nabla\left(\boldsymbol{u}_\varepsilon - \sum_{k=0}^{K} \varepsilon^k N^k(\cdot/\varepsilon)\cdot\nabla^k \boldsymbol{v}^*_{\varepsilon,K}\right)\right|\right|_{L^2(D_\varepsilon,\mathbb{R}^{d\times d})} \leq C_K(\boldsymbol{f})\varepsilon^{K+2},$$

$$\left|\left|p_\varepsilon - \left(q^*_{\varepsilon,K} + \sum_{k=0}^{K-1} \varepsilon^{k-1}\boldsymbol{\beta}^k(\cdot/\varepsilon)\cdot\nabla^k \boldsymbol{v}^*_{\varepsilon,K}\right)\right|\right|_{L^2(D_\varepsilon)} \leq C_K(\boldsymbol{f})\varepsilon^{K+1}.$$

Remark 4.16. As the reader may expect, error bounds with the same order of convergence hold for the truncation at order K of the "classical" ansatz (1.7), see [26, 46] up to the order $K = 1$, and in Proposition 7.37 of [33] at all orders.

5. Low volume fraction limits when the scaling η of the obstacle vanishes.

In this section, we provide evidences that (1.6) is "well-behaved" in the sense that it has the potential to capture the homogenized regimes (1.2)–(1.4) in the low volume fraction limit where the size of the obstacles vanish. Our results supporting this claim are obtained by analyzing the asymptotics of the tensors $\mathcal{X}^{k*}$, M^k and $\mathbb{D}_K^{2K+2}$ as the scaling ratio η converges to 0.

In this whole subsection, we assume for simplicity, that the space dimension is greater than 3:

$$d \geq 3.$$

Similar results are expected to hold in dimension $d = 2$ but would require a different treatment, as e.g. in [6, 41]. The hole ηT is assumed to be an non-empty open subset strictly included in the unit cell for any $\eta \leq 1$ (it does not touch the boundary): $\eta T \subset\subset P$.

Let us recall the definition of the Deny-Lions (or Beppo-Levi) space denoted by $\mathcal{D}^{1,2}(\mathbb{R}^d\backslash T, \mathbb{R}^d)$ (the reader is referred to [6, 4, 8] and also [48], p.59. for more details).

DEFINITION 5.1 (Deny-Lions space). *The Deny-Lions space $\mathcal{D}^{1,2}(\mathbb{R}^d\backslash T, \mathbb{R}^d)$ is the completion of the space of smooth vector fields by the L^2 norm of their gradients:*

$$\mathcal{D}^{1,2}(\mathbb{R}^d\backslash T, \mathbb{R}^d) := \overline{\mathcal{D}(\mathbb{R}^d\backslash T, \mathbb{R}^d)}^{||\nabla\cdot||_{L^2(\mathbb{R}^d\backslash T,\mathbb{R}^d)}},$$

where $\mathcal{D}(\mathbb{R}^d\backslash T, \mathbb{R}^d)$ is the space of compactly supported smooth vector fields. When $d \geq 3$, it is admits the following characterization:

$$\mathcal{D}^{1,2}(\mathbb{R}^d\backslash T, \mathbb{R}^d) = \{\boldsymbol{v} \text{ measurable } \mid ||\boldsymbol{v}||_{L^{2d/(d-2)}(\mathbb{R}^d\backslash T,\mathbb{R}^d)} < +\infty \text{ and } ||\nabla\boldsymbol{v}||_{L^2(\mathbb{R}^d\backslash T,\mathbb{R}^{d\times d})} < +\infty\}.$$

For any $1 \leq j \leq d$, we consider the unique solution $(\boldsymbol{\Psi}_j, \sigma_j)$ to the exterior Stokes problem

$$
(5.1) \qquad \begin{cases} -\Delta \boldsymbol{\Psi}_j + \nabla \sigma_j = 0 \text{ in } \mathbb{R}^d \backslash T \\ \operatorname{div}(\boldsymbol{\Psi}_j) = 0 \text{ in } \mathbb{R}^d \backslash T \\ \Psi_j = 0 \text{ on } \partial T \\ \boldsymbol{\Psi}_j \to \boldsymbol{e}_j \text{ at } \infty \\ \sigma_j \in L^2(\mathbb{R}^d \backslash T). \end{cases}
$$

The convergence condition $\boldsymbol{\Psi}_j \to \boldsymbol{e}_j$ at infinity must be understood in the sense that $\boldsymbol{\Psi}_j - \boldsymbol{e}_j$ belongs to $\mathcal{D}^{1,2}(\mathbb{R}^d \backslash T, \mathbb{R}^d)$. Similarly, the pressures $(\sigma_j)_{1 \leq j \leq d}$ are uniquely determined by the condition $\sigma_j \in L^2(\mathbb{R}^d \backslash T)$ (see e.g. Lemma 1.1, article V. of [36]). We denote by $F := (F_{ij})_{1 \leq i,j \leq d}$ the matrix collecting the drag force components:

$$
(5.2) \qquad F_{ij} := \int_{\mathbb{R}^d \backslash T} \nabla \boldsymbol{\Psi}_i : \nabla \boldsymbol{\Psi}_j \mathrm{d}x = -\int_{\partial T} \boldsymbol{e}_j \cdot (\nabla \boldsymbol{\Psi}_i - \sigma_i I) \cdot \boldsymbol{n} \mathrm{d}s,
$$

where the normal $\boldsymbol{n}$ is pointing *inward* T.

5.1. Technical estimates in the growing periodic domain $\eta^{-1} P \backslash T$. In all this section, vector fields of the rescaled cell $\eta^{-1} P$ are indicated by a tilde $\widetilde{}$ notation. For a given vector field $\widetilde{\boldsymbol{v}} \in L^2(\eta^{-1} P, \mathbb{R}^d)$, we denote by $\langle \widetilde{\boldsymbol{v}} \rangle$ the average $\langle \widetilde{\boldsymbol{v}} \rangle := \eta^d \int_{\eta^{-1} P} \widetilde{\boldsymbol{v}}(y) \mathrm{d}y$.

Let us recall that for any $\boldsymbol{v} \in H^1(P \backslash (\eta T), \mathbb{R}^d)$, if $\widetilde{\boldsymbol{v}}$ is the rescaled function defined by $\widetilde{\boldsymbol{v}}(y) := v(\eta y)$ in the rescaled cell $\eta^{-1} P \backslash T$, then the L^2 norms of $\boldsymbol{v}$, $\widetilde{\boldsymbol{v}}$ and of their gradients are related by the following identities:

$$
\begin{aligned}
||\boldsymbol{v}||_{L^2(P \backslash (\eta T), \mathbb{R}^d)} &= \eta^{d/2} ||\widetilde{\boldsymbol{v}}||_{L^2(\eta^{-1} P \backslash T, \mathbb{R}^d)} \\
||\nabla \boldsymbol{v}||_{L^2(P \backslash (\eta T), \mathbb{R}^{d \times d})} &= \eta^{d/2 - 1} ||\nabla \widetilde{\boldsymbol{v}}||_{L^2(\eta^{-1} P \backslash T, \mathbb{R}^{d \times d})}.
\end{aligned}
$$

The asymptotic behaviors of the tensors $\mathcal{X}^{k*}, M^k$ are obtained by following the methodology of [6, 41, 34], which relies on several technical results stated in this part.

LEMMA 5.2. *Assume $d \geq 3$. There exists a constant $C > 0$ independent of $\eta > 0$ such that for any $\widetilde{\boldsymbol{v}} \in H^1(\eta^{-1} P \backslash T, \mathbb{R}^d)$ which vanishes on the hole ∂T and which is $\eta^{-1} P$ periodic, the following inequalities hold:*

$$
(5.3) \qquad ||\widetilde{\boldsymbol{v}}||_{L^2(\eta^{-1} P \backslash T, \mathbb{R}^d)} \leq C \eta^{-d/2} ||\nabla \widetilde{\boldsymbol{v}}||_{L^2(\eta^{-1} P \backslash T, \mathbb{R}^{d \times d})},
$$

$$
(5.4) \qquad |\langle \widetilde{\boldsymbol{v}} \rangle| \leq C ||\nabla \widetilde{\boldsymbol{v}}||_{L^2(\eta^{-1} P \backslash T, \mathbb{R}^{d \times d})},
$$

$$
(5.5) \qquad ||\widetilde{\boldsymbol{v}} - \langle \widetilde{\boldsymbol{v}} \rangle||_{L^2(\eta^{-1} P \backslash T, \mathbb{R}^" d)} \leq C \eta^{-1} ||\nabla \widetilde{\boldsymbol{v}}||_{L^2(\eta^{-1} P \backslash T, \mathbb{R}^{d \times d})},
$$

$$
(5.6) \qquad ||\widetilde{\boldsymbol{v}} - \langle \widetilde{\boldsymbol{v}} \rangle||_{L^{2d/(d-2)}(\eta^{-1} P \backslash T, \mathbb{R}^d)} \leq C ||\nabla \widetilde{\boldsymbol{v}}||_{L^2(\eta^{-1} P \backslash T, \mathbb{R}^{d \times d})}.
$$

Proof. See [41, 45, 6, 42]. ∎

LEMMA 5.3. *Consider $\boldsymbol{h} \in L^2(\eta^{-1} P \backslash T, \mathbb{R}^d)$ and $g \in L^2(\eta^{-1} P \backslash T)$ a function satisfying $\int_{\eta^{-1} P \backslash T} g \mathrm{d}x = 0$. Let $(\boldsymbol{v}, \phi) \in H^1(\eta^{-1} P \backslash T, \mathbb{R}^d) \times L^2(\eta^{-1} P \backslash T)$ be the* *unique solution to the following Stokes system:*

$$
(5.7) \qquad \begin{cases} -\Delta \boldsymbol{v} + \nabla \phi = \boldsymbol{h} \ \textit{in} \ \eta^{-1}P\backslash T \\ \quad \operatorname{div}(\boldsymbol{v}) = g \ \textit{in} \ \eta^{-1}P\backslash T \\ \displaystyle\int_{\eta^{-1}P\backslash T} \phi \mathrm{d}x = 0 \\ \quad \boldsymbol{v} = 0 \ \textit{on} \ \partial T \\ \quad \boldsymbol{v} \ \textit{is} \ \eta^{-1}P\textit{–periodic.} \end{cases}
$$

There exists a constant $C > 0$ independent of η, $\boldsymbol{h}$ and g such that

$$
(5.8) \quad ||\nabla \boldsymbol{v}||_{L^2(\eta^{-1}P\backslash T, \mathbb{R}^{d\times d})} + ||\phi||_{L^2(\eta^{-1}P\backslash T)} \\ \leq C(\eta^{-1}||\boldsymbol{h} - \langle \boldsymbol{h}\rangle||_{L^2(\eta^{-1}P\backslash T, \mathbb{R}^d)} + \eta^{-d}|\langle \boldsymbol{h}\rangle| + ||g||_{L^2(\eta^{-1}P\backslash T)}).
$$

Proof. From Lemma 2.2.4 in [7], for any $\eta > 0$, there exists a linear "Bogovskii's" operator $B_\eta : L^2(P\backslash(\eta T)) \to H^1(P\backslash(\eta T), \mathbb{R}^d)$ satisfying for any $\phi \in L^2(P\backslash(\eta Y))$ such that $\int_{P\backslash(\eta T)} \phi \mathrm{d}y = 0$:

(i) $\operatorname{div}(B_\eta \phi) = \phi$,

(ii) $B_\eta \phi = 0$ on $\partial(\eta T)$,

(iii) $B_\eta \phi$ is P–periodic,

(iv) $||\nabla(B_\eta \phi)||_{L^2(P\backslash(\eta T), \mathbb{R}^{d\times d})} \leq C||\phi||_{L^2(P\backslash(\eta T))}$ for a constant C independent of η and ϕ.

For any $\widetilde{\phi} \in L^2(\eta^{-1}P\backslash T)$ such that $\int_{\eta^{-1}P\backslash T} \widetilde{\phi} \mathrm{d}y = 0$, we define

$$
\widetilde{B}_\eta(\widetilde{\phi}) := \eta^{-1}\left[B_\eta(\widetilde{\phi}(\eta^{-1}\cdot))(\eta\cdot)\right].
$$

The operator $\widetilde{B}_\eta : L^2(\eta^{-1}P\backslash T) \to H^1(\eta^{-1}P\backslash T, \mathbb{R}^d)$ satisfies the following properties: for any $\widetilde{\phi} \in L^2(\eta^{-1}P\backslash T)$ such that $\int_{\eta^{-1}P\backslash T} \widetilde{\phi} \mathrm{d}x = 0$,

(i) $\operatorname{div}(\widetilde{B}_\eta \widetilde{\phi}) = \widetilde{\phi}$ in $\eta^{-1}P\backslash T$,

(ii) $\widetilde{B}_\eta \widetilde{\phi} = 0$ on ∂T,

(iii) $\widetilde{B}_\eta \widetilde{\phi}$ is $\eta^{-1}P$–periodic,

(iv) $||\nabla(\widetilde{B}_\eta \widetilde{\phi})||_{L^2(\eta^{-1}P\backslash T, \mathbb{R}^{d\times d})} \leq C||\widetilde{\phi}||_{L^2(\eta^{-1}P\backslash T)}$ for a constant C independent of η and ϕ.

The proof follows then classically along the lines of Corollary 4.4. Upon an integration by parts and by using Lemma 5.2, it is readily obtained with $\boldsymbol{w} := \boldsymbol{v} - \widetilde{B}_\eta g$:

$$
\begin{aligned}
||\nabla \boldsymbol{w}||^2_{L^2(\eta^{-1}P\backslash T, \mathbb{R}^{d\times d})} &= \int_{\eta^{-1}P\backslash T} \boldsymbol{h} \cdot \boldsymbol{w} \mathrm{d}y \\
&= \int_{\eta^{-1}P\backslash T} (\boldsymbol{h} - \langle \boldsymbol{h}\rangle) \cdot (\boldsymbol{w} - \langle \boldsymbol{w}\rangle) \mathrm{d}y + \int_{\eta^{-1}P\backslash T} \langle \boldsymbol{h}\rangle \cdot \langle \boldsymbol{w}\rangle \mathrm{d}y \\
&\leq C\big(||\boldsymbol{h} - \langle \boldsymbol{h}\rangle||_{L^2(\eta^{-1}P\backslash T, \mathbb{R}^d)}||\boldsymbol{w} - \langle \boldsymbol{w}\rangle||_{L^2(\eta^{-1}P\backslash T, \mathbb{R}^d)} + \eta^{-d}|\langle \boldsymbol{h}\rangle|\,|\langle \boldsymbol{w}\rangle|\big) \\
(5.9) \qquad &\leq C\big(\eta^{-1}||\boldsymbol{h} - \langle \boldsymbol{h}\rangle||_{L^2(\eta^{-1}P\backslash T, \mathbb{R}^d)} + \eta^{-d}|\langle \boldsymbol{h}\rangle|\big)||\nabla \boldsymbol{w}||_{L^2(\eta^{-1}P\backslash T, \mathbb{R}^{d\times d})}
\end{aligned}
$$

for a constant $C > 0$ independent of η and $\boldsymbol{h}$. This implies

$$
\begin{aligned}
||\nabla \boldsymbol{v}||_{L^2(\eta^{-1}P\backslash T)} &\leq ||\nabla \boldsymbol{w}||_{L^2(\eta^{-1}P\backslash T, \mathbb{R}^{d\times d})} + ||\nabla(\widetilde{B}_\eta g)||_{L^2(\eta^{-1}P\backslash T, \mathbb{R}^{d\times d})} \\
&\leq C\big(||\nabla \boldsymbol{w}||_{L^2(\eta^{-1}P\backslash T, \mathbb{R}^{d\times d})} + ||g||_{L^2(\eta^{-1}P\backslash T)}\big),
\end{aligned}
$$

whence the bound on $||\nabla \boldsymbol{v}||_{L^2(\eta^{-1}P\backslash T,\mathbb{R}^{d\times d})}$ by using (5.9). The bound for the pressure is obtained by writing

$$\int_{\eta^{-1}P\backslash T}\phi^2\mathrm{d}x = \int_{\eta^{-1}P\backslash T}\phi\,\mathrm{div}(B_\eta\phi)\mathrm{d}x$$
$$= -\int_{\eta^{-1}P\backslash T}\nabla\phi\cdot B_\eta\phi\mathrm{d}x = \int_{\eta^{-1}P\backslash T}(\nabla\boldsymbol{v}:\nabla(B_\eta\phi) - \boldsymbol{h}\cdot(B_\eta\phi))\mathrm{d}x,$$

from where (5.8) follows analogously. ∎

5.2. Asymptotic convergences of homogenized tensors in the low volume fraction limit $\eta\to 0$. The asymptotics of the corrector tensors $(\boldsymbol{\mathcal{X}}_j^0,\alpha_j^0)$ and of $\mathcal{X}^{0*}$ (defined in (3.3) and (3.7)) have been obtained in of Theorem 3.1 in [6]. The following proposition extends this result to the whole family of tensors $(\boldsymbol{\mathcal{X}}^k,\alpha_j^k)_{k\in\mathbb{N}}$ and $(\mathcal{X}^{k*})_{k\in\mathbb{N}}$.

PROPOSITION 5.4. *Assume $d\geq 3$. For any $k\geq 0$ and $1\leq j\leq d$, denote by $(\widetilde{\boldsymbol{\mathcal{X}}}_j^{2k},\widetilde{\alpha}_j^{2k})$ and $(\widetilde{\boldsymbol{\mathcal{X}}}_j^{2k+1},\widetilde{\alpha}_j^{2k+1})$ the rescaled tensors in $\eta^{-1}P\backslash T$ defined by*

$$\begin{cases}\widetilde{\boldsymbol{\mathcal{X}}}_j^{2k}(x) := \eta^{(d-2)(k+1)}\boldsymbol{\mathcal{X}}_j^{2k}(\eta x)\\ \widetilde{\alpha}_j^{2k}(x) := \eta^{(d-2)(k+1)-1}\alpha_j^{2k}(\eta x)\end{cases}\qquad \begin{cases}\widetilde{\boldsymbol{\mathcal{X}}}_j^{2k+1}(x) := \eta^{(d-2)(k+1)}\boldsymbol{\mathcal{X}}_j^{2k+1}(\eta x)\\ \widetilde{\alpha}_j^{2k+1}(x) := \eta^{(d-2)(k+1)-1}\alpha_j^{2k+1}(\eta x)\end{cases}$$

for any $x\in\eta^{-1}P\backslash T$. Then:
1. there exists a constant C independent of $\eta>0$ such that

$$\forall\eta>0,\ ||\nabla\widetilde{\boldsymbol{\mathcal{X}}}_j^{2k}||_{L^2(\eta^{-1}P\backslash T,\mathbb{R}^{d\times d})} + ||\widetilde{\alpha}_j^{2k}||_{L^2(\eta^{-1}P\backslash T)}\leq C,$$

$$\forall\eta>0,\ ||\nabla\widetilde{\boldsymbol{\mathcal{X}}}_j^{2k+1}||_{L^2(\eta^{-1}P\backslash T,\mathbb{R}^{d\times d})} + ||\widetilde{\alpha}_j^{2k+1}||_{L^2(\eta^{-1}P\backslash T)}\leq C;$$

2. the following convergences hold as $\eta\to 0$:

$$(5.10)\qquad (\widetilde{\boldsymbol{\mathcal{X}}}_i^{2k},\widetilde{\alpha}_i^{2k})\rightharpoonup(c_{ij}^{2k}\boldsymbol{\Psi}_j, c_{ij}^{2k}\sigma_j)\quad \textit{weakly in } H^1_{loc}(\mathbb{R}^d\backslash T,\mathbb{R}^d)\times L^2_{loc}(\mathbb{R}^d\backslash T),$$

$$(5.11)\qquad (\widetilde{\boldsymbol{\mathcal{X}}}_i^{2k+1},\widetilde{\alpha}_i^{2k+1})\rightharpoonup(0,0)\quad \textit{weakly in } H^1_{loc}(\mathbb{R}^d\backslash T,\mathbb{R}^d)\times L^2_{loc}(\mathbb{R}^d\backslash T),$$

$$(5.12)\qquad \mathcal{X}^{2k*}\sim\frac{1}{\eta^{(d-2)(k+1)}}c^{2k},$$

$$(5.13)\qquad \mathcal{X}^{2k+1*} = o\left(\frac{1}{\eta^{(d-2)(k+1)}}\right),$$

where c_{ij}^{2k} denotes the coefficients of the matrix valued tensor $c^{2k} := (c_{ij}^{2k})_{1\leq i,j\leq d}$ of order $2k$ given by

$$c^{2k} := F^{-(k+1)}J^{2k}\ \textit{with}\ J^{2k} = \overbrace{I\otimes I\otimes\cdots\otimes I}^{k\ times}.$$

Proof. The result is proved by induction on k.

1. *Case $2k$ with $k=0$.* The tensor $(\widetilde{\boldsymbol{\mathcal{X}}}_i^0,\widetilde{\alpha}_i^0)$ satisfies

$$(5.14)\qquad \begin{cases}-\Delta\widetilde{\boldsymbol{\mathcal{X}}}_i^0 + \nabla\widetilde{\alpha}_i^0 = \eta^d\boldsymbol{e}_i \text{ in } \mathbb{R}^d\backslash T\\ \mathrm{div}(\widetilde{\boldsymbol{\mathcal{X}}}_i^0) = 0 \text{ in } \mathbb{R}^d\backslash T,\end{cases}$$

as well as the other boundary conditions of (5.7). Lemma 5.3 implies then

$$||\nabla\widetilde{\boldsymbol{\mathcal{X}}}_i^0||_{L^2(\eta^{-1}P\backslash T,\mathbb{R}^{d\times d})}+||\widetilde{\alpha}_i^0||_{L^2(\eta^{-1}P\backslash T)}\leq C\eta^{-d}\eta^d|\langle\boldsymbol{e}_i\rangle|\leq C.$$

From (5.4), we also obtain that $\langle\widetilde{\boldsymbol{\mathcal{X}}}_i^0\rangle$ is bounded. Hence, up to extracting a subsequence, there exists a constant matrix $c^0:=(c^0_{ij})_{1\leq i,j\leq d}$, and fields $(\widehat{\boldsymbol{\Psi}}_i^0,\widehat{\sigma}_i^0)_{1\leq i\leq d}$ such that

$$\langle\widetilde{\boldsymbol{\mathcal{X}}}_i^0\rangle\cdot\boldsymbol{e}_j\to c^0_{ij},$$

$$(\widetilde{\boldsymbol{\mathcal{X}}}_i^0,\widetilde{\alpha}_i^0)\rightharpoonup(\widehat{\boldsymbol{\Psi}}_i^0,\widehat{\sigma}_i^0)\text{ weakly in }H^1_{loc}(\eta^{-1}P\backslash T,\mathbb{R}^d)\times L^2_{loc}(\eta^{-1}P\backslash T).$$

Multiplying (5.14) by a compactly supported test function $\boldsymbol{\Phi}\in\mathcal{C}_c^\infty(\mathbb{R}^d\backslash T)$ and integrating by parts yields

$$\int_{\eta^{-1}P\backslash T}(\nabla\widetilde{\boldsymbol{\mathcal{X}}}_i^0:\nabla\boldsymbol{\Phi}-\widetilde{\alpha}_i^0\mathrm{div}(\boldsymbol{\Phi}))\mathrm{d}x=\int_{\eta^{-1}P\backslash T}\eta^d\boldsymbol{\Phi}\cdot\boldsymbol{e}_i\mathrm{d}x.$$

Passing to the limit as $\eta\to0$ implies then

$$\begin{cases}-\Delta\widehat{\boldsymbol{\Psi}}_i^0+\nabla\widehat{\sigma}_i^0=0\text{ in }\mathbb{R}^d\backslash T\\ \mathrm{div}(\boldsymbol{\Psi}_i^0)=0\text{ in }\mathbb{R}^d\backslash T\\ \boldsymbol{\Psi}_i^0=0\text{ on }\partial T.\end{cases}$$

By applying the point (5.6) of Lemma 5.2 and by using the lower semi-continuity of the Lebesgue space norms, we infer $(\widehat{\boldsymbol{\Phi}}_i^0-c^0_{ij}\boldsymbol{e}_j,\widehat{\sigma}_i^0)\in\mathcal{D}^{1,2}(\mathbb{R}^d\backslash T)\times L^2(\mathbb{R}^d\backslash T)$ (see the proof of Theorem 3.1 in [6] for a detailed justification). By linearity, it is then necessary that $(\widehat{\Phi}_i^0,\widehat{\sigma}_i^0)=(c^0_{ij}\boldsymbol{\Psi}_j,c^0_{ij}\sigma_j)$ where $(\boldsymbol{\Psi}_j,\sigma_j)$ are the solution to the exterior problem (5.1). In order to identify the coefficient c^0_{ij}, we integrate (5.1) by parts against the test function $\boldsymbol{\Phi}=\boldsymbol{e}_j$ then yields

$$0=\eta^d\int_{\eta^{-1}P\backslash T}\delta_{ij}\mathrm{d}x+\int_{\partial T}\boldsymbol{e}_j\cdot(\nabla\widetilde{\boldsymbol{\mathcal{X}}}_i^0-\widetilde{\alpha}_i^0I)\cdot\boldsymbol{n}\mathrm{d}x.$$

Passing to the limit as $\eta\to0$ by using the continuity of the drag force with respect to the weak convergence and (5.2) yields then

$$0=\delta_{ij}+\int_{\partial T}\boldsymbol{e}_j\cdot(\nabla\widehat{\boldsymbol{\Phi}}_i^0-\widehat{\sigma}_i^0)\cdot\boldsymbol{n}\mathrm{d}x=\delta_{ij}-c^0_{ip}F_{pj}.$$

This implies $c^0=F^{-1}$ as claimed and the convergence of the whole sequence by uniqueness of the limit. The asymptotic for $\mathcal{X}^{0*}$ as $\eta\to0$ follows by the change of variable $y=\eta x$:

$$\mathcal{X}^{0*}_{ij}=\boldsymbol{e}_i\cdot\int_{P\backslash(\eta T)}\boldsymbol{\mathcal{X}}_j^0\mathrm{d}y=\eta^{2-d}\eta^d\boldsymbol{e}_i\cdot\int_{\eta^{-1}P\backslash T}\widetilde{\boldsymbol{\mathcal{X}}}_j^0\mathrm{d}y\sim\eta^{2-d}\langle\widetilde{\boldsymbol{\mathcal{X}}}_j^0\rangle\cdot\boldsymbol{e}_i\sim\eta^{2-d}c^0_{ji}.$$

2. *Case* $2k+1$ *with* $k=0$. The tensor $(\widetilde{\boldsymbol{\mathcal{X}}}_i^1,\widetilde{\alpha}_i^1)$ satisfies

(5.15)
$$\begin{cases}-\Delta\widetilde{\boldsymbol{\mathcal{X}}}_i^1+\nabla\widetilde{\alpha}_i^1=\eta(2\partial_l\widetilde{\boldsymbol{\mathcal{X}}}_i^0-\widetilde{\alpha}_j^0\boldsymbol{e}_l)\otimes e_l\text{ in }\eta^{-1}P\backslash T\\ \mathrm{div}(\widetilde{\boldsymbol{\mathcal{X}}}_i^1)=-\eta(\widetilde{\boldsymbol{\mathcal{X}}}_j^0-\langle\widetilde{\boldsymbol{\mathcal{X}}}_j^0\rangle)\cdot\boldsymbol{e}_l\otimes e_l\text{ in }\eta^{-1}P\backslash T.\end{cases}$$

Applying Lemma 5.3 and remarking that $\langle 2\partial_l \widetilde{\boldsymbol{\mathcal{X}}}_i^0 - \widetilde{\alpha}_i^0 \boldsymbol{e}_l \rangle = 0$, we obtain

$$||\nabla \widetilde{\boldsymbol{\mathcal{X}}}_i^1||_{L^2(\eta^{-1}P\backslash T,\mathbb{R}^{d\times d})} + ||\widetilde{\alpha}_i^1||_{L^2(\eta^{-1}P\backslash T)} \leq C.$$

Integrating (5.15) by parts against a compactly supported test function $\boldsymbol{\Phi} \in \mathcal{C}^c(\mathbb{R}^d\backslash T)$ and passing to the limit as $\eta \to 0$, we obtain similarly the existence of a matrix valued tensor $c^1 := (c_{ij}^1))_{1\leq i,j\leq d}$ (of order 1) such that, up to the extraction of a subsequence:

$$\langle \widetilde{\boldsymbol{\mathcal{X}}}_i^1 \rangle \cdot \boldsymbol{e}_j \to c_{ij}^1,$$

$$(\widetilde{\boldsymbol{\mathcal{X}}}_i^1, \widetilde{\alpha}_i^1) \rightharpoonup (c_{ij}^1 \Psi_j, c_{ij}^1 \sigma_j) \text{ weakly in } H^1_{loc}(\eta^{-1}P\backslash T, \mathbb{R}^d) \times L^2_{loc}(\eta^{-1}P\backslash T).$$

Integrating (5.15) by parts against the test function $\boldsymbol{e}_j$ and passing to the limit as $\eta \to 0$ yields in this situation $0 = c_{ij}^1 F_{pj}$ whence $c^1 = 0$.

3. *General case.* Assuming that the result holds till rank k, the differential equations satisfied by the rescaled tensors in $\eta^{-1}P\backslash T$ read:

$$(5.16)\qquad \begin{cases} -\Delta \widetilde{\boldsymbol{\mathcal{X}}}_i^{2k+2} + \nabla \widetilde{\alpha}_i^{2k+2} = \eta^{d-1}(\partial_l \widetilde{\boldsymbol{\mathcal{X}}}_i^{2k+1} - \widetilde{\alpha}_i^{2k+1}\boldsymbol{e}_l) \otimes e_l + \eta^d \widetilde{\boldsymbol{\mathcal{X}}}_i^{2k} \otimes I \\ \operatorname{div}(\widetilde{\boldsymbol{\mathcal{X}}}_i^{2k+2}) = -\eta^{d-1}(\widetilde{\boldsymbol{\mathcal{X}}}_i^{2k+1} - \langle \widetilde{\boldsymbol{\mathcal{X}}}_i^{2k+1} \rangle) \cdot \boldsymbol{e}_l \otimes e_l. \end{cases}$$

$$(5.17)\qquad \begin{cases} -\Delta \widetilde{\boldsymbol{\mathcal{X}}}_i^{2k+3} + \nabla \widetilde{\alpha}_i^{2k+3} = \eta(\partial_l \widetilde{\boldsymbol{\mathcal{X}}}_i^{2k+2} - \widetilde{\alpha}_i^{2k+2}\boldsymbol{e}_l) \otimes e_l + \eta^d \widetilde{\boldsymbol{\mathcal{X}}}_i^{2k+1} \otimes I \\ \operatorname{div}(\widetilde{\boldsymbol{\mathcal{X}}}_i^{2k+3}) = -\eta(\widetilde{\boldsymbol{\mathcal{X}}}_i^{2k+2} - \langle \widetilde{\boldsymbol{\mathcal{X}}}_i^{2k+2} \rangle) \cdot \boldsymbol{e}_l \otimes e_l. \end{cases}$$

Using Lemma 5.2, Lemma 5.3 and the point 1. of the proposition at rank k, we readily obtain

$$||\nabla \widetilde{\boldsymbol{\mathcal{X}}}_i^{2k+2}||_{L^2(\eta^{-1}P\backslash T,\mathbb{R}^{d\times d})} + ||\widetilde{\alpha}_i^{2k+2}||_{L^2(\eta^{-1}P\backslash T)} \leq C,$$

$$||\nabla \widetilde{\boldsymbol{\mathcal{X}}}_i^{2k+3}||_{L^2(\eta^{-1}P\backslash T,\mathbb{R}^{d\times d})} + ||\widetilde{\alpha}_i^{2k+3}||_{L^2(\eta^{-1}P\backslash T)} \leq C.$$

Repeating the above arguments, we obtain, up to the extraction of a subsequence, the existence of matrix valued tensors c^{2k+2} and c^{2k+3} such that

$$\langle \widetilde{\boldsymbol{\mathcal{X}}}_i^{2k+2} \rangle \cdot \boldsymbol{e}_j \rightharpoonup c_{ij}^{2k+2} \text{ and } \langle \widetilde{\boldsymbol{\mathcal{X}}}_i^{2k+3} \rangle \cdot \boldsymbol{e}_j \rightharpoonup c_{ij}^{2k+3},$$

$$(\widetilde{\boldsymbol{\mathcal{X}}}_i^{2k+2}, \widetilde{\alpha}_i^{2k+2}) \rightharpoonup (c_{ij}^{2k+2}\boldsymbol{\Psi}_j, c_{ij}^{2k+2}\sigma_j) \text{ weakly in } H^1_{loc}(\mathbb{R}^d\backslash T, \mathbb{R}^d) \times L^2_{loc}(\mathbb{R}^d\backslash T),$$

$$(\widetilde{\boldsymbol{\mathcal{X}}}_i^{2k+3}, \widetilde{\alpha}_i^{2k+3}) \rightharpoonup (c_{ij}^{2k+3}\boldsymbol{\Psi}_j, c_{ij}^{2k+3}\sigma_j) \text{ weakly in } H^1_{loc}(\mathbb{R}^d\backslash T, \mathbb{R}^d) \times L^2_{loc}(\mathbb{R}^d\backslash T).$$

The last step consists in integrating (5.16) and (5.17) by part against the test function $\boldsymbol{e}_j$ and to pass to the limit as $\eta \to 0$ in order to identify c_{ij}^{2k+2} and c_{ij}^{2k+3}. Performing this computation as above yields

$$0 = c_{ij}^{2k} \otimes I - c_{ip}^{2k+2} F_{pj},$$

$$0 = c_{ij}^{2k+1} \otimes I - c_{ip}^{2k+3} F_{pj}$$

from where we infer $c^{2k+2} = c^{2k}F^{-1} \otimes I$, $c^{2k+3} = c^{2k+1}F^{-1} \otimes I$, whence the result (recall $c^1 = 0$ from the point 2. of the proof). ∎

Using the identity of (3.20), we obtain the asymptotics for the coefficients M^k of the infinite order homogenized equation (1.9).

COROLLARY 5.5. *Assume $d \geq 3$. The following convergences hold for the matrix valued tensors M^k as $\eta \to 0$:*

$$M^0 \sim \eta^{d-2} F, \tag{5.18}$$

$$M^1 = o\left(\eta^{d-2}\right), \tag{5.19}$$

$$M^2 \to -I, \tag{5.20}$$

$$\forall k \geq 1,\ M^{2k} = o\left(\frac{1}{\eta^{(d-2)(k-1)}}\right), \tag{5.21}$$

$$\forall k \geq 1,\ M^{2k+1} = o\left(\frac{1}{\eta^{(d-2)(k-1)}}\right). \tag{5.22}$$

Proof. We replace the asymptotics of (5.4) in the explicit formula for the tensors M^k given in (3.20). (5.18) is an immediate consequence of $M^0 = (\mathcal{X}^{0*})^{-1}$. The convergence (5.20) is obtained by writing, according to (3.20):

$$\begin{aligned} M^2 &= -(\mathcal{X}^{0*})^{-1} \otimes \mathcal{X}^{2*} \otimes (\mathcal{X}^{0*})^{-1} + (\mathcal{X}^{0*})^{-1} \otimes \mathcal{X}^{1*} \otimes (\mathcal{X}^{0*})^{-1} \otimes \mathcal{X}^{1*} \otimes (\mathcal{X}^{0*})^{-1} \\ &= -\frac{\eta^{2(d-2)}}{\eta^{2(d-2)}} F \otimes c^2 \otimes F + o\left(\frac{\eta^{3(d-2)}}{\eta^{2(d-2)}}\right) = -(FF^{-2}F) \otimes I + o(\eta^{d-2}) \\ &= -I + o(\eta^{d-2}). \end{aligned}$$

For M^{2k+1} with $k \geq 0$, we use (3.20) and we observe that, for any $0 \leq p \leq 2k+1$ and indices $1 \leq i_1 \dots i_p \leq 2k+1$ such that $i_1 + \dots + i_p = 2k+1$, there exists at least one odd index i_q with $1 \leq q \leq p$. Using (5.12) and (5.13), we arrive at

$$\begin{aligned} &(\mathcal{X}^{0*})^{-1} \otimes \mathcal{X}^{i_1*} \otimes \dots \otimes (\mathcal{X}^{0*})^{-1} \otimes \mathcal{X}^{i_p*} \otimes (\mathcal{X}^{0*})^{-1} \\ &\qquad = o\left(\frac{\eta^{(p+1)(d-2)}}{\eta^{(p+\lfloor i_1/2 \rfloor + \dots + \lfloor i_p/2 \rfloor)(d-2)}}\right) = o\left(\frac{1}{\eta^{(d-2)(k-1)}}\right), \end{aligned}$$

which implies (5.19) and (5.22). For M^{2k} with $k > 1$, we separate the summands of (3.20) into two categories. For a given p indices such that $1 \leq p \leq 2k$ and $i_1 + \dots + i_p = 2k$, there are only two possibilities:
1. either there exists at least one odd index i_q, in that case the above reasoning implies as well

$$(\mathcal{X}^{0*})^{-1} \otimes \mathcal{X}^{i_1*} \otimes \dots \otimes (\mathcal{X}^{0*})^{-1} \mathcal{X}^{i_p*} \otimes (\mathcal{X}^{0*})^{-1} = o\left(\frac{1}{\eta^{(d-2)(k-1)}}\right).$$

2. or all indices $i_1 + \dots + i_p$ are even, in that case we may write, as $\eta \to 0$:

$$\begin{aligned} &(\mathcal{X}^{0*})^{-1} \otimes \mathcal{X}^{i_1*} (\mathcal{X}^{0*})^{-1} \otimes \dots \otimes (\mathcal{X}^{0*})^{-1} \mathcal{X}^{i_p*} \otimes (\mathcal{X}^{0*})^{-1} \\ &\qquad \sim \frac{\eta^{(d-2)(p+1)}}{\eta^{(d-2)(p+k)}} F \otimes c^{i_1} \otimes \dots \otimes F \otimes c^{i_p} \otimes F \\ &\qquad \sim \frac{1}{\eta^{(d-2)(k-1)}} (FF^{-(i_1/2+1)}) \times \dots \times (FF^{-(i_p/2+1)}) F J^{2k} \\ &\qquad \sim \frac{1}{\eta^{(d-2)(k-1)}} F^{-(k-1)} \otimes J^{2k}. \end{aligned}$$

Note that in the latter case, the asymptotic does not depend on the choice of indices $i_1+\cdots+i_p=2k$. Therefore, by isolating the terms featuring only even indices $i_1:=2j_1,\dots,i_p:=2j_p$ in (3.20), we obtain finally

$$M^{2k}=\frac{1}{\eta^{(d-2)(k-1)}}F^{-(k-1)}\otimes J^{2k}\left(\sum_{p=1}^{2k}(-1)^p\sum_{\substack{2j_1+\dots+2j_p=2k\\1\leq j_1,\dots,j_p\leq k}}1\right)+o\left(\frac{1}{\eta^{(d-2)(k-1)}}\right).$$

The asymptotic (5.21) follows from the fact that the summation over p in the above expression is zero (see e.g. the end of the proof of Corollary 6 in [34]). ◻

Remark 5.6. We have therefore obtained the following asymptotic estimates for the coefficients $\varepsilon^{k-2}M^k$ of the infinite order homogenized equation (1.9) as $\eta\to 0$:

$$\begin{aligned}\varepsilon^{-2}M^0&\sim(\eta^{d-2}/\varepsilon^2)F,\\ \varepsilon^{-1}M^1&=o\left(\varepsilon(\eta^{d-2}/\varepsilon^2)\right),\\ \varepsilon^0M^2&\to -I,\\ \varepsilon^{2k-2}M^{2k}&=o\left(\left(\varepsilon^2/\eta^{d-2}\right)^{k-1}\right)\text{ for }k\geq 1,\\ \varepsilon^{2k-1}M^{2k+1}&=o\left(\varepsilon\left(\varepsilon^2/\eta^{d-2}\right)^{k-1}\right)\text{ for }k\geq 1.\end{aligned}$$

These asymptotics bring into play the ratio ε^2/η^{d-2} and so the critical scaling $\eta_{\text{crit}}\sim\varepsilon^{2/(d-2)}$. They imply thus the"coefficient-wise" convergence of (1.9) to the Brinkman regime (1.3) at the critical rate $\eta\sim\varepsilon^{2/(d-2)}$, in which case $\varepsilon^{-2}M^0\to F$ and $\varepsilon^{k-2}M^k\to 0$ for any $k>2$. Note that $\varepsilon^0M^2\to -I$ whatever the rate of convergence at which $\eta\to 0$. The Darcy regimes (1.4) and (1.5) correspond to the situation where $\eta^{d-2}/\varepsilon^2\to+\infty$; in that case the zeroth order term $\varepsilon^{-2}M^0$ is dominant.

Finally, the leading coefficients of the Stokes regime (1.2) are retrieved for $\eta=o(\varepsilon^{2/(d-2)})$, since in this case, $\varepsilon^{-2}M^0\to 0$, $\varepsilon^{-1}M^1\to 0$ and $\varepsilon^0M^0\to -I$. However the present analysis is not sufficient to conclude that the coefficients $\varepsilon^{k-2}M^k$ of order $k>2$ converge to zero in the subcritical regime $\eta=o(\varepsilon^{2/(d-2)})$. Indeed $\varepsilon^{k-2}M^k$ is just bounded by $(\varepsilon^2/\eta^{d-2})^k$, a quantity which can potentially blow up for too small values of η. This matter is to be adressed in a future work through a more accurate analysis of the rate of convergence of the coefficients $\mathcal{X}^{2k*}$ and $\mathcal{X}^{2k+1*}$ in the asymptotic (5.12) and (5.13). To date, let us note that Jing obtained recently $\mathcal{X}^{0*}=F/\eta^{d-2}+O(1)$ in the scalar case (proof of the Lemma 5.1 in [41]) by using layer potential techniques.

Remark 5.7. From the estimates of (5.4), we obtain that the coefficients $(\mathbb{D}^k_K)$ of (1.6) satisfy the same asymptotic convergences of Corollary 5.5. Indeed, by using the definition (3.21) we find that there exists a constant $C>0$ independent of η such that for any $K\geq 0$, $K+1\leq k\leq 2K+1$ and $1\leq j\leq d$:

$$||\nabla \boldsymbol{N}^K_j||_{L^2(Y,\mathbb{R}^{d\times d})}\leq C\eta^{-(d-2)\lfloor K/2\rfloor}\eta^{d/2-1}$$

$$||\beta^{k-K-1}_j||_{L^2(Y,\mathbb{R}^d)}\leq C\eta^{-(d-2)\lfloor\frac{k-K-1}{2}\rfloor+1}\eta^{d/2}.$$

Applying the inequality $\lfloor x\rfloor+\lfloor y\rfloor\leq\lfloor x+y\rfloor$, we obtain that the coefficients $\mathbb{A}^k_{ij}$ of Proposition 4.10 satisfy for $K+1\leq k\leq 2K+1$,

$$|\mathbb{A}^k_{ij}|\leq C\eta^d\eta^{-\lfloor\frac{k-1}{2}\rfloor(d-2)}\leq C\eta^2M^k_{ij}.$$

Hence the discrepancy induced by the coefficients $\mathbb{A}_{ij}^k$ is small and $\mathbb{D}_{K,ij}^k$ also satisfies (5.21) and (5.22) for $K+1 \leq k \leq 2K+1$. Similarly (see Corollary 7 of [34]), we may show that $\mathbb{D}_0^2 \to -I$. The remaining coefficients $\mathbb{D}_K^k$ are equal to M^k.

Acknowledgments. This work was supported by the Association Nationale de la Recherche et de la Technologie (ANRT) [grant number CIFRE 2017/0024] and by the project ANR-18-CE40-0013 SHAPO financed by the French Agence Nationale de la Recherche (ANR). The author is grateful towards G. Allaire, C. Dapogny and S. Fliss for insightful discussions and their helpful comments and revisions.

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