Entropy solutions for a two-phase transition model for vehicular traffic with metastable phase and time depending point constraint on the density flow

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Abstract

We consider a macroscopic two-phase transition model for vehicular traffic flow subject to a point constraint on the density flux. The two phases correspond to light and heavy traffic and their dynamics are described respectively by the Lighthill-Whitham-Richards model and the Aw-Rascle-Zhang model. Their intersection, the so-called metastable phase, is assumed to be non empty. The discrete in time point constraint mechanism, inducing flux limitation at bottlenecks, is explored within this two-phase model.

We introduce a new definition of admissible solutions for the Cauchy problem, for which we prove existence and we provide a characterization. In particular, these admissible solutions attain the maximal flux allowed by the constraint whenever it is enforced, which guarantees compatibility of the constructed solutions with the modeling assumption imposed at the level of the Riemann solver. These results rely on the wave-front tracking method and on adaptation of the specific entropies and renormalization properties introduced in *Andreainov, Donadello, Rosini, M3AS (2016)* while dealing with the Aw-Rascle-Zhang model with point constraint.

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Key words. Conservation laws, phase transitions, Lighthill-Whitham-Richards model, Aw-Rascle-Zhang model, point constraint on the density flux, wave-front tracking, entropy conditions.

1 Introduction

Our work explores a particular instance of so-called phase transition models used in macroscopic descriptions of road traffic. It is a continuation and an extension of the work [9]. Here we sharpen the definition of solution and prove an existence result for a constrained two-phase transition model of hyperbolic conservation laws introduced in [9, 19]; we also consider a non-local variant of the model giving account of capacity drop phenomena.

1.1 Motivations

The model we deal with (see (7) below for for the short-cut PDE formulation and Definition 2.1 for the precise meaning given to it) is situated at the crossroads of two lines of research in macroscopic traffic modeling and model analysis. The first line consists in combining the classical LWR scalar equation (Lighthill, Whitham and Richards [30, 33]) and the well-known ARZ system (Aw, Rascle and Zhang [8, 36]), within a unique model featuring transitions between the "free flow" phase Ω_{f} described by LWR and the "congested flow" phase Ω_{c} described by ARZ. The interest of this approach resides in the fact that LWR, as any first order model, does not capture completely the dynamics observed at high traffic flow densities, while ARZ features a degeneracy and instability when density approaches zero, see [25]. Indeed, within the LWR framework the fundamental flow equation

$$\rho_t + (v\,\rho)_x = 0 \tag{1}$$

(ρ and v representing the density and the velocity of the flow, respectively) is closed with the functional dependence $v = v(\rho)$, that appears as heuristically justified only at low densities; whereas the ARZ description features the closure relation $v = w - p(\rho)$, where the velocity combines the reaction of agents to the surronding density, encoded in the function p, and the so-called "Lagrangian marker" w. The latter is (formally) transported along the flow as

$$w_t + v\,w_x = 0, \tag{2}$$

leading to a strictly hyperbolic 2×2 system, at least away from the vacuum $\rho = 0$. We give a historical account on such *phase transition models* in § 1.2 below. Next, the second line of research that underlies our work is the introduction, within macroscopic traffic models (1), of point constraints formally given as

$$(v\,\rho)|_{x=x_0} \leqslant \mathtt{F}, \tag{3}$$

meaning that the location $x = x_0$ is considered as a bottleneck, where the flow is limited to the maximal value $\mathtt{F}$. The value $\mathtt{F}$ is a function of time that can be given *a priori* or it can be computed in a time-discrete way from the solution itself, thus giving rise to non-local models with point constraint. The interest of the family of models (1), (3) resides in their ability to take into account small-scale inhomogeneities of the flow (traffic lights, tollgates, construction sites or obstacles on the road). The time dependence of $\mathtt{F}$, which is technically very demanding (cf. [4] for the ARZ case) allows to model the traffic in presence, for instance, of traffic lights or construction sites operating only during a part of the day. In the case of non-local dependence of $\mathtt{F}$ on ρ, these models allow for possible adaptive management of such situations; moreover, they are able to reproduce capacity drop and its avatars like "Faster is Slower", the Braess paradoxes, see [1], and self-organization phenomena, see [2, 7]. We refer to [4, 5, 16, 22] for LWR and ARZ with point constraints, and to [2–4] for LWR with non-local point constraints and applications. Note that moving bottlenecks with local or non-local constraints can further be considered [20, 24, 28, 29, 31, 35].

1.2 Positioning of the present model with respect to the existing literature

An informal way to describe our model (7) is to say that it combines the fundamental flow conservation relation (1), the point constraint (3) with specific assumptions on $\mathtt{F}$, and (roughly speaking) transport equation (2) for the Lagrangian marker w along the flow. The closure relation is provided by linking the Lagrangian marker w to the state variables (ρ, v) in two distinct ways in the *free phase* $\Omega_{\rm f}$ and in the *congested phase* $\Omega_{\rm c}$ - compatible choices being made in the *metastable phase* $\Omega_{\rm f} \cap \Omega_{\rm c}$.

In the congested phase the traffic is governed by a 2×2 system of conservation laws (a so-called *second-order* macroscopic model), whereas in the free phase it is governed by a scalar conservation law (a so-called *first-order* macroscopic model). The two phases are coupled via *phase transitions*, namely discontinuities between two states belonging to different phases and satisfying the Rankine-Hugoniot conditions. As a matter of fact, the coupling in phase transition models is usually prescribed in terms of the Riemann solver, i.e. of the local behavior of the solution at the points where the transitions occur rather than in pure PDE terms: this is a typical approach to non-classical solutions of hyperbolic systems of conservation laws, relying on the property of finite propagation speed. Prescribing a Riemann solver directly encodes the underlying modeling assumptions, as the conditions for phase transitions and the action of the constraint in the case of our model. We refer to [16] for the founding example of LWR with point constraint, and [23] for a wide family of discontinuous-flux conservation laws defined via the Riemann solver approach.

The complete analysis of the Riemann problem is a necessary ingredient in the construction of a converging sequence of approximate solutions via the wave front tracking algorithm, but is not sufficient alone to characterize the weak solution obtained in the limit.

The question of characterization of such Riemann-solver-based solutions in the form of weak and entropy formulations (or more precisely, "adapted entropy" formulations) is of particular interest, because it permits to apply PDE analytical techniques and build even more complex models on the top of the well-understood ones (like [2, 3], based upon [16]). As successful examples of such characterization, we refer to [5, 16] and [6] for LWR with point constraints and for discontinuous-flux conservation laws, respectively; see also [4] for the "Kruzhkov-like" entropy characterization of solutions of the ARZ model, originally defined in [8] by means of the Riemann solver. Providing a characterization for model (7) in PDE terms is the main goal of our work.

The first two-phase model has been introduced by Colombo in [15]. Later, Goatin proposed in [25] a two-phase model which couples the ARZ model for the congested phase $\Omega_{\rm c}$, with the LWR model for the free-flow phase $\Omega_{\rm f}$, see also [10] for its generalization.

Both Colombo [15] and Goatin [25] assume that $\Omega_{\rm c} \cap \Omega_{\rm f} = \emptyset$. The first two-phase model with a *metastable phase* $\Omega_{\rm c} \cap \Omega_{\rm f} \neq \emptyset$ has been introduced in [18]. Metastability is a well-known situation in models of fluid dynamics (see, e.g., [26]); in the context of traffic flows, we refer to [34, Figure 1] for empirical evidences. Existence results for Cauchy problems for different phase transition models have already been established in [10, 13, 17, 25] for the case without point constraints, and in [9] for the case with point constraint; the Riemann problem has recently been studied in the case of moving constraints [31].

In the present paper we assume that metastable phase is present, i.e., $\Omega_{\rm c} \cap \Omega_{\rm f} \neq \emptyset$. For this reason, see [15, Remark 2], we assume that $\Omega_{\rm f}$ is characterized by a unique value V of the velocity. In other words, the scalar conservation law describing the LWR dynamics of the free phase can also be seen as the mere transport equation with constant velocity V. This choice is in accordance with typical experimental data for low traffic densities [34, Figure 1]. We consider vehicles having different features encoded in the "Lagrangian marker" w proper to the ARZ family of models, hence we allow the density flux function to vanish at different densities. This is a feature that our model shares with the phase transition models of [10, 25], whereas in the models of [13, 15] the density flux function vanishes at a unique maximal density.

Combining phase transitions with a point constraint on the flux (3) located at $x = 0$ as in [9, 11, 19], we impose that at the location $x = 0$ (the bottleneck location, that we often call *interface* in the sequel) the density flux of the solution is lower than a value $\mathtt{F}$. However, differently from [9, 11, 19], here $\mathtt{F}$ can be time dependent (restricted to be piecewise constant, with sufficiently large level of passing capacity). In particular our setting allows to deal with constraint functions whose values, updated at fixed times, depend on the past evolution of the solution: we refer to Section 5 for precise assumptions of $\mathtt{F}$ as function of t and ρ. While models with continuously varying constraints allow to reproduce capacity drop leading to non-monotone empirical features of real traffic flows (see [1, 3]), here we highlight the situations where the constraint and its variations do not result from the intrinsic disorganization (or, on the contrary, from a self-organization, see [2, 7]) of the flow at high densities. Indeed, having in mind traffic management, we underline that the constraint and its variations under the form considered here naturally arise from operation of bottlenecks (such as toll gates or traffic lights) at discrete times as a function of data collected non-locally in time upstream the flow.

1.3 Results, technical foundations and outline of the paper

With respect to the work [9] of which our paper is a follow-up, our contribution is twofold.

First, we show that the classical wave-front tracking algorithm (see, e.g., [27]) gives at the limit an entropy solution in the sense of Definition 2.1 - the definition being strengthened with respect to [9]. This allows us to characterize the bottleneck flow in a sharper way than what was achieved in [9] (in particular, the technical assumption [9, (2.14)], which corresponds to (29) given below, can actually be bypassed). As a consequence, we can claim that non-classical shocks occur precisely at the level $\mathtt{F}$ of the flux $f = v\,\rho$ imposed by the constraint (3); moreover, the Lagrangian marker w in (2) does not increase across the non-classical shock. Note that these two properties underlie the definition of the Riemann solver, that is the same as in [9]. Thus, the sharp characterization of admissible solutions adequately reflects the modeling assumptions imposed while prescribing the Riemann solver.

Let us briefly describe the specific technical tools that allow us to achieve this flux characterization at the constraint. First, we rely upon a localized version (Definition 2.1 **(S.3)**) of the renormalization property which, with the reference to (1), (2) can be roughly stated as

$$\text{“ } \rho_t + (v\,\rho)_x = 0,\ (\rho\,w)_t + (v\,\rho\,w)_x = 0 \quad \Longrightarrow \quad \forall g \in \mathbf{C^0}(\mathbb{R},\mathbb{R}) \quad (\rho\,g(w))_t + (v\,\rho\,g(w))_x = 0 \text{ ”}$$

(cf. [4,32] and Remark 2.2). Second, in order to make sense of the point constraint (3) (Definition 2.1 **(S.5)**), we indagate the value of the limit density flux at the constraint position through a classical application of the Green theorem. Note that the proof of our existence result exploits, in the passage to the limit, the careful choice of the entropy inequalities (or, more precisely, of the contribution of the point constraint to these inequalities) and relies again upon the renormalization property and upon the Green theorem.

Second, our existence result opens way to the study of modeling situations where a non-local point constraint is imposed at the bottleneck, even if we had to impose an additional restriction on the range of possible values for the constraint function $\mathtt{F}$ in (3), see assumption **(H.1)** in Theorem 2.8. We restrict our attention to constraints updated at discrete times; note that analogous models based on LWR were constructed and analyzed in [2] (along with continuous-time models, which extensions to (7) we are unable to analyse in depth). Our restriction to piecewise constant in time constraints $\mathtt{F}$ is motivated by technical reasons. In particular, since our existence proof relies on wave-front tracking approximations, we need to control the increase in total variation of the approximate solutions at any time at which the constraint level changes. In practice this is possible provided changes only occur a locally finite number of times, and the updated constraint level lays in the interval where metastable states exist. In our opinion, the most promising direction to overcome these technical restriction requires different compactness techniques, e.g. of the compensated compactness type; work in this direction is the subject of ongoing research.

The paper is organized as follows. In Section 2 we introduce the needed notations, the model and the main result of the paper, which concerns constraint given beforehand. We defer the proofs of our main results to Sections 3 and 4. In Section 5 we briefly indicate how our results can be applied to a particular class of non-local point constraints.

2 Model and main result

In this section we introduce some notations, see Figure 1, state the two-phase transition model (7), fix the notion of entropy solution (see Definition 2.1) and formulate our main result in Theorem 2.8.

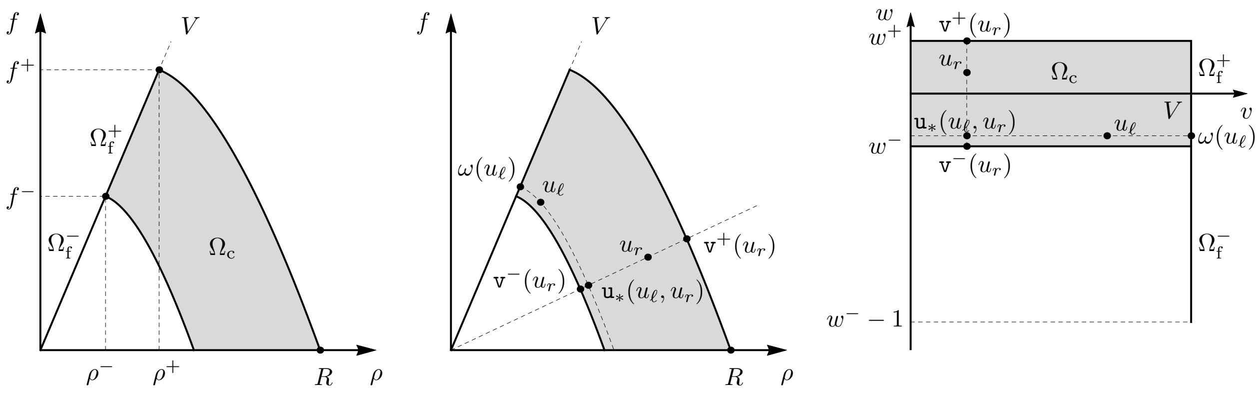

Figure 1: Notations.

Let $\rho \geqslant 0$ and $v \geqslant 0$ be the density and the velocity of the vehicles, respectively. Denote $u \doteq (\rho, v)$ and let

$$f(u) \doteq v\,\rho$$

be the density flux. If $V > 0$ is the unique velocity in the free-flow phase Ω_f and ρ^+ is the maximal density in Ω_f, then

$$\Omega_f \doteq \left\{u \in \mathbb{R}_+^2 : \rho \leqslant \rho^+,\ v = V\right\},$$

where $\mathbb{R}_+ \doteq [0, \infty)$. We further consider that the segment

$$[\rho^-, \rho^+] \times \{V\} = \Omega_f \cap \Omega_c$$

is the metastable phase ("metastable phase" meaning the intersection between the free phase Ω_f and the congested phase Ω_c, as typical in the literature on phase transition models), then the ARZ formalism [8, 36] leads us to set

$$\Omega_c \doteq \left\{u \in \mathbb{R}_+^2 : v \leqslant V,\ w^- \leqslant w \doteq v + p(\rho) \leqslant w^+\right\},$$

where $w^\pm \doteq p(\rho^\pm) + V$. In the congested phase Ω_c governed by ARZ, the *anticipation factor* $p \in \mathbf{C^2}((0, \infty); \mathbb{R})$ is a nonlinearity whose role in the modeling is to take into account drivers' reactions to the state of traffic in front of them; the closure relation $w = v + p(\rho)$ links the state variables (ρ, v) and the Lagrangian marker w of the ARZ phase. We assume that

$$p'(\rho) > 0, \qquad 2\,p'(\rho) + p''(\rho)\,\rho > 0 \qquad \text{for every } \rho > 0, \tag{4}$$

and

$$v < p'(\rho)\,\rho \text{ for every } (\rho, v) \in \Omega_c. \tag{5}$$

Typical choices for p are $p(\rho) \doteq \rho^\gamma$ with $\gamma > V/(w^- - V)$, see [8], and $p(\rho) \doteq V_{\rm ref} \ln(\rho/\rho_{\max})$ with $V_{\rm ref} > V$ and $\rho_{\max} > 0$, see [25]. Denote $f^\pm \doteq V\,\rho^\pm$ and let $R \doteq p^{-1}(w^+) > 0$ be the maximal density (in the congested phase). Define

$$\Omega \doteq \Omega_f \cup \Omega_c, \qquad \Omega_f^- \doteq \left\{u \in \Omega_f : \rho \in [0, \rho^-)\right\}, \qquad \Omega_f^+ \doteq \left\{u \in \Omega_f : \rho \in [\rho^-, \rho^+]\right\}, \qquad \Omega_c^- \doteq \Omega_c \setminus \Omega_f^+.$$

Notice that the metastable phase $\Omega_f \cap \Omega_c$ coincides with Ω_f^+. We extend the Lagrangian marker $w\colon \Omega_c \to [w^-, w^+]$ defined by $w(u) \doteq p(\rho) + v$ by introducing $\mathtt{w}\colon \Omega \to [w^- - 1, w^+]$ and $\mathtt{W}\colon \Omega \to [w^-, w^+]$ defined by

$$\mathtt{w}(u) \doteq \begin{cases} v + p(\rho) & \text{if } u \in \Omega_c, \\ w^- - 1 + \dfrac{\rho}{\rho^-} & \text{if } u \in \Omega_f^-, \end{cases} \qquad \mathtt{W}(u) \doteq \begin{cases} v + p(\rho) & \text{if } u \in \Omega_c, \\ w^- & \text{if } u \in \Omega_f^-. \end{cases} \tag{6}$$

Both these extensions are exploited below to introduce several quantities; for instance, they are both needed to define $\mathtt{u}_*$ in (19c). In poor words, $\mathtt{w}$ is involved when we need to distinguish in the (extended) (v, w)-coordinates states u belonging to $\Omega_{\rm f}^-$, whereas we use $\mathtt{W}$ if this is not the case. Remark (2.4) below specifies the dynamics of $\mathtt{W}(u)$ within our model.

In this paper we study the constrained Cauchy problem for the phase transition model

Free flow (linearly degenerate LWR)

$$\begin{cases} u \in \Omega_{\rm f}, \\ \rho_t + (\rho\, V)_x = 0, \\ v = V, \end{cases}$$

Congested flow (ARZ)

$$\begin{cases} u \in \Omega_{\rm c}, \\ \rho_t + (\rho\, v)_x = 0, \\ \big(\rho\, \mathtt{W}(u)\big)_t + \big(\rho\, \mathtt{W}(u)\, v\big)_x = 0, \end{cases} \tag{7a}$$

with initial condition

$$u(0,x) = \overline{u}(x) \tag{7b}$$

and local point constraint on the density flux at $x = 0$

$$f\big(u(t,0_\pm)\big) \leqslant \mathtt{F}(t), \tag{7c}$$

where $\mathtt{F}\colon (0,\infty) \to [0, f^+]$ is a given function. Note that in Section 5 we will explain how to deal with piecewise constant constraint functions $\mathtt{F}$ such that, if t_i and t_{i+1} are two consequent times at which $\mathtt{F}$ is discontinuous, the value of $\mathtt{F}(t)$ for $t \in (t_i, t_{i+1})$ depend on $u_{|[0,t_i]\times\mathbb{R}}$.

For $u \in \Omega$, $k \in [0, V]$ and $F \in [0, f^+]$ we define

$$\mathtt{N}_F^k(u) \doteq \begin{cases} f(u)\, \mathtt{n}_F^k\big(\mathtt{W}(u)\big) & \text{if } F \neq 0, \\ k & \text{if } F = 0, \end{cases} \qquad \mathtt{n}_F^k(W) \doteq \left[\frac{k}{F} - \frac{1}{p^{-1}(W-k)}\right]_+, \tag{8}$$

and introduce the entropy-entropy flux pair

$$\mathtt{E}^k(u) \doteq \begin{cases} 0 & \text{if } v \geqslant k, \\ \dfrac{\rho}{p^{-1}\big(\mathtt{W}(u)-k\big)} - 1 & \text{if } v < k, \end{cases} \qquad \mathtt{Q}^k(u) \doteq \begin{cases} 0 & \text{if } v \geqslant k, \\ \dfrac{f(u)}{p^{-1}\big(\mathtt{W}(u)-k\big)} - k & \text{if } v < k, \end{cases} \tag{9}$$

which is obtained by adapting the entropy-entropy flux pair introduced in [4] for the ARZ model.

Definition 2.1. *Let $\overline{u} \in \mathbf{BV}(\mathbb{R};\Omega)$. Let $\mathtt{F} \in \mathbf{L^\infty}\big((0,\infty);[0,f_+]\big)$. We say that $u \in \mathbf{L^\infty}\big((0,\infty);\mathbf{BV}(\mathbb{R};\Omega)\big) \cap \mathbf{C^0}\big(\mathbb{R}_+;\mathbf{L^1_{loc}}(\mathbb{R};\Omega)\big)$ is an admissible solution to constrained Cauchy problem* (7) *if the following holds:*

(S.1) *Initial condition* (7b) *holds for a.e. $x \in \mathbb{R}$, namely (bearing in mind the time continuity with $\mathbf{L^1_{loc}}$ values)*

$$u(0,x) = \overline{u}(x) \qquad \text{for a.e. } x \in \mathbb{R}.$$

(S.2) *The function $u \doteq (\rho, v)$ provides a weak solution to the mass conservation equation, namely, for any $\phi \in \mathbf{C_c^\infty}\big((0,\infty)\times\mathbb{R};\mathbb{R}\big)$ we have (recalling the notation $f(u) = \rho\, v$)*

$$\int_0^\infty \int_{\mathbb{R}} \big(\rho\, \phi_t + f(u)\, \phi_x\big)\, \mathrm{d}x\, \mathrm{d}t = 0. \tag{10}$$

(S.3) *The function $\mathtt{W}(u)$ defined in* (6) *satisfies the weak formulation and the renormalization property away from the constraint, namely, for any $g \in \mathbf{C}([w^-, w^+];\mathbb{R})$ and $\phi \in \mathbf{C_c^\infty}\big((0,\infty)\times\mathbb{R};\mathbb{R}\big)$ such that $\phi(\cdot, 0) \equiv 0$, we have that*

$$\int_0^\infty \int_{\mathbb{R}} \Big(\mathtt{E}_g(u)\, \phi_t + \mathtt{Q}_g(u)\, \phi_x\Big)\, \mathrm{d}x\, \mathrm{d}t \;=\; 0, \tag{11}$$

where $\mathtt{E}_g(u) \doteq \rho\, g\big(\mathtt{W}(u)\big)$ and $\mathtt{Q}_g(u) \doteq v\, \mathtt{E}_g(u)$.

(S.4) *Entropy inequalities are satisfied up to the constraint, namely, for any $k \in [0, V]$ and $\phi \in \mathbf{C_c^\infty}((0,\infty)\times\mathbb{R};\mathbb{R})$ such that $\phi \geqslant 0$ we have*

$$\int_0^\infty \left(\int_{\mathbb{R}} \Big(\mathtt{E}^k(u)\, \phi_t + \mathtt{Q}^k(u)\, \phi_x\Big)\, \mathrm{d}x + \mathtt{N}^k_{\mathtt{F}(t)}\big(u(t,0_-)\big)\, \phi(t,0)\right) \mathrm{d}t \geqslant 0. \tag{12}$$

(S.5) *The constraint condition* (7c) *holds for a.e. $t > 0$, namely*

$$f\big(u(t,0_\pm)\big) \leqslant \mathtt{F}(t) \qquad \text{for a.e. } t > 0.$$

Remark 2.2. *It is possible to prove, using the delicate theory of [32], that as soon as* (11) *holds with* $g = Id$ *(which corresponds to the mere weak formulation) and* (10) *holds (that is, the field* $(\rho, \rho v)$ *is divergence-free), the formulation* (11) *holds with arbitrary* g. *We include the renormalization property* (10) *into the definition because of its importance in the derivation of entropy inequalities and also because it is easily proved at the level of approximate solutions constructed with the wave-front tracking algorithm.*

In (12)*, the notation* $\mathtt{E}^k(u)$, $\mathtt{Q}^k(u)$, $\mathtt{N}^k_{\mathtt{F}(t)}(u)$ *for "Kruzhkov-like" entropies, the associated entropy fluxes and the associated constraint-related interface terms, respectively, is borrowed from our work [4] on ARZ with point constraint. This choice reflects the fact that, in view of the linear degeneracy of LWR considered in the free phase, we can merely rely upon entropy characterization of solution admissibility borrowed from the ARZ playground. In* (11)*, the notation* $\mathtt{E}_g(u)$, $\mathtt{Q}_g(u)$ *for conserved entropies and the associated entropy fluxes also stems from [4]. We use it in order to underline the inerpretation of the renormalization property* (11) *within the usual entropy dissipation paradigm of hyperbolic conservation laws. We refer to [4, Section 2.3] for detailed description of entropies of ARZ.*

Before turning to further comments on Definition 2.1, in the following proposition we state precisely which discontinuities are admissible for the solutions to (7).

Proposition 2.3. *Let* u *be a solution of constrained Cauchy problem* (7) *in the sense of Definition 2.1. Assume that* $\mathtt{F}$ *is a piecewise constant function. Then* u *has the following properties:*

- *At any Lipschitz curve of discontinuity* $x = \delta(t)$ *of* u*, the first Rankine-Hugoniot jump condition holds for a.e.* $t > 0$*:*
$$\Big[\rho\big(t, \delta(t)_+\big) - \rho\big(t, \delta(t)_-\big)\Big]\, \dot\delta(t) = f\big(u(t, \delta(t)_+)\big) - f\big(u(t, \delta(t)_-)\big), \tag{13}$$
and if $\delta(t) \neq 0$*, then it satisfies also the second Rankine-Hugoniot jump condition*
$$\begin{aligned}&\Big[\rho\big(t, \delta(t)_+\big)\, \mathtt{W}\Big(u\big(t, \delta(t)_+\big)\Big) - \rho\big(t, \delta(t)_-\big)\, \mathtt{W}\Big(u\big(t, \delta(t)_-\big)\Big)\Big]\, \dot\delta(t)\\ &= f\Big(u\big(t, \delta(t)_+\big)\Big)\, \mathtt{W}\Big(u\big(t, \delta(t)_+\big)\Big) - f\Big(u\big(t, \delta(t)_-\big)\Big)\, \mathtt{W}\Big(u\big(t, \delta(t)_-\big)\Big).\end{aligned} \tag{14}$$

- *Any discontinuity of* u *away from the constraint location* $x = 0$ *is classical, i.e., it satisfies the Lax entropy inequalities.*

- *Non-classical discontinuities of* u *may occur only at* $x = 0$*, and in this case the (density) flux* $f(u(t, 0_\pm))$ *at* $x = 0$ *equals the maximal flux* $\mathtt{F}$ *allowed by the constraint.*
Moreover, whatever be the nature of the shock at $x = 0$*, it holds*
$$f(u(t, 0_\pm))\Big(\mathtt{W}\big(u(t, 0_-)\big) - \mathtt{W}\big(u(t, 0_+)\big)\Big) \geqslant 0 \ \textit{ for a.e. } t > 0. \tag{15}$$

The proof is deferred to Section 4.

Remark 2.4. *While the formal writing* (7a) *does not describe the dynamics* $\mathtt{W}(u)$ *in the free phase nor at the phase transitions, Definition 2.1**(S.3)** states that* $\mathtt{W}(u)$ *plays the role of a globally defined Lagrangian marker for our phase transition model* (7)*, up to the possible lack of conservativity at the constraint location. That is, a solution* u *to the constrained Cauchy problem* (7) *does not satisfy in general the second Rankine-Hugoniot condition* (14) *along* $x = 0$ *(condition that rewrites as*
$$\rho(t, 0_-)\, \mathtt{W}\big(u(t, 0_-)\big)\, v(t, 0_-) = \rho(t, 0_+)\, \mathtt{W}\big(u(t, 0_+)\big)\, v(t, 0_+) \qquad \textit{for a.e. } t > 0$$
for jumps along the interface). Indeed the (extended) linearized momentum $\rho\, \mathtt{W}(u)$ *is conserved across (classical) shocks and phase transitions, but in general it is not conserved across non-classical shocks even if they are between states in* $\Omega_{\rm c}$*. As a consequence, a solution to* (7) *taking values in* $\Omega_{\rm c}$ *is not necessarily a weak solution to the* 2×2 *system of conservation laws in* (7a) *for the congested flow. For this reason in* (11) *we consider test functions* ϕ *such that* $\phi(\cdot, 0) \equiv 0$*. This is in the same spirit of the solutions considered in [11, 19–22] for traffic through locations with reduced capacity.*

However, differently from [9], in this paper we do not require in (12) *that* $\phi(\cdot, 0) \equiv 0$*. This allows us to better characterize the (density) flux at* $x = 0$ *associated to non-classical shocks. In fact, we can ensure that the flux of the non-classical shocks of any solution is equal to the maximal flux* $\mathtt{F}$ *allowed by the constraint without requiring the technical assumption [9, (2.14)], as pointed out in Proposition 2.3 below. We can also ensure that the lack of conservativity in the generalized momentum equation is sign-definite, namely* $\int_{\mathbb{R}} \rho(t,x)\, \mathtt{W}(u(t,x))\, \mathrm{d}x$ *is non-increasing with* t*: this is an immediate consequence of the above inequality* (15).

Note that the assumption of piecewise constancy of $\mathtt{F}$ can be weakened via localisation arguments (cf. [4, Prop. 3.2]); however, we do not go beyond the piecewise constant setting in the rest of this paper, because of the difficulty of controlling the variation of the approximate solutions constructed with wave-front tracking.

Remark 2.5. *The characterization of non-classical shocks at $x = 0$ by the constraint saturation condition $f(u(t, 0_\pm)) = \mathtt{F}(t)$, achieved in Proposition 2.3 for the model at hand, was the cornerstone of the uniqueness results in the LWR point constrained models, [5, 16], and it holds true for the constrained ARZ model, [4]. Note that for phase transition models without the metastable phase, the constraint saturation property can not be true in all situations, see [12].*

In order to describe precisely non-classical shocks and define the functional designed to control the total variation of the wave-front tracking solutions, let us introduce $v_F^\pm \in [0, V]$ and $w_F \in [w^- - 1, w^+]$ defined by the following conditions, see Figure 2:

$$\begin{array}{llll}
\text{if } F = f^+ : & v_F^+ \doteq V, & v_F^- \doteq V, & w_F \doteq w^+, \\
\text{if } F \in [f^-, f^+) : & v_F^+ \doteq V, & v_F^- + p(F/v_F^-) = w^+, & w_F \doteq p\,(F/V) + V, \\
\text{if } F \in (0, f^-) : & v_F^+ + p(F/v_F^+) = w^-, & v_F^- + p(F/v_F^-) = w^+, & w_F \doteq w^- - 1 + \dfrac{F}{f^-}, \\
\text{if } F = 0 : & v_F^+ \doteq 0, & v_F^- \doteq 0, & w_F \doteq w^- - 1.
\end{array}$$

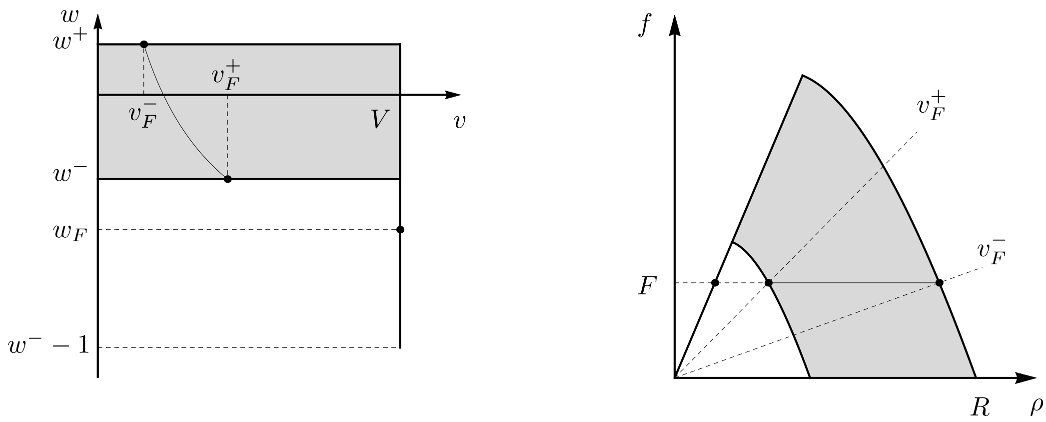

Figure 2: Geometrical meaning of w_F, $v_F^\pm$ and Ξ_F in the case $F \in (0, f^-)$. The curve in the figure on the left is the graph of Ξ_F, which corresponds to the horizontal solid segment in the figure on the right.

For any $F \in (0, f^+)$, let $\Xi_F : [v_F^-, v_F^+] \to [w^-, w^+]$ be given by $\Xi_F(v) \doteq v + p(F/v)$, see Figure 2. Notice that Ξ_F is strictly decreasing by (5) and is strictly convex by (4).

For any $F \in (0, f^+)$, let $[w^- - 1, w^+] \ni w \mapsto \hat{\mathtt{u}}(w, F) = (\hat{\mathtt{r}}(w, F), \hat{\mathtt{v}}(w, F)) \in \Omega_{\rm c}$ and $[0, V] \ni v \mapsto \check{\mathtt{u}}(v, F) = (\check{\mathtt{r}}(v, F), \check{\mathtt{v}}(v, F)) \in \Omega$ be defined in the (v, w)-coordinates by

$$\begin{aligned}
&\hat{\mathtt{v}}(w,F) \doteq \begin{cases} \Xi_F^{-1}(w) & \text{if } w > \max\{w^-, w_F\}, \\ v_F^+ & \text{if } w_F < w \leqslant w^-, \\ V & \text{if } w \leqslant w_F, \end{cases} &&
\hat{\mathtt{w}}(w,F) \doteq \begin{cases} w & \text{if } w > \max\{w^-, w_F\}, \\ w^- & \text{if } w_F < w \leqslant w^-, \\ w_F & \text{if } w \leqslant w_F, \end{cases} \\
&\check{\mathtt{v}}(v,F) \doteq \begin{cases} V & \text{if } v > v_F^+, \\ v & \text{if } v \in [v_F^-, v_F^+], \\ v_F^- & \text{if } v < v_F^-, \end{cases} &&
\check{\mathtt{w}}(v,F) \doteq \begin{cases} w_F & \text{if } v > v_F^+, \\ \Xi_F(v) & \text{if } v \in [v_F^-, v_F^+], \\ w^+ & \text{if } v < v_F^-, \end{cases}
\end{aligned} \tag{16}$$

where $\hat{\mathtt{w}} \doteq \mathtt{w} \circ \hat{\mathtt{u}}$ and $\check{\mathtt{w}} \doteq \mathtt{w} \circ \check{\mathtt{u}}$, see Figures 3 and 4.

The following lemma collects some useful properties of the maps defined above. In particular, it explains why we limit our study to the case of $\mathtt{F}$ taking values in $[f_-, f_+]$.

Lemma 2.6.

1. *For any $F \in (0, f^+)$, $w \in [w^- - 1, w^+]$ and $v \in [0, V]$ we have*

$$f\big(\hat{\mathtt{u}}(w, F)\big) = f\big(\check{\mathtt{u}}(v, F)\big) = F.$$

2. *The maps* $w \mapsto \hat{\mathtt{u}}(w, F)$ *and* $v \mapsto \check{\mathtt{u}}(v, F)$ *are Lipschitz continuous if and only if* $F \geqslant f^-$. *The Lipschitz constant is then uniform with respect to* $F \in [f_-, f_+]$.

3. *If* $F < f^-$, *then* $w \mapsto \hat{\mathtt{u}}(w, F)$ *and* $v \mapsto \check{\mathtt{u}}(v, F)$ *are only left-continuous.*

4. $\hat{\mathtt{w}}(w, F) \geqslant w$ *and* $\check{\mathtt{v}}(v, F) \geqslant v$.

5. $w \mapsto \hat{\mathtt{w}}(w, F)$ *and* $v \mapsto \check{\mathtt{v}}(v, F)$ *are non-decreasing, while* $w \mapsto \hat{\mathtt{v}}(w, F)$ *and* $v \mapsto \check{\mathtt{w}}(v, F)$ *are non-increasing.*

Proof. We focus on the proof of 2, which is crucial for the analysis of increase of the Glimm-like functionals providing the variation control. The other properties can be assessed analogously, upon examination of the definitions. The uniform in F Lipschitz continuity of $\hat{w}$ and $\check{v}$ is obvious. The uniform Lipschitz regularity of $\hat{v}$ and of $\check{w}$ requires uniform bounds on $\Xi_F'(v)$ and $(\Xi_F^{-1})'(w)$ for $v \in [v_F^-, v_F^+]$ and $w \geqslant \max\{w^-, w_F\}$. As $w = \Xi_F(v) = v + p(F/v)$ in these calculations, v cannot lie too close to zero. One readily computes

$$\Xi_F'(v) = 1 - p'(F/v)F/v^2 = \frac{v - p'(\rho_F(v))\rho_F(v)}{v},$$

where $\rho_F(v) = F/v$. By assumption (5) (notice that the inequality $v - p(\rho_F(v))\rho_F(v) > 0$ is strict and can be stengthened to $v - p(\rho_F(v))\rho_F(v) \geqslant \delta > 0$ since p' is continuous and the domain Ω_c is compact), the claimed uniform bounds follow. □

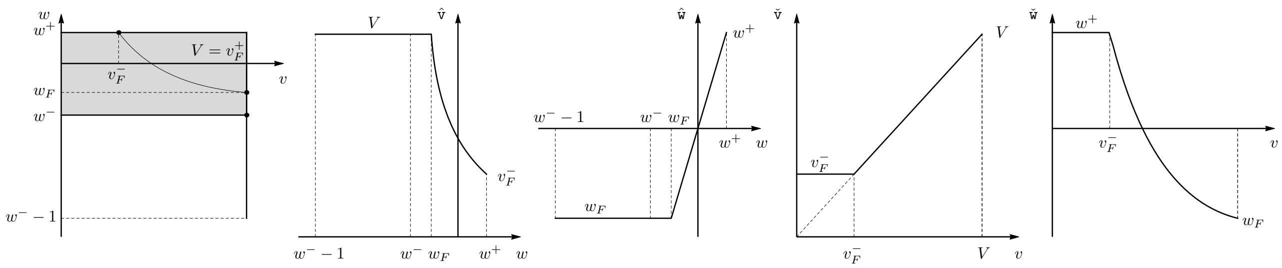

Figure 3: Geometrical meaning of $\hat{\mathtt{u}}$ and $\check{\mathtt{u}}$ defined in (16) in the case $F \in (f^-, f^+)$.

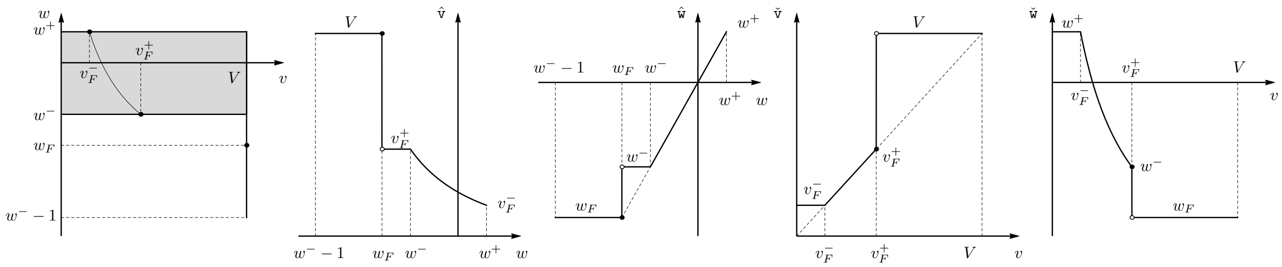

Figure 4: Geometrical meaning of $\hat{\mathtt{u}}$ and $\check{\mathtt{u}}$ defined in (16) in the case $F \in (0, f^-)$.

Denote by TV_+ and TV_- the positive and negative total variations, respectively. For any $u \colon \mathbb{R} \to \Omega$ and $F \in (0, f^+)$ let

$$\begin{aligned} \hat{\Upsilon}(u, F) &\doteq \mathrm{TV}_+\Big(\hat{\mathtt{v}}\big(\mathtt{w}(u), F\big); (-\infty, 0)\Big) + \mathrm{TV}_-\Big(\hat{\mathtt{w}}\big(\mathtt{w}(u), F\big); (-\infty, 0)\Big), \\ \check{\Upsilon}(u, F) &\doteq \mathrm{TV}_+\Big(\check{\mathtt{v}}(v, F); (0, \infty)\Big) + \mathrm{TV}_-\Big(\check{\mathtt{w}}(v, F); (0, \infty)\Big). \end{aligned} \tag{17}$$

Remark 2.7. *Due to Lemma 2.6, if* $F \in [f^-, f^+]$ *and* $u \in \mathbf{BV}(\mathbb{R}; \Omega)$, *then* $\hat{\Upsilon}(u; F) + \hat{\Upsilon}(u; F)$ *are uniformly bounded by a constant times the total variation of* u.

We can now state our main result. Note that it contains essentailly the two cases: constant constraint $\mathtt{F} \equiv F(0) \in [0, f_+]$ with an additional restriction on nonlinear variations of $\overline{u}$, and piecewise constant constraint taking values above the threshold f_-. Note that the assumption $\mathtt{F}(t) \geqslant f_-$ corresponds to the presence of metastable states verifying the imposed constraint.

Theorem 2.8. *Let* $\overline{u} \in \mathbf{L^1} \cap \mathbf{BV}(\mathbb{R}; \Omega)$. *Assume that* $\mathtt{F} \in \mathbf{PC}\big((0, \infty); [0, f^+]\big)$ *satisfies one of the following conditions:*

(H.1) $\mathtt{F}$ *takes its values in* $[f^-, f^+]$*;*

(H.2) $\mathtt{F}(t) \equiv \mathtt{F}(0) \in [0, f^-)$ *and* $\hat{\Upsilon}(\overline{u}) + \check{\Upsilon}(\overline{u})$ *is finite.*

Then the approximate solutions u_n *constructed in Section 3.4 converge to a solution* $u \in \mathbf{C^0}(\mathbb{R}_+; \mathbf{L^1_{loc}}(\mathbb{R}; \Omega))$ *of constrained Cauchy problem* (7) *in the sense of Definition 2.1. Moreover for all* $t,\ s \in \mathbb{R}_+$ *the following estimates hold*

$$\mathrm{TV}\big(u(t)\big) \leqslant K, \qquad \|u(t) - u(s)\|_{\mathbf{L^1}(\mathbb{R};\Omega)} \leqslant L\,|t-s|, \qquad \|u(t)\|_{\mathbf{L^\infty}(\mathbb{R};\Omega)} \leqslant R + V, \tag{18}$$

where K *and* L *are constants that depend on* $\overline{u}$ *and* $\mathtt{F}$*. Furthermore, non-classical discontinuities of* u *can occur only at the constraint location* $x = 0$*, and in this case the (density) flow at* $x = 0$ *and time* $t > 0$ *is the maximal flow* $\mathtt{F}(t)$ *allowed by the constraint, moreover,* (15) *is fulfilled.*

As in [4, 10, 17], the proof of the above theorem is based on the wave-front tracking algorithm, see [14, 27] and the references therein. The details of the proof are deferred to Section 3.

Corollary 2.9. *The conclusion of Theorem 2.8 still holds true under the following hypothesis*

- $\overline{u} \in \mathbf{L^1} \cap \mathbf{BV}(\mathbb{R}; \Omega)$*;*
- $\mathtt{F} \in \mathbf{PC}\big((0, \infty); [0, f^+]\big)$ *and there exists* t_1 *such that*
  - F *takes values in* $[f_-, f_+]$ *for all* $t \geqslant t_1$*;*
  - $\mathtt{F}(t) \equiv \mathtt{F}(0) \in [0, f^-)$ *for all* $t \in [0, t_1)$*, and* $\hat{\Upsilon}(\overline{u}) + \check{\Upsilon}(\overline{u})$ *is finite.*

3 Proof of Theorem 2.8

In this section we prove Theorem 2.8. Note that the justification of the very particular situation of Corollary 2.9 is immediate, the **BV** control being guaranteed by the successive application of the arguments used for the case **(H.2)** and then for the case **(H.1)** of Theorem 2.8.

3.1 Lax curves

In the (ρ, f)-plane the Lax curves in Ω_c of the first and second characteristic families passing through $\bar{u} = (\bar{\rho}, \bar{v}) \in \Omega_c$ are respectively described by the graphs of the maps

$$\begin{aligned}
\big[p^{-1}\big(\mathtt{W}(\bar{u}) - V\big), p^{-1}\big(\mathtt{W}(\bar{u})\big)\big] &\ni \rho \mapsto \mathfrak{L}_{\mathtt{W}(\bar{u})}(\rho) \doteq f\big(\rho, \mathtt{W}(\bar{u}) - p(\rho)\big) \in \mathbb{R}_+, \\
\big[p^{-1}(w^- - \bar{v}), p^{-1}(w^+ - \bar{v})\big] &\ni \rho \mapsto \bar{v}\,\rho \in \mathbb{R}_+.
\end{aligned}$$

Conditions (4) and (5) ensure that for any $w \in [w^-, w^+]$ the map $\rho \mapsto \mathfrak{L}_w(\rho) = (w - p(\rho))\,\rho$ is strictly concave and strictly decreasing in $[p^{-1}(w - V), p^{-1}(w)]$.

We introduce the following functions, see Figure 1:

$$\omega \colon \Omega_c \to \Omega_f^+, \qquad u = \omega(\bar{u}) \iff \begin{cases} \mathtt{w}(u) = \mathtt{w}(\bar{u}), \\ v = V, \end{cases} \tag{19a}$$

$$\mathtt{v}^\pm \colon \Omega \to \Omega_c, \qquad u = \mathtt{v}^\pm(\bar{u}) \iff \begin{cases} \mathtt{W}(u) = w^\pm, \\ v = \bar{v}, \end{cases} \tag{19b}$$

$$\mathtt{u}_* \colon \Omega^2 \to \Omega_c, \qquad u = \mathtt{u}_*(u_\ell, u_r) \iff \begin{cases} \mathtt{w}(u) = \mathtt{W}(u_\ell), \\ v = v_r, \end{cases} \tag{19c}$$

$$\Lambda \colon \big\{(u_\ell, u_r) \in \Omega^2 : \rho_\ell \neq \rho_r\big\} \to \mathbb{R}, \qquad \Lambda(u_\ell, u_r) \doteq \frac{f(u_r) - f(u_\ell)}{\rho_r - \rho_\ell}. \tag{19d}$$

Notice that:

- the point $\omega(\bar{u})$ is the intersection of Ω_f^+ and the Lax curve of the first characteristic family passing through $\bar{u}$;
- for any $w \in [w^-, w^+]$ the point $(p^{-1}(w), 0)$ is the intersection of the Lax curve of the first characteristic family corresponding to w and the segment $\{(\rho, v) \in \Omega_c : v = 0\}$;

- the point $\mathtt{v}^{\pm}(\bar{u})$ is the intersection of the Lax curve of the second characteristic family passing through $\bar{u}$ and $\{u \in \Omega_{\rm c} : \mathtt{w}(u) = w^{\pm}\}$;
- for any u_ℓ, $u_r \in \Omega_{\rm c}$ the point $\mathtt{u}_*(u_\ell, u_r)$ is the intersection between the Lax curve of the first characteristic family passing through u_ℓ and the Lax curve of the second characteristic family passing through u_r;
- $\Lambda(u_\ell, u_r)$ is the speed of a discontinuity (u_ℓ, u_r), that in the (ρ, f)-coordinates coincides with the slope of the segment connecting u_ℓ and u_r.

Observe that by definition $\mathtt{v}^{\pm}(\bar{u}) = \mathtt{u}_*((p^{-1}(w^{\pm}), 0), \bar{u})$ and $\omega(\bar{u}) = \mathtt{u}_*(\bar{u}, (0, V))$.

3.2 Riemann solvers

For completeness, we recall the definitions of the Riemann solvers $\mathcal{R}$ and $\mathcal{R}_F$ introduced in [10] and [19], associated to Riemann problem (7a) and to constrained Riemann problem (7a), (7c) with $\mathtt{F} \equiv F$ constant belonging to $[0, f^+]$, respectively, and used in Section 3.3 to define the approximate Riemann solvers $\mathcal{R}_n$ and $\mathcal{R}_{F,n}$.

We recall that Riemann problems for (7a) are Cauchy problems with initial condition of the form

$$u(0,x) = \begin{cases} u_\ell & \text{if } x < 0, \\ u_r & \text{if } x \geqslant 0. \end{cases} \tag{20}$$

Definition 3.1. *The Riemann solver $\mathcal{R}\colon \Omega^2 \to \mathbf{L}^\infty(\mathbb{R}; \Omega)$ associated to Riemann problem* (7a), (20) *is defined as follows.*

(R.1) If $u_\ell, u_r \in \Omega_{\rm f}$, then $\mathcal{R}[u_\ell, u_r]$ consists of a contact discontinuity (u_ℓ, u_r) with speed of propagation V.

(R.2) If $u_\ell, u_r \in \Omega_{\rm c}$, then $\mathcal{R}[u_\ell, u_r]$ consists of a 1-wave $(u_\ell, \mathtt{u}_*(u_\ell, u_r))$ and of a 2-contact discontinuity $(\mathtt{u}_*(u_\ell, u_r), u_r)$.

(R.3) If $u_\ell \in \Omega_{\rm c}^-$ and $u_r \in \Omega_{\rm f}^-$, then $\mathcal{R}[u_\ell, u_r]$ consists of a 1-rarefaction $(u_\ell, \omega(u_\ell))$ and a contact discontinuity $(\omega(u_\ell), u_r)$.

(R.4) If $u_\ell \in \Omega_{\rm f}^-$ and $u_r \in \Omega_{\rm c}^-$, then $\mathcal{R}[u_\ell, u_r]$ consists of a phase transition $(u_\ell, \mathtt{v}^-(u_r))$ and a 2-contact discontinuity $(\mathtt{v}^-(u_r), u_r)$.

Since $(t,x) \mapsto \mathcal{R}[u_\ell, u_r](x/t)$ does not in general satisfy constraint condition (7c) with $\mathtt{F} \equiv F$ constant belonging to $[0, f^+]$, we introduce

$$\begin{aligned} \mathcal{D}_F \doteq\ & \big\{(u_\ell, u_r) \in \Omega \times \Omega : f\big(\mathcal{R}[u_\ell, u_r](t, 0_\pm)\big) \leqslant F\big\} \\ =\ & \big\{(u_\ell, u_r) \in \Omega_{\rm f} \times \Omega_{\rm f} : f(u_\ell) \leqslant F\big\} \\ & \cup \big\{(u_\ell, u_r) \in \Omega_{\rm c} \times \Omega : f\big(\mathtt{u}_*(u_\ell, u_r)\big) \leqslant F\big\} \\ & \cup \big\{(u_\ell, u_r) \in \Omega_{\rm f}^- \times \Omega_{\rm c}^- : \min\big\{f(u_\ell), f\big(\mathtt{v}^-(u_r)\big)\big\} \leqslant F\big\}, \\ \mathcal{D}_F^{\complement} \doteq\ & \Omega^2 \setminus \mathcal{D}_F, \end{aligned}$$

and the constrained Riemann solver $\mathcal{R}_F$ in the following definition.

Definition 3.2. *The constrained Riemann solver $\mathcal{R}_F\colon \Omega^2 \to \mathbf{L}^\infty(\mathbb{R}; \Omega)$ associated to constrained Riemann problem* (7a), (7c), (20) *with $\mathtt{F} \equiv F$ constant belonging to $[0, f^+]$ is defined as*

$$\mathcal{R}_F[u_\ell, u_r](x) \doteq \begin{cases} \mathcal{R}[u_\ell, u_r](x) & \text{if } (u_\ell, u_r) \in \mathcal{D}_F, \\ \begin{cases} \mathcal{R}[u_\ell, \hat{\mathtt{u}}_\ell](x) & \text{if } x < 0, \\ \mathcal{R}[\check{\mathtt{u}}_r, u_r](x) & \text{if } x > 0, \end{cases} & \text{if } (u_\ell, u_r) \in \mathcal{D}_F^{\complement}, \end{cases}$$

where $\hat{\mathtt{u}}_\ell \doteq \hat{\mathtt{u}}(\mathtt{w}(u_\ell), F) \in \Omega_{\rm c}$ and $\check{\mathtt{u}}_r \doteq \check{\mathtt{u}}(v_r, F) \in \Omega$ are defined by (16).

In Figure 5 we clarify the selection criterion (16) for $\hat{\mathtt{u}}_\ell$ and $\check{\mathtt{u}}_r$ in the case $(u_\ell, u_r) \in \mathcal{D}_F^{\complement}$ and $F \in (0, f^-)$. We point out that $\hat{\mathtt{u}}_\ell$ and $\check{\mathtt{u}}_r$ satisfy the following general properties.

If $(u_\ell, u_r) \in \mathcal{D}_F^{\complement}$, then $\mathtt{w}(u_\ell) > \mathtt{w}(\check{\mathtt{u}}_r)$ and $v_r > \hat{\mathtt{v}}_\ell$.
If $(u_\ell, u_r) \in \mathcal{D}_F^{\complement}$ and $u_\ell \in \Omega_{\rm f}^-$, then $\mathtt{w}(\hat{\mathtt{u}}_\ell) = w^-$.
If $(u_\ell, u_r) \in \mathcal{D}_F^{\complement}$ and $u_r \in \Omega_{\rm f}$, then $\check{\mathtt{v}}_r = V$.

We recall that both $\mathcal{R}$ and $\mathcal{R}_F$ are $\mathbf{L^1_{loc}}$-continuous, see [19, Propositions 2 and 3].

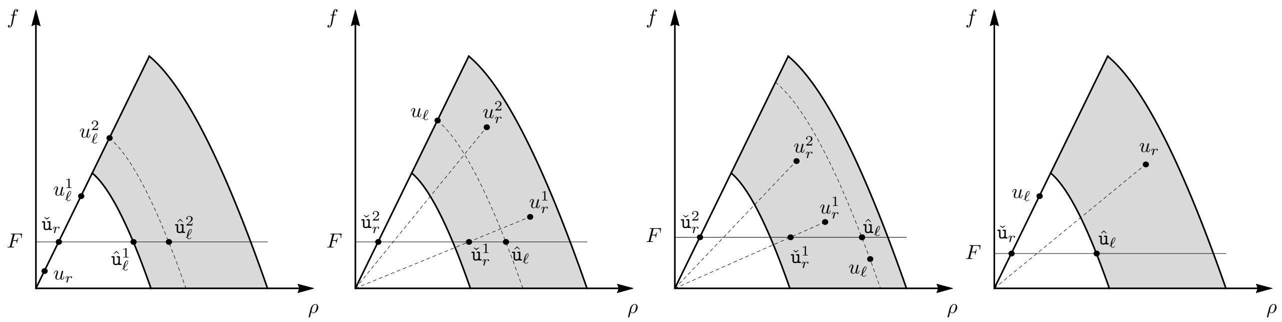

Figure 5: The selection criterion (16) for $\hat{\mathtt{u}}_\ell \doteq \hat{\mathtt{u}}(\mathtt{w}(u_\ell), F)$ and $\check{\mathtt{u}}_r \doteq \check{\mathtt{u}}(v_r, F)$ exploited in Definition 3.2 in the case $(u_\ell, u_r) \in \mathcal{D}_\ell^{\complement}$ and $F \in (0, f^-)$. In the first picture u_ℓ^1, u_ℓ^2 represent the left state in two different cases and $\hat{\mathtt{u}}_\ell^1$, $\hat{\mathtt{u}}_\ell^2$ are the corresponding $\hat{\mathtt{u}}_\ell$. Second and third pictures have analogous meaning, with u_r^1, u_r^2 and $\check{\mathtt{u}}_r^1$, $\check{\mathtt{u}}_r^2$.

3.3 The approximate Riemann solvers

For simplicity here and in the following we assume that $n \in \mathbb{N}$ is sufficiently large. For any $F \in [0, f^+]$ we introduce below a grid $\mathcal{G}_{F,n}$ in Ω and approximate Riemann solvers $\mathcal{R}_n$, $\mathcal{R}_{F,n} : \mathcal{G}_{F,n} \times \mathcal{G}_{F,n} \to \mathbf{PC}(\mathbb{R}; \mathcal{G}_{F,n})$.

The grid

We introduce in Ω a grid $\mathcal{G}_{F,n} \doteq \Omega \cap \mathcal{P}_{F,n}$, see Figure 6, with $\mathcal{P}_{F,n}$ given in the (v, w)-coordinates by

$$\left(\cup_{i=0}^{M\cdot 2^n} \left\{v^i\right\}\right) \times \left(\cup_{i=0}^{N\cdot 2^n} \left\{w^i\right\}\right),$$

where M, N, v^i and w^i, are defined as follows:

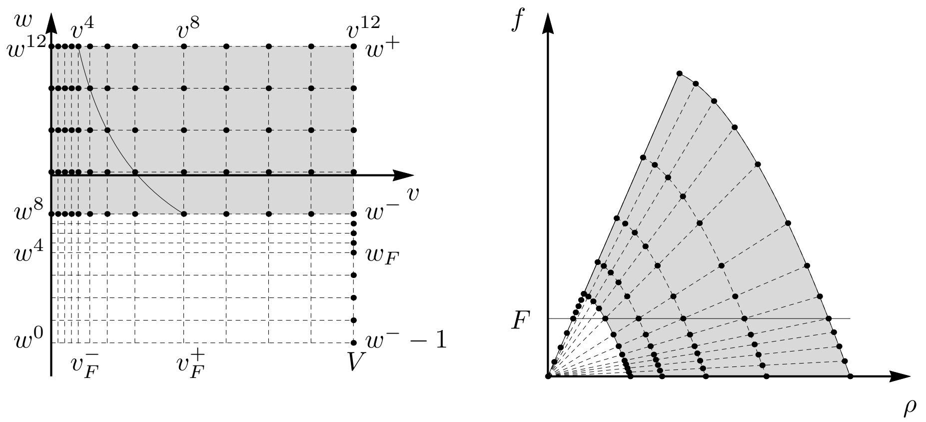

Figure 6: The grid $\mathcal{G}_{F,n}$ corresponding to $F \in (0, f^-)$ and $n = 2$. The curve in the figure on the left is the support of Ξ_F, which corresponds to (a portion of) the horizontal line in the figure on the right.

- If $F = 0$, then we let $M = 1$, $N = 2$,

$$w^i \doteq \begin{cases} w^- - 1 + i\, 2^{-n} & \text{if } i \in \{0, \ldots, 2^n\}, \\ w^- + (i - 2^n)\, 2^{-n}\, (w^+ - w^-) & \text{if } i \in \{2^n + 1, \ldots, 2 \cdot 2^n\}, \end{cases}$$

and

$$v^i \doteq i\, 2^{-n}\, V \quad \text{if } i \in \{0, \ldots, 2^n\}.$$

- If $F \in (0, f^-)$, then we let $M = 3$, $N = 3$,

$$w^i \doteq \begin{cases} w^- - 1 + i\, 2^{-n}\, (w_F - w^- + 1) & \text{if } i \in \{0, \ldots, 2^n\}, \\ w_F + (i - 2^n)\, 2^{-n}\, (w^- - w_F) & \text{if } i \in \{2^n + 1, \ldots, 2 \cdot 2^n\}, \\ w^- + (i - 2 \cdot 2^n)\, 2^{-n}\, (w^+ - w^-) & \text{if } i \in \{2 \cdot 2^n + 1, \ldots, 3 \cdot 2^n\}, \end{cases}$$

and

$$v^i \doteq \begin{cases} i\, 2^{-n}\, v_F^- & \text{if } i \in \{0, \ldots, 2^n\}\,, \\ \Xi_F^{-1}(w^{4 \cdot 2^n - i}) & \text{if } i \in \{2^n + 1, \ldots, 2 \cdot 2^n\}\,, \\ v_F^+ + (i - 2 \cdot 2^n)\; 2^{-n}\, (V - v_F^+) & \text{if } i \in \{2 \cdot 2^n + 1, \ldots, 3 \cdot 2^n\}\,. \end{cases}$$

- If $F \in [f^-, f^+]$, then we let $M = 2$, $N = 3$,

$$w^i \doteq \begin{cases} w^- - 1 + i\, 2^{-n} & \text{if } i \in \{0, \ldots, 2^n\}\,, \\ w^- + (i - 2^n)\; 2^{-n}\, (w_F - w^-) & \text{if } i \in \{2^n + 1, \ldots, 2 \cdot 2^n\}\,, \\ w_F + (i - 2 \cdot 2^n)\; 2^{-n}\; (w^+ - w_F) & \text{if } i \in \{2 \cdot 2^n + 1, \ldots, 3 \cdot 2^n\}\,, \end{cases}$$

and

$$v^i \doteq \begin{cases} i\, 2^{-n}\, v_F^- & \text{if } i \in \{0, \ldots, 2^n\}\,, \\ \Xi_F^{-1}(w^{4 \cdot 2^n - i}) & \text{if } i \in \{2^n + 1, \ldots, 2 \cdot 2^n\}\,. \end{cases}$$

Notice that if $F \in \{f^-, f^+\}$, then we do not necessarily have $w^i \neq w^{i+1}$.

The approximate Riemann solvers

An approximate solution $u_n \in \mathbf{PC}(\mathbb{R}; \mathcal{G}_{F,n})$ to (7) is constructed by applying the Riemann solvers $\mathcal{R}_n$, $\mathcal{R}_{F,n}$: $\mathcal{G}_{F,n} \times \mathcal{G}_{F,n} \to \mathbf{PC}(\mathbb{R}; \mathcal{G}_{F,n})$, in which rarefactions are replaced by piecewise constant rarefaction fans. More precisely, for any $(u_\ell, u_r) \in \mathcal{G}_{F,n} \times \mathcal{G}_{F,n}$ such that $\mathtt{w}_\ell = \mathtt{w}_r$ and $v_\ell = v^h < v_r = v^{h+k}$, we let

$$\mathcal{R}_n[u_\ell, u_r](\xi) \doteq \begin{cases} u_\ell & \text{if } \xi \leqslant \Lambda(u_\ell, u_1), \\ u_j & \text{if } \Lambda(u_{j-1}, u_j) < \xi \leqslant \Lambda(u_j, u_{j+1}),\ 1 \leqslant j \leqslant k-1, \\ u_r & \text{if } \xi > \Lambda(u_{k-1}, u_r), \end{cases}$$

where $u_0 \doteq u_\ell$, $u_k \doteq u_r$ and $u_j \in \mathcal{G}_{F,n}$ is such that $v_j \doteq v^{h+j}$ and $w_j = \mathtt{w}_\ell$. The Riemann solver $\mathcal{R}_{F,n}$ is defined as follows:

1. If $f\left(\mathcal{R}_n[u_\ell, u_r](0_\pm)\right) \leqslant F$, then $\mathcal{R}_{F,n}[u_\ell, u_r] \doteq \mathcal{R}_n[u_\ell, u_r]$.
2. If $f\left(\mathcal{R}_n[u_\ell, u_r](0_\pm)\right) > F$, then

$$\mathcal{R}_{F,n}[u_\ell, u_r](\xi) \doteq \begin{cases} \mathcal{R}_n[u_\ell, \hat{\mathtt{u}}_\ell](\xi) & \text{if } \xi < 0, \\ \mathcal{R}_n[\check{\mathtt{u}}_r, u_r](\xi) & \text{if } \xi \geqslant 0. \end{cases}$$

3.4 The approximate solution

In this section we apply a wave-front tracking algorithm to construct an approximate solution u_n in the space $\mathbf{PC}$ of piecewise constant functions taking finitely many values.

By assumption $\mathtt{F} \in \mathbf{PC}\big((0, \infty); [0, f^+]\big)$. Therefore there exist $F_i \in [0, f^+]$ and $t_i \geqslant 0$, $i \in \{0, 1, \ldots, N\}$, such that

$$F_i \neq F_{i+1}, \qquad t_i < t_{i+1}, \qquad \mathtt{F}(t) = F_i \quad \forall t \in (t_i, t_{i+1}),$$

with $t_0 = 0$ and $t_{N+1} = \infty$.

We recall that in the assumptions of theorem 2.8 we allow $F \equiv F(0) \in [0, f_+]$ when $N = 0$, while for $N \geqslant 1$ we assume $F_i \in [f_-, f_+]$ for $0 \leqslant i \leqslant N$. As stated in Corollary 2.9, one can also consider the intermediate situation in which $N \geqslant 1$, $F_0 \in [0, f_-)$ and $F_i \in [f_-, f_+]$ for $1 \leqslant i \leqslant N$.

An approximate solution $u_n \in \mathbf{PC}(\mathbb{R}_+ \times \mathbb{R}; \cup_{i=0}^N \mathcal{G}_{F_i,n})$ to (7) can be constructed as follows, using at every time step the projection on the corresponding grid. As a first step we consider the grid $\mathcal{G}_{F_0,n}$ and approximate the initial datum $\overline{u}$ with $\overline{u}_n^0 \in \mathbf{PC}(\mathbb{R}; \mathcal{G}_{F_0,n})$ such that

$$\begin{aligned} &\|\overline{v}_n^0\|_{\mathbf{L}^\infty} \leqslant \|\overline{v}^0\|_{\mathbf{L}^\infty}, \quad \mathrm{TV}(\overline{v}_n^0) \leqslant \mathrm{TV}(\overline{v}^0), \quad \|\overline{v}_n^0 - \overline{v}^0\|_{\mathbf{L}^1(K)} \leqslant \frac{C(K)}{2^n}, \\ &\|\overline{\mathtt{w}}_n^0\|_{\mathbf{L}^\infty} \leqslant \|\overline{\mathtt{w}}^0\|_{\mathbf{L}^\infty}, \quad \mathrm{TV}(\overline{\mathtt{w}}_n^0) \leqslant \mathrm{TV}(\overline{\mathtt{w}}^0), \quad \|\overline{\mathtt{w}}_n^0 - \overline{\mathtt{w}}^0\|_{\mathbf{L}^1(K)} \leqslant \frac{C(K)}{2^n}, \end{aligned} \tag{21}$$

where for every compact subset K of $\mathbb{R}$, $C(K)$ is a constant independent of n. Here

$$\overline{\mathtt{w}}_n^0 \doteq \mathtt{w}(\overline{u}_n^0).$$

The approximate solution u_n^0 is then obtained by gluing together the approximate solutions computed by applying $\mathcal{R}_{F_0,n}$ at $x=0$ at time $t=0$ and at any time a wave-front reaches $x=0$, and by applying $\mathcal{R}_n$ at any discontinuity of $\overline{u}_n^0$ away from $x=0$ or at any interaction between wave-fronts taking place away from $x=0$.

At time $t=t_1$ we restart the above construction by updating the constraint to F_1 and by using $u_n^0(t_1,\cdot)$ as initial datum. More precisely, we consider the grid $\mathcal{G}_{F_1,n}$ and approximate $u_n^0(t_1,\cdot) \in \mathbf{PC}(\mathbb{R};\mathcal{G}_{F_0,n})$ by $\overline{u}_n^1 \in \mathbf{PC}(\mathbb{R};\mathcal{G}_{F_1,n})$ such that

$$\|\overline{v}_n^1\|_{\mathbf{L}^\infty} \leqslant \|v_n^0(t_1,\cdot)\|_{\mathbf{L}^\infty}, \quad \mathrm{TV}(\overline{v}_n^1) \leqslant \mathrm{TV}(v_n^0(t_1,\cdot)), \quad \|\overline{v}_n^1 - v_n^0(t_1,\cdot)\|_{\mathbf{L}^1(K)} \leqslant \frac{C(K)}{2^n},$$

$$\|\overline{\mathtt{w}}_n^1\|_{\mathbf{L}^\infty} \leqslant \|\mathtt{w}_n^0(t_1,\cdot)\|_{\mathbf{L}^\infty}, \quad \mathrm{TV}(\overline{\mathtt{w}}_n^1) \leqslant \mathrm{TV}(\mathtt{w}_n^0(t_1,\cdot)), \quad \|\overline{\mathtt{w}}_n^1 - \mathtt{w}_n^0(t_1,\cdot)\|_{\mathbf{L}^1(K)} \leqslant \frac{C(K)}{2^n},$$

where

$$\overline{\mathtt{w}}_n^1 \doteq \mathtt{w}(\overline{u}_n^1).$$

The approximate solution u_n^1 is then obtained by gluing together the approximate solutions computed by applying $\mathcal{R}_{F_1,n}$ at $x=0$ at time $t=t_1$ and at any time a wave-front reaches $x=0$, and by applying $\mathcal{R}_n$ at any discontinuity of $\overline{u}_n^1$ away from $x=0$ or at any interaction between wave-fronts taking place away from $x=0$.

More in general, at time $t=t_i$ we update the constraint to F_i, consider the grid $\mathcal{G}_{F_i,n}$, approximate $u_n^{i-1}(t_i,\cdot) \in \mathbf{PC}(\mathbb{R};\mathcal{G}_{F_{i-1},n})$ with $\overline{u}_n^i \in \mathbf{PC}(\mathbb{R};\mathcal{G}_{F_i,n})$ such that

$$\begin{aligned}
&\|\overline{v}_n^i\|_{\mathbf{L}^\infty} \leqslant \|v_n^{i-1}(t_i,\cdot)\|_{\mathbf{L}^\infty}, \quad \mathrm{TV}(\overline{v}_n^i) \leqslant \mathrm{TV}(v_n^{i-1}(t_i,\cdot)), \quad \|\overline{v}_n^i - v_n^{i-1}(t_i,\cdot)\|_{\mathbf{L}^1(K)} \leqslant \frac{C(K)}{2^n},\\
&\|\overline{\mathtt{w}}_n^i\|_{\mathbf{L}^\infty} \leqslant \|\mathtt{w}_n^{i-1}(t_i,\cdot)\|_{\mathbf{L}^\infty}, \quad \mathrm{TV}(\overline{\mathtt{w}}_n^i) \leqslant \mathrm{TV}(\mathtt{w}_n^{i-1}(t_i,\cdot)), \quad \|\overline{\mathtt{w}}_n^i - \mathtt{w}_n^{i-1}(t_i,\cdot)\|_{\mathbf{L}^1(K)} \leqslant \frac{C(K)}{2^n},
\end{aligned} \tag{22}$$

where

$$\overline{\mathtt{w}}_n^i \doteq \mathtt{w}(\overline{u}_n^i).$$

The approximate solution u_n^i is then obtained by gluing together the approximate solutions computed by applying $\mathcal{R}_{F_i,n}$ at $x=0$ at time $t=t_i$ and at any time a wave-front reaches $x=0$, and by applying $\mathcal{R}_n$ at any discontinuity of $\overline{u}_n^1$ away from $x=0$ or at any interaction between wave-fronts taking place away from $x=0$.

By iterating the above procedure we obtain the approximate solution

$$u_n(t,x) = \sum_{i=0}^{N} u_n^i(t-t_i,x) \cdot \mathbb{1}_{(t_i,t_{i+1}]}(t). \tag{23}$$

As usual, in order to extend the construction globally in time we have to ensure that only finitely many interactions may occur in finite time. In Section 3.5 we prove that $u_n(t,\cdot)$ is well defined for all $t>0$ and belongs to $\mathbf{PC}(\mathbb{R}_+\times\mathbb{R};\cup_{i=0}^N\mathcal{G}_{F_i,n})$. Finally, in Section 3.6 we prove that u_n converges (up to a subsequence) in $\mathbf{L}^1_{\mathbf{loc}}$ to a limit u, which results to be an admissible solution to (7) in the sense of Definition 2.1.

3.5 A priori estimates

In this section we prove the main a priori estimates on the sequence of approximate solutions $\{u_n\}_n$ defined in (23). In Proposition 3.4 we state that $u_n(t,\cdot)$ takes values in $\mathcal{G}_{F_i,n}$ for any $t\in(t_i,t_{i+1}]$ and estimate $\mathrm{TV}\big(u_n(t,\cdot)\big)$ uniformly in n and t. This together with Proposition 3.6 guarantee that the number of interactions and the number of the discontinuities of u_n are both bounded globally in time.

We choose to study the total variation in the (v,w)-coordinates rather than in the (ρ,v)-coordinates. This choice is convenient to describe the grid, the approximate Riemann solvers and ease the forthcoming analysis, because the total variation of u_n in these coordinates does not increase after any interaction away from $x=0$. Furthermore, the entropy pairs defined in (9) in the (v,w)-coordinates are well defined, but in the (ρ,v)-coordinates are multi-valued at the vacuum.

Observe that any Contact Discontinuity (CD) has non-negative speed (of propagation), any Shock (S) or Rarefaction Shock (RS) has negative speed, all the Non-classical Shocks (NSs) are stationary and the speed of all the possible Phase Transitions (PTs) ranges in the interval $(-f^-/(p^{-1}(w^-)-\rho^-),V)$. Below we say that

(u_ℓ, u_r) is a null wave if $u_\ell = u_r$. Notice that if (u_ℓ, u_r) is a PT then $u_\ell \in \Omega_{\rm f}^-$ and $u_r \in \Omega_{\rm c}^-$, moreover if (u_ℓ, u_r) is a PT with $\mathtt{w}_r > w^-$ then $\rho_\ell = 0$.

Let u_n be an approximate solution of the form (23). For any $t > 0$, let $\sharp_n^i(t)$ and $\sharp_n(t)$ be the number of waves/discontinuities of $u_n^i(t,\cdot)$ and $u_n(t,\cdot)$, respectively. By definition (23) we have

$$\sharp_n(t) = \sum_{i=0}^{N} \sharp_n^i(t - t_i) \cdot \mathbb{1}_{(t_i, t_{i+1}]}(t). \tag{24}$$

Lemma 3.3. *We have that $\sharp_n \colon (0,\infty) \to \mathbb{R}_+$ is uniformly bounded.*

Proof. We have by construction that $\overline{u}_n^i \in \mathbf{PC}(\mathbb{R}; \mathcal{G}_{F_i,n})$ and by assumption that $F_i \in [0, f^+]$. We can therefore apply [9, Proposition 4.1] and obtain that $\sharp_n^i \colon (0,\infty) \to \mathbb{R}_+$, $i \in \{1,\ldots,N\}$, are uniformly bounded. Hence by (24) the proof is complete. □

We introduce $\mathcal{T}_n^i, \mathcal{T}_n \colon (0,\infty) \to \mathbb{R}_+$ and $\mathcal{T}^0 \geqslant 0$ defined as

$$\begin{aligned} \mathcal{T}_n^i(t) &\doteq \mathrm{TV}\big(v_n^i(t,\cdot)\big) + \mathrm{TV}\big(\mathtt{w}_n^i(t,\cdot)\big) + 2\hat{\Upsilon}_n^i(t) + 2\check{\Upsilon}_n^i(t), \\ \mathcal{T}_n(t) &\doteq \sum_{i=0}^{N} \mathcal{T}_n^i(t - t_i) \cdot \mathbb{1}_{(t_i,t_{i+1}]}(t), \\ \mathcal{T}^0 &\doteq \mathrm{TV}(\overline{v}) + \mathrm{TV}(\overline{\mathtt{w}}) + 2\hat{\Upsilon}^0 + 2\check{\Upsilon}^0, \end{aligned} \tag{25}$$

where for any $i \in \{0,\ldots,N\}$ we define

$$\begin{aligned} &\mathtt{w}_n^i \doteq \mathtt{w}(u_n^i), && \hat{\Upsilon}_n^i(t) \doteq \hat{\Upsilon}\big(u_n^i(t,\cdot), F_i\big), && \check{\Upsilon}_n^i(t) \doteq \check{\Upsilon}\big(u_n^i(t,\cdot), F_i\big), \\ &\overline{\mathtt{w}} \doteq \mathtt{w}(\overline{u}), && \hat{\Upsilon}^0(t) \doteq \hat{\Upsilon}(\overline{u}, F_0), && \check{\Upsilon}^0(t) \doteq \check{\Upsilon}(\overline{u}, F_0). \end{aligned}$$

Recall that $\hat{\Upsilon}$ and $\check{\Upsilon}$ are defined in (17). Conventionally, we assume that u_n^i, $i \in \{1,\ldots,N\}$, are left continuous in time, i.e., $u_n^i(t,\cdot) \equiv u_n^i(t_-,\cdot)$. Then by definition (23) we have that also u_n is left continuous in time. Hence by definition (25) also $\mathcal{T}_n$ is left continuous in time.

The next lemma gives uniform bounds on the total variation of the approximate solution u_n. For convenience, denote by C the uniform Lipschitz constant which existence is claimed in Lemma 2.6. We also need to guarantee that only finite number of fronts is generated by the algorithm in finite time. In order to count emerging fronts, let $\varepsilon_n^i > 0$ be the minimal (v,w)-distance between two points in the grid $\mathcal{G}_{F_i,n}$, namely

$$\varepsilon_n^i \doteq \min_{\substack{u^1,\, u^2 \in \mathcal{G}_{F_i,n} \\ u^1 \neq u^2}} \Big\{ \max\Big\{ |v^1 - v^2|, |\mathtt{w}(u^1) - \mathtt{w}(u^2)| \Big\} \Big\},$$

and define $\varepsilon_n \doteq \min\big\{\varepsilon_n^i : i \in \{0,\ldots,N\}\big\}$.

Proposition 3.4. *Fix $n \in \mathbb{N}$ sufficiently large. We have that:*

- **(a)** *for all $i \in \{0,\ldots,N\}$, the map $(t_i, t_{i+1}) \ni t \mapsto \mathcal{T}_n(t) \in \mathbb{R}_+$ is non-increasing and decreases by at least ε_n any time the number of waves increases;*
- **(b)** *$(0,\infty) \ni t \mapsto \mathcal{T}_n(t) \in \mathbb{R}_+$ is uniformly bounded by $(1+2C)^N \cdot \mathcal{T}^0$ with respect to $t > 0$ and $n \in \mathbb{N}$;*
- **(c)** *for all $i \in \{0,\ldots,N\}$, $u_n(t,\cdot) \in \mathbf{PC}(\mathbb{R}; \mathcal{G}_{F_i,n})$ for all $t \in (t_i, t_{i+1}]$.*

Proof. By [9, Proposition 4.1] we have that

- **(i)** the map $(0,\infty) \ni t \mapsto \mathcal{T}_n^i(t) \in \mathbb{R}_+$ is non-increasing and decreases by at least ε_n^i any time the number of waves increases;
- **(ii)** $u_n^i(t,\cdot) \in \mathbf{PC}(\mathbb{R}; \mathcal{G}_{F_i,n})$ for all $t > 0$.

Therefore by (23) and (25) properties **(a)** and **(c)** hold true. Property **(b)** follows readily from the definition of $\mathcal{T}$, the requirement that $F_i \in [f_-, f_+]$ for $i \geqslant 1$, and Lemma 2.6. □

Remark 3.5. *At this point, let us make apparent the difficulty in extension of our result to the case where* F_i *need not be restricted to values above the threshold* f^-. *Consider, e.g., the first re-meshing time* t_1. *Under the assumption* $F_1 \in [f^-, f^+]$, *we control* $\hat{\Upsilon}(\overline{u}_n^1, F_1)$, $\check{\Upsilon}(\overline{u}_n^1, F_1)$ *by the variation of* $\overline{u}_n^1$ *via the uniform Lipschitz constant of the maps* $\hat{u}, \check{u}$ *(see Lemma 2.6). When* $F_1 \in [0, f^-)$, *the discontinuity of the maps* $\hat{u}, \check{u}$ *makes this control impossible. There remains the eventuality of controlling* $\hat{\Upsilon}(\overline{u}_n^1, F_1)$, $\check{\Upsilon}(\overline{u}_n^1, F_1)$ *by the values* $\hat{\Upsilon}(u_n^0(t_1, \cdot), F_0)$, $\check{\Upsilon}(u_n^0(t_1, \cdot), F_0)$, *being understood that* $\overline{u}_n^1$ *is a projection of* $u_n^0(t_1, \cdot)$ *due to re-meshing. At this point, it is the change of the constraint level from* F_0 *to* F_1 *that creates a major difficulty: we are unable to control, e.g.,* $\hat{\Upsilon}(u_n^0(t_1, \cdot), F_1)$ *by* $\hat{\Upsilon}(u_n^0(t_1, \cdot), F_0)$ *without artificial restrictions. Note that the technique that was developed for handling the analogous difficulty in the case of the ARZ system (see [4]) does not extend to our case, due to the more complex definition of the interaction potentials* $\hat{\Upsilon}, \check{\Upsilon}$ *and to the fact that* $\mathtt{W}(u)$ *may fail to satisfy the conservation equation at* $x = 0$.

Beside the bound on the number of wave-fronts proved in Proposition 3.4, we need to bound also the number of interactions. This is the aim of the next proposition, which together with Proposition 3.4 ensure the global existence of u_n.

Proposition 3.6. *For any fixed* $n \in \mathbb{N}$ *sufficiently large, we have that the number of interactions occurring in the interval of time* $(0, \infty)$ *is bounded. In particular* u_n *is globally defined.*

Proof. By [9, Proposition 3.2] we have that the number of interactions involved in the construction of each u_n^i is bounded. Therefore the statement of the proposition follows directly from the definition (23) of u_n. □

3.6 Convergence

The convergence is proved by following the traditional method of proving compactness via Helly's theorem.

We observe that

$$|\rho_\ell - \rho_r| \leqslant L_\rho \left(|v_\ell - v_r| + |\mathtt{w}_\ell - \mathtt{w}_r|\right),$$

where $L_\rho \doteq \max\{\rho^-, \|1/p'\|_{\mathbf{L}^\infty([p^{-1}(w^-), p^{-1}(w^+)];\mathbb{R})}\}$, because

$$\rho_{\ell,r} = \begin{cases} p^{-1}(\mathtt{w}_{\ell,r} - v_{\ell,r}) & \text{if } \mathtt{w}_{\ell,r} \in [w^-, w^+], \\ (\mathtt{w}_{\ell,r} + 1 - w^-)\, \rho^- & \text{if } \mathtt{w}_{\ell,r} \in [w^- - 1, w^-). \end{cases}$$

As a consequence $\mathrm{TV}(\rho) \leqslant L_\rho\, (\mathrm{TV}(v) + \mathrm{TV}(\mathtt{w}))$, hence

$$\mathrm{TV}(u) \leqslant (1 + L_\rho) \big(\mathrm{TV}(v) + \mathrm{TV}(\mathtt{w})\big).$$

Moreover, by Proposition 3.4 we have that for any $t > 0$

$$\mathrm{TV}\big(v_n(t, \cdot)\big) + \mathrm{TV}\big(\mathtt{w}_n(t, \cdot)\big) \leqslant \mathcal{T}_n(t) \leqslant C\, \mathcal{T}^0,$$

and therefore

$$\mathrm{TV}(u_n(t, \cdot)) \leqslant K \doteq (1 + L_\rho)\, C\, \mathcal{T}^0. \tag{26}$$

Proposition 3.7. *Fix* $i \in \{0, 1, \ldots, N\}$. *For any* $s, t \in (t_i, t_{i+1})$ *we have*

$$\|u_n(t, \cdot) - u_n(s, \cdot)\|_{\mathbf{L}^1(\mathbb{R};\Omega)} \leqslant L\, |t - s|, \tag{27}$$

with $L \doteq K \max\{V, R\, p'(R)\}$ *which does not depend on* i *or* n.

Proof. If no interaction occurs for times between s and t, then

$$\begin{aligned} \|u_n(t, \cdot) - u_n(s, \cdot)\|_{\mathbf{L}^1(\mathbb{R};\Omega)} \leqslant & \sum_{i \in \mathsf{D}(t)} \left|(t - s)\, \dot{\delta}_n^i(t) \left(\rho_n\big(t, \delta_n^i(t)_-\big) - \rho_n\big(t, \delta_n^i(t)_+\big)\right)\right| \\ & + \sum_{i \in \mathsf{D}(t)} \left|(t - s)\, \dot{\delta}_n^i(t) \left(v_n\big(t, \delta_n^i(t)_-\big) - v_n\big(t, \delta_n^i(t)_+\big)\right)\right| \leqslant L\, |t - s|, \end{aligned}$$

where $\delta_n^i(t) \in \mathbb{R}$, $i \in \mathsf{D}(t) \subset \mathbb{N}$, are the positions of the discontinuities of $u_n(t, \cdot)$. The case when one or more interactions take place for times between t and s is similar, because by the finite speed of propagation of the waves the map $(t_i, t_{i+1}) \ni t \mapsto u_n(t, \cdot)$ is $\mathbf{L^1_{loc}}$-continuous across interaction times. □

In general u_n is not Lipschitz continuous in time with respect to the $\mathbf{L^1}$-norm in space, namely, (27) does not hod true for all $s, t > 0$. Indeed, the approximation at time t_i of $u_n^{i-1}(t_i, \cdot)$ with $\overline{u}_n^i$ satisfies conditions listed in (22), which do not guarantee the continuity across t_i. This prevents an application of Helly's theorem directly to u_n. However, if we set $\Delta_n^i(x) \doteq u_n(t_i^+, x) - u_n(t_i^-, x)$, we observe that the functions u_n^{Lip} defined as

$$u_n^{\mathrm{Lip}}(t,x) \doteq u_n(t,x) - \sum_{i=1}^{N} \Delta_n^i(x) \cdot \mathbb{1}_{[t_i,\infty)}(t),$$

are Lipschitz continuous in time. The most significative situation is with $t_{i+1} > t > t_i > s > t_{i-1}$ for some $i \in \{1, \ldots, N\}$, and in this case

$$\begin{aligned}\|u_n^{\mathrm{Lip}}(t,\cdot) - u_n^{\mathrm{Lip}}(s,\cdot)\|_{\mathbf{L^1}(\mathbb{R};\Omega)} &= \|u_n(t,\cdot) - \Delta_n^i(\cdot) - u_n(s,\cdot)\|_{\mathbf{L^1}(\mathbb{R};\Omega)} \\ &\leqslant \|u_n(t,\cdot) - u_n(t_i^+, x)\|_{\mathbf{L^1}(\mathbb{R};\Omega)} + \|u_n(t_i^-, x) - u_n(s,\cdot)\|_{\mathbf{L^1}(\mathbb{R};\Omega)} \leqslant L\,|t-s|.\end{aligned}$$

All other cases are similar.

From the construction of the solutions u_n detailed in Section 3.4 and in particular from (22) (note that the constant $C(K)$ in (22) depends neither on n nor on F_i) we have that $u_n^{\mathrm{Lip}} - u_n$ converges to the null function in $\mathbf{L^1_{loc}}$ as n goes to infinity. At the same time the sequence $\{u_n^{\mathrm{Lip}}\}_n$ satisfies all the requirement of Helly's Theorem, so that it converges (up to a subsequence) in $\mathbf{L^1_{loc}}(\mathbb{R}_+ \times \mathbb{R}; \Omega)$ to a function $u \in \mathbf{L^\infty}(\mathbb{R}_+; \mathbf{BV}(\mathbb{R}; \Omega)) \cap \mathbf{C^0}(\mathbb{R}_+; \mathbf{L^1_{loc}}(\mathbb{R}; \Omega))$ and the limit satisfies the estimates listed in (18). Although the a.e. convergence is enough for the sake of the proof of Theorem 2.8, its extension sketched in Section 5 requires convergence of $\{u_n\}_n$ in the topology of $\mathbf{L^\infty}(\mathbb{R}_+; \mathbf{L^1_{loc}}(\mathbb{R}; \Omega))$. Observe that the above arguments do guarantee this convergence.

3.7 Characterization of the limit

We start by focusing on the renormalization property and the way it is used to handle entropy inequalities: these are the key arguments of the characterization of admissible discontinuities at the constraint. First, based on the fact that wave-front tracking solutions are piecewise constant weak solutions of the problem except at times $t = t_i$ and eventually at the constraint location $x = 0$, we readily assess the local renormalization property for approximate solutions.

Proposition 3.8. *The approximate solution u_n satisfies the renormalization property* (11) *with test functions supported in* $\{(t,x) : t_i < t < t_{i+1},\ x \neq 0\}$.

Proof. Let $\phi \in \mathbf{C_c^\infty}\big((t_i, t_{i+1}); \mathbb{R}_*\big)$ be a test function. Due to the discrete nature of u_n, without loss of generality we can assume that its support intersects only one discontinuity curve $x = \delta(t)$ of u_n. We denote $u_+^n \doteq u_n(t, \delta(t)_+)$ and so on, and for simplicity of notation we drop the subscript n. The Rankine-Hugoniot conditions (13), (14) along $x = \delta(t)$ give

$$\begin{aligned}&\begin{cases}\rho_+\,\dot\delta(t) - f(u_+) = \rho_-\,\dot\delta(t) - f(u_-),\\ \big[\rho_+\,\dot\delta(t) - f(u_+)\big]\,\mathtt{W}(u_+) = \big[\rho_-\,\dot\delta(t) - f(u_-)\big]\,\mathtt{W}(u_-),\end{cases}\\ \Longleftrightarrow &\begin{cases}\rho_+\,\dot\delta(t) = f(u_+),\\ \rho_-\,\dot\delta(t) = f(u_-),\end{cases} \quad\text{or}\quad \begin{cases}\mathtt{W}(u_+) = \mathtt{W}(u_-),\\ \rho_+\,\dot\delta(t) - f(u_+) = \rho_-\,\dot\delta(t) - f(u_-).\end{cases}\end{aligned}$$

In both the cases, for any continuous function $g \colon [w_-, w_+] \to \mathbb{R}$ we have

$$\big[\rho_+\,\dot\delta(t) - f(u_+)\big]\,g\big(\mathtt{W}(u_+)\big) = \big[\rho_-\,\dot\delta(t) - f(u_-)\big]\,g\big(\mathtt{W}(u_-)\big).$$

As a consequence we have that u satisfies the renormalization property in $\mathbb{R}^2_- \cup \mathbb{R}^2_+$. □

Now, we can adapt to the present framework [4, Proposition 3.1].

Proposition 3.9. *For any test function $\phi \in \mathbf{C_c^\infty}\big((0,\infty) \times \mathbb{R}; \mathbb{R}\big)$ such that* $\mathrm{supp}(\phi) \subset (t_i, t_{i+1}) \times \mathbb{R}$*, for an $i \in \{1, \ldots, N\}$, we have that*

$$\int_0^\infty \mathtt{N}^k_{\mathtt{F}(t)}\big(u_n(t, 0_-)\big)\,\phi(t,0)\,\mathrm{d}t = \int_0^\infty \int_{-\infty}^0 \Big(\rho_n\,\mathtt{N}^k_{\mathtt{F}(t)}(u_n)\,\psi_t\,\xi + f(u_n)\,\mathtt{N}^k_{\mathtt{F}(t)}(u_n)\,\psi\,\xi_x\Big)\,\mathrm{d}x\,\mathrm{d}t$$

where $\mathtt{N}^k_F(u)$ is defined by (8)*, $\psi(t) = \phi(t,0)$ and ξ is an arbitrary $\mathbf{C_c^\infty}(\mathbb{R};\mathbb{R})$ test function such that $\xi(0) = 1$.*

Proof. By Proposition 3.8 with $g(W) \doteq \mathtt{n}_F^k(W)$, see (8), we have that u_n satisfies the equation

$$\left(\rho_n\, \mathtt{n}_F^k\big(\mathtt{W}(u_n)\big)\right)_t + \left(f(u_n)\, \mathtt{n}_F^k\big(\mathtt{W}(u_n)\big)\right)_x = 0,$$

in the sense of distributions in $(t_i, t_{i+1}) \times (-\infty, 0)$. As a consequence by the Gauss-Green formula we have

$$\begin{aligned}\int_0^\infty \int_{-\infty}^0 \left(\rho_n\, \mathtt{n}_F^k\big(\mathtt{W}(u_n)\big)\, \psi_t\, \xi + f(u_n)\, \mathtt{n}_F^k\big(\mathtt{W}(u_n)\big)\, \psi\, \xi_x\right) \mathrm{d}x\, \mathrm{d}t &= \int_0^\infty f\big(u_n(t, 0_-)\big)\, \mathtt{n}_F^k\Big(\mathtt{W}\big(u_n(t, 0_-)\big)\Big)\, \psi\, \mathrm{d}x\, \mathrm{d}t \\ &= \int_0^\infty \mathtt{N}_F^k\Big(\mathtt{W}\big(u_n(t, 0_-)\big)\Big)\, \psi\, \mathrm{d}x\, \mathrm{d}t.\end{aligned}$$

□

Proposition 3.10. *Let* $\overline{u} \in \mathbf{L^1} \cap \mathbf{BV}(\mathbb{R}; \Omega)$ *and* $\mathtt{F} \in \mathbf{PC}\big((0, \infty); [0, f^+]\big)$ *satisfy* ***(H.1)*** *or* ***(H.2)****. If u is a limit of the sequence of approximate solutions $\{u_n\}_n$ constructed in Section 3.4, then u is a solution to constrained Cauchy problem* (7) *in the sense of Definition 2.1.*

Proof. We consider separately the conditions listed in Definition 2.1.

(S.1) Initial condition (7b) holds by (21), (27) and the $\mathbf{L^1_{loc}}$-convergence of u_n to u.

(S.2) We prove now (10), that is for any test function $\phi \in \mathbf{C_c^\infty}\big((0, \infty) \times \mathbb{R}; \mathbb{R}\big)$ we have

$$\int_0^\infty \int_{\mathbb{R}} \Big(\rho\, \phi_t + f(u)\, \phi_x\Big)\, \mathrm{d}x\, \mathrm{d}t = 0.$$

Choose $T > 0$ such that $\phi(t, x) = 0$ whenever $t \geqslant T$. Since u_n is uniformly bounded and f is uniformly continuous on bounded sets, it is sufficient to prove that

$$\int_0^T \int_{\mathbb{R}} \Big(\rho_n\, \phi_t + f(u_n)\, \phi_x\Big)\, \mathrm{d}x\, \mathrm{d}t \to 0. \tag{28}$$

By the Gauss-Green formula the double integral above can be written as

$$\int_0^T \sum_{j \in \mathsf{D}(t)} \left(\dot{\delta}_n^j(t)\, \Delta\rho_n^j(t) - \Delta f_n^j(t)\right) \phi\big(t, \delta_n^j(t)\big)\, \mathrm{d}t + \sum_{i=1}^N \int_{\mathbb{R}} \big(\rho_n(t_i^-, x) - \rho_n(t_i^+, x)\big)\, \phi(t_i, x)\, \mathrm{d}x,$$

where

$$\Delta\rho_n^j(t) \doteq \rho_n\big(t, \delta_n^j(t)_+\big) - \rho_n\big(t, \delta_n^j(t)_-\big), \qquad \Delta f_n^j(t) \doteq f\Big(u_n\big(t, \delta_n^j(t)_+\big)\Big) - f\Big(u_n\big(t, \delta_n^j(t)_-\big)\Big).$$

By construction any discontinuity of $u_n(t, \cdot)$ satisfies the first Rankine-Hugoniot condition (13), therefore

$$\dot{\delta}_n^j(t)\, \Delta\rho_n^j(t) - \Delta f_n^j(t) = 0, \qquad j \in \mathsf{D}(t).$$

Moreover we have

$$\int_{\mathbb{R}} \sum_{i=1}^N \big(\rho_n(t_i^-, x) - \rho_n(t_i^+, x)\big)\, \phi(t_i, x)\, \mathrm{d}x \leqslant \|\phi\|_{\mathbf{L^\infty}}\, L_\rho \sum_{i=1}^N \Big(\|\overline{v}_n^i - v_n^{i-1}(t_i, \cdot)\|_{\mathbf{L^1_{loc}}} + \|\overline{\mathtt{w}}_n^i - \mathtt{w}_n^{i-1}(t_i, \cdot)\|_{\mathbf{L^1_{loc}}}\Big)$$

and (28) is trivial.

(S.3) Property (11) follows by Proposition 3.8, with the contribution of the restart times t_i controlled in the same way as in the above proof of property **(S.2)**.

(S.4) We prove now (12), namely that for any $k \in [0, V]$ and $\phi \in \mathbf{C_c^\infty}((0, \infty) \times \mathbb{R}; \mathbb{R})$ such that $\phi \geqslant 0$ we have

$$\int_0^\infty \left(\int_{\mathbb{R}} \Big(\mathtt{E}^k(u)\, \phi_t + \mathtt{Q}^k(u)\, \phi_x\Big)\, \mathrm{d}x + \mathtt{N}_{\mathtt{F}(t)}^k\big(u(t, 0_-)\big)\, \phi(t, 0)\right) \mathrm{d}t \geqslant 0,$$

where $\mathtt{N}_F^k(u)$, $\mathtt{E}^k$ and $\mathtt{Q}^k$ are defined in (8) and (9). From Proposition 3.9 follows that for all $t \notin \{t_1, \ldots, t_N\}$

$$\lim_{n \to \infty} \int_0^\infty \mathtt{N}_{\mathtt{F}(t)}^k\big(u_n(t, 0_-)\big)\, \phi(t, 0)\, \mathrm{d}t = \int_0^\infty \mathtt{N}_{\mathtt{F}(t)}^k\big(u(t, 0_-)\big)\, \phi(t, 0)\, \mathrm{d}t, \tag{29}$$

$$\lim_{n\to\infty}\int_0^\infty \mathtt{N}^k_{\mathtt{F}(t)}\big(u_n(t,0_+)\big)\,\phi(t,0)\,\mathrm{d}t = \int_0^\infty \mathtt{N}^k_{\mathtt{F}(t)}\big(u(t,0_+)\big)\,\phi(t,0)\,\mathrm{d}t.$$

Choose $T>0$ such that $\phi(t,x)=0$ whenever $t\geqslant T$. By the a.e. convergence of u_n to u and the uniform continuity of $\mathtt{E}^k$ and $\mathtt{Q}^k$, it is sufficient to prove that

$$\liminf_{n\to\infty}\int_0^T\int_{\mathbb{R}}\Big(\mathtt{E}^k(u_n)\,\phi_t+\mathtt{Q}^k(u_n)\,\phi_x\Big)\,\mathrm{d}x\,\mathrm{d}t\geqslant 0. \tag{30}$$

By the Gauss-Green formula the double integral above can be written as

$$\int_0^T\sum_{j\in\mathtt{D}(t)}\Big(\dot\delta_n^j(t)\,\Delta\mathtt{E}_n^{k,j}(t)-\Delta\mathtt{Q}_n^{k,j}(t)\Big)\,\phi\big(t,\delta_n^j(t)\big)\,\mathrm{d}t+\sum_{i=1}^N\int_{\mathbb{R}}\Big(\mathtt{E}^k(u_n(t_i^-,x))-\mathtt{E}^k(u_n(t_i^+,x))\Big)\phi(t_i,x)\,\mathrm{d}x$$

$$\geqslant\int_0^T\sum_{j\in\mathtt{D}(t)}\Big(\dot\delta_n^j(t)\,\Delta\mathtt{E}_n^{k,j}(t)-\Delta\mathtt{Q}_n^{k,j}(t)\Big)\,\phi\big(t,\delta_n^j(t)\big)\,\mathrm{d}t-\sum_{i=1}^N\int_{\mathbb{R}}\Big|\mathtt{E}^k(u_n(t_i^-,x))-\mathtt{E}^k(u_n(t_i^+,x))\Big|\phi(t_i,x)\,\mathrm{d}x.$$

Since the last sum above converges to zero as n goes to infinity, to prove (30) it is sufficient to show that

$$\liminf_{n\to\infty}\int_0^T\sum_{j\in\mathtt{D}(t)}\Big(\dot\delta_n^j(t)\,\Delta\mathtt{E}_n^{k,j}(t)-\Delta\mathtt{Q}_n^{k,j}(t)\Big)\,\phi\big(t,\delta_n^j(t)\big)\,\mathrm{d}t\geqslant 0, \tag{31}$$

where

$$\Delta\mathtt{E}_n^{k,j}(t)\doteq\mathtt{E}^k\Big(u_n\big(t,\delta_n^j(t)_+\big)\Big)-\mathtt{E}^k\Big(u_n\big(t,\delta_n^j(t)_-\big)\Big),\qquad \Delta\mathtt{Q}_n^{k,j}(t)\doteq\mathtt{Q}^k\Big(u_n\big(t,\delta_n^j(t)_+\big)\Big)-\mathtt{Q}^k\Big(u_n\big(t,\delta_n^j(t)_-\big)\Big).$$

To estimate the integral in (31) we have to distinguish the following cases.

- If the j-th discontinuity is a PT, then we let $x\doteq\delta_n^j(t)$ and observe that

$$\begin{aligned}&\rho_n(t,x_-)<\min\big\{\rho_n(t,x_+),p^{-1}(w^--k)\big\}, && \dot\delta_n^j(t)=\Lambda\big(u_n(t,x_-),u_n(t,x_+)\big),\\ &v_n(t,x_-)=V>v_n(t,x_+), && \mathtt{W}\big(u_n(t,x_-)\big)=w^-\leqslant\mathtt{w}\big(u_n(t,x_+)\big)=\mathtt{W}\big(u_n(t,x_+)\big),\end{aligned}$$

hence

$$\Delta\mathtt{E}_n^{k,j}(t)=\begin{cases}\dfrac{\rho_n(t,x_+)}{\rho_{n,+}^k}-1 & \text{if } v_n(t,x_+)<k\leqslant V,\\ 0 & \text{if } k\leqslant v_n(t,x_+),\end{cases}$$

$$-\Delta\mathtt{Q}_n^{k,j}(t)=\begin{cases}k-\dfrac{f\big(u_n(t,x_+)\big)}{\rho_{n,+}^k} & \text{if } v_n(t,x_+)<k\leqslant V,\\ 0 & \text{if } k\leqslant v_n(t,x_+),\end{cases}$$

where $\rho_{n,+}^k\doteq p^{-1}(\mathtt{w}(u_n(t,x_+))-k)$. If $v_n(t,x_+)<k\leqslant V$, then

$$\begin{aligned}&\dot\delta_n^j(t)\,\Delta\mathtt{E}_n^{k,j}(t)-\Delta\mathtt{Q}_n^{k,j}(t)\\ &=\Lambda\big(u_n(t,x_-),u_n(t,x_+)\big)\left[\frac{\rho_n(t,x_+)}{\rho_{n,+}^k}-1\right]+k-\frac{f\big(u_n(t,x_+)\big)}{\rho_{n,+}^k}\\ &=\underbrace{\left[\frac{\rho_n(t,x_+)}{\rho_{n,+}^k}-1\right]}_{>0}\underbrace{\Big[\Lambda\big(u_n(t,x_-),u_n(t,x_+)\big)-\Lambda\big((\rho_{n,+}^k,k),u_n(t,x_+)\big)\Big]}_{>0}>0.\end{aligned}$$

- If the j-th discontinuity is a CD, then we let $x\doteq\delta_n^j(t)$ and observe that $\dot\delta_n^j(t)=v_n(t,x_-)=v_n(t,x_+)$ implies that $\dot\delta_n^j(t)\,\Delta\mathtt{E}_n^{k,j}(t)-\Delta\mathtt{Q}_n^{k,j}(t)=0$.
- If the j-th discontinuity is a S, then we let $x\doteq\delta_n^j(t)$ and observe that

$$\begin{aligned}&\rho_n(t,x_-)<\rho_n(t,x_+), && f\big(u_n(t,x_-)\big)>f\big(u_n(t,x_+)\big), && \dot\delta_n^j(t)=\Lambda\big(u_n(t,x_-),u_n(t,x_+)\big)<0,\\ &v_n(t,x_-)>v_n(t,x_+), && && \mathtt{w}_\pm\doteq\mathtt{w}\big(u_n(t,x_-)\big)=\mathtt{w}\big(u_n(t,x_+)\big)\geqslant w^-,\end{aligned}$$

hence

$$\Delta \mathtt{E}_n^{k,j}(t) = \begin{cases} \dfrac{\rho_n(t,x_+) - \rho_n(t,x_-)}{p^{-1}(\mathtt{w}_\pm - k)} & \text{if } v_n(t,x_+) < v_n(t,x_-) < k, \\ \dfrac{\rho_n(t,x_+)}{p^{-1}(\mathtt{w}_\pm - k)} - 1 & \text{if } v_n(t,x_+) < k \leqslant v_n(t,x_-), \\ 0 & \text{if } k \leqslant v_n(t,x_+) < v_n(t,x_-), \end{cases}$$

$$-\Delta \mathtt{Q}_n^{k,j}(t) = \begin{cases} \dfrac{f\big(u_n(t,x_-)\big) - f\big(u_n(t,x_+)\big)}{p^{-1}(\mathtt{w}_\pm - k)} & \text{if } v_n(t,x_+) < v_n(t,x_-) < k, \\ k - \dfrac{f\big(u_n(t,x_+)\big)}{p^{-1}(\mathtt{w}_\pm - k)} & \text{if } v_n(t,x_+) < k \leqslant v_n(t,x_-), \\ 0 & \text{if } k \leqslant v_n(t,x_+) < v_n(t,x_-). \end{cases}$$

If $k > v_n(t,x_-)$ or $k \leqslant v_n(t,x_+)$, then it is immediate to see that $\dot{\delta}_n^j(t)\, \Delta \mathtt{E}_n^{k,j}(t) - \Delta \mathtt{Q}_n^{k,j}(t) = 0$. Furthermore, if $v_n(t,x_+) < k \leqslant v_n(t,x_-)$, then

$$\begin{aligned} &\dot{\delta}_n^j(t)\, \Delta \mathtt{E}_n^{k,j}(t) - \Delta \mathtt{Q}_n^{k,j}(t) \\ &= \Lambda\big(u_n(t,x_-), u_n(t,x_+)\big) \left[\frac{\rho_n(t,x_+)}{p^{-1}(\mathtt{w}_\pm - k)} - 1\right] + k - \frac{f\big(u_n(t,x_+)\big)}{p^{-1}(\mathtt{w}_\pm - k)} \\ &= \underbrace{\left[\frac{\rho_n(t,x_+)}{p^{-1}(\mathtt{w}_\pm - k)} - 1\right]}_{>0} \underbrace{\left[\Lambda\big(u_n(t,x_-), u_n(t,x_+)\big) - \Lambda\Big(\big(p^{-1}(\mathtt{w}_\pm - k), k\big), u_n(t,x_+)\Big)\right]}_{>0} > 0. \end{aligned}$$

- If the j-th discontinuity is a RS, then we let $x \doteq \delta_n^j(t)$ and observe that

$$\begin{aligned} &\rho_n(t,x_-) > \rho_n(t,x_+), && f\big(u_n(t,x_-)\big) < f\big(u_n(t,x_+)\big), && \dot{\delta}_n^j(t) = \Lambda\big(u_n(t,x_-), u_n(t,x_+)\big) < 0, \\ &v_n(t,x_-) < v_n(t,x_+), && && \mathtt{w}_\pm \doteq \mathtt{w}\big(u_n(t,x_-)\big) = \mathtt{w}\big(u_n(t,x_+)\big) \geqslant w^-, \end{aligned}$$

hence

$$\Delta \mathtt{E}_n^{k,j}(t) = \begin{cases} \dfrac{\rho_n(t,x_+) - \rho_n(t,x_-)}{p^{-1}(\mathtt{w}_\pm - k)} & \text{if } v_n(t,x_-) < v_n(t,x_+) < k, \\ \dfrac{\rho_n(t,x_-)}{p^{-1}(\mathtt{w}_\pm - k)} - 1 & \text{if } v_n(t,x_-) < k \leqslant v_n(t,x_+), \\ 0 & \text{if } k \leqslant v_n(t,x_-) < v_n(t,x_+), \end{cases}$$

$$-\Delta \mathtt{Q}_n^{k,j}(t) = \begin{cases} \dfrac{f\big(u_n(t,x_-)\big) - f\big(u_n(t,x_+)\big)}{p^{-1}(\mathtt{w}_\pm - k)} & \text{if } v_n(t,x_-) < v_n(t,x_+) < k, \\ \dfrac{f\big(u_n(t,x_-)\big)}{p^{-1}(\mathtt{w}_\pm - k)} - k & \text{if } v_n(t,x_-) < k \leqslant v_n(t,x_+), \\ 0 & \text{if } k \leqslant v_n(t,x_-) < v_n(t,x_+). \end{cases}$$

If $k > v_n(t,x_+)$ or $k \leqslant v_n(t,x_-)$, then it is immediate to see that $\dot{\delta}_n^j(t)\, \Delta \mathtt{E}_n^{k,j}(t) - \Delta \mathtt{Q}_n^{k,j}(t) = 0$. Furthermore, if $v_n(t,x_-) < k \leqslant v_n(t,x_+)$, then

$$\begin{aligned} &\dot{\delta}_n^j(t)\, \Delta \mathtt{E}_n^{k,j}(t) - \Delta \mathtt{Q}_n^{k,j}(t) \\ &= \Lambda\big(u_n(t,x_-), u_n(t,x_+)\big) \left[\frac{\rho_n(t,x_-)}{p^{-1}(\mathtt{w}_\pm - k)} - 1\right] + \frac{f\big(u_n(t,x_-)\big)}{p^{-1}(\mathtt{w}_\pm - k)} - k \\ &= \underbrace{\left[\frac{\rho_n(t,x_-)}{p^{-1}(\mathtt{w}_\pm - k)} - 1\right]}_{>0} \underbrace{\left[\Lambda\big(u_n(t,x_-), u_n(t,x_+)\big) + \Lambda\Big(u_n(t,x_-), \big(p^{-1}(\mathtt{w}_\pm - k), k\big)\Big)\right]}_{<0} \\ &\geqslant -\frac{2}{\rho^-}\, p^{-1}(\mathtt{w}_\pm)\, p'\big(p^{-1}(\mathtt{w}_\pm)\big) \left[\rho_n(t,x_-) - \rho_n(t,x_+)\right] \end{aligned}$$

because $\rho_n(t,x_-) > p^{-1}(\mathtt{w}_\pm - k) \geqslant \rho_n(t,x_+) \geqslant \rho^-$ and because by the concavity of $\mathfrak{L}_{\mathtt{w}_\pm}(\rho) = (\mathtt{w}_\pm - p(\rho))\,\rho$ we have

$$\begin{aligned} 0 &> \Lambda\big(u_n(t,x_-),u_n(t,x_+)\big) > \Lambda\Big(u_n(t,x_-),\big(p^{-1}(\mathtt{w}_\pm - k),k\big)\Big) \\ &> \mathfrak{L}'_{\mathtt{w}_\pm}\big(\rho_n(t,x_-)\big) = \mathtt{w}_\pm - p\big(\rho_n(t,x_-)\big) - \rho_n(t,x_-)\,p'\big(\rho_n(t,x_-)\big) \\ &\geqslant \mathfrak{L}'_{\mathtt{w}_\pm}\big(p^{-1}(\mathtt{w}_\pm)\big) = -p^{-1}(\mathtt{w}_\pm)\,p'\big(p^{-1}(\mathtt{w}_\pm)\big). \end{aligned}$$

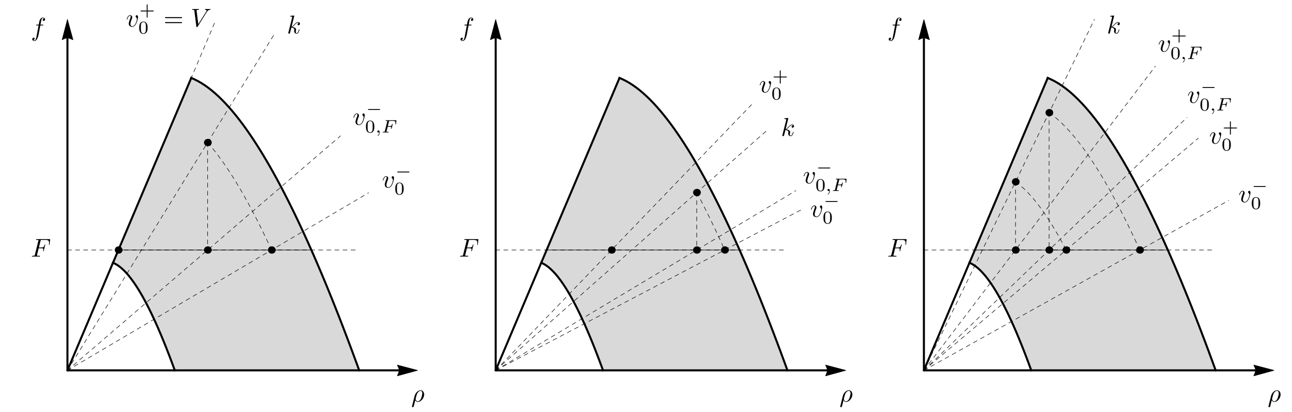

Figure 7: Above $F \in (f^-, f^+)$, $v_0^\pm \doteq v_n(t,0_\pm)$ and $v_{0,F}^\pm \doteq F/p^{-1}(\mathtt{W}(u_n(t,0_\pm)) - k)$. With the first two pictures we show that if $v_0^- < k < v_0^+$, then $v_{0,F}^- < k$. In the last picture we consider the case $v_0^- < v_0^+ < k$ and show that $v_{0,F}^- < v_{0,F}^+ < k$.

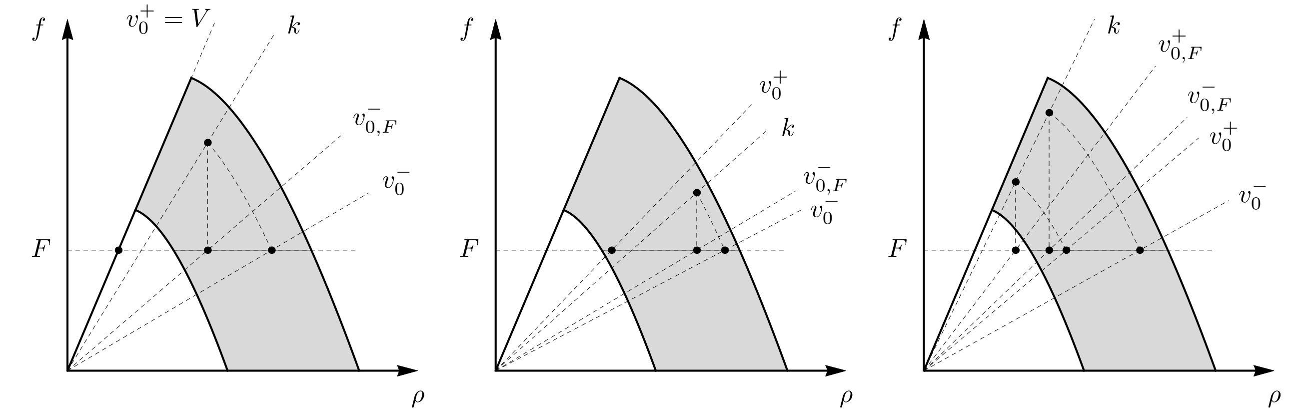

Figure 8: Above $F \in (0, f^-)$, $v_0^\pm \doteq v_n(t,0_\pm)$ and $v_{0,F}^\pm \doteq F/p^{-1}(\mathtt{W}(u_n(t,0_\pm)) - k)$. With the first two pictures we show that if $v_0^- < k < v_0^+$, then $v_{0,F}^- < k$. In the last picture we consider the case $v_0^- < v_0^+ < k$ and show that $v_{0,F}^- < v_{0,F}^+ < k$.

- If the j-th discontinuity is a NS occuring at $x = 0$, then

$$\begin{aligned} &\delta_n^j(t) = 0, && f\big(u_n(t,0_\pm)\big) = F, && v_F^- \leqslant v_n(t,0_-) < v_n(t,0_+), \\ &\dot\delta_n^j(t) = 0, && && \mathtt{w}\big(u_n(t,0_-)\big) = \mathtt{W}\big(u_n(t,0_-)\big) \geqslant \mathtt{W}\big(u_n(t,0_+)\big), \end{aligned}$$

hence

$$-\Delta\mathtt{Q}_n^{k,j}(t) = \begin{cases} \dfrac{F}{p^{-1}\Big(\mathtt{w}\big(u_n(t,0_-)\big) - k\Big)} - \dfrac{F}{p^{-1}\Big(\mathtt{W}\big(u_n(t,0_+)\big) - k\Big)} & \text{if } v_n(t,0_-) < v_n(t,0_+) < k, \\ \dfrac{F}{p^{-1}\Big(\mathtt{w}\big(u_n(t,0_-)\big) - k\Big)} - k & \text{if } v_n(t,0_-) < k \leqslant v_n(t,0_+), \\ 0 & \text{if } k \leqslant v_n(t,0_-) < v_n(t,0_+), \end{cases}$$

$$\mathtt{N}_F^k\big(u_n(t,0_-)\big) = \begin{cases} \left[k - \dfrac{F}{p^{-1}\Big(\mathtt{W}\big(u_n(t,0_-)\big) - k\Big)}\right]_+ & \text{if } F \neq 0,\\ k & \text{if } F = 0. \end{cases}$$

Notice that if $F = 0$, then $u_n(t,0_+) = (0,V)$ and $u_n(t,0_-) \in [p^{-1}(w^-), R] \times \{0\}$. We observe, see Figures 7 and 8, that $-\Delta\mathtt{Q}_n^{k,j}(t) < 0$ and that $-\Delta\mathtt{Q}_n^{k,j}(t) + \mathtt{N}_F^k(u_n(t,0_-)) \geqslant 0$ and therefore

$$\Big[\dot\delta_n^j(t)\,\Delta\mathtt{E}_n^{k,j}(t) - \Delta\mathtt{Q}_n^{k,j}(t)\Big]\,\phi\big(t,\delta_n^j(t)\big) + \mathtt{N}_F^k\big(u(t,0_-)\big)\,\phi(t,0) = \Big[-\Delta\mathtt{Q}_n^{k,j}(t) + \mathtt{N}_F^k\big(u_n(t,0_-)\big)\Big]\phi(t,0) \geqslant 0.$$

Thus, as in the cases above, we can conclude that (31) holds true. We underline that beside the NSs, the only possible stationary discontinuities at $x = 0$ are PTs and CDs, however in both of these cases we have $f(u_n(t,0_-)) = 0$ and therefore $\mathtt{N}_F^k(u_n(t,0_-)) = 0$.

The above case by case study shows that

$$\begin{aligned}
&\liminf_{n\to\infty} \int_0^T \int_{\mathbb{R}} \Big[\mathtt{E}^k(u_n)\,\phi_t + \mathtt{Q}^k(u_n)\,\phi_x\Big]\,\mathrm{d}x\,\mathrm{d}t\\
&= \liminf_{n\to\infty} \int_0^T \sum_{i\in\mathsf{RS}_n(t)} \Big[\dot\delta_n^j(t)\,\Delta\mathtt{E}_n^{k,j}(t) - \Delta\mathtt{Q}_n^{k,j}(t)\Big]\,\phi\big(t,\delta_n^j(t)\big)\,\mathrm{d}t\\
&\geqslant -\frac{2}{\rho^-} \max_{\rho\in[p^{-1}(w^-),R]} \big|\rho\,p'(\rho)\big| \liminf_{n\to\infty} \int_0^T \sum_{i\in\mathsf{RS}_n(t)} \Big[\rho_n\big(t,\delta_n^j(t)_-\big) - \rho_n\big(t,\delta_n^j(t)_+\big)\Big]\,\phi\big(t,\delta_n^j(t)\big)\,\mathrm{d}t\\
&\geqslant -\frac{2\,T}{\rho^-}\,\|\phi\|_{\mathbf{L}^\infty}\,K \max_{\rho\in[\rho^-,R]}\big|\rho\,p'(\rho)\big| \doteq -M,
\end{aligned}$$

where $\delta_n^j(t) \in \mathbb{R}$, $i \in \mathsf{RS}_n(t) \subset \mathbb{N}$, are the positions of the RSs of $u_n(t,\cdot)$ and K is defined in (26).
We claim that for any fixed $h > 0$, there exists a dense set $\mathcal{K}_h$ of values of k in $[0,V]$ such that

$$\liminf_{n\to\infty} \int_0^T \sum_{i\in\mathsf{RS}_n(t)} \Big[\dot\delta_n^j(t)\,\Delta\mathtt{E}_n^{k,j}(t) - \Delta\mathtt{Q}_n^{k,j}(t)\Big]\,\phi\big(t,\delta_n^j(t)\big)\,\mathrm{d}t \geqslant -\frac{1}{h}.$$

To prove it we fix a, $b \in [0,V]$ with $a < b$ and show that there exists $k \in (a,b)$ such that the above estimate is satisfied. Let $l \doteq \lceil 2(M\,h+1)/(b-a)\rceil$ and introduce the set

$$\mathcal{K}_h \doteq \frac{2\,\mathbb{N}+1}{l} \cap (a,b).$$

Let $\mathcal{E}_n > 0$ be the maximal (v,w)-distance between two "consecutive" points in the grid $\mathcal{G}_{F,n}$ having the same w-coordinate, namely, with a slight abuse of notations, we let

$$\mathcal{E}_n \doteq \max_{\substack{(v^j,w),\,(v^{j+1},w)\in\mathcal{G}_{F,n}\\ v^j\neq v^{j+1}}} (v^{j+1} - v^j).$$

Let $\mathfrak{n}_h \in \mathbb{N}$ be sufficiently large so that $\mathcal{E}_{\mathfrak{n}_h} < 2/l$. Take $n \geqslant \mathfrak{n}_h$. We claim that for any $j \in \mathsf{RS}_n(t)$ we have

$$\mathcal{K}_h \cap \Big(v_n\big(t,\delta_n^j(t)_-\big), v_n\big(t,\delta_n^j(t)_+\big)\Big)$$

has at most one element. Indeed, if $\mathcal{K}_h$ has more than one element then for any $i \in \mathsf{RS}_n(t)$ we have

$$v_n\big(t,\delta_n^j(t)_+\big) - v_n\big(t,\delta_n^j(t)_-\big) \leqslant \mathcal{E}_n < \frac{2}{l} = \min_{\substack{k^1,\,k^2\in\mathcal{K}_h\\ k^1\neq k^2}} |k^1 - k^2|.$$

As a consequence the sum

$$\sum_{k\in\mathcal{K}_h} \Big[\dot\delta_n^j(t)\,\Delta\mathtt{E}_n^{k,j}(t) - \Delta\mathtt{Q}_n^{k,j}(t)\Big]$$

has at most one nonzero element; moreover

$$-m\,\Big(\rho_n\big(t,\delta_n^j(t)_-\big) - \rho_n\big(t,\delta_n^j(t)_+\big)\Big) \leqslant \sum_{k\in\mathcal{K}_h} \Big[\dot\delta_n^j(t)\,\Delta\mathtt{E}_n^{k,j}(t) - \Delta\mathtt{Q}_n^{k,j}(t)\Big],$$

where

$$m \doteq \frac{2}{\rho^-} \max_{\rho \in [\rho^-, R]} \left|\rho\, p'(\rho)\right| = \frac{M}{T\, C\, \|\phi\|_{\mathbf{L}^\infty}}.$$

Therefore we find

$$\sum_{i \in \mathsf{RS}_n(t)} \sum_{k \in \mathcal{K}_h} \left[\dot{\delta}_n^j(t)\, \Delta\mathtt{E}_n^{k,j}(t) - \Delta\mathtt{Q}_n^{k,j}(t)\right] \geqslant -m\, K.$$

By exchanging the sums, multiplying by the test function and integrating in time we get

$$\sum_{k \in \mathcal{K}_h} \int_0^T \sum_{i \in \mathsf{RS}_n(t)} \left[\dot{\delta}_n^j(t)\, \Delta\mathtt{E}_n^{k,j}(t) - \Delta\mathtt{Q}_n^{k,j}(t)\right] \phi\big(t, \delta_n^j(t)\big)\, \mathrm{d}t \geqslant -M.$$

Moreover, by construction we have that $\mathcal{K}_h$ is a non-empty set with a finite number of elements (it has at most $h\, M$ elements), hence

$$h\, M \max_{k \in \mathcal{K}_h} \left[\int_0^T \sum_{i \in \mathsf{RS}_n(t)} \left[\dot{\delta}_n^j(t)\, \Delta\mathtt{E}_n^{k,j}(t) - \Delta\mathtt{Q}_n^{k,j}(t)\right] \phi\big(t, \delta_n^j(t)\big)\, \mathrm{d}t\right] \geqslant -M.$$

In conclusion we proved that there exists $k \in \mathcal{K}_h \subseteq (a, b)$ such that the above estimate is satisfied for any $n \geqslant \mathfrak{n}_h$; therefore, since $\mathcal{K}_h$ has a finite number of elements, we have

$$\liminf_{n \to \infty} \int_0^T \sum_{i \in \mathsf{RS}_n(t)} \left[\dot{\delta}_n^j(t)\, \Delta\mathtt{E}_n^{k,j}(t) - \Delta\mathtt{Q}_n^{k,j}(t)\right] \phi\big(t, \delta_n^j(t)\big)\, \mathrm{d}t \geqslant -\frac{1}{h}.$$

Since a and b are arbitrary, the above estimate holds true for a dense set of values of k in $[0, V]$.

Actually, the above estimate holds for any k in $[0, V]$ because the term in brackets in the above formula is continuous with respect to k. Finally, for the arbitrariness of h, we have that

$$\liminf_{n \to \infty} \int_0^T \sum_{i \in \mathsf{RS}_n(t)} \left[\dot{\delta}_n^j(t)\, \Delta\mathtt{E}_n^{k,j}(t) - \Delta\mathtt{Q}_n^{k,j}(t)\right] \phi\big(t, \delta_n^j(t)\big)\, \mathrm{d}t \geqslant 0$$

and this concludes the proof of (30).

(S.5) We prove now that (7c) holds for a.e. $t > 0$, namely

$$f\big(u(t, 0_\pm)\big) \leqslant F \qquad \text{for a.e. } t > 0.$$

By construction $f(u_n(t, 0_\pm)) \leqslant F$ for any $t > 0$, namely the approximate solutions satisfy (7c). Since weak convergence preserves pointwise inequalities, it is sufficient to prove that $f(u_n(t, 0_\pm))$ weakly converges to $f(u(t, 0_\pm))$. If ϕ is a smooth test function of time with compact support in $(0, \infty)$ and φ is a smooth test function of space with compact support and such that $\varphi(0) = 1$, then

$$\int_0^\infty f\big(u_n(t, 0_-)\big)\, \phi(t)\, \mathrm{d}t = \int_0^\infty \int_{-\infty}^0 \Big[\rho_n(t, x)\, \dot{\phi}(t)\, \varphi(x) + f\big(u_n(t, x)\big)\, \phi(t)\, \dot{\varphi}(x)\Big] \mathrm{d}x\, \mathrm{d}t.$$

The right-hand side passes to the limit, yielding the analogous expression with u_n replaced by u. By using again the Gauss-Green formula, one finally finds that

$$\lim_{n \to \infty} \int_0^\infty f\big(u_n(t, 0_-)\big)\, \phi(t)\, \mathrm{d}t = \int_0^\infty f\big(u(t, 0_-)\big)\, \phi(t)\, \mathrm{d}t.$$

As a consequence $f(u_n(t, 0_-))$ weakly converges to $f(u(t, 0_-))$, hence $f(u(t, 0_-)) \leqslant F$ for a.e. $t > 0$. At last, since we already proved that u satisfies the first Rankine-Hugoniot condition, we have $f(u(t, 0_-)) = f(u(t, 0_+))$, hence $f(u(t, 0_\pm)) \leqslant F$ for a.e. $t > 0$. □

4 Proof of Proposition 2.3

We limit ourself to the proof of the last statement, which deals with non-classical discontinuities occurring at $x = 0$. Moreover, we consider only the case with $\mathtt{F} \equiv F$ constant belonging to $[0, f^+]$; the general case is analogous. The statement is obvious if $F = 0$, due to **(S.5)** and the fact that $f(u) \geqslant 0$. We can therefore assume that $F > 0$ and that $x \mapsto u(t_0, x)$ has a (stationary) non-classical shock (u_ℓ, u_r), with $v_\ell < v_r$ and $f(u_\ell) = f(u_r) \doteq f \leqslant F$. We want to prove that $f = F$. Consider the test function

$$\phi(t,x) \doteq \left[\int_{|x|-\varepsilon}^{\infty} \varphi_\varepsilon(z)\,\mathrm{d}z\right]\left[\int_{t-t_0+\varepsilon}^{t-t_0+2\varepsilon} \varphi_\varepsilon(z)\,\mathrm{d}z\right],$$

where φ_ε is a smooth approximation of the Dirac mass centered at 0_+, $\delta^D_{0_+}$, namely

$$\varphi_\varepsilon \in \mathbf{C_c^\infty}(\mathbb{R};\mathbb{R}_+),\ \varepsilon > 0,\ \mathrm{supp}(\delta_\varepsilon) \subseteq [0,\varepsilon],\ \|\varphi_\varepsilon\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} = 1,\ \varphi_\varepsilon \to \delta^D_{0_+}.$$

Observe that as ε goes to zero

$$\begin{aligned}
&\phi(t_0,x) \equiv 0 \to 0,\\
&\phi(t,0) = \int_{t-t_0+\varepsilon}^{t-t_0+2\varepsilon} \varphi_\varepsilon(z)\,\mathrm{d}z \to \delta^D_{t_{0-}}(t),\\
&\phi_t(t,x) = \left[\int_{|x|-\varepsilon}^{\infty} \varphi_\varepsilon(z)\,\mathrm{d}z\right]\Big[\varphi_\varepsilon(t-t_0+2\varepsilon) - \varphi_\varepsilon(t-t_0+\varepsilon)\Big] \to 0,\\
&\chi_{\mathbb{R}_\pm}(x)\,\phi_x(t,x) \to \mp\,\delta^D_{0_\pm}(x)\,\delta^D_{t_{0-}}(t).
\end{aligned}$$

Then by (12) for all k belonging to the interval $(\hat{v}(w_\ell,F), \check{v}(v_r,F))$ we have

$$\begin{aligned}
&\mathtt{Q}^k(u_\ell) - \mathtt{Q}^k(u_r) + f\left[\frac{k}{F} - \frac{1}{p^{-1}\big(\mathtt{W}(u_\ell)-k\big)}\right]_+\\
&= \left[\frac{f}{p^{-1}\big(\mathtt{W}(u_\ell)-k\big)} - k\right] + f\left[\frac{k}{F} - \frac{1}{p^{-1}\big(\mathtt{W}(u_\ell)-k\big)}\right] = \left[\frac{f}{F} - 1\right]k \geqslant 0.
\end{aligned}$$

Since $f \leqslant F$, the above estimate implies that $f = F$ and this concludes the proof of the constraint saturation claim.

It remains to prove that $\mathtt{W}(u_n(t,\cdot))$ may only have decreasing jumps at $x = 0$, in the precise sense (15). To do so, let us observe that whatever be the jump in u_n at time t across $x = 0$, there holds

$$f\big(u_n(t,0_-)\big) = f\big(u_n(t,0_+)\big) \geqslant 0 \ \text{ and } \ \mathtt{W}\big(u_n(t,0_-)\big) \geqslant \mathtt{W}\big(u_n(t,0_+)\big). \tag{32}$$

It is possible to pass to the limit in (32) arguing in an indirect way. Indeed, we have the following weak form for comparing the fluxes of the generalized momentum at $x = 0^-$ and $x = 0^+$: for all $\phi \in \mathbf{C^0_c}(t_i, t_{i+1}); \mathbb{R}_+)$

$$\int_0^T f\big(u_n(t,0_-)\big)\mathtt{W}\big(u_n(t,0_-)\big)\phi(t,0)\,\mathrm{d}t \geqslant \int_0^T f\big(u_n(t,0_+)\big)\mathtt{W}\big(u_n(t,0_+)\big)\phi(t,0)\,\mathrm{d}t. \tag{33}$$

Arguing as in Proposition 3.9 or in the proof of the property **(S.5)** of the main theorem, we use the Gauss-Green theorem to convert each of the integrals at $x = 0^\pm$ into volumic terms (with integrals over $(0,T) \times \mathbb{R}_\pm$), and then pass to the limit as $n \to \infty$ using the stong convergence of u_n. This argument shows that (33) is inherited at the limit where u_n is replaced by u. Localization with the test function ensures (15).

5 Time-discrete non-local constraints in the phase transition model

Assume that we are given a map $\mathcal{Q} : \mathcal{E} \mapsto \mathcal{F}$, with

$$\mathcal{E} \doteq \mathbf{L^\infty}\big((0,\infty); \mathbf{BV}(\mathbb{R};\Omega)\big) \cap \mathbf{PC}\big(\mathbb{R}_+; \mathbf{L^1_{loc}}(\mathbb{R};\Omega)\big), \quad \mathcal{F} \doteq \mathbf{PC}\big((0,\infty); [f^-, f^+]\big).$$

We supply $\mathcal{E}$ with the $\mathbf{L^\infty}\big(\mathbb{R}_+; \mathbf{L^1_{loc}}(\mathbb{R};\Omega)\big)$ topology, while for $\mathcal{F}$ we consider the topology of pointwise a.e. convergence. We assume that $\mathcal{Q}$ satisfies the following properties:

(Q.1) The discontinuities of $\mathcal{Q}[u]$ can only occur at times $t_i = i\Delta$, for some fixed $\Delta > 0$ (the minimal switching time) ;

(Q.2) The value of $\mathcal{Q}[u](t)$ only depends on $u_{|_{[0,t]\times\mathbb{R}}}$;

(Q.3) The operator $\mathcal{Q}$ is continuous with respect to the above mentioned topologies on $\mathcal{E}$ and $\mathcal{F}$.

Examples of such operators and the underlying modeling motivations are detailed in [2, Section 1.4.1]. Typically, they reproduce the adaptation of the constraint level made at discrete times $t_i = i\Delta$ in response to the upstream averaged density of agents measured by continuous or discrete in time observations.

Let us sketch the extension of the preceding theory to such non-locally constrained problems; nontrivial details are discussed at each point.

- The definition of solution to such models is exactly the same as Definition 2.1 with the additional requirement that for all $t > 0$ there hold $\mathtt{F}(t) = \mathcal{Q}[u](t)$.
- The construction of solutions is fully analogous to the one of Section 3.4 with the only difference that the constraint level F_i^n on the interval $(t_i, t_{i+1}) = (i\Delta, (i+1)\Delta)$ (with the explicit dependence of the constraint level F_i on the approximation parameter n) is computed as $F_i^n \doteq \mathcal{Q}[u_n](t_i)$; note that **(Q.2)** makes this choice meaningful. This corresponds to the standard splitting procedure for approximation of coupled problems. At the practical level, the operator $\mathcal{Q}$ can also be discretized (see [4] for typical examples and for the general view on consistency of such discretizations), however in principle the wave-front tracking procedure allows for the computation of $\mathcal{Q}$ on u_n.
- The dependence of F_i^n on n is handled by the basic compactness argument, being understood that $F_i^n \in [f^-, f^+]$. Let us stress that the switching time Δ is independent of n. We denote by F_i the limit (along the suitable subsequence) of F_i^n. We can consider a finite number of switches, or even afford for $i \in \mathbb{N}$ upon using the diagonal extraction argument.
- The uniformity of the bounds on the total variation of u_n (at every fixed time horizon) is ensured by the uniformity of the Lipschitz constant in Lemma 2.6.
- In the passage to the limit, we have to care about the convergence $F_i^n \to F_i$. The constraint level $\mathtt{F}$ appears explicitly at points **(S.4)**,**(S.5)** of Definition 2.1. Handling the dependence of $\mathtt{F}^n$ on n is easy due to the continuity of $\mathtt{N}_F^k(u)$ with respect to $f \in [f^-, f^+]$, and to the obvious possibility to pass to the limit in **(S.5)**.
- Finally, the link $F^n(t) = \mathcal{Q}[u_n](t)$ is preserved at the limit, for a.e. $t > 0$, due to **(Q.3)** and the convergence of (the suitable subsequence of) $\{u_n\}_n$ in the $\mathbf{L}^\infty\big(\mathbb{R}_+; \mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}; \Omega)\big)$ topology, see the conclusion of §3.6. In the case $\mathcal{Q}$ is replaced by fully discrete approximations $\mathcal{Q}^n$, consistency properties are also required at this step of the argumentation. For the sake of conciseness, we do not pursue this line here.

To sum up, the result of Theorem 2.8 readily extends to $[f^-, f^+]$-valued non-local constraints verifying the structural properties **(Q.1)**, **(Q.2)**, **(Q.3)**.
While the two latter conditions are natural for the whole class of traffic models with non-local point constraint, the assumption **(Q.1)** and the restriction $F \geqslant f^-$ result from the technical limitations of our BV-based approach. Further work on this kind of models requires either smoother interaction potential terms (the Υ terms) in the Glimm-like functional $\mathcal{T}$, or less restrictive compactness tools such as compensated compactness.

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