Bérenger's PML on Rectangular Domains for Pauli's Equations

Laurence Halpern, Jeffrey Rauch

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Laurence Halpern *   Jeffrey Rauch †

Abstract

This article proves the well posedness of the boundary value problem that arises when Bérenger's PML agorithm is applied to Pauli's equations. As in standard practice, the computational domain is rectangular and the absorptions are positive near the boundary and zero in the interior so are always x-dependent. At the boundary of the rectangle, the natural absorbing boundary conditions are imposed. The estimates allow exponential growth in time, but have no loss of derivatives for the physical quantities. The analysis proceeds by Laplace transform. Existence is proved for a carefully constructed boundary value problems for a complex stretched Helmholtz equation. Uniqueness is reduced by an analyticity argument to a result in [15]. This is the first stability proof for Bérenger's algorithm with x-dependent absorptions on a domain whose boundary is not smooth.

Keywords. Hyperbolic boundary value problem, PML, trihedral corner, dissipative boundary conditions, PML, Bérenger, Pauli system.

AMS Subject Classification. 35F46, 35J25, 35L20, 35L53, 35Q40, 65N12.

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1 Introduction

This paper analyses initial boundary value problems that arise when one uses Bérenger's perfectly matched absorbing layers in the time domain. The

*Université Sorbonne Paris Nord, LAGA, CNRS, UMR 7539, F-93430, Villetaneuse, France, halpern@math.univ-paris13.fr

†Department of Mathematics, University of Michigan, Ann Arbor 48109 MI, USA rauch@umich.edu.

most common configuration is a rectangular region of interest surrounded by a larger rectanglular computational domain, In the region between the rectangles, Bérenger's perfectly matched layers are interposed. Boundary conditions at the exterior boundary are imposed that are designed to be weakly reflecting. In addition to perfect matching, an advantage of the PML strategy is its ease of implementation including at the corners. To our knowlege the present work is the first to prove well posedness for such a PML with non constant absorptions σ_j in the presence of corners. That problem poses two fundamental challenges.

• Even for a system with a very simple energy estimate like Pauli's equations, the Bérenger split system does not have simple energy estimates. One does not have a simple computation of energy flux through the boundary.

• The external boundary is rectangular. The smooth faces are characteristic for the split equations. Worse still the rectangle has edges and trihedral corners. The analysis of boundary value problems at trihedral corners is challenging even without the extra difficulties posed by Bérenger's layers.

The Pauli system shares the Lorentz invariance, symmetry, and three dimensionality of Maxwell's equations. It has two advantages. It is a 2×2 system as opposed to a 6×6 system. More importantly, the generator is elliptic. The analysis extends with almost no modifications to the Dirac system. The Pauli operator is

$$L := \partial_t + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_1 + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_2 + \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \partial_3 := \partial_t + \sum_{j=1}^{3} A_j \partial_j. \quad (1.1)$$

Introduce the notations with $\xi \in \mathbb{C}^3$,

$$L(\partial_t, \partial_x) \ := \ \partial_t \ + \ A(\partial_x), \qquad A(\xi) \ := \ A_1\xi_1 \ + \ A_2\xi_2 \ + \ A_3\xi_3 \, . \qquad (1.2)$$

Definition 1.1 *For $A \in \mathrm{Hom}(\mathbb{C}^k)$, with spectrum disjoint from $i\mathbb{R}$, $\mathcal{E}^+(A)$ (resp. $\mathcal{E}^-(A)$) denotes the spectral subpace coresponding to eigenvalues with strictly positive (resp. strictly negative) real part. Denote by $\pi^\pm(A)$ the corresponding spectral projections onto those spaces. As our interest is the Pauli system, $\mathcal{E}^\pm(\xi)$ and $\pi^\pm(\xi)$ are shorthands for $\mathcal{E}^\pm(A(\xi))$ and $\pi^\pm(A(\xi))$ for $\xi \in \mathbb{C}^3$ so that $A(\xi)$ has no purely imaginary eigenvalues.*

Definition 1.2 *Denote by $\mathcal{Q} = \mathcal{Q}(L_1, L_2, L_3)$ the rectangle*

$$\mathcal{Q} \ := \ \big\{ x \in \mathbb{R}^3 \ : \ |x_j| < L_j/2, \quad j = 1, 2, 3 \big\}.$$

$\mathcal{Q}$ has six open faces G_j with $1 \le j \le 6$. For $i = 1,2,3$,

$$G_i := \big\{x_i = -L_i/2, \text{ and, } |x_j| < L_j/2 \text{ for } j \neq i\big\},$$
$$G_{i+3} := \big\{x_i = L_i/2, \text{ and, } |x_j| < L_j/2 \text{ for } j \neq i\big\}.$$

For a point $x \in G_j$, $\nu(x)$ denotes the outward unit normal to $\mathcal{Q}$.

The split equations involve non negative **absorption coefficients** *$\sigma_j \in C_0^\infty(\mathbb{R})$ for $j = 1,2,3$.*

Example 1.1 *For the* **usual implementations of the PML method***, the layers are concentrated in a band of width ρ about $\partial\mathcal{Q}$ where the σ_j need not vanish. It is anticipated that the waves are damped there. The goal is to compute accurately in the smaller rectangle $\Pi\{|x_j| < (L_j/2) - \rho\}$ where $\sigma_j = 0$. This smaller rectangle contains the support of f and the region where accurate values are desired.*

The Pauli system is

$$\Big(\partial_t \;+\; A_1\partial_1 \;+\; A_2\partial_2 \;+\; A_3\partial_3\Big)u \;=\; f \quad \text{on} \quad \mathcal{Q}. \tag{1.3}$$

The source term f vanishes on a neighborhood of $\partial\mathcal{Q}$. In the standard implementation, f is supported in the smaller interior rectangle $\Pi\{|x_j| < (L_j/2) - \rho\}$.

Definition 1.3 *Bérenger's method has unknown that is a triple (U^1, U^2, U^3) with U^j taking values in $\mathbb{C}^2$. On $\mathbb{R}\times\mathcal{Q}$, (U^1, U^2, U^3) satisfy the* **split equations***,*

$$\begin{aligned}
\big(\partial_t + \sigma_1(x_1)\big)U^1 \;+\; A_1\partial_1\Big(U^1 + U^2 + U^3\Big) \;&=\; f_1\,,\\
\big(\partial_t + \sigma_2(x_2)\big)U^2 \;+\; A_2\partial_2\Big(U^1 + U^2 + U^3\Big) \;&=\; f_2\,,\\
\big(\partial_t + \sigma_3(x_3)\big)U^3 \;+\; A_3\partial_3\Big(U^1 + U^2 + U^3\Big) \;&=\; f_3\,.
\end{aligned} \tag{1.4}$$

The j^{th} equation has the ∂_j derivative. The f_j are constrained to satisfy $f = \sum_j f_j$ and to vanish on a neighborhood of $\partial\mathcal{Q}$. A choice respecting the symmetry of the problem is $f_j = f/3$ for $j = 1,2,3$.

In the idealized implementation with unbounded computation domain with $\mathcal{Q} = \mathbb{R}^3$ and

$$\operatorname{supp} f \subset \Pi\big\{|x_j| \;<\; (L_j/2) - \rho\big\}, \quad \text{and,} \quad \sigma_j = 0 \quad \text{on} \quad \Pi\big\{|x_j| \;<\; (L_j/2) - \rho\big\}\,,$$

perfect matching is the property that if U^j denotes the solution on $\mathbb{R}^3$ of the split equations and u the solution on $\mathbb{R}^3$ of (1.3) with $U^j = u = 0$ for $t < 0$, then $U^1 + U^2 + U^3 = u$ on $\Pi\{|x_j| < (L_j/2) - \rho\}$ (see [14]). When a finite computation domain is used, the boundary of $\mathcal{Q}$ is not perfectly transparent, inducing errors. In favorable cases like the Pauli system, waves are expected to decay exponentially in the layers so little signal reaches $\partial\mathcal{Q}$ and the reflections cause small errors. In practice rather thin layers suffice. With x-dependent absorptions and computations in the time domain (as opposed to time harmonic problems), proving exponential decay in the layers is an outstanding open problem.

The split equations are not symmetric and they have a lower order term that depends on x through the absorption coefficients σ_j. They do not have simple *a priori* estimates showing that they yield a well posed pure initial value problem. Petit-Bergez [22, 14] proved that since the Pauli system generates a C_0-semigroup on $L^2(\mathbb{R}^3)$ and has elliptic generator, it follows that the split equations on $\mathbb{R}^3$ also generate a C_0-semigroup. This contrasts to the loss of one derivative for the split Maxwell equations proved by Arbarbanel and Gottlieb [1].

The Pauli system is symmetric hyperbolic. The most strongly dissipative boundary condition for the Pauli system is $u \in \mathcal{E}^+(\nu)$. Thanks to the symmetry and ellipticity there is an M_0 so that for that $M > M_0$ and $f \in e^{Mt}L^2(\mathbb{R} \times \mathcal{Q})$ the boundary value problem $Lu = f$ with boundary condition $u \in \mathcal{E}^+(\nu)$ on the $\mathbb{R} \times G_j$ has a unique solution $u \in e^{Mt}L^2(\mathbb{R} \times \mathcal{Q})$ with $u \in e^{Mt}L^2(\mathbb{R} \times \partial\mathcal{Q})$ (see Part I of [15]).

The next result asserts a similar conclusion for the split equation with the absorbing boundary condition $U^1 + U^2 + U^3 \in \mathcal{E}^+(\nu)$ on the G_j. It is the main result of the paper.

Uniqueness folllows by an analyticity argument from [15]. Ours is the first existence theorem for the split equations with nonconstant σ_j in domains whose boundary is not smooth. Since cubes with non constant σ is standard practice it is the first justification, beyond ample practical experience, that the usual Bérenger algorithms are stable.

Theorem 1.4 *There are strictly positive constants C, M so that if $\lambda > M$, $K \subset \mathcal{Q}$ is compact, $f \in e^{\lambda t}L^2(\mathbb{R} \times \mathcal{Q})$ with support in $[0, \infty[\times K$, then there is one and only one*

$$\big(U^1, U^2, U^3\big) \in e^{\lambda t}L^2(\mathbb{R} \times \mathcal{Q}) \quad \text{with,} \quad \nabla_{t,x}(U^1 + U^2 + U^3) \in e^{\lambda t}L^2(\mathbb{R} \times \mathcal{Q}),$$

supported in $t \geq 0$ that satisfies (1.4), and the boundary condition $U^1 + U^2 +$

$U^3 \in \mathcal{E}^+(\nu)$ on each G_j. The function $U^1 + U^2 + U^3$ satisfies,

$$\begin{aligned} &\big\|e^{-\lambda t}\big\{\lambda(U^1+U^2+U^3)\,,\,\nabla_{t,x}(U^1+U^2+U^3)\big\}\big\|_{L^2(\mathbb{R}\times\mathcal{Q})} \\ &+\lambda^{-1/2}\big\|e^{-\lambda t}\big\{\lambda(U^1+U^2+U^3)\,,\,\partial_t(U^1+U^2+U^3)\big\}\big\|_{L^2(\mathbb{R}\times\partial\mathcal{Q})} \\ &\qquad\qquad\qquad\qquad \leq\ C\,\big\|e^{-\lambda t}\{\lambda\, f, \nabla_{t,x} f\}\big\|_{L^2(\mathbb{R}\times\mathcal{Q})}. \end{aligned} \tag{1.5}$$

The split unknowns satisfy the weaker estimate

$$\big\|e^{-\lambda t}\{\lambda U^j\,,\,\partial_t U^j\}\big\|_{L^2(\mathbb{R}\times\mathcal{Q})}\ \leq\ C\,\big\|e^{-\lambda t}\big\{\lambda f\,,\,\nabla_{t,x} f\big\}\big\|_{L^2(\mathbb{R}\times\mathcal{Q})}\,. \tag{1.6}$$

Remark 1.1 **i.** *It is reasonable to think of of the Bérenger algorithm as a method that inputs f and outputs $U^1 + U^2 + U^3$. Estimate 1.5 shows that the output satisfies bounds as strong as strictly dissipative boundary value problems for symmetric hyperbolic systems. This behavior is known for the pure initial value problem (see for example Theorem 1.3 of Bécache and Joly [5] for the split Maxwell equations with constant σ_j) and for Bérenger transmission problem even with variable σ_j (see [14]).*

ii. *The estimate for the U^j is weaker. The two sides have the same number of derivatives. What is missing is an estimate for $\nabla_x U^j$. For the pure initial value problem for for constant coefficient split Maxwell equations on $\mathbb{R}^3$ this was observed by Arbarbanel and Gottlieb [1] . For initial data satisfying the divergence relations, the loss of derivatives does not occur, even for variable and even discontinuous σ_j (see [15] that includes numerical experiments).*

iii. *The estimates of Theorem 1.4 permit exponential growth in time. Even for sources compactly supported in time. Practical experience with Bérenger's method for Maxwell's equations and other models closely tied to the wave equation (e.g. Pauli) show no growth in time even with variable σ_j. Interesting bounds uniform in time are proved for the case of constant σ_j for sufficiently regular solutions by Bécache-Joly, Diaz-Joly, and Baffet-Grote-Imperiale-Kachanovska [5, 10, 3]. It is not known whether the uniform bounds of these authors imply that the $L^2(\mathcal{Q})$ norm $U^1 + U^2 + U^3$ or any of its derivatives is uniformly bounded. Uniform bounds in time is an important and wide open problem. The present paper can be viewed as the construction of norm estimates resembling those of the preceeding authors, albeit at the expense of exponential growth and substantial difficulty. Appelo-Hagstrom-Kreiss [2] analyse the problem of exponential growth with constant parameters by explicit formulas in Fourier. They propose stabilization methods. Variable coefficients and corner domains are beyond that strategy.*

The paper is organized as follows. Section 2 presents the Pauli system and most importantly the stretched Pauli system that is satisfied by the Laplace transform of $\widehat{U}^1 + \widehat{U}^2 + \widehat{U}^3$. Theorems 2.5 and 2.8 assert existence and uniqueness for the boundary value problems for the stretched system on $\mathcal{Q}$ as well as smoothed versions $\mathcal{Q}_\delta$ converging to $\mathcal{Q}$ as $\delta \to 0$. With important δ-independent estimates. Most of the section is devoted to showing that the solutions on $\mathcal{Q}_\delta$ satisfy an additional boundary condition stated in Corollary 2.15.

The second boundary condition yields a boundary value problem of Helmholtz type satisfied by the solution of the stretched equations. Theorem 2.8 is proved by solving that boundary value problem in Section 3. The estimates are derived by the energy method tied to a family of complex quadratic forms $\mathcal{A}$. The real and imaginary parts play key roles. The family is singular in the limit $\delta \to 0$. The lower and upper bounds for $\mathcal{A}$ have different growth rates for $|\tau|$ large. In spite of this apparent failure of uniform coercivity, estimates uniform in δ and τ are proved. A key is that on solutions, the form $\mathcal{A}$ is much smaller than its upper bound.

Section 4 derives the main theorems from the Helmholtz existence results. Section 4.1 proves Theorem 2.8. This proof has the interesting converse proof that the solution of the Helmholtz boundary value problem solves the stretched boudary value problem from which it was derived. The holomorhpy in τ is crucial. Section 4.2 proves Theorem 2.5 by passing to the limit $\delta \to 0$. Section 4.3 derives the main result, Theorem 1.4.

The proof is technical. The hypothesis $\sigma_j \in C^\infty$ avoids some inessential difficulties. The proof uses H^2 regularity for the Helmholtz problem on $\mathcal{Q}_\delta$. Absorptions $\sigma_j \in L^\infty$ suffice for H^1 regularity and σ_j lipschitzian is sufficient for H^2. Standard practive involves such lipschitzian absorptions. We do not treat this straight forward strengthening of the results here.

2 The Pauli and stretched systems

2.1 Pauli system and its symbol

The coefficients of the Pauli system satisfy,

$$A_j^2 \;=\; I, \qquad A_iA_j + A_jA_i \;=\; 0 \quad \text{for} \quad i \neq j\,. \tag{2.1}$$

These identities imply the connections to the Laplacian,

$$\Big(\sum_j A_j\partial_j\Big)^2 \;=\; \Delta\,, \qquad \Big(\sum A_j\partial_j - \tau\Big)\Big(\sum A_j\partial_j + \tau\Big) \;=\; \Delta - \tau^2\,. \quad (2.2)$$

Proposition 2.1 **i.** *For all* $(\tau,\xi)\in\mathbb{C}^{1+3}$,

$$\det L(\tau,\xi) \;=\; \tau^2 \;-\; \sum_{j=1}^{3}\xi_j^2\,. \qquad (2.3)$$

ii. *For* $\xi\in\mathbb{R}^3\backslash 0$, *the* 2×2 *hermitian symmetric matrix* $A(\xi)$ *has eigenvalues* $\pm|\xi|$ *with one dimensional eigenspaces*

$$\mathcal{E}^-(A(\xi)) \;=\; \mathbb{C}\big(\xi_1-|\xi|\,,\,\xi_2-i\xi_3\big), \quad \text{and,} \quad \mathcal{E}^+(A(\xi)) \;=\; \mathbb{C}\big(\xi_1+|\xi|\,,\,\xi_2-i\xi_3\big).$$

iii. *For all* $\xi,\eta\in\mathbb{C}^3$,

$$A(\xi)A(\eta)+A(\eta)A(\xi) \;=\; 2\Big(\sum_i \xi_i\eta_i\Big)I\,. \qquad (2.4)$$

Proof. **i.** For $\xi\in\mathbb{C}^3$,

$$A(\xi) \;=\; \begin{pmatrix} \xi_1 & \xi_2+i\,\xi_3 \\ \xi_2-i\,\xi_3 & -\xi_1 \end{pmatrix}.$$

The trace vanishes and the determinant is equal to $\sum\xi_j^2$. This implies **i**.
ii. For $\xi\in\mathbb{R}^3\setminus 0$,

$$A(\xi)\mp|\xi|\,I \;=\; \begin{pmatrix} \xi_1\mp|\xi| & \xi_2+i\,\xi_3 \\ \xi_2-i\,\xi_3 & -\xi_1\mp|\xi| \end{pmatrix}.$$

Both matrices are singular. The kernel of of $A(\xi)-|\xi|I$ is $\mathcal{E}^+(\xi)$. The range, orthogonal to the kernel, is $\mathcal{E}^-(\xi)$. The columns of the matrix is each a basis of the range. The first column yields the formula for $\mathcal{E}^-(\xi)$ in **ii.** The other choice of sign yields $\mathcal{E}^-(\xi)$.
iii. Expand

$$\Big(\sum_i A_i\xi_i\Big)\Big(\sum_j A_j\eta_j\Big) \;=\; \sum_{i,j} A_iA_j\xi_i\eta_j\,.$$

Symmetrizing yields,

$$A(\xi)A(\eta)+A(\eta)A(\xi) \;=\; \sum_{i,j} A_iA_j\xi_i\eta_j \;+\; \sum_{i,j} A_iA_j\eta_i\xi_j\,.$$

In the last sum interchange the roll of i, j to find

$$A(\xi)A(\eta) + A(\eta)A(\xi) \;=\; \sum_{i,j} A_i A_j \xi_i \eta_j \;+\; \sum_{i,j} A_j A_i \eta_j \xi_i \,.$$

Separate out the terms with $i = j$ to find

$$A(\xi)A(\eta) + A(\eta)A(\xi) \;=\; 2\sum_i A_i^2\, \xi_i \eta_i \;+\; \sum_{i\neq j} \Big(A_i A_j \;+\; A_j A_i\Big)\eta_i \xi_j \,.$$

Equation (2.1) yields (2.4). □

Example 2.1 *Define* $\mathcal{Z} := \big\{\xi \in \mathbb{C}^3 : \sum_j \xi_j^2 = 0\big\}$. *For* $\xi \in \mathbb{C}^3 \setminus \mathcal{Z}$, spec $A(\xi)$ *consists of two simple eigenvalues differing by a factor* -1.

The eigenvalues $\pm|\xi|$ *for* $\xi \in \mathbb{R}^3 \setminus 0$ *extend to holomorphic eigenvalues* $\lambda^\pm(\xi) = \pm(\sum \xi_j^2)^{1/2}$ *on*

$$\Big\{\xi \in \mathbb{C}^3 \setminus 0 \;:\; |\mathrm{Im}\,\xi| < |\mathrm{Re}\,\xi|\Big\} \tag{2.5}$$

where the square root is defined to be the root with strictly positive real part.

Proposition 2.2 i. *The eigenprojections* $\pi^\pm(\xi)$ *for* $\xi \in \mathbb{R}^3 \setminus 0$ *extend to holomorphic functions on* (2.5) *satisfying with notation from Example 2.1,*

$$\pi^\pm(\xi)\, A(\xi) \;=\; A(\xi)\, \pi^\pm(\xi) \;=\; \pm\, \lambda^\pm(\xi)\; \pi^\pm(\xi)\,. \tag{2.6}$$

They are given by

$$\pi^\pm(\xi) \;=\; \frac{1}{2}\, \Big(\sum \xi_j^2\Big)^{-1/2} \Big(A(\xi) \pm \big(\sum \xi_j^2\big)^{1/2}\, I\Big)\,.$$

ii. *For* ξ, η *belonging to* (2.5),

$$\pi^\pm(\eta)\, A(\xi)\, \pi^\pm(\eta) \;=\; \big(\sum \xi_j^2\big)^{-1/2} \big(\sum \xi_i \eta_i\big)\, \pi^\pm(\eta)\,. \tag{2.7}$$

Proof. i. The formulas

$$A(\xi) = \lambda^+\big(\pi^+(A(\xi)) - \pi^-(A(\xi))\big), \quad \text{and,} \quad I \;=\; \pi^+(A(\xi)) + \pi^-(A(\xi)).$$

imply the formulas for $\pi^\pm(A(\xi))$ in **i.**

ii. Multiply (2.4) on the left and right by $\pi^\pm(\eta)$ to find

$$2\,\pi^\pm(\eta)\Big(\sum \xi_i \eta_i\Big)\pi^\pm(\eta) \;=\; \pi^\pm(\eta)A(\xi)A(\eta)\pi^\pm(\eta) \;+\; \pi^\pm(\eta)A(\eta)A(\xi)\pi^\pm(\eta)\,.$$

Use (2.6) twice and $\pi^{\pm}(\eta)^2 = \pi^{\pm}(\eta)$ to find,

$$
\begin{aligned}
2\Big(\sum \xi_i \eta_i\Big)\pi^{\pm}(\eta) \;&=\; \pi^{\pm}(\eta)A(\xi)\lambda^{\pm}(\eta)\,\pi^{\pm}(\eta) \;+\; \lambda^{\pm}(\eta)\,\pi^{\pm}(\eta)A(\xi)\pi^{\pm}(\eta) \\
&=\; 2\,\lambda^{\pm}(\eta)\,\pi^{\pm}(\eta)A(\xi)\pi^{\pm}(\eta)\,.
\end{aligned}
$$

This completes the proof. □

2.2 The stretched system, Theorem 2.5, Theorem 2.8

For (1.4), consider the Laplace transformed equations,

$$
\begin{aligned}
\big(\tau + \sigma_1(x_1)\big)\widehat{U}^1 \;+\; A_1\partial_1\Big(\widehat{U}^1 + \widehat{U}^2 + \widehat{U}^3\Big) \;&=\; \widehat{f}_1\,, \\
\big(\tau + \sigma_2(x_2)\big)\widehat{U}^2 \;+\; A_2\partial_2\Big(\widehat{U}^1 + \widehat{U}^2 + \widehat{U}^3\Big) \;&=\; \widehat{f}_2\,, \qquad (2.8)\\
\big(\tau + \sigma_3(x_3)\big)\widehat{U}^3 \;+\; A_3\partial_3\Big(\widehat{U}^1 + \widehat{U}^2 + \widehat{U}^3\Big) \;&=\; \widehat{f}_3\,.
\end{aligned}
$$

Define for $j = 1, 2, 3$,

$$
\widetilde{\partial}_j \;:=\; \frac{\tau}{\tau + \sigma_j(x_j)}\,\frac{\partial}{\partial x_j}\,, \qquad u \;:=\; \widehat{U}^1 + \widehat{U}^2 + \widehat{U}^3\,. \qquad (2.9)
$$

These definitions yield the **stretched equation**

$$
\Big(\tau \;+\; A_1\widetilde{\partial}_1 \;+\; A_2\widetilde{\partial}_2 \;+\; A_3\widetilde{\partial}_3\Big)u \;=\; F \;:=\; \sum_{j=1}^{3} \frac{\tau}{\tau + \sigma_j(x_j)}\hat{f}_j\,. \qquad (2.10)
$$

Definition 2.3 *The operator*

$$
L(\tau, \widetilde{\partial}_x) \;:=\; \tau \;+\; \sum_{j=1}^{3} A_j\widetilde{\partial}_j \;=\; \tau \;+\; \sum_{j=1}^{3} A_j \frac{\tau}{\tau + \sigma_j(x_j)}\,\frac{\partial}{\partial x_j}
$$

is called the **stretched Pauli operator***.*

Equation (2.10) resembles the Laplace transform of the original system. For τ real and positive it comes from the original transformed system by a change of variable, called *coordinate stretching* (see Section 2.3.2, and Chew-Weedon [8]). This plays a key role in the proof of Theorem 1.4.

The stretched equations are sometimes expressed using auxiliary variables ψ_j defined as the solutions of

$$
(\partial_t + \sigma_j(x_j))\psi_j \;=\; \partial_t u, \qquad \psi_j = 0 \quad \text{for} \quad t < 0.
$$

Then $\widetilde{\partial}_j\widehat{u} \;=\; \partial_j\widehat{\psi}_j\,.$

Definition 2.4 **i.** *If $\Omega \subset \mathbb{R}^3$ is open and $K \subset \Omega$ is compact,*

$$C^\infty_K(\Omega) \ := \ \Big\{ f \in C^\infty(\Omega) \ ; \ \mathrm{supp}\, f \subset K \Big\}.$$

ii. *Similarly,*

$$L^2_K(\Omega) \ := \ \Big\{ f \in L^2(\Omega) \ ; \ \mathrm{supp}\, f \subset K \Big\}.$$

iii. *The space $H^{-1}_0(\Omega)$ is defined to be $\mathrm{Hom}(H^1(\Omega); \mathbb{C})$, the set of continuous linear functionals on $H^1(\Omega)$.*

Theorem 2.5 *There exist C, M_1 so that for all compact $K \subset \mathcal{Q}$, $M \geq M_1$, and, holomorphic $F : \{\mathrm{Re}\,\tau > M\} \to L^2_K(\mathcal{Q})$, there is a unique holomorphic function $u : \{\mathrm{Re}\,\tau > M\} \to H^1(\mathcal{Q})$ satisfying the stretched boundary value problem on $\mathcal{Q}$,*

$$L(\tau, \widetilde{\partial}_x) u \ = \ F \quad \text{on} \quad \mathcal{Q}, \qquad u \in \mathcal{E}^+\big(\nu\big) \quad \text{on} \quad G_j, \quad 1 \leq j \leq 6.$$

It satisfies for all $\mathrm{Re}\,\tau > M$,

$$(\mathrm{Re}\,\tau) \, \big\| u \big\|_{L^2(\mathcal{Q})} + (\mathrm{Re}\,\tau)^{1/2} \, \big\| u \big\|_{L^2(\partial\mathcal{Q})} + \big\| \nabla_x \big\|_{L^2(\mathcal{Q})} \\ \leq \ C \Big\| \Big(\sum_j A_j \widetilde{\partial}_j - \tau \Big) F \, \Big\|_{H^{-1}_0(\mathcal{Q})} .$$

Remark 2.1 **i.** *This Theorem and the uniqueness theorem in Part I of [15] are the main ingredients in proving Theorem 1.4.*
ii. *For $F \in L^2_K(\mathcal{Q})$, and $\phi \in H^1(\mathcal{Q})$,*

$$\Big\langle \Big(\sum_j A_j \widetilde{\partial}_j - \tau \Big) F \ , \ \phi \Big\rangle \ = \ \Big\langle F, \Big(- \sum_j A_j \widetilde{\partial}_j - \tau \Big) \phi \Big\rangle.$$

Therefore

$$\Big\| \Big(\sum_j A_j \widetilde{\partial}_j - \tau \Big) F \, \Big\|_{H^{-1}_0(\mathcal{Q})} \ \lesssim \ \|F\|_{L^2(\Omega)}.$$

This estimate does not extend to functions whose support reaches the boundary. For example for compactly supported $F \in C^\infty(\overline{\mathcal{Q}})$,

$$\big\langle \partial_j F \, , \, \phi \big\rangle \ = \ \int_{\mathcal{Q}} \partial_j F \ \phi \ dx \ = \ - \int_{\mathcal{Q}} F \, \partial_j \phi \ dx \ + \ \int_{\partial\mathcal{Q}} F \, \phi \, \nu_j \ d\Sigma.$$

The size in $H^{-1}_0(\mathcal{Q})$ depends on $F|_{\partial\mathcal{Q}}$.

Theorem 2.5 is proved by solving the stretched equation on smoothed truncated domains and passing to the limit.

Definition 2.6 *The singular set of the boundary of $\mathcal{Q}$ is*

$$\mathcal{S} \ := \ \Big\{ x \in \partial\mathcal{Q} \, : \, \exists i \neq j, \ x \in \overline{G}_i \cap \overline{G}_j \Big\}.$$

Introduce for $0 < \delta < 1$ bounded smooth approximations $\mathcal{Q}_\delta$ of $\mathcal{Q}$. Smooth the edges and corners of $\mathcal{Q}$ on a $\delta/2$-neighborhood of $\mathcal{S}$ to yield bounded smooth convex sets $\mathcal{Q}_\delta$. Do this so that for $\delta_1 < \delta_2$, $\mathcal{Q}_{\delta_1} \supset \mathcal{Q}_{\delta_2}$.

The symbol Ω is often used to denote elements of the family of sets $\{\mathcal{Q}_\delta : \delta \in]0,1[\}$.

Definition 2.7 *For τ with $\operatorname{Re}\tau > 0$ and $\nu \in \mathbb{R}^3$ define*

$$\widetilde{\nu}(\tau, x) \ := \ \Big(\frac{\nu_1 \, \tau}{\tau + \sigma_1(x)} \, , \, \frac{\nu_2 \, \tau}{\tau + \sigma_2(x)} \, , \, \frac{\nu_3 \, \tau}{\tau + \sigma_3(x)} \Big)$$

In the next discussion this is used with ν equal to the unit normal to $\partial\Omega$. Next choose a boundary condition for the stretched equations on $\mathcal{Q}_\delta$. On the flat parts of $\partial\mathcal{Q}_\delta$ one has $u \in \mathcal{E}^+(\nu)$. On the curved parts of the boundary and for $\tau > 0$ and real, the stretched problem is symmetric hyperbolic and the normal matrix is $A(\widetilde{\nu}(\tau, x))$. The maximally dissipative condition is $u \in \mathcal{E}^+(\widetilde{\nu})$. If $u(\tau)$ is holomorphic and satisfies this condition for $\tau > 0$ then by analytic continuation it holds for general τ. Therefore, $u \in \mathcal{E}^+(\widetilde{\nu})$ is the natural maximally dissipative condition for τ complex.

The main result for the stretched system on $\mathcal{Q}_\delta$ is the following.

Theorem 2.8 *There exist C, M_1 so that for all δ, $M \geq M_1$ compact $K \subset \mathcal{Q}_\delta$, and holomorphic $F : \{\operatorname{Re}\tau > M\} \to C^\infty_K(\mathcal{Q}_\delta)$ there is a unique holomorphic $u^\delta : \{\operatorname{Re}\tau > M\} \to H^2(\mathcal{Q}_\delta)$ satisfying*

$$L(\tau, \widetilde{\partial}_x) u^\delta \ = \ F, \quad \text{on} \quad \mathcal{Q}_\delta, \qquad u^\delta|_{\partial\mathcal{Q}_\delta} \ \in \ \mathcal{E}^+\big(A(\widetilde{\nu}(\tau, x))\big). \tag{2.11}$$

The solution satisfies the $H^1(\mathcal{Q}_\delta)$ bound with constant independent of τ, δ,

$$(\operatorname{Re}\tau) \left\| u^\delta \right\|_{L^2(\mathcal{Q}_\delta)} + (\operatorname{Re}\tau)^{1/2} \left\| u^\delta \right\|_{L^2(\partial\mathcal{Q}_\delta)} + \left\| \nabla_x u^\delta \right\|_{L^2(\mathcal{Q}_\delta)} \\ \leq \ C \left\| \Big(\sum_j A_j \widetilde{\partial}_j - \tau \Big) F(\tau) \right\|_{H_0^{-1}(\mathcal{Q}_\delta)}.$$

Strategy of proof. Theorem 2.8 is proved by solving carefully constructed Helmholtz equations and boundary conditions on $\mathcal{Q}_\delta$. The boundary conditions, automatically satisfied by solutions of the stretched problems, are identified in the next section. On $\mathcal{Q}_\delta$ the solutions are smooth. The smoothness is used to prove that the solution of the Helmholtz problem on $\mathcal{Q}_\delta$ solves the stretched equations, proving Theorem 2.8. Taking the limit $\delta \to 0$ yields Theorem 2.5.

2.3 Second boundary condition for the Helmholtz BVP

This section concern solutions of the stretched Pauli boundary value problem. Theorem 2.8 is proved by solving a Helmholtz boundary value problem. The hypotheses of Theorem 2.8 yield the boundary condition $u \in \mathcal{E}^+(\widetilde{\nu})$. Corollary 2.15 of this section yields a crucial second boundary condition. Example 3.1 shows that it is a natural boundary condition for a weak formulation. Section 4.1 includes a proof of the converse implication that the Helmholtz equation plus the two boundary conditions imply the stretched Pauli equations.

2.3.1 Neumann identity for the unstretched Pauli system

Definition 2.9 *For $\underline{x} \in \partial\Omega$ the* **Weingarten map** *(see for example [16]) is the real self adjoint map of the tangent space $T_{\underline{x}}(\partial\Omega)$ to itself that is the differential of the unit normal ν. It maps $\mathbf{v} \ni T_{\underline{x}}(\partial\Omega) \;\to\; \mathbf{v}\cdot\nabla\nu$. Its eigenvalues are the* **principal curvatures** *of $\partial\Omega$ at $\underline{x}$. The* **mean curvature***, denoted $H_\Omega(\underline{x})$, is the average of the two principal curvatures.*

Extend ν to a smooth unit vector field defined on a neighborhood of $\partial\Omega$ so as to be constant on normal lines to the boundary. Then $\pi^\pm(\nu(x))$ is well defined and smooth for x in a neighborhood of $\partial\Omega$.

The identity of the next proposition is simple in case of flat boundaries. To prove Theorem 2.8 it is needed on the curved parts $\mathcal{Q}_\delta$.

Proposition 2.10 *If $u \in H^2(\Omega)$ satisfies the boundary condition $\pi^-\big(\nu\big)\,u = 0$ on $\partial\Omega$, then,*

$$\pi^+\big(\nu\big)\sum_{j=1}^{3} A_j\partial_j u \;=\; \pi^+(\nu)\big(\nu\cdot\partial_x + 2H_\Omega\big)u, \quad \text{on} \quad \partial\Omega. \tag{2.12}$$

Proof of Proposition 2.10. An invariance argument shows that it is sufficient to treat the case where $\underline{x} = 0$, $\nu(\underline{x}) = (-1, 0, 0)$ and the x_j-axes for $j \geq 2$ are principal curvature directions of $\partial\Omega$.

Denote by $\mathbf{k}_j$, $j = 1, 2, 3$ the standard basis for $\mathbb{R}^3$. The principal curvatures corresponding to the tangent directions $\mathbf{k}_2$ and $\mathbf{k}_3$ are denoted κ_2 and κ_3. The mean curvature is $H := (\kappa_2 + \kappa_3)/2$. At $\underline{x}$ the outward unit normal is $-\mathbf{k}_1$. At $\underline{x}$ the principal curvature formulas are $\partial_2\nu = -\kappa_2\mathbf{k}_1$ and $\partial_3\nu = -\kappa_3\mathbf{k}_1$.

First simplifications of the left hand side of (2.12). The operator on the left is $\pi^+(\nu)(A_1\partial_1 + A_2\partial_2 + A_3\partial_3)$. On the x_1 axis, $\nu = (-1, 0, 0)$, so $\pi^+(\nu(x))A_1 = -\pi^+(\nu(x))$. On that axis the operator is

$$-\pi^+(\nu)\partial_1 + \pi^+(\nu)\big(A_2\partial_2 + A_3\partial_3\big) = \nu \cdot \partial_x + \pi^+(\nu)\big(A_2\partial_2 + A_3\partial_3\big). \tag{2.13}$$

Second simplifications. Consider the two summands $\pi^+(\nu)A_j\partial_j u$ with $j \geq 2$. On the x_1-axis, part **ii** of Proposition 2.2 implies that

$$\pi^+(\nu)\, A_2\, \pi^+(\nu) \;=\; \pi^+(\nu)\, A_3\, \pi^+(\nu) \;=\; 0\,. \tag{2.14}$$

Using the boundary condition yields

$$\partial_j u = \partial_j\Big(\pi^+(\nu) + \pi^-(\nu)u\Big) = \partial_j\big(\pi^+(\nu)u\big) \quad \text{at} \quad \underline{x}. \tag{2.15}$$

For $j \in \{2, 3\}$ if Z is a vector field on a neighborhood of $\underline{x}$ that is tangent to the boundary and satisfies $Z(\underline{x}) = \partial_j$ then

$$\partial_j u(\underline{x}) \;=\; Z\big(u|_{\partial\Omega}\big)(\underline{x})\,.$$

Since $\pi^+(\nu)u = u$ on the boundary it follows that

$$\partial_j u(\underline{x}) \;=\; Z\big(\pi^+(\nu)u|_{\partial\Omega}\big)(\underline{x}) \;=\; \Big(\partial_j\big(\pi^+(\nu)u\big)\Big)(\underline{x})\,.$$

Using (2.14) in the last of the following equalities yields

$$\begin{aligned} \pi^+(\nu)\, A_j\partial_j u(\underline{x}) \;&=\; \pi^+(\nu)\, A_j\Big(\partial_j\big[\pi^+(\nu)u\big]\Big)(\underline{x}) \\ &=\; \pi^+(\nu)\, A_j\Big(\partial_j\pi^+(\nu)\, u(\underline{x}) \;+\; \pi^+(\nu)\, \partial_j u(\underline{x})\Big) \\ &=\; \pi^+(\nu)\, A_j\big(\partial_j\pi^+(\nu)\big)u(\underline{x})\,. \end{aligned} \tag{2.16}$$

The perturbation theory step. Use perturbation theory to compute the term $\partial_j \pi^+$ in the last expression. Denote by $Q(\xi)$ the partial inverse of $A(\xi) - |\xi| I$ associated to the eigenvalue $+|\xi|$. It is defined by

$$Q(\xi)\big(A(\xi) - |\xi| I\big) \;=\; I - \pi^+(\xi), \qquad Q(\xi)\,\pi^+(\xi) \;=\; 0\,.$$

Writing

$$A(\xi) - |\xi|\, I = \big(|\xi|\pi^+ - |\xi|\pi^-\big) \;-\; \big(|\xi|\pi^+ + |\xi|\pi^-\big) \;=\; -2|\xi|\pi^-$$

shows that $Q = (-2|\xi|)^{-1}\pi^-(\xi)$.

First order perturbation theory (see Theorem 3.I.2 in [24], or formulas (II.2.13), (II.2.33) in [18]) implies that

$$\frac{\partial}{\partial x_j}\Big(\pi^+\big(A(\nu)\big)\Big) \;=\; -\,\pi^+(\nu)\left(\frac{\partial A(\nu)}{\partial x_j}\right)Q(\nu) \;-\; Q(\nu)\Big(\frac{\partial A(\nu)}{\partial x_j}\Big)\pi^+(\nu)\,.$$

Endgame. When this is injected in (2.16) the contribution of the first term vanishes thanks to (2.14). Turn next to

$$\frac{\partial}{\partial x_j} A(\nu(x)) \;=\; A\Big(\frac{\partial \nu}{\partial x_j}\Big)\,.$$

The principal curvature formulas imply that at $\underline{x}$,

$$\frac{\partial \nu}{\partial x_j} \;=\; \kappa_j(\underline{x})\,\mathbf{e}_j\,, \quad \text{for} \quad j = 2,3\,, \qquad \text{so,} \qquad A\Big(\frac{\partial \nu}{\partial x_j}\Big) \;=\; \kappa_j(\underline{x})\, A_j\,.$$

Therefore (2.16) yields

$$\pi^+(\nu)\, A_j \partial_j u(\underline{x}) \;=\; \kappa_j(\underline{x})\, \pi^+(\nu)\, A_j\, \pi^-(\nu) A_j\, \pi^+(\nu)\,.$$

Compute using (2.14) and omitting the argument $\nu(\underline{x})$ for ease of reading yields

$$\pi^+ A_j \pi^- A_j \pi^+ = \pi^+ A_j \big(\pi^- + \pi^+\big) A_j \pi^+ = \pi^+ A_j\, A_j \pi^+ = \pi^+\pi^+ = \pi^+\,.$$

Therefore

$$\pi^+(\nu(\underline{x}))\, A_j \partial_j u(\underline{x}) = \kappa_j(\underline{x})\, \pi^+(\nu(\underline{x}))\, u, \quad \text{for} \quad j = 2,3. \tag{2.17}$$

The sum of the terms (2.17) is equal to $(\kappa_2 + \kappa_3)\,\pi^+ u = 2H_\Omega\,\pi^+ u$. This yields

$$\pi^+(\nu(\underline{x}))\Big(\nu \cdot \nabla_x + 2H_\Omega(\underline{x})\Big) u\,.$$

This completes the proof of (2.12). □

2.3.2 Transverse identity for stretched Pauli for $\tau \in]m, \infty[$

Definition 2.11 **i.** *For $\tau \in \mathbb{C} \setminus 0$ the coordinate stretchings $X_j(\tau, x_j)$ are defined as the solutions of*

$$\frac{dX_j}{dx_j} \;=\; \frac{\tau + \sigma_j(x_j)}{\tau}\,, \qquad X_j(0) \;=\; 0\,. \tag{2.18}$$

ii. *For real $\tau > 0$, $\partial_j X_j > 0$ and $x \mapsto X(\tau, x)$ is a diffeomorphism from $\mathbb{R}^3$ onto itself. Denote by $\underline{\Omega} \subset \mathbb{R}^d_X$ the image of $\Omega \subset \mathbb{R}^d_x$.*

Example 2.2 *In the standard implementation of Example 1.1, the σ_j vanish on $\cap_j \{|x_j| \le (L_j/2) - \rho\}$. Therefore for all τ, X is equal to the identity on that set.*

Compute for $\tau > 0$,

$$\frac{\partial}{\partial x_j} = \sum_k \frac{\partial X_k}{\partial x_j}\,\frac{\partial}{\partial X_k} = \frac{\tau + \sigma_j(x_j)}{\tau}\,\frac{\partial}{\partial X_j}, \qquad \frac{\tau}{\tau + \sigma_j(x_j)}\,\frac{\partial}{\partial x_j} = \frac{\partial}{\partial X_j}. \tag{2.19}$$

Equation 2.19 gives a geometric interpretation of the stretched operator $L(\tau, \widetilde{\partial})$ for real $\tau > 0$. It shows that $\widetilde{\partial}_j$ in the x coordinates is equal to $\partial/\partial X_j$ in the X coordinates. Therefore if $u(x)$ and $v(X)$ are related by $v(X(\tau, x)) = u(x)$ then $L(\tau, \widetilde{\partial})u(x) = \big(L(\tau, \partial_X)v\big)(X(\tau, x))$.
To find the conormals to $\underline{\Omega}$, compute

$$\sum_j \nu_j dx_j = \sum_j \nu_j \sum_k \frac{\partial x_j}{\partial X_k} dX_k = \sum_j \nu_j \frac{\partial x_j}{\partial X_j} dX_j = \sum_j \frac{\nu_j\, \tau}{\tau + \sigma_j}\; dX_j\,.$$

$\sum_j \nu_j dx_j$ annihilates the tangent space to $\partial\Omega$ at x. The map $x \to X$ takes the tangent space to Ω to the tangent space to $\underline{\Omega}$, Therefore, $\sum_j \nu_j \tau/(\tau + \sigma_j)\, dX_j$ annihilates the tangent space to $\underline{\Omega}$ at $X(x)$. It is therefore a conormal to $\underline{\Omega}$. The unit conormal $\nu_{\underline{\Omega}}(X)$ is

$$\nu_{\underline{\Omega}}(X) \;=\; \bigg(\sum_j \frac{\nu_j^2(x(X))\,\tau^2}{(\tau + \sigma_j(x(X)))^2}\bigg)^{-1/2} \bigg(\sum_{j=1}^{3} \frac{\nu_j(x(X))\,\tau}{\tau + \sigma_j(x(X))}\, dX_j\bigg).$$

Definition 2.12 *For $\mathrm{Re}\,\tau > 0$ and x on a neighborhood of $\partial\Omega$ define the first order differential operator V by*

$$V(\tau, x, \partial) \;:=\; \Big(\sum_j \frac{\nu_j^2\,\tau^2}{(\tau + \sigma_j)^2}\Big)^{-1/2}\; \sum_j \frac{\nu_j\,\tau^2}{(\tau + \sigma_j)^2}\,\frac{\partial}{\partial x_j}\,. \tag{2.20}$$

Remark 2.2 i. *For* $\tau \in]0, \infty[$ V *is a unit vector field transverse to* $\partial\Omega$ *since its scalar product with the unit outward normal* $\nu \cdot \partial$ *is strictly postive.*

ii. *For complex* τ *the coefficients of* V *are complex so it need not be a vector field.*

iii. *There is an* $R > 0$ *independent of* δ *so that for* $|\tau| > R$, $\partial\Omega$ *is non characteristic for* V. Indeed, $V - \nu \cdot \partial$ has coefficients $O(1/\tau)$ and the boundary is noncharacteristic for $\nu \cdot \partial$.

Corollary 2.13 *There is an* $m > 0$ *so that if* $\tau \in]m, \infty[$ *and* $u \in H^2(\Omega)$ *satisfies the boundary condition*

$$u \in \mathcal{E}^+\big(\widetilde{\nu}(\tau, x)\big) \quad \text{on} \quad \partial\Omega, \tag{2.21}$$

then with $V(\tau, x, \partial)$ *from* (2.20),

$$\pi^+(\widetilde{\nu}) \sum_{j=1}^{3} A_j \widetilde{\partial}_j u \;=\; \pi^+(\widetilde{\nu}) \Big(V(\tau, \underline{x}, \partial) + 2H_{\underline{\Omega}}(X(\tau, \underline{x}))\Big) u \quad \text{on} \quad \partial\Omega\,. \tag{2.22}$$

Remark 2.3 *The normal matrix of the stretched system is equal to* $A(\widetilde{\nu})$. *For positive* τ, *the boundary condition in* (2.21) *is the natural maximally absorbing one.*

Proof. Corollary 2.13 follows from Proposition 2.10. Define $v : \underline{\Omega} \to \mathbb{C}^2$ by $v(X) := u(x(X))$. Since u satisfies the stretched Pauli system on a neighborhood of $\partial\Omega$, (2.19) implies that that v satisfies the unstretched Pauli system on a neighborhood of $\partial\underline{\Omega}$.

The unstretched differential equation satisfied by v has principle symbol $\sum_j A_j\, \partial/\partial X_j$. The symbol at any outward conormal vector to $\underline{\Omega}$ is equal to a positive multiple of $\sum_j A_j \nu_j \tau/(\tau + \sigma_j)$. This sum is equal to the symbol of the stretched operator on Ω at the conormal ν to Ω. Thus the positive eigenspace of the unstretched symbol at $\nu_{\underline{\Omega}}(X)$ is equal to the positive eigenspace of the stretched operator at $\nu_\Omega(x)$.

The boundary condition satisfied by u asserts that

$$u \;\in\; \mathcal{E}^+\big(A(\widetilde{\nu})\big) \;=\; \mathcal{E}^+\big(A(\nu_{\underline{\Omega}})\big)\,.$$

Therefore v satisfies the boundary condition $v|_{\partial\underline{\Omega}} \in \mathcal{E}^+\big(A(\nu_{\underline{\Omega}})\big)$. The function v on $\underline{\Omega}$ therefore satisfies the hypotheses of Proposition 2.10 on $\underline{\Omega}$. That Proposition implies that for $X \in \partial\underline{\Omega}$,

$$\pi^+\big(\widetilde{\nu}(x(X))\big) \sum_{j=1}^{3} A_j \widetilde{\partial}_j u \;=\; \pi^+\big(\widetilde{\nu}(x(X))\big) \Big(\nu_{\underline{\Omega}} \cdot \partial_X + 2H_{\underline{\Omega}}(X)\Big) v\,.$$

Equation (2.19) shows that

$$\nu_{\underline{\Omega}} \cdot \partial_X \;=\; \Big(\sum_j \frac{\nu_j^2}{(\tau+\sigma_j)^2} \Big)^{-1/2} \sum_j \frac{\nu_j}{\tau+\sigma_j}\, \frac{\tau}{\tau+\sigma_j}\, \frac{\partial}{\partial x_j} \;=\; V\,.$$

Inserting in the preceding equation yields (2.22). □

2.3.3 Transverse identity for stretched Pauli for $\tau \notin \mathbb{R}$

Part **iv** of the next proposition derives the key identity for complex τ. It follows from the real case by analytic continuation.

Proposition 2.14 i. *There is an $R_1 > 1$ so that for $|\tau| > R_1$ the spectrum of $A(\widetilde{\nu}(\tau, x))$ consists of one simple eigenvalue in $|z-1| < 1$ and a second in $|z-(-1)| < 1$. Then the map $\tau \mapsto \pi^{\pm}\big(A\big(\widetilde{\nu}(\tau, x)\big)$ is analytic in $|\tau| > R_1$.*

ii. *There is an $R_2 \geq R_1$ so that the function $\tau \mapsto \nu_{\underline{\Omega}}(X(\tau, x))$ from $]m, \infty[$ to $C^\infty(\partial\Omega)$ has a holomorphic extension to $\{|\tau| > R_2\}$.*

iii. *There is an $R_3 \geq R_2$ so that the function $\tau \mapsto H_{\underline{\Omega}}(X(\tau, x))$ from $]m, \infty[$ to $C^\infty(\partial\Omega)$ has a holomorphic extension to $\{|\tau| > R_3\}$.*

iv. *If $|\tau| > R_3$ and $u \in H^2(\Omega)$ satisfies $u \in \mathcal{E}^+(\widetilde{\nu})$ on $\partial\Omega$, then (2.22) holds.*

Proof. i. For $|\tau|$ large one has uniformly for $x \in \partial\Omega$,

$$\widetilde{\nu} \;=\; \left(\frac{\nu_1\,\tau}{\tau+\sigma_1}\,,\, \frac{\nu_2\,\tau}{\tau+\sigma_2}\,,\, \frac{\nu_3\,\tau}{\tau+\sigma_3} \right) \;=\; \nu \,+\, O(|\tau|^{-1})\,,$$

The assertions in **i** follows from Part **ii** of Proposition 2.1.

ii. It suffices to construct the analytic continuation for points in a neighborhood of each $\underline{X} \in \partial\underline{\Omega}$. Suppose that $\underline{X} = X(\tau, \underline{x})$ with $\tau > 0$ and $\underline{x} \in \partial\Omega$ and X the stretching transformation defined by (2.18). The map $\tau \mapsto X(\tau, \cdot)$ is holomorphic on $\tau \neq 0$ with values in $C^\infty(\partial\Omega; \mathbb{C})$. In addition, $\partial X/\partial x = I + O(1/\tau)$, so $\partial X/\partial x$ is invertible for $|\tau| > R$.

Suppose that $x(u_1, u_2)$ is a parametrization of a neighborhood of $\underline{x}$ in $\partial\Omega$. Then for $\tau > 0$, $X(\tau, x(u_1, u_2))$ is a parametrization of a neighborhood of $\underline{X}$ in $\partial\underline{\Omega}$. For those τ the tangent space to $\partial\underline{\Omega}$ is spanned by the independent vectors $\partial X(\tau, x(u))/\partial u_i$, $1 = 1, 2$. Thanks to the invertibility of $\partial X/\partial x$, the formula

$$\text{Span}\,\Big\{ \frac{\partial X(\tau, x(u))}{\partial u_1}\,,\, \frac{\partial X(\tau, x(u))}{\partial u_2} \Big\} \;=\; \text{Span}\,\Big\{ \frac{\partial X}{\partial x}\frac{\partial x}{\partial u_1}\,,\, \frac{\partial X}{\partial x}\frac{\partial x}{\partial u_2} \Big\} \qquad (2.23)$$

shows that the tangent space has a holomorphic conitnuation to $|\tau| > R$.

For real τ a normal vector to $\underline{\Omega}$ as $X(\tau, x(u))$ is given by

$$\frac{\partial X}{\partial x}\frac{\partial x}{\partial u_1} \ \wedge \ \frac{\partial X}{\partial x}\frac{\partial x}{\partial u_2}.$$

It is nonvanishing because $\partial X/\partial x$ is invertible and the vectors $\partial x/\partial u_j$ are independent. The unit normal vector is given by

$$\nu(X(\tau, x(u)) \ = \ \frac{\frac{\partial X}{\partial x}\frac{\partial x}{\partial u_1} \ \wedge \ \frac{\partial X}{\partial x}\frac{\partial x}{\partial u_2}}{\left[\sum_i \left(\left(\frac{\partial X}{\partial x}\frac{\partial x}{\partial u_1} \ \wedge \ \frac{\partial X}{\partial x}\frac{\partial x}{\partial u_2}\right)_i\right)^2\right]^{1/2}}$$

Since $\partial X/\partial x = I + O(1/\tau)$ it follows that one can choose $R > 0$ so that

$$\sum_i \left(\left(\frac{\partial X}{\partial x}\frac{\partial x}{\partial u_1} \ \wedge \ \frac{\partial X}{\partial x}\frac{\partial x}{\partial u_2}\right)_i\right)^2$$

has strictly positive real part for $|\tau| > R$. With that choice the expression for $\nu(X(\tau, x(u))$ yields an analytic continuation of the unit normal vector to $|\tau| > R$. For nonreal values of τ, $\nu(X(\tau, x(u))$ need not be real and need not be of unit length.

iii. For τ real the Weingarten map is the map from $T_{\underline{X}}(\partial\underline{\Omega})$ to itself that maps the two basis vectors as follows,

$$\frac{\partial X(\tau, x(u))}{\partial u_j} \quad \to \quad \frac{\partial \nu_{\underline{\Omega}}(X(\tau, x(u))}{\partial u_j}, \qquad j = 1, 2. \tag{2.24}$$

The holomorphic extension of ν implies that the Weingarten map extends holomorphically to a family of linear map of the holomorphic family of two dimensional spaces (2.23) to itself.

For τ real the mean curvature $H_{\underline{\Omega}}$ is equal to one half of the trace of the Weingarten map. The preceding paragraph shows that this trace has a holomorphic continuation proving **iii.**

iv. The difference of the two sides of (2.22) is holomorphic in $|\tau| > R_3$. Corollary 2.13 implies that it vanishes for τ on the real axis and larger than M. By analytic continuation it vanishes on $|\tau| > R_3$. □

The next corollary gives the desired second boundary condition.

Corollary 2.15 *If $u \in H^2(\Omega)$ satisfies $L(\tau, \widetilde{\partial})u = 0$ on $\partial\Omega$ and $u \in \mathcal{E}^+(\widetilde{\nu})$ on $\partial\Omega$, then*

$$\pi^+(\widetilde{\nu})\Big(V(\tau, x, \partial) + \tau + 2H_{\underline{\Omega}}(X(\tau, \underline{x}))\Big)u = 0 \quad \text{on} \quad \partial\Omega\,. \tag{2.25}$$

Proof. Equation (2.22) implies that

$$\pi^+(\widetilde{\nu})L(\tau,\widetilde{\partial})u \;=\; \pi^+(\widetilde{\nu})\,\Big(V(\tau,x,\partial)+\tau+2H_{\underline{\Omega}}(X(\tau,\underline{x}))\Big)u \quad \text{on} \quad \partial\Omega\,.$$

Since $L(\tau,\widetilde{\partial})u=0$ on $\partial\Omega$, u satisfies (2.25). ☐

3 The Pauli-Helmoltz system

3.1 The Helmholtz operator

Derive a Helmholtz equation that is satisfied by all solutions of the stretched equations. For $i\neq j$ the anticommutation formulas (2.1) imply that

$$A_i\widetilde{\partial}_i\,A_j\widetilde{\partial}_j \;+\; A_j\widetilde{\partial}_j\,A_i\widetilde{\partial}_i \;=\; 0, \qquad \text{for}, \qquad i\neq j\,.$$

Indeed, when the derivatives fall on variable coefficients they yield zero. Define

$$\widetilde{\partial}_j^2 \;:=\; \Big(\frac{\tau}{\tau+\sigma_j(x_j)}\,\partial_j\Big)\Big(\frac{\tau}{\tau+\sigma_j(x_j)}\,\partial_j\Big)$$

where the order of the operators is important. The following stretched versions of (2.4) hold,

$$\begin{aligned}\Big(\sum_j A_j\widetilde{\partial}_j\Big)^2 \;&=\; -\sum_j \widetilde{\partial}_j^2\,,\\ \Big(\sum A_j\widetilde{\partial}_j-\tau\Big)\Big(\sum A_j\widetilde{\partial}_j+\tau\Big) \;&=\; \sum_j \widetilde{\partial}_j^2 \;-\; \tau^2\,.\end{aligned} \tag{3.1}$$

The second equation in (3.1) shows that where a function u satisfies $L(\tau,\widetilde{\partial})u=0$, it satisfies the elliptic equation $\big(\sum_j\widetilde{\partial}_j^2-\tau^2\big)u=0$. The next lemma constructs a divergence form equation.

Definition 3.1 *Define*

$$p\big(\tau,x,\partial\big)u \;:=\; \sum_{j=1}^{3}\,\partial_j\,\frac{(\tau+\sigma_{j+1}(x_{j+1}))(\tau+\sigma_{j+2}(x_{j+2}))}{\tau(\tau+\sigma_j(x_j))}\,\partial_j u\,, \tag{3.2}$$

and

$$\Pi(\tau,x) \;:=\; \prod_{i=1}^{3}\,\frac{\tau+\sigma_i(x_i)}{\tau}\,. \tag{3.3}$$

Lemma 3.2 *As operators on $H^2_{loc}(\mathcal{Q})$,*

$$\Pi(\tau,x)\Big(\sum_j A_j\widetilde{\partial}_j - \tau\Big)\Big(\sum_j A_j\widetilde{\partial}_j + \tau\Big) \;=\; p(\tau,x,\partial) \;-\; \tau^2\,\Pi(\tau,x). \tag{3.4}$$

Proof. Expanding the product on the left using the anticommutation relations (2.1) yields

$$\sum_j \,\Big(\prod_{i=1}^3 \frac{\tau+\sigma_i(x_i)}{\tau}\Big)\,\Big(\frac{\tau}{\tau+\sigma_j(x_j)}\,\partial_j\Big)\Big(\frac{\tau}{\tau+\sigma_j(x_j)}\,\partial_j\Big) \;-\; \tau^2\,\prod_{i=1}^3 \frac{\tau+\sigma_i(x_i)}{\tau}\,.$$

The factor before the first derivative on the left is equal to

$$\Big(\prod_{i=1}^3 \frac{\tau+\sigma_i(x_i)}{\tau}\Big)\,\frac{\tau}{\tau+\sigma_j(x_j)} \;=\; \frac{\big(\tau+\sigma_{j+1}(x_{j+1})\big)\big(\tau+\sigma_{j+2}(x_{j+2})\big)}{\tau^2}\,.$$

This function does not depend on x_j so commutes with ∂_j.

$$\begin{aligned}\Big(\prod_{i=1}^3 \frac{\tau+\sigma_i(x_i)}{\tau}\Big)\,\Big(\frac{\tau}{\tau+\sigma_j(x_j)}\,\partial_j\Big)\Big(\frac{\tau}{\tau+\sigma_j(x_j)}\,\partial_j\Big)&\\ \;=\; \partial_j\Big(\frac{\big(\tau+\sigma_{j+1}(x_{j+1})\big)\big(\tau+\sigma_{j+2}(x_{j+2})\big)}{\tau^2}\;\frac{\tau}{\tau+\sigma_j(x_j)}\Big)\partial_j&\\ \;=\; \partial_j\Big(\frac{\big(\tau+\sigma_{j+1}(x_{j+1})\big)\big(\tau+\sigma_{j+2}(x_{j+2})\big)}{\tau\,\big(\tau+\sigma_j(x_j)\big)}\Big)\partial_j\,.&\end{aligned}$$

This completes the proof. □

Remark 3.1 **i.** *The factors in the product on the left of* (3.4) *are*

$$\sum_j A_j\widetilde{\partial}_j + \tau \;=\; L\big(\tau,\widetilde{\partial}\big)\,,\quad \text{and,}\quad \sum_j A_j\widetilde{\partial}_j - \tau \;=\; L\big(-\tau,\widetilde{\partial}\big)\,. \tag{3.5}$$

ii. *Since*

$$\Big|\Pi(\tau,x)-1\Big| \;=\; \Big|\prod_{i=1}^3 \frac{\tau+\sigma_i(x_i)}{\tau}\;-1\Big| \;\lesssim\; \frac{1}{|\tau|} \tag{3.6}$$

the coefficients of the operator on the right of (3.4) *differ from those of the classical Helmholtz operator $\Delta-\tau^2$ by $O(|\tau|^{-1})$.*

Definition 3.3 • *For vectors* α, β *in* $\mathbb{C}^k$ *define* $\alpha \cdot \beta := \sum_j \alpha_j \beta_j$.

• *Define the continuous bilinear form* $a : H^1(\mathcal{Q}; \mathbb{C}^2) \times H^1(\mathcal{Q}; \mathbb{C}^2) \to \mathbb{C}$ *associated to* $-p$ *by*

$$a(u,v) = \int_{\mathcal{Q}} \sum_{j=1}^{3} \frac{(\tau + \sigma_{j+1}(x_{j+1}))(\tau + \sigma_{j+2}(x_{j+2}))}{\tau(\tau + \sigma_j(x_j))} \partial_j u \cdot \partial_j v \; dx\,. \qquad (3.7)$$

• *If* $\Omega \subset \mathcal{Q}$ *is open the formula with integration over* Ω *defines a continuous form from* $H^1_{loc}(\Omega) \times H^1_{compact}(\Omega) \to \mathbb{C}$.

Remark 3.2 **i.** *If* $u \in H^1_{loc}(\Omega)$ *and* $f \in H^{-1}_{loc}(\Omega)$ *then* u *satisfies* $pu = f$ *on* Ω *if and only if*

$$\forall \phi \in C_0^\infty(\Omega), \qquad a(u, \phi) \;=\; -\int_\Omega f \cdot \phi \; dx\,.$$

ii. *Multiplying numerator and denominator of the coefficient of* ∂_j *in* (3.7) *by* $\tau + \sigma_j$ *shows that*

$$a(u,v) \;=\; \int_\Omega \Pi(\tau,x) \; \sum_{j=1}^{3} \frac{\tau^2}{(\tau+\sigma_j)^2} \; \partial_j u \cdot \partial_j v \; dx\,. \qquad (3.8)$$

iii. *If* $u \in H^2(\Omega)$, *an integration by parts yields*

$$a(u,v) = -\int_\Omega p\, u \cdot v \; dx + \int_{\partial\Omega} \Pi(\tau,x) \sum_{j=1}^{3} \frac{\nu_j\, \tau^2}{(\tau+\sigma_j)^2} \, \partial_j u \cdot v \, d\Sigma. \qquad (3.9)$$

To solve the stretched equation, start by using (3.4) to show that any solution must satisfies the Helmholtz equation

$$\Big(p(\tau, x, \partial) - \tau^2\, \Pi(\tau, x)\Big) u \;=\; \Pi(\tau, x) \; \Big(\sum A_j \widetilde{\partial}_j - \tau\Big) F. \qquad (3.10)$$

Remark 3.3 *There is an extensive literature on using the PML technology for the solution of time harmonic scattering problems for the wave equation beginning with Collino-Monk and Lassas-Somersalo [9, 19, 20, 6, 7]. All depend on choosing* σ_j *constant outside a compact set and then relying on an explicit Green's function for the Helmholtz operator* $\tau^2 - p$ *with* $\tau = i\omega$ *and* x *outside that compact set. Rellich's Uniqueness Theorem and the exponential decay of the Green's function drives the analysis. The operator* p *and the* *form $a(\cdot,\cdot)$ appear in those articles. Variable σ_j, corners, and absorbing boundary conditions at trihedral corners have no analogue in their work. This time harmonic work is related to the method of complex scaling in Scattering Theory introduced by Balslev-Coombes [4] and raised to high art by Sjöstrand and a brilliant school (see [11]).*

3.2 The Helmholtz boundary value problem, Theorem 3.7

Equation (3.10) is supplemented by boundary conditions. The goal is to prove Theorem 2.8 so have $u \in \mathcal{E}^+(\widetilde{\nu})$, equivalently $\pi^-(\widetilde{\nu})u = 0$. Corollary 2.15 provides the second boundary condition. The present section is devoted to studying the resulting Helmholtz boundary value problem,

$$\begin{aligned} \Big(\tau^2\,\Pi(\tau,x)\ -\ p(\tau,x,\partial)\Big)u &= f \quad \text{on} \quad \mathcal{Q}_\delta\,,\\ \pi^-(\widetilde{\nu}(\tau,x))u &= g_1 \quad \text{on} \quad \partial\mathcal{Q}_\delta\,, \qquad (3.11)\\ \pi^+(\widetilde{\nu}(\tau,x))\Big(V(\tau,x,\partial)+\tau+2H_{\underline{\Omega}}(X(\tau,\underline{x}))\Big)u &= g_2 \quad \text{on} \quad \partial\mathcal{Q}_\delta\,. \end{aligned}$$

Here g_1 and g_2 are functions on $\partial\Omega$ that take values in $\mathcal{E}^-(A(\widetilde{\nu}(\tau,x))$ and $\mathcal{E}^+(A(\widetilde{\nu}(\tau,x))$ respectively.

Definition 3.4 *For $S \in \operatorname{Hom}\mathbb{C}^k$ denote by $S^\dagger$ the transposed matrix so that $Su\cdot v = u\cdot S^\dagger v$ for all vectors $u, v \in \mathbb{C}^k$.*

For $|\operatorname{Im}\xi| < |\operatorname{Re}\xi|$, $A(\xi)$ has two eigenvalues $\lambda^\pm(\xi)$ and spectral representation

$$A(\xi) = \lambda^+\pi^+(\xi) + \lambda^-\pi^-(\xi), \quad \text{so,} \quad A(\xi)^\dagger = \lambda^+\pi^+(\xi)^\dagger + \lambda^-\pi^-(\xi)^\dagger.$$

Therefore $\lambda^\pm$ are eigenvalues of $A(\xi)^\dagger$ and $\pi^\pm(\xi)^\dagger$ are the corresponding spectral projections.

Definition 3.5 *Define the* **transposed boundary value problem** *as,*

$$\begin{aligned} \Big(\tau^2\,\Pi(\tau,x)\ -\ p(\tau,x,\partial)\Big)u &= f \quad \text{on} \quad \mathcal{Q}_\delta\,,\\ \pi^-(\widetilde{\nu}(\tau,x))^\dagger u &= g_1 \quad \text{on} \quad \partial\mathcal{Q}_\delta\,, \qquad (3.12)\\ \pi^+(\widetilde{\nu}(\tau,x))^\dagger\Big(V(\tau,x,\partial)+\tau+2H_{\underline{\Omega}}(X(\tau,\underline{x}))\Big)u &= g_2 \quad \text{on} \quad \partial\mathcal{Q}_\delta\,. \end{aligned}$$

The g_1 and g_2 are functions on $\partial\Omega$ taking values in $\mathcal{E}^-(A(\widetilde{\nu})^\dagger)$ and $\mathcal{E}^+(A(\widetilde{\nu})^\dagger)$ respectively.

The annihilator of the range of the direct problem is equal to the nullspace of the transposed problem (see Section 3.3.3).

Lemma 3.6 *There is an $R>0$ independent of δ to that for $|\tau|>R$ the boundary value problems (3.11) and (3.12) satisfy Lopatinski's condition for all $x\in\partial\mathcal{Q}_\delta$.*

Proof. Treat only (3.11). The proof for the other is nearly identical.

The Lopatinski condition concerns only the leading order parts of the operators. In addition as $|\tau|\to\infty$, $p(\tau,x,\partial)\to\Delta$ and $V(\tau,x,\partial)\to\nu_{\mathcal{Q}_\delta}\cdot\partial$. Thus it suffices to prove Lopatinski's condition for the constant coefficient half space problems,

$$\begin{aligned}\Delta\, u \;&=\; f \quad \text{on} \quad \nu\cdot x<0\,,\\ \pi^-(\nu)u \;&=\; g_1 \quad \text{on} \quad \nu\cdot x=0\,,\\ \pi^+(\nu)\big(\nu\cdot\partial_x u\big) \;&=\; g_2 \quad \text{on} \quad \nu\cdot x=0\,.\end{aligned}\tag{3.13}$$

Though (3.11) is not rotation invariant, (3.13) is rotation invariant. It therefore suffices to consider (3.13) with $\nu=(-1,0,0)$. That yields the boundary value problem

$$\begin{aligned}\Delta\, u \;&=\; f \quad \text{on} \quad x_1>0\,,\\ u_1 \;&=\; g_1 \quad \text{on} \quad x_1=0\,,\\ \partial_{x_1}u_2 \;&=\; g_2 \quad \text{on} \quad x_1=0\,.\end{aligned}\tag{3.14}$$

This is the Dirichlet problem for u_1 and the Neumann problem for u_2. Lopatinski's condition is known for each of them. □

Theorem 3.7 *There is an $M>0$ so that if $\operatorname{Re}\tau>M$ and $0<\delta<1$, then the continuous linear map*

$$H^2(\mathcal{Q}_\delta)\ni u\mapsto(f,g_1,g_2)\in L^2(\mathcal{Q}_\delta)\times H^{3/2}(\partial\mathcal{Q}_\delta;\mathcal{E}^-(\widetilde{\nu}))\times H^{1/2}(\partial\mathcal{Q}_\delta;\mathcal{E}^+(\widetilde{\nu}))$$

defined by (3.11) is one to one and onto.

Beginning the proof of Theorem 3.7. The theory of elliptic boundary value problems satisfying Lopatinski's condition implies the following facts.

- The kernel of the map is a finite dimensional subset of $C^\infty(\overline{\mathcal{Q}_\delta})$.
- The range is closed with finite codimension.
- The annihilator of the range is a subspace of $C^\infty(\overline{\mathcal{Q}_\delta})\times C^\infty(\partial\mathcal{Q}_\delta)\times C^\infty(\partial\mathcal{Q}_\delta)$.

To prove the theorem it suffices to prove that the kernel and the annihilator of the range are both trivial.

3.3 Main *a priori* estimate, Theorem 3.8

Theorem 3.8 *There are constants C, M independent of $\delta \in]0,1[$ and $\tau \in \{\operatorname{Re}\tau > M\}$ so that if $u \in H^2(\mathcal{Q}_\delta)$ satisfies the direct problem (3.11) (resp. the transposed problem (3.12)) with $g_1 = 0$ and $g_2 = 0$ then*

$$|\tau|\,(\operatorname{Re}\tau)\,\big\|u\big\|^2_{L^2(\mathcal{Q}_\delta)} + |\tau|\,\big\|u\big\|^2_{L^2(\partial\mathcal{Q}_\delta)} + \big\|\nabla u\big\|^2_{L^2(\mathcal{Q}_\delta)} \le C\,\big\|f\big\|^2_{H_0^{-1}(\mathcal{Q}_\delta)}. \tag{3.15}$$

The proof relies on two estimates for a bilinear form $\mathcal{A}$ associated to the boundary value problem. The first is a lower bound for $\mathcal{A}(u,\overline{u})$ that holds for all $u \in H^1(\Omega)$. The second is an upper bound that relies on the boundary conditions.

Definition 3.9 • *Using the analytic continuation $H_{\underline{\Omega}}(X(\tau,x))$ from Part* **iii** *of Proposition 2.14, define $\Phi, \beta \in C^\infty(\{\operatorname{Re}\tau > M\}\times]0,1[\times\partial\Omega)$ by*

$$\begin{aligned}\Phi(\tau,x) \;&:=\; \Pi(\tau,x)\Big(\sum_j \frac{\nu_j^2\tau^2}{(\tau+\sigma_j)^2}\Big)^{1/2},\\ \beta(\tau,\delta,x) \;&:=\; \tau + 2H_{\underline{\Omega}}\big(X(\tau,x)\big).\end{aligned} \tag{3.16}$$

• *With $\operatorname{Re}\tau > M$ and $a(u,v)$ from (3.7), define continuous bilinear forms $\mathcal{A}(\tau,\cdot,\cdot) : H^1(\Omega)\times H^1(\Omega) \to \mathbb{C}$ by*

$$\mathcal{A}(\tau,u,v) \;:=\; a(u\,,\,v) + \int_\Omega \tau^2\,\Pi(\tau,x)\;u\cdot v\;dx + \int_{\partial\Omega}\Phi\,\beta\,u\,\cdot\,v\;d\Sigma\,. \tag{3.17}$$

The dependence of $\mathcal{A}$ on Ω and therefore δ is suppressed. Similarly, the dependence of $\mathcal{A}$ on τ is usually not indicated. There is a simple δ independent upper bound for $|\tau| > 1$,

$$\begin{aligned}\big|\mathcal{A}(u,v)\big| \lesssim \Big[&\Big(|\tau|^2\,\|u\|^2_{L^2(\Omega)} + \||\beta|^{1/2}\,u\|^2_{L^2(\partial\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}\Big)^{1/2}\\ &\cdot\Big(|\tau|^2\,\|v\|^2_{L^2(\Omega)} + \||\beta|^{1/2}\,v\|^2_{L^2(\partial\Omega)} + \|\nabla v\|^2_{L^2(\Omega)}\Big)^{1/2}\Big].\end{aligned} \tag{3.18}$$

The term $H_{\underline{\Omega}}(X(\tau,x))$ in $\mathcal{A}$ is equal to zero except for a δ neighborhood of $\mathcal{S}$ where it attains values $\sim 1/\delta$.

The connection with the boundary value problems is the following identity. It shows that boundary conditions in terms of $(V+\beta)u$, as in (2.25), (3.11), and (3.12) arise as natural boundary conditions.

Lemma 3.10 *If $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$, define $f := (\tau^2 \Pi - p)u$. Then,*

$$\mathcal{A}(\tau, u, v) - \int_\Omega f \cdot v\, dx = \int_{\partial\Omega} \Phi(\tau, x) \big(V + \beta(\tau, \delta, x)\big) u \cdot v \; d\Sigma\,. \tag{3.19}$$

Proof. The differential operator appearing in the boundary term of Green's formula (3.9) is related to the operator $V(\tau, x, \partial)$ associated to the natural boundary condition for the stretched Pauli system by

$$\begin{aligned}\Pi(\tau, x) \sum_{j=1}^{3} \frac{\nu_j\, \tau^2}{(\tau + \sigma_j)^2}\, \partial_j &= \Pi(\tau, x) \Big(\sum_{j=1}^{3} \Big(\frac{\nu_j\, \tau^2}{(\tau + \sigma_j)^2} \Big)^2 \Big)^{1/2} V(\tau, x, \partial) \\ &= \Phi(\tau, x)\, V(\tau, x, \partial)\end{aligned}$$

Equation (3.9) shows that

$$\begin{gathered} a(u\,, v) + \int_\Omega \tau^2\, \Pi(\tau, x)\; u \cdot v\; dx - \int_\Omega f \cdot v\, dx = \\ \int_{\partial\Omega} \Pi(\tau, x) \sum_{j=1}^{3} \frac{\nu_j\, \tau^2\, \partial_j u}{(\tau + \sigma_j)^2} \cdot v\; d\Sigma = \int_{\partial\Omega} \Phi(\tau, x)\; V u \cdot v\; d\Sigma\,. \end{gathered} \tag{3.20}$$

Adding $\int_{\partial\Omega} \Phi\, \beta\, u \cdot v\, d\Sigma$ to both sides proves (3.19). □

Example 3.1 *If on $\partial\Omega$, u satisfies*

$$\pi^-(\widetilde{\nu}) u = 0, \quad \text{and,} \quad \Big(V + \tau + 2 H_{\underline{\Omega}}(X(\tau, \underline{x})) \Big) u = 0,$$

and v satisfies $\pi^+(\widetilde{\nu})^\dagger v = 0$, then the boundary term in the lemma vanishes. This yields a weak formulation, and, a mixed finite element approach to the boundary value problem for u.

3.3.1 Lower bound for $|\mathcal{A}(u, \overline{u})|$.

Proposition 3.11 *There are constants $C, M > 0$ independent of $\delta \in]0, 1[$ so that for any $\tau \in \big\{ \operatorname{Re} \tau \geq M \big\}$, and $u \in H^1(\mathcal{Q}_\delta)$,*

$$|\tau|\, (\operatorname{Re} \tau)\, \|u\|^2_{L^2(\mathcal{Q}_\delta)} + \|\, |\beta|^{1/2}\, u\|^2_{L^2(\partial\mathcal{Q}_\delta)} + \|\nabla_x u\|^2_{L^2(\mathcal{Q}_\delta)} \leq C \big| \mathcal{A}(u, \overline{u}) \big|. \tag{3.21}$$

Remark 3.4 *In (3.36), we show that $H_{\underline{\Omega}} = H_\Omega + O(1/\tau)$. Since $\beta = \tau + 2\, H_{\underline{\Omega}}(\tau, x)$ it follows that there is an M independent of δ to that for $\operatorname{Re} \tau > M$*

$$|\tau| + H_\Omega(x) \;\leq\; |\beta(\tau, \delta, x)| \;\leq\; |\tau| + 3 H_\Omega(x)\,.$$

Proof. Step 1. $\mathcal{A}_0$ and its real and imaginary parts. Denote by $\mathcal{A}_0$ the form that one would have if $\sigma_j = 0$ for all j,

$$\mathcal{A}_0(\tau, u, v) \;:=\; \int_{\mathcal{Q}_\delta} \tau^2\; u\cdot v\; dx \;+\; \int_{\partial\mathcal{Q}_\delta} \beta\, u\cdot v\; d\Sigma \;+\; \int_{\mathcal{Q}_\delta} \nabla_x u\cdot \nabla_x v\; dx\,.$$

Therefore,

$$\mathcal{A}_0(\tau, u, \overline{u}) \;:=\; \int_{\mathcal{Q}_\delta} \tau^2\; |u|^2\; dx \;+\; \int_{\partial\mathcal{Q}_\delta} \beta\; |u|^2\; d\Sigma \;+\; \int_{\mathcal{Q}_\delta} \big|\nabla_x u\big|^2\; dx\,.$$

Prove the corresponding estimate for $\mathcal{A}_0$ holds on $\{\operatorname{Re}\tau > 0, |\tau| > M_1\}$. Compute exact expressions for the real and imaginary parts of $\mathcal{A}_0$. The real part is

$$\begin{aligned}\operatorname{Re}\mathcal{A}_0(u,\overline{u}) \;&=\; \Big((\operatorname{Re}\tau)^2 - (\operatorname{Im}\tau)^2\Big)\big\|u\big\|^2_{L^2(\mathcal{Q}_\delta)} \\ &\quad+\; \big\|(\operatorname{Re}\beta)^{1/2} u\big\|^2_{L^2(\partial\mathcal{Q}_\delta)} + \big\|\nabla_x u\big\|^2_{L^2(\mathcal{Q}_\delta)}\,.\end{aligned} \tag{3.22}$$

Use $\operatorname{Im}\tau^2 = 2\;(\operatorname{Im}\tau)(\operatorname{Re}\tau)$ to find,

$$\begin{aligned}\operatorname{Im}\int_{\mathcal{Q}_\delta} \tau^2\; |u|^2\; dx \;&=\; (\operatorname{Im}\tau)\int_{\mathcal{Q}_\delta} 2\operatorname{Re}\tau\; |u|^2\, dx\,,\\ \operatorname{Im}\int_{\partial\mathcal{Q}_\delta} \beta\; |u|^2\; d\Sigma \;&=\; (\operatorname{Im}\tau)\int_{\partial\mathcal{Q}_\delta} |u|^2\; d\sigma\,.\end{aligned}$$

Combining shows that for $0 \neq \operatorname{Im}\tau$,

$$\frac{\operatorname{Im}\mathcal{A}_0(u,\overline{u})}{\operatorname{Im}\tau} \;=\; 2\operatorname{Re}\tau\,\big\|u\big\|^2_{L^2(\mathcal{Q}_\delta)} + \big\|u\big\|^2_{L^2(\partial\mathcal{Q}_\delta)}\,. \tag{3.23}$$

Step 2. Proof for $\mathcal{A}_0$. • The bound (3.21) for $\mathcal{A}_0$ is proved by combining (3.22) and (3.23). Care is needed where the terms on the right of (3.22) do not have the same sign. Where $|\operatorname{Im}\tau| < \operatorname{Re}\tau/2$, (3.22) implies directly the bound (3.21) for $\mathcal{A}_0$.

• It suffices to consider the complementary set $\{|\operatorname{Im}\tau| \geq \operatorname{Re}\tau/2\}$. In that parameter range (3.23) implies

$$\operatorname{Re}\tau\,\|u\|^2_{L^2(\mathcal{Q}_\delta)} + \|u\|^2_{L^2(\partial\mathcal{Q}_\delta)} \;\lesssim\; \frac{|\operatorname{Im}\mathcal{A}_0(u,\overline{u})|}{|\tau|}\,. \tag{3.24}$$

Multiplying by $|\tau|$ yields

$$|\tau|\,(\operatorname{Re}\tau)\,\|u\|^2_{L^2(\mathcal{Q}_\delta)} + |\tau|\,\|u\|^2_{L^2(\partial\mathcal{Q}_\delta)} \;\lesssim\; |\operatorname{Im}\mathcal{A}_0(u,\overline{u})|\,. \tag{3.25}$$

• The parameter range $|\operatorname{Im}\tau| \geq \operatorname{Re}\tau/2$ is divided into two subsets. The first is where $\operatorname{Re}\tau > |\operatorname{Im}\tau|$. On this set (3.22) implies that

$$\begin{aligned}\big\|\nabla_x u\big\|^2_{L^2(\mathcal{Q}_\delta)} + \big\||\beta|^{1/2}\,u\big\|_{\partial\mathcal{Q}_\delta} \;\lesssim\; \big\|\nabla_x u\big\|^2_{L^2(\mathcal{Q}_\delta)} + \big\|(\operatorname{Re}\beta)^{1/2}\,u\big\|^2_{\partial\mathcal{Q}_\delta} \\ \lesssim\; \operatorname{Re}\tau\,\big\|u\big\|^2_{L^2(\partial\mathcal{Q}_\delta)} + \operatorname{Re}\mathcal{A}_0(u,\overline{u}) \;\lesssim\; \big|\mathcal{A}(u\,,\,\overline{u})\big|\,,\end{aligned}$$

where (3.25) is used in the third inequality. Combining the last two estimates yields (3.21).

• There remains the parameter range $|\operatorname{Im}\tau| \leq \operatorname{Re}\tau \leq 2|\operatorname{Im}\tau|$. In this range (3.25) implies

$$|\tau|^2\,\big\|u\big\|^2_{L^2(\mathcal{Q}_\delta)} + |\tau|\,\big\|u\big\|^2_{L^2(\partial\mathcal{Q}_\delta)} \lesssim \big|\operatorname{Im}\mathcal{A}_0(u,\overline{u})\big|, \tag{3.26}$$

In particular

$$\Big|(\operatorname{Re}\tau)^2 - (\operatorname{Im}\tau)^2\Big|\,\big\|u\big\|^2_{L^2(\mathcal{Q}_\delta)} \;\lesssim\; \big|\mathcal{A}_0(u,\overline{u})\big|\,.$$

Using this in (3.22) yields for $|\tau| > M_1$,

$$\big\|\nabla_x u\big\|^2_{L^2(\mathcal{Q}_\delta)} + \big\||\beta|^{1/2}\,u\big\|_{\partial\mathcal{Q}_\delta} \;\lesssim\; \big|\mathcal{A}_0(u,\overline{u})\big|\,. \tag{3.27}$$

Adding (3.26) and (3.27) completes the proof of (3.21) for $\mathcal{A}_0$.

Step 3. Perturbation argument. For $\tau \neq 0$, $\tau + \sigma_j(x_j) = \tau(1 + \sigma_j/\tau))$. Write

$$\begin{aligned}a(u,\overline{u}) - a_0(u,\overline{u}) = \int_\Omega \Big(\frac{(\tau+\sigma_{j+1})(\tau+\sigma_{j+2})}{\tau(\tau+\sigma_j)} - 1\Big)|\partial_j u|^2\,dx \\ + \tau^2\int_\Omega \big(\Pi(\tau,x) - 1\big)|u|^2\,dx + \int_{\partial\Omega}\big(\Phi(\tau,x)-1\big)|\beta|\,|u|^2\,d\Sigma\,.\end{aligned}$$

In each term on the right the prefactors in parentheses are $O(1/\tau)$. Applied to $\mathcal{A}$ this yields

$$\begin{aligned}\big|\mathcal{A}(u,\overline{u}) - \mathcal{A}_0(u,\overline{u})\big| \;\lesssim \\ |\tau|\|u\|^2_{L^2(\mathcal{Q}_\delta)} + \|u\|^2_{L^2(\partial\mathcal{Q}_\delta)} + \frac{1}{|\tau|}\|\nabla_x u\|^2_{L^2(\mathcal{Q}_\delta)} \lesssim \frac{1}{\operatorname{Re}\tau}\,\big|\mathcal{A}_0(u\,,\,\overline{u})\big|,\end{aligned} \tag{3.28}$$

where inequality (3.21) for $\mathcal{A}_0$ is used in the last inequality. The triangle inequality and estimate (3.28) imply

$$\big|\mathcal{A}(u\,,\,\overline{u})\big| \;\geq\; \mathcal{A}_0(u\,,\,\overline{u}) \;-\; \big|\mathcal{A}(u\,,\,\overline{u}) - \mathcal{A}_0(u\,,\,\overline{u})\big| \;\geq\; \Big(1 - \frac{c}{\operatorname{Re}\tau}\Big)\;\big|\mathcal{A}_0(u\,,\,\overline{u})\big|.$$

For $\operatorname{Re}\tau$ large, the factor in front is strictly positive. Therefore estimate (3.21) follows from the corresponding estimate for $\mathcal{A}_0$. This completes the proof of Proposition 3.11. □

3.3.2 Upper bound for $|\mathcal{A}(u,\overline{u})|$, proof of Theorem 3.8

Proposition 3.12 *If $u \in H^2(\Omega)$ is a solution of the Helmholtz boundary value problem (3.11) (resp. the transposed problem (3.12)) with $g_1 = 0$ and $g_2 = 0$, then with constant independent of $\delta \in]0,1[$ and $|\tau| > 1$,*

$$\big|\mathcal{A}(u,\overline{u})\big| \lesssim \|f\|_{H_0^{-1}(\Omega)} \|u\|_{H^1(\Omega)} + \frac{1}{|\tau|}\Big(\big\||\beta|^{1/2}\, u\big\|^2_{L^2(\partial\Omega)} + \|u\|^2_{H^1(\Omega)}\Big).$$

Proof of Proposition 3.12. For (3.11) write

$$\big(V+\beta\big)u = (\pi^+(\widetilde{\nu}) + \pi^-(\widetilde{\nu}))\big(V+\beta\big)u \;=\; \pi^-(\widetilde{\nu})\big(V+\beta\big)u.$$

For the transposed boundary value problem (3.12) write

$$\big(V+\beta\big)u \;=\; (\pi^+(\widetilde{\nu})^\dagger + \pi^-(\widetilde{\nu})^\dagger)\big(V+\beta\big)u \;=\; \pi^-(\widetilde{\nu})^\dagger\big(V+\beta\big)u.$$

Continuing the computation for (3.11), Lemma 3.10 yields for $u, v \in H^1(\Omega)$,

$$\mathcal{A}(u,v) \;=\; \int_\Omega f \,\cdot\, v\, dx \;-\; \int_{\partial\Omega} \Phi\, \pi^-(\widetilde{\nu})\, \big(V(\tau,x,\partial) + \beta\big)u \,\cdot\, v\; d\Sigma.$$

With $v = \overline{u}$ this is

$$\mathcal{A}(u,\overline{u}) \;=\; \int_\Omega f \,\cdot\, \overline{u}\, dx \;-\; \int_{\partial\Omega} \Phi\, \pi^-(\widetilde{\nu})\, \big(V(\tau,x,\partial) + \beta\big)u \,\cdot\, \overline{u}\; d\Sigma. \tag{3.29}$$

The first term on the right in (3.29) satisfies

$$\Big|\int_\Omega f \,\cdot\, \overline{u}\; dx\Big| \;\le\; \|f\|_{H_0^{-1}(\Omega)}\; \|u\|_{H^1(\Omega)}\,. \tag{3.30}$$

The norm in $H_0^{-1}(\Omega)$ appears and not the norm in $H^{-1}(\Omega)$ because $\overline{u}$ need not vanish on $\partial\Omega$.

The difficult step is to derive an upper bound for

$$\int_{\partial\Omega} \Phi\, \pi^-(\widetilde{\nu})\big(V(\tau,x,\partial) + \beta\big)u \,\cdot\, \overline{u}\; d\Sigma.$$

The boundary condition $\pi^-(\widetilde{\nu})u = 0$ implies $\pi^+(\widetilde{\nu})u = u$ so,

$$\int_{\partial\Omega} \Phi\, \pi^-(\widetilde{\nu})\big(V(\tau,x,\partial)+\beta\big)u \cdot \overline{u}\, d\Sigma = \int_{\partial\Omega} \Phi\, \pi^-(\widetilde{\nu})\big(V(\tau,x,\partial)+\beta\big)u \cdot \overline{\pi^+(\widetilde{\nu})u}\, d\Sigma.$$

Write

$$\overline{\pi^+(\widetilde\nu)u} \;=\; \overline{\pi^+(\widetilde\nu)}\,\overline{u} \;=\; \pi^+(\widetilde\nu)^\dagger\,\overline{u} \;+\; \Big(\overline{\pi^+(\widetilde\nu)} - \pi^+(\widetilde\nu)^\dagger\Big)\overline{u}\,.$$

When this is inserted the $(\pi^+(\widetilde\nu))^\dagger\overline{u}$ term yields zero. Therefore

$$\begin{aligned}
&\int_{\partial\Omega} \Phi\,\pi^-(\widetilde\nu)\big(V(\tau,x,\partial)+\beta\big)u\,\cdot\,\overline{u}\;d\Sigma \\
&\qquad= \int_{\partial\Omega} \Phi\,\pi^-(\widetilde\nu)\big(V(\tau,x,\partial)+\beta\big)u\,\cdot\,\Big(\overline{\pi^+(\widetilde\nu)} - \pi^+(\widetilde\nu)^\dagger\Big)\overline{u}\;d\Sigma \\
&\qquad= \int_{\partial\Omega} \Phi\,\big(V(\tau,x,\partial)+\beta\big)u\,\cdot\,\pi^-(\widetilde\nu)^\dagger\Big(\overline{\pi^+(\widetilde\nu)} - \pi^+(\widetilde\nu)^\dagger\Big)\overline{u}\;d\Sigma \\
&\qquad= \int_{\partial\Omega} \Phi\,\big(V(\tau,x,\partial)+\beta\big)u\,\cdot\,w\;d\Sigma
\end{aligned} \tag{3.31}$$

with

$$w := \pi^-(\widetilde\nu)^\dagger\Big(\overline{\pi^+(\widetilde\nu)} - \pi^+(\widetilde\nu)^\dagger\Big)\overline{u}.$$

For the transposed problem the difficult boundary term is

$$\int_{\partial\Omega} \Phi\,\pi^-(\widetilde\nu)^\dagger\big(V(\tau,x,\partial)+\beta\big)u\,\cdot\,\overline{u}\;d\Sigma = \int_{\partial\Omega} \Phi\,\big(V(\tau,x,\partial)+\beta\big)u\,\cdot\,w\;d\Sigma$$

with

$$w := \pi^-(\widetilde\nu)\Big(\overline{\pi^+(\widetilde\nu)^\dagger} - \pi^+(\widetilde\nu)\Big)\overline{u}.$$

The estimates in the two cases are virtually identical. The details are presented only for the direct problem. For the direct problem define $\mathsf{m} \in C^\infty(\{\operatorname{Re}\tau > M\} \times \partial\Omega$ by

$$\mathsf{m}(\tau,x) := \tau\;\pi^-(\widetilde\nu)^\dagger\;\big(\overline{\pi^+(\widetilde\nu)} - \pi^+(\widetilde\nu)^\dagger\big), \qquad \text{so,}\quad w = \frac{1}{\tau}\,\mathsf{m}\,\overline{u}\,. \tag{3.32}$$

Need an upper bound for (3.31). Equation (3.32) shows that this is equal to

$$\frac{1}{\tau}\Big(\int_{\partial\Omega} \Phi V u\cdot \mathsf{m}\overline{u}\;dx \;+\; \Phi\;\beta\;u\cdot\mathsf{m}\,\overline{u}\;d\Sigma\Big). \tag{3.33}$$

The next lemma gathers estimates for V and m.

Lemma 3.13 *There are constants C, M so that for all $\operatorname{Re}\tau > M$, and, $0<\delta<1$, the following hold.*

i. $\operatorname{supp}\mathsf{m} \;\subset\; \big\{x\in\partial\Omega \;:\; \operatorname{dist}(x,\mathcal{S})<\delta\big\}$.

ii. $\|\mathsf{m}(\tau,x)\|_{L^\infty(\partial\Omega)} \ \le\ C.$

iii. $\left\|\nabla_x \mathsf{m}(\tau,x)\right\|_{L^\infty(\partial\Omega)} \ \le\ C\,|\beta|\,.$

iv. *For all* $u\in H^{1/2}(\partial\Omega)$,

$$\|\mathsf{m}\,u\|_{H^{1/2}(\partial\Omega)} \ \lesssim\ \|\,|\beta(\tau,x)|^{1/2}\,u\|_{L^2(\partial\Omega)} + \|u\|_{H^{1/2}(\partial\Omega)}.$$

v. *For all* $u\in H^1(\Omega)$, $\|Vu\|_{H^{-1/2}(\partial\Omega)} \ \le\ C\,\|u\|_{H^1(\Omega)}$.

Proof of Lemma. i. For most points $x\in\partial\Omega$, one has $x\in G_j$ for some j, $\nu=-\mathbf{e}_j$, and $A(\nu)=-A_j$. At those points $A(\nu)$ is real and hermitian symmetric and $\widetilde{\nu}$ is parallel to ν. Therefore,

$$\pi^\pm(\nu) \ =\ \pi^\pm(\nu)^\dagger \ =\ \overline{\pi^\pm(\nu)} \ =\ \pi^\pm(\widetilde{\nu}) \ =\ \pi^\pm(\widetilde{\nu})^\dagger \ =\ \overline{\pi^\pm(\widetilde{\nu})}\,. \tag{3.34}$$

It follows that $\mathsf{m}=0$ at such points. This shows that m is supported on the rounded edges of $\partial\Omega$ proving **i.**

ii. Compute

$$\frac{\tau}{\tau+\sigma_j} \ =\ \frac{1}{1+\sigma_j/\tau} \ =1-\frac{\sigma_j}{\tau}+\Big(\frac{\sigma_j}{\tau}\Big)^2 -\ \cdots\,.$$

It follows that as $|\tau|\to\infty$,

$$\widetilde{\nu}-\nu \ =\ O(1/|\tau|),\quad \text{so,}\quad \pi^+(\widetilde{\nu})-\pi^+(\nu) \ =\ O(1/|\tau|).$$

To estimate the size of m write

$$\overline{\pi^+(\widetilde{\nu})}-\pi^+(\widetilde{\nu})^\dagger \ =\ \Big(\overline{\pi^+(\widetilde{\nu})}-\overline{\pi^+(\nu)}\Big) \ +\ \Big(\overline{\pi^+(\nu)}-\pi^+(\widetilde{\nu})^\dagger\Big)\,.$$

The first summand is $O(1/\tau)$. Equation (3.34) implies that the second is equal to $\pi^+(\nu)^\dagger-\pi^+(\widetilde{\nu})^\dagger$ so is also $O(1/|\tau|)$. If follows that m is bounded uniformly in τ,δ, proving **ii.**

iii. Use the notations from Proposition 2.14. Then $\tau\mapsto\nu(\tau,\cdot)$ is analytic in $|\tau|>R$ with values in $C^\infty(\partial\Omega)$.

Expand the stretchings in $z=1/\tau$ about $z=0$. The transformation satisfies

$$\frac{dX_j(\tau,x_j)}{dx_j} \ =\ \frac{\tau+\sigma_j(x_j)}{\tau} \ =\ 1+z\,\sigma_j(x_j)\,,\qquad X_j(\tau,0)\ =\ 0\,. \tag{3.35}$$

Thus X is analytic on a neighborhood of $z=0$ with $X(0,x)=x$. The derivative with respect to x satisfies $D_xX=I+O(z)$. It follows that

$$\nu(\tau,x)=\nu(\infty,x)+O(1/\tau),\quad \text{and,}\quad \nabla_x\nu(\tau,x)=\nabla_x\nu(\infty,x)+O(z).$$

At $\tau = \infty$ the $\nabla_x \nu$ restricted to the tangent space is the Weingarten map of $\partial\Omega$ from Definition 2.9. At $\tau = \infty$, the eigenvalues are nonnegative. Therefore

$$\begin{aligned}
H_{\underline{\Omega}}(\tau, x) &= H_\Omega(x) + O(1/\tau), \\
|\nabla_x \nu(\infty, x)| &\lesssim \max\{\kappa_1, \kappa_2\} \leq 2H_\Omega(x), \\
|\nabla_x \nu(\tau, x)| &\lesssim |H_{\underline{\Omega}}(\tau, x)| + |\tau|^{-1} \lesssim |\beta(\tau, \delta, x)|\,.
\end{aligned} \tag{3.36}$$

Since $|\nabla_x \mathsf{m}| \lesssim |\nabla_x \nu|$ this proves **iii.**

iv. Estimates **ii, iii** imply that with constants independent of τ, δ and all u,

$$\begin{aligned}
\|\mathsf{m}\, u\|_{L^2(\partial\Omega)} &\lesssim \|u\|_{L^2(\partial\Omega)}, \\
\|\mathsf{m}\, u\|_{H^1(\partial\Omega)} &\lesssim \|\,|\beta|\, u\|_{L^2(\partial\Omega)} + \|u\|_{H^1(\partial\Omega)}\,.
\end{aligned} \tag{3.37}$$

To prove the second, apply the product rule with vector fields ∂ that are tangent to the boundary to find $\partial(\mathsf{m}u) = \mathsf{m}\partial u + (\partial \mathsf{m})u$. Therefore

$$\|\partial(\mathsf{m}u)\|_{L^2(\partial\Omega)} \leq \|\mathsf{m}\|_{L^\infty(\partial\Omega)} \|\partial u\|_{L^2(\partial\Omega)} + \|(\partial \mathsf{m})\, u\|_{L^2(\partial\Omega)}\,.$$

Using **iii** in the second summand proves (3.37)

Denote by $\Delta_{\partial\Omega}$ the Laplace-Betrami operator of $\partial\Omega$. The estimates (3.37) are the cases $\theta = 0, 1$ of

$$\|\mathsf{m}\, u\|_{H^\theta(\partial\Omega)} \lesssim \|(|\beta(\tau, x)| + |\Delta_\Omega|^{1/2})^\theta u\|_{L^2(\partial\Omega)}.$$

Interpolation implies the estimate for $0 \leq \theta \leq 1$. Use the case $\theta = 1/2$. For self adjoint $B_j \geq 0$ with B_1 bounded and $u \in \mathcal{D}(B_2)$,

$$\begin{aligned}
\big\|\sqrt{B_1 + B_2}\, u\big\|^2 &= \big(\sqrt{B_1 + B_2}\, u, \sqrt{B_1 + B_2}\, u\big) = ((B_1 + B_2)u, u) \\
&= (B_1 u, u) + (B_2 u, u) = \big\|\sqrt{B_1}\, u\big\|^2 + \big\|\sqrt{B_2}\, u\big\|^2\,.
\end{aligned}$$

With $B_1 = |\beta(\tau, x)|$ and $B_2 = |\Delta_\Omega|^{1/2}$ this yields

$$\|(|\beta(\tau, x)| + |\Delta_\Omega|^{1/2})^{1/2} u\|^2_{L^2(\partial\Omega)} = \||\beta(\tau, x)|^{1/2} u\|^2_{L^2(\partial\Omega)} + \||\Delta_\Omega|^{1/4} u\|^2_{L^2(\partial\Omega)}.$$

Using this in the $\theta = 1/2$ estimate proves **iv.**

v. With constants indpendent of δ, τ with $|\tau| > R$, one has for all $u \in H^1(\Omega)$,

$$\int_\Omega \big|\nabla_x u\big|^2\, dx \leq C\Big(-\operatorname{Re} \int_\Omega p(\tau, x, \partial)\, u \cdot \overline{u}\, dx \Big).$$

It follows that For $|\tau| > R$ and $0 < \delta < 1$, the operator $1 - p(\tau, x, \partial)$ is an isomorphism of $H^1(\Omega)$ to $H_0^{-1}(\Omega)$, and with constants independent of τ, δ,

$$\|u\|_{H^1(\Omega)} \lesssim \|(1-p)u\|_{H_0^{-1}(\Omega)} \lesssim \|u\|_{H^1(\Omega)} .$$

Therefore,

$$\begin{aligned} \|pu\|_{H_0^{-1}(\Omega)} &\leq \|(1-p)u\|_{H_0^{-1}(\Omega)} + \|u\|_{H_0^{-1}(\Omega)} \\ &\lesssim \|u\|_{H^1(\Omega)} + \|u\|_{H_0^{-1}(\Omega)} \lesssim \|u\|_{H^1(\Omega)} . \end{aligned} \tag{3.38}$$

Using (2.20), (3.9), and (3.16) shows that for all $u, v \in H^1(\Omega)$

$$a(u,v) - \int_\Omega \big(p(\tau,x,\partial)\, u\big) \cdot v \, dx = \int_{\partial\Omega} \Phi(\tau,x)\ Vu \cdot v \, d\Sigma .$$

For $\phi \in H^{1/2}(\partial\Omega)$ choose $v \in H^1(\Omega)$ with $\|v\|_{H^1(\Omega)} \lesssim \|\phi\|_{H^{1/2}(\partial\Omega)}$ to find,

$$\begin{aligned} \Big| \int_{\partial\Omega} \Phi(\tau,x)\ Vu \cdot \phi \, d\Sigma \Big| &= \Big| a(u,v) - \int_\Omega (pu) \cdot v \, dx \Big| \\ &\lesssim \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + \|pu\|_{H_0^{-1}(\Omega)} \|v\|_{H^1(\Omega)} \\ &\lesssim \big(\|\nabla u\|_{L^2(\Omega)} + \|pu\|_{H_0^{-1}(\Omega)} \big) \|\phi\|_{H^{1/2}(\partial\Omega)} . \end{aligned}$$

Using this in the upper bound for $|\int \Phi V u \cdot \phi \, d\Sigma|$, shows that

$$\Big| \int_{\partial\Omega} \Phi(\tau,x)\ Vu \cdot \phi \, d\Sigma \Big| \lesssim \|u\|_{H^1(\Omega)}\ \|\phi\|_{H^{1/2}(\partial\Omega)} .$$

Since Φ and $1/\Phi$ as well as their derivatives are uniformly bounded, this proves **v.** $\square$

End of proof of Proposition 3.12. The second term on the right in (3.33) is estimated as

$$\Big| \int_{\partial\Omega} \Phi\ \beta\ u \cdot \mathsf{m}\, \overline{u} \, d\Sigma \Big| \lesssim \int_{\partial\Omega} |\beta|\ |u|^2 \, d\Sigma = \| |\beta|^{1/2} u \|^2_{L^2(\partial\Omega)} . \tag{3.39}$$

The first summand is estimated as

$$\Big| \int_{\partial\Omega} \Phi V u \cdot \mathsf{m} \overline{u} \, dx \Big| \lesssim \|Vu\|_{H^{-1/2}(\partial\Omega)}\ \|\mathsf{m}\overline{u}\|_{H^{1/2}(\partial\Omega)} . \tag{3.40}$$

For $\|\mathsf{m}\, u\|_{H^{1/2}(\partial\Omega)}$ use Part **iv** of the lemma in (3.40) to find,

$$\begin{gathered}\Big|\int_{\partial\Omega} \Phi V u\cdot \mathsf{m}\overline{u}\ dx\Big| \ \lesssim \\ \big(\|p\, u\|_{H_0^{-1}(\Omega)} + \|\nabla u\|_{L^2(\Omega)}\big)\big(\|\,|\beta|^{1/2}u\|_{L^2(\partial\Omega)} + \|u\|_{H^{1/2}(\partial\Omega)}\big).\end{gathered} \tag{3.41}$$

Use this, (3.38), and, $\|u\|_{H^{1/2}(\partial\Omega)} \lesssim \|u\|_{H^1(\Omega)}$ in (3.41) to find,

$$\Big|\int_{\partial\Omega} \Phi V u\cdot \mathsf{m}\overline{u}\ dx\Big| \ \lesssim\ \|\,|\beta|^{1/2}u\|^2_{L^2(\partial\Omega)} + \|u\|^2_{H^1(\Omega)}\,. \tag{3.42}$$

Adding the estimates (3.30), (3.39), and (3.42) for the three terms on the right of (3.29) proves Proposition 3.12. □

Proof of Theorem 3.8. Combine the lower and upper bounds for $|\mathcal{A}(u,\overline{u})|$ from Propositions 3.11 and 3.12 to find,

$$\begin{gathered}|\tau|\,(\mathrm{Re}\,\tau)\,\|u\|^2_{L^2(\mathcal{Q}_\delta)} + \||\beta|^{1/2}u\|^2_{L^2(\partial\mathcal{Q}_\delta)} + \|\nabla_x u\|^2_{L^2(\mathcal{Q}_\delta)}\ \leq \\ C\,\|f\|_{H_0^{-1}(\Omega)}\|u\|_{H^1(\Omega)} + \frac{C}{|\tau|}\Big(\||\beta|^{1/2}u\|^2_{L^2(\partial\Omega)} + \|u\|^2_{H^1(\Omega)}\Big).\end{gathered}$$

Choose $M = 2C$. Then for $\mathrm{Re}\,\tau > M$, the second summand on the right can be absorbed in the left hand side. This proves the Theorem. □

3.3.3 End of proof of Theorem 3.7

Proof that the map $u\mapsto(f,g_1,g_2)$ has trivial kernel. If $u\in C^\infty(\overline{\mathcal{Q}_\delta})$ is in the kernel, it follows that $u\in H^2(\mathcal{Q}_\delta)$ and satisfies the homogeneous boundary value problem with sources f, g_1, g_2 equal to zero. Theorem 3.8 implies that $u=0$.

Proof that the annihilator of the range, is $\{0\}$. • Derive the following Green's identity for $u,v\in H^2(\Omega)$,

$$\begin{aligned}&\int_\Omega \big(\tau^2\Pi(\tau,x) - p(\tau,x,\partial)\big)u\cdot v\ dx - u\cdot\big(\tau^2\Pi(\tau,x) - p(\tau,x,\partial)\big)v\ dx \\ &= -\int_{\partial\Omega}\Phi(\tau,x)\Big((V+\beta(\tau,x))u\cdot v - u\cdot(V+\beta(\tau,x))v\Big)\,d\Sigma.\end{aligned} \tag{3.43}$$

To prove this, subtract (3.19) from the same identity with u and v interchanged.

- Equations for the annihilators. The function

$$(\underline{u}, \underline{g}_1, \underline{g}_2) \ \in \ C^\infty(\overline{\Omega}) \times C^\infty(\partial\Omega; \mathcal{E}^-) \times C^\infty(\partial\Omega; \mathcal{E}^+)$$

annihilates the range if and only if $\forall u \in H^2(\Omega)$,

$$\begin{aligned} \int_\Omega \big(\tau^2 \Pi(\tau, x) - p(\tau, x, \partial)\big) u \cdot \underline{u} \ dx \ + \ \int_{\partial\Omega} \pi^-(\widetilde{\nu}) u \cdot \underline{g}_1 \ d\Sigma \\ + \int_{\partial\Omega} \pi^+(\widetilde{\nu}) \Big(V + \tau + 2H_{\underline{\Omega}}\Big) u \cdot \underline{g}_2 \, d\Sigma \ = \ 0. \end{aligned} \tag{3.44}$$

The operator $\tau^2\Pi(\tau, x) - p$ is equal to its own transpose. Therefore, taking u that vanish on a neighborhood of $\partial\Omega$ implies that

$$\big(\tau^2 \Pi(\tau, x) - p(\tau, x, \partial)\big) \underline{u} \ = \ 0 \qquad \text{on} \quad \Omega\,. \tag{3.45}$$

This together with (3.43) shows that (3.44) holds if and only if

$$\begin{aligned} 0 \ = \ \int_{\partial\Omega} \pi^+(\widetilde{\nu}) \Big(V + \tau + 2H_{\underline{\Omega}}\Big) u \ \cdot \ \underline{g}_2 \ + \ \pi^-(\widetilde{\nu}) u \cdot \underline{g}_1 \\ - \Phi(\tau, x) \big((V + \tau + 2H_{\underline{\Omega}}) u \cdot \underline{u} - u \cdot (V + \tau + 2H_{\underline{\Omega}}) \underline{u}\big) \ d\Sigma. \end{aligned} \tag{3.46}$$

Equation (3.46) is used first on test functions u that satisfy $(V + \tau + 2H_{\underline{\Omega}})u = 0$ on $\partial\Omega$. That constraint leaves $u|_{\partial\Omega}$ arbitrary. Of those test functions first consider those that satisfy $\pi^-(\widetilde{\nu})u|_{\partial\Omega} = 0$. For those one finds

$$\int_{\partial\Omega} \Phi(\tau, x) \, u \cdot (V + \tau + 2H_{\underline{\Omega}}) \underline{u} \ d\Sigma \ = \ 0\,.$$

Since the Φ factor is scalar and nowhere vanishing it follows that for arbitrary $\phi \in C^\infty(\partial\Omega)$,

$$\int_{\partial\Omega} \pi^+(\widetilde{\nu}) \phi \ \cdot \ (V + \tau + 2H_{\underline{\Omega}}) \underline{u} \ d\Sigma \ = \ 0\,.$$

This shows that $\underline{u}$ satisfies the transposed boundary condition

$$\pi^+(\widetilde{\nu})^\dagger \, \big(V + \tau + 2H_{\underline{\Omega}}\big) \underline{u} \ = \ 0, \qquad \text{on} \quad \partial\Omega. \tag{3.47}$$

Next take u satisfying $\pi^+(\widetilde{\nu})u|_{\partial\Omega} = 0$. Then $u|_{\partial\Omega} = \pi^-(\widetilde{\nu})u$. This yields

$$\int_{\partial\Omega} \Phi(\tau, x) \Big(\pi^-(\widetilde{\nu}) u \cdot (V + \tau + 2H_{\underline{\Omega}}) \underline{u}\Big) \ + \ \pi^-(\widetilde{\nu}) u \cdot \underline{g}_1 \, d\Sigma\,.$$

The set of functions $\pi^-(\widetilde{\nu})u|_{\partial\Omega}$ includes the set of $\pi^-(\widetilde{\nu})\psi$ for an arbitrary $\psi \in C^\infty(\partial\Omega;\mathbb{C}^2)$. It follows that on $\partial\Omega$,

$$\pi^-(\widetilde{\nu})^\dagger\bigg(\Phi(\tau,x)\Big(V+\tau+2H_{\underline{\Omega}}\Big)\,\underline{u}\ +\ \underline{g}_1\bigg)=0\quad\text{on}\quad\partial\Omega. \tag{3.48}$$

Next extract the information from test functions that satisfy $u|_{\partial\Omega}=0$. For such test functions, $\big[V+\tau+2H]_{\partial\Omega}$ can be chosen as an an arbitrary element $\psi \in C^\infty(\partial\Omega;\mathbb{C}^2)$. This yields

$$-\int_{\partial\Omega}\Phi(\tau,x)\,\psi\cdot\underline{u}\;d\Sigma+\int_{\partial\Omega}\pi^+(\widetilde{\nu})\psi\ \cdot\ \underline{g}_2\,d\Sigma\ =\ 0.$$

First take those ψ that satisfy $\pi^+(\widetilde{\nu})\psi = 0$. That is equivalent to $\psi = \pi^-(\widetilde{\nu})\phi$ for arbitrary ϕ. That yields

$$\int_{\partial\Omega}\Phi(\tau,x)\,\pi^-(\widetilde{\nu})\phi\cdot\underline{u}\;d\Sigma\ =\ 0\,.$$

This is equivalent to the Dirichlet boundary condition for $\underline{u}$,

$$\pi^-(\widetilde{\nu})^\dagger\underline{u}\ =\ 0,\quad\text{on}\quad\partial\Omega. \tag{3.49}$$

Finally, consider ψ with $\pi^-(\widetilde{\nu})\psi=0$. Equivalently $\psi=\pi^+(\widetilde{\nu})\phi$ for arbitrary ϕ. This yields

$$\int_{\partial\Omega}\pi^+(\widetilde{\nu})\phi\cdot\Big(-\Phi(\tau,x)\underline{u}\ +\ \underline{g}_2\Big)\;d\Sigma\ =\ 0.$$

Since ϕ is arbitrary this is equivalent to

$$\pi^+(\widetilde{\nu})^\dagger\Big(-\Phi(\tau,x)\underline{u}\ +\ \underline{g}_2\Big)=0\quad\text{on}\quad\partial\Omega. \tag{3.50}$$

• Proof that $\underline{u}=0$, $\underline{g}_1=0$, and $\underline{g}_2=0$. The three equations (3.45), (3.47), and (3.49) assert that $\underline{u}$ is a smooth solution of the transposed boundary value problem with zero sources. Theorem 3.8 implies that $\underline{u}=0$.

From the fact that $\underline{u}=0$, (3.48) implies that $(\pi^-)^\dagger\underline{g}_1=0$. In addition $\underline{g}_1$ takes values in $\mathcal{E}^-(\widetilde{\nu})$. There is an R_2 so that for $|\tau|>R_1$, $\pi^-(\widetilde{\nu})^\dagger$ is injective on $\mathcal{E}^-(\widetilde{\nu}(\tau,x))$ for all $x\in\partial\Omega$. For those τ, conclude that $\underline{g}_1=0$.

An entirely analogous argument using (3.50) shows that $\underline{g}_2=0$. This completes the proof that the annihilator of the range is equal to $\{0\}$. □

3.4 Analyticity in τ of the Helmholtz solution

Use the shorthand $\mathcal{E}^\pm(\tau,x)$ for $\mathcal{E}^+(\widetilde{\nu}(\tau,x))$. The vector spaces $\mathcal{E}^\pm(\tau,x)$ depends analytically on τ. The next example shows that defining what it means to depend analytically on τ has pitfalls.

Example 3.2 i. *The subspace $\mathbb{U}(\tau)\subset\mathbb{C}^2$ spanned by $(1,\tau^2)$ depends analytically on τ for any reasonable definition including the one below.*

ii. *The unit vectors spanning $\mathbb{U}(\tau)$ are*

$$e^{i\theta(\tau)}\ \frac{(1,\tau^2)}{(1+|\tau|^4)^{1/2}}, \qquad \theta\in\mathbb{R}.$$

No choice of θ makes this holomorphic.

iii. *Orthogonal projection onto $\mathbb{U}(\tau)$ has matrix equal to*

$$\frac{1}{1+|\tau|^4}\begin{pmatrix}1 & \overline{\tau}^2\\ \tau^2 & |\tau|^4\end{pmatrix}.$$

It is not a holomorphic function of τ.

The analytic dependence of $\mathcal{E}^\pm(\tau,x)$ is expressed as follows. For each (τ,x), $\mathbb{C}^2=\mathcal{E}^+(\tau,x)\oplus\mathcal{E}^-(\tau,x)$. For τ near a fixed $\underline{\tau}$ and all $x\in\partial\Omega$, $\pi^+(\widetilde{\nu})$ is an isomorphism from $\mathcal{E}^+(\tau,x)\to\mathcal{E}^+(\underline{\tau},x)$. Define the linear transformation $R^+(\tau,x)\in\text{Hom}(\mathbb{C}^2)$ to be the inverse of this isomorphism for $v\in\mathcal{E}^+(\underline{\tau},x)$ and equal to zero on $\mathcal{E}^-(\underline{\tau},x)$. An analogous definition yields $R^-(\tau,x)$. Then $R^\pm(\tau,x)\in\text{Hom}(\mathcal{E}^\pm(\underline{\tau},x):\mathbb{C}^2)$ depend analytically on τ. For τ near $\underline{\tau}$ and all $x\in\partial\Omega$,

$$\mathcal{E}^+(\tau,x)\ =\ R^+(\tau,x)\,\mathcal{E}^+(\underline{\tau},x)\,.$$

This is a local trivializaion of $\mathcal{E}^+(\tau,x)$ that depends analytically on τ. Considering different $\underline{\tau}$ the change of trivialization formulas are analytic in τ. This is the definition of analytic dependence.

The boundary value problem (3.11) has source terms g_j that takes values in $\mathcal{E}^\pm(\tau,x)$. The local representation allows one to suppress the τ dependence as follows. For τ near $\underline{\tau}$, a section g_1 of $\mathcal{E}^+(\tau,x)$ is uniquely represented as $R^+(\tau,x)\underline{g}$ where $\underline{g}$ is takes values in the τ dependent space $\mathcal{E}^+(\underline{\tau},x)$. The boundary value problem takes the form

$$\begin{aligned}\big(\tau^2\,\Pi(\tau,x)\ -\ p(\tau,x,\partial)\big)u\ &=\ f \quad \text{on} \quad \Omega,\\ \pi^+(\widetilde{\nu}(\tau,x))u &= R^-(\tau,x)\underline{g}_1 \quad \text{on} \quad \partial\Omega,\\ \pi^+(\widetilde{\nu}(\tau,x))\big(V+\tau+2H_{\underline{\Omega}}(X(\tau\underline{x}))\big)u &= R^+(\tau,x)\underline{g}_2 \quad \text{on} \quad \partial\Omega.\end{aligned} \tag{3.51}$$

Here $\underline{g}_1$ takes values in $\mathcal{E}^-(\underline{\tau}, x)$ and $\underline{g}_2$ takes values in $\mathcal{E}^+(\underline{\tau}, x)$. In this form, the source terms $\underline{g}_j$ belong to a τ-independent space and the coefficients of the operators depend differentiably on τ, x and analytically on τ.

Definition 3.14 *A τ-dependent section $g_1(\tau) \in H^{3/2}(\mathcal{E}^-(\tau, x))$ depends analytically on τ when the corresponding functions $\underline{g}_1(\tau) \in H^{3/2}(\mathcal{E}^-(\underline{\tau}, x))$ depend analytically on τ. A similar definition applies for $g_2(\tau) \in H^{1/2}(\mathcal{E}^-(\tau, x))$.*

Theorem 3.15 *If the source terms*

$$(f, g_1, g_2) \ \in \ L^2(\Omega) \times H^{3/2}(\mathcal{E}^-(\tau, x)) \times H^{1/2}(\mathcal{E}^+(\tau, x))$$

depend analytically on τ on $\operatorname{Re}\tau > M$, then the correponding solution $u(\tau, \cdot)$ of (3.11) is an analytic function of τ with values in $H^2(\Omega)$.

Proof. Standard elliptic theory shows that writing $\tau = a + ib$ the map $a, b \mapsto u$ is infinitely differentiable with values in $H^2(\Omega)$. The derivatives satisfy the system obtained by differentiating, with respect to a, b, the system and boundary conditions satisfied by u.

To prove analyticity it suffices to show that $w := \partial u / \partial \overline{\tau} = 0$. Since all the coefficients and the f, g_1, g_2 are analytic, differentiating the boundary value problem with respect to $\overline{\tau}$ shows that w satisfies

$$\begin{aligned} \Big(\tau^2\,\Pi(\tau, x) \ - \ p(\tau, x, \partial)\Big) w \ &= \ 0 \quad \text{on} \quad \mathcal{Q}_\delta, \\ \pi^-(\widetilde{\nu}(\tau, x)) w \ &= \ 0 \quad \text{on} \quad \partial\mathcal{Q}_\delta, \\ \pi^+(\widetilde{\nu}(\tau, x)) \Big(V(\tau, x, \partial) + \tau + 2H_{\underline{\Omega}}(X(\tau, \underline{x}))\Big) w \ &= \ 0 \quad \text{on} \quad \partial\mathcal{Q}_\delta. \end{aligned}$$

Theorem 3.8, implies that $w = 0$. □

4 Proofs of the Main Theorems

4.1 The stretched equation on $\mathcal{Q}_\delta$, Theorem 2.8

Proof of Theorem 2.8. Uniqueness. Multiply the differential equation $L(\tau, \widetilde{\partial}) u^\delta = F$ from (2.11) by $\Pi(\tau, x)\big(\tau - \sum A_j \widetilde{\partial}_j\big)$ and use (3.4) to find the first line in the Helmholtz boundary value problem

$$\begin{aligned} \big(\tau^2 \Pi(\tau, x) \ - \ p(\tau, x, \partial)\big) u^\delta \ &= \ \Pi(\tau, x)\big(\tau - \sum A_j \widetilde{\partial}_j\big) F\,, \\ \pi^-(\widetilde{\nu})\, u^\delta \ &= \ 0, \quad \text{on} \quad \partial\mathcal{Q}_\delta, \\ \pi^+(\widetilde{\nu}) \big(V(\tau, x, \partial) + \tau + 2H_{\underline{\mathcal{Q}_\delta}}\big) u^\delta \ &= \ 0, \quad \text{on} \quad \partial\mathcal{Q}_\delta. \end{aligned} \tag{4.1}$$

The second line is part of (3.4). The last line follows from part **iv** of Proposition 2.14 since $F = 0$ on a neighborhood of $\partial\mathcal{Q}_\delta$ and $u^\delta \in H^2(\mathcal{Q}_\delta)$.

The hypotheses of Theorem 3.8 are satisfied. Apply the estimate of that Theorem with $f = 0$ to conclude that $u = 0$.

Existence. For $\mathrm{Re}\,\tau > M$, Theorem 3.7 implies that the boundary value problem (4.1) has a unique solution $u^\delta \in H^2(\mathcal{Q}_\delta)$. Theorem 3.15 implies that u is holomorphic with values in $H^2(\mathcal{Q}_\delta)$. Theorem 3.8 implies that with constant independent of δ,

$$|\tau|\,(\mathrm{Re}\,\tau)\,\big\|u^\delta\big\|^2_{L^2(\mathcal{Q}_\delta)} + |\tau|\,\big\|u^\delta\big\|^2_{L^2(\partial\mathcal{Q}_\delta)} + \big\|\nabla u^\delta\big\|^2_{L^2(\mathcal{Q}_\delta)} \le C\big\|F\big\|^2_{H_0^{-1}(\mathcal{Q}_\delta)}. \tag{4.2}$$

To complete the proof it suffices to show that u^δ satisfies the stretched boundary value problem (2.10) on $\mathcal{Q}_\delta$. Need to reverse the steps that lead from the stretched equations to the Helmholtz boundary value problem.

Define

$$w \;:=\; \big(A(\widetilde{\partial}) \,+\, \tau\big)u \;\in\; H^1(\mathcal{Q}_\delta).$$

Need to show that the stretched equation, $w = F$, is satisfied.

The Helmholtz equation implies that $w - F \in H^1(\mathcal{Q}_\delta)$ satisfies

$$\big(A(\widetilde{\partial}) \,-\, \tau\big)\Big(w \,-\, F\Big) \;=\; 0\,, \qquad \text{on} \quad \mathcal{Q}_\delta\,. \tag{4.3}$$

Part **iv** of Proposition 2.14 shows that the derivative boundary condition satisfied by u^δ is equivalent to

$$\pi^+\big(A(\widetilde{\nu})\big)\; A(\widetilde{\nu})^{-1}\;\big(w - F\big) \;=\; 0 \qquad \text{on} \quad \partial\mathcal{Q}_\delta\,.$$

Since $\pi^+(A(\widetilde{\nu}))$ and $A(\widetilde{\nu})$ commute, this is equivalent to

$$\pi^+\big(A(\widetilde{\nu})\big)\;\big(w - F\big) \;=\; 0 \qquad \text{on} \quad \partial\mathcal{Q}_\delta\,. \tag{4.4}$$

When τ is real and large, the pair of equations (4.3), (4.4) is a strictly dissipative boundary value problem with vanishing sources on the smooth domain $\mathcal{Q}_\delta$ with noncharacteristic boundary. Friedrich's Theorem [12, 13, 21, 23] implies that $w - F = 0$ for τ large and real.

The map $\tau \mapsto (w - F)(\tau)$ is holomorphic for $\mathrm{Re}\,\tau$ large. It vanishes on $]m, \infty[$ for m large. By analytic continuation, it follows that $w - F = 0$ for all $\mathrm{Re}\,\tau > M$.

Thus the stretched equation is satisfied on $\mathcal{Q}_\delta$ for $\mathrm{Re}\,\tau > M$. This completes the proof of existence. □

Remark 4.1 *This recalls the proof of perfect matching in [14], where the perfection is inherited from real values by analytic continuation.*

4.2 The stretched equation on $\mathcal{Q}$, Theorem 2.5

Proof of Theorem 2.5. Uniqueness. An analyticity argument reduces it to the difficult uniqueness theorem of Part I of [15].

The Laplace transform of the solution with vanishing data is holomorphic in $\mathrm{Re}\,\tau$ large. To prove that it vanishes it is sufficient to prove that it vanishes for $\tau \in]m, \infty[$ for m large.

For τ real and large, the stretched equation, $L(\tau, \widetilde{\partial})\widehat{u}(\tau) = 0$ is symmetric positive in the sense of Friedrichs, that is

$$L(\tau, \widetilde{\partial}) + L(\tau, \widetilde{\partial}) \ \geq\ C_1(\mathrm{Re}\,\tau - C_2)I, \qquad C_1 > 0\,.$$

In addition, $\widehat{u}(\tau)$ satisfies strictly dissipative boundary conditions on each smooth faces G_j, and, has square integrable traces on the G_j.

The stretched equation is elliptic on the faces G_j. The uniqueness theorem for such strictly dissipative elliptic problems with trihedral corners from Part I of [15] implies that $\widehat{u}(\tau) = 0$.

Existence. Use Theorem 2.8. Solve on $\mathcal{Q}_\delta$ and pass to the limit $\delta \to 0$. At the same time one must smooth the source term f in order to apply Theorem 2.8.

Choose $0 < \underline{\epsilon} < \mathrm{dist}\,(K, \partial\mathcal{Q})/2$. Define K' to be the set of points at distance $\underline{\epsilon}$ from K. Then $K' \subset \mathcal{Q}$ is compact. For $\epsilon < \underline{\epsilon}$, define $F_\epsilon := j_\epsilon * F$ where j_ϵ is a smooth mollification kernel on $\mathbb{R}^3$ with support in the ball of radius ϵ at the origin. The source term $F_\epsilon \in C^\infty_{K'}(\mathcal{Q})$. For δ sufficiently small $K' \subset \mathcal{Q}_\delta$ and Theorem 2.8 applies.

Define $\delta(n) = 2^{-n}$, and $u^{\delta(n)} \in H^2(\mathcal{Q}_{\delta(n)})$ to be the solution from Theorem 2.8 with source term equal to $F_{\delta(n)}$. Then with C independent of n,

$$\begin{aligned} &|\tau|\,(\mathrm{Re}\,\tau)\,\big\|u^{\delta(n)}\big\|^2_{L^2(\mathcal{Q}_{\delta(n)})} + |\tau|\,\big\|u^{\delta(n)}\big\|^2_{L^2(\partial\mathcal{Q}_{\delta(n)})} \\ &\quad + \big\|\nabla_x u^{\delta(n)}\big\|^2_{L^2(\mathcal{Q}_{\delta(n)})} \ \leq\ C\Big\|\big(\sum A_j\widetilde{\partial}_j - \tau\big)F_{\delta(n)}\Big\|^2_{H_0^{-1}(\mathcal{Q}_{\delta(n)})}\,. \end{aligned} \tag{4.5}$$

The right hand side is bounded with a constant independent of n,

$$\Big\|\big(\sum A_j\widetilde{\partial}_j - \tau\big)F_{\delta(n)}\Big\|_{H_0^{-1}(\mathcal{Q}_{\delta(n)})} \ \leq\ C\|F\|_{L^2_K(\mathcal{Q})}\,.$$

Extract a subsequence that converges weakly in $H^1(\mathcal{Q}_{\delta(1)})$ to a limit v_1. Extract a further subsequence that converges weakly in $H^1(\mathcal{Q}_{\delta(2)})$ to a limit v_2. And so forth. For each $n > 1$, one has $v_n = v_{n-1}$ on $\mathcal{Q}_{\delta(n-1)}$. Define

$v \in H^1(\mathcal{Q})$ by $v = v_n$ on $\mathcal{Q}_{\delta(n)}$. Using that $\mathcal{Q}_{\delta(n)} \nearrow \mathcal{Q}$ and $\partial\mathcal{Q}_{\delta(n)} \cap G_j \nearrow G_j$ conclude that

$$|\tau|\,(\operatorname{Re}\tau)\,\big\|v\big\|^2_{L^2(\mathcal{Q})} + |\tau|\,\big\|v\big\|^2_{L^2(\partial\mathcal{Q})} + \big\|\nabla_x v\big\|^2_{L^2(\mathcal{Q})} \le C\big\|F\big\|^2_{L^2_K(\mathcal{Q})}\,. \tag{4.6}$$

By the Cantor diagonal process, extract a subsequence, denoted u_k so that for each n, u_k converges weakly to v in $H^1(\mathcal{Q}_{\delta(n)})$.

The differential equation $L(\tau,\widetilde{\partial})v = f$ on $\mathcal{Q}$ follows from the equations $L(\tau,\widetilde{\partial})u_k = F_k$ on $\mathcal{Q}_{\delta(n(k))}$ on passing to the limit $k \to \infty$. Similarly, the boundary condition

$$\pi^+(\nu)v \;=\; 0, \qquad \text{on} \quad G_j$$

follows on passing to the limit in

$$\pi^+(\nu)\,u_{\delta(n)}\big|_{G_j\cap\partial\mathcal{Q}_{\delta(n)}} \;=\; 0.$$

For any $\underline{\delta} > 0$ the holomorphy of $\tau \mapsto v(\tau)$ from $\operatorname{Re}\tau > M$ to $L^2(\mathcal{Q}_{\underline{\delta}})$ follows from the fact that it is the weak limit of bounded family of holomorphic functions. Therefore, for any δ, $v : \{\operatorname{Re}\tau > M\} \to L^2(\mathcal{Q}_\delta)$ is holomorphic.

To show that v is holomorphic with values in $L^2(\mathcal{Q})$ it is sufficient to show that $\tau \mapsto \ell(v(\tau))$ is holomorphic for each ℓ in the dual of of $H^1(\mathcal{Q})$.

Since $v \in L^\infty\big(\{\operatorname{Re}\tau > M\}\,;\,H^1(\mathcal{Q})\big)$, it suffices to show that $\ell(v(\tau))$ is holomorphic for ℓ in a dense subset. Indeed if ℓ is the limit of ℓ_j for which the result is true, estimate

$$\Big|\ell(v(\tau)) \;-\; \ell_j(v(\tau))\Big| \;\le\; \|\ell - \ell_j\| \sup_{\operatorname{Re}\tau > M} \|v(\tau)\|_{H^1(\mathcal{Q})}, \quad \text{on} \;\; \operatorname{Re}\tau > M.$$

This proves that $\ell(v(\tau))$ is the uniform limit of the holomorphic functions $\ell_j(v(\tau))$.

Take the dense set to be the linear functionals $v \mapsto \int v\cdot\phi\,dx$ with $\phi \in C_0^\infty(\mathcal{Q})$. For each such ϕ, $\phi \in C_0^\infty(\mathcal{Q}_\delta)$ for δ small. That $\ell(v(\tau))$ is holomorphic then folllows from the fact that v is holomorphic with values in $H^1(\mathcal{Q}_\delta)$. This completes the proof of the Theorem. ☐

4.3 Bérenger's equation on $\mathbb{R}_t \times \mathcal{Q}$, Theorem 1.4

To pass from estimates for the Laplace transform to space time estimates use the classical Paley-Wiener Theorem for functions with values in a Hilbert space H (see [17]).

Theorem 4.1 *The Laplace transforms of functions $F \in e^{Mt} L^2(\mathbb{R}\,;\,H)$ with $\operatorname{supp} F \subset \{t \geq 0\}$ are exactly the functions $G(\tau)$ holomorphic in $\operatorname{Re}\tau > M$ with values in H and so that*

$$\sup_{\lambda > M} \int_{\operatorname{Re}\tau = \lambda} \big\|\widehat{F}(\tau)\big\|_H^2 \; |d\tau| \; < \; \infty\,.$$

In this case the function $\widehat{F}(\tau)$ has trace at $\operatorname{Re}\tau = M$ that satisfies

$$\int e^{-2Mt}\, \|F(t)\|_H^2 \, dt = \sup_{\lambda > M} \int_{\operatorname{Re}\tau = \lambda} \big\|\widehat{F}(\tau)\big\|_H^2 \; |d\tau| = \int_{\operatorname{Re}\tau = M} \big\|\widehat{F}(\tau)\big\|_H^2 \; |d\tau|\,.$$

Proof of Theorem 1.4. Uniqueness. Need to show that if U^1, U^2, U^3 is a solution with source $f = 0$, then $U^j = 0$. Denote by $\widehat{U}^j$ the Laplace transform that is holomorphic in $\{\operatorname{Re}\tau > M\}$ with values in $L^2(\mathcal{Q})$.

The function $v(\tau) := \sum \widehat{U}^j$ is holomorphic with values in $H^1(\mathcal{Q})$ and satisfies the stretched equation

$$\tau \; v \; + \; \sum A_j \widetilde{\partial}_j v \; = \; 0\,.$$

In addition the boundary condition satisfied by $\sum U^j$ implies that v satisfies the boundary condition

$$v|_{G_j} \; \in \; \mathcal{E}^+(\nu)\,, \qquad j = 1, 2, 3.$$

The uniqueness part of Theorem 2.5 implies that v vanishes identically on $\{\operatorname{Re}\tau > M\}$.

The Laplace transform of the split equation yields

$$\big(\tau + \sigma_1(x_1)\big)\widehat{U}^j \; = \; -A_1 \partial_1 v \; = \; 0\,.$$

This implies that $\widehat{U}^j$ vanishes and therefore that $U^j = 0$. This completes the proof of uniqueness.

Existence. The solution $u(t,x)$ is constructed by finding its Laplace transform. Denote by $U^1(t,x)$, $U^2(t,x)$, and, $U^3(t,x)$ the unknowns to be found. Denote by $v(\tau, x)$ the function of τ that will be the Laplace transform of $U^1(t,x) + U^2(t,x) + U^3(t,x)$. Recall that in the split equations, $f_j = f/3$.

Define $v(\tau, x)$ to be the solution of the stretched equation

$$\tau \; v \; + \; A_j \widetilde{\partial}_j v \; = \; \widehat{f}(\tau) \sum_{j=1}^{3} \frac{1}{\tau + \sigma_j(x)}\,. \tag{4.7}$$

constructed in Theorem 2.5. It is holomorphic in $\text{Re}\,\tau > M$ with values in $H^1(\mathcal{Q})$ and satisfies

$$
\begin{aligned}
(\text{Re}\,\tau)\big\|v(\tau)\big\|_{L^2(\mathcal{Q})} + (\text{Re}\,\tau)^{1/2}\big\|\,v(\tau)|_{\partial O}\big\|_{L^2(\partial\mathcal{Q})} \\
+\ \big\|\nabla_x v(\tau)\big\|_{L^2(\mathcal{Q})} \ \le\ C\,\big\|\widehat{f}(\tau)\big\|_{L^2_K(\mathcal{Q})}\,.
\end{aligned}
\tag{4.8}
$$

Define V^j destined to be the Laplace transforms of the U^j by the analogue of (2.8),

$$
\begin{aligned}
\big(\tau+\sigma_1(x_1)\big)V^1 \ +\ A_1\partial_1 v \ &=\ \widehat{f}/3\,,\\
\big(\tau+\sigma_2(x_2)\big)V^2 \ +\ A_2\partial_2 v \ &=\ \widehat{f}/3\,,\\
\big(\tau+\sigma_3(x_3)\big)V^3 \ +\ A_3\partial_3 v \ &=\ \widehat{f}/3\,.
\end{aligned}
\tag{4.9}
$$

Multiplying line j of (4.9) by $\tau/(\tau+\sigma_j(x_j))$ yields

$$
\tau\,V^j + A_j\widetilde{\partial}_j v \ =\ \frac{\widehat{f_j}}{3(\tau+\sigma_j)}\,, \quad j=1,2,3\,.
$$

Summing yields

$$
\tau\Big(V^1+V^2+V^3\Big)\ +\ \sum A_j\widetilde{\partial}_j v \ =\ \widehat{f}\ \sum_{j=1}^{3}\frac{1}{\tau+\sigma_j(x)}\,.
$$

Subtracting from (4.7) yields

$$
\tau\Big(V^1+V^2+V^3-v\Big)\ =\ 0 \quad \text{so,} \quad v=V^1+V^2+V^3.
$$

The Paley-Wiener theorem implies that

$$
\sup_{\lambda>M}\int \|\widehat{f}(\tau)\|^2\ |d\tau|\ \le\ \int e^{2Mt}\|f(t)\|^2_{L^2_K(\mathcal{Q})}\ dt.
$$

Equation (4.8) together with the Paley-Wiener Theorem implies that v is the Laplace transform of a function $u\in e^{Mt}L^2(\mathbb{R};H^1(\mathcal{Q}))$ supported in $t\ge 0$. Moreover,

$$
\begin{aligned}
\int_0^\infty e^{2Mt}\Big(M\big\|u(t)\big\|^2_{L^2(\mathcal{Q})} + M^{1/2}\big\|\,u(t)|_{\partial O}\big\|^2_{L^2(\partial\mathcal{Q})} + \big\|\,\nabla_x u(t)\big\|^2_{L^2(\partial\mathcal{Q})}\Big)\,dt \\
\lesssim \int_0^\infty e^{2Mt}\|f(t)\|^2_{L^2_K(\mathcal{Q})}\,dt.
\end{aligned}
$$

Similarly the Paley-Wiener Theorem implies that $V^j(\tau)$ is the Laplace transform of a function $U^j(t) \in e^{Mt}L^2(\mathbb{R}; L^2(\mathcal{Q}))$ supported in $t \geq 0$ and satisfying

$$\int_0^\infty e^{2Mt} \left\| MU^j(t), \partial_t U^j(t) \right\|^2_{L^2(\mathcal{Q})} dt \;\lesssim\; \int_0^\infty e^{2Mt} \|f(t)\|_{L^2_K(\mathcal{Q})} dt.$$

The fact that $v = \sum V^j$ implies that $u = \sum U^j$. Equation (4.9) implies that (U^1, U^2, U^3) satisfies the Bérenger split equations. The last two estimates are exactly those required in Theorem 1.4.

Denoting by $\mathbb{L}$ the Laplace transform, one has

$$\mathbb{L}\big(\pi^-(\nu)\, u|_{G_j}\big) \;=\; \pi^-(\nu)\big(\mathbb{L}(u|_{G_j})\big) \;=\; \pi^-(\nu)\, v|_{G_j} \;=\; 0\,.$$

This proves the boundary condition $\pi^-(\nu)\, u|_{G_j} = 0$.

This completes the proof that the U^j satisfy the boundary value problem and estimates of Theorem 1.4. □

References

[1] S. Abarbanel and D. Gottlieb. On the construction and analysis of absorbing layers in CEM. *Appl. Numer. Math.*, 27:331–340, 1998.

[2] D. Appelö, T. Hagström, and G. Kreiss. Perfectly matched layers for hyperbolic systems: General formulation, well-posedness and stability. *SIAM J. Appl. Math.*, 67:1–23, 2006.

[3] D. H. Baffet, M. J. Grote, S. Imperiale, and M. Kachanovska. Energy decay and stability of a perfectly matched layer for the wave equation. *Journal of Scientific Computing*, 81(3):2237–2270, 2019.

[4] E. Balslev and J.-M. Combes. Spectral properties of many-body schrödinger operators with dilatation-analytic interactions. *Communications in Mathematical Physics*, 22(4):280–294, 1971.

[5] E. Bécache and P. Joly. On the analysis of Bérenger's perfectly matched layers for Maxwell's equations. *M2AN Math. Model. Numer. Anal.*, 36:87–119, 2002.

[6] J. H. Bramble and J. E. Pasciak. Analysis of a cartesian PML approximation to acoustic scattering problems in R2 and R3. *Journal of Computational and Applied Mathematics*, 247:209 – 230, 2013.

[7] Z. Chen, X. Xiang, and X. Zhang. Convergence of the PML method for elastic wave scattering problems. *Math. Comput.*, 85(302):2687–2714, 2016.

[8] W. Chew and W. Weedon. A 3d perfectly matched medium from modified Maxwell's equations with stretched coordinates. *IEEE Microwave and optical technology letters*, 17:599–604, 1995.

[9] F. Collino and P. Monk. The perfectly matched layer in curvilinear coordinates. *SIAM J. Sci. Comput*, 19(6):2061–2090, 1998.

[10] J. Diaz and P. Joly. A time domain analysis of PML models in acoustics. *Compt. Methods Appl. Mech. ENgrg*, 195:3820–3853, 2006.

[11] S. Dyatlov and M. Zworski. *Mathematical theory of scattering resonances*, volume 200. American Mathematical Soc., 2019.

[12] K. O. Friedrichs. Symmetric hyperbolic linear differential equations. *Commun. Pure Appl. Math.*, 7:345–392, 1954.

[13] K. O. Friedrichs. Symmetric positive linear differential equations. *Commun. Pure Appl. Math.*, 11:333–418, 1958.

[14] L. Halpern, S. Petit-Bergez, and J. Rauch. The analysis of matched layers. *Confluentes Math.*, 3(2):159–236, 2011.

[15] L. Halpern and J. Rauch. Hyperbolic boundary value problems with trihedral corners. *Discrete Contin. Dyn. Syst.*, 36(8):4403–4450, 2016.

[16] N. J. Hicks. *Notes on differential geometry*, volume 3. van Nostrand Princeton, 1965.

[17] E. Hille and R. S. Phillips. *Functional analysis and semi-groups*, volume 31. American Mathematical Soc., 1996.

[18] T. Kato. *Perturbation theory for linear operators*, volume 132. Springer Science & Business Media, 2013.

[19] M. Lassas and E. Somersalo. On the existence and convergence of the solution of PML equations. *Computing*, 60(3):229–241, 1998.

[20] M. Lassas and E. Somersalo. Analysis of the PML equations in general convex geometry. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, 131(5):1183–1207, 2001.

[21] P. D. Lax and R. S. Phillips. Local boundary conditions for dissipative symmetric linear differential operators. *Comm. Pure Appl. Math.*, 13:427–455, 1960.

[22] S. Petit-Bergez. *Problèmes faiblement bien posés : discrétisation et applications*. PhD thesis, Université Paris 13, 2006. http://tel.archives-ouvertes.fr/tel-00545794/fr/.

[23] J. Rauch. Symmetric positive systems with boundary characteristic of constant multiplicity. *Transactions of the American Mathematical Society*, 291(1):167–187, 1985.

[24] J. Rauch. *Hyperbolic Partial Differential Equations and Geometric Optics*, volume 133. American Mathematical Soc., 2012.