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Almost sure asymptotics for Riemannian random waves

Louis Gass

June 14, 2020

Abstract

We consider the Riemannian random wave model of Gaussian linear combinations of Laplace eigenfunctions on a general compact Riemannian manifold. With probability one with respect to the Gaussian coefficients, we establish that, both for large band and monochromatic models, the process properly rescaled and evaluated at an independently and uniformly chosen point X on the manifold, converges in distribution under the sole randomness of X towards an universal Gaussian field as the frequency tends to infinity. This result is reminiscent of Berry's conjecture and extends the celebrated central limit Theorem of Salem–Zygmund for trigonometric polynomials series to the more general framework of compact Riemannian manifolds. We then deduce from the above convergence the almost-sure asymptotics of the nodal volume associated with the random wave. To the best of our knowledge, these asymptotics were only known in expectation and not in the almost sure sense due to the lack of sufficiently accurate variance estimates. This in particular addresses a question of S. Zelditch regarding the almost sure equidistribution of nodal lines.

Contents

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1 Introduction and main results

1.1 Introduction

The central limit theorem by Salem and Zygmund [30] asserts that, when properly rescaled and evaluated at a uniform random point on the circle, a generic real trigonometric polynomial converges in distribution towards a Gaussian random variable. This classical result was recently revisited in [3] where the authors established both a quantitative version and a functional version of Salem–Zygmund theorem and then use these results to deduce the almost sure asymptotics of the number of zeros of random trigonometric polynomials with symmetric coefficients. The goal of the present article is to extend the latter results to the general Riemannian framework and more particularly to the so-called Riemannian random wave model, where random trigonometric polynomials are naturally replaced by random, Gaussian, linear combinations of Laplace eigenfunctions.

The study of nodal sets associated with Laplace eigenfunctions is the object of a vast literature, in particular thanks to Yau's conjecture, see [33, 13] and [27, 25, 26] for recent breakthroughs. The introduction of probabilistic models in this context has numerous motivations among which quantum chaos heuristics [35] and Berry's conjecture [8], which roughly states that under the hypothesis of a chaotic geodesic flow, a Laplace eigenfunction of high energy statistically behaves like a universal Euclidean random wave. The most common probabilistic model then consists in considering random linear combinations of Laplace eigenfunctions, whose coefficients are independent and identically distributed standard Gaussian variables, see for instance [29], [32] in the case of toral and spherical harmonics or [34] for the case of a general Riemannian manifold.

The literature then covers the asymptotics behavior of natural geometric observables associated with the nodal sets, such as their volume, their number of connected components [31] and other topological invariants, see [15] or [24]. Note that most of these results concerns the asymptotics of such quantities in expectation, sometimes accompanied with concentration estimates, e.g. variance estimates. To the best of the author's knowledge, there aren't any results concerning the almost sure asymptotics of random nodal sets on general Riemannian manifolds (without extracting a subsequence of eigenvalues), which is precisely the object of this paper.

Indeed, we consider here a generic Gaussian combination of Laplace eigenfunctions and this combination being fixed, we evaluate it at a uniform and independent random point on the manifold. Under the sole randomness of this evaluation point, we then prove that when properly normalized and localized in the neighboring of the point, the random field statistically converges towards an explicit universal Euclidean random wave, see Section 2 below for precise statements. This result is thus in line with Berry's conjecture and generalizes Salem–Zygmund's central limit theorem to the Riemannian framework. Our method is inspired by [3] and makes a crucial use of Weyl type estimates and some decorrelation estimates of the limit field.

Starting from a stochastic representation formula of the nodal volume, in the spirit of Bourgain's derandomization technique [9, 10], we then deduce from the above convergence, the almost-sure asymptotics of the nodal volume of a Riemannian random wave to an explicit universal limit. This last result answers question raised by S.Zelditch in [34] about the almost sure convergence of random nodal measure. Moreover, it allows to recover and reinforce the asymptotics in expectation obtained so far in the literature, see e.g. [24, 12]. Note that our approach is only based upon the almost sure convergence in distribution of the random field and some uniform moment bounds, and it does not require any variance nor concentration estimates.

1.2 Geometric and probabilistic setting

In order to state our main results, let us describe the geometric and probabilistic contexts and fix our notations.

1.2.1 Geometric setting

Let $(\mathcal{M}, g)$ be a closed, compact manifold without boundary of dimension d. Without loss of generality we will assume that the associated volume measure μ is normalized i.e. $\mu(\mathcal{M}) = 1$. It is naturally equipped with the Laplace–Beltrami operator denoted Δ. The second order differential operator Δ is autoadjoint and has compact resolvent. Spectral theory asserts the existence of an orthonormal basis $(\varphi_n)_{n\in\mathbb{N}}$ of eigenfunctions of Δ associated to the eigenvalues $(-\lambda_n^2)_{n\in\mathbb{N}}$ (ordered and indexed with multiplicity). For all $n \in \mathbb{N}$,

$$\Delta\varphi_n = -\lambda_n^2\varphi_n \quad \text{and} \quad \int_{\mathcal{M}} \varphi_n^2 \mathrm{d}\mu = 1.$$

Given $x, y \in \mathcal{M}$ and $\lambda \in \mathbb{R}_+$, we define

$$K_\lambda(x,y) = \sum_{\lambda_n\le\lambda} \varphi_n(x)\varphi_n(y) \quad \text{and} \quad K_\lambda(x) := \sum_{\lambda_n\le\lambda} \varphi_n^2(x),$$

the two-point spectral kernel projector on the eigenspace generated by the eigenfunctions up to order λ. Integrating the function $x \mapsto K_\lambda(x)$ on $\mathcal{M}$ we obtain

$$K(\lambda) := \mathrm{Card}\,\{n \in \mathbb{N} \mid \lambda_n \le \lambda\} = \int_{\mathcal{M}} K_\lambda(x)\mathrm{d}\mu(x),$$

the eigenvalue counting function. A fundamental tool in spectral analysis is the local Weyl law, first proved by Hörmander in [17]. It describes the precise asymptotics of the two-point spectral projector. Let σ_d be the volume of the unit ball in $\mathbb{R}^d$:

$$\sigma_d = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)},$$

and define for $x \in \mathbb{R}^d$ the function

$$\mathcal{B}_d(\|x\|) := \frac{1}{\sigma_d}\int_{|\xi|\le 1} e^{i\langle x,\xi\rangle}\mathrm{d}\xi.$$

It is well-defined since the right-hand since is invariant by rotation. The local Weyl law asserts that uniformly on $x, y \in \mathcal{M}$,

$$K_\lambda(x,y) = \frac{\sigma_d}{(2\pi)^d}\lambda^d\mathcal{B}_d(\lambda.\,\mathrm{dist}(x,y)) + O(\lambda^{d-1}). \tag{1}$$

The limit kernel $\mathcal{B}_d$ only depends on the dimension d. It is related to the Bessel function of the first kind $\mathcal{J}$ by the formula

$$\mathcal{B}_d(\|x\|) = \frac{1}{\sigma_d}\left(\frac{2\pi}{\|x\|}\right)^{d/2}\mathcal{J}_{\frac{d}{2}}(\|x\|).$$

The result of Hörmander goes beyond since the Weyl asymptotics is also true in the $\mathcal{C}^\infty$ topology. For an arbitrary number of derivatives in x and y, one has

$$\partial_{\alpha,\beta}K_\lambda(x,y) = \frac{\sigma_d}{(2\pi)^d}\lambda^d\partial_{\alpha,\beta}\left[\mathcal{B}_d(\lambda.\,\mathrm{dist}(x,y))\right] + O(\lambda^{d+\alpha+\beta-1}), \tag{2}$$

and the remainder is also uniform and x and y. Taking $x = y$ in the local Weyl law, one gets the following classical Weyl law on the number of eigenvalues of magnitude lower than λ:

$$K_\lambda(x) = \frac{\sigma_d}{(2\pi)^d}\lambda^d + O(\lambda^{d-1}) \quad \text{and} \quad K(\lambda) = \frac{\sigma_d}{(2\pi)^d}\lambda^d + O(\lambda^{d-1}), \tag{3}$$

from which one can deduce a first-order asymptotics for the n-th eigenvalue given by

$$\lambda_n \simeq 2\pi \left(\frac{n}{\sigma_d}\right)^{1/d}. \tag{4}$$

The remainder in the Weyl law is a widely discussed topic and can be improved in many cases, see the general survey [21]. In the torus case $\mathbb{T}^2$, it is the famous Gauss circle problem which is deeply intertwined with number theory, see [18] or [19] for a general survey. The remainder is sharp for the d-sphere, where the eigenvalues are explicit, see [32]. Under the hypotheses that the geodesics are almost surely aperiodic (see [20] for a precise statement), the remainder in Weyl law (3) is in fact a $o(\lambda^{d-1})$, which implies the following finer asymptotic:

$$k(\lambda) := \mathrm{Card}\left\{n \in \mathbb{N} \mid \lambda_n \in [\lambda, \lambda+1]\right\} = \frac{\sigma_d}{(2\pi)^d} d\lambda^{d-1} + o(\lambda^{d-1}).$$

Along the same lines, under some geodesic conditions on $\mathcal{M}$ defined in [11] and [12] (which roughly states that almost-surely, a geodesic never return to its starting point), the authors proved a finer remainder for the local Weyl Law in the $\mathcal{C}^\infty$ topology. We define

$$k_\lambda(x,y) := \sum_{\lambda_n \in [\lambda, \lambda+1]} \varphi_n(x)\varphi_n(y),$$

the two-point spectral kernel projector on the eigenspace generated by the eigenfunctions associated with eigenvalues between λ and $\lambda + 1$, and

$$\mathcal{S}_d : \|x\| \mapsto \frac{1}{d\sigma_d} \int_{|\xi|=1} e^{i\langle x,\xi\rangle} \mathrm{d}\xi = \mathcal{B}_{d-2}(\|x\|).$$

One has the following asymptotics, valid in the $\mathcal{C}^\infty$ topology:

$$k_\lambda(x,y) = \frac{\sigma_d}{(2\pi)^d} d\lambda^{d-1} \mathcal{S}_d(\lambda.\,\mathrm{dist}(x,y)) + o(\lambda^{d-1}), \tag{5}$$

and the remainder is uniform in $x, y \in \mathcal{M}$ (we refer to [12] for a thorough discussion on the required hypotheses for (5) to hold true).

1.2.2 Probabilistic models

Let us now describe our main probabilistic models, classically known as the (large band) Riemannian random wave model and monochromatic (or band-limited) Riemannian random wave model. Let us consider $(a_n)_{n\geq 0}$ a sequence of independent and identically distributed standard Gaussian random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P}_a)$. We will denote by $\mathbb{E}_a$ the associated expectation. The two models are defined as the following Gaussian combination of eigenfunctions:

$$f_\lambda : x \mapsto \frac{1}{\sqrt{K(\lambda)}} \sum_{\lambda_n \leq \lambda} a_n \varphi_n(x) \quad \text{and} \quad \widetilde{f}_\lambda : x \mapsto \frac{1}{\sqrt{k(\lambda)}} \sum_{\lambda_n \in]\lambda, \lambda+1]} a_n \varphi_n(x).$$

In the monochromatic regime, we will always assume that the manifold $\mathcal{M}$ is chosen such that the asymptotic estimate (5) is satisfied uniformly on $x, y \in \mathcal{M}$. This condition implies that $d \geq 2$, and we will use this fact in the proofs. We could also have introduced an intermediate band regime, see for instance [7], for which $\lambda_n \in]\alpha\lambda, \lambda]$ and $\alpha \in [0,1[$, but for the simplicity of the statements, we choose to focus on the two "extreme" cases defined above. These processes give a probabilistic interpretation of the projector kernels introduced above since they coincide with the covariance kernel of theses processes. For all $x, y \in \mathcal{M}$ we have indeed

$$\mathbb{E}_a[f_\lambda(x)f_\lambda(y)] = \frac{K_\lambda(x,y)}{K(\lambda)} \quad \text{and} \quad \mathbb{E}_a[\widetilde{f}_\lambda(x)\widetilde{f}_\lambda(y)] = \frac{k_\lambda(x,y)}{k(\lambda)}.$$

Consider the canonical Euclidean space $\mathbb{R}^d$. For all $x \in \mathcal{M}$ we define

$$I_x : \mathbb{R}^d \longrightarrow T_x\mathcal{M},$$

an isometry between $\mathbb{R}^d$ and the tangent space at x. We only require the mapping $x \mapsto I_x$ to be measurable. For the torus $\mathbb{T}^d$ we can choose for I_x the canonical isometry, but in all generality there is no canonical choice (nor even a continuous choice) of a family $(I_x)_{x\in\mathcal{M}}$. Denoting $\exp_x$ the Riemannian exponential based at $x \in \mathcal{M}$ we define

$$\Phi_x := \exp_x \circ I_x.$$

This map allows us to define a rescaled and flattened version of f_λ and $\widetilde{f}_\lambda$ (or any function on $\mathcal{M}$) around some point $x \in \mathcal{M}$ by setting

$$\begin{aligned} g_\lambda^x :\ & \mathbb{R}^d \longrightarrow \mathbb{R} \\ & v \longrightarrow f_\lambda\left[\Phi_x\left(\frac{v}{\lambda}\right)\right] \end{aligned} \qquad\qquad \begin{aligned} \widetilde{g}_\lambda^x :\ & \mathbb{R}^d \longrightarrow \mathbb{R} \\ & v \longrightarrow \widetilde{f}_\lambda\left[\Phi_x\left(\frac{v}{\lambda}\right)\right]. \end{aligned}$$

In the literature the processes g_λ^x and $\widetilde{g}_\lambda^x$ have already been studied, see for instance [7, 12, 34]. Thanks to the Weyl law, they converge in distribution (at a fixed point x) towards an isotropic Gaussian process whose covariance function is given by the function $\mathcal{B}_d$ and $\mathcal{S}_d$ respectively. In particular the limit process only depends on the topological dimension d and is independent of the base manifold $\mathcal{M}$.

Let us now consider a random variable X, which is equidistributed on the manifold $\mathcal{M}$ and **independent** of the coefficients sequence $(a_n)_{n\geq 0}$. For consistency, we denote by $\mathbb{P}_X$ the (uniform) distribution of X and $\mathbb{E}_X$ the associated expectation. Randomizing on the spatial parameter x we define the following processes on $\mathbb{R}^d$:

$$g_\lambda^X : v \mapsto f_\lambda\left(\Phi_X\left(\frac{v}{\lambda}\right)\right) \quad \text{and} \quad \widetilde{g}_\lambda^X : v \mapsto \widetilde{f}_\lambda\left(\Phi_X\left(\frac{v}{\lambda}\right)\right), \quad v \in \mathbb{R}^d. \tag{6}$$

1.3 Statement of the results and outline of the proofs

The first main result of the article is the following functional central limit theorem which generalizes [3, Thm. 3] to the case of a general compact Riemannian manifold.

Theorem 1.1. *Almost surely with respect to the probability $\mathbb{P}_a$, the two processes $(g_\lambda^X(v))_{v\in\mathbb{R}^d}$ and $(\widetilde{g}_\lambda^X(v))_{v\in\mathbb{R}^d}$ converge in distribution under $\mathbb{P}_X$ with respect to the $\mathcal{C}^\infty$ topology, towards isotropic Gaussian processes $(g_\infty(v))_{v\in\mathbb{R}^d}$ and $(\widetilde{g}_\infty(v))_{v\in\mathbb{R}^d}$ with respective covariance functions*

$$\mathbb{E}_X[g_\infty(u)g_\infty(v)] = \mathcal{B}_d(\|u-v\|) \qquad \text{and} \qquad \mathbb{E}_X[\widetilde{g}_\infty(u)\widetilde{g}_\infty(v)] = \mathcal{S}_d(\|u-v\|).$$

Let us emphasize that in the literature these kind of results are known only under Gaussian expectation. The concentration result obtained in [12] allows the authors to prove a similar result up to a subsequence of polynomial growth. Our result is new in the sense that the sole randomization on the uniform random variable X suffices to recover the asymptotic behavior of g_λ (without extracting a subsequence), and open the door to almost-sure results concerning functionals of f_λ, as demonstrates the next Theorem 1.2 concerning almost-sure asymptotics of the nodal volume. Moreover, it corroborates Berry conjecture, which roughly states that under the hypothesis of a chaotic geodesic flow, a Laplace eigenfunction of high energy statistically behaves like a universal Euclidean random wave.

The proof of Theorem 1.1 is the object of the next Section 2 and it is based upon convergence of characteristic functions. Taking the expectation under $\mathbb{P}_a$, the Gaussian framework allows us to make – technical but explicit – computations of characteristic functions. By a Borel–Cantelli argument we recover an almost sure convergence under $\mathbb{P}_a$. The proof could certainly be applied to more general settings as it uses mostly the following two main ingredients :

- The local Weyl law, which gives the limit distribution of g_λ^x (as a Gaussian process) towards the Gaussian process g_∞.
- The statistical decorrelation of Lemma 2.4, which roughly states that if X and Y are independent uniform random variables on $\mathcal{M}$, then the associated Gaussian processes g_λ^X and g_λ^Y statistically decorrelate as λ goes to $+\infty$. It is a consequence of the decaying rate of the limit kernel $\mathcal{B}_d$.

As usual, a proof of convergence for stochastic processes splits into two parts. The convergence of finite dimensional distributions given by Theorem 2.1, and a tightness property given by Theorem 2.6.

The second main result of the article is the following almost-sure asymptotics of the nodal volume associated with the random fields f_λ and $\widetilde{f}_\lambda$. Almost surely, the nodal sets $\{f_\lambda = 0\}$ and $\{\widetilde{f}_\lambda = 0\}$ are random smooth submanifolds of codimension one. We denote by $\mathcal{H}^{d-1}$ the $(d-1)-$dimensional Hausdorff measure, and let B be a ball in $\mathbb{R}^d$ of Euclidean volume one.

Theorem 1.2. *Almost surely with respect to the sequence $(a_k)_{k\geq 0}$,*

$$\lim_{\lambda\to+\infty} \frac{\mathcal{H}^{d-1}(\{f_\lambda = 0\})}{\lambda} = \mathbb{E}_X[\mathcal{H}^{d-1}(\{g_\infty = 0\} \cap B)],$$

and

$$\lim_{\lambda\to+\infty} \frac{\mathcal{H}^{d-1}(\{\widetilde{f}_\lambda = 0\})}{\lambda} = \mathbb{E}_X[\mathcal{H}^{d-1}(\{\widetilde{g}_\infty = 0\} \cap B)].$$

This result improves the result[34, Thm. 1] or [24, Thm. 1.1] about the convergence of nodal volume in expectation under $\mathbb{P}_a$. Passing from an almost-sure convergence to a convergence in expectation is a short corollary of our proof (see Corollary 3.13). It positively answers the question raised by S. Zelditch in [34, Cor. 2], about the asymptotics of random nodal measure. In that context, Theorem 1.2 only addresses the almost-sure asymptotics of the total nodal volume, but our proof could be extended to the random nodal volume contained in any ball, from which follows the almost-sure asymptotics of random nodal measure. In [24] is considered the more general framework of random submanifolds. Here we only focused on the case of hypersurfaces, but the same scheme of proof could have been applied to show the almost-sure asymptotics of the nodal volume of random submanifolds.

The right-hand side in Theorem 1.2 can be explicitly computed by the Kac–Rice formula for random fields (see the Remark 3.3) and have in fact

$$\lim_{\lambda\to+\infty}\frac{\mathcal{H}^{d-1}(\{f_\lambda=0\})}{\lambda}=\frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{d+2}}\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\quad\text{and}\quad\lim_{\lambda\to+\infty}\frac{\mathcal{H}^{d-1}(\{\widetilde{f}_\lambda=0\})}{\lambda}=\frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{d}}\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}.$$

The proof relies on the connection between the nodal volumes of the processes f_λ and g_λ^X given by Lemma 3.2 which states

$$\frac{\mathcal{H}^{d-1}(\{f_\lambda=0\})}{\lambda}\simeq\mathbb{E}_X[\mathcal{H}^{d-1}(\{g_\lambda^X=0\}\cap B)]=:\mathbb{E}_X[Z_\lambda].$$

By Theorem 1.2 and the continuity of the random nodal volume for the $\mathcal{C}^1$-topology, the continuous mapping theorem asserts that the nodal volume of g_λ^X on B, denoted Z_λ, converges in distribution towards the nodal volume of g_∞.

To recover convergence of expectations and thus Theorem 1.2, it is then sufficient to prove the uniform integrability of the family $(Z_\lambda)_{\lambda>0}$. Unfortunately the process g_λ^X is not Gaussian under $\mathbb{P}_X$, and sufficient conditions for the boundedness of power moments in the literature (as the ones given in [5]) are too restrictive for our purpose. The approach we use to prove finiteness of all positive moments in Theorem 3.11 do not rely on the Kac–Rice formula, ill-devised for non Gaussian processes, but on more geometric considerations.

Thanks to a variant of the Crofton formula given by Lemma 3.9, we can relate the nodal volume of g_λ^X to the anti-concentration of g_λ^X around zero on deterministic points. The anti-concentration bound is given in Lemma 3.4 by the finiteness of a small negative moment of g_λ^X. The proof of the existence of a negative moment uses the explicit rate of convergence of characteristic function given in Lemma 2.7. It allows us to rewrite the convergence in term of the so-called smooth Wasserstein distance in Lemma 3.5 (following the approach in [5]), which is a stronger notion of convergence than the convergence in distribution.

Throughout the different proofs, C will denote a generic constant which does not depend on λ nor the sequence $(a_n)_{n>0}$, and $C(\omega)$ will denote a constant which does not depend on λ but may depend on the sequence $(a_n)_{n>0}$ (generally, a constant that comes from a Borel–Cantelli argument).

At last, we will prove the above theorems mostly in the long band regime, that is for the process g_λ^X, but the proofs apply almost verbatim in the monochromatic regime. The minor differences arising between the two cases will be detailed in the proofs.

2 Salem–Zygmund CLT for Riemannian random waves

In this section we give the proof of Theorem 1.1, and a few corollary results which will be of use in the study of the almost sure asymptotics of nodal volume in next Section 3. As usual, the proof of the functional convergence splits into the convergence of finite dimensional marginals and some tightness estimates.

2.1 Finite dimensional convergence and decorrelation estimates

We first establish a quantitative version of the convergence of the finite dimensional marginals of g_λ^X (resp. $\widetilde{g}_\lambda^X$) towards those of g_∞ (resp. $\widetilde{g}_\infty$), the rate of convergence depending on the ambient dimension. In small dimension we need to take into account small correctives, which reflects the slow decay of the limit kernel $\mathcal{B}_d$ (resp. $\mathcal{S}_d$) at infinity. Recall that the monochromatic regime is considered only when the refined local Weyl law (5) is valid, which implies $d \geq 2$. We set

$$\eta(\lambda) = \begin{cases} \log \lambda & \text{in the large band regime and } d = 1, \\ 1 & \text{in the large band regime and } d \geq 2, \\ \sqrt{\lambda} & \text{in the monochromatic regime and } d = 2, \\ 1 & \text{in the monochromatic regime and } d \geq 3. \end{cases} \tag{7}$$

Fix an integer $p \geq 1$, $v = (v_1, \ldots, v_p) \in (\mathbb{R}^n)^p$ and $t = (t_1, \ldots, t_p) \in \mathbb{R}^p$, and define in the large band regime

$$N_\lambda(v,t) := \sum_{i=1}^p t_i\, g_\lambda^X(v_i) \quad \text{and} \quad N_\infty(v,t) := \sum_{i=1}^p t_i\, g_\infty(v_i),$$

and respectively in the monochromatic regime

$$N_\lambda(v,t) := \sum_{i=1}^p t_i\, \widetilde{g}_\lambda^X(v_i) \quad \text{and} \quad N_\infty(v,t) := \sum_{i=1}^p t_i\, \widetilde{g}_\infty(v_i).$$

We will simply write N_λ and N_∞ when appropriate. Note that these linear combinations are Gaussian random variables under $\mathbb{P}_a$. We prove that the characteristic function of N_λ under $\mathbb{P}_X$ converges to the one of N_∞ as λ goes to infinity.

Theorem 2.1. *Almost surely with respect to the probability $\mathbb{P}_a$, in the large band regime,*

$$\forall t \in \mathbb{R}^p, \forall v \in \mathbb{R}^p, \lim_{\lambda \to +\infty} \mathbb{E}_X\left[e^{iN_\lambda(v,t)}\right] = \mathbb{E}_X\left[e^{iN_\infty(v,t)}\right] = \exp\left(-\frac{1}{2}\sum_{i,j=1}^p t_i t_j \mathcal{B}_d(\|v_i - v_j\|)\right),$$

and in the monochromatic regime,

$$\forall t \in \mathbb{R}^p, \forall v \in \mathbb{R}^p, \lim_{\lambda \to +\infty} \mathbb{E}_X\left[e^{iN_\lambda(v,t)}\right] = \mathbb{E}_X\left[e^{iN_\infty(v,t)}\right] = \exp\left(-\frac{1}{2}\sum_{i,j=1}^p t_i t_j \mathcal{S}_d(\|v_i - v_j\|)\right).$$

Since $N_\lambda(v,t)$ is a Gaussian random variable under $\mathbb{P}_a$, the explicit formula for the characteristic function of a Gaussian variable gives

$$\mathbb{E}_X\left[e^{iN_\lambda(v,t)}\right] = e^{-\frac{1}{2}\mathbb{E}[N_\lambda(v,t)^2]}, \quad \text{and similarly,} \quad \mathbb{E}_X\left[e^{iN_\infty(v,t)}\right] = e^{-\frac{1}{2}\mathbb{E}[N_\infty(v,t)^2]}.$$

In order to quantify the convergence rate, for any integer $q > 0$, we set

$$\Delta_\lambda^{(q)} := \mathbb{E}_a\left[\left|\mathbb{E}_X\left[e^{iN_\lambda(v,t)}\right] - \mathbb{E}_X\left[e^{iN_\infty(v,t)}\right]\right|^{2q}\right]. \tag{8}$$

Let K be a compact subset of $\mathbb{R}^d$. In the following we will assume that the vectors $v_1, \ldots, v_p$ belong to K.

Theorem 2.2. *There is a constant C depending only on $\mathcal{M}$, K and q, such that*

$$\Delta_\lambda^{(q)} \leq C(1 + \|t\|)^{4q}\left(\frac{\eta(\lambda)}{\lambda}\right)^q. \tag{9}$$

The proof of Theorem 2.1 is a direct consequence of the second assertion in Theorem 2.5 at the end of this section, but observe that Theorem 2.2 implies a weak version of Theorem 2.1 and gives the core idea of the proof. Indeed, let us recall from Equation (4) that the sequence $(\lambda_n)_{n\geq 0}$ of eigenvalues grows as $Cn^{1/d}$. Fix some $t \in \mathbb{R}^p$ and let $\varepsilon > 0$. Markov inequality implies that

$$\mathbb{P}_a\left(\left|\mathbb{E}_X\left[e^{iN_{\lambda_n}(v,t)}\right] - \mathbb{E}_X\left[e^{iN_\infty(v,t)}\right]\right| > \lambda_n^\varepsilon\sqrt{\frac{\eta(\lambda_n)}{\lambda_n}}\right) \leq \frac{\Delta_{\lambda_n}^{(q)}}{\lambda_n^{2q\varepsilon}}\left(\frac{\lambda_n}{\eta(\lambda_n)}\right)^q = O\left(n^{-2q\varepsilon/d}\right).$$

For $q > d/(2\varepsilon)$, the left-hand term is summable and Borel–Cantelli Lemma implies the existence a constant $C(\omega, v, t)$ such that

$$\left|\mathbb{E}_X\left[e^{iN_\lambda(v,t)}\right] - \mathbb{E}_X\left[e^{iN_\infty(v,t)}\right]\right| \leq C(\omega, v, t)\frac{\sqrt{\eta(\lambda)}}{\lambda^{\frac{1}{2}-\varepsilon}}. \tag{10}$$

In particular, for a fixed $t \in \mathbb{R}^p$ and $v \in \mathbb{R}^p$, this proves the convergence in distribution of $N_\lambda(v,t)$ towards $N_\infty(v,t)$, almost surely with respect to the probability $\mathbb{P}_a$. Note that Theorem 2.1 states that the convergence holds almost surely under $\mathbb{P}_a$, simultaneously for all $t \in \mathbb{R}^d$ and $v \in K^p$, and thus requires the inversion of quantifiers. We deal with this issue in Theorem 2.5 at the end of Section 2, which makes explicit the dependence of $C(\omega, v, t)$ in Equation (10) with respect to v and t.

Proof of Theorem 2.2. Define

$$\widetilde{\Delta}_\lambda^{(q)} := \mathbb{E}_a\left[\left|\mathbb{E}_X\left[e^{iN_\lambda(v,t)}\right] - \mathbb{E}_a\mathbb{E}_X\left[e^{iN_\lambda(v,t)}\right]\right|^{2q}\right]. \tag{11}$$

By triangular inequality, we have

$$\Delta_\lambda^{(q)} \leq 4^q\left(\widetilde{\Delta}_\lambda^{(q)} + \left|e^{-\frac{1}{2}\mathbb{E}_X[N_\infty(v,t)^2]} - \mathbb{E}_X\left[e^{-\frac{1}{2}\mathbb{E}_a[N_\lambda(v,t)^2]}\right]\right|^{2q}\right).$$

Using the 1−Lipschitz regularity of $x \mapsto e^{-x}$, we then get

$$\Delta_\lambda^{(q)} \leq 4^q\widetilde{\Delta}_\lambda^{(q)} + 4^{q-1}\left|\mathbb{E}_X\left[N_\infty(v,t)^2\right] - \mathbb{E}_X\mathbb{E}_a\left[N_\lambda(v,t)^2\right]\right|^{2q}. \tag{12}$$

The last term in Equation (12) can be evaluated as follows. The following direct computation is done is the large band regime with limit kernel $\mathcal{B}_d$, but it remain true in the monochromatic regime with limit kernel $\mathcal{S}_d$. We have first

$$\begin{aligned}
\mathbb{E}_a[N_\lambda^2] &= \mathbb{E}_a\left[\left(\sum_{i=1}^p t_i g_\lambda^X(v_i)\right)^2\right] \\
&= \sum_{i,j=1}^p t_i t_j \frac{1}{K(\lambda)}\sum_{\lambda_n\leq\lambda}\varphi_n\left[\Phi_X\left(\frac{v_i}{\lambda}\right)\right]\varphi_n\left[\Phi_X\left(\frac{v_j}{\lambda}\right)\right] \\
&= \sum_{i,j=1}^p t_i t_j \frac{K_\lambda\left(\Phi_X\left(\frac{v_i}{\lambda}\right), \Phi_X\left(\frac{v_j}{\lambda}\right)\right)}{K(\lambda)}.
\end{aligned}$$

Using Weyl law and the fact that v lives in a compact set, we obtain

$$\begin{aligned}
&\left|\mathbb{E}_a\left[N_\lambda^2\right] - \mathbb{E}_X[N_\infty^2]\right| \leq \sum_{i,j=1}^p |t_i||t_j|\left|\frac{K_\lambda\left(\Phi_X\left(\frac{v_i}{\lambda}\right), \Phi_X\left(\frac{v_j}{\lambda}\right)\right)}{K(\lambda)} - \mathcal{B}_d(\|v_i - v_j\|)\right| \\
&\leq \sum_{i,j=1}^p |t_i||t_j|\left|\mathcal{B}_d\left[\lambda\,\mathrm{dist}\left(\Phi_X\left(\frac{v_i}{\lambda}\right), \Phi_X\left(\frac{v_j}{\lambda}\right)\right)\right] - \mathcal{B}_d(\|v_i - v_j\|)\right| + \|t\|^2 O\left(\frac{1}{\lambda}\right) \\
&\leq \sum_{i,j=1}^p |t_i||t_j|\left|\lambda\,\mathrm{dist}\left(\Phi_X\left(\frac{v_i}{\lambda}\right), \Phi_X\left(\frac{v_j}{\lambda}\right)\right) - \|v_i - v_j\|\right| + \|t\|^2 O\left(\frac{1}{\lambda}\right).
\end{aligned}$$

The last line is justified by the fact that $\mathcal{B}_d$ (resp. $\mathcal{S}_d$) is Lipschitz continuous. The differential of the exponential map at 0 is the identity, which implies the following asymptotic, uniformly on v in a compact subset:

$$\left|\lambda \operatorname{dist}\left(\Phi_X\left(\frac{v_i}{\lambda}\right), \Phi_X\left(\frac{v_j}{\lambda}\right)\right) - \|v_i - v_j\|\right| = O\left(\frac{1}{\lambda}\right),$$

and we deduce

$$\left|\mathbb{E}_a\left[N_\lambda^2\right] - \mathbb{E}_X[N_\infty^2]\right| = \|t\|^2 O\left(\frac{1}{\lambda}\right).$$

Injecting this estimate in Equation (12), we get

$$\Delta_\lambda^{(q)} \leq 4^q \widetilde{\Delta}_\lambda^{(q)} + \|t\|^{4q} O\left(\frac{1}{\lambda^{2q}}\right).$$

The conclusion of Theorem 2.2 then follows from the following lemma. □

Lemma 2.3. *There is a constant C depending only on $\mathcal{M}$, K and q, such that*

$$\widetilde{\Delta}_\lambda^{(q)} \leq C(1 + \|t\|^{4q}) \left(\frac{\eta(\lambda)}{\lambda}\right)^q. \tag{13}$$

The proof of Lemma 2.3 is rather technical and for the sake of readability, it is postponed until Section A.1 of the Appendix. To give the reader a taste of the arguments involved, the proof is essentially based on explicit computations of characteristic functions and the key argument is the following decorrelation Lemma 2.4. With the same notations as above, let Y be a uniform random variable in $\mathcal{M}$, independent of X and of the Gaussian coefficients (a_k). Let us set

$$N_\lambda^X := \sum_{j=1}^p t_j g_\lambda^X(v_j), \quad N_\lambda^Y := \sum_{j=1}^p t_j g_\lambda^Y(v_j),$$

in the large band regime and respectively in the monochromatic regime

$$N_\lambda^X := \sum_{j=1}^p t_j \widetilde{g}_\lambda^X(v_j), \quad N_\lambda^Y := \sum_{j=1}^p t_j \widetilde{g}_\lambda^Y(v_j).$$

Lemma 2.4. *There is a constant C depending only on $\mathcal{M}$ and K, such that*

$$\mathbb{E}_X\left[\left|\mathbb{E}_a\left[N_\lambda^X N_\lambda^Y\right]\right|\right] \leq C\|t\|^2 \frac{\eta(\lambda)}{\lambda}.$$

Proof of Lemma 2.4. An explicit computation gives

$$\begin{aligned}\left|\mathbb{E}_a\left[N_\lambda^X N_\lambda^Y\right]\right| &= \left|\sum_{i,j=1}^p t_i t_j \frac{1}{K_\lambda} \sum_{\lambda_n \leq \lambda} \varphi_n\left[\Phi_X\left(\frac{v_i}{\lambda}\right)\right] \varphi_n\left[\Phi_Y\left(\frac{v_j}{\lambda}\right)\right]\right| \\ &\leq \sum_{i,j=1}^p |t_i||t_j| \left|\mathcal{B}_d\left(\lambda. \operatorname{dist}(\Phi_X\left(\frac{v_i}{\lambda}\right), \Phi_Y\left(\frac{v_j}{\lambda}\right)\right)\right| + \|t\|^2 O\left(\frac{1}{\lambda}\right),\end{aligned}$$

and the remainder is uniform on X, Y. Again, the above computation is done in the large band regime with limit kernel $\mathcal{B}_d$, but it holds in the monochromatic regime with limit kernel $\mathcal{S}_d$. Define

$$c_\lambda := \lambda.\operatorname{dist}\left(\Phi_X\left(\frac{v_i}{\lambda}\right), \Phi_Y\left(\frac{v_j}{\lambda}\right)\right) - \lambda \operatorname{dist}(X, Y).$$

By triangle inequality, $|c_\lambda|$ is bounded by $2|K|$, where $|K|$ is the diameter of the compact subset K in which lives $v_1, \dots, v_p$. It follows that

$$\left|\mathbb{E}_a\left[N_\lambda^X N_\lambda^Y\right]\right| \leq \sum_{i,j=1}^p |t_i||t_j| \left|\mathcal{B}_d\left(\lambda.\operatorname{dist}(X,Y) + c_\lambda\right)\right| + \|t\|^2 O\left(\frac{1}{\lambda}\right).$$

Taking the expectation with respect to X we obtain

$$\begin{aligned}
\mathbb{E}_X\left[\left|\mathbb{E}_a\left[N_\lambda^X N_\lambda^Y\right]\right|\right] &\leq \int_M \sum_{i,j=1}^p |t_i||t_j| \left|\mathcal{B}_d\left(\lambda.\operatorname{dist}(x,Y) + c_\lambda\right)\right| \mathrm{d}\mu(x) + O\left(\frac{\|t\|^2}{\lambda}\right) \\
&\leq \sum_{i,j=1}^p |t_i||t_j| \left(\int_{\operatorname{dist}(x,Y)\leq\varepsilon} \left|\mathcal{B}_d\left(\lambda.\operatorname{dist}(x,Y) + c_\lambda\right)\right| \mathrm{d}\mu(x) \quad + \right. \\
&\qquad \left. \int_{\operatorname{dist}(x,Y)>\varepsilon} \left|\mathcal{B}_d\left(\lambda.\operatorname{dist}(x,Y) + c_\lambda\right)\right| \mathrm{d}\mu(x)\right) + O\left(\frac{\|t\|^2}{\lambda}\right) \\
&\leq \sum_{i,j=1}^p |t_i||t_j|(I_1 + I_2) + O\left(\frac{\|t\|^2}{\lambda}\right),
\end{aligned} \tag{14}$$

where I_1 and I_2 are the two integrals appearing in the last expression. For ε small enough we can pass in local polar coordinates into the first integral I_1. We obtain

$$\begin{aligned}
I_1 &\leq d\,\sigma_d \int_0^\varepsilon \sup_{c\in[-2,2]} \left|\mathcal{B}_d\left(\lambda r + c\right)\right| (1 + O(r^2)) r^{d-1} \mathrm{d}r \\
&\leq \frac{C}{\lambda^d} \int_0^{\lambda\varepsilon} \sup_{c\in[-2,2]} \left|\mathcal{B}_d\left(u + c\right)\right| u^{d-1} \mathrm{d}u.
\end{aligned} \tag{15}$$

We use the following asymptotics for $\mathcal{B}_d$ and $\mathcal{S}_d$ at infinity:

$$\mathcal{B}_d(u) = C u^{-\frac{d+1}{2}} \sin\left(u - \frac{d-1}{4}\pi\right) + O\left(u^{-\frac{d+3}{2}}\right),$$

$$\mathcal{S}_d(u) = C u^{-\frac{d-1}{2}} \sin\left(u - \frac{d-3}{4}\pi\right) + O\left(u^{-\frac{d+3}{2}}\right).$$

Injecting these asymptotics into expression (15) we obtain the four following cases:

$$I_1 = \begin{cases} O(\log\lambda/\lambda) & \text{in the large band regime and } d = 1 \\ O(1/\lambda) & \text{in the large band regime and } d \geq 2, \\ O(1/\sqrt{\lambda}) & \text{in the monochromatic regime and } d = 2, \\ O(1/\lambda) & \text{in the monochromatic regime and } d \geq 3. \end{cases}$$

For the second integral and λ large enough, we use the fact that $|c_\lambda| \leq 2|K|$ and the asymptotic formula for $\mathcal{B}_d$ (resp. $\mathcal{S}_d$) to obtain

$$I_2 \leq \sup_{t\geq\varepsilon} \sup_{c\in[-2,2]} |\mathcal{B}_d(\lambda t + c)|,$$

from which we deduce

$$I_2 = \begin{cases} O(1\sqrt{\lambda}) & \text{in the monochromatic regime and } d = 2, \\ O(1/\lambda) & \text{else.} \end{cases}$$

Finally we recover from inequality (14) and the definition (7) of $\eta(\lambda)$ that

$$\mathbb{E}_X\left[\left|\mathbb{E}_a\left[N_\lambda^X N_\lambda^Y\right]\right|\right] \leq C\|t\|^2\frac{\eta(\lambda)}{\lambda}.$$

□

In the following application of Theorem 1.1 to nodal volume, we will need finer estimates on the constant $C(\omega, v, t)$ in Equation (10). The Borel–Cantelli Lemma does not allow to track the dependence of $C(\omega, v, t)$ with respect to the parameters v and t. It is the content of the following theorem, proved in Appendix A.2. The proof relies of Sobolev injections in order to control the supremum norm by some $W^{k,1}$ norm, which is more convenient to work with when taking the expectation under $\mathbb{P}_a$.

Theorem 2.5. *Fix $\varepsilon > 0$. There is a constant $C(\omega)$ depending only K and ε, such that*

$$\sup_{v\in K}\left|\mathbb{E}_X\left[e^{itg_\lambda^X(v)}\right] - e^{-\frac{t^2}{2}}\right| \leq C(\omega)(1+|t|^{2+\varepsilon})\frac{\sqrt{\eta(\lambda)}}{\lambda^{\frac{1}{2}-\varepsilon}}.$$

And more generally,

$$\sup_{v\in K}\left|\mathbb{E}_X\left[e^{iN_\lambda(v,t)}\right] - e^{-\frac{1}{2}\mathbb{E}_X[N_\infty(v,t)^2]}\right| \leq C(\omega)(1+\|t\|^{2+\varepsilon})\frac{\sqrt{\eta(\lambda)}}{\lambda^{\frac{1}{2}-\varepsilon}}.$$

In the worst case, $\eta(\lambda) = \sqrt{\lambda}$, so it holds independently from the dimension that

$$\sup_{v\in K}\left|\mathbb{E}_X\left[e^{itg_\lambda^X(v)}\right] - e^{-\frac{t^2}{2}}\right| \leq C(\omega)\frac{1+|t|^{2+\varepsilon}}{\lambda^{\frac{1}{4}-\varepsilon}}.$$

2.2 Tightness estimates

We now turn to the proof of the tightness for the family $(g_\lambda^X)_{\lambda>0}$. In the following, we set

$$\delta := (\sigma_d)^{-1/d} \qquad \text{and} \qquad B := B(0,\delta), \tag{16}$$

the Euclidean ball centered at zero with radius δ. Recall that the quantity σ_d is the volume of the unit ball in $\mathbb{R}^d$. The parameter δ is naturally chosen such that the ball B has unit volume. The following theorem holds true for balls of any radius but the notations are simplified for radius δ.

Theorem 2.6. *Almost surely with respect to the probability $\mathbb{P}_a$, the family of stochastic processes $(g_\lambda^X)_{\lambda>0}$ is tight with respect to the Frechet topology on $\mathcal{C}^\infty(B)$.*

The tightness in $\mathcal{C}^1$ topology is sufficient for the rest of the article but the proof of $\mathcal{C}^\infty$ tightness does not cost any more calculations. The proof is short once we proved the following lemma.

Lemma 2.7. *Let p be a positive integer, and α a d-dimensional multi-index. There is a constant $C(\omega)$ depending only p and α such that*

$$\mathbb{E}_X\left[\int_B |\partial_\alpha g_\lambda^X(v)|^{2p}\mathrm{d}v\right] \leq C(\omega).$$

The proof of Lemma 2.7 is given in the Appendix B and relies on hypercontractivity and a Borel–Cantelli argument.

Proof of Theorem 2.6. By Kolmogorov tightness criterion for stochastic processes (see [23, p. 39]) in dimension d with $\mathcal{C}^\infty$ topology, it suffices to show that for every multi-index of differentiation β, for some $p > d/2$, and for all $u, v \in B$,

$$\mathbb{E}_X\left[\left|\partial_\beta g_\lambda^X(v) - \partial_\beta g_\lambda^X(u)\right|^{2p}\right] \le C(\omega)\|v-u\|^{2p}.$$

We use the mean-value Theorem and Sobolev injection to get

$$\begin{aligned}
\mathbb{E}_X\left[\left(\frac{\partial_\beta g_\lambda^X(v) - \partial_\beta g_\lambda^X(u)}{\|v-u\|}\right)^{2p}\right] &\le C\sum_{k=1}^d \mathbb{E}_X\left[\left(\sup_{u\in B}\left|\partial_k\partial_\beta g_\lambda^X\right|\right)^{2p}\right]\\
&\le C\sum_{k=1}^d \mathbb{E}_X\left[\left(\|\partial_k\partial_\beta g_\lambda^X\|_{W^{d+1,1}}\right)^{2p}\right]\\
&\le C\sum_{|\alpha|\le|\beta|+d+2} \mathbb{E}_X\left[\left(\int_B\left|\partial_\alpha g_\lambda^X(u)\right|\mathrm{d}u\right)^{2p}\right]\\
&\le C\sum_{|\alpha|\le|\beta|+d+2} \mathbb{E}_X\left[\int_B\left|\partial_\alpha g_\lambda^X(u)\right|^{2p}\mathrm{d}u\right].
\end{aligned}$$

From Lemma 2.7, we have then

$$\mathbb{E}_X\left[\int_B\left|\partial_\alpha g_\lambda^X(u)\right|^{2p}\mathrm{d}u\right] \le C(\omega),$$

hence the result. □

3 Almost sure asymptotics of nodal volume

As already mentioned above, almost surely in the random coefficients, the nodal sets $\{f_\lambda = 0\}$ and $\{\widetilde{f}_\lambda = 0\}$ associated to the random wave models are random smooth submanifolds of codimension one. The object of this section is to give the proof of Theorem 1.2 on the almost sure asymptotics of the associated nodal volume.

3.1 A Stochastic representation formula

The first step in the proof of Theorem 1.2 consists in connecting the zeros of f_λ (resp. $\widetilde{f}_\lambda$) to the zeros of g_λ^X (resp. $\widetilde{g}_\lambda^X$). This is the object of Lemma 3.2 below. We first recall a variant of co-area formula (see [14, p. 248]). Let $f : \mathcal{M} \to \mathbb{R}$ be a smooth function, $\varphi : \mathbb{R} \to \mathbb{R}$ be a positive measurable function, and A a compact subset of $\mathcal{M}$. Then

$$\int_A \varphi(f(x))\|\nabla_x f\|\mathrm{d}\mu(x) = \int_{\mathbb{R}} \varphi(y)\mathcal{H}^{d-1}(\{f=y\}\cap A)\mathrm{d}y.$$

The following lemma is widely known in the literature but we could not find any explicit proof, and we give here a proof for sake of completeness.

Lemma 3.1. *Suppose that* 0 *is a regular value of* f *on* A*. Then the mapping*

$$y \mapsto \mathcal{H}^{d-1}(\{f=y\}\cap A)$$

is continuous in a neighborhood of 0 *and the following formula holds true:*

$$\mathcal{H}^{d-1}(\{f=0\}\cap A)=\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_A \mathbb{1}_{\{|f(x)|<\varepsilon\}}\|\nabla_x f\|\mathrm{d}\mu(x).$$

Proof. Define $g_y : x\mapsto f(x)-y$. The function g_y converges to f in $\mathcal{C}^1$ topology when $y\to 0$. Since 0 is a regular value of f, then [2, Thm. 3] implies that the application

$$y\mapsto \mathcal{H}^{d-1}(\{g_y=0\}\cap A), \tag{17}$$

is continuous in a neighborhood of 0, which proves the first part of the theorem. Now we choose $\varphi_\varepsilon=\frac{1}{2\varepsilon}\mathbb{1}_{]-\varepsilon,\varepsilon[}$ in the co-area formula. By continuity of the mapping (17), we recover the announced formula, letting ε go to zero. □

Recall the definition of δ in (16).

Lemma 3.2. *Let $f:\mathcal{M}\to\mathbb{R}$ a smooth function such that* 0 *is a regular value of f. Then*

$$\frac{\mathcal{H}^{d-1}(\{f=0\})}{\lambda^d}=\left(1+O\left(\frac{1}{\lambda^2}\right)\right)\mathbb{E}_X\left[\mathcal{H}^{d-1}\left(\{f=0\}\cap B\left(X,\frac{\delta}{\lambda}\right)\right)\right],$$

and the Big-Oh does not depend on the function f.

Proof. From Lemma 3.1, we have

$$\mathbb{E}_X\left[\mathcal{H}^{d-1}\left(\{f=0\}\cap B\left(X,\frac{\delta}{\lambda}\right)\right)\right]=\mathbb{E}_X\left[\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{B\left(X,\frac{\delta}{\lambda}\right)}\mathbb{1}_{\{|f(x)|<\varepsilon\}}\|\nabla_x f\|\mathrm{d}\mu(x)\right]. \tag{18}$$

From the inequality

$$\frac{1}{2\varepsilon}\int_{B\left(X,\frac{\delta}{\lambda}\right)}\mathbb{1}_{\{|f(x)|<\varepsilon\}}\|\nabla_x f\|\mathrm{d}\mu(x)\leq\frac{1}{2\varepsilon}\int_{\mathcal{M}}\mathbb{1}_{\{|f(x)|<\varepsilon\}}\|\nabla_x f\|\mathrm{d}\mu(x),$$

and using Lemma 3.1 (with $A=\mathcal{M}$), we deduce that the last quantity is continuous in ε, thus bounded by a constant. We can apply dominated convergence in (18) to obtain

$$\begin{aligned}\mathbb{E}_X\left[\mathcal{H}^{d-1}\left(\{f=0\}\cap B\left(X,\frac{\delta}{\lambda}\right)\right)\right]&=\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{\mathcal{M}}\int_{\mathcal{M}}\mathbb{1}_{\{\mathrm{dist}(x,y)<\frac{\delta}{\lambda}\}}\mathbb{1}_{\{|f(x)|<\varepsilon\}}\|\nabla_x f\|\mathrm{d}\mu(y)\mathrm{d}\mu(x)\\&=\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{\mathcal{M}}\mathrm{Vol}_{\mathcal{M}}\left(B\left(x,\frac{\delta}{\lambda}\right)\right)\mathbb{1}_{\{|f(x)|<\varepsilon\}}\|\nabla_x f\|\mathrm{d}\mu(x).\end{aligned}$$

Standard comparison theorem for geodesic ball asserts that uniformly on x,

$$\begin{aligned}\mathrm{Vol}_{\mathcal{M}}\left(B\left(x,\frac{\delta}{\lambda}\right)\right)&=\mathrm{Vol}_{\mathbb{R}^d}\left(B\left(0,\frac{\delta}{\lambda}\right)\right)\left(1+O\left(\frac{1}{\lambda^2}\right)\right)\\&=\frac{1}{\lambda^d}\left(1+O\left(\frac{1}{\lambda^2}\right)\right),\end{aligned}$$

from which we deduce

$$\begin{aligned}\mathbb{E}_X\left[\mathcal{H}^{d-1}\left(\{f=0\}\cap B\left(X,\frac{\delta}{\lambda}\right)\right)\right]&=\frac{1}{\lambda^d}\left(1+O\left(\frac{1}{\lambda^2}\right)\right)\lim_{\varepsilon\to 0}\int_{\mathcal{M}}\frac{\mathbb{1}_{\{|f(x)|<\varepsilon\}}}{2\varepsilon}\|\nabla_x f\|\mathrm{d}\mu(y)\mathrm{d}\mu(x)\\&=\frac{1}{\lambda^d}\left(1+O\left(\frac{1}{\lambda^2}\right)\right)\mathcal{H}^{d-1}(\{f=0\}).\end{aligned}$$

□

Note that, alternatively, we could have proved the asymptotic representation formula given by Lemma 3.2 using the closed Kac–Rice formula for manifolds in [22].

3.2 Application of the Central Limit Theorem

The next step in the proof of Theorem 1.2 then consists in using the central limit theorem as established in Section 2. We define the mapping

$$\begin{aligned} \Phi_x^{(\lambda)} : B &\longrightarrow \mathcal{M} \\ v &\longrightarrow \Phi_x\left(\frac{v}{\lambda}\right). \end{aligned}$$

Choosing $f = f_\lambda$ in Lemma 3.2 and recalling the relation (6) between g_λ and f_λ, we obtain

$$\begin{aligned} \frac{\mathcal{H}^{d-1}(\{f_\lambda = 0\})}{\lambda^d} &= \mathbb{E}_X\left[\mathcal{H}^{d-1}\left(\{f_\lambda = 0\} \cap B\left(X, \frac{\delta}{\lambda}\right)\right)\right]\left(1 + O\left(\frac{1}{\lambda^2}\right)\right) \\ &= \mathbb{E}_X\left[\mathcal{H}^{d-1}\left[\Phi_X^{(\lambda)}\left(\{g_\lambda^X = 0\} \cap B\right)\right]\right]\left(1 + O\left(\frac{1}{\lambda^2}\right)\right). \end{aligned} \tag{19}$$

The mapping $\Phi_x^{(\lambda)}$ is a diffeomorphism onto its image for λ small enough and uniformly on $x \in \mathcal{M}$. The exponential map is a local diffeomorphism and its differential at zero is the identity. We deduce that the mapping $\Phi_x^{(\lambda)}$ is bi-Lipschitz, and uniformly on $x \in \mathcal{M}$,

$$\operatorname{Lip}\left(\Phi_x^{(\lambda)}\right) = \frac{1}{\lambda}\left(1 + O\left(\frac{1}{\lambda}\right)\right) \quad \text{and} \quad \operatorname{Lip}\left((\Phi_x^{(\lambda)})^{-1}\right) = \lambda\left(1 + O\left(\frac{1}{\lambda}\right)\right).$$

Using scaling properties of Hausdorff measures under bi-Lipschitz mappings we obtain

$$\mathcal{H}^{d-1}\left[\Phi_X^{(\lambda)}\left(\{g_\lambda^X = 0\} \cap B\right)\right] = \frac{1}{\lambda^{d-1}}\mathcal{H}^{d-1}\left[\{g_\lambda^X = 0\} \cap B\right]\left(1 + O\left(\frac{1}{\lambda}\right)\right),$$

and from expression (19) in follows that

$$\frac{\mathcal{H}^{d-1}(\{f_\lambda = 0\})}{\lambda} = \mathbb{E}_X\left[\mathcal{H}^{d-1}\left(\{g_\lambda^X = 0\} \cap B\right)\right]\left(1 + O\left(\frac{1}{\lambda}\right)\right). \tag{20}$$

The function $g \mapsto \mathcal{H}^{d-1}(\{g = 0\} \cap B)$ is continuous on the set of functions that are regular at point 0, endowed with the $\mathcal{C}^1$ topology. The limit process g_∞ is non-degenerate since the limit kernels $\mathcal{B}_d$ and $\mathcal{S}_d$ are positive definite covariance functions, and Bulinskaya Lemma (see [6, p. 34]) asserts that $\mathbb{P}_a$-almost surely, the point 0 is a regular value for the process f_λ (and hence for g_λ^X) for λ large enough, say $\lambda > \lambda_0$. Since there are only a countable number of eigenvalues, then $\mathbb{P}_a$-almost surely, 0 is a regular value for the whole family of functions $(f_\lambda)_{\lambda > \lambda_0}$, and hence for the whole family of stochastic processes $(g_\lambda^X)_{\lambda > \lambda_0}$. Define

$$Z_\lambda := \mathcal{H}^{d-1}(\{g_\lambda^X = 0\} \cap B) \qquad \text{and} \qquad Z_\infty := \mathcal{H}^{d-1}(\{g_\infty = 0\} \cap B).$$

The continuous mapping theorem and the convergence in distribution of Theorem 1.1 imply the following convergence in distribution under $\mathbb{P}_X$:

$$\mathbb{P}_a - \text{a.s.}, \qquad Z_\lambda \overset{\mathbb{P}_X}{\Longrightarrow} Z_\infty. \tag{21}$$

Theorem 1.2 is proved if we can pass to the convergence of expectations under $\mathbb{P}_X$ in (21), according to the stochastic representation formula (20). Passing to the expectation follows from the uniform integrability (with respect to $\mathbb{P}_X$) of the family of random variables $(Z_\lambda)_{\lambda > 0}$. This last point is the object of the next Sections 3.3 and 3.4.

Remark 3.3. The quantity $\mathbb{E}_X[\mathcal{H}^{d-1}(\{g_\infty = 0\} \cap B)]$ in Theorem 1.2 has an explicit value, thanks to the Kac–Rice formula. We roughly sketch the proof here (see [6, p. 177] for more details). Taking the expectation in the co-area formula gives

$$\int_{\mathbb{R}} \varphi(y)\mathbb{E}_X\left[\mathcal{H}^{d-1}(\{g_\infty = y\} \cap B)\right]\mathrm{d}y = \int_B \mathbb{E}_X\left[\varphi(g_\infty(x))\|\nabla_x g_\infty\|\right]\mathrm{d}\mu(x).$$

The Gaussian process g_∞ is stationary, hence its law does not depend on the point x. The Gaussian variables $g_\infty(x), \partial_1 g_\infty(x), \dots \partial_d g_\infty(x)$ are independents. Hence,

$$\int_{\mathbb{R}} \varphi(y)\mathbb{E}_X\left[\mathcal{H}^{d-1}(\{g_\infty = y\} \cap B)\right] \mathrm{d}y = \mathrm{Vol}(B)\mathbb{E}_X[\varphi(g_\infty)]\mathbb{E}_X[\|\nabla g_\infty\|]$$
$$= \mathbb{E}_X[\|\nabla g_\infty\|]\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \varphi(y)e^{-\frac{y^2}{2}}\mathrm{d}y,$$

and we deduce that for almost all $y \in \mathbb{R}$,

$$\mathbb{E}_X\left[\mathcal{H}^{d-1}(\{g_\infty = y\} \cap B)\right] = \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\mathbb{E}_X[\|\nabla g_\infty\|].$$

It is actually true for all $y \in \mathbb{R}$, and this is the difficult part of the proof which we do not detail. An direct computation gives

$$\mathbb{E}_X\left[(\partial_1 g_\infty)^2\right] = \dots = \mathbb{E}_X\left[(\partial_d g_\infty)^2\right] = \frac{1}{d+2},$$

and

$$\mathbb{E}_X[\|\nabla g_\infty\|] = \sqrt{\frac{2}{d+2}}\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}.$$

Taking $y = 0$ we deduce

$$\mathbb{E}_X\left[\mathcal{H}^{d-1}(\{g_\infty = 0\} \cap B)\right] = \frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{d+2}}\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)},$$

When $d = 1$ we recover the classical asymptotics $\frac{1}{\pi\sqrt{3}}$ for the number of real roots of a random trigonometric polynomial. For the process $\widetilde{g}_\infty$, we have

$$\mathbb{E}_X\left[(\partial_1 g_\infty)^2\right] = \dots = \mathbb{E}_X\left[(\partial_d g_\infty)^2\right] = \frac{1}{d},$$

which gives

$$\mathbb{E}_X\left[\mathcal{H}^{d-1}(\{\widetilde{g}_\infty = 0\} \cap B)\right] = \frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{d}}\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}.$$

3.3 Negative moment estimates for the random field

The uniform integrability of the volume of the nodal set can be deduced from anti-concentration of the stochastic process g_λ^X around zero. If the manifold were real-analytic, it would be sufficient to have the finiteness of a logarithmic moment, which is the approach taken in [3], see Remark 3.12 below. Since we consider here $\mathcal{C}^\infty$ manifolds, we need a stronger control, given by the following lemma.

Lemma 3.4. *Let $\nu < \frac{1}{40d}$. There is a constant $C(v, \omega)$ such that*

$$\sup_{\lambda>0}\mathbb{E}_X[|g_\lambda^X(v)|^{-\nu}] < C(v, \omega).$$

Let $\alpha > 0$, $\varepsilon > 0$ and $(v_i)_{i\in\mathbb{N}}$ be any sequence in B. There is a constant $C(\omega)$ (also depending on α, ε and the sequence $(v_i)_{i\in\mathbb{N}}$) such that

$$\sup_{\lambda>0}\int_1^{+\infty}\frac{1}{t^{1+\alpha+\varepsilon}}\sum_{i=0}^{\lceil t^\alpha\rceil}\mathbb{E}_X[|g_\lambda^X(v_i)|^{-\nu}]\,\mathrm{d}t < C(\omega).$$

The second technical assertion is a refinement of the first one and will be used in the final step of the proof of uniform integrability. It compensates the fact that the constant $C(v,\omega)$ may depend on v, see also Remark 3.8 below.

The proof of Lemma 3.4 relies on the two following lemmas, which relate the speed of convergence of characteristic functions given in Theorem 2.5 to more classical distances on the space of measures. The first lemma compares the Kolmogorov distance and the so-called smooth Wasserstein distance.

Lemma 3.5. *Given two random variables X, Y, and $\alpha \in \mathbb{N}$, we set*

$$\mathrm{Wass}_{(\alpha)}(X,Y) := \sup\left\{\mathbb{E}[|\phi(X)-\phi(Y)|] \,\Big|\, \phi \in \mathcal{C}^\alpha(\mathbb{R}),\ \|\phi\|_\infty \le 1, \ldots, \|\phi^{(\alpha)}\|_\infty \le 1\right\},$$

and

$$\mathrm{Kol}(X,Y) := \sup_{t\in\mathbb{R}} |\mathbb{P}(X \le t) - \mathbb{P}(Y \le t)|.$$

If Y has a density bounded by M, there is a constant C depending only on M and α such that :

$$\mathrm{Kol}(X,Y) \le \min\left(1, C\ \mathrm{Wass}_{(\alpha)}(X,Y)^{\frac{1}{\alpha+1}}\right).$$

Proof. Fix some $t \in \mathbb{R}$. Let $0 < \varepsilon < 1$, and consider $\varphi \in \mathcal{C}^\alpha(\mathbb{R})$ a nonincreasing function such that

$$\varphi(x) = \begin{cases} 1 & \text{if } x \le 0 \\ 0 & \text{if } x \ge 1 \end{cases}.$$

Define $\varphi_\varepsilon : x \mapsto \varphi((x-t)/\varepsilon)$, which is an upper $\mathcal{C}^\alpha$ approximation of $\mathbb{1}_{]-\infty,t]}$. Then

$$\mathbb{P}(X \le t) - \mathbb{P}(Y \le t) \le (\mathbb{E}[\varphi_\varepsilon(X)] - \mathbb{E}[\varphi_\varepsilon(Y)]) + (\mathbb{E}[\varphi_\varepsilon(Y)] - \mathbb{P}(Y \le t)).$$

For the first term, observe that $\|\varphi_\varepsilon^{(k)}\|_\infty = \varepsilon^{-k}\|\varphi^{(k)}\|_\infty$, and thus there is a constant C such that

$$\mathbb{E}[\varphi_\varepsilon(X)] - \mathbb{E}[\varphi_\varepsilon(Y)] \le \frac{C}{\varepsilon^\alpha}\,\mathrm{Wass}_{(\alpha)}(X,Y).$$

For the second term,

$$\mathbb{E}[\varphi_\varepsilon(Y)] - \mathbb{P}(Y \le t) \le M\varepsilon,$$

We can make the same computations with a lower $\mathcal{C}^\alpha$ approximation of $\mathbb{1}_{]-\infty,t]}$, which gives a similar lower bound on the quantity $\mathbb{P}(X \le t) - \mathbb{P}(Y \le t)$. Optimizing in ε we obtain the desired bound. ∎

The second lemma relates the smooth Wasserstein distance and the rate of convergence of characteristic functions. A general form of the theorem can be found in [7], but we will sketch the proof here for completeness.

Lemma 3.6. *Let $(X_n)_{n\ge 0}$ a sequence of random variables converging in distribution towards a random variable X. Assume that for some exponents $m \in \mathbb{N}$ and $\alpha \in \mathbb{R}_+$ there is a constant C such that*

$$\left|\mathbb{E}\left[e^{itX_n}\right] - \mathbb{E}\left[e^{itX}\right]\right| \le C\frac{1+|t|^m}{n^\alpha},$$

and for some exponent $\beta > 0$:

$$\sup_{n\in\mathbb{N}} \mathbb{E}[|X_n|^\beta] < +\infty.$$

Then there is a constant C depending on m, α, β such that :

$$\mathrm{Wass}_{(m+1)}(X_n, X) \leq C n^{-\frac{2\alpha\beta}{2\beta+1}}.$$

Proof. Let ϕ be a function in $\mathcal{S}(\mathbb{R})$, supported on the compact $[-(M+1), M+1]$. Using Plancherel isometry we have (the constant C may change from line to line)

$$\begin{aligned}
|\mathbb{E}[\phi(X_n) - \phi(X)]| &\leq \frac{1}{2\pi} \int_{\mathbb{R}} \left| \mathbb{E}\left[e^{itX_n}\right] - \mathbb{E}\left[e^{itX}\right] \right| |\widehat{\phi}(t)| \mathrm{d}t \\
&\leq \frac{C}{n^\alpha} \int_{\mathbb{R}} (1 + |t|^m) |\widehat{\phi}(t)| \mathrm{d}t \\
&\leq \frac{C}{n^\alpha} \int_{\mathbb{R}} (1 + |t|^{m+1}) |\widehat{\phi}(t)| \frac{1}{1+|t|} \mathrm{d}t \\
&\leq \frac{C}{n^\alpha} \int_{\mathbb{R}} |\widehat{\phi}(t)| \frac{\mathrm{d}t}{1+|t|} \quad + \quad \frac{C}{n^\alpha} \int_{\mathbb{R}} |t|^{m+1} |\widehat{\phi}(t)| \frac{\mathrm{d}t}{1+|t|} \\
&\leq \frac{C}{n^\alpha} \sqrt{\int_{\mathbb{R}} |\widehat{\phi}(t)|^2 \mathrm{d}t} \quad + \quad \frac{C}{n^\alpha} \sqrt{\int_{\mathbb{R}} |t|^{2m+2} |\widehat{\phi}(t)|^2 \mathrm{d}t} \quad \text{Jensen} \\
&\leq \frac{C}{n^\alpha} \|\phi\|_2 + \frac{C}{n^\alpha} \|\phi^{(m+1)}\|_2 \qquad \text{Plancherel} \\
&\leq C \frac{\sqrt{M+1}}{n^\alpha} \left(\|\phi\|_\infty + \|\phi^{(m+1)}\|_\infty \right).
\end{aligned} \tag{22}$$

By standard approximation argument, the inequality is true for every $\phi \in \mathcal{C}^{m+1}(\mathbb{R})$ with support in $[-(M+1), M+1]$. Suppose now that ϕ does not have compact support. Let χ_M a smooth function with support in $[-(M+1), M+1]$ such that $\chi_M = 1$ on $[-M, M]$. Set $\phi_M = \phi.\chi_M$. We write

$$|\mathbb{E}[\phi(X_n) - \phi(X)]| \leq |\mathbb{E}[\phi_M(X_n) - \phi_M(X)]| + \mathbb{P}(X_n > M) + \mathbb{P}(X > M).$$

From inequality (22) and Markov inequality applied to the function $x \mapsto |x|^\beta$,

$$|\mathbb{E}[\phi(X_n) - \phi(X)]| \leq C \frac{\sqrt{M+1}}{n^\alpha} \left(\|\phi_M\|_\infty + \|\phi_M^{(m+1)}\|_\infty \right) + \frac{C}{M^\beta}.$$

Using Leibniz rule, we have

$$\|\phi_M\|_\infty \leq \|\phi\|_\infty \qquad \text{and} \qquad \|\phi_M^{(m+1)}\|_\infty \leq C_m \sup_{k \leq m+1} \|\phi^{(k)}\|_\infty.$$

Choosing

$$M = \left(\frac{n^\alpha}{\sup_{k \leq m+1} \|\phi^{(k)}\|_\infty} \right)^{\frac{1}{\beta + \frac{1}{2}}}$$

and under the requirement that $M > 1$, we obtain

$$|\mathbb{E}[\phi(X_n) - \phi(X)]| \leq C n^{-\frac{2\alpha\beta}{2\beta+1}} \left(\sup_{k \leq m+1} \|\phi^{(k)}\|_\infty \right)^{\frac{2\beta}{2\beta+1}},$$

from which it follows that

$$\mathrm{Wass}_{(m+1)}(X_n, X) \leq C n^{-\frac{2\alpha\beta}{2\beta+1}}.$$

□

Remark 3.7. Denote $\mathrm{Wass}^X_{(\alpha)}$ is the smooth Wasserstein distance under $\mathbb{P}_X$, and let N be a standard Gaussian random variable. Lemma 3.6 and the rate of convergence given by Theorem 2.5 imply that for every $\varepsilon > 0$ the existence of a constant $C(\omega)$ independent of $v \in B$, such that

$$\mathrm{Wass}^X_{(4)}(g^X_\lambda(v), N) \leq \frac{C(\omega)}{\lambda^{\frac{1}{4}-\varepsilon}}.$$

The moment condition is satisfied for every $\beta > 0$ and uniformly in $v \in B$, by Sobolev injection and Lemma 2.7.

We are now in position to give the proof of Lemma 3.4 on the negative moment of the random field g_λ.

Proof of Lemma 3.4 . We define $\phi : x \mapsto |x|^{-\nu}$. Let ϕ_M be a $\mathcal{C}^\infty(\mathbb{R})$ approximation of ϕ, which coincide on $\mathbb{R} \setminus [-\frac{1}{M}, \frac{1}{M}]$. We can choose the function ϕ_M such that for all $p \in \mathbb{N}$, $|\phi_M^{(p)}\|_\infty \leq C_p M^{\nu+p}$ (see Figure 1).

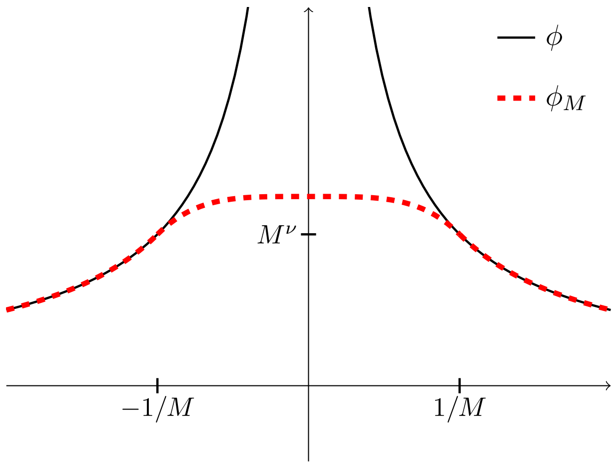

Figure 1: The functions ϕ and ϕ_M.

Let N be a standard Gaussian random variable under $\mathbb{P}_X$. We write

$$\begin{aligned}\mathbb{E}_X\left[|g^X_\lambda(v)|^{-\nu} - |N|^{-\nu}\right] &= \underbrace{\mathbb{E}_X[\phi(g^X_\lambda(v)) - \phi_M(g^X_\lambda(v))]}_{\Delta_1} + \underbrace{\mathbb{E}_X[\phi_M(g^X_\lambda(v)) - \phi_M(N)]}_{\Delta_2} \\ &+ \underbrace{\mathbb{E}_X[\phi(N) - \phi_M(N)]}_{\Delta_3}.\end{aligned}$$

For the term Δ_3, we use Cauchy–Schwarz inequality to obtain

$$\mathbb{E}_X[\phi(N) - \phi_M(N)] \leq \mathbb{E}_X\left[(\phi - \phi_M)(N)\mathbb{1}_{|N| \leq \frac{1}{M}}\right] \leq \sqrt{\frac{\mathbb{E}_X[|N|^{-2\nu}]}{M}} = \frac{C}{\sqrt{M}}. \tag{23}$$

For the term Δ_2, we use the smooth Wasserstein estimate in Lemma 3.6 and Remark 3.7. We have

$$|\mathbb{E}_X[\phi_M(g^X_\lambda(v)) - \phi_M(N)]| \leq \max_{p \leq 4} \|\phi^{(p)}_M\|_\infty \, \mathrm{Wass}_{(4)}(g^X_\lambda, N) \leq C\frac{M^{\nu+4}}{\lambda^{\frac{1}{4}-\varepsilon}}. \tag{24}$$

For the more difficult term Δ_1, we use Cauchy–Schwarz inequality to obtain

$$\mathbb{E}_X[\phi(g^X_\lambda(v)) - \phi_M(g^X_\lambda(v))] \leq \sqrt{\mathbb{E}_X[|g^X_\lambda(v)|^{-2\nu}]}\,.\sqrt{\mathbb{P}_X\left(|g^X_\lambda(v)| < \frac{1}{M}\right)}. \tag{25}$$

For the right-hand term, using Kolmogorov distance and Lemma 3.5 we have

$$\begin{aligned}\mathbb{P}_X\left(|g_\lambda^X(v)| < \frac{1}{M}\right) &\le \mathbb{P}\left(|N| < \frac{1}{M}\right) + 2\,\mathrm{Kol}(g_\lambda^X(v), N)\\ &\le \frac{C}{M} + C\,\mathrm{Wass}_{(4)}(g_\lambda^X(v), N)^{\frac{1}{5}}\\ &\le \frac{C}{M} + \frac{C}{\lambda^{\frac{1}{20}-\varepsilon}}.\end{aligned}$$

For the right-hand term we fix $\theta = \nu d + \varepsilon$ with $\varepsilon > 0$, and

$$p = \frac{1}{2\nu + \frac{\varepsilon}{d}}.$$

The exponent p satisfies

$$2\nu p < 1 \qquad \text{and} \qquad 2\theta p > d.$$

We compute

$$\begin{aligned}\mathbb{P}_a\left(\mathbb{E}_X[|g_\lambda^X(v)|^{2\nu}| > \lambda^{2\theta}\right) &\le \frac{\mathbb{E}_a\left[\mathbb{E}_X[|g_\lambda^X(v)|^{-2\nu}]^p\right]}{\lambda^{2p\theta}}\\ &\le \frac{\mathbb{E}_X\mathbb{E}_a[|g_\lambda^X(v)|^{-2\nu p}]}{\lambda^{2p\theta}}.\end{aligned}$$

Recall that g_λ is a Gaussian variable under $\mathbb{P}_a$, whose variance approaches 1 uniformly in X and v. Since $2\nu p < 1$ we obtain

$$\mathbb{P}_a\left(\mathbb{E}_X[|g_{\lambda_n}^X(v)|^{-2\nu}] > \lambda_n^{2\theta}\right) \le C\frac{\mathbb{E}_X\left[\mathbb{E}_a\left[(g_{\lambda_n}^X(v))^2\right]^{-\nu p}\right]}{\lambda_n^{2\theta p}} \le \frac{C}{\lambda_n^{2\theta p}}.$$

Since $\lambda_n \simeq Cn^{1/d}$ the left-hand side is summable and Borel–Cantelli lemma asserts the existence of a constant $C(v,\omega)$ such that

$$\mathbb{E}_X[|g_{\lambda_n}^X(v)|^{-2\nu}] \le C(v,\omega)\lambda_n^{2\theta}.$$

Finally, bounding the terms in (25) we obtain

$$\mathbb{E}_X[\phi(g_\lambda^X(v)) - \phi_M(g_\lambda^X(v))] \le C(v,\omega)\lambda^{d\nu+\varepsilon}\sqrt{\frac{1}{M} + \frac{1}{\lambda^{\frac{1}{20}-\varepsilon}}}. \tag{26}$$

Adding the bounds on Δ_1, Δ_2 and Δ_3 given by the expressions (23), (24) and (26), we obtain the following bound:

$$\mathbb{E}_X\left[|g_\lambda^X(v)|^{-\nu}\right] \le \mathbb{E}_X[|N|^{-\nu}] + \frac{C}{\sqrt{M}} + C\frac{M^{\nu+4}}{\lambda^{\frac{1}{4}-\varepsilon}} + C(v,\omega)\lambda^{d\nu+\varepsilon}\sqrt{\frac{1}{M} + \frac{1}{\lambda^{\frac{1}{20}-\varepsilon}}}.$$

We choose $\nu < \frac{1}{40d}$, and $M = \lambda^{1/20}$. Since a Gaussian random variable has bounded negative moments for exponents $\nu > -1$, we deduce

$$\sup_{\lambda>0}\mathbb{E}_X\left[|g_\lambda^X(v)|^{-\nu}\right] \le C(v,\omega). \tag{27}$$

It remains to prove the second technical part of Lemma 3.4. We cannot directly apply the first bound since the constant obtained in (27) may depend on v. Mimicking the previous computation, we write

$$\int_1^{+\infty}\frac{1}{t^{\alpha+2}}\sum_{i=0}^{\lfloor t^\alpha\rfloor}\mathbb{E}_X[|g_\lambda^X(v_i)|^{-\nu}]\,\mathrm{d}t = \Delta_1 + \Delta_2 + \Delta_3.$$

Estimates (23) and (24) for Δ_1 and Δ_2 remain unchanged. For the quantity Δ_3, we keep the previous notations. We have, using Markov inequality in the first line, and Hölder inequality in the second line,

$$
\begin{aligned}
\mathbb{P}_a\left(\int_1^{+\infty}\frac{1}{t^{1+\alpha+\varepsilon}}\sum_{i=0}^{\lfloor t^\alpha\rfloor}\mathbb{E}_X[|g_\lambda^X(v_i)|^{-2\nu}]\,\mathrm{d}t>\lambda^{2\theta}\right) &\leq \frac{1}{\lambda^{2p\theta}}\mathbb{E}_a\left[\left(\int_1^{+\infty}\frac{1}{t^\alpha}\sum_{i=0}^{\lfloor t^\alpha\rfloor}\mathbb{E}_X[|g_\lambda^X(v_i)|^{-2\nu}]\,\frac{\mathrm{d}t}{t^{1+\varepsilon}}\right)^p\right]\\
&\leq \frac{C}{\lambda^{2p\theta}}\int_1^{+\infty}\frac{\lfloor t^\alpha\rfloor^{p-1}}{t^{p\alpha+1+\varepsilon}}\sum_{i=0}^{\lfloor t^\alpha\rfloor}\mathbb{E}_X[\mathbb{E}_a[|g_\lambda^X(v_i)|^{-2\nu p}]\,\mathrm{d}t\\
&\leq \frac{C}{\lambda^{2p\theta}}\int_1^{+\infty}\frac{\lfloor t^\alpha\rfloor^{p}}{t^{p\alpha+1+\varepsilon}}\mathrm{d}t\\
&\leq \frac{C}{\lambda^{2p\theta}}.
\end{aligned}
$$

The end of the proof remains unchanged. □

Remark 3.8. The dependence in v of the constant $C(v,\omega)$ given in Equation (27) is not entirely satisfactory, and is a consequence of Borel–Cantelli lemma in Equation (26). We were not able to give a bound on the quantity

$$\mathbb{E}_a\left[\sup_{v\in B}\mathbb{E}_X\left[|g_\lambda^X(v)|^{-\nu}\right]\right].$$

It does not impact the rest of the article since the second part of Lemma 3.6 suffices to carry on our computations, but let us give a little more insight about what happens from a measure-theoretic point of view.

The Sobolev trick we used before to obtain the uniformity on v does not apply here due to the lack of regularity of the function $x\mapsto |x|^{-\nu}$. Nevertheless it may happen in particular cases that we can recover uniformity. If we are on a torus $\mathbb{T}^d$ endowed with any flat metric, we can choose for the isometry I_x the canonical embedding into $\mathbb{R}^d$ and the mapping Φ_x is the usual sum. If X is a uniform random variable on $\mathbb{T}^d$, then so is $X+v$ for any $v\in\mathbb{T}^d$. It follows that under $\mathbb{P}_X$ and for all $v,v'\in\mathbb{R}^d$,

$$g_\lambda^X(v)\overset{\mathcal{L}}{=}g_\lambda^X(v'),$$

and quantities such as $\mathbb{E}_X\left[|g_\lambda^X(v)|^{-\nu p}\right]$ do not depend on v, which gives the uniformity in v. Denote by μ_v the pushforward of the measure μ under the mapping $x\mapsto\Phi_x(v)$. For all $f\in\mathcal{C}^0(\mathcal{M})$,

$$\int_{\mathcal{M}} f(\Phi_x(v))\mathrm{d}\mu(x)=\int_{\mathcal{M}} f(x)\mathrm{d}\mu_v(x).$$

In the torus case, μ_v is the canonical measure and does not depend on the parameter v. In all generality, few can be said about μ_v. It does not always admit a density with respect to the Riemannian measure since the function $x\mapsto\Phi_x(v)$ may have support on a 1-dimensional subspace for an ill-chosen choice of isometries $(I_x)_{x\in\mathcal{M}}$. Nevertheless, if the measure μ_v has a density h_v belonging to $L^p(\mathcal{M})$ space for some $p>1$ and uniformly on $v\in B$, then

$$
\begin{aligned}
\int_{\mathcal{M}}|g_\lambda^X(v)|^{-\nu}\mathrm{d}\mu(x) &= \int_{\mathcal{M}}|f_\lambda(x)|^{-\nu}h_{\frac{v}{\lambda}}(x)\mathrm{d}\mu(x)\\
&\leq\left(\int_{\mathcal{M}}|f_\lambda(x)|^{-\nu q}\right)^{1/q}\sup_{v\in B}\|h_v\|_p\qquad\text{with }\frac{1}{p}+\frac{1}{q}=1.
\end{aligned}
$$

Taking the expectation under $\mathbb{P}_a$, and choosing $\nu<\frac{1}{q}$, we obtain

$$\mathbb{E}_a\left[\sup_{v\in B}\mathbb{E}_X\left[|g_\lambda^X(v)|^{-\nu}\right]\right]\leq\sup_{v\in B}\|h_v\|_p\int_{\mathcal{M}}\mathbb{E}_a\left[|f_\lambda(x)|^{-\nu q}\right]^{1/q}\mathrm{d}x<+\infty.$$

At last, given a smooth compact Riemannian manifold $\mathcal{M}$, it is always possible to construct a family of isometries $(I_x)_{x\in\mathcal{M}}$ such that the family $(\mu_v)_{v\in B}$ has a density uniformly bounded in L^∞.

3.4 Uniform moment estimates for the nodal volume

In order to complete the proof of the uniform integrability of nodal volume, we now introduce a geometric lemma which relates the nodal volume of a function to the number of zero of this function on a straight line passing through predefined points. It is a variant of the Crofton formula (see [1] for a general presentation of the various Crofton formulæ), and a d-dimensional extension of [3, Thm. 6].

Lemma 3.9. *Let E be a $\mathcal{C}^2$-hypersurface in $\mathbb{R}^d$, intersecting the cube $D=[0,a]^d$. Assume that E has bounded curvature on D. Then there exists a segment S passing through one of the vertices of the cube D and such that*

$$\mathrm{Card}(E\cap S\cap D)\geq c\,\frac{\mathcal{H}^{d-1}(E\cap D)}{a^{d-1}},\quad \textit{with}\quad c=\frac{1}{2^{d+1}d}.$$

Proof. Both sides are dimensionless and it suffices to prove the assertion for $a=1$. We can assume that $\mathcal{H}^{d-1}(E\cap\partial D)=0$, else we could find a segment S passing through one of the vertices and such that $\mathcal{H}^1(E\cap S\cap\partial D)>0$, and in that case the result is true.

We will prove Lemma 3.9 by a probabilistic method. We denote $(A_j)_{1\leq j\leq 2^d}$ the vertices of the cube. Let P be a point chosen uniformly randomly on the cube $[0,1]^d$. Let (A_jP) be the random line passing through the points A_j and P. We will in fact prove that

$$\mathbb{E}\left[\sum_{j=1}^{2^d}\mathrm{Card}\left\{E\cap(A_jP)\cap D\right\}\right]\geq\frac{1}{2d}\mathcal{H}^{d-1}(E\cap D),\tag{28}$$

which implies the result, since for some realization of P and some j we must have

$$\mathrm{Card}(E\cap(A_jP)\cap D)\geq\frac{1}{2^{d+1}d}\mathcal{H}^{d-1}(E\cap D).$$

Since we assumed that $\mathcal{H}^{d-1}(E\cap\partial D)=0$ we can suppose that $E\subset\mathring{D}$. Since the manifold E has bounded curvature, it is a doubling space and Vitali–Lebesgue covering theorem (see [16, p. 4]) asserts that for all $r_0>0$, we can find a disjoint family of (relatively compact) geodesic balls $(E_{r_n})_{n\geq 0}$ in E such that the geodesic ball E_{r_n} has radius $r_n<r_0$, and such that

$$\mathcal{H}^{d-1}\left(E\setminus\left(\bigsqcup_{n\in\mathbb{N}}E_{r_n}\right)\right)=0.$$

By linearity of both sides of (28) and monotone convergence, it is sufficient to prove the inequality (28) by replacing E with E_r, a small (relatively compact) geodesic ball of radius $r<r_0$ centered at some point $x\in E$. For r sufficiently small, the geodesic ball E_r is comparable to a $\mathbb{R}^{d-1}$-ball. More precisely, set $B_r(x)=\exp_x^{-1}(E_r)$. Riemannian volume comparison theorems asserts that

$$\mathcal{H}^{d-1}(B_r(x))=\mathcal{H}^{d-1}(E_r)(1+o(r_0)),$$

and the estimate is uniform on E by the curvature bound assumption. We will prove that for some $j\in\{1,\dots,2^d\}$,

$$\mathbb{E}\left[\sum_{j=1}^{2^d}\mathrm{Card}\left\{E_r\cap(A_jP)\right\}\right]\geq\mathbb{E}\left[\mathrm{Card}\left\{E_r\cap(A_jP)\right\}\right]\geq\frac{1}{2d}\mathcal{H}^{d-1}(E_r).$$

Let $\mathbf{n}_x$ denote a normal unit vector at x. A little geometry shows that we can choose j such that

$$|\langle \overrightarrow{A_j x}, \mathbf{n}_x \rangle| \geq \frac{1}{2}. \tag{29}$$

For r_0 small enough and uniformly on E, the hypersurface E_r is almost flat and the line $(A_j P)$ has at most one point of intersection with E_r. The opposite would imply that for some $y \in E_r$ the line $(A_j y)$ is tangent to E_r at some point y. That is $\langle \overrightarrow{A_j y}, \mathbf{n}_y \rangle = 0$. But it contradicts the inequality (29) and the continuity on E_r of the mapping

$$x \mapsto |\langle \overrightarrow{A_j x}, \mathbf{n}_x \rangle|.$$

The uniformity of E comes from the fact that the modulus of continuity of this application is controlled by the curvature of E. We deduce

$$\mathbb{E}\left[\text{Card}\left\{E_r \cap (A_j P)\right\}\right] = \mathbb{P}\left(\text{Card}\left\{E_r \cap (A_j P)\right\} \neq \emptyset\right).$$

Uniformly on x in E, we can find $r' = r + o(r_0)$ such that every line that passes through A_j and intersects the $d-1$-dimensional ball $B_{r'}(x) \subset T_x E$, also passes through E_r. Indeed, the central projection of E_r onto $T_x E$ with center of projection A_j is almost a $\mathbb{R}^{d-1}$-ball and must contain a ball of radius $r' = r + o(r_0)$ (see Figure 2).

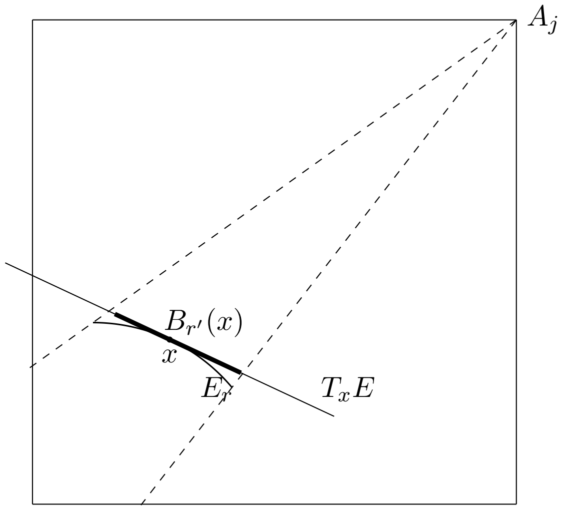

Figure 2: Construction of $B_{r'}(x)$.

We deduce

$$\mathbb{P}\left(\text{Card}\left\{E_r \cap (A_j P)\right\} \neq \emptyset\right) \geq \mathbb{P}\left(\text{Card}\left\{B_{r'}(x) \cap (A_j P)\right\} \neq \emptyset\right).$$

But the right hand side is easy to estimate. It is the volume of the cone in $[0,1]^d$, based at A_j and generated by the ball $B_{r'}(x)$. The formula base $\times$ height$/d$ gives

$$\begin{aligned}
\mathbb{P}\left(\text{Card}\left\{B_{r'}(x) \cap (A_j P)\right\} \neq \emptyset\right) &\geq \frac{\mathcal{H}^{d-1}(B_{r'}(x))}{d} |\langle \overrightarrow{A_j x}, \mathbf{n}_x \rangle| \\
&\geq \frac{1}{2d} \mathcal{H}^{d-1}(B_r(x))(1 + o(r_0)) \\
&\geq \frac{1}{2d} \mathcal{H}^{d-1}(E_r)(1 + o(r_0)).
\end{aligned}$$

Patching up the above estimates we recover

$$\mathbb{E}\left[\sum_{j=1}^{2^d} \text{Card}\left\{E \cap (A_j P) \cap D\right\}\right] \geq \frac{1}{2d} \mathcal{H}^{d-1}(E \cap D)(1 + o(r)),$$

and letting r go to zero we deduce the result. $\square$

Remark 3.10. If $g : \mathbb{R}^d \to \mathbb{R}$ is a smooth function and 0 is a regular value of g, then $g^{-1}(\{0\})$ is a smooth manifold and we can apply Lemma 3.9 to deduce the existence of a segment S passing through one of the vertices of the cube $D = [0, a]^d$ and such that

$$\mathrm{Card}(\{g = 0\} \cap S \cap D) \geq c\, \frac{\mathcal{H}^{d-1}(\{g = 0\} \cap D)}{a^{d-1}}. \tag{30}$$

Denote g_S its restriction on S, and suppose that g_S cancels at least p times at points $\mathbf{w}_1, \ldots, \mathbf{w}_p$. By the generalized Rolle lemma, for all $v \in S$, there exists a point c_v in S such that

$$|g(v)| = \frac{\prod_{j=1}^p \|v - \mathbf{w}_j\|}{p!} \left| g_S^{(p)}(c_v) \right|.$$

Hence if the segment S passes through the vertex v_j on the cube D then

$$\begin{aligned} |g(v_j)| &\leq \frac{\mathcal{H}^1(S)^p}{p!} \sum_{|\alpha|=p} \|\partial_\alpha g\|_\infty \\ &\leq C a^p \sum_{|\alpha|=p} \|\partial_\alpha g\|_\infty. \end{aligned}$$

To sum up, if we have

$$\mathcal{H}^{d-1}(\{g = 0\} \cap D) \geq \frac{a^{d-1}}{c} p,$$

then for at least one of the vertices v_j of the cube,

$$|g(v_j)| \leq C a^p \sum_{|\alpha|=p} \|\partial_\alpha g\|_\infty.$$

In [4, Thm. 5.2] the authors proved the finiteness of moments of nodal volume under the requirement of joint bounded density of k first derivatives. This hypothesis is too strong for our purpose, since our process under $\mathbb{P}_X$ depends only on the randomness of X and we cannot expect a joint bounded density of the first derivatives.

Theorem 3.11. *$\mathbb{P}_a$-almost surely, the family of random variables $(Z_\lambda)_{\lambda>0}$ is uniformly integrable. More precisely, for all $\gamma > 0$,*

$$\sup_{\lambda>0} \mathbb{E}_X[Z_\lambda^{1+\gamma}] \leq C_\gamma(\omega).$$

Conjointly with the convergence in distribution of the nodal volume, it implies the convergence of all moments of Z_λ to those of Z_∞.

Proof of Theorem 3.11. For all $A > 0$ (to be fixed later),

$$\begin{aligned} \mathbb{E}_X[Z_\lambda^{1+\gamma}] &= (1+\gamma) \int_0^{+\infty} t^\gamma \mathbb{P}_X(Z_\lambda > t) \mathrm{d}t \\ &\leq C_A + (1+\gamma) \int_A^{+\infty} t^\gamma \mathbb{P}_X(Z_\lambda > t) \mathrm{d}t, \end{aligned} \tag{31}$$

hence we need to estimate the quantity $\mathbb{P}_X(Z_\lambda > t)$ for all t greater than some constant A. Up to embedding the ball B in a cube we will consider that the vector v lives in a hypercube (of size 1 for simplicity).

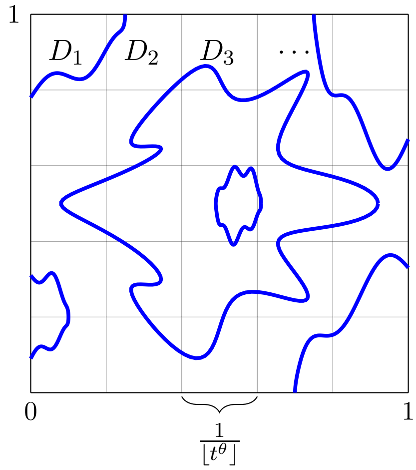

Figure 3: The grid defined on $[0,1]^d$.

Consider a rectangular grid on $[0,1]^d$ of size $\frac{1}{\lfloor t^\theta \rceil}$ with $\theta = 1-\varepsilon$. The hypercube $[0,1]^d$ is split into $\lfloor t^\theta \rceil^d$ smaller cubes, which we number by $(D_i)_{1\le i\le \lfloor t^\theta \rceil^d}$ (see Figure 3).
Let $(v_j)_{1\le j\le \lceil t^\theta \rceil^d}$ be the vertices of the grid. For all $i \in \{1,\dots,\lfloor t^\theta \rceil^d\}$, let $(v_{ij})_{1\le j\le 2^d}$ be the vertices of the i-th cube, and

$$Z_\lambda^{(i)} := \mathcal{H}^{d-1}(\{g_\lambda^X = 0\}) \cap D_i),$$

be the volume of zeros contained in the i-th cube. By the pigeonhole principle,

$$\{Z_\lambda > t\} \subset \bigcup_{i=1}^{\lfloor t^\theta \rceil^d} \left\{ Z_\lambda^{(i)} > \frac{t}{\lfloor t^\theta \rceil^d} \right\}. \tag{32}$$

Let p be an integer to be fixed later. We use Lemma 3.9 and then Remark 3.10, keeping the same notations, to deduce that if $t \ge A$, with

$$A := \left(\frac{p}{c}\right)^{1/(1-\theta)}.$$

and $a = \lfloor t^\theta \rceil^{-1}$, then

$$\left\{ Z_\lambda^{(i)} > \frac{t}{\lfloor t^\theta \rceil^d} \right\} \subset \left\{ Z_\lambda^{(i)} > \frac{p}{c\lfloor t^\theta \rceil^{d-1}} \right\} \subset \bigcup_{j=1}^{2^d} \left\{ |g_\lambda^X(v_{ij})| \le \frac{C}{\lfloor t^\theta \rceil^p} \sum_{|\alpha|=p} \|\partial_\alpha g_\lambda^X\|_\infty \right\}.$$

Fix $k \ge 1$. Taking the expectation with respect to $\mathbb{P}_X$ we obtain

$$\begin{aligned}
\mathbb{P}_X\left(Z_\lambda^{(i)} > \frac{t}{\lfloor t^\theta \rceil^d} \right) &\le \sum_{j=1}^{2^d} \mathbb{P}_X\left(|g_\lambda(v_{ij})| \le \frac{C}{\lfloor t^\theta \rceil^p} \sum_{|\alpha|=p} \|\partial_\alpha g_\lambda^X\|_\infty \right) \\
&\le \sum_{j=1}^{2^d} \mathbb{P}_X\left(|g_\lambda^X(v_{ij})| \le \frac{t^\varepsilon}{\lfloor t^\theta \rceil^p} \right) + 2^d \mathbb{P}_X\left(C \sum_{|\alpha|=p} \|\partial_\alpha g_\lambda^X\|_\infty > t^\varepsilon \right) \\
&\le \left(\frac{t^\varepsilon}{\lfloor t^\theta \rceil^p}\right)^\nu \sum_{j=1}^{2^d} \mathbb{E}_X[|g_\lambda^X(v_{ij})|^{-\nu}] + C\frac{\mathbb{E}_X\left[\sum_{|\alpha|=p} \|\partial_\alpha g_\lambda^X\|_\infty^k\right]}{t^{k\varepsilon}} \qquad (33) \\
&\le \left(\frac{t^\varepsilon}{\lfloor t^\theta \rceil^p}\right)^\nu \sum_{j=1}^{2^d} \mathbb{E}_X[|g_\lambda^X(v_{ij})|^{-\nu}] + \frac{C_k(\omega)}{t^{k\varepsilon}}.
\end{aligned}$$

In the last line we used the Sobolev injection and estimate of Lemma 2.7 to bound the right hand side. Taking the expectation in expression (32) and using the union bound we obtain

$$\mathbb{P}_X(Z_\lambda > t) \le C_k(\omega)\frac{\lfloor t^\theta \rfloor^d}{t^{k\varepsilon}} + \left(\frac{t^\varepsilon}{\lfloor t^\theta \rfloor^p}\right)^\nu 2^d \sum_{i=1}^{\lceil t^\theta \rceil^d} \mathbb{E}_X[|g_\lambda^X(v_i)|^{-\nu}].$$

Recalling expression (31) we obtain

$$\mathbb{E}_X[Z_\lambda^{1+\gamma}] \le C + C_k(\omega)\int_A^{+\infty} t^\gamma \frac{\lfloor t^\theta \rfloor^d}{t^{k\varepsilon}}\mathrm{d}t + C\int_A^{+\infty} t^\gamma \left(\frac{t^\varepsilon}{\lfloor t^\theta \rfloor^p}\right)^\nu \sum_{j=1}^{\lceil t^\theta \rceil^d} \mathbb{E}_X[|g_\lambda^X(v_i)|^{-\nu}]\mathrm{d}t.$$

Choosing k and p such that

$$k > \frac{\gamma + \theta d + 1}{\varepsilon} \qquad \text{and} \qquad \nu(p\theta - \varepsilon) - (d\theta + \gamma) > 1,$$

we can apply the second part of Lemma 3.4 to deduce the existence of a constant $C(\omega)$ such that

$$\mathbb{E}_X[Z_\lambda^{1+\gamma}] \le C(\omega).$$

$\square$

Remark 3.12. If we were in an analytic setting, we could use the same argument as the one in [3, Thm. 9], which roughly relies on the convergence of Taylor expansion of eigenfunctions. In the $\mathcal{C}^\infty$ setting we can only apply the generalized Rolle lemma with a fixed p, and it explains why we used the partitioning of the cube $[0,1]^d$. A careful analysis of the proof shows that it requires a manifold of finite regularity $\mathcal{C}^k$ for $k = 81d$ (the constant is far from optimal and a few improvements could have been made throughout the proof).

Corollary 3.13. *For all $p \ge 1$:*

$$\lim_{\lambda \to +\infty} \frac{\mathbb{E}_a\left[\left(\mathcal{H}^{d-1}(\{f_\lambda = 0\})\right)^p\right]}{\lambda^p} = \left(\frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{d+2}}\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\right)^p,$$

and

$$\lim_{\lambda \to +\infty} \frac{\mathbb{E}_a\left[\left(\mathcal{H}^{d-1}(\{\widetilde{f}_\lambda = 0\})\right)^p\right]}{\lambda^p} = \left(\frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{d}}\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\right)^p.$$

Proof. Passing from almost-sure convergence to convergence in expectation is a consequence of the dominated convergence. It suffices to show that

$$\sup_{\lambda > 0} \mathbb{E}_a\left[(\mathbb{E}_{\mathbf{X}}[Z_\lambda])^p\right] < +\infty,$$

which can bee seen by raising to the power p and taking the expectation under $\mathbb{P}_a$ in Equation (33). In more direct way, all the almost-sure estimates are deduced from Borel-Cantelli lemma and Markov inequality applied to the power function with arbitrary large exponent, and thus remain true under expectation. A similar argument holds for higher moments. $\square$

To conclude this paper, we emphasize that all our arguments are encoded into the local Weyl law in $\mathcal{C}^\infty$ topology, and decaying properties of the limit kernel $\mathcal{B}_d$ (or $\mathcal{S}_d$). There is no doubt that the approach taken here could be applied to similar settings.

A Proof of decorrelation estimates

In this first part of the Appendix, we give the proof of Lemma 2.3 and Theorem 2.5 stated in Section 2.

A.1 Proof of Lemma 2.3

Let $X_1, \ldots, X_{2q}$ be independents copies of X. The expectation with respect to the random variables $X_1, \ldots, X_{2q}$ will be noted $\mathbb{E}_{\mathbf{X}}$. To enhance the dependence with respect to X_k, we set for all $k \in \{1, \ldots, q\}$,

$$N_\lambda^{(k)} = \sum_{j=1}^{p} t_j g_\lambda^{X_k}(v_j),$$

and for all $k \in \{q+1, \ldots, 2q\}$,

$$N_\lambda^{(k)} = -\sum_{j=1}^{p} t_j g_\lambda^{X_k}(v_j).$$

Then, for $k \neq l$, applying Lemma 2.4 with $X = X_k$ and $Y = X_l$, we have uniformly in X_l

$$\mathbb{E}_{X_k}\left[\left|\mathbb{E}_a\left[N_\lambda^{(k)} N_\lambda^{(l)}\right]\right|\right] = \|t\|^2 O\left(\frac{\eta(\lambda)}{\lambda}\right). \tag{34}$$

The following lemma, based on an explicit computation of Gaussian characteristic functions and integral Taylor formula, gives an explicit expression of $\widetilde{\Delta}_\lambda^{(q)}$, which is the object of Lemma 2.3. For $s \in [0,1]^{2q}$, let

$$\begin{aligned} f(s) \quad &:= \sum_{k=1}^{2q} \mathbb{E}_a\left[\left(N_\lambda^{(k)}\right)^2\right] + \sum_{\substack{k,l=1\\k\neq l}}^{2q} s_k s_l \mathbb{E}_a\left[N_\lambda^{(k)} N_\lambda^{(l)}\right] \\ &= \mathbb{E}_a\left[\left(\sum_{k=1}^{2q} s_k N_\lambda^{(k)}\right)^2\right] + \sum_{k=1}^{2q}\left(1 - s_k^2\right)\mathbb{E}_a\left[\left(N_\lambda^{(k)}\right)^2\right] \end{aligned}.$$

Lemma A.1. *We have*

$$\widetilde{\Delta}_\lambda^{(q)} = \mathbb{E}_{\boldsymbol{X}}\left[\int_{[0,1]^{2q}} \partial_1 \ldots \partial_{2q}\left(\exp\left(-\frac{1}{2}f\right)\right)(s)\mathrm{d}s\right].$$

Proof of Lemma A.1. From mutual independence of the family $(X_1, \ldots, X_{2q})$,

$$\widetilde{\Delta}_\lambda^{(q)} = \mathbb{E}_{\mathbf{X}}\mathbb{E}_a\left[\prod_{k=1}^{2q}\left(e^{iN_\lambda^{(k)}} - \mathbb{E}_a\left[e^{iN_\lambda^{(k)}}\right]\right)\right].$$

We define

$$\widetilde{\Delta}_\lambda^{(q)}(s) := \mathbb{E}_{\mathbf{X}}\mathbb{E}_a\left[\left(\prod_{k=1}^{2q}\mathbb{E}_a\left[e^{i\sqrt{1-s_k^2}N_\lambda^{(k)}}\right]\right)\left(\prod_{k=1}^{2q}\left(e^{is_k N_\lambda^{(k)}} - \mathbb{E}_a\left[e^{is_k N_\lambda^{(k)}}\right]\right)\right)\right]. \tag{35}$$

Developing the product, and using the characteristic function of a Gaussian random variable, a direct computation shows that

$$\widetilde{\Delta}_\lambda^{(q)}(s) := \mathbb{E}_{\mathbf{X}}\left[\exp\left(-\frac{1}{2}\sum_{k=1}^{2q}\mathbb{E}_a\left[\left(N_\lambda^{(k)}\right)^2\right]\right)\sum_{A\subset\{1,\dots,2q\}}(-1)^{|A|}\exp\left(-\frac{1}{2}\sum_{\substack{k,l\in A\\ k\neq l}} s_k s_l \mathbb{E}_a\left[N_\lambda^{(k)}N_\lambda^{(l)}\right]\right)\right]. \tag{36}$$

If $s_k = 0$ for some $k\in\{1,\dots,2q\}$, then from expression (35),

$$\widetilde{\Delta}_\lambda^{(q)}(s) = 0.$$

In other words, the function $s\mapsto \Delta_\lambda^{(q)}(s)$ cancels if one of its coordinates is zero. By integral Taylor formula,

$$\widetilde{\Delta}_\lambda^{(q)} = \widetilde{\Delta}_\lambda^{(q)}(1,\dots,1) = \int_{[0,1]^{2q}} \partial_1\dots\partial_{2q}\Delta_\lambda^{(q)}(s)\mathrm{d}s.$$

But from expression (36), the only term that depends on all coordinates (and thus won't be canceled after differentiation) is the term corresponding to $A=\{1,\dots,2q\}$, which is

$$\mathbb{E}_{\mathbf{X}}\left[\exp\left(-\frac{1}{2}\sum_{k=1}^{2q}\mathbb{E}_a\left[\left(N_\lambda^{(k)}\right)^2\right]\right)\exp\left(-\frac{1}{2}\sum_{\substack{k,l=1\\ k\neq l}}^{2q} s_k s_l \mathbb{E}_a\left[N_\lambda^{(k)}N_\lambda^{(l)}\right]\right)\right] = \mathbb{E}_{\mathbf{X}}\left[\exp\left(-\frac{1}{2}f(s)\right)\right].$$

□

Now, for a set A, denote $\Pi(A)$ the collection of partitions of A into groups of two elements. A direct computation shows that

$$\partial_1\dots\partial_{2q}\left(\exp\left(-\frac{1}{2}f\right)\right) = \exp\left(-\frac{1}{2}f\right)\sum_{\substack{A\subset\{1,\dots,2q\}\\ |A|\text{ even}}}(-1)^{\frac{|A|}{2}}\left(\sum_{B\in\Pi(A)}\prod_{(k,l)\in B}\mathbb{E}_a\left[N_\lambda^{(k)}N_\lambda^{(l)}\right]\right)$$
$$\times\prod_{k\in A^c}\left(\sum_{\substack{l=1\\ k\neq l}}^{2q} s_l\mathbb{E}_a\left[N_\lambda^{(k)}N_\lambda^{(l)}\right]\right).$$

We deduce, using the mutual independence of the family $X_1,\dots,X_{2q}$ and the estimate 34,

$$\begin{aligned}
\widetilde{\Delta}_\lambda^{(q)} &\le C\mathbb{E}_{\mathbf{X}}\left[\sum_{\substack{A\subset\{1,\dots,2q\}\\ |A|\text{ even}}}\left(\sum_{B\in\Pi(A)}\prod_{(k,l)\in B}\left|\mathbb{E}_a\left[N_\lambda^{(k)}N_\lambda^{(l)}\right]\right|\right)\prod_{k\in A^c}\left(\sum_{\substack{l=1\\ k\neq l}}^{2q}\left|\mathbb{E}_a\left[N_\lambda^{(k)}N_\lambda^{(l)}\right]\right|\right)\right]\\
&\le C\sum_{\substack{A\subset\{1,\dots,2q\}\\ |A|\text{ even}}}\left(\sum_{B\in\Pi(A)}\prod_{(k,l)\in B}\mathbb{E}_{\mathbf{X}}\left[\left|\mathbb{E}_a\left[N_\lambda^{(k)}N_\lambda^{(l)}\right]\right|\right]\right)\mathbb{E}_{\mathbf{X}}\left[\prod_{k\in A^c}\left(\sum_{\substack{l=1\\ k\neq l}}^{2q}\left|\mathbb{E}_a\left[N_\lambda^{(k)}N_\lambda^{(l)}\right]\right|\right)\right]\\
&\le C\sum_{\substack{A\subset\{1,\dots,2q\}\\ |A|\text{ even}}}\left(\|t\|^2\frac{\eta(\lambda)}{\lambda}\right)^{\frac{|A|}{2}}\mathbb{E}_{\mathbf{X}}\left[\prod_{k\in A^c}\left(\sum_{\substack{l=1\\ k\neq l}}^{2q}\left|\mathbb{E}_a\left[N_\lambda^{(k)}N_\lambda^{(l)}\right]\right|\right)\right].
\end{aligned} \tag{37}$$

To give a bound on the right-hand term and thus establish Lemma 2.3, we use the following lemma whose proof again relies on the decorrelation estimates of Lemma 2.4.

Lemma A.2. *There is a constant C depending only on $\mathcal{M}$, K and q such that*

$$\mathbb{E}_{\boldsymbol{X}}\left[\prod_{k\in A^c}\left(\sum_{\substack{l=1\\ l\neq k}}^{2q}\left|\mathbb{E}_a\left[N_\lambda^{(k)}N_\lambda^{(l)}\right]\right|\right)\right]\leq C(1+\|t\|^4)^{q-\frac{|A|}{2}}\left(\frac{\eta(\lambda)}{\lambda}\right)^{q-\frac{|A|}{2}}.$$

Proof of Lemma A.2. Assume without loss of generality that $A^c=\{1,\dots,2m\}$. We compute

$$\mathbb{E}_{\mathbf{X}}\left[\prod_{k=1}^{2m}\left(\sum_{\substack{l=1\\ k\neq l}}^{2q}\left|\mathbb{E}_a\left[N_\lambda^{(k)}N_\lambda^{(l)}\right]\right|\right)\right]=\sum_{\substack{l_1,\dots,l_{2m}=1\\ l_k\neq k}}^{2q}\underbrace{\mathbb{E}_{\mathbf{X}}\left[\prod_{k=1}^{2m}\left|\mathbb{E}_a\left[N_\lambda^{(k)}N_\lambda^{(l_k)}\right]\right|\right]}_{\Delta_\ell}.$$

Now fix $\ell=(l_1,\dots,l_{2m})$. Consider the following graph G_ℓ with vertices in $\{1,\dots,2q\}$: two vertices k and l are connected if the term $\left|\mathbb{E}_a\left[N_\lambda^{(k)}N_\lambda^{(l)}\right]\right|$ appears into the expression Δ_ℓ. If the graph G_ℓ is disconnected, we can use independence of the random variables $X_1,\dots X_{2q}$, and we are left to show the aforementioned bound for connected graphs. Thanks to Weyl law, there is a constant C such that

$$\left|\mathbb{E}_a\left[N_\lambda^{(k)}N_\lambda^{(l)}\right]\right|\leq C\|t\|^2,$$

and we can assume (up to bounding a one of the terms in the product) that G_ℓ is a tree (with $2m-1$ edges). Suppose without loss of generality that 1 is a leaf of the tree attached to 2. Then

$$\begin{aligned}\Delta_\ell&\leq C\|t\|^2\,\mathbb{E}_{X_2,\dots,X_{2q}}\left[\mathbb{E}_{X_1}\left[\left|\mathbb{E}_a\left[N_\lambda^{(1)}N_\lambda^{(2)}\right]\right|\right]\prod_{\substack{(k,l)\in G_\ell\\ (k,l)\neq(1,2)}}\left|\mathbb{E}_a\left[N_\lambda^{(k)}N_\lambda^{(l)}\right]\right|\right]\\&\leq C\|t\|^4 O\left(\frac{\eta(\lambda)}{\lambda}\right)\mathbb{E}_{X_2,\dots,X_{2q}}\left[\prod_{\substack{(k,l)\in G_\ell\\ (k,l)\neq(1,2)}}\left|\mathbb{E}_a\left[N_\lambda^{(k)}N_\lambda^{(l)}\right]\right|\right],\end{aligned}$$

after the estimate 34. Repeating the procedure leaf by leaf we obtain the bound

$$\mathbb{E}_{\mathbf{X}}\left[\prod_{k=1}^{2m}\left|\mathbb{E}_a\left[N_\lambda^{(k)}N_\lambda^{(l_k)}\right]\right|\right]=\|t\|^{4m}O\left(\left(\frac{\eta(\lambda)}{\lambda}\right)^{2m-1}\right)=\|t\|^{4m}O\left(\left(\frac{\eta(\lambda)}{\lambda}\right)^{m}\right).$$

We used the fact that $2m-1\geq m$, with equality when $m=1$. That is, the worst case is attained for graphs G_l with m connected components, for instance when $G_\ell=\{(1,2),(3,4)\dots,(2m-1,2m)\}$. □

A.2 Proof of Theorem 2.5

The proof of Theorem 2.5 is rather technical, and relies on the following Sobolev injection for a smooth domain Ω:

$$W^{d+1,1}(\Omega)\subset L_\infty(\Omega).$$

It allows us to bound the supremum norm by the $W^{d+1,1}$ Sobolev norm, which is interchangeable with the expectation under $\mathbb{P}_a$. We only detail the proof of the first assertion for simplicity. The second assertion is the generalization to the case $t\in\mathbb{R}^p$, and its proof follows the same lines.

Proof of Theorem 2.5. Let B_K be a ball containing the compact K. Let $t \mapsto h(t)$ be a non-negative symmetric function, and non-increasing on $\mathbb{R}_+$. For any smooth function $f : B_K \times \mathbb{R} \to \mathbb{R}$ with $f(v,0) = 0$,

$$\begin{aligned}\sup_{v\in K}\sup_{t\in\mathbb{R}} h(t)|f(v,t)| &\leq \sup_{t\in\mathbb{R}} h(t) \int_{[0,t]} \sup_{v\in B_K} |\partial_t f(v,s)| \mathrm{d}s \\ &\leq \sup_{t\in\mathbb{R}} \int_{[0,t]} h(s) \sup_{v\in B_K} |\partial_t f(v,s)| \mathrm{d}s \\ &\leq C \int_{-\infty}^{+\infty} h(t) \|\partial_t f(v,t)\|_{W^{d+1,1}(B_K)} \mathrm{d}t. \end{aligned} \tag{38}$$

We set

$$f(v,t) = \left|\mathbb{E}_X\left[e^{itg_\lambda^X(v)}\right] - e^{-\frac{t^2}{2}}\right|^{2q} \qquad \text{and} \qquad h(t) = \frac{1}{(1+|t|^{2+\varepsilon})^{2q}}.$$

in (38). By Fubini theorem,

$$\mathbb{E}_a\left[\sup_{v\in K}\sup_{t\in\mathbb{R}} h(t) f(v,t)\right] \leq C \sum_{|\alpha|\leq d+1} \int_{-\infty}^{+\infty} h(t) \int_{B_K} \mathbb{E}_a\left[|\partial_\alpha \partial_t f(v,t)|\right] \mathrm{d}v\mathrm{d}t. \tag{39}$$

It remains to estimate the integrand. Using the derivative of the power function, we have

$$\partial_\alpha\partial_t f(v,t) = g(v,t)\left|\mathbb{E}_X\left[e^{itg_\lambda^X(v)}\right] - e^{-\frac{t^2}{2}}\right|^{2(q-d-2)},$$

for some function g to be explicited. Using Cauchy–Schwarz inequality

$$\mathbb{E}_a\left[|\partial_\alpha\partial_t f(v,t)|\right] \leq \sqrt{\mathbb{E}_a[g(v,t)^2]} \, . \, \sqrt{\mathbb{E}_a\left[\left|\mathbb{E}_X\left[e^{itg_\lambda^X(v)}\right] - e^{-\frac{t^2}{2}}\right|^{4(q-d-2)}\right]}.$$

According to Theorem 2.2, there is a constant C independent of t and λ such that

$$\sqrt{\mathbb{E}_a\left[\left|\mathbb{E}_X\left[e^{itg_\lambda^X(v)}\right] - e^{-\frac{t^2}{2}}\right|^{4(q-d-2)}\right]} \leq C(1+|t|^{4(q-d-2)})\left(\frac{\eta(\lambda)}{\lambda}\right)^{q-d-2}.$$

We will show that for some polynomial P of degree m independent of q,

$$\mathbb{E}_a[g(v,t)^2] \leq P(t). \tag{40}$$

We will establish this fact in the end of the proof. Injecting this into the expression (39), we obtain

$$\mathbb{E}_a\left[\sup_{v\in K}\sup_{t\in\mathbb{R}} h(t)f(v,t)\right] \leq C\left(\frac{\eta(\lambda)}{\lambda}\right)^{q-d-2} \int_{-\infty}^{+\infty} \frac{(1+|t|^{\frac{m}{2}})(1+|t|^{4(q-d-2)})}{(1+|t|^{2+\varepsilon})^{2q}} \mathrm{d}t,$$

and for q large enough, the integral is bounded. The end of the proof is the same as in the remark following Theorem 2.2. We have by Markov inequality and q large enough,

$$\begin{aligned}\mathbb{P}_a\left(\sup_{t\in\mathbb{R}}\sup_{v\in K} \frac{\left|\mathbb{E}_X\left[e^{itg_\lambda^X(v)}\right] - e^{-\frac{t^2}{2}}\right|}{1+|t|^{2+\varepsilon}} > \lambda^\varepsilon\left(\frac{\eta(\lambda)}{\lambda}\right)^{1/2}\right) &\leq \left(\frac{\lambda}{\eta(\lambda)\lambda^{2\varepsilon}}\right)^q \mathbb{E}_a\left[\sup_{v\in K}\sup_{t\in\mathbb{R}} h(t)f(v,t)\right] \\ &\leq C\frac{1}{\lambda^{2q\varepsilon}}\left(\frac{\eta(\lambda)}{\lambda}\right)^{d+2}.\end{aligned}$$

For q large enough the right-hand term is summable, and Borel–Cantelli lemma implies the existence a constant $C(\omega)$ depending only on K and ε such that

$$\left|\mathbb{E}_X\left[e^{itg_\lambda^X(v)}\right]-e^{-\frac{t^2}{2}}\right|\leq C(\omega)(1+|t|^{2+\varepsilon})\frac{\sqrt{\eta(\lambda)}}{\lambda^{\frac{1}{2}-\varepsilon}}.$$

It remains to show the estimate (40). We have

$$\left|\mathbb{E}_X\left[e^{itg_\lambda^X(v)}\right]-e^{-\frac{t^2}{2}}\right|^{2q}=\mathbb{E}_{\mathbf{X}}\left[\prod_{k=1}^{2q}\left(e^{\pm itg_\lambda^{X_k}(v)}-e^{-\frac{t^2}{2}}\right)\right].$$

From this expression we deduce that

$$g(v,t)=\mathbb{E}_{\mathbf{X}}\left[F\left(t,\left(\partial_\alpha g_\lambda^{X_k}\right)_{\substack{|\alpha|\leq d'\\ 1\leq k\leq 2q}}\right)\right],$$

with F a function bounded by a polynomial of degree $2(d+2)$ in its arguments. But the partial derivatives of g_λ are still Gaussian under $\mathbb{P}_a$, and the local Weyl law (see also the expression (41)) implies the existence of a universal constant C such that

$$\mathbb{E}_a\left[\left(\partial_\alpha g_\lambda^{X_k}\right)^{2p}\right]=\frac{(2p)!}{2^p p!}\mathbb{E}_a\left[\left(\partial_\alpha g_\lambda^{X_k}\right)^2\right]^p\leq\frac{(2p)!}{2^p p!}C,$$

It implies that

$$\mathbb{E}_a\left[g(v,t)^2\right]\leq P(t),$$

for some polynomial in t whose degree is independent of q.

□

B Proof of tightness estimates

Proof of Lemma 2.7. Let $x,y\in\mathcal{M}$. We have

$$\mathbb{E}_a\left[\partial_\alpha g_\lambda^x(u)\,\partial_\alpha g_\lambda^y(v)\right]=\frac{1}{K(\lambda)\lambda^{2|\alpha|}}\sum_{\lambda_n\leq\lambda}\left(\partial_\alpha\left(\varphi_n\circ\Phi_x\right)\left(\frac{u}{\lambda}\right)\right)\left(\partial_\alpha\left(\varphi_n\circ\Phi_y\right)\left(\frac{v}{\lambda}\right)\right).$$

Setting

$$x_u=\Phi_x\left(\frac{u}{\lambda}\right)\qquad\text{and}\qquad y_v=\Phi_y\left(\frac{v}{\lambda}\right),$$

and using the fact that $\mathrm{d}\exp_x=\mathrm{Id}$, we obtain

$$\mathbb{E}_a\left[\partial_\alpha g_\lambda^x(u)\,\partial_\alpha g_\lambda^y(v)\right]=\frac{1}{K(\lambda)\lambda^{2|\alpha|}}\left(\partial_{\alpha,\alpha}K_\lambda(x_u,y_v)\right)+\text{lower order terms}.$$

Recalling that the kernel $\mathcal{B}_d$ (resp. $\mathcal{S}_d$) is the $\mathcal{C}^\infty$ scaling limit of the spectral projector we have

$$\mathbb{E}_a\left[\partial_\alpha g_\lambda^X(u)\,\partial_\alpha g_\lambda^Y(v)\right]=\partial_{\alpha,\alpha}\left[\mathcal{B}_d(\lambda.\operatorname{dist}(x_u,y_v))\right]+O\left(\frac{1}{\lambda}\right).$$

We briefly describe the $\mathcal{C}^\infty$ extension of decorrelation estimates given by Lemma 2.4. The proof is very similar and we refer to the proof of Lemma 2.4 for more details. Firstly, by the local Weyl law in the $\mathcal{C}^\infty$ topology, we have uniformly on $x\in\mathcal{M}$ and $u\in B$,

$$\mathbb{E}_a\left[\left(\partial_\alpha g_\lambda^X(u)\right)^2\right]=C_\alpha+O\left(\frac{1}{\lambda}\right)\quad\text{and}\quad C_\alpha=\partial_{\alpha,\alpha}\mathcal{B}_d(\|u-v\|)|_{u=v=0}.\tag{41}$$

Secondly, take X and Y two independent uniform random variables on $\mathcal{M}$, and $k \geq 1$. As in Lemma 2.4, we write

$$\mathbb{E}_X\left[\mathbb{E}_a[\partial_\alpha g_\lambda^X(u)\,\partial_\alpha g_\lambda^Y(v)]^{2k}\right] = I_1 + I_2,$$

with

$$I_1 = \int_{\mathrm{dist}(x,Y)>\varepsilon} \mathbb{E}_a[\partial_\alpha g_\lambda^X(u)\,\partial_\alpha g_\lambda^Y(v)]^{2k}\mathrm{d}x \quad \text{and} \quad I_2 = \int_{\mathrm{dist}(x,Y)<\varepsilon} \mathbb{E}_a[\partial_\alpha g_\lambda^X(u)\,\partial_\alpha g_\lambda^Y(v)]^{2k}\mathrm{d}x.$$

Using a similar argument as in Lemma 2.4, we deduce that uniformly on $u, v \in B$,

$$\mathbb{E}_X\left[\mathbb{E}_a[\partial_\alpha g_\lambda^X(u)\,\partial_\alpha g_\lambda^Y(v)]^{2k}\right] = O\left(\frac{\eta(\lambda)}{\lambda}\right). \tag{42}$$

Define

$$W_X : u \mapsto \partial_\alpha g_\lambda^X(u) \quad \text{and} \quad W_Y : u \mapsto \partial_\alpha g_\lambda^Y(u).$$

The joint process (W_X, W_Y) is Gaussian under $\mathbb{P}_a$. We fix $u, v \in B$ and set

$$\rho(u,v) = \frac{\mathbb{E}_a[W_X(u)W_Y(v)]^2}{\mathbb{E}_a[W_X(u)^2]\mathbb{E}_a[W_Y(v)^2]}.$$

A direct Gaussian computation shows that

$$\mathbb{E}[W_X(u)^{2p}] = \frac{(2p)!}{2^p p!}\mathbb{E}\left[W_X(u)^2\right]^p \quad \text{and} \quad \mathbb{E}[W_Y(v)^{2p}] = \frac{(2p)!}{2^p p!}\mathbb{E}\left[W_Y(v)^2\right]^p,$$

and

$$\frac{\mathbb{E}[(W_X(u)W_Y(v))^{2p}]}{\mathbb{E}_a\left[W_X(u)^{2p}\right]\mathbb{E}_a\left[W_Y(v)^{2p}\right]} = \sum_{k=0}^{p} \frac{\binom{2p+2k}{2p}\binom{2p}{p+k}}{\binom{2p}{p}}\rho(u,v)^k(1-\rho(u,v))^{p-k} := Q_p(\rho(u,v)). \tag{43}$$

From identity (43) we compute

$$\mathbb{E}_a\left[\left(\mathbb{E}_X\left[\int_B W_X(u)^{2p}\mathrm{d}u\right] - \frac{(2p)!}{2^p p!}(C_\alpha)^p\right)^2\right] = \left(\frac{(2p)!}{2^p p!}\right)^2(\Delta_1 + \Delta_2), \tag{44}$$

with

$$\Delta_1 := \left(\mathbb{E}_X\left[\int_B \mathbb{E}_a\left[W_X(u)^2\right]^p\mathrm{d}u\right] - (C_\alpha)^p\right)^2,$$

and

$$\Delta_2 := \int_B\int_B \mathbb{E}_{X,Y}\left[\mathbb{E}_a\left[W_X(u)^2\right]^p\mathbb{E}_a\left[W_Y(v)^2\right]^p(Q_p(\rho(u,v)) - 1)\right]\mathrm{d}u\mathrm{d}v.$$

From Equation (41) we have

$$\Delta_1 = O\left(\frac{1}{\lambda^2}\right).$$

As for the term Δ_2, we use the fact that $\mathbb{E}_a\left[W_X(u)^2\right]$ is bounded above and below by positive constants for λ large enough, from equation (41). We develop the polynomial Q_p and we use equation (42) to obtain

$$\begin{aligned}
\mathbb{E}_{X,Y}\left[\mathbb{E}_a\left[W_X(u)^2\right]^p\mathbb{E}_a\left[W_Y(v)^2\right]^p(Q_p(\rho(u,v)) - 1)\right] &\leq C\,\mathbb{E}_{X,Y}\left[|Q_p(\rho(u,v)) - 1|\right] \\
&\leq C\mathbb{E}_{X,Y}\left[\sum_{k=1}^{p}|p_k|\rho(u,v)^k\right] \\
&= O\left(\frac{\eta(\lambda)}{\lambda}\right).
\end{aligned}$$

Since the estimate is uniform on u, v we deduce

$$\Delta_2 = O\left(\frac{\eta(\lambda)}{\lambda}\right).$$

Injecting this estimate into identity (44) we obtain

$$\mathbb{E}_a\left[\left(\mathbb{E}_X\left[\int_B W_X(u)^{2p}\mathrm{d}u\right] - \frac{(2p)!}{2^p p!}(C_\alpha)^p\right)^2\right] = O\left(\frac{\eta(\lambda)}{\lambda}\right),$$

The quantity inside the square is a polynomial of degree at most $2p$ in the Gaussian random variables $(a_n)_{n\geq 0}$, and hence belongs to a finite fixed sum of Wiener chaos. The hypercontractivity property asserts that for such a polynomial, all the L^q norms for $q \geq 2$ are equivalents, which in our case implies that for every $q \geq 2$,

$$\mathbb{E}_a\left[\left(\mathbb{E}_X\left[\int_B W_X(u)^{2p}\mathrm{d}u\right] - \frac{(2p)!}{2^p p!}(C_\alpha)^p\right)^q\right] = O\left(\left(\frac{\eta(\lambda)}{\lambda}\right)^{q/2}\right).$$

For more details on Wiener chaos and hypercontractivity we refer the reader to the book [28]. Borel–Cantelli lemma implies the existence for every $\varepsilon > 0$ of a constant $C(\omega)$ independent of λ such that

$$\left|\mathbb{E}_X\left[\int_B W_X(u)^{2p}\mathrm{d}u\right] - \frac{(2p)!}{2^p p!}(C_\alpha)^p\right| \leq C(\omega)\frac{\sqrt{\eta(\lambda)}}{\lambda^{\frac{1}{2}-\varepsilon}},$$

which in turn implies the existence of a constant $\widetilde{C}(\omega)$ such that

$$\sup_{\lambda>0}\mathbb{E}_X\left[\int_B |\partial_\alpha g_\lambda^X(u)|^{2p}\mathrm{d}u\right] \leq \widetilde{C}(\omega).$$

□

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