Carleman estimates and some inverse problems for the coupled quantitative thermoacoustic equations by boundary data. Part II: some inverse problems

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Michel Cristofol[*], Shumin Li[†‡] and Yunxia Shang[§]

Abstract. In this paper, we consider Carleman estimates and inverse problems for the coupled quantitative thermoacoustic equations. In the previous Part I paper, we established Carleman estimates for the coupled quantitative thermoacoustic equations by assuming that the coefficients satisfy suitable conditions. We apply Carleman estimates in the previous Part I paper to some inverse problems for the coupled quantitative thermoacoustic equations and prove stability estimates of the Hölder type.

Keywords. Inverse problems, thermoacoustic equations, coupled

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1 Introduction and main results

1.1 Introduction

We investigate the so called quantitative thermoacoustic tomography process (e.g. [1, 5, 16, 18] and their references). According to [5], assuming that the variations

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in temperature and pressure are weak and neglecting the nonlinear effects, we obtain the system

$$\begin{cases} \partial_t^2 p - \rho v_s^2 \mathrm{div}\left(\dfrac{1}{\rho}\nabla p\right) - \Gamma\partial_t\{\mathrm{div}\,(\kappa\nabla\theta)\} = \Gamma\partial_t\Pi_a, \\ \partial_t\theta - \dfrac{1}{\rho C_p}\mathrm{div}\,(\kappa\nabla\theta) - \dfrac{\theta_0\varsigma}{\rho C_p}\partial_t p = \dfrac{\Pi_a}{\rho C_p}, \qquad \text{in } Q, \end{cases} \tag{1.1}$$

for the temperature rise θ and the pressure perturbation p from the equilibrium steady state depending on $(x,t) = (x_1,\cdots,x_n,t) \in Q \triangleq \Omega \times (0,T)$. Here $n \in \mathbb{N}^*$ and Ω is a bounded domain in $\mathbb{R}^n$ with the boundary $\partial\Omega \in C^3$. Throughout this paper, we set $\partial_j = \frac{\partial}{\partial x_j}$, $\partial_t = \frac{\partial}{\partial t}$, $\partial_j^2 = \frac{\partial^2}{\partial x_j^2}$, $\partial_j\partial_k = \frac{\partial^2}{\partial x_j \partial x_k}$, $\triangle = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$, $\partial_t^2 = \frac{\partial^2}{\partial t^2}$, $1 \le j,k \le n$. We assume that the mass density at steady state ρ, the acoustic wave velocity v_s and the isobar specific heat capacity C_p are given strictly positive functions of x and independent of t, the Grüneisen parameter Γ, the background temperature θ_0 and the volume thermal expansivity ς are given non-negative functions of x and independent of t. Finally, the absorbed energy Π_a is an unknown function of x and t which can be written in the form: $\Pi_a(x,t) = \mu_a(x)R(x,t)$ with $\mu_a(x,t)$ corresponds to the absorption coefficient and $R(x,t)$ is the fluence and the thermal conductivity κ is an unknown strictly positive functions of x and independent of t.

We set

$$\Theta(x,t) = \partial_t\theta(x,t) \tag{1.2}$$

Differentiating the second equation in (1.1) with respect to t, we obtain

$$\begin{cases} \partial_t^2 p - \rho v_s^2 \mathrm{div}\left(\dfrac{1}{\rho}\nabla p\right) - \Gamma\mathrm{div}\,(\kappa\nabla\Theta) = \Gamma(x)\partial_t\Pi_a, \\ \partial_t\Theta - \dfrac{1}{\rho C_p}\mathrm{div}\,(\kappa\nabla\Theta) - \dfrac{\theta_0\varsigma}{\rho C_p}\partial_t^2 p = \dfrac{1}{\rho C_p}\partial_t\Pi_a, \qquad \text{in } Q. \end{cases} \tag{1.3}$$

We will assume that Θ and p satisfy initial and boundary conditions

$$\Theta(x,0) = \Theta_0(x), \quad p(x,0) = p_0(x), \quad \partial_t p(x,0) = p_1(x), \qquad x \in \Omega \tag{1.4}$$

and

$$\Theta(x,t) = b(x,t), \qquad p(x,t) = h(x,t), \qquad (x,t) \in \Sigma \triangleq \partial\Omega \times (0,T). \tag{1.5}$$

The methodology used in this paper is based on Carleman estimates. Bugkheim and Klibanov [4] have initiated the use of Carleman estimates for proving the uniqueness in several inverse coefficient problems. For the development of this approach, there are many works. We refer to some of them (e.g., [2], [3], [6], [7], [8], [9], [10], [11], [12], [13], [14], [17]). The main advantages of this approach concern the finite number of observations required, the obtention of stability inequalities between the coefficient to be reconstructed and the observation data. Nevertheless, some drawback concerns the case of parabolic models: indeed, we need the additive observation of the solution of the problem on all the domain at one strictly positive time. The case of system of operators with respect to the scalar case is less addressed and the case of hybrid problem involving strong coupling between the second order terms in the two equations induces technical difficulties. We are interested in the so-called Thermo-Acoustic-Tomography (TAT in short) and as we aim to solve our different inverse problems using the observation of only one component, the pressure P or the temperature θ, we need to develop specific Carleman estimates (see Lemma 1.1). The TAT model answers to the current focuses of interests towards achieving better contrasted image with higher spectral resolution (e.g. functional imaging in medical applications). This is why multiwaves or hybrid systems are proposed as models. As already mentioned, mathematically, solving these inverse problems is even more complicated because they require a hard coupling between PDE of different nature : parabolic and hyperbolic. The system (1.1) propose a model where the thermal effects are fully kept. Indeed, the TAT approach provides a complete model coupling heat transfer and pressure equations.

The outline of this paper is the following. In the Section 2 we prove two key Carleman estimates involving the observation of only one component. The section 3 is concerned by the proof of the reconstruction Theorem 1.1 and the reconstruction Theorem 1.2.

1.2 Settings and hypothesis

Therefore, in a first part of this paper, we are going to derive two Carleman estimates for the following strongly coupled hyperbolic-parabolic system involving the observation of only one component.

$$\begin{cases} \partial_t^2 p(x,t) - a_1(x)\Delta p(x,t) - a_2(x)\Delta\Theta(x,t) = f(x,t), \\ \partial_t\Theta(x,t) - a_3(x)\Delta\Theta(x,t) - a_4(x)\partial_t^2 p(x,t) = g(x,t), \quad \text{in} \quad Q, \end{cases} \tag{1.6}$$

where $f(x,t),\ g(x,t) \in L^2(Q),\ a_j(x) \in C^2(\overline{\Omega})$ $(j = 1,2,3,4)$ are real-valued functions. We will assume that Θ and p satisfy initial condition (1.4) and boundary conditions (1.5).

Let $(x \cdot x')$ denote the scalar product in $\mathbb{R}^n$. Let $\nu = \nu(x) = (\nu_1(x), \cdots, \nu_n(x))$ denote the outward unit normal vector to $\partial\Omega$ at x. We assume that $\omega \subset \Omega$ is a subdomain of Ω satisfy

$$\overline{\partial\Omega \setminus \partial\omega} \subset \left\{ x \in \partial\Omega \middle| \left((x - x_0) \cdot \nu(x) \right) < 0 \right\} \tag{1.7}$$

with some $x_0 = (x_0^1, x_0^2, \cdots, x_0^n) \in \mathbb{R}^n \setminus \overline{\Omega}$. Let $T > 0$ be given. Denote $t_0 = \frac{T}{2}$.

We introduce two sets which are concerned with the coefficients $a_j(x)$, $j = 1$, 2, 3, 4:

$$\begin{aligned} \mathcal{U} = \mathcal{U}_{\sigma_0,\sigma_1,M_0,M_1,M_2} = \Bigg\{ & \big(a_1(x), a_2(x), a_3(x), a_4(x)\big) \in \left(C^2(\overline{\Omega})\right)^4 \Bigg| \\ & a_1(x) \geq \sigma_1,\ a_3(x) \geq \sigma_1,\ a_2(x) \geq 0,\ a_4(x) \geq 0,\ \forall x \in \overline{\Omega}, \\ & \|a_j\|_{C(\overline{\Omega})} \leq M_0,\ \|a_j\|_{C^1(\overline{\Omega})} \leq M_1, \|a_j\|_{C^2(\overline{\Omega})} \leq M_2,\ j = 1, 2, 3, 4, \\ & 3a_1 - 2\left((x - x_0) \cdot \nabla a_1\right) + \frac{a_1}{(1 + \frac{a_2 a_4}{a_3})} \left(2(x - x_0) \cdot \nabla \left(\frac{a_2 a_4}{a_3} \right) \right) \geq \sigma_0 \Bigg\} \end{aligned} \tag{1.8}$$

where the constants $M_0 > 1,\ M_1 > 0,\ M_2 > 0,\ \sigma_0 > 0,\ M_0 > \sigma_1 > 0$ are given. Denote

$$m = \inf_{x \in \overline{\Omega}} |x - x_0|^2, \quad M = \sup_{x \in \overline{\Omega}} |x - x_0|^2, \quad \text{and} \quad \mathcal{D} = \sqrt{M - m}. \tag{1.9}$$

We assume that $\big(a_1(x), a_2(x), a_3(x), a_4(x)\big) \in \mathcal{U} = \mathcal{U}_{\sigma_0,\sigma_1,M_0,M_1,M_2}$. Denote

$$\begin{aligned} \alpha_1 =& 228 n M_0^3 M_1 M^{\frac{1}{2}} + 20 M_0^4 - 8\sigma_1^2, \quad \alpha_2 = \left(132 n M_1 M^{\frac{1}{2}} + 16 M_0 \right) M_0^3, \\ \alpha_3 =& \min \left\{ \frac{\sigma_0 \sigma_1}{\left(\frac{2M_0^4 \alpha_1}{\sigma_1^2} + \alpha_2 \right)}, \frac{\sigma_1}{8M}, \frac{\sigma_1^4}{16 M_0^8} \right\}, \\ \alpha_4 =& 6 M_0^2 \alpha_3 + \frac{2M_0^2}{\sigma_1},\ \alpha_5 = 2 M_0^2 \alpha_3 + 2 \left(1 + \frac{M_0^2}{\sigma_1} + \frac{3M_0^3}{\sigma_1^2} \right) M_1 + \frac{2M_0^2}{\sigma_1}, \\ \alpha_6 =& \frac{4 M_0^3 \alpha_3}{\sigma_1} + 2 \left(1 + \frac{M_0^2}{\sigma_1} + \frac{3M_0^3}{\sigma_1^2} \right) n M_1, \\ \alpha_7 =& \frac{16 M_0^8}{\sigma_1^7} \left(1 + \frac{M_0^2}{\sigma_1} \right)^2,\ \alpha_8 = 2 \left(1 + \frac{M_0^2}{\sigma_1} \right)^2, \\ \alpha_9 =& \frac{-\mathcal{D}\alpha_6 + \sqrt{\mathcal{D}^2 \alpha_6^2 + \sigma_1 \left(\mathcal{D}^2 \alpha_7 + \alpha_8 \right)}}{4 \left(\mathcal{D}^2 \alpha_7 + \alpha_8 \right)}. \end{aligned} \tag{1.10}$$

We choose $\beta > 0$ such that

$$0 < \beta < \min \left\{ \alpha_9^2,\ \frac{\alpha_1^2 \alpha_3^2}{16 \alpha_4^2 \mathcal{D}^2},\ \frac{\sigma_0^2 \sigma_1^2}{16 \alpha_5^2 \mathcal{D}^2},\ \frac{m^2 \sigma_1^3}{2M_0 \left(\sigma_1 + M_0^2 \right) \mathcal{D}^2} \right\}. \tag{1.11}$$

We will prove two Carleman estimate for (1.6) with the exponential weight function $\mathrm{e}^{2s\varphi}$ where

$$\varphi(x,t) = \mathrm{e}^{\lambda\psi(x,t)}, \quad \psi(x,t) = |x - x_0|^2 - \beta\,(t - t_0)^2 + \beta t_0^2, \quad \forall (x,t) \in Q, \tag{1.12}$$

and $\lambda > 0$ is a suitably large constant.

We set $\nabla = (\partial_1, \cdots, \partial_n)$, $\nabla_{x,t} = (\partial_1, \cdots, \partial_n, \partial_t)$, $|\nabla w|^2 = \sum_{k=1}^n |\partial_k w|^2$, $|\nabla_{x,t} w|^2 = |\nabla w|^2 + |\partial_t w|^2$, and so on. $L^2(Q)$, $H^2(\Omega)$, etc. denote usual Sobolev spaces. We further set

$$\mathcal{H}^{2,1}(Q) = \left\{u \in L^2(Q); \partial_j u, \partial_j^2 u, \partial_j\partial_k u, \partial_t u \in L^2(Q),\ j,k = 1,\cdots,n\right\},$$
$$\mathcal{H}^{2,2}(Q) = \left\{u \in L^2(Q); \partial_j u, \partial_j^2 u, \partial_j\partial_k u, \partial_t u, \partial_t^2 u \in L^2(Q),\ j,k = 1,\cdots,n\right\},$$

and $\mathcal{W} = \mathcal{H}^{2,1}(Q) \times \mathcal{H}^{2,2}(Q)$.

1.3 Main results

A first serie of results concerning new Carleman estimates for (1.6) involving the observation of only one component of the system (1.6) is proved in Lemma 1.1. More precisely the Carleman estimate (1.15) involves only the observation of the pressure and the Carleman estimate (1.16) involves only the observation of the temperature. The proofs are based on the results established in the paper [15] by Li and Shang.

Lemma 1.1. *Let $(\Theta, p) \in \mathcal{W}$ satisfy (1.6) and*

$$\Theta(x,t) = 0, \quad p(x,t) = 0, \qquad (x,t) \in \Sigma \triangleq \partial\Omega \times (0,T) \tag{1.13}$$

$$\Theta(x,0) = \Theta(x,T) = 0,\ \partial_t^j p(x,0) = \partial_t^j p(x,T) = 0,\ x \in \overline{\Omega},\ j = 0,1. \tag{1.14}$$

We assume that $(a_1, a_2, a_3, a_4) \in \mathcal{U}$, and that (1.11) holds. Moreover, we assume that exists a constant $\sigma_2 > 0$ such that $a_2 \geq \sigma_2$ on $\overline{\Omega}$. Then there exists a constant $\eta(\beta) > 0$ such that for any $T \in \left(0, \frac{2(\mathcal{D}+\eta)}{\sqrt{\beta}}\right)$, there exists a constant $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$, there exist constants $s_0(\lambda) > 0$ and $K_1 = K_1(s_0, \lambda_0, \beta, \Omega, T, m, M, M_0, M_1, M_2, \sigma_0, \sigma_1, \sigma_2) > 0$ such that

$$\int_Q \left(s^3\lambda^4\varphi^3\Theta^2 + \frac{1}{s\varphi}|\partial_t\Theta|^2 + s\lambda\varphi|\nabla\Theta|^2 + s^3\lambda^4\varphi^3 p^2 + s\lambda\varphi|\nabla_{x,t}p|^2\right)\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t$$
$$\leq K_1 \int_Q \left(f^2 + g^2\right)\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t$$

$$+ K_1 s^3\lambda^4\Phi^3(\lambda)\mathrm{e}^{2s\Phi(\lambda)} \int_{Q_\omega} \left(f^2 + g^2 + \sum_{j=0}^{2} |\partial_t^j p|^2 + |\triangle p|^2 \right) \mathrm{d}x\mathrm{d}t, \tag{1.15}$$

for all $s \geq s_0$*, where* $\Phi(\lambda) = \mathrm{e}^{\lambda\left(M^2+\beta t_0^2\right)}$*, and* $Q_\omega \triangleq \omega \times (0,T)$*.*

Besides, if there exists a constant $\sigma_3 > 0$ *such that* $a_4 \geq \sigma_3$ *on* $\overline{\Omega}$*, then there exists* $K_2 = K_2(s_0,\ \lambda_0,\ \beta,\ \Omega,\ T,\ m,\ M,\ M_0,\ M_1,\ M_2,\ \sigma_0,\ \sigma_1,\ \sigma_3) > 0$ *such that*

$$\begin{aligned}
&\int_Q \left(s^3\lambda^4\varphi^3\Theta^2 + \frac{1}{s\varphi}|\partial_t\Theta|^2 + s\lambda\varphi|\nabla\Theta|^2 + s^3\lambda^4\varphi^3 p^2 + s\lambda\varphi|\nabla_{x,t}p|^2 \right) \mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t \\
&\leq K_2 \int_Q \left(f^2 + g^2\right) \mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t \\
&\quad + K_2 s^3\lambda^4\Phi^3(\lambda)\mathrm{e}^{2s\Phi(\lambda)} \int_{Q_\omega} \left(g^2 + \Theta^2 + |\partial_t\Theta|^2 + |\triangle\Theta|^2\right) \mathrm{d}x\mathrm{d}t,
\end{aligned} \tag{1.16}$$

for all $s \geq s_0$*.*

Then, we will consider different inverse coefficient problems. We are interested to recover the unknown coefficient(s) using as less as possible of observation and we focus on the observation of only one component of the system (1.1). We are able to use for each case the observation of the temperature $\theta(x,t)$ (or the pressure $p(x,t)$) as well as the observation of the pressure $p(x,t)$ (or the temperature $\theta(x,t)$). We refer to [1] for the choice of the coefficients of interest to be reconstructed. Among all the coefficients appearing in the model (1.1), the absorption coefficient $\mu_a(x)$ is one of the most studied. Then due to the existing relations between a lot of coefficients, the reconstruction of the thermal conductivity $\kappa(x)$ seems to be relevant.

First, we address the inverse source problem of determining $\mu_a = \mu_a(x)$, $x \in \Omega$ from the interior measurement

$$\Theta(x,t_0), \qquad x \in \Omega,$$

and the measurement in a partial subboundary layer

$$\Theta(x,t), \qquad (x,t) \in Q_\omega.$$

We assume that

$$\partial_t\Pi_a(x,t) = \mu_a(x)R(x,t)$$

where $\mu_a(x)$ is an unknown function of x and independent of t and $R(x,t)$ and $\kappa(x)$ are given functions. We rewrite accordingly the system (1.3)

$$\begin{cases} \partial_t^2 p - \rho v_s^2 \mathrm{div}\left(\dfrac{1}{\rho}\nabla p\right) - \Gamma \mathrm{div}\,(\kappa\nabla\Theta) = \mu_a(x)R_1(x,t), \\ \partial_t\Theta - \dfrac{1}{\rho C_p}\mathrm{div}\,(\kappa\nabla\Theta) - \dfrac{\theta_0\varsigma}{\rho C_p}\partial_t^2 p = \mu_a(x)R_2(x,t), \qquad \text{in } Q. \end{cases} \tag{1.17}$$

where $R_1(x,t) = \Gamma(x)R(x,t)$ and $R_2(x,t) = \frac{1}{\rho(x)C_p(x)}R(x,t)$.

Assume

$$\begin{cases} v_s, \rho, \Gamma, \kappa, C_p, \theta_0, \varsigma \in C^2(\overline{\Omega}), \\ v_s > \sqrt{\sigma_1} > 0,\ \rho > 0,\ C_p > 0,\ \kappa > 0,\ \Gamma \geq 0,\ \theta_0\varsigma \geq 0 \quad \text{on } \overline{\Omega}, \end{cases} \tag{1.18}$$

and

$$\left(v_s^2, \Gamma\kappa, \frac{\kappa}{\rho C_p}, \frac{\theta_0\varsigma}{\rho C_p}\right) \in \mathcal{U}. \tag{1.19}$$

Moreover, we consider an arbitrarily fixed function $\eta_0(x) \in C^2(\Omega)$.

Theorem 1.1. *Let (Θ, p) satisfy (1.3), let $(\hat{\Theta}, \hat{p})$ satisfy (1.3) where μ_a is replaced by $\hat{\mu_a}$, let (Θ, p) and $(\hat{\Theta}, \hat{p})$ satisfy the same boundary condition (1.5) and let (Θ, p) satisfy initial conditions* (1.4) *and let $(\hat{\Theta}, \hat{p})$ satisfy initial condition* (3.2). *Assume that $\Theta, \hat{\Theta}, p, \hat{p} \in W^{5,\infty}(Q)$, $R_1, R_2 \in W^{3,\infty}(Q)$ and $\partial_t R_2(x, t_0) \neq 0$ for all $x \in \overline{\Omega}$. Assume $v_s, \Gamma, C_p, \theta_0, \varsigma$ satisfy (1.18) and (1.19). We further assume that exists a constant $\sigma_3 > 0$ such that $\frac{\theta_0\varsigma}{\rho C_p} \geq \sigma_3$ on $\overline{\Omega}$. Then there exists a constant $\beta = \beta(\sigma_0, \sigma_1, M_0, M_1, M_2) > 0$ such that for any $T > \frac{2\mathcal{D}}{\sqrt{\beta}}$, there exist constants $C > 0$ and $\tau \in (0,1)$ such that*

$$\|\mu_a - \hat{\mu_a}\|_{L^2(\Omega)} \leq CF^\tau, \tag{1.20}$$

for all $\mu_a, \hat{\mu_a}$ satisfying

$$\mu_a = \hat{\mu_a} = \eta_0 \text{ on } \omega, \tag{1.21}$$

where

$$F = \left\|\partial_t(\Theta - \hat{\Theta})(\cdot, t_0)\right\|_{H^2(\Omega)} + \sum_{k=0}^{4} \|\partial_t^k(\Theta - \hat{\Theta})\|_{L^2((0,T),H^2(\omega))}. \tag{1.22}$$

Then, we address the inverse problem of determining $\kappa = \kappa(x)$, $x \in \Omega$ from the interior measurement

$$p(x, t_0), \qquad x \in \Omega,$$

and the measurement in a subboundary layer

$$p(x,t), \qquad (x,t) \in Q_\omega \triangleq \omega \times (0,T),$$

assuming that $\mu_a(x)$ is known and we consider an arbitrarily fixed function $\eta_1(x) \in C^2(\Omega)$.

Theorem 1.2. *Let* (Θ, p) *satisfy (1.3), let* $(\hat{\Theta}, \hat{p})$ *satisfy (1.3) where* κ *is replaced by* $\hat{\kappa}$*, let* (Θ, p) *and* $(\hat{\Theta}, \hat{p})$ *satisfy the same boundary condition (1.5) and let* (Θ, p) *satisfy initial conditions* (1.4) *and let* $(\hat{\Theta}, \hat{p})$ *satisfy initial condition* (3.9)*. Assume that* $\Theta, \hat{\Theta}, p, \hat{p} \in W^{5,\infty}(Q)$ *and*

$$\left(\nabla\hat{\theta}\left(x, \frac{T}{2}\right) \cdot (x - x_0)\right) \neq 0 \text{ for all } x \in \overline{\Omega}. \tag{1.23}$$

Assume $v_s, \Gamma, C_p, \theta_0, \varsigma$ *satisfy (1.18) and (1.19). We further assume that exists a constant* $\sigma_2 > 0$ *such that* $\Gamma\kappa \geq \sigma_2$ *on* $\overline{\Omega}$*. Then there exists a constant* $\beta = \beta(\sigma_0, \sigma_1, M_0, M_1, M_2) > 0$ *such that for any* $T > \frac{2\mathcal{D}}{\sqrt{\beta}}$*, there exist constants* $C > 0$ *and* $\tau \in (0, 1)$ *such that*

$$\|\kappa - \hat{\kappa}\|_{H^1(\Omega)} \leq CF_1^\tau, \tag{1.24}$$

for all $\kappa, \hat{\kappa}$ *satisfying*

$$\kappa = \hat{\kappa} = \eta_1 \text{ and } \nabla\kappa = \nabla\hat{\kappa} = \nabla\eta_1 \text{ on } \omega \cup \partial\Omega, \tag{1.25}$$

where

$$F_1 = G_1 + \|p - \hat{p}\|_{H^5(Q_\omega)},$$

with
$G_1^2 = \|\partial_t(p - \hat{p})(\cdot, t_0)\|^2_{H^3(\Omega)} + \|Y(\cdot, t_0)\|^2_{H^5(\Omega)}$ *where* $Y(x, t) = \int_0^t (p - \hat{p})(x, t')\mathrm{d}t'$.

2 Carleman estimates for thermoacoustic system

Recall the following result established in [15] by Li and Shang.

Theorem 2.1. *Let* $(\Theta, p) \in \mathcal{W}$ *satisfy (1.6) and*

$$\Theta(x, t) = 0, \quad p(x, t) = 0, \qquad (x, t) \in \Sigma \triangleq \partial\Omega \times (0, T)$$

$$\Theta(x, 0) = \Theta(x, T) = 0,\ \partial_t^j p(x, 0) = \partial_t^j p(x, T) = 0,\ x \in \overline{\Omega},\ j = 0, 1.$$

We assume that $\big(a_1, a_2, a_3, a_4\big) \in \mathcal{U}$*, and that (1.11) holds. Then there exists a constant* $\eta(\beta) > 0$ *such that for any* $T \in \left(0, \frac{2(\mathcal{D}+\eta)}{\sqrt{\beta}}\right)$*, there exists a constant* $\lambda_0 > 0$ *such that for all* $\lambda > \lambda_0$*, there exist constants* $s_0(\lambda) > 0$ *and* $K = K(s_0,$ $\lambda_0,\ \beta,\ \Omega,\ T,\ m,\ M,\ M_0,\ M_1,\ M_2,\ \sigma_0,\ \sigma_1) > 0$ *such that*

$$\int_Q \left(s^3\lambda^4\varphi^3\Theta^2 + \frac{1}{s\varphi}|\partial_t\Theta|^2 + s\lambda\varphi|\nabla\Theta|^2 + s^3\lambda^4\varphi^3 p^2 + s\lambda\varphi|\nabla_{x,t}p|^2\right) \mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t$$

$$
\begin{aligned}
&\leq K \int_Q \left(f^2+g^2\right) \mathrm{e}^{2s\varphi} \mathrm{d}x\mathrm{d}t \\
&+ C\int_{Q_\omega} \left\{ s^3\lambda^4\varphi^3\left(\Theta^2+p^2\right) + \frac{1}{s\varphi}|\partial_t\Theta|^2 + s^3\lambda^3\varphi^3\left|\partial_t p\right|^2 \right\} \mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t,
\end{aligned}
\tag{2.1}
$$

for all $s \geq s_0$.

By Theorem 2.1, we prove Lemma 1.1.

Proof. By (2.1), we have

$$
\begin{aligned}
&\int_Q \left(s^3\lambda^4\varphi^3\Theta^2 + \frac{1}{s\varphi}|\partial_t\Theta|^2 + s\lambda\varphi|\nabla\Theta|^2 + s^3\lambda^4\varphi^3 p^2 + s\lambda\varphi|\nabla_{x,t}p|^2 \right) \mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t \\
&\leq K \int_Q \left(f^2+g^2\right) \mathrm{e}^{2s\varphi} \mathrm{d}x\mathrm{d}t \\
&+ Cs^3\lambda^4\Phi^3(\lambda)\mathrm{e}^{2s\Phi(\lambda)} \int_{Q_\omega} \left(\Theta^2+p^2+|\partial_t\Theta|^2+|\partial_t p|^2\right) \mathrm{d}x\mathrm{d}t,
\end{aligned}
\tag{2.2}
$$

for all $s \geq s_0$, where $\Phi(\lambda) = \mathrm{e}^{\lambda\left(M^2+\beta t_0^2\right)}$. Here and henceforth, C denotes generic positive constants which are dependent on n, Ω, T, β, m, M, M_j $(j=0,1,2)$, σ_0, σ_1, σ_2, σ_3, s_0, λ_0, but independent of s and λ. By a usual density argument, we can assume that $(\Theta, p) \in C^\infty(Q) \times C^\infty(Q)$.

We assume that there exists a constant $\sigma_2 > 0$ such that $a_2(x) \geq \sigma_2$ for all $x \in \overline{\Omega}$ and prove (1.15). By (1.6), we have (see [15])

$$
a_7\partial_t^2 p - a_1\Delta p - a_8\partial_t\Theta = f - a_8 g, \tag{2.3}
$$

where

$$
a_7 = 1 + \frac{a_2a_4}{a_3} \qquad \text{and} \quad a_8 = \frac{a_2}{a_3}. \tag{2.4}
$$

By $\left(a_1,a_2,a_3,a_4\right) \in \mathcal{U}$ and $a_2(x) \geq \sigma_2$ for all $x \in \overline{\Omega}$, we have $a_8(x) \geq \frac{\sigma_2}{M_0}$ for all $x \in \overline{\Omega}$. Therefore, by (2.3), we have

$$
\int_{Q_\omega} |\partial_t\Theta|^2 \mathrm{d}x\mathrm{d}t \leq C \int_{Q_\omega} \left(f^2+g^2+\left|\partial_t^2 p\right|^2 + |\triangle p|^2\right) \mathrm{d}x\mathrm{d}t, \tag{2.5}
$$

for all $s \geq s_0$. Furthermore, by (1.14) and Poincarés inequality, we have

$$
\int_0^T \Theta^2 \mathrm{d}t \leq C \int_0^T |\partial_t\Theta|^2 \mathrm{d}t, \text{ for all } s \geq s_0.
$$

Integrating it over ω and using (2.5), we obtain

$$\begin{aligned}\int_{Q_\omega}\Theta^2\mathrm{d}x\mathrm{d}t &\le C\int_{Q_\omega}\left|\partial_t\Theta\right|^2\mathrm{d}x\mathrm{d}t\\ &\le C\int_{Q_\omega}\left(f^2+g^2+\left|\partial_t^2 p\right|^2+\left|\triangle p\right|^2\right)\mathrm{d}x\mathrm{d}t, \qquad \text{for all } s\ge s_0. \end{aligned} \tag{2.6}$$

Substituting (2.5) and (2.6) into (2.2), we obtain (1.15).

Besides, we assume that there exists a constant $\sigma_3>0$ such that $a_4\ge\sigma_3$ on $\overline{\Omega}$ and prove (1.16). We have (2.2) and then stimate $\int_{Q_\omega}\left(p^2+\left|\partial_t p\right|^2\right)\mathrm{d}x\mathrm{d}t$. We multiply the second equation in (1.6) by $(\partial_t p)\,\mathrm{e}^{2t}$ and integrate it over Q_ω. Integrating by parts and using (1.14), we have

$$\begin{aligned}&\int_{Q_\omega} g\left(\partial_t p\right)\mathrm{e}^{2t}\mathrm{d}x\mathrm{d}t\\
&=\int_{Q_\omega}\left(\partial_t\Theta-a_3\triangle\Theta\right)\left(\partial_t p\right)\mathrm{e}^{2t}\mathrm{d}x\mathrm{d}t-\frac{1}{2}\int_{Q_\omega}a_4\left\{\partial_t\left(\left(\partial_t p\right)^2\right)\right\}\mathrm{e}^{2t}\mathrm{d}x\mathrm{d}t\\
&-\frac{1}{2}\sigma_3\int_{Q_\omega}\left\{\partial_t\left(p^2\right)\right\}\mathrm{e}^{2t}\mathrm{d}x\mathrm{d}t+\sigma_3\int_{Q_\omega}p\left(\partial_t p\right)\mathrm{e}^{2t}\mathrm{d}x\mathrm{d}t\\
&=\int_{Q_\omega}\left(\partial_t\Theta-a_3\triangle\Theta\right)\left(\partial_t p\right)\mathrm{e}^{2t}\mathrm{d}x\mathrm{d}t+\int_{Q_\omega}a_4\left|\partial_t p\right|^2\mathrm{e}^{2t}\mathrm{d}x\mathrm{d}t\\
&+\sigma_3\int_{Q_\omega}p^2\mathrm{e}^{2t}\mathrm{d}x\mathrm{d}t+\sigma_3\int_{Q_\omega}p\left(\partial_t p\right)\mathrm{e}^{2t}\mathrm{d}x\mathrm{d}t\\
&\ge-\frac{\sigma_3}{8}\int_{Q_\omega}\left|\partial_t p\right|^2\mathrm{e}^{2t}\mathrm{d}x\mathrm{d}t-C\int_{Q_\omega}\left(\left|\partial_t\Theta\right|^2+\left|\triangle\Theta\right|^2\right)\mathrm{e}^{2t}\mathrm{d}x\mathrm{d}t\\
&+\sigma_3\int_{Q_\omega}\left(\left|\partial_t p\right|^2+p^2\right)\mathrm{e}^{2t}\mathrm{d}x\mathrm{d}t-\frac{\sigma_3}{2}\int_{Q_\omega}\left(\left|\partial_t p\right|^2+p^2\right)\mathrm{e}^{2t}\mathrm{d}x\mathrm{d}t\\
&\ge\frac{3\sigma_3}{8}\int_{Q_\omega}\left(\left|\partial_t p\right|^2+p^2\right)\mathrm{e}^{2t}\mathrm{d}x\mathrm{d}t-C\int_{Q_\omega}\left(\left|\partial_t\Theta\right|^2+\left|\triangle\Theta\right|^2\right)\mathrm{e}^{2t}\mathrm{d}x\mathrm{d}t,\end{aligned}$$

$$\int_{Q_\omega} g\left(\partial_t p\right)\mathrm{e}^{2t}\mathrm{d}x\mathrm{d}t\le\frac{\sigma_3}{4}\int_{Q_\omega}\left|\partial_t p\right|^2\mathrm{e}^{2t}\mathrm{d}x\mathrm{d}t+C\int_{Q_\omega}g^2\mathrm{e}^{2t}\mathrm{d}x\mathrm{d}t,\ \text{for all } s\ge s_0.$$

Therefore, we have

$$\begin{aligned}\frac{\sigma_3}{8}\int_{Q_\omega}\left(\left|\partial_t p\right|^2+p^2\right)\mathrm{d}x\mathrm{d}t&\le\frac{\sigma_3}{8}\int_{Q_\omega}\left(\left|\partial_t p\right|^2+p^2\right)\mathrm{e}^{2t}\mathrm{d}x\mathrm{d}t\\
&\le C\int_{Q_\omega}\left(g^2+\left|\partial_t\Theta\right|^2+\left|\triangle\Theta\right|^2\right)\mathrm{e}^{2t}\mathrm{d}x\mathrm{d}t\\
&\le C\mathrm{e}^{2T}\int_{Q_\omega}\left(g^2+\left|\partial_t\Theta\right|^2+\left|\triangle\Theta\right|^2\right)\mathrm{d}x\mathrm{d}t,\ \text{for all } s\ge s_0.\end{aligned}\tag{2.7}$$

Substituting (2.7) into (2.2), we obtain (1.16).

The proof of Lemma 1.1 is complete. □

3 Inverse problems

In this section, we shall prove Theorem 1.1 and 1.2 for inverse problems.

We recall $t_0 = \frac{T}{2}$, $\varphi(x,t) = \mathrm{e}^{\lambda\psi(x,t)}$, $\psi(x,t) = |x-x_0|^2 - \beta(t-t_0)^2 + \beta t_0^2$, $\mathcal{D} = \sqrt{M-m}$, $T > \frac{2\mathcal{D}}{\sqrt{\beta}}$. We have

$$\varphi(x,t_0) \geq \mathrm{e}^{\lambda\left(\inf_{x\in\overline{\Omega}}|x-x_0|^2+\beta t_0^2\right)} \geq d \triangleq \mathrm{e}^{\lambda\left(\inf_{x\in\overline{\Omega}}|x-x_0|^2+\mathcal{D}^2\right)}, \quad x \in \overline{\Omega},$$

and

$$\varphi(x,T) = \varphi(x,0) < \mathrm{e}^{\lambda\sup_{x\in\overline{\Omega}}|x-x_0|^2} = d, \quad x \in \overline{\Omega}.$$

For any given sufficiently small $\varepsilon_0 > 0$, $\exists\delta = \delta(\varepsilon_0) > 0$, such that

$$\varphi(x,t) \leq d - \varepsilon_0, \quad (x,t) \in \overline{\Omega} \times ([0,2\delta] \cup [T-2\delta,T]).$$

We take a cut-off function $\chi(t) \in C^\infty(\mathbb{R})$ satisfying $0 \leq \chi \leq 1$ with

$$\chi(t) = \begin{cases} 0, & t \in [0,\delta] \cup [T-\delta,T], \\ 1, & t \in [2\delta, T-2\delta]. \end{cases} \tag{3.1}$$

3.1 Inverse source problem of determining the absorption coefficient

In this part, we aim to determine the absorption coefficient μ_a from a single measurement of $\Theta(x,t)$ and we are going to detail the proof of Theorem 1.1.

Proof. Let $(\Theta(x,t), p(x,t))$ satisfy (1.17), (1.4)–(1.5) and $(\hat{\Theta}(x,t), \hat{p}(x,t))$ satisfy (1.17) in which the coefficient μ_a is replaced by $\hat{\mu_a}$. We will assume that $\hat{\Theta}$ and $\hat{p}$ satisfy initial conditions

$$\hat{\Theta}(x,0) = \hat{\Theta}_0(x), \quad \hat{p}(x,0) = \hat{p}_0(x), \quad \partial_t\hat{p}(x,0) = \hat{p}_1(x), \qquad x \in \Omega \tag{3.2}$$

and the boundary conditions (1.5).

Now let $y = p - \hat{p}$, $z = \Theta - \hat{\Theta}$ and note $m_a = \mu_a - \hat{\mu_a}$. We get from (1.17)

$$\begin{cases} \partial_t^2 y - \rho v_s^2 \mathrm{div}\left(\dfrac{1}{\rho}\nabla y\right) - \Gamma \mathrm{div}(\kappa\nabla z) = m_a(x)R_1(x,t), \\ \partial_t z - \dfrac{1}{\rho C_p}\mathrm{div}(\kappa\nabla z) - \dfrac{\theta_0\varsigma}{\rho C_p}\partial_t^2 y = m_a(x)R_2(x,t), \qquad \text{in } Q, \end{cases} \tag{3.3}$$

with boundary conditions:

$$y(x,t)=0,\ z(x,t)=0,\ \ (x,t)\in\partial\Omega\times(0,T).$$

Set $u_1=\chi\partial_t^2y,\ u_2=\chi\partial_t^3y,\ v_1=\chi\partial_t^2z,\ v_2=\chi\partial_t^3z$. Then, taking the time derivative of system (3.3) and multiplying by χ we get

$$\begin{cases}\partial_t^2u_k-v_s^2\Delta u_k-\Gamma\kappa\Delta v_k=\chi\, m_a(x)\partial_t^{k+1}R_1+\Upsilon\left(\nabla u_k,\nabla v_k\right)\\ \qquad+\left(\partial_t^2\chi\right)\left(\partial_t^{k+1}y\right)+2\left(\partial_t\chi\right)\left(\partial_t^{k+2}y\right),\\ \partial_tv_k-\dfrac{\kappa}{\rho C_p}\Delta v_k-\dfrac{\Theta_0\varsigma}{\rho C_p}\partial_t^2u_k=\chi\, m_a(x)\partial_t^{k+1}R_2+\left(\nabla\kappa\cdot\nabla v_k\right)\\ \qquad+\left(\partial_t\chi\right)\left(\partial_t^{k+1}z\right)-\dfrac{\Theta_0\varsigma}{\rho C_p}\left\{\left(\partial_t^2\chi\right)\left(\partial_t^{k+1}y\right)+2\left(\partial_t\chi\right)\left(\partial_t^{k+2}y\right)\right\},\ \text{in } Q,\\ u_k(x,t)=0,\ v_k(x,t)=0,\qquad (x,t)\in\partial\Omega\times(0,T),\\ u_k(x,0)=v_k(x,0)=\partial_tu_k(x,0)=\partial_tv_k(x,0)\\ \qquad=u_k(x,T)=v_k(x,T)=\partial_tu_k(x,T)=\partial_tv_k(x,T)=0,\quad x\in\overline{\Omega},\end{cases}$$

for $k=1,2$, where $\Upsilon\left(\nabla y,\nabla z\right)=\rho v_s^2\left(\nabla\left(\frac{1}{\rho}\right)\cdot\nabla y\right)+\Gamma\left(\nabla\kappa\cdot\nabla z\right)$.

Fixing $\lambda>1$ large, applying (1.16) in Lemma 1.1 and noting the definition of χ, we can obtain, for $k=1,2$,

$$\begin{aligned}&\int_Q\chi^2\left\{s^3\left(|\partial_t^{k+1}z|^2+|\partial_t^{k+1}y|^2\right)+s\left(|\nabla\left(\partial_t^{k+1}z\right)|^2+|\nabla_{x,t}\left(\partial_t^{k+1}y\right)|^2\right)\right\}\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t\\ &\le C\int_Q\left\{s^3\left(v_k^2+u_k^2\right)+s\left(|\nabla v_k|^2+|\nabla_{x,t}u_k|^2+|\left(\partial_t\chi\right)\partial_t^{k+1}y|^2\right)\right\}\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t\\ &\le C\int_Q\left\{s^3\left(v_k^2+u_k^2\right)+s\left(|\nabla v_k|^2+|\nabla_{x,t}u_k|^2\right)\right\}\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t+CsM_3\mathrm{e}^{2s(d-\varepsilon_0)}\\ &\le CsM_3\mathrm{e}^{2s(d-\varepsilon_0)}+C\int_Q\chi^2\, m_a^2\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t+C\int_Q\left(|\nabla v_k|^2+|\nabla u_k|^2\right)\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t\\ &\quad+C\mathrm{e}^{Cs}\int_{Q_\omega}\left\{m_a^2+\sum_{j=2}^{3}\left|\partial_t^j\nabla z\right|^2+|\partial_t\chi|^2\left(\sum_{j=2}^{3}\left|\partial_t^jz\right|^2+\sum_{j=3}^{4}\left|\partial_t^jy\right|^2\right)\right.\\ &\quad\left.+\left|\partial_t^2\chi\right|^2\sum_{j=2}^{3}\left|\partial_t^jy\right|^2+\sum_{j=2}^{4}\left|\partial_t^jz\right|^2+\sum_{j=2}^{3}\left|\partial_t^j\Delta z\right|^2\right\}\mathrm{d}x\mathrm{d}t,\end{aligned}\tag{3.4}$$

for all large $s>s_0$, with

$$M_3=\sum_{j=2}^{4}\int_Q\left(\left|\partial_t^jp\right|^2+\left|\partial_t^j\hat{p}\right|^2\right)\mathrm{d}x\mathrm{d}t+\sum_{j=2}^{3}\int_Q\left(\left|\partial_t^j\Theta\right|^2+\left|\partial_t^j\hat{\Theta}\right|^2\right)\mathrm{d}x\mathrm{d}t.$$

Taking $s_0 > 0$ large enough, the third term in the right hand side of (3.4) can be absorbed by the left hand side. By the second equation in (3.3) and the hypothesis of Theorem 1.1, we have,

$$\partial_t^{k+2} y = \frac{\rho C_p}{\theta_0 \varsigma} \left\{ \partial_t^{k+1} z - \frac{1}{\rho C_p} \mathrm{div}\left(\kappa \nabla \partial_t^k z\right) - m_a(x) \partial_t^k R_2(x,t) \right\}, \text{ in } Q,\ k = 0,1,2.$$

Therefore, we have

$$\begin{aligned} &\int_{Q_\omega} \left\{ |\partial_t \chi|^2 \sum_{j=3}^{4} |\partial_t^j y|^2 + \left|\partial_t^2 \chi\right|^2 \sum_{j=2}^{3} |\partial_t^j y|^2 \right\} \mathrm{d}x\mathrm{d}t \\ &\leq C \int_{Q_\omega} \left(m_a^2 + \sum_{j=1}^{3} \left|\partial_t^j z\right|^2 + \sum_{j=0}^{2} \left|\nabla \partial_t^j z\right|^2 + \sum_{j=0}^{2} \left|\Delta \partial_t^j z\right|^2 \right) \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{3.5}$$

Let $t_0 = \frac{T}{2}$.

$$\begin{aligned} &m_a(x) \partial_t R_2(\cdot, t_0) = \partial_t^2 z(\cdot, t_0) - \frac{\Theta_0 \varsigma}{\rho C_p} \partial_t^3 y(\cdot, t_0) - \frac{\kappa}{\rho C_p} \Delta \partial_t z(\cdot, t_0) \\ &- \left(\nabla \kappa \cdot \nabla \partial_t z(\cdot, t_0)\right), \quad \text{on } \overline{\Omega}. \end{aligned}$$

Integrating in space after multiplication by $\mathrm{e}^{2s\varphi(x,t_0)}$ we get:

$$\begin{aligned} &\int_\Omega \left|m_a(x) \partial_t R_2(x,t_0)\right|^2 \mathrm{e}^{2s\varphi(x,t_0)} \mathrm{d}x \\ &\leq C \int_\Omega \left\{ \left|\partial_t^2 z(x,t_0)\right|^2 + \left|\partial_t^3 y(x,t_0)\right|^2 \right\} \mathrm{e}^{2s\varphi(x,t_0)} \mathrm{d}x + C \mathrm{e}^{Cs} \|\partial_t z(\cdot,t_0)\|_{H^2(\Omega)}^2, \end{aligned}$$

for all $s > s_0$. Then by integration in time and after introducing $\chi(t)$, we have

$$\begin{aligned} &\int_\Omega \left|m_a(x) \partial_t R_2(x,t_0)\right|^2 \mathrm{e}^{2s\varphi(x,t_0)} \mathrm{d}x \\ &\leq C \int_0^{t_0} \int_\Omega \partial_t \left\{ \left(\left|\chi \partial_t^2 z\right|^2 + \left|\chi \partial_t^3 y\right|^2 \right) \mathrm{e}^{2s\varphi} \right\} \mathrm{d}x\mathrm{d}t + C \mathrm{e}^{Cs} \|\partial_t z(\cdot,t_0)\|_{H^2(\Omega)}^2 \\ &\leq C \int_Q s \left(\left|\chi \partial_t^2 z\right|^2 + \left|\chi \partial_t^3 y\right|^2 \right) \mathrm{e}^{2s\varphi} \mathrm{d}x\mathrm{d}t \\ &\quad + \frac{C}{s} \int_Q \left(\left|\chi \partial_t^3 z\right|^2 + \left|\chi \partial_t^4 y\right|^2 \right) \mathrm{e}^{2s\varphi} \mathrm{d}x\mathrm{d}t \\ &\quad + C M_3 \mathrm{e}^{2s(d-\varepsilon_0)} + C \mathrm{e}^{Cs} \|\partial_t z(\cdot,t_0)\|_{H^2(\Omega)}^2, \quad s > s_0. \end{aligned}$$

Then thanks to (3.4) and (1.21) we obtain

$$\int_\Omega \left|m_a(x) \partial_t R_2(x,t_0)\right|^2 \mathrm{e}^{2s\varphi(x,t_0)} \mathrm{d}x$$

$$\leq CM_3 \mathrm{e}^{2s(d-\varepsilon_0)} + \frac{C}{s^2} \int_Q \chi^2 \, m_a^2 \mathrm{e}^{2s\varphi} \mathrm{d}x\mathrm{d}t + C\mathrm{e}^{Cs} F^2$$
$$\leq CM_3 \mathrm{e}^{2s(d-\varepsilon_0)} + \frac{CT}{s^2} \int_\Omega m_a^2 \mathrm{e}^{2s\varphi(x,t_0)} \mathrm{d}x + C\mathrm{e}^{Cs} F^2, \tag{3.6}$$

for all large $s > s_0$ with

$$F = \|\partial_t z(\cdot, t_0)\|_{H^2(\Omega)}^2 + \int_{Q_\omega} \left(\sum_{j=1}^{4} \left|\partial_t^j z\right|^2 + \sum_{j=0}^{3} |\nabla \partial_t^j z|^2 + \sum_{j=0}^{3} \left|\Delta \partial_t^j z\right|^2 \right) \mathrm{d}x\mathrm{d}t.$$

For the second inequality in (3.6), we have used

$$\varphi(x,t) \leq \varphi(x,t_0) \quad \text{for all } (x,t) \in \overline{Q}. \tag{3.7}$$

By the hypothesis of Theorem 1.1, the second term in the right hand side can be absorbed by the left hand side in (3.6) for all $s > s_0$ large enough. Hence, taking $s_1 > s_0$ large enough and since $\varphi(x,t_0) \geq d$ on $\overline{\Omega}$, for all $s > s_1$, we obtain

$$\int_\Omega |m_a(x)|^2 \, \mathrm{d}x \leq C\mathrm{e}^{-2sd} \int_\Omega |m_a(x) \partial_t R_2(x,t_0)|^2 \, \mathrm{e}^{2s\varphi(x,t_0)} \mathrm{d}x \leq CM_3 \mathrm{e}^{-2s\varepsilon_0} + C\mathrm{e}^{Cs} F^2.$$

Therefore, we have

$$\int_\Omega |m_a(x)|^2 \, \mathrm{d}x \leq C\mathrm{e}^{Cs_1} \left(M_3 \mathrm{e}^{-2s\varepsilon_0} + \mathrm{e}^{Cs} F^2 \right), \quad \text{for all } s > 0.$$

Assume $F \neq 0$. Choosing $s > 0$ such that $M_3 \mathrm{e}^{-2s\varepsilon_0} = \mathrm{e}^{Cs} F^2$, we get the desired result.

□

Remark 3.1. *Following equation (2.12) in [1] the knowledge of μ_a allows us to recover the conductivity $\sigma(x)$ via its Fourier transform.*

Remark 3.2. *A similar estimate using the observation of only the component p could be established.*

3.2 Inverse problem of determining the thermal conductivity

In this part, we aim to determine the thermal conductivity $\kappa(x)$ from a single measurement of $p(x,t)$ and we are going to detail the proof of Theorem 1.2.

Proof. First, we rewrite the system (1.3)

$$\begin{cases} \partial_t^2 p - \rho v_s^2 \mathrm{div}\left(\dfrac{1}{\rho}\nabla p\right) - \Gamma \mathrm{div}\,(\kappa\nabla\Theta) = F_1(x,t), \\ \partial_t\Theta - \dfrac{1}{\rho C_p}\mathrm{div}\,(\kappa\nabla\Theta) - \dfrac{\theta_0\varsigma}{\rho C_p}\partial_t^2 p = F_2(x,t), \qquad \text{in } Q, \end{cases} \tag{3.8}$$

where $F_1(x,t) = \Gamma(x)\partial_t\Pi_a$ and $F_2(x,t) = \frac{1}{\rho C_p}\partial_t\Pi_a$ are assumed to be known. Let $(\Theta(x,t), p(x,t))$ satisfy (3.8) and $(\hat{\Theta}(x,t), \hat{p}(x,t))$ satisfy (3.8) in which the coefficient κ is replaced by $\hat{\kappa}$. We will assume that $\hat{\Theta}$ and $\hat{p}$ satisfy initial conditions

$$\hat{\Theta}(x,0) = \Theta_0(x), \quad \hat{p}(x,0) = \hat{p_0}(x), \quad \partial_t\hat{p}(x,0) = p_1(x), \qquad x\in\Omega \tag{3.9}$$

and the boundary conditions (1.5).

Now let $y = p - \hat{p}$, $z = \Theta - \hat{\Theta}$ and note $K = \kappa - \hat{\kappa}$. We get from (3.8)

$$\begin{cases} \partial_t^2 y - v_s^2\Delta y - \Gamma\kappa\Delta z = \Upsilon\left(\nabla y, \nabla z\right) + \Gamma\mathrm{div}\left(K\nabla\hat{\Theta}\right), \\ \rho C_p\partial_t z - \kappa\Delta z - \theta_0\varsigma\partial_t^2 y = (\nabla\kappa\cdot\nabla z) + \mathrm{div}\left(K\nabla\hat{\Theta}\right), \quad \text{in} \quad Q, \\ y(x,t) = 0,\ z(x,t) = 0, \qquad (x,t)\in\partial\Omega\times(0,T), \end{cases} \tag{3.10}$$

where $\Upsilon\left(\nabla y,\nabla z\right) = \rho v_s^2\left(\nabla\left(\frac{1}{\rho}\right)\cdot\nabla y\right) + \Gamma\left(\nabla\kappa\cdot\nabla z\right)$.

Let $\chi(t)$ be the same as in §3.1. Setting as previously $u_1 = \chi\partial_t y$, $u_2 = \chi\partial_t^2 y$, $u_3 = \chi\partial_t^3 y$, $v_1 = \chi\partial_t z$, $v_2 = \chi\partial_t^2 z$, $v_3 = \chi\partial_t^3 z$, then taking the time derivative of system (3.10) and multiplying by χ we get

$$\begin{cases} \partial_t^2 u_k - v_s^2\Delta u_k - \Gamma\kappa\Delta v_k = \left(\partial_t^2\chi\right)\left(\partial_t^k y\right) + 2\left(\partial_t\chi\right)\left(\partial_t^{k+1}y\right) \\ \qquad + \Upsilon\left(\nabla u_k,\nabla v_k\right) + \chi\Gamma\mathrm{div}\left(K\nabla\partial_t^k\hat{\Theta}\right), \\ \partial_t v_k - \dfrac{\kappa}{\rho C_p}\Delta v_k - \dfrac{\Theta_0\varsigma}{\rho C_p}\partial_t^2 u_k = \left(\partial_t\chi\right)\left(\partial_t^k z\right) - \dfrac{\Theta_0\varsigma}{\rho C_p}\left\{\left(\partial_t^2\chi\right)\left(\partial_t^k y\right) + 2\left(\partial_t\chi\right)\left(\partial_t^{k+1}y\right)\right\} \\ \qquad + \dfrac{1}{\rho C_p}\left(\nabla\kappa\cdot\nabla v_k\right) + \dfrac{1}{\rho C_p}\chi\mathrm{div}\left(K\nabla\partial_t^k\hat{\Theta}\right), \quad \text{in} \quad Q, \\ u_k(x,t) = 0,\ v_k(x,t) = 0, \qquad (x,t)\in\partial\Omega\times(0,T), \\ u_k(x,0) = v_k(x,0) = \partial_t u_k(x,0) = \partial_t v_k(x,0) \\ \qquad = u_k(x,T) = v_k(x,T) = \partial_t u_k(x,T) = \partial_t v_k(x,T) = 0, \quad x\in\overline{\Omega}, \end{cases}$$

for $k = 1,2,3$.

Fixing $\lambda > 1$ large, applying (1.15) in Lemma 1.1 and noting the definition of χ, we can obtain, for $k = 1,2,3$,

$$\int_Q \chi^2\left\{s^3\left(|\partial_t^k z|^2 + |\partial_t^k y|^2\right) + s\left(|\nabla\left(\partial_t^k z\right)|^2 + |\nabla\left(\partial_t^k y\right)|^2\right)\right\}\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t$$

$$
\begin{aligned}
&= \int_Q \left\{ s^3 \left(v_k^2 + u_k^2 \right) + s \left(|\nabla v_k|^2 + |\nabla u_k|^2 \right) \right\} \mathrm{e}^{2s\varphi} \mathrm{d}x\mathrm{d}t \\
&\leq \int_Q \left\{ s^3 \left(v_k^2 + u_k^2 \right) + s \left(|\nabla v_k|^2 + |\nabla_{x,t} u_k|^2 \right) \right\} \mathrm{e}^{2s\varphi} \mathrm{d}x\mathrm{d}t \\
&\leq C M_4 \mathrm{e}^{2s(d-\varepsilon_0)} + C \int_Q \chi^2 \left(K^2 + |\nabla K|^2 \right) \mathrm{e}^{2s\varphi} \mathrm{d}x\mathrm{d}t \\
&+ C \int_Q \left(|\nabla v_k|^2 + |\nabla u_k|^2 \right) \mathrm{e}^{2s\varphi} \mathrm{d}x\mathrm{d}t \\
&+ C \mathrm{e}^{Cs} \int_{Q_\omega} \left(K^2 + |\nabla K|^2 + \sum_{j=1}^{5} |\partial_t^j y|^2 + \sum_{j=1}^{3} |\partial_t^j \Delta y|^2 \right) \mathrm{d}x\mathrm{d}t \\
&+ C \mathrm{e}^{Cs} \int_{Q_\omega} \left(\sum_{j=1}^{3} |\nabla \partial_t^j y|^2 + \sum_{j=1}^{3} \left| \left(\partial_t^2 \chi \right) \left(\partial_t^j y \right) \right|^2 + \sum_{j=1}^{3} \left| \left(\partial_t \chi \right) \left(\partial_t^{j+1} y \right) \right|^2 \right) \mathrm{d}x\mathrm{d}t \\
&+ C \mathrm{e}^{Cs} \int_{Q_\omega} \left(\sum_{j=1}^{3} |\nabla \partial_t^j z|^2 + \sum_{j=1}^{3} \left| \left(\partial_t \chi \right) \left(\partial_t^j z \right) \right|^2 \right) \mathrm{d}x\mathrm{d}t, \qquad (3.11)
\end{aligned}
$$

for all large $s > s_0$, with

$$
M_4 = \sum_{j=1}^{4} \left(\left\| \partial_t^j p \right\|_{L^2(Q)}^2 + \left\| \partial_t^j \hat{p} \right\|_{L^2(Q)}^2 \right) + \sum_{j=1}^{3} \left(\left\| \partial_t^j \Theta \right\|_{L^2(Q)}^2 + \left\| \partial_t^j \hat{\Theta} \right\|_{L^2(Q)}^2 \right).
$$

Taking $s_0 > 0$ large enough, the third term in the right hand side of (3.4) can be absorbed by the left hand side.

Let $t_0 = \frac{T}{2}$. From system (3.10), we can write

$$
\begin{aligned}
&\operatorname{div} \left(K \nabla \hat{\Theta} \left(\cdot, t_0 \right) \right) = \rho C_p \partial_t z(\cdot, t_0) - \Theta_0 \varsigma \partial_t^2 y(\cdot, t_0) - \kappa \Delta z(\cdot, t_0) \\
&- \left(\nabla \kappa \cdot \nabla z(\cdot, t_0) \right), \quad \text{on } \overline{\Omega},
\end{aligned}
$$

To get rid of the term in $\Delta z(\cdot, t_0)$, we come back to the system (3.10) and we eliminate the terms in κ and in Δz to obtain:

$$
\Gamma \rho C_p \partial_t z = \partial_t^2 y - v_s^2 \Delta y + \Gamma \theta_0 \varsigma \partial_t^2 y - \rho v_s^2 \left(\nabla \left(\frac{1}{\rho} \right) \cdot \nabla y \right), \text{ in } Q. \qquad (3.12)
$$

By (1.4) and (3.9), we have $z(x,0) = \partial_t y(x,0) = 0$. Setting $Y(x,t) = \int_0^t y(x,t')\mathrm{d}t'$, integrating (3.12) with respect to t from 0 to t, and noting the hypothesis of Theorem 1.2 we have

$$
z = \frac{1 + \Gamma \theta_0 \varsigma}{\Gamma \rho C_p} \partial_t y - \frac{v_s^2}{\Gamma \rho C_p} \Delta Y - \frac{v_s^2}{\Gamma C_p} \left(\nabla \left(\frac{1}{\rho} \right) \cdot \nabla Y \right). \qquad (3.13)
$$

Therefore, we get estimates of $\|\Delta z(\cdot,t_0)\|^2_{L^2(\Omega)}$ and of $\|\nabla\Delta z(\cdot,t_0)\|^2_{L^2(\Omega)}$ in terms of norms of $\partial_t y(\cdot,t_0)$ and $Y(\cdot,t_0)$ in the form

$$\|\Delta z(\cdot,t_0)\|^2_{L^2(\Omega)} \le C\left(\|\Delta\partial_t y(\cdot,t_0)\|^2_{L^2(\Omega)} + \|Y(\cdot,t_0)\|^2_{H^4(\Omega)}\right),$$

and

$$\|\nabla\Delta z(\cdot,t_0)\|^2_{L^2(\Omega)} \le C\left(\|\Delta\partial_t y(\cdot,t_0)\|^2_{H^1(\Omega)} + \|Y(\cdot,t_0)\|^2_{H^5(\Omega)}\right),$$

and we set

$$\|\partial_t y(\cdot,t_0)\|^2_{H^3(\Omega)} + \|Y(\cdot,t_0)\|^2_{H^5(\Omega)} = G_1^2 \quad \text{and} \quad G_1 \ge 0.$$

By (3.13), we further have

$$\|\left(\nabla\kappa\cdot\nabla z(\cdot,t_0)\right)\|^2_{L^2(\Omega)} + \|\nabla\left(\nabla\kappa\cdot\nabla z(\cdot,t_0)\right)\|^2_{L^2(\Omega)} \le CG_1^2.$$

Then we deduce

$$\begin{aligned}
&\int_\Omega \left\{\left|\operatorname{div}\left(K\nabla\hat\Theta\,(x,t_0)\right)\right|^2 + \left|\nabla\operatorname{div}\left(K\nabla\hat\Theta\,(x,t_0)\right)\right|^2\right\} \mathrm{e}^{2s\varphi(x,t_0)}\mathrm{d}x \\
&\le C\int_\Omega \left\{|\partial_t z(x,t_0)|^2 + \left|\partial_t^2 y(x,t_0)\right|^2\right\} \mathrm{e}^{2s\varphi(x,t_0)}\mathrm{d}x \\
&\quad + C\int_\Omega \left\{|\nabla\partial_t z(x,t_0)|^2 + \left|\nabla\partial_t^2 y(x,t_0)\right|^2\right\} \mathrm{e}^{2s\varphi(x,t_0)}\mathrm{d}x + C\mathrm{e}^{Cs}G_1^2. \\
&\le C\int_0^{t_0}\int_\Omega \partial_t\left\{\left(|\chi\partial_t z|^2 + \left|\chi\partial_t^2 y\right|^2 + |\chi\nabla\partial_t z|^2 + \left|\chi\nabla\partial_t^2 y\right|^2\right)\mathrm{e}^{2s\varphi}\right\}\mathrm{d}x\mathrm{d}t \\
&\quad + C\mathrm{e}^{Cs}G_1^2 \\
&\le C\int_Q s\left(|\chi\partial_t z|^2 + \left|\chi\partial_t^2 y\right|^2 + |\chi\nabla\partial_t z|^2 + \left|\chi\nabla\partial_t^2 y\right|^2\right)\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t \\
&\quad + C\int_Q \frac{1}{s}\left(\left|\chi\partial_t^2 z\right|^2 + \left|\chi\partial_t^3 y\right|^2 + \left|\chi\nabla\partial_t^2 z\right|^2 + \left|\chi\nabla\partial_t^3 y\right|^2\right)\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t \\
&\quad + CM_5\mathrm{e}^{2s(d-\varepsilon_0)} + C\mathrm{e}^{Cs}G_1^2, \qquad s > s_0,
\end{aligned} \tag{3.14}$$

where

$$M_5 = \left\|\partial_t^2 p\right\|^2_{H^1(Q)} + \left\|\partial_t^2\hat p\right\|^2_{H^1(Q)} + \|\partial_t\Theta\|^2_{H^1(Q)} + \left\|\partial_t\hat\Theta\right\|^2_{H^1(Q)}.$$

Moreover, by (3.13), we can obtain

$$\int_{Q_\omega}\left(\sum_{j=1}^3 |\nabla\partial_t^j z|^2 + \sum_{j=1}^3 \left|(\partial_t\chi)\left(\partial_t^j z\right)\right|^2\right)\mathrm{d}x\mathrm{d}t$$

$$\leq C\int_{Q_\omega}\left(\sum_{j=0}^{4}|\nabla\partial_t^j y|^2+\sum_{j=0}^{2}|\nabla\Delta\partial_t^j y|^2+\sum_{j=2}^{4}|\partial_t^j y|^2+\sum_{j=0}^{2}\sum_{k,l=1}^{n}|\partial_k\partial_l\partial_t^j y|^2\right)\mathrm{d}x\mathrm{d}t,$$

for all $s>s_0$. From this last inequality and from (3.11) and (3.14) we deduce

$$\begin{aligned}&\int_\Omega\left\{\left|\mathrm{div}\left(K\nabla\hat{\Theta}\,(x,t_0)\right)\right|^2+\left|\nabla\mathrm{div}\left(K\nabla\hat{\Theta}\,(x,t_0)\right)\right|^2\right\}\mathrm{e}^{2s\varphi(x,t_0)}\mathrm{d}x\\&\leq C\,(M_4+M_5)\,\mathrm{e}^{2s(d-\varepsilon_0)}+C\int_Q\chi^2\,\left(K^2+|\nabla K|^2\right)\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t\\&\quad+C\mathrm{e}^{Cs}G_1^2+C\mathrm{e}^{Cs}\int_{Q_\omega}\left(K^2+|\nabla K|^2+\sum_{j=0}^{4}|\nabla\partial_t^j y|^2+\sum_{j=0}^{2}|\nabla\Delta\partial_t^j y|^2\right.\\&\qquad\left.+\sum_{j=1}^{5}|\partial_t^j y|^2+\sum_{j=0}^{3}\sum_{k,l=1}^{n}|\partial_k\partial_l\partial_t^j y|^2\right)\mathrm{d}x\mathrm{d}t,\quad\text{for all large } s>s_0.\end{aligned}$$

We recall the hypothesis (1.25) on κ. Noting (1.23), applying Lemma 6.2 by Yamamoto [19], and using (3.7), we have

$$\begin{aligned}&s^2\int_\Omega\left(K^2+|\nabla K|^2\right)\mathrm{e}^{2s\varphi(x,t_0)}\mathrm{d}x\\&\leq\int_\Omega\left\{\left|\mathrm{div}\left(K\nabla\hat{\Theta}\,(x,t_0)\right)\right|^2+\left|\nabla\mathrm{div}\left(K\nabla\hat{\Theta}\,(x,t_0)\right)\right|^2\right\}\mathrm{e}^{2s\varphi(x,t_0)}\mathrm{d}x\\&\leq C\,(M_4+M_5)\,\mathrm{e}^{2s(d-\varepsilon_0)}+CT\int_\Omega\left(K^2+|\nabla K|^2\right)\mathrm{e}^{2s\varphi(x,t_0)}\mathrm{d}x+C\mathrm{e}^{Cs}G_1^2\\&\quad+C\mathrm{e}^{Cs}\int_{Q_\omega}\left(\sum_{j=0}^{4}|\nabla\partial_t^j y|^2+\sum_{j=0}^{2}|\nabla\Delta\partial_t^j y|^2\right.\\&\qquad\left.+\sum_{j=1}^{5}|\partial_t^j y|^2+\sum_{j=0}^{3}\sum_{k,l=1}^{n}|\partial_k\partial_l\partial_t^j y|^2\right)\mathrm{d}x\mathrm{d}t,\quad\text{for all large } s>s_0.\end{aligned}$$

Then taking $s_2>s_0$ large enough and since $\varphi\,(x,t_0)\geq d$ on $\overline{\Omega}$, we get

$$\begin{aligned}&\int_\Omega\left(K^2+|\nabla K|^2\right)\mathrm{d}x\leq\left(s^2-CT\right)\mathrm{e}^{-2sd}\int_\Omega\left(K^2+|\nabla K|^2\right)\mathrm{e}^{2s\varphi(x,t_0)}\mathrm{d}x\\&\leq C\,(M_4+M_5)\,\mathrm{e}^{-2s\varepsilon_0}+C\mathrm{e}^{Cs}F_1^2,\quad\text{for all } s>s_2.\end{aligned}$$

We end the proof mimicking the end of the proof of Theorem 1.1. $\square$

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