Nonlinear boundary value problems relative to one dimensional heat equation

Laurent Veron

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Laurent Véron*

To Julian with high esteem and sincere friendship

Abstract

We consider the problem of existence of a solution u to $\partial_t u - \partial_{xx} u = 0$ in $(0,T) \times \mathbb{R}_+$ subject to the boundary condition $-u_x(t,0) + g(u(t,0)) = \mu$ on $(0,T)$ where μ is a measure on $(0,T)$ and g a continuous nondecreasing function. When $p > 1$ we study the set of self-similar solutions of $\partial_t u - \partial_{xx} u = 0$ in $\mathbb{R}_+ \times \mathbb{R}_+$ such that $-u_x(t,0) + u^p = 0$ on $(0,\infty)$. At end, we present various extensions to a higher dimensional framework.

Key Words: Nonlinear heat flux; Singularities; Radon measures; Marcinkiewicz spaces.

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1 Introduction

Let $g : \mathbb{R} \mapsto \mathbb{R}$ be a continuous nondecreasing function. Set $Q^T_{\mathbb{R}_+} = (0,T) \times \mathbb{R}_+$ for $0 < T \leq \infty$ and $\partial_\ell Q^T_{\mathbb{R}_+} = \overline{\mathbb{R}_+} \times \{0\}$. The aim of this article is to study the following 1-dimensional heat equation with a nonlinear flux on the parabolic boundary

$$\begin{aligned} u_t - u_{xx} &= 0 && \text{in } Q^T_{\mathbb{R}_+} \\ -u_x(.,0) + g(u(.,0)) &= \mu && \text{in } [0,T) \\ u(0,.) &= \nu && \text{in } \mathbb{R}_+, \end{aligned} \tag{1.1}$$

where ν, μ are Radon measures in $\mathbb{R}_+$ and $[0,T)$ respectively. A related problem in $Q^\infty_{\mathbb{R}_+}$ for which there exist explicit solutions is the following,

$$\begin{aligned} u_t - u_{xx} &= 0 && \text{in } Q^\infty_{\mathbb{R}_+} \\ -u_x(t,0) + |u|^{p-1}u(t,0) &= 0 && \text{for all } t > 0 \\ \lim_{t\to 0} u(t,x) &= 0 && \text{for all } x > 0 \end{aligned} \tag{1.2}$$

where $p > 1$. Problem (1.2) is invariant under the transformation T_k defined for all $k > 0$ by

$$T_k[u](t,x) = k^{\frac{1}{p-1}} u(k^2 t, kx). \tag{1.3}$$

This leads naturaly to look for existence of self-similar solutions under the form

$$u_s(t,x) = t^{-\frac{1}{2(p-1)}} \omega\left(\frac{x}{\sqrt{t}}\right). \tag{1.4}$$

Putting $\eta = \frac{x}{\sqrt{t}}$, ω satisfies

$$\begin{aligned} -\omega'' - \frac{1}{2}\eta\omega' - \frac{1}{2(p-1)}\omega &= 0 \qquad \text{in } \mathbb{R}_+ \\ -\omega'(0) + |\omega|^{p-1}\omega(0) &= 0 \\ \lim_{\eta\to\infty} \eta^{\frac{1}{p-1}}\omega(\eta) &= 0. \end{aligned} \tag{1.5}$$

Brezis, Terman and Peletier opened the study of self-similar solutions of nonlinear heat equations in proving in [4] the existence of a positive strongly singular function satisfying

$$u_t - \Delta u + |u|^{p-1}u = 0 \qquad \text{in } \mathbb{R}_+ \times \mathbb{R}^n, \tag{1.6}$$

and vanishing at $t = 0$ on $\mathbb{R}^n \setminus \{0\}$. They called it the *very singular solution*. Their method of construction is based upon the study of an ordinary differential equation with a phase space analysis. A new and more flexible method based upon variational analysis has been provided by [6]. Other singular solutions of (1.6) in different configurations such as boundary singularities have been studied in [11]. We set $K(\eta) = e^{\eta^2/4}$ and

$$L^2_K(\mathbb{R}_+) = \left\{\phi \in L^1_{loc}(\mathbb{R}_+) : \int_{\mathbb{R}_+} \phi^2 K dx := \|\phi\|^2_{L^2_K} < \infty\right\}, \tag{1.7}$$

and, for $k \geq 1$,

$$H^k_K(\mathbb{R}_+) = \left\{\phi \in L^2_K(\mathbb{R}_+) : \sum_{\alpha=0}^{k} \left\|\phi^{(\alpha)}\right\|^2_{L^2_K} := \|\phi\|^2_{H^k_K} < \infty\right\}. \tag{1.8}$$

Let us denote by $\mathcal{E}$ the subset of $H^1_K(\mathbb{R}_+)$ of weak solutions of (1.5) that is the set of functions satisfying

$$\int_0^\infty \left(\omega'\zeta' - \frac{1}{2(p-1)}\omega\zeta\right) K(\eta)d\eta + \left(|\omega|^{p-1}\omega\zeta\right)(0) = 0, \tag{1.9}$$

and by $\mathcal{E}_+$ the subset of nonnegative solutions. The next result gives the structure of $\mathcal{E}$.

Theorem 1.1 *1- If* $p \geq 2$*, then* $\mathcal{E} = \{0\}$*.*
2- If $1 < p \leq \frac{3}{2}$*, then* $\mathcal{E}_+ = \{0\}$
3 - If $\frac{3}{2} < p < 2$ *then* $\mathcal{E} = \{\omega_s, -\omega_s, 0\}$ *where* ω_s *is the unique positive solution of* (1.5)*. Furthermore for any* $\epsilon > 0$ *there exists* $c_\epsilon > 0$ *such that*

$$c_\epsilon \eta^{\frac{1}{p-1}-1-\epsilon} \leq e^{\frac{\eta^2}{4}} \omega_s(\eta) \leq c\eta^{\frac{1}{p-1}-1} \quad \text{for all } \ \eta > 0. \tag{1.10}$$

Whenever it exists the function u_s defined in (1.4) is the limit, when $\ell \to \infty$ of the positive solutions $u_{\ell\delta_0}$ of

$$\begin{aligned} u_t - u_{xx} &= 0 && \text{in } \ Q^\infty_{\mathbb{R}_+} \\ -u_x(t,.) + |u|^{p-1}u(t,.) &= \ell\delta_0 && \text{in } \ [0,T) \\ \lim_{t\to 0} u(t,x) &= 0 && \text{for all } \ x \in \mathbb{R}_+. \end{aligned} \tag{1.11}$$

When such a function u_s does not exits the sequence $\{u_{\ell\delta_0}\}$ tends to infinity. This is a charateristic phenomenon of an underlying fractional diffusion associated to the linear equation

$$\begin{aligned} u_t - u_{xx} &= 0 && \text{in } \ Q^\infty_{\mathbb{R}_+} \\ -u_x(.,0) &= \mu && \text{in } \ [0,\infty) \\ u(0,.) &= 0 && \text{in } \ \mathbb{R}_+. \end{aligned} \tag{1.12}$$

More generaly we consider problem (1.1). We define the set $\mathbb{X}(Q^T_{\mathbb{R}_+})$ of test functions by

$$\mathbb{X}(Q^T_{\mathbb{R}_+}) = \left\{\zeta \in C^{1,2}_c([0,T) \times [0,\infty)) : \zeta_x(t,0) = 0 \ \text{ for } t \in [0,T]\right\}. \tag{1.13}$$

Definition 1.2 *Let ν, μ be Radon measures in $\mathbb{R}_+$ and $[0,T)$ respectively. A function u defined in $\overline{Q^T_{\mathbb{R}_+}}$ and belonging to $L^1_{loc}(\overline{Q^T_{\mathbb{R}_+}}) \cap L^1(\partial_\ell Q^T_{\mathbb{R}_+}; dt)$ such that $g(u) \in L^1(\partial_\ell Q^T_{\mathbb{R}_+}; dt)$ is a weak solution of* (1.1) *if for every $\zeta \in \mathbb{X}(Q^T_{\mathbb{R}_+})$ there holds*

$$-\int_0^T\int_0^\infty (\zeta_t + \zeta_{xx}) u dx dt + \int_0^T (g(u)\zeta)(t,0) dt = \int_0^\infty \zeta d\nu(x) + \int_0^T \zeta(t,0) d\mu(t). \tag{1.14}$$

We denote by $E(t,x)$ the Gaussian kernel in $\mathbb{R}_+ \times \mathbb{R}$. The solution of

$$\begin{aligned} v_t - v_{xx} &= 0 && \text{in } Q^\infty_{\mathbb{R}_+} \\ -v_x &= \delta_0 && \text{in } \overline{\mathbb{R}_+} \\ v(0,.) &= 0 && \text{in } \mathbb{R}_+, \end{aligned} \tag{1.15}$$

has explicit expression

$$v(t,x) = 2E(t,x) = \frac{1}{\sqrt{\pi t}} e^{-\frac{x^2}{4t}}. \tag{1.16}$$

If $x, y > 0$ and $s < t$ we set $\tilde{E}(t-s,x,y) = E(t-s,x-y) + E(t-s,x+y)$. When $\nu \in \mathfrak{M}^b(\mathbb{R}_+)$ and $\mu \in \mathfrak{M}^b(\overline{\mathbb{R}_+})$ the solution of

$$\begin{aligned} v_t - v_{xx} &= 0 && \text{in } Q^\infty_{\mathbb{R}_+} \\ -v_x(.,0) &= \mu && \text{in } \overline{\mathbb{R}_+} \\ u(0,.) &= \nu && \text{in } \mathbb{R}_+, \end{aligned} \tag{1.17}$$

is given by

$$\begin{aligned} v_{\nu,\mu}(t,x) &= \int_0^\infty \tilde{E}(t,x,y) d\nu(y) + 2\int_0^t E(t-s,x) d\mu(s) \\ &= \mathcal{E}_{\mathbb{R}_+}[\nu](t,x) + \mathcal{E}_{\mathbb{R}_+\times\{0\}}[\mu](t,x) = \mathcal{E}_{Q^\infty_{\mathbb{R}_+}}[(\nu,\mu)](t,x). \end{aligned} \tag{1.18}$$

We prove the following existence and uniqueness result.

Theorem 1.3 *Let $g : \mathbb{R} \mapsto \mathbb{R}$ be a continuous nondecreasing function such that $g(0) = 0$. If g satisfies*

$$\int_1^\infty (g(s) - g(-s)) s^{-3} ds < \infty, \tag{1.19}$$

then for any bounded Borel measures ν in $\mathbb{R}_+$ and μ in $[0,T)$, there exists a unique weak solution $u := u_{\nu,\mu} \in L^1(Q^T_{\mathbb{R}_+})$ of (1.1)*. Furthermore the mapping $(\nu,\mu) \mapsto u_{\nu,\mu}$ is nondecreasing.*

When $g(s) = |s|^{p-1}s$, condition (1.19) is satisfied if

$$0 < p < 2. \tag{1.20}$$

The above result is still valid under minor modifications if $\mathbb{R}_+$ is replaced by a bounded interval $I := (a, b)$, and problem (1.1) by

$$\begin{aligned} u_t - u_{xx} &= 0 && \text{in } Q_I^T \\ u_x(.,b) + g(u(.,b)) &= \mu_1 && \text{in } [0,T) \\ -u_x(.,a) + g(u(.,a)) &= \mu_2 && \text{in } [0,T) \\ u(0,.) &= \nu && \text{in } (a,b), \end{aligned} \tag{1.21}$$

where ν, μ_j $(j = 1, 2)$ are Radon measures in I and $(0, T)$ respectively.

In the last section we present the scheme of the natural extensions of this problem to a multidimensional framework

$$\begin{aligned} u_t - \Delta u &= 0 && \text{in } Q^T_{\mathbb{R}^n_+} \\ -u_{x_n} + g(u) &= \mu && \text{in } \partial_\ell Q^T_{\mathbb{R}^n_+} \\ u(0,.) &= \nu && \text{in } \mathbb{R}^n_+, \end{aligned} \tag{1.22}$$

The construction of solutions with measure data can be generalized but there are some difficulties in the obtention of self-similar solutions. The equation with a source flux

$$\begin{aligned} u_t - \Delta u &= 0 && \text{in } Q^T_{\mathbb{R}^n_+} \\ u_{x_n} + g(u) &= 0 && \text{in } \partial_\ell Q^T_{\mathbb{R}^n_+} \\ u(0,.) &= \nu && \text{in } \mathbb{R}^n_+, \end{aligned} \tag{1.23}$$

has been studied by several authors, in particular Fila, Ishige, Kawakami and Sato [7], [8], [9]. Their main concern deals with global existence of solutions.

2 Self-similar solutions

2.1 The symmetrization

We define the operator $\mathcal{L}_K$ in $C_0^2(\mathbb{R})$ by

$$\mathcal{L}_K(\phi) = -K^{-1}(K\phi')'.$$

The operator $\mathcal{L}_K$ has been thouroughly studied in [6]. In particular

$$\inf\left\{\int_{-\infty}^{\infty} \phi'^2 K(\eta)\eta : \int_{-\infty}^{\infty} \phi^2 K(\eta)d\eta = 1\right\} = \frac{1}{2}. \tag{2.1}$$

The above infimum is achieved by $\phi_1 = (4\pi)^{-\frac{1}{2}}K^{-1}$ and $\mathcal{L}_K$ is an isomorphism from $H^1_K(\mathbb{R})$ onto its dual $(H^1_K(\mathbb{R}))' \sim H^{-1}_K(\mathbb{R})$. Finally $\mathcal{L}_K^{-1}$ is compact from $L^2_K(\mathbb{R})$ into $H^1_K(\mathbb{R})$, which implies that $\mathcal{L}_K$ is a Fredholm self-adjoint operator with

$$\sigma(\mathcal{L}_K) = \left\{\lambda_j = \tfrac{1+j-1}{2} : j = 1, 2, ...\right\},$$

and

$$ker\,(\mathcal{L}_K - \lambda_j I_d) = span\left\{\phi_1^{(j)}\right\}.$$

If ϕ is defined in $\mathbb{R}_+$, $\tilde{\phi}(x) = \phi(-x)$ is the symmetric with respect to 0 while $\phi^*(x) = -\phi(-x)$ is the antisymmetric with respect to 0. The operator $\mathcal{L}_K$

restricted to $\mathbb{R}_+$ is denoted by $\mathcal{L}_K^+$. The operator $\mathcal{L}_K^{+,N}$ with Neumann condition at $x = 0$ is again a Fredholm operator. This is also valid for the operator $\mathcal{L}_K^{+,D}$ with Dirichlet condition at $x = 0$. Hence, if ϕ is an eigenfunction of $\mathcal{L}_K^{+,N}$, then $\tilde{\phi}$ is an eigenfunction of $\mathcal{L}_K$ in $L_K^2(\mathbb{R})$. Similarly, if ϕ is an eigenfunction of $\mathcal{L}_K^{+,D}$, then ϕ^* is an eigenfunction of $\mathcal{L}_K$ in $L_K^2(\mathbb{R})$. Conversely, any even (resp. odd) eigenfunction of $\mathcal{L}_K$ in $L_K^2(\mathbb{R})$ satisfies Neumann (resp. Dirichlet) boundary condition at $x = 0$. Hence its restiction to $L_K^2(\mathbb{R}_+)$ is an eigenfunction of $\mathcal{L}_K^{+,N}$ (resp. $\mathcal{L}_K^{+,D}$). Since $\phi_1^{(j)}$ is even (resp. odd) if and only if j is even (resp. odd), we derive

$$H_K^{1,0}(\mathbb{R}_+) = \bigoplus_{\ell=1}^{\infty} span\left\{\phi_1^{(2\ell+1)}\right\}, \tag{2.2}$$

and

$$H_K^{1}(\mathbb{R}_+) = \bigoplus_{\ell=0}^{\infty} span\left\{\phi_1^{(2\ell)}\right\}. \tag{2.3}$$

Note that $\phi \in H_K^1(\mathbb{R}_+)$ such that $\phi_x(0) = 0$ (resp. $\phi(0) = 0$) implies $\tilde{\phi} \in H_K^1(\mathbb{R})$ (resp. $\phi^* \in H_K^1(\mathbb{R})$). Furthermore, ϕ_1 is an eigenfunction of $\mathcal{L}_K^+$ in $H_K^1(\mathbb{R}_+^n)$ with Neumann boundary condition on $\partial\mathbb{R}_+^n$ while $\partial_{x_n}\phi_1$ is an eigenfunction of $\mathcal{L}_K^+$ in $H_K^1(\mathbb{R}_+^n)$ with Dirichlet boundary condition on $\partial\mathbb{R}_+^n$. We list below two important properties of $H_K^1(\mathbb{R}_+)$ valid for any $\beta > 0$. Actually they are proved in [6, Prop. 1.12] with $H_{K^\beta}^1(\mathbb{R})$ but the proof is valid with $H_{K^\beta}^1(\mathbb{R}_+)$.

$$\begin{array}{lll} (i) & \phi \in H_{K^\beta}^1(\mathbb{R}_+) \Longrightarrow K^{\frac{\beta}{2}}\phi \in C^{0,\frac{1}{2}}(\mathbb{R}_+) & \\ (ii) & H_{K^\beta}^1(\mathbb{R}_+) \hookrightarrow L_{K^\beta}^2(\mathbb{R}_+) \quad \text{is compact for all } n \geq 1. & \end{array} \tag{2.4}$$

2.2 Proof of Theorem 1.1-(i)-(ii)

Assume $p \geq 2$, then $\frac{1}{2(p-1)} \leq \frac{1}{2}$. If ω is a weak solution, then

$$\int_0^\infty \left(\omega'^2 - \frac{1}{2(p-1)}\omega^2\right) K d\eta + |\omega|^{p+1}(0) = 0$$

If $\frac{1}{2} > \frac{1}{2(p-1)}$ we deduce that $\omega = 0$. Furthermore, when $\frac{1}{2} = \frac{1}{2(p-1)}$ then

$$|\omega|^{p+1}(0) = 0.$$

If ω is nonzero, it is an eigenfunction of $\mathcal{L}_K^{+,D}$. Since the first eigenvalue is 1 it would imply $1 = \frac{1}{2(p-1)} \leq \frac{1}{2}$, contradiction.

Assume $1 < p \leq \frac{3}{2}$ and ω is a nonnegative weak solution. We take $\zeta(\eta) = \eta e^{-\frac{\eta^2}{4}} = -2\phi_1'(\eta)$, then

$$\int_0^\infty \left(-\zeta'' - \frac{1}{2(p-1)}\zeta\right) \omega K(\eta) d\eta + \zeta'(0)\omega^p(0) = 0$$

Since $-\zeta'' = \zeta\lfloor_{\mathbb{R}_+} > 0$ and $\zeta'(0) = \phi_1(0) = 1$, we derive $\omega\zeta = 0$ if $1 > \frac{1}{2(p-1)}$ and $\omega(0) = 0$ if $1 = \frac{1}{2(p-1)}$. Hence $\omega'(0) = 0$ by the equation and $\omega \equiv 0$ by the Cauchy-Lipschitz theorem. □

2.3 Proof of Theorem 1.1-(iii)

We define the following functional on $H^1_K(\mathbb{R}^n_+)$

$$J(\phi) = \frac{1}{2}\int_0^\infty \left(\phi'^2 - \frac{1}{2(p-1)}\phi^2\right) K d\eta + \frac{1}{p+1}|\phi(0)|^{p+1}. \tag{2.5}$$

Lemma 2.1 *The functional J is lower semicontinuous in $H^1_K(\mathbb{R}_+)$. It tends to infinity at infinity and achieves negative values.*

Proof. We write

$$J(\psi) = J_1(\psi) - J_2(\psi) = J_1(\psi) - \frac{1}{2(p-1)} \|\psi\|^2_{L^2_K}.$$

Clearly J_1 is convex and J_2 is continuous in the weak topology of $H^1_K(\mathbb{R}_+)$ since the imbedding of $H^1_K(\mathbb{R}_+)$ into $L^2_K(\mathbb{R}_+)$ is compact. Hence J is weakly semicontinuous in $H^1_K(\mathbb{R}_+)$.

Let $\epsilon > 0$, then

$$J(\epsilon\phi_1) = \left(\frac{1}{4} - \frac{1}{4(p-1)}\right)\frac{\epsilon^2\sqrt{\pi}}{2} + \frac{\epsilon^{p+1}}{p+1}.$$

Since $1 < p < 2$, $\frac{1}{4} - \frac{1}{4(p-1)} < 0$. Hence $J(\epsilon\phi_1) < 0$ for ϵ small enough, thus J achieves negative values on $H^1_K(\mathbb{R}_+)$.

If $\psi \in H^1_K(\mathbb{R}_+)$ it can be written in a unique way under the form $\psi = a\phi_1 + \psi_1$ where $a = 2\sqrt{\pi}\psi(0)$ and $\psi_1 \in H^{1,0}_K(\mathbb{R}_+)$. Hence, for any $\epsilon > 0$,

$$\begin{aligned}
J(\psi) = {} & \frac{1}{2}\int_0^\infty \left(\psi_1'^2 - \frac{1}{2(p-1)}\psi_1^2\right) K d\eta + \frac{a^2}{2}\int_0^\infty \left(\phi_1'^2 - \frac{1}{2(p-1)}\phi_1^2\right) K d\eta \\
& + a\int_0^\infty \left(\psi_1'\phi_1' - \frac{1}{2(p-1)}\psi_1\phi_1\right) K d\eta + \frac{1}{p+1}|a|^{p+1} \\
\geq {} & \frac{2p-3}{4(p-1)}\int_0^\infty \psi_1'^2 K d\eta - \frac{a\epsilon}{2}\int_0^\infty \left(\psi_1'^2 + \frac{1}{2(p-1)}\psi_1^2\right) K d\eta \\
& + \frac{a^2(p-2)\sqrt{\pi}}{4(p-1)} - \frac{ap\sqrt{\pi}}{4(p-1)\epsilon} + \frac{1}{p+1}|a|^{p+1}.
\end{aligned}$$

Note that $\|\psi\|^2_{H^1_K} \leq 4\left(\|\psi_1'\|^2_{L^2_K} + a^2\right)$. Since $2p-3 > 0$, we can take $\epsilon > 0$ small enough in order that

$$\lim_{\|\psi\|_{H^1_K}\to\infty} J(\psi) = \infty. \tag{2.6}$$

□

End of the proof of Theorem 1.1-(iii). By Lemma 2.1 the functional J achieves its minimum in $H^1_K(\mathbb{R}_+)$ at some $\omega_s \neq 0$, and ω_s can be assumed to be non-negative since J is even. By the strong maximum principle $\omega_s > 0$, and by the method used in the proof of [13, Proposition 1] is is easy to prove that positive solutions belong to $H^2_K(\mathbb{R}_+)$. Assume that $\tilde{\omega}$ is another positive solution, then

$$\int_0^\infty \left(\frac{(K\omega_s')'}{\omega_s} - \frac{(K\tilde{\omega}_s')'}{\tilde{\omega}_s}\right)(\omega_s^2 - \tilde{\omega}_s^2)d\eta = 0.$$

Integration by parts, easily justified by regularity, yields

$$\begin{aligned}\int_0^\infty \left(\frac{(K\omega_s')'}{\omega_s} - \frac{(K\tilde{\omega}_s')'}{\tilde{\omega}_s}\right)(\omega_s^2 - \tilde{\omega}_s^2)d\eta &= \left[K\omega_s'\left(\omega_s - \frac{\tilde{\omega}_s^2}{\omega_s}\right) - K\tilde{\omega}_s'\left(\frac{\omega_s^2}{\tilde{\omega}_s} - \tilde{\omega}_s\right)\right]_0^\infty \\ &\quad - \int_0^\infty \left(\omega_s - \frac{\tilde{\omega}_s^2}{\omega_s}\right)' K\omega_s' d\eta + \int_0^\infty \left(\frac{\omega_s^2}{\tilde{\omega}_s} - \tilde{\omega}_s\right)' K\omega_s' d\eta \\ &= -\left(\omega_s^{p-1} - \tilde{\omega}_s^{p-1}\right)\left(\omega_s^2 - \tilde{\omega}_s^2\right)(0) \\ &\quad - \int_0^\infty \left(\left(\frac{\omega_s'\tilde{\omega}_s - \omega_s\tilde{\omega}_s'}{\tilde{\omega}_s}\right)^2 + \left(\frac{\omega_s\tilde{\omega}_s' - \tilde{\omega}_s\omega_s'}{\omega_s}\right)^2\right) d\eta.\end{aligned}$$

This implies that $\omega_s = \tilde{\omega}_s$. The proof of (1.10) is similar as the proof of estimate (2.5) in [11, Theorem 4.1]. □

3 Problem with measure data

3.1 The regular problem

Set $G(r) = \int_0^r g(s)ds$. We consider the functional J in $L^2(\mathbb{R}_+)$ with domain $D(J) = H^1(\mathbb{R}_+)$ defined by

$$J(u) = \frac{1}{2}\int_0^\infty u_x^2 dx + G(v(0)).$$

It is convex and lower semicontinuous in $L^2(\mathbb{R}_+)$ and its subdifferential ∂J sastisfies

$$\int_0^\infty \partial J(u)\zeta dx = \int_0^\infty u_x\zeta_x dx + g(u(0))\zeta(0)$$

for all $\zeta \in H^1(\mathbb{R}_+)$. Therefore

$$\int_0^\infty \partial J(u)\zeta dx = -\int_0^\infty u_{xx}\zeta dx + (g(u(0)) - u_x(0))\zeta(0).$$

Hence

$$\partial J(u) = -u_{xx} \quad \text{for all } u \in D(\partial J) = \{v \in H^1(\mathbb{R}_+) : v_x(0) = g(v(0))\}. \tag{3.1}$$

The operator ∂J is maximal monotone, hence it generates a semi-group of contractions. Furthermore, for any $u_0 \in L^2(\mathbb{R}_+)$ and $F \in L^2(0,T;L^2(L^2(\mathbb{R}_+))$ there exists a unique strong solution to

$$\begin{aligned} U_t + \partial J(U) = F \quad &\text{a.e. on } (0,T) \\ U(0) = u_0 \quad & \end{aligned} \tag{3.2}$$

Proposition 3.1 *Let $\mu \in H^1(0,T)$ and $\nu \in L^2(\mathbb{R}_+)$. Then there exists a unique function $u \in C([0,T]; L^2(\mathbb{R}_+)$ such that $\sqrt{t}u_{xx} \in L^2((0,T)\times\mathbb{R}_+)$ which satisfies* (3.3). *The mapping $(\mu,\nu) \mapsto u := u_{\mu,\nu}$ is non-decreasing and u is a weak solution in the sense that it satisfies* (1.14).

Proof. Let $\eta \in C_0^2([0,\infty))$ such that $\eta(0) = 0$, $\eta'(0) = 1$. If $f \in H^1(0,T)$, $\nu \in L^2(\mathbb{R}_+)$, and u is a solution of

$$\begin{aligned} u_t - u_{xx} &= 0 && \text{in } Q^T_{\mathbb{R}_+} \\ -u_x(.,0) + g(u(.,0)) &= \mu(t) && \text{in } [0,T) \\ u(0,.) &= \nu && \text{in } \mathbb{R}_+, \end{aligned} \tag{3.3}$$

where $\nu \in L^2(\mathbb{R}_+)$, then the function $v(t,x) = u(t,x) - \mu(t)\eta(x)$ satisfies

$$\begin{aligned} v_t - v_{xx} &= F && \text{in } Q^T_{\mathbb{R}_+} \\ -v_x(.,0) + g(v(.,0)) &= 0 && \text{in } [0,T) \\ v(0,.) &= \nu - \mu(0)\eta && \text{in } \mathbb{R}_+, \end{aligned} \tag{3.4}$$

with $F(t,x) = -(\mu'(t)\eta(x) + \mu(t)\eta''(x))$. The proof of the existence follows by using [2, Theorem 3.6].
Next, let $(\tilde\mu,\tilde\nu) \in H^1(0,T) \times L^2(\mathbb{R}_+)$ such that $\tilde\mu \le \mu$ and $\tilde\nu \le \nu$ and let $\tilde u = u_{\tilde\mu,\tilde\nu}$, then

$$\frac{1}{2}\frac{d}{dt}\int_0^\infty (\tilde u - u)_+^2 dx + \int_0^\infty (\partial_x(\tilde u - u)_+)^2 dx - (\tilde\mu(t) - \mu(t))(\tilde u(t,0) - u(t,0))_+$$
$$+ (g(\tilde u(t,0)) - g(u(t,0))))(\tilde u(t,0) - u(t,0)) = 0$$

Then

$$\int_0^\infty (\tilde u - u)_+^2 dx\lfloor_{t=0} \Longrightarrow \int_0^\infty (\tilde u - u)_+^2 dx = 0 \quad \text{on } [0,T].$$

We can also use (1.18) to express the solution of (3.3):

$$u(t,x) = \int_0^\infty \tilde E(t,x,y)\nu(y)dy + 2\int_0^t E(t-s,x)(\mu(s) - g(u(s,0)))ds.$$

In particular, if $g(0) = 0$, then

$$|u(t,x)| \le \int_0^\infty \tilde E(t,x,y)|\nu(y)|dy + 2\int_0^t E(t-s,x)|\mu(s)|ds.$$

The proof of (1.14) follows since u is a strong solution. □

Next, we prove that the problem is well-posed if $\mu \in L^1(0,T)$.

Proposition 3.2 *Assume $\{\nu_n\} \subset C_c(\mathbb{R}_+)$ and $\{\mu_n\} \subset C^1([0,T])$ are Cauchy sequences in $L^1(\mathbb{R}_+)$ and $L^1(0,T)$ respectively. Then the sequence $\{u_n\}$ of solutions of*

$$\begin{aligned} u_{n\,t} - u_{n\,xx} &= 0 && \text{in } Q^T_{\mathbb{R}_+} \\ -u_{n\,x}(.,0) + g(u_n(.,0)) &= \mu_n(t) && \text{in } [0,T) \\ u_n(0,.) &= \nu_n && \text{in } \mathbb{R}_+ \end{aligned} \tag{3.5}$$

converges in $C([0,T]; L^1(\mathbb{R}_+)$ to a function u which satisfies (1.14).

Proof. For $\epsilon > 0$ let p_ϵ be an odd C^1 function defined on $\mathbb{R}$ such that $p'_\epsilon \geq 0$ and $p_\epsilon(r) = 1$ on $[\epsilon, \infty)$, and put $j_\epsilon(r) = \int_0^r p_\epsilon(s)ds$. Then

$$\frac{d}{dt}\int_0^\infty j_\epsilon(u_n - u_m)dx + \int_0^\infty (u_{n\,x} - u_{m\,x})^2 p'_\epsilon(u_n - u_m)dx$$
$$+ \left(g(u_n(t,0)) - g(u_m(t,0))\right) p_\epsilon(u_n(t,0) - u_m(t,0))$$
$$= \left(\mu_n(t) - \mu_m(t)\right) p_\epsilon(u_n(t,0) - u_m(t,0)).$$

Hence

$$\int_0^\infty j_\epsilon(u_n - u_m)(t,x)dx + \left(g(u_n(t,0)) - g(u_m(t,0))\right) p_\epsilon(u_n(t,0) - u_m(t,0))$$
$$\leq \int_0^\infty j_\epsilon(\nu_n - \nu_m)dx + \left(\mu_n(t) - \mu_m(t)\right) p_\epsilon(u_n(t,0) - u_m(t,0)).$$

Letting $\epsilon \to 0$ implies $p_\epsilon \to sgn_0$, hence for any $t \in [0,T]$,

$$\begin{aligned}\int_0^\infty |u_n - u_m|(t,x)dx + |g(u_n(t,0)) - g(u_m(t,0)| \\ \leq \int_0^\infty |\nu_n - \nu_m|dx + |\mu_n(t) - \mu_m(t)|.\end{aligned} \tag{3.6}$$

Therefore $\{u_n\}$ and $\{g(u_n(.,0)\}$ are Cauchy sequences in $C([0,T]; L^1(\mathbb{R}_+))$ and $C([0,T])$ respectively with limit u and $g(u)$ and $u = u_{\nu,\mu}$ satisfies (1.14). If we assume that $(\nu, \tilde{\nu})$ and $(\mu, \tilde{\mu})$ are couples of elements of $L^1(\mathbb{R}_+)$ and $L^1(0,T)$ respectively and if $u = u_{\nu,\mu}$ and $\tilde{u} = u_{\tilde{\nu},\tilde{\mu}}$, there holds by the above technique,

$$\begin{aligned}\int_0^\infty |u - \tilde{u}|(t,x)dx + |g(u(t,0)) - g(\tilde{u}(t,0)| \\ \leq \int_0^\infty |\tilde{\nu} - \tilde{\nu}|dx + |\tilde{\mu}(t) - \tilde{\mu}(t)| \quad \text{for all } t \in [0,T].\end{aligned} \tag{3.7}$$

□

The following lemma is a parabolic version of an inequality due to Brezis.

Lemma 3.3 *Let $\nu \in L^1(\mathbb{R}_+)$ and $\mu \in L^1(0,T)$ and v be a function defined in $[0,T) \times \mathbb{R}_+$, belonging to $L^1(Q^T_{\mathbb{R}_+}) \cap L^1(\partial_\ell Q^T_{\mathbb{R}_+})$ and satisfying*

$$-\int_0^T\int_0^\infty (\zeta_t + \zeta_{xx})v\,dxdt = \int_0^T \zeta(.,0)\mu dt + \int_0^\infty \nu\zeta dx. \tag{3.8}$$

Then for any $\zeta \in \mathbb{X}(Q^T_{\mathbb{R}_+})$, $\zeta \geq 0$, there holds

$$-\int_0^T\int_0^\infty (\zeta_t + \zeta_{xx})|v|dxdt \leq \int_0^\infty \zeta(.,0)sign(v)\mu dt + \int_0^\infty |\nu|\zeta dx. \tag{3.9}$$

Similarly

$$-\int_0^T\int_0^\infty (\zeta_t + \zeta_{xx})v_+dxdt \leq \int_0^\infty \zeta(.,0)sign_+(v)\mu dt + \int_0^\infty \nu_+\zeta dx. \tag{3.10}$$

Proof. Let p_ϵ be the approximation of $sign_0$ used in Proposition 3.2 and η_ϵ be the solution of

$$\begin{aligned} -\eta_{\epsilon\, t} - \eta_{\epsilon\, xx} = p_\epsilon(v) \qquad & \text{in } Q^T_{\mathbb{R}_+} \\ \eta_{\epsilon\, x}(.,0) = 0 \qquad & \text{in } [0,T] \\ \eta_\epsilon(0,.) = 0 \qquad & \text{in } \mathbb{R}_+. \end{aligned}$$

Then $|\eta_\epsilon| \leq \eta^*$ where η^* satisfies

$$\begin{aligned} -\eta^*_t - \eta^*_{xx} = 1 \qquad & \text{in } Q^T_{\mathbb{R}_+} \\ \eta^*_x(.,0) = 0 \qquad & \text{in } [0,T] \\ \eta^*(0,.) = 0 \qquad & \text{in } \mathbb{R}_+. \end{aligned}$$

Although η_ϵ does not belong to $\mathbb{X}(Q^T_{\mathbb{R}_+})$ (it is not in $C^{1,2}([0,T)\times\mathbb{R}_+)$, it is an admissible test function and we deduce that there exists a unique solution to (3.8). Thus v is given by expression (1.18).

In order to prove (3.9), we can assume that μ and ν are smooth, $\zeta \in \mathbb{X}(Q^T_{\mathbb{R}_+})$, $\zeta \geq 0$ and set $h_\epsilon = p_\epsilon(v)\zeta$ and $w_\epsilon = vp_\epsilon(v)$, then

$$\begin{aligned} \int_0^\infty h_{\epsilon\, xx} v dx &= \int_0^\infty \left(2p'_\epsilon(v)v_x\zeta_x + p_\epsilon(v)\zeta_{xx} + \zeta(p_\epsilon(v))_{xx}\right) v dx \\ &= \int_0^\infty \left(2vp'_\epsilon(v)v_x\zeta_x - w_{\epsilon\, x}\zeta_x - (v\zeta)_x(p_\epsilon(v))_x\right) dx \\ &\qquad\qquad - \zeta(t,0)v(t,0)p'_\epsilon(v(t,0))v_x(t,0) \\ &= -\int_0^\infty \left(\zeta_x(j_\epsilon(v))_x + \zeta p'(v)_\epsilon v_x^2\right) dx - \zeta(t,0)v(t,0)p'_\epsilon(v(t,0))v_x(t,0) \\ &= -\int_0^\infty \left(\zeta p'(v)_\epsilon v_x^2 - j_\epsilon(v)\zeta_{xx}\right) dx - \zeta(t,0)v(t,0)p'_\epsilon(v(t,0))v_x(t,0), \end{aligned} \tag{3.11}$$

and

$$\int_0^T h_{\epsilon\, t} v dt = \int_0^T (p_\epsilon(v)\zeta_t + p'_\epsilon(v)\zeta v_t) v dt. \tag{3.12}$$

Since v is smooth

$$\begin{aligned} 0 &= \int_0^T\int_0^\infty (v_t - v_{xx})h_\epsilon dx dt \\ &= -\int_0^T\int_0^\infty (h_{\epsilon\, t} + h_{\epsilon\, xx}) v dx dt - \int_0^\infty h_\epsilon(0,x)\nu(x)dx \\ &\qquad - \int_0^T \left[p_\epsilon(v(t,0)) - v(t,0)p'_\epsilon(v(t,0))\right]\zeta(t,0)\mu(t)dt. \end{aligned}$$

Therefore, using (3.9) and (3.10),

$$\begin{aligned} -\int_0^T\int_0^\infty \left((j_\epsilon v)\zeta_{xx} + vp_\epsilon(v)\zeta_t\right) dx dt + \int_0^T\int_0^\infty \left(\zeta p'_\epsilon(v)v_x^2 - vp'_\epsilon(v)v_t\zeta\right) dx dt \\ = \int_0^\infty h_\epsilon(0,x)\nu(x)dx + \int_0^T h_\epsilon(t,0)\mu(t)dt. \end{aligned} \tag{3.13}$$

Put $\ell_\epsilon(s)=\int_0^s rp'_\epsilon(r)dr$, then $|\ell_\epsilon(s)\leq c\epsilon^{-1}s^2\chi_{_{[-\epsilon,\epsilon]}}(s)|$. Since

$$\int_0^T\int_0^\infty \zeta vp'_\epsilon(v)v_tdxdt=-\int_0^\infty \ell_\epsilon(v(0,x))\zeta(x)dx-\int_0^T\int_0^\infty \zeta_t\ell_\epsilon(v)dxdt,$$

and ζ has compact support, it follows that

$$\lim_{\epsilon\to 0}\int_0^T\int_0^\infty \zeta vp'_\epsilon(v)v_tdxdt=0.$$

Letting $\epsilon\to 0$ in (3.13), we derive (3.9) for smooth v. Using Proposition 3.2 completes the proof of (3.9). The proof of (3.10) is similar. □

Remark. Inequalities (3.9) and (3.10) hold if $\zeta(t,x)$ does not vanish if $|x|\geq R$ for some R but if it satisfies

$$\lim_{x\to\infty}\sup_{t\in[0,T]}\left(\zeta(t,x)+|\zeta_x(t,x)|\right)=0. \tag{3.14}$$

The proof follows by replacing $\zeta(t,x)$ by $\zeta(t,x)\eta_n(x)$ where $\eta_n\in C_c^\infty(\mathbb{R}_+)$ with $0\leq\eta_n\leq 1$, $\eta_n(x)=1$ on $[0,n]$, $\eta_n(x)=0$ on $[n+1,\infty)$, $|\eta'_n|\leq 2$, $|\eta''_n|\leq 4$. Then $\eta_n\zeta\in\mathbb{X}(Q^T_{\mathbb{R}_+})$ by letting $n\to\infty$ and the proof follows by letting $n\to\infty$.

3.2 Proof of Theorem 1.3

We give first some *heat-ball* estimates relative to our problem. For $r>0$, $x\in\mathbb{R}_+$ and $t\in\mathbb{R}$ we set

$$e(t,x;r)=\left\{(s,y)\in(0,T)\times\mathbb{R}_+ : s\leq t,\ \tilde E(t-s,x,y)\geq r\right\}. \tag{3.15}$$

Since

$$e(t,x;r)\subset[t-\tfrac{1}{4\pi er^2},t]\times[x-\tfrac{1}{r\sqrt{\pi e}},x+\tfrac{1}{r\sqrt{\pi e}}],$$

there holds

$$|e(t,x;r)|\leq\frac{1}{2r^3(\pi e)^{\frac{3}{2}}}, \tag{3.16}$$

and if

$$e^*(t;r)=\left\{s\in(0,T) : s\leq t,\ E(t-s,0,0)\geq r\right\}, \tag{3.17}$$

then we have

$$e^*(t;r)\subset[t-\tfrac{1}{4\pi er^2},t]\Longrightarrow|e^*(t;r)|\leq\frac{1}{4r^2\pi e}. \tag{3.18}$$

If G is a measured space, λ a positive measure on G and $q>1$, $M^q(G,\lambda)$ is the Marcinkiewicz space of measurable functions $f:G\mapsto\mathbb{R}$ satisfying for some constant $c>0$ and all measurable set $E\subset G$,

$$\int_E|f|d\lambda\leq c\left(\lambda(E)\right)^{\frac{1}{p'}}, \tag{3.19}$$

and

$$\|f\|_{M^q(G,\lambda)}=\inf\{c>0 \text{ s.t. } (3.29) \text{ holds}\}.$$

Lemma 3.4 *Assume μ,ν are bounded measure in $\overline{\mathbb{R}_+}$ and $\mathbb{R}_+$ respectively and u is the solution of* (1.17) *given by* (1.18) *and $v_{\nu,\mu}$ is the solution of* (1.17). *Then*

$$\|v_{\nu,\mu}\|_{M^3(Q^T_{\mathbb{R}_+})} + \left\|v_{\nu,\mu}\lfloor_{\partial Q^T_{\mathbb{R}_+}}\right\|_{M^2(\partial Q^T_{\mathbb{R}_+})} \leq c\left(\|\mu\|_{\mathfrak{M}(\partial Q^T_{\mathbb{R}_+})} + \|\nu\|_{\mathfrak{M}(Q^T_{\mathbb{R}_+})}\right). \tag{3.20}$$

Proof. First we consider $v_{0,\mu}$

$$v_{0,\mu}(t,x) = 2\int_0^t E(t-s,x)d\mu(s).$$

If $F \subset [0,T]$ is a Borel set, than for any $\tau > 0$

$$\begin{aligned}
\int_F E(t-s,0)ds &= \int_{F\cap\{E\leq\tau\}} E(t-s,0)ds + \int_{F\cap\{E>\tau\}} E(t-s,0)ds \\
&\leq \tau|F| + \int_{\{E>\tau\}} E(t-s,0)ds \\
&\leq \tau|F| - \int_\tau^\infty \lambda d|e^*(t,\lambda)| \\
&\leq \tau|F| + \int_\tau^\infty \lambda d|e^*(t,\lambda)| \\
&\leq \tau|F| + \frac{1}{4\pi e\tau}.
\end{aligned}$$

If we choose $\tau^2 = \frac{1}{4\pi e|F|}$, we derive

$$\int_F E(t-s,0)ds \leq \frac{|F|^{\frac{1}{2}}}{\sqrt{\pi e}}. \tag{3.21}$$

If $F \subset (0,T)$ is a Borel set then

$$\left|\int_F v_{0,\mu}(t,0)dt\right| = 2\left|\int_0^t\int_F E(t-s,0)dtd\mu(s)\right| \leq \frac{2|F|^{\frac{1}{2}}}{\sqrt{\pi e}}\|\mu\|_{\mathfrak{M}(\partial Q^T_{\mathbb{R}_+})}.$$

This proves that

$$\left\|v_{0,\mu}\lfloor_{\partial Q^T_{\mathbb{R}_+}}\right\|_{M^2(\partial Q^T_{\mathbb{R}_+})} \leq c\|\mu\|_{\mathfrak{M}(\partial Q^T_{\mathbb{R}_+})}. \tag{3.22}$$

Similarly, if $G \subset [0,T]\times[0,\infty)$ is a Borel set, then

$$\int_G \tilde{E}(t-s,x,0)ds \leq \frac{2|G|^{\frac{1}{3}}}{\sqrt{\pi e}}, \tag{3.23}$$

and

$$\|v_{0,\mu}\|_{M^3(Q^T_{\mathbb{R}_+})} \leq c\|\mu\|_{\mathfrak{M}(\partial Q^T_{\mathbb{R}_+})}. \tag{3.24}$$

In the same way we prove that

$$\|v_{\nu,0}\|_{M^3(Q^T_{\mathbb{R}_+})} + \left\|v_{\nu,0}\lfloor_{\partial Q^T_{\mathbb{R}_+}}\right\|_{M^2(\partial Q^T_{\mathbb{R}_+})} \leq c\,\|\nu\|_{\mathfrak{M}(Q^T_{\mathbb{R}_+})}. \tag{3.25}$$

This ends the proof. □

Proof of Theorem 1.3

Uniqueness. Assume u and $\tilde{u}$ are solutions of (1.1), then $w = u - \tilde{u}$ satisfies

$$\begin{aligned} w_t - w_{xx} &= 0 && \text{in } Q^T_{\mathbb{R}_+} \\ -w_x(.,0) + g(u(.,0)) - g(\tilde{u}(.,0)) &= 0 && \text{in } [0,T) \\ w(0,.) &= 0 && \text{in } \mathbb{R}_+. \end{aligned} \tag{3.26}$$

Applying (3.9), we obtain

$$-\int_0^T\int_0^\infty (\zeta_t + \zeta_{xx})|w|dxdt + \int_0^\infty (g(u(.,0)) - g(\tilde{u}(.,0)))sign(w)\zeta(t,0)dt \leq 0,$$

for any $\zeta \in \mathbb{X}^T_{\mathbb{R}_+}$ with $\zeta \geq 0$. Let $\theta \in C^1_c(Q^T_{\mathbb{R}_+})$, $\eta \geq 0$, we take ζ to be the solution of

$$\begin{aligned} -\zeta_t - \zeta_{xx} &= \theta && \text{in } (0,T)\times\mathbb{R}_+ \\ \zeta_x(t,0) &= 0 && \text{in } (0,T) \\ \zeta(T,x) &= 0 && \text{in } (0,\infty). \end{aligned}$$

Then ζ satisfies (3.14), hence

$$\int_0^T\int_0^\infty \theta|w|dxdt + \int_0^\infty (g(u(.,0)) - g(\tilde{u}(.,0)))sign(w)\zeta(t,0)dt \leq 0.$$

This implies $w = 0$.

Existence. Without loss of generality we can assume that μ and ν are nonnegative. Let $\{\nu_n\} \subset C_c(\mathbb{R}_+)$ and $\{\mu_n\} \subset C_c([\mathbb{R}_+]0,T))$ converging to ν and μ in the sense of measures and let u_n be the solution of (3.5). Then from (3.7),

$$\int_0^T\int_0^\infty |u_n|dxdt + \int_0^T |g(u_n(t,0))|dt \leq T\int_0^\infty |\nu_n|dx + \int_0^T |\mu_n|dt. \tag{3.27}$$

Therefore u_n and $g(u_n(.,0))$ remain bounded respectively in $L^1(Q^T_{\mathbb{R}_+})$ and in $L^1(0,T)$. Furthermore, by Lemma 3.4, u_n remains bounded in $M^3(Q^T_{\mathbb{R}_+})$ and in $M^2(\partial Q^T_{\mathbb{R}_+})$. We can also write u_n under the form

$$\begin{aligned} u_n(t,x) &= \int_0^\infty \tilde{E}(t,x,y)\mu_n(y)dy + 2\int_0^t E(t-s,x)(\nu_n(t) - g(u_n(t,0)))ds \\ &= A_n(t,x) + B_n(t,x). \end{aligned} \tag{3.28}$$

Since we can perform the even reflexion through $y = 0$, the mapping

$$(t,x) \mapsto A_n(t,x) := \int_0^\infty \tilde{E}(t,x,y)\mu_n(y)dy,$$

is relatively compact in $C^m_{loc}(\overline{Q^T_{\mathbb{R}_+}})$ for any $m \in \mathbb{N}^*$. Hence we can extract a subsequence $\{u_{n_k}\}$ which converges uniformly on every compact subset of $(0,T] \times [0,\infty)$, hence a.e. on $(0,T]$ for the 1-dimensional Lebesque measure. Concerning the boundary term

$$(t,x) \mapsto B_n(t,x) := \int_0^t E(t-s,x)(\nu_n(t) - g(u_n(t,0)))ds,$$

it is relatively compact on every compact subset of $[0,T] \times (0,\infty)$. If $x = 0$, then

$$B_n(t,0) = \int_0^t (\nu_n(t) - g(u_n(t,0)))\frac{ds}{\sqrt{\pi(t-s)}}.$$

Since $\|\nu_n(.) - g(u_n(.,0))\|_{L^1(0,T)}$, $t \mapsto B_n(t,0)$ is uniformly integrable on $(0,T)$, hence relatively compact by the Frechet-Kolmogorov Theorem. Therefore there exists a subsequence, still denoted by $\{n_k\}$ such that $B_{n_k}(t,0)$ converges for almost all $t \in (0,T)$. This implies that the sequence of function $\{u_{n_k}\}$ defined by (3.28) converges in $\overline{Q^T_{\mathbb{R}_+}}$ up to a set $\Theta \cup \Lambda$ where $\Theta \subset Q^T_{\mathbb{R}_+}$ is neglectable for the 2-dimensional Lebesgue measure and $\Lambda \subset \partial_\ell Q^T_{\mathbb{R}_+}$ neglectable for the 1-dimensional Lebesgue measure.

From Lemma 3.4, $(u_{n,k}\lfloor_{Q^T_{\mathbb{R}_+}}, u\lfloor_{\partial_\ell Q^T_{\mathbb{R}_+}})$ converges in $L^1_{loc}(Q^T_{\mathbb{R}_+}) \times L^1(\partial_\ell Q^T_{\mathbb{R}_+})$ and the convergence of each of the components holds also almost everywhere (up to a subsequence). Since $u_{n,k}$ is a weak solution, it satisfies for any $\zeta \in \mathbb{X}(Q^T_{\mathbb{R}_+})$

$$\begin{aligned} -\int_0^T\int_0^\infty (\zeta_t + \zeta_{xx})u_{n,k}dxdt + \int_0^T (g(u_{n,k})\zeta)(t,0)dt \\ = \int_0^\infty \zeta\nu_{n,k}(x)dx + \int_0^T \zeta(t,0)\mu_{n,k}(t)dt. \end{aligned} \tag{3.29}$$

In order to prove the convergence of $g(u_{n,k}(t,0))$, we use Vitali's convergence theorem and the assumption (1.19). Let $F \subset [0,T]$ be a Borel set. Using the fact that $0 \leq u_{n,k} \leq v_{\nu_{n,k},\mu_{n,k}}$ and the estimate of Lemma 3.4, we have for any $\lambda > 0$,

$$\begin{aligned} \int_F |g(u_{n,k}(t,0))|dt &\leq \int_{F\cap\{u_{n,k}(t,0)\leq\lambda\}} |g(u_{n,k}(t,0))|dt + \int_{\{u_{n,k}(t,0)>\lambda\}} |g(u_{n,k}(t,0))|dt \\ &\leq g(\lambda)|F| - \int_\lambda^\infty \sigma d|\{t : |g(u_{n,k}(t,0))| > \sigma\}| \\ &\leq g(\lambda)|F| + c\int_\lambda^\infty |g(\sigma)|\sigma^{-3}ds, \end{aligned}$$

where c depends of $\|\mu\|_{\mathfrak{M}(\partial Q^T_{\mathbb{R}_+})} + \|\nu\|_{\mathfrak{M}(Q^T_{\mathbb{R}_+})}$. For $\epsilon > 0$ given, we chose λ large enough so that the integral term above is smaller than ϵ and then $|F|$ such that $g(\lambda)|F| + \leq \epsilon$. Hence $\{g(u_{n,k}(.,0))\}$ is uniformly integrable. Therefore up to a subsequence, it converges to $g(u(.,0))$ in $L^1(0,T)$. Clearly u satisfies

$$\begin{aligned} -\int_0^T\int_0^\infty (\zeta_t + \zeta_{xx})udxdt + \int_0^T (g(u)\zeta)(t,0)dt \\ = \int_0^\infty \zeta\nu(x)dx + \int_0^T \zeta(t,0)\mu(t)dt, \end{aligned} \tag{3.30}$$

which ends the existence proof.

Monotonicity. If $\nu \geq \tilde{\nu}$ and $\mu \geq \tilde{\mu}$; we can choose the approximations such that $\nu_n \geq \tilde{\nu}_n$ and $\mu_n \geq \tilde{\mu}_n$. It follows from (3.10) that $u_{\nu_n,\mu_n} \geq u_{\tilde{\nu}_n,\tilde{\mu}_n}$. Choosing the same subsequence $\{n_k\}$, the limits u, $\tilde{u}$ are in the same order. The conclusion follows by uniqueness. $\square$

3.3 The case $g(u) = |u|^{p-1}u$

Condition (1.19) is satisfied if $p < 2$. If this condition holds there exists a solution $u_{\ell\delta_0} = u_{0,\ell\delta_0}$ and the mapping $\ell \mapsto u_{\ell\delta_0}$ is increasing.

Theorem 3.5 *(i) If $1 < p \leq \frac{3}{2}$, $u_{\ell\delta_0}$ tends to ∞ when $k \to \infty$.*

(ii) If $\frac{3}{2} < p < 2$, $u_{\ell\delta_0}$ converges to U_{ω_s} defined by $U_{\omega_s}(t,x) = t^{-\frac{1}{2(p-1)}}\omega_s(\frac{x}{\sqrt{t}})$, when $k \to \infty$.

Proof. By uniqueness and using (1.3), there holds

$$T_k[u_{\ell\delta_0}] = u_{k^{\frac{2-p}{p-1}}\ell\delta_0}, \tag{3.31}$$

for any $k, \ell > 0$. Since $\ell \mapsto u_{\ell\delta_0}$ is increasing, its limit u_∞, when $\ell \to \infty$, satisfies

$$T_k[u_\infty] = u_\infty. \tag{3.32}$$

Hence u_∞ is a positive self-similar solution of (1.2), provided it exists. Hence $u_\infty = U_{\omega_s}$ if $\frac{3}{2} < p < 2$. If $1 < p \leq \frac{3}{2}$, $u_{k\delta_0}$ admits no finite limit when $k \to \infty$ which ends the proof. $\square$

Remark. As a consequence of this result, no a priori estimate of Brezis-Friedman type (parabolic Keller-Osserman) exists for a nonnegative function $u \in C^{2,1}(\overline{Q^\infty_{\mathbb{R}_+}} \setminus \{(0,0)\}$ solution of

$$\begin{aligned} u_t - u_{xx} &= 0 && \text{in } Q^\infty_{\mathbb{R}_+} \\ -u_x(.,0) + |u|^{p-1}u(.,0) &= 0 && \text{for all } t > 0 \\ u(0,x) &= 0 && \text{for all } x > 0. \end{aligned} \tag{3.33}$$

when $1 < p \leq \frac{3}{2}$. When $\frac{3}{2} < p < 2$ it is expected that

$$u(t,x) \leq \frac{c}{(|x|^2 + t)^{\frac{1}{2(p-1)}}}. \tag{3.34}$$

The type of phenomenon (i) in Theorem 3.5 is characteristic of fractional diffusion. It has already been observed in [5, Theorem 1.3] with equations

$$u_t + (-\Delta)^\alpha + t^\beta u^p = 0 \qquad \text{in } \mathbb{R}_+ \times \mathbb{R}^N u((0,.) = k\delta_0 \quad \text{in } \mathbb{R}^N \tag{3.35}$$

when $0 < \alpha < 1$ is small and $p > 1$ is close to 1.

4 Extension and open problems

The natural extension is to replace a one dimensional domain by a mutidimenional one. The main open problem is the question of a priori estimate as stated in the last remark above.

4.1 Self-similar solutions

Let $\eta = (\eta_1, ..., \eta_n)$ be the coordinates in $\mathbb{R}^n$ and denote $\mathbb{R}^n_+ = \{\eta = (\eta_1, ..., \eta_n) = (\eta', \eta_n) : \eta_n > 0\}$. We set $K(\eta) = e^{\frac{|\eta|^2}{4}}$ and $K'(\eta') = e^{\frac{|\eta'|^2}{4}}$. Similarly to Section 2 we define $\mathcal{L}_K$ in $C^2_0(\mathbb{R}^n)$ by

$$\mathcal{L}_K(\phi) = -K^{-1}div(K\nabla\phi). \tag{4.1}$$

If $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$, we set $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$. We denote by ϕ_1 the function K^{-1}. Then the set of eigenvalues of $\mathcal{L}_K$ is the set of numbers $\left\{\lambda_k = \frac{n+k}{2} : k \in \mathbb{N}\right\}$ with corresponding set of eigenspaces

$$N_k = span\left\{D^\alpha\phi_1 : |\alpha| = k\right\}.$$

The operators $\mathcal{L}^{+,N}_K$ and $\mathcal{L}^{+,D}_K$ are defined acoordingly in $H^1_K(\mathbb{R}^n_+)$ and $H^{1,0}_K(\mathbb{R}^n_+)$ respectively and $\sigma(\mathcal{L}^{+,N}_K) = \left\{\frac{n+k}{2} : k \in \mathbb{N}\right\}$ and $\sigma(\mathcal{L}^{+,D}_K) = \left\{\frac{n+k}{2} : k \in \mathbb{N}^*\right\}$ Furthermore

$$N_{k,N} = ker\left(\mathcal{L}^{+,N}_K - \tfrac{n+k}{2}I_d\right) = span\left\{D^\alpha\phi_1 : |\alpha| = k, \alpha_n = 2\ell \text{ with } \ell \in \mathbb{N}\right\}, \tag{4.2}$$

and

$$N_{k,D} = ker\left(\mathcal{L}^{+,D}_K - \tfrac{n+k}{2}I_d\right) = span\left\{D^\alpha\phi_1 : |\alpha| = k, \alpha_n = 2\ell + 1 \text{ with } \ell \in \mathbb{N}\right\}. \tag{4.3}$$

Since $\mathcal{L}^{+,N}_K$ and $\mathcal{L}^{+,D}_K$ are Fredholm operators,

$$H^1_K(\mathbb{R}^n_+) = \bigoplus_{k=0}^{\infty} N_{k,N} \text{ and } H^{1,0}_K(\mathbb{R}^n_+) = \bigoplus_{k=1}^{\infty} N_{k,D}. \tag{4.4}$$

We define the following functional on $H^1_K(\mathbb{R}^n_+)$

$$J(\phi) = \frac{1}{2}\int_{\mathbb{R}^n_+}\left(|\nabla\phi|^2 - \frac{1}{2(p-1)}\phi^2\right)Kd\eta + \frac{1}{p+1}\int_{\partial\mathbb{R}^n_+}|\phi|^{p+1}K'd\eta'. \tag{4.5}$$

The critical points of J satisfies

$$\begin{aligned} -\Delta\omega - \frac{1}{2}\eta.\nabla\omega - \frac{1}{2(p-1)}\omega = 0 \qquad & \text{in } \mathbb{R}^n_+ \\ -\omega_{\eta_n} + |\omega|^{p-1}\omega = 0 \qquad & \text{in } \partial\mathbb{R}^n_+. \end{aligned} \tag{4.6}$$

If ω is a solution of (4.6), the function

$$u_\omega(t,x) = t^{-\frac{1}{2(p-1)}}\omega\left(\frac{x}{\sqrt{t}}\right) \tag{4.7}$$

satisfies

$$\begin{aligned} u_{\omega\,t} - \Delta u_\omega = 0 \qquad & \text{in } Q^\infty_{\mathbb{R}^n_+} := (0,\infty) \times \mathbb{R}^n_+ \\ -u_{\omega\,x_n} + |u_\omega|^{p-1}u_\omega = 0 \qquad & \text{in } \partial_\ell Q^\infty_{\mathbb{R}^n_+} := (0,\infty) \times \partial\mathbb{R}^n_+. \end{aligned} \tag{4.8}$$

Here we have set $\mathbb{R}^n_+ = \{x = (x_1, ..., x_n) = (x', x_n) : x_n > 0\}$. We denote by $\mathcal{E}$ the subset $H^1_K(\mathbb{R}^n_+) \cap L^p(\partial\mathbb{R}^n_+; d\eta')$ of solutions of (4.6) and by $\mathcal{E}_+$ the subset of positive solutions. As for the case $n = 1$ we have the following non-existence result

Proposition 4.1 *1- If* $p \geq 1 + \frac{1}{n}$*, then* $\mathcal{E} = \{0\}$.
2- If $1 < p \leq 1 + \frac{1}{n+1}$*, then* $\mathcal{E}_+ = \{0\}$

The proof is similar to the one of Theorem 1.1. Hence the existence is to be found in the range $1 + \frac{1}{n+1} < p < 1 + \frac{1}{n}$.

Conjecture *Assume* $1 + \frac{1}{n+1} < p < 1 + \frac{1}{n}$*, then the functional* J *is bounded from below in* $H^1_K(\mathbb{R}^n_+) \cap L^p_{K'}(\partial\mathbb{R}^n_+)$*. Furthermore* $J(\phi)$ *tends to infinity when* $\|\phi\|_{H^1_K(\mathbb{R}^n_+)} + \left\|\phi\lfloor_{\partial\mathbb{R}^n_+}\right\|_{L^{p+1}_{K'}(\partial\mathbb{R}^n_+)}$ *tends to infinity.*

4.2 Problem with measure data

The method for proving Theorem 1.3 can be adapted to prove the following n-dimensional result

Theorem 4.2 *Let* $g : \mathbb{R} \mapsto \mathbb{R}$ *be a nondecreasing continuous function such that* $g(0) = 0$ *and*

$$\int_1^\infty (g(s) - g(-s))s^{-\frac{2n+1}{n}} ds < \infty, \tag{4.9}$$

then for any bounded Radon measures ν *in* $\mathbb{R}^n_+$ *and* μ *in* $(0,T) \times \partial\mathbb{R}^n_+$*, there exists a unique Borel function* $u := u_{\nu,\mu}$ *defined in* $\overline{Q_T^{\mathbb{R}^n_+}} := [0,T] \times \mathbb{R}^n_+$ *such that* $u \in L^1(Q_T^{\mathbb{R}^n_+})$, $u\lfloor_{(0,T)\times\partial\mathbb{R}^n_+} \in L^1((0,T) \times \partial\mathbb{R}^n_+)$ *and* $g(u) \in L^1((0,T) \times \partial\mathbb{R}^n_+)$ *solution of*

$$\begin{aligned} u_t - \Delta u &= 0 && \text{in } Q^T_{\mathbb{R}^n_+} \\ -u_{x_n} + g(u) &= \mu && \text{in } \partial_\ell Q^T_{\mathbb{R}^n_+} \\ u(0,.) &= \nu && \text{in } \mathbb{R}^n_+, \end{aligned} \tag{4.10}$$

in the sense that

$$\iint_{Q^T_{\mathbb{R}^n_+}} (-\partial_t\zeta - \Delta\zeta)udxdt + \iint_{\partial_\ell Q^T_{\mathbb{R}^n_+}} g(u)\zeta dx'dt = \int_{\mathbb{R}^n_+} \zeta d\nu + \iint_{\partial_\ell Q^T_{\mathbb{R}^n_+}} \zeta d\mu, \tag{4.11}$$

for all $\zeta \in C^{1,2}_c(\overline{Q^T_{\mathbb{R}^n_+}})$ *such that* $\zeta_{x_n} = 0$ *on* $(0,T) \times \partial\mathbb{R}^n_+$ *and* $\zeta(T,.) = 0$*. Furthermore* $(\nu,\mu) \mapsto u_{\nu,\mu})$ *is nondecreasing.*

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