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EXISTENCE AND UNIQUENESS OF STRONG SOLUTIONS FOR THE SYSTEM OF INTERACTION BETWEEN A COMPRESSIBLE NAVIER-STOKES-FOURIER FLUID AND A DAMPED PLATE EQUATION

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ABSTRACT. The article is devoted to the mathematical analysis of a fluid-structure interaction system where the fluid is compressible and heat conducting and where the structure is deformable and located on a part of the boundary of the fluid domain. The fluid motion is modeled by the compressible Navier-Stokes-Fourier system and the structure displacement is described by a structurally damped plate equation. Our main results are the existence of strong solutions in an $L^p - L^q$ setting for small time or for small data. Through a change of variables and a fixed point argument, the proof of the main results is mainly based on the maximal regularity property of the corresponding linear systems. For small time existence, this property is obtained by decoupling the linear system into several standard linear systems whereas for global existence and for small data, the maximal regularity property is proved by showing that the corresponding linear coupled *fluid-structure* operator is $\mathcal{R}$ -sectorial.

Keywords. fluid-structure interaction, compressible Navier–Stokes–Fourier system, maximal regularity, $\mathcal{R}$ -sectorial operators, strong solutions

AMS subject classifications. 35Q30, 76D05, 76N10

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1. INTRODUCTION

In this work, we study the interaction between a viscous compressible heat conducting fluid and a viscoelastic structure located on a part of the fluid domain boundary. More precisely, we consider a smooth bounded domain $\mathcal{F} \subset \mathbb{R}^3$ such that its boundary $\partial\mathcal{F}$ contains a flat part $\Gamma_S := \mathcal{S} \times \{0\}$, where $\mathcal{S}$ is a smooth bounded domain of $\mathbb{R}^2$. We also set

$$\Gamma_0 = \partial\mathcal{F} \setminus \overline{\Gamma_S}.$$

The set Γ_0 is rigid and remains unchanged whereas on the flat part, we assume that there is a plate that can deform only in the transversal direction, and if we denote by η the corresponding displacement, then Γ_S is transformed into

$$\Gamma_S(\eta) := \left\{ [x_1, x_2, \eta(x_1, x_2)]^\top ; [x_1, x_2]^\top \in \mathcal{S} \right\}.$$

In our study, we consider only displacements η regular enough and satisfying the boundary conditions (the plate is clamped):

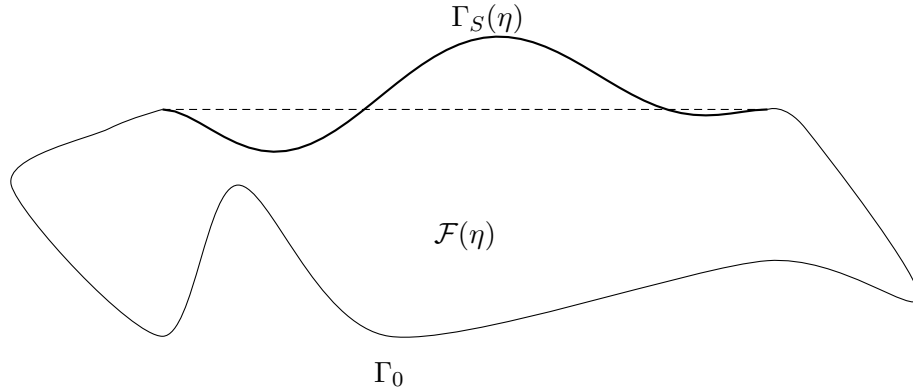
$$\eta = \nabla_s \eta \cdot n_S = 0 \quad \text{on } \partial\mathcal{S} \quad (1.1)$$

and a condition insuring that the deformed plate does not have any contact with the other part of the boundary of the fluid domain:

$$\Gamma_0 \cap \Gamma_S(\eta) = \emptyset. \quad (1.2)$$

We have denoted by n_S the unitary exterior normal to $\partial\mathcal{S}$ and in the whole article we add the index s in the gradient and in the Laplace operators if they apply to functions defined on $\mathcal{S} \subset \mathbb{R}^2$ (and we keep the usual notation for functions defined on a domain of $\mathbb{R}^3$).

With the above notations and hypotheses, $\Gamma_0 \cup \overline{\Gamma_S(\eta)}$ corresponds to a closed simple and regular surface whose interior is the fluid domain $\mathcal{F}(\eta)$. In what follows, we consider that η is also a function of time and its evolution is governed by a damped plate equation.



In $\mathcal{F}(\eta(t))$, we assume that there is a viscous compressible heat conducting fluid and we denote by $\tilde{\rho}$, $\tilde{v}$, and $\tilde{\vartheta}$ respectively its density, velocity and temperature. The equations modeling the evolution of these quantities can be written as follows:

$$\begin{cases} \partial_t \tilde{\rho} + \operatorname{div}(\tilde{\rho} \tilde{v}) = 0, & t > 0, x \in \mathcal{F}(\eta(t)), \\ \tilde{\rho}(\partial_t \tilde{v} + (\tilde{v} \cdot \nabla) \tilde{v}) - \operatorname{div} \mathbb{T}(\tilde{v}, \tilde{\pi}) = 0, & t > 0, x \in \mathcal{F}(\eta(t)), \\ c_v \tilde{\rho} (\partial_t \tilde{\vartheta} + \tilde{v} \cdot \nabla \tilde{\vartheta}) + \tilde{\pi} \operatorname{div} \tilde{v} - \kappa \Delta \tilde{\vartheta} = \alpha (\operatorname{div} \tilde{v})^2 + 2\mu |\mathbb{D} \tilde{v}|^2 & t > 0, x \in \mathcal{F}(\eta(t)), \\ \partial_{tt} \eta + \Delta_s^2 \eta - \Delta_s \partial_t \eta = \mathbb{H}_\eta(\tilde{v}, \tilde{\pi}) & t > 0, s \in \mathcal{S}, \end{cases} \quad (1.3)$$

with the boundary conditions

$$\begin{cases} \tilde{v}(t, s, \eta(t, s)) = \partial_t \eta(t, s) e_3 & t > 0, s \in \mathcal{S}, \\ \tilde{v} = 0 & t > 0, x \in \Gamma_0, \\ \frac{\partial \tilde{\vartheta}}{\partial \tilde{n}}(t, x) = 0 & t > 0, x \in \partial \mathcal{F}(\eta(t)), \\ \eta = \nabla_s \eta \cdot n_S = 0 & t > 0, s \in \partial \mathcal{S}, \end{cases} \quad (1.4)$$

and the initial conditions

$$\begin{cases} \eta(0, \cdot) = \eta_1^0, \quad \partial_t \eta(0, \cdot) = \eta_2^0 & \text{in } \mathcal{S}, \\ \tilde{\rho}(0, \cdot) = \tilde{\rho}^0, \quad \tilde{v}(0, \cdot) = \tilde{v}^0, \quad \tilde{\vartheta}(0, \cdot) = \tilde{\vartheta}^0 & \text{in } \mathcal{F}(\eta_1^0). \end{cases} \quad (1.5)$$

In the above system (e_1, e_2, e_3) is the canonical basis of $\mathbb{R}^3$, the fluid stress tensor is defined by

$$\mathbb{T}(\tilde{v}, \tilde{\pi}) = 2\mu \mathbb{D}(\tilde{v}) + (\alpha \operatorname{div} \tilde{v} - \tilde{\pi}) I_3, \quad \mathbb{D}(\tilde{v}) = \frac{1}{2} (\nabla \tilde{v} + \nabla \tilde{v}^\top),$$

and the pressure law is given by

$$\tilde{\pi} = R_0 \tilde{\rho} \tilde{\vartheta} + \pi_0. \quad (1.6)$$

The above physical constants satisfy

$$R_0 > 0, \quad \mu > 0 \text{ (viscosity)}, \quad \alpha + \frac{2}{3}\mu > 0, \quad \kappa > 0, \quad c_v > 0, \quad \pi_0 \in \mathbb{R}. \quad (1.7)$$

For any matrix $A, B \in \mathcal{M}_d(\mathbb{R})$, we use the canonical scalar product and norm:

$$A : B = \sum_{i,j} a_{ij} b_{ij}, \quad |A| = \sqrt{A : A}.$$

We have set

$$\nabla_s = [\partial_{y_1}, \partial_{y_2}]^\top, \quad \Delta_s = \partial_{y_1}^2 + \partial_{y_2}^2.$$

The function $\mathbb{H}$ is defined by

$$\mathbb{H}_\eta(\tilde{v}, \tilde{\pi}) = -\sqrt{1 + |\nabla_s \eta|^2} (\mathbb{T}(\tilde{v}, \tilde{\pi}) \tilde{n})|_{\Gamma_S(\eta(t))} \cdot e_3, \quad (1.8)$$

where

$$\tilde{n} = \frac{1}{\sqrt{1 + |\nabla_s \eta|^2}} [-\nabla_s \eta, 1]^\top,$$

is the unit normal to $\Gamma_S(\eta(t))$ outward $\mathcal{F}(\eta(t))$. Let us mention that the boundary conditions (1.4) are obtained by assuming that the fluid does not slip on the boundaries and that the plate is thermally insulated.

Fluid-structure interaction problems have been an active area of research among the engineers, physicist and mathematicians over the last few decades due to the numerous practical applications and the corresponding scientific challenges. The type of model considered in this article appears in the

design of many engineering structures, e.g aircraft and bridges etc., ([4]) as well as in biomechanics ([7]).

Let us mention some related works from the literature. In the last two decades, there has been considerable number of works on similar fluid-structure systems where the fluid is modelled by incompressible flows. We refer to, for instance [23] and references therein for a concise description of recent progress regarding incompressible flows interacting with deformable structure (beam or plate) located on a part of the fluid domain boundary. Moreover, in some recent articles ([22, 5, 6]) existence and uniqueness of strong solutions (either local in time or for small initial data) were proved without the additional damping term (i.e., without the term $-\Delta_s \partial_t \eta$) in the beam/plate equation.

Concerning compressible fluids interacting with plate/beam equations through boundary of the fluid domain, there are only few results available in the literature. Global existence of weak solutions until the structure touches the boundary of the fluid domain were proved in [19, 9]. Local in time existence of strong solutions in the corresponding $2D/1D$ case was recently obtained in [33]. Well-posedness and stability of linear compressible fluid-structure systems were studied in [10, 4].

Let us mention that all the above mentioned works correspond to a “Hilbert” space framework. In this article, we are interested in studying existence and uniqueness of strong solutions, local in time or global in time for small initial data, within an “ $L^p - L^q$ ” framework. More precisely, we look for solutions in the spaces of functions which are L^p with respect to time and L^q with respect to space variable, with arbitrary $p, q > 1$. In the context of fluid-solid interaction problems, there are only few articles available in the literature that studies well-posedness in an $L^p - L^q$ framework. Let us mention [20, 32] (viscous incompressible fluid and rigid bodies), [25, 31, 24] (viscous compressible fluid and rigid bodies) and [30, 13] (viscous incompressible fluid interacting with viscoelastic structure located at the boundary of the fluid domain). In fact, this article is a compressible counterpart of our previous work [30].

The main novelties that we bring in this article are :

- The full nonlinear free boundary system coupling viscous compressible Navier-Stokes-Fourier system and a viscoelastic structure located on a part of the fluid domain has not, at the best of our knowledge, been studied in the literature.
- The existence and uniqueness results are proved in $L^p - L^q$ setting.
- Global in time existence for small initial data seems to be a new result for such coupled systems.

Let us emphasize that using the $L^p - L^q$ setting allows us to weaken the regularity on the initial conditions (see for instance [33]). Moreover, this “ $L^p - L^q$ ” framework is interesting even for studies in fluid-structure interaction problems done in the “ $L^2 - L^2$ ” framework: let us quote for instance the uniqueness of weak solutions ([21, 8]), the asymptotic behavior for large time ([17, 18]), and the asymptotic behavior for small structures ([28]).

1.1. Notation. To state our main results, we need to introduce some notations for the functional spaces. For $\Omega \subset \mathbb{R}^n$ is an open set, $q > 1$ and $k \in \mathbb{N}$, we denote by $L^q(\Omega)$ and $W^{k,q}(\Omega)$ the standard Lebesgue and Sobolev spaces respectively. $W^{s,q}(\Omega)$, with $q > 1$ and $s \in \mathbb{R}_+^*$, denotes the usual Sobolev-Slobodeckij space. Moreover, $W_0^{k,q}(\Omega)$ is the completion of $C_c^\infty(\Omega)$ with respect to the $W^{k,q}(\Omega)$ norm. Let $k, m \in \mathbb{N}$, $k < m$. For $1 \leq p < \infty$, $1 \leq q < \infty$, we consider the standard definition of the Besov spaces by real interpolation of Sobolev spaces

$$B_{q,p}^s(\mathcal{F}) = \left(W^{k,q}(\mathcal{F}), W^{m,q}(\mathcal{F}) \right)_{\theta,p} \quad \text{where } s = (1 - \theta)k + \theta m, \quad \theta \in (0, 1).$$

We refer to [1] and [38] for a detailed presentation of the Besov spaces. We denote by C_b^k is the set of continuous and bounded functions with derivatives continuous and bounded up to the order k . For

$s \in (0, 1)$ and a Banach space U , $F_{p,q}^s(0, T, U)$ stands for U valued Lizorkin-Triebel space. For precise definition of such spaces we refer to [38]. If $T \in (0, \infty]$, we set

$$\begin{aligned} W_{p,q}^{1,2}((0, T); \mathcal{F}) &= L^p(0, T; W^{2,q}(\mathcal{F})) \cap W^{1,p}(0, T; L^q(\mathcal{F})), \\ W_{p,q}^{2,4}((0, T); \mathcal{S}) &= L^p(0, T; W^{4,q}(\mathcal{S})) \cap W^{1,p}(0, T; W^{2,q}(\mathcal{S})) \cap W^{2,p}(0, T; L^q(\mathcal{S})), \\ W_{p,q}^{1,2}((0, T); \mathcal{S}) &= L^p(0, T; W^{2,q}(\mathcal{S})) \cap W^{1,p}(0, T; L^q(\mathcal{S})). \end{aligned}$$

We have the following embeddings (see, for instance, [3, Theorem 4.10.2, p.180]),

$$W_{p,q}^{1,2}((0, T); \mathcal{F}) \hookrightarrow C_b^0([0, T]; B_{q,p}^{2(1-1/p)}(\mathcal{F})), \quad (1.9)$$

$$W_{p,q}^{2,4}((0, T); \mathcal{S}) \hookrightarrow C_b^0([0, T]; B_{q,p}^{2(2-1/p)}(\mathcal{S})) \cap C_b^1([0, T]; B_{q,p}^{2(1-1/p)}(\mathcal{S})). \quad (1.10)$$

In particular, in what follows, we use the following norm for $W_{p,q}^{1,2}((0, T); \mathcal{F})$:

$$\|f\|_{W_{p,q}^{1,2}((0, T); \mathcal{F})} := \|f\|_{L^p(0, T; W^{2,q}(\mathcal{F}))} + \|f\|_{W^{1,p}(0, T; L^q(\mathcal{F}))} + \|f\|_{C_b^0([0, T]; B_{q,p}^{2(1-1/p)}(\mathcal{F}))}$$

and we proceed similarly for the two other spaces.

We also introduce functional spaces with time decay. We write for any $\beta \in \mathbb{R}$

$$\mathbb{E}_\beta : \mathbb{R} \rightarrow \mathbb{R}, t \mapsto e^{\beta t}.$$

We denote by $L_\beta^p(0, \infty)$ the space $\mathbb{E}_{-\beta} L^p(0, \infty)$, that is the set of functions f such that $t \mapsto e^{\beta t} f(t)$ is in $L^p(0, \infty)$. The corresponding norm is

$$\|f\|_{L_\beta^p(0, \infty)} := \|\mathbb{E}_\beta f\|_{L^p(0, \infty)}.$$

We proceed similarly for all spaces on $(0, \infty)$ or on $[0, \infty)$.

Finally, we also need to introduce functional spaces for the fluid density, velocity and temperature depending on the displacement η of the structure. Assume $T \in (0, \infty]$ and that $\eta \in W_{p,q}^{2,4}((0, T); \mathcal{S})$ satisfies (1.1) and (1.2). We show in Section 3 that there exists a mapping $X = X_\eta$ such that $X(t, \cdot)$ is a C^1 -diffeomorphism from $\mathcal{F}$ onto $\mathcal{F}(\eta(t))$ and for any function $\tilde{f}$ defined for $t \in (0, T)$ and $x \in \mathcal{F}(\eta(t))$, we then define

$$f(t, y) := \tilde{f}(t, X(t, y)) \quad (t \in (0, T), y \in \mathcal{F}).$$

Then we define the following sets as follows

$$\begin{aligned} \tilde{f} &\in W^{r,p}(0, T; W^{s,q}(\mathcal{F}(\eta(\cdot)))) \quad \text{if} \quad f \in W^{r,p}(0, T; W^{s,q}(\mathcal{F})), \\ \tilde{f} &\in W_{p,q}^{1,2}((0, T); \mathcal{F}(\eta(\cdot))) \quad \text{if} \quad f \in W_{p,q}^{1,2}((0, T); \mathcal{F}), \\ \tilde{f} &\in C^0([0, T]; B_{q,p}^{2(1-1/p)}(\mathcal{F}(\eta(\cdot)))) \quad \text{if} \quad f \in C^0([0, T]; B_{q,p}^{2(1-1/p)}(\mathcal{F})) \end{aligned}$$

and a similar definition for all the other spaces.

1.2. Statement of the main results. Let us give the conditions we require on (p, q) and on the initial data for the system (1.3)–(1.8):

$$2 < p < \infty, \quad 3 < q < \infty, \quad \frac{1}{p} + \frac{1}{2q} \neq \frac{1}{2}, \quad (1.11)$$

$$\eta_1^0 \in B_{q,p}^{2(2-1/p)}(\mathcal{S}), \quad \eta_2^0 \in B_{q,p}^{2(1-1/p)}(\mathcal{S}), \quad \tilde{\rho}^0 \in W^{1,q}(\mathcal{F}(\eta_1^0)), \quad \min_{\mathcal{F}(\eta_1^0)} \tilde{\rho}^0 > 0, \quad (1.12)$$

$$\tilde{v}^0 \in B_{q,p}^{2(1-1/p)}(\mathcal{F}(\eta_1^0))^3, \quad \tilde{\vartheta}^0 \in B_{q,p}^{2(1-1/p)}(\mathcal{F}(\eta_1^0)), \quad (1.13)$$

with the compatibility conditions

$$\eta_1^0 = \nabla \eta_1^0 \cdot n_S = \eta_2^0 = 0 \text{ on } \partial \mathcal{S}, \quad \tilde{v}^0 = 0 \text{ on } \Gamma_0, \quad \tilde{v}^0 = \eta_2^0 e_3 \text{ on } \Gamma_S(\eta_1^0), \quad (1.14)$$

$$\nabla \eta_2^0 \cdot n_S \text{ on } \partial \mathcal{S} \quad \text{and} \quad \frac{\partial \tilde{\vartheta}^0}{\partial n} = 0 \text{ on } \partial \mathcal{F}(\eta_1^0), \quad \text{if} \quad \frac{1}{p} + \frac{1}{2q} < \frac{1}{2}. \quad (1.15)$$

Note that, all the traces in the above relation makes sense for our choice of p and q (see for instance, [38, p. 200]).

We also need a geometrical condition on the initial deformation. Using that $\mathcal{F}$ is a smooth domain, there exist two smooth surfaces $\eta_- : \mathcal{S} \rightarrow \mathbb{R}_-^*$, $\eta_+ : \mathcal{S} \rightarrow \mathbb{R}_+^*$ such that

$$\left\{ [y_1, y_2, y_3]^\top \in \mathcal{S} \times \mathbb{R} ; y_3 \in (\eta_-(y_1, y_2), 0) \right\} \subset \mathcal{F}, \quad (1.16)$$

$$\left\{ [y_1, y_2, y_3]^\top \in \mathcal{S} \times \mathbb{R} ; y_3 \in (0, \eta_+(y_1, y_2)) \right\} \subset \mathbb{R}^3 \setminus \overline{\mathcal{F}}. \quad (1.17)$$

Then our geometrical condition on the initial deformation writes

$$\eta_- < \eta_1^0 < \eta_+ \quad \text{in } \mathcal{S}. \quad (1.18)$$

This yields in particular that $\Gamma_0 \cap \Gamma_S(\eta_1^0) = \emptyset$. According to the geometry, we can in some situation remove the condition $\eta_1^0 < \eta_+$. Note that this condition is not a smallness condition, η_+ and η_- do not need to be small.

Our main results are the following two theorems. The first one is the local in time existence and uniqueness :

Theorem 1.1. *Assume (p, q) satisfies (1.11) and that $[\tilde{\rho}^0, \tilde{v}^0, \tilde{\vartheta}^0, \eta_1^0, \eta_2^0]^\top$ satisfies (1.12)–(1.15) and (1.18). Then there exists $T > 0$, depending only on initial data, such that the system (1.3)–(1.8) admits a unique strong solution $[\tilde{\rho}, \tilde{v}, \tilde{\vartheta}, \eta]^\top$ satisfying*

$$\begin{aligned} \tilde{\rho} &\in W^{1,p}(0, T; W^{1,q}(\mathcal{F}(\eta(\cdot)))) , \quad \tilde{v} \in W_{p,q}^{1,2}((0, T); \mathcal{F}(\eta(\cdot)))^3, \\ \tilde{\vartheta} &\in W_{p,q}^{1,2}((0, T); \mathcal{F}(\eta(\cdot))), \quad \eta \in W_{p,q}^{2,4}((0, T); \mathcal{S}), \\ \Gamma_0 \cap \Gamma_S(\eta(t)) &= \emptyset \quad (t \in [0, T]), \quad \tilde{\rho}(t, x) > 0 \quad (t \in [0, T], x \in \overline{\mathcal{F}(\eta(t))}). \end{aligned}$$

Our second main result states the global existence and uniqueness under a smallness condition on the initial data. Let $\bar{\rho}$ and $\bar{\vartheta}$ be two given positive constants. Let us take in the pressure law (1.6)

$$\pi_0 = -R_0 \bar{\rho} \bar{\vartheta}. \quad (1.19)$$

With the above choice of π_0 , $[\tilde{\rho}, \tilde{v}, \tilde{\vartheta}, \eta]^\top = [\bar{\rho}, 0, \bar{\vartheta}, 0]^\top$ is a steady state solution to the system (1.3)–(1.8).

Then our result states as follows:

Theorem 1.2. *Assume (p, q) satisfies (1.11) and assume that $\bar{\rho}$ and $\bar{\vartheta}$ are two given positive constants such that (1.19) holds. Then there exist $\beta > 0$ and $R > 0$ such that for any $[\tilde{\rho}^0, \tilde{v}^0, \tilde{\vartheta}^0, \eta_1^0, \eta_2^0]^\top$ satisfying (1.12)–(1.15), and*

$$\begin{aligned} \|\tilde{\rho}^0 - \bar{\rho}\|_{W^{1,q}(\mathcal{F}(\eta_1^0))} + \|\tilde{v}^0\|_{B_{q,p}^{2(1-1/p)}(\mathcal{F}(\eta_1^0))^3} + \|\tilde{\vartheta}^0 - \bar{\vartheta}\|_{B_{q,p}^{2(1-1/p)}(\mathcal{F}(\eta_1^0))} \\ + \|\eta_1^0\|_{B_{q,p}^{2(2-1/p)}(\mathcal{S})} + \|\eta_2^0\|_{B_{q,p}^{2(1-1/p)}(\mathcal{S})} \leq R, \end{aligned} \quad (1.20)$$

the system (1.3)–(1.8) admits a unique strong solution $[\tilde{\rho}, \tilde{v}, \tilde{\vartheta}, \eta]^\top$ satisfying

$$\begin{aligned}\tilde{\rho} &\in C_b^0([0, \infty); W^{1,q}(\mathcal{F}(\eta(\cdot))))), \quad \nabla \tilde{\rho} \in W_\beta^{1,p}(0, \infty; L^q(\mathcal{F}(\eta(\cdot))))^3, \quad \partial_t \tilde{\rho} \in L_\beta^p(0, \infty; W^{1,q}(\mathcal{F}(\eta(\cdot))))), \\ \tilde{v} &\in W_{p,q,\beta}^{1,2}((0, T); \mathcal{F}(\eta(\cdot)))^3, \\ \tilde{\vartheta} &\in C_b^0([0, \infty); B_{q,p}^{2(1-1/p)}(\mathcal{F}(\eta(\cdot))))), \quad \nabla \tilde{\vartheta} \in L_\beta^p(0, \infty; W^{1,q}(\mathcal{F}(\eta(\cdot))))^3, \quad \partial_t \tilde{\vartheta} \in L_\beta^p(0, \infty; L^q(\mathcal{F}(\eta(\cdot))))), \\ \eta &\in C_b^0([0, \infty); B_{q,p}^{2(2-1/p)}(\mathcal{S})), \quad \eta \in L_\beta^p(0, \infty; W^{4,q}(\mathcal{S})) + L^\infty(0, \infty; W^{4,q}(\mathcal{S})), \\ \partial_t \eta &\in W_{p,q,\beta}^{1,2}((0, \infty); \mathcal{S}),\end{aligned}$$

and

$$\Gamma_0 \cap \Gamma_S(\eta(t)) = \emptyset \quad (t \in [0, \infty)), \quad \tilde{\rho}(t, x) > 0 \quad (t \in [0, \infty), x \in \overline{\mathcal{F}(\eta(t))}).$$

Remark 1.3. Let us make the following remarks on the above results:

- (1) Note that, in Theorem 1.1 we do not need initial displacement of the plate η_1^0 to be zero. This is a difference with respect to previous works, for instance [33] or our previous work [30] (with an incompressible fluid). Here we manage to handle this case by modifying our change of variables (see Section 3.1).
- (2) In Theorem 1.1 and Theorem 1.2, we do not have any “loss of regularity” at initial time. More precisely, we obtain the continuity of the solution with respect to time in the same space where the initial data belong. Due to the coupling between the fluid system and the structure equations, some results in the literature are stated with this loss of regularity: for instance in [33, Theorem 1.7], there is a loss of order 1/2 in the space regularity for the fluid velocity at initial time.
- (3) As explained above since we work in the “ $L^p - L^q$ ” framework, we need less regularity on the initial conditions than in the Hilbert case done by [33]. More precisely, in [33] the author assumes that the initial conditions satisfy

$$\eta_1^0 = 0, \quad \eta_2^0 \in W^{3,2}(\mathcal{S}), \quad \tilde{\rho}^0 \in W^{2,2}(\mathcal{F}(\eta_1^0)), \quad \tilde{v}^0 \in W^{3,2}(\mathcal{F}(\eta_1^0))^3,$$

with the corresponding compatibility conditions.

- (4) Theorem 1.1 and Theorem 1.2 can be adapted to the 2D/1D case, that is where $\mathcal{F}$ is a regular bounded domain in $\mathbb{R}^2$ such that $\partial\mathcal{F}$ contains a flat part $\Gamma_S = \mathcal{S} \times \{0\}$, where $\mathcal{S}$ is an open bounded interval of $\mathbb{R}$. In that case we can take $p, q \in (2, \infty)$ such that $\frac{1}{p} + \frac{1}{2q} \neq \frac{1}{2}$.
- (5) Instead of taking heat conducting fluid, we can also consider barotropic fluid model, i.e., the system (1.3) without the temperature equation and with the pressure law $\tilde{\pi} = \tilde{\rho}^\gamma$, for some constant $\gamma > 1$. In that case, we can take $1 < p < \infty$ and $n < q < \infty$ ($n = 2$ or 3, the dimension of the fluid domain) such that $\frac{1}{p} + \frac{1}{2q} \neq 1$.

The proofs of Theorem 1.1 and Theorem 1.2 follow a standard approach in the literature on well-posedness for fluid-solid interaction systems. One of the main difficulties in studying fluid-structure models is that the fluid system is written in the deformed configuration (in Eulerian variables) whereas the structure equations are written in the reference configuration (in Lagrangian variables). Since the fluid domain $\mathcal{F}(\eta(t))$ depends on the structure displacement, which is one of unknowns, we first reformulate the problem in a fixed domain. This is achieved thanks to a combination of a *geometric* change of variables (defined through the initial displacement of the structure) and a *Lagrangian* change of coordinates. With this combined change of variables, we reformulate the problem in the reference domain $\mathcal{F}$. In most of the existing literature, a geometric change of variables via the displacement of

the fluid-structure interface is used to rewrite the problem in a fixed domain ([29, 22, 5, 30]). However, in the context of compressible fluid-structure systems, it is more convenient to use a Lagrangian (see for instance [24]) or a combination of geometric and Lagrangian change of coordinates ([25]). In fact, such transformations allow us to use basic contraction mapping theorem. More precisely, this transformation eliminates the difficult term $\tilde{v} \cdot \nabla \tilde{\rho}$ from the density equation.

Next, we associate the original nonlinear problem to a linear one involving the non-homogeneous terms. In the case of the local in time existence, this linear system can be partially decoupled (see system (3.24)-(3.27)). The $L^p - L^q$ regularity of such linear system over finite time interval is obtained by combining various existing maximal $L^p - L^q$ results for parabolic systems. One of the difficulties is that due to the non-zero initial displacement of the beam, we are dealing with linear operators involving variable coefficients. For the global existence part, we use a “monolithic” type approach, which means that the linearized system in consideration is still a coupled system of fluid and structure equations (see system (4.20)-(4.22)). A crucial step is to show the maximal $L^p - L^q$ property of the associated *fluid-structure* linear operator in the infinite time horizon. This is achieved by showing that this operator is $\mathcal{R}$ -sectorial and generates an exponentially stable semigroup in a suitable function space. Finally, for both the existence for small time and the existence for small initial conditions, we end the proof by using the Banach fixed point theorem.

The plan of the paper is as follows. In Section 2, we recall some results concerning $\mathcal{R}$ -sectorial operators that are used both for the proofs of Theorem 1.1 and Theorem 1.2. Then, we prove Theorem 1.1 in Section 3. In Section 3.1, we introduce the combination of Lagrangian and geometric change of coordinates to reformulate the original problem in the reference configuration. Local in time existence for the system written in reference configuration is stated in Theorem 3.1. In Section 3.2, we prove the maximal $L^p - L^q$ regularity of a linearized system, whereas in Section 3.3, we derive estimates for the nonlinear terms in order to prove Theorem 3.1 by using the Banach fixed point theorem. Section 4 is devoted to the proof of Theorem 1.2. In Section 4.1 we apply the same change of variables than in Section 3.1 with some slight modifications and then linearize the system around a constant steady state. The global in time existence for small initial data for the system written in the reference configuration is stated in Theorem 4.1. In Section 4.2, we introduce the so-called *fluid-structure* operator and we show that it is an $\mathcal{R}$ -sectorial operator and in Section 4.3 that it generates an exponentially stable semigroup in a suitable function space. The maximal $L^p - L^q$ regularity of the linearized system is proved in Section 4.4. Finally, in Section 4.5 we show Theorem 4.1. by using the Banach fixed point theorem.

2. SOME BACKGROUND ON $\mathcal{R}$ -SECTORIAL OPERATORS

We recall here some definitions and properties related to $\mathcal{R}$ -sectorial operators. First, let us give the definition of $\mathcal{R}$ -boundedness ($\mathcal{R}$ for Randomized) for a family of operators (see, for instance, [40, 11, 27]):

Definition 2.1. Assume $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces and $\mathcal{E} \subset \mathcal{L}(\mathcal{X}, \mathcal{Y})$. We say that $\mathcal{E}$ is $\mathcal{R}$ -bounded if there exist $p \in [1, \infty)$ and a constant $C > 0$, such that for any integer $N \geq 1$, any $T_1, \dots, T_N \in \mathcal{E}$, any independent Rademacher random variables $r_1, \dots, r_N$, and any $x_1, \dots, x_N \in \mathcal{X}$,

$$\left(\mathbb{E} \left\| \sum_{j=1}^N r_j T_j x_j \right\|_{\mathcal{Y}}^p \right)^{1/p} \leq C \left(\mathbb{E} \left\| \sum_{j=1}^N r_j x_j \right\|_{\mathcal{X}}^p \right)^{1/p}.$$

The $\mathcal{R}_p$ -bound of $\mathcal{E}$ on $\mathcal{L}(\mathcal{X}, \mathcal{Y})$, denoted by $\mathcal{R}_p(\mathcal{E})$, is the smallest constant C in the above inequality.

Let us recall that a Rademacher random variable is a symmetric random variables with value in $\{-1, 1\}$ and that $\mathbb{E}$ denotes the expectation of a random variable. Note that the above definition is independent of $p \in [1, \infty)$ (see, for instance, [11, p.26]). The $\mathcal{R}_p$ -bound has the following properties (see, for instance, Proposition 3.4 in [11]):

$$\mathcal{R}_p(\mathcal{E}_1 + \mathcal{E}_2) \leq \mathcal{R}_p(\mathcal{E}_1) + \mathcal{R}_p(\mathcal{E}_2), \quad \mathcal{R}_p(\mathcal{E}_1 \mathcal{E}_2) \leq \mathcal{R}_p(\mathcal{E}_1) \mathcal{R}_p(\mathcal{E}_2). \quad (2.1)$$

For any $\beta \in (0, \pi)$, we consider the sector $\mathcal{R}$ -sectorial operators:

$$\Sigma_\beta = \{\lambda \in \mathbb{C} \setminus \{0\} ; |\arg(\lambda)| < \beta\}. \quad (2.2)$$

We can introduce the definition of :

Definition 2.2 (sectorial and $\mathcal{R}$ -sectorial operators). *Let $A : \mathcal{D}(A) \rightarrow \mathcal{X}$ be a densely defined closed linear operator on the Banach space $\mathcal{X}$. The operator A is $(\mathcal{R})$ -sectorial of angle $\beta \in (0, \pi)$ if*

$$\Sigma_\beta \subset \rho(A)$$

and if the set

$$R_\beta = \{\lambda(\lambda - A)^{-1} ; \lambda \in \Sigma_\beta\}$$

is $(\mathcal{R})$ -bounded in $\mathcal{L}(\mathcal{X})$.

We denote by $M_\beta(A)$ (respectively $\mathcal{R}_\beta(A)$) the bound (respectively the $\mathcal{R}$ -bound) of R_β . One can replace in the above definitions R_β by the set

$$\widetilde{R}_\beta = \{A(\lambda - A)^{-1} ; \lambda \in \Sigma_\beta\}.$$

In that case, we denote the uniform bound and the $\mathcal{R}$ -bound by $\widetilde{M}_\beta(A)$ and $\widetilde{\mathcal{R}}_\beta(A)$.

The following result, due to [40] (see also [11, p.45]), shows the important relation between the notion of $\mathcal{R}$ -sectoriality and the maximal regularity of type L^p :

Theorem 2.3. *Assume $\mathcal{X}$ is a UMD Banach space and that $A : \mathcal{D}(A) \rightarrow \mathcal{X}$ is a densely defined, closed linear operator on $\mathcal{X}$. Then the following assertions are equivalent:*

- (1) *For any $T \in (0, \infty]$ and for any $f \in L^p(0, T; \mathcal{X})$, the Cauchy problem*

$$u' = Au + f \quad \text{in } (0, T), \quad u(0) = 0 \quad (2.3)$$

admits a unique solution u with $u', Au \in L^p(0, T; \mathcal{X})$ and there exists a constant $C > 0$ such that

$$\|u'\|_{L^p(0, T; \mathcal{X})} + \|Au\|_{L^p(0, T; \mathcal{X})} \leq C \|f\|_{L^p(0, T; \mathcal{X})}.$$

- (2) *A is $\mathcal{R}$ -sectorial of angle $> \frac{\pi}{2}$.*

In the above definition, we recall that $\mathcal{X}$ is a UMD Banach space if the Hilbert transform is bounded in $L^p(\mathbb{R}; \mathcal{X})$ for $p \in (1, \infty)$. In particular, the closed subspaces of $L^q(\Omega)$ for $q \in (1, \infty)$ are UMD Banach spaces. We refer the reader to [3, pp.141–147] for more information on UMD spaces.

Combining the above theorem with [15, Theorem 2.4] and [37, Theorem 1.8.2], we can consider the following Cauchy problem

$$u' = Au + f \quad \text{in } (0, \infty), \quad u(0) = u_0. \quad (2.4)$$

Corollary 2.4. *Assume $\mathcal{X}$ is a UMD Banach space, $1 < p < \infty$ and A is a closed, densely defined operator in $\mathcal{X}$ with domain $\mathcal{D}(A)$. Let us suppose also that A is a $\mathcal{R}$ -sectorial operator of angle $> \frac{\pi}{2}$ and that the semigroup generated by A has negative exponential type. Then for any $u_0 \in (\mathcal{X}, \mathcal{D}(A))_{1-1/p, p}$ and for any $f \in L^p(0, \infty; \mathcal{X})$, the system (2.4) admits a unique solution in $L^p(0, \infty; \mathcal{D}(A)) \cap W^{1,p}(0, \infty; \mathcal{X})$.*

Finally, we will need the following result ([26, Corollary 2]) on the perturbation theory of $\mathcal{R}$ -sectoriality.

Proposition 2.5. *Suppose A is a $\mathcal{R}$ -sectorial operator of angle β on a Banach space $\mathcal{X}$. Assume that $B : \mathcal{D}(B) \rightarrow \mathcal{X}$ is a linear operator such that $\mathcal{D}(A) \subset \mathcal{D}(B)$ and such that there exist $a, b \geq 0$ satisfying*

$$\|Bx\|_{\mathcal{X}} \leq a\|Ax\|_{\mathcal{X}} + b\|x\|_{\mathcal{X}} \quad (x \in \mathcal{D}(A)). \quad (2.5)$$

If

$$a < \frac{1}{\widetilde{M}_{\beta}(A)\widetilde{\mathcal{R}}_{\beta}(A)} \quad \text{and} \quad \lambda > \frac{bM_{\beta}(A)\widetilde{\mathcal{R}}_{\beta}(A)}{1 - a\widetilde{M}_{\beta}(A)\widetilde{\mathcal{R}}_{\beta}(A)},$$

then $A + B - \lambda$ is $\mathcal{R}$ -sectorial of angle β .

3. LOCAL IN TIME EXISTENCE

The aim of this section is to prove Theorem 1.1.

3.1. Change of variables and Linearization. In this subsection, we consider a change of variables to transform the moving domain $\mathcal{F}(\eta(t))$ into the fixed domain $\mathcal{F}$. For this we use the Lagrangian change of variables to write everything in $\mathcal{F}(\eta_1^0)$ and a geometric change of variables to transform $\mathcal{F}(\eta_1^0)$ into $\mathcal{F}$. Let us start with the second one.

First using that $\mathcal{F}$ is smooth, there exist an open bounded neighborhood $\widetilde{\mathcal{S}}$ of $\overline{\mathcal{S}}$ in $\mathbb{R}^2$, $\widetilde{\varepsilon} > 0$ and $\widetilde{\eta} : \widetilde{\mathcal{S}} \rightarrow \mathbb{R}$ smooth such that

$$(\widetilde{\mathcal{S}} \times [-\widetilde{\varepsilon}, \widetilde{\varepsilon}]) \cap \partial\mathcal{F} = \{(s, \widetilde{\eta}(s)), s \in \widetilde{\mathcal{S}}\}.$$

We have in particular that $\widetilde{\eta} \equiv 0$ in $\mathcal{S}$. From (1.16), (1.17), we can extend η_- and η_+ with

$$\{[y_1, y_2, y_3]^{\top} \in \widetilde{\mathcal{S}} \times \mathbb{R} ; y_3 \in (\eta_-(y_1, y_2), \widetilde{\eta}(y_1, y_2))\} \subset \mathcal{F},$$

$$\{[y_1, y_2, y_3]^{\top} \in \widetilde{\mathcal{S}} \times \mathbb{R} ; y_3 \in (\widetilde{\eta}(y_1, y_2), \eta_+(y_1, y_2))\} \subset \mathbb{R}^3 \setminus \overline{\mathcal{F}}.$$

Using (1.13)–(1.14) and that $q > 3$, we can extend η_1^0 by 0 in $\mathbb{R}^2 \setminus \overline{\mathcal{S}}$ with $\eta_1^0 \in W^{2,q}(\mathbb{R}^2)$. Then (1.18) yields the existence of $\varepsilon \in (0, 1)$ such that

$$\eta_-(1 - \varepsilon) < \eta_1^0 < \eta_+(1 - \varepsilon) \quad \text{in } \widetilde{\mathcal{S}}.$$

We consider $\chi \in C_c^\infty(\mathbb{R}^3)$ such that

$$\text{supp } \chi \subset \{[y_1, y_2, y_3]^{\top} \in \widetilde{\mathcal{S}} \times \mathbb{R} ; y_3 \in (\eta_-(y_1, y_2), \eta_+(y_1, y_2))\},$$

$$\chi \equiv 1 \quad \text{in } \{[y_1, y_2, y_3]^{\top} \in \mathcal{S} \times \mathbb{R} ; y_3 \in ((1 - \varepsilon)\eta_-(y_1, y_2), (1 - \varepsilon)\eta_+(y_1, y_2))\}.$$

We also define

$$\Lambda(y_1, y_2, y_3) = \eta_1^0(y_1, y_2)\chi(y_1, y_2, y_3)e_3 \quad [y_1, y_2, y_3]^{\top} \in \mathbb{R}^3$$

and we consider

$$\begin{cases} \zeta'(t, y) = \Lambda(\zeta(t, y)), \\ \zeta(0, y) = y \in \mathbb{R}^3. \end{cases} \quad (3.1)$$

Then

$$X^0 := \zeta(1, \cdot) \quad (3.2)$$

is a C^1 -diffeomorphism such that

$$\begin{aligned} X^0 &\equiv \text{Id} \quad \text{in } \mathbb{R}^3 \setminus \{[y_1, y_2, y_3]^{\top} \in \mathcal{S} \times \mathbb{R} ; y_3 \in (\eta_-(y_1, y_2), \eta_+(y_1, y_2))\} \\ X^0(\Gamma_S(0)) &= \Gamma_S(\eta_1^0), \end{aligned}$$

$$\begin{aligned}
X^0 \left(\left\{ [y_1, y_2, y_3]^\top \in \mathcal{S} \times \mathbb{R} ; y_3 \in (\eta_-(y_1, y_2), 0) \right\} \right) \\
= \left\{ [y_1, y_2, y_3]^\top \in \mathcal{S} \times \mathbb{R} ; y_3 \in (\eta_-(y_1, y_2), \eta_1^0(y_1, y_2)) \right\}.
\end{aligned}$$

In particular, X^0 is a C^1 -diffeomorphism such that $X^0(\mathcal{F}) = \mathcal{F}(\eta_1^0)$ and such that $X^0 = \text{Id}$ on Γ_0 .

We consider the characteristics X associated with the fluid velocity $\tilde{v}$:

$$\begin{cases} \partial_t X(t, y) = \tilde{v}(t, X(t, y)) & (t > 0), \\ X(0, y) = X^0(y), & y \in \overline{\mathcal{F}}. \end{cases} \quad (3.3)$$

Assume that X is a C^1 -diffeomorphism from $\overline{\mathcal{F}}$ onto $\overline{\mathcal{F}(\eta(t))}$ for all $t \in (0, T)$. For each $t \in (0, T)$, we denote by $Y(t, \cdot) = [X(t, \cdot)]^{-1}$ the inverse of $X(t, \cdot)$. We consider the following change of variables

$$\begin{aligned} \rho(t, y) &= \tilde{\rho}(t, X(t, y)), & v(t, y) &= \tilde{v}(t, X(t, y)), \\ \vartheta(t, y) &= \tilde{\vartheta}(t, X(t, y)), & \pi &= R_0 \rho \vartheta + \pi_0, \end{aligned} \quad (3.4)$$

for $(t, y) \in (0, T) \times \mathcal{F}$. In particular,

$$\tilde{\rho}(t, x) = \rho(t, Y(t, x)), \quad \tilde{v}(t, x) = v(t, Y(t, x)), \quad \tilde{\vartheta}(t, x) = \vartheta(t, Y(t, x)),$$

for $(t, x) \in (0, T) \times \mathcal{F}(\eta(t))$. We introduce the notation

$$\mathbb{B}_X := \text{Cof } \nabla X, \quad \delta_X := \det \nabla X, \quad \mathbb{A}_X := \frac{1}{\delta_X} \mathbb{B}_X^\top \mathbb{B}_X, \quad (3.5)$$

$$\mathbb{B}^0 := \mathbb{B}_{X^0}, \quad \delta^0 := \delta_{X^0}, \quad \mathbb{A}^0 := \mathbb{A}_{X^0}. \quad (3.6)$$

This change of variables transforms (1.3)–(1.8) into the following system for $[\rho, v, \vartheta, \eta]^\top$:

$$\begin{cases} \partial_t \rho + \frac{\rho^0}{\delta^0} \nabla v : \mathbb{B}^0 = F_1 & \text{in } (0, T) \times \mathcal{F}, \\ \rho(0, \cdot) = \rho^0 & \text{in } \mathcal{F}, \end{cases} \quad (3.7)$$

$$\begin{cases} \partial_t v - \mathcal{L}v = F_2 & \text{in } (0, T) \times \mathcal{F}, \\ v = 0 & \text{on } (0, T) \times \Gamma_0, \\ v = \partial_t \eta e_3 & \text{on } (0, T) \times \Gamma_S, \\ v(0, \cdot) = v^0 & \text{in } \mathcal{F}, \end{cases} \quad (3.8)$$

$$\begin{cases} \partial_t \vartheta - \frac{\kappa}{c_v \rho^0 \delta^0} \text{div}(\mathbb{A}^0 \nabla \vartheta) = F_3 & \text{in } (0, T) \times \mathcal{F}, \\ \mathbb{A}^0 \nabla \vartheta \cdot n = G & \text{on } (0, T) \times \partial \mathcal{F}, \\ \vartheta(0, \cdot) = \vartheta^0 & \text{in } \mathcal{F}, \end{cases} \quad (3.9)$$

$$\begin{cases} \partial_{tt} \eta + \Delta_s^2 \eta - \Delta_s \partial_t \eta = H & \text{in } (0, T) \times \mathcal{S}, \\ \eta = \nabla \eta \cdot n_S = 0 & \text{on } (0, T) \times \partial \mathcal{S}, \\ \eta(0, \cdot) = \eta_1^0, \quad \partial_t \eta(0, \cdot) = \eta_2^0 & \text{in } \mathcal{S}, \end{cases} \quad (3.10)$$

where we have used the following notation

$$\rho^0 := \tilde{\rho}^0 \circ X^0, \quad v^0 := \tilde{v}^0 \circ X^0, \quad \vartheta^0 := \tilde{\vartheta}^0 \circ X^0, \quad (3.11)$$

$$\mathcal{L}v = \frac{1}{\rho^0 \delta^0} \text{div } \mathbb{T}^0(v), \quad \mathbb{T}^0(v) := \mu \nabla v \mathbb{A}^0 + \frac{\mu + \alpha}{\delta^0} \mathbb{B}^0 (\nabla v)^\top \mathbb{B}^0 \quad (3.12)$$

$$F_1(\rho, v, \vartheta, \eta) := \frac{\rho^0}{\delta^0} \nabla v : \mathbb{B}^0 - \frac{\rho}{\delta_X} \nabla v : \mathbb{B}_X \quad (3.13)$$

$$F_2(\rho, v, \vartheta, \eta) := \frac{1}{\rho^0 \delta^0} \left[(\rho^0 \delta^0 - \rho \delta_X) \partial_t v + \mu \operatorname{div} (\nabla v (\mathbb{A}_X - \mathbb{A}^0)) \right. \\ \left. + (\mu + \alpha) \operatorname{div} \left[\frac{1}{\delta_X} \mathbb{B}_X (\nabla v)^\top \mathbb{B}_X - \frac{1}{\delta^0} \mathbb{B}^0 (\nabla v)^\top \mathbb{B}^0 \right] + R_0 \mathbb{B}_X \nabla (\rho \vartheta) \right] \quad (3.14)$$

$$F_3(\rho, v, \vartheta, \eta) := \frac{1}{c_v \rho^0 \delta^0} \left[c_v (\rho^0 \delta^0 - \rho \delta_X) \partial_t \vartheta + \kappa \operatorname{div} ((\mathbb{A}_X - \mathbb{A}^0) \nabla \vartheta) \right. \\ \left. + \frac{\alpha}{\delta_X} (\mathbb{B}_X : \nabla v)^2 + \frac{\mu}{2\delta_X} \left| \nabla v \mathbb{B}_X^\top + \mathbb{B}_X \nabla v^\top \right|^2 - (R_0 \rho \vartheta + \pi_0) \nabla v : \mathbb{B}_X \right] \quad (3.15)$$

$$G(\rho, v, \vartheta, \eta) = (\mathbb{A}^0 - \mathbb{A}_X) \nabla \vartheta \cdot n \quad (3.16)$$

$$H(\rho, v, \vartheta, \eta) = -\frac{\mu}{\delta_X} (\nabla v \mathbb{B}_X^\top + \mathbb{B}_X \nabla v^\top) \begin{bmatrix} -\nabla_s \eta \\ 1 \end{bmatrix} \cdot e_3 - \frac{\alpha}{\delta_X} \nabla v : \mathbb{B}_X + R_0 \rho \vartheta + \pi_0. \quad (3.17)$$

The characteristics X defined in (3.3) can now be written as

$$X(t, y) = X^0(y) + \int_0^t v(r, y) \, dr, \quad (3.18)$$

for every $y \in \mathcal{F}$ and $t \geq 0$.

The hypotheses (1.12)–(1.15) on the initial conditions are transformed into the following conditions

$$\rho^0 \in W^{1,q}(\mathcal{F}), \quad \min_{\overline{\mathcal{F}}} \rho^0 > 0, \quad (3.19)$$

$$\eta_1^0 \in B_{q,p}^{2(2-1/p)}(\mathcal{S}), \quad \eta_1^0 = \nabla_s \eta_1^0 \cdot n_S = 0 \text{ on } \mathcal{S}, \quad (3.20)$$

$$v^0 \in B_{q,p}^{2(1-1/p)}(\mathcal{F})^3, \quad \vartheta^0 \in B_{q,p}^{2(1-1/p)}(\mathcal{F}), \quad \eta_2^0 \in B_{q,p}^{2(1-1/p)}(\mathcal{S}), \quad (3.21)$$

$$v^0 = 0 \text{ on } \Gamma_0, \quad v^0 = \eta_2^0 e_3 \text{ on } \Gamma_S, \quad \eta_2^0 = 0 \text{ on } \partial \mathcal{S}, \quad (3.22)$$

$$\nabla \eta_2^0 \cdot n_S = 0 \text{ on } \partial \mathcal{S} \quad \text{and} \quad \mathbb{A}^0 \nabla \vartheta^0 \cdot n = 0 \quad \text{on } \partial \mathcal{F} \quad \text{if} \quad \frac{1}{p} + \frac{1}{2q} < \frac{1}{2}. \quad (3.23)$$

Here n is the unit normal to $\partial \mathcal{F}$ outward to $\mathcal{F}$. The regularity properties in (3.19) and (3.21) can be obtained from (1.12), (1.13) by applying [30, Lemma 2.1]. Using the above change of variables, our main result in Theorem 1.1 can be rephrased as

Theorem 3.1. *Assume (p, q) satisfies (1.11) and that $[\rho^0, v^0, \vartheta^0, \eta_1^0, \eta_2^0]^\top$ satisfies (3.19)–(3.23) and (1.18). Then there exists $T > 0$ such that the system (3.7)–(3.18) admits a unique strong solution*

$$[\rho, v, \vartheta, \eta]^\top \in W^{1,p}(0, T; W^{1,q}(\mathcal{F})) \times (W_{p,q}^{1,2}((0, T); \mathcal{F}))^3 \times W_{p,q}^{1,2}((0, T); \mathcal{F}) \times W_{p,q}^{2,4}((0, T); \mathcal{S})$$

Moreover,

$$\min_{[0,T] \times \overline{\mathcal{F}}} \rho > 0, \quad \Gamma_0 \cap \Gamma_S(\eta(t)) = \emptyset \quad (t \in [0, T]),$$

and for all $t \in [0, T]$, $X(t, \cdot) : \mathcal{F} \rightarrow \mathcal{F}(\eta(t))$ is a C^1 -diffeomorphism.

3.2. Maximal L^p - L^q regularity of a linear system. The proof of Theorem 3.1 relies on the Banach fixed point theorem and on maximal L^p - L^q estimates of a linearized system. By replacing the nonlinear terms F_1, F_2, F_3, G and H in (3.7)–(3.10) by given source terms f_1, f_2, f_3, g and h we obtain the following linear system

$$\begin{cases} \partial_t \rho + \frac{\rho^0}{\delta^0} \nabla v : \mathbb{B}^0 = f_1 & \text{in } (0, T) \times \mathcal{F}, \\ \rho(0, \cdot) = \hat{\rho}^0 & \text{in } \mathcal{F}, \end{cases} \quad (3.24)$$

$$\begin{cases} \partial_t v - \mathcal{L}v = f_2 & \text{in } (0, T) \times \mathcal{F}, \\ v = 0 & \text{on } (0, T) \times \Gamma_F, \\ v = \partial_t \eta e_3 & \text{on } (0, T) \times \Gamma_S, \\ v(0, \cdot) = v^0 & \text{in } \mathcal{F}, \end{cases} \quad (3.25)$$

$$\begin{cases} \partial_t \vartheta - \frac{\kappa}{c_v \rho^0 \delta^0} \operatorname{div} (\mathbb{A}^0 \nabla \vartheta) = f_3 & \text{in } (0, T) \times \mathcal{F}, \\ \mathbb{A}^0 \nabla \vartheta \cdot n = g & \text{on } (0, T) \times \partial \mathcal{F}, \\ \vartheta(0, \cdot) = \vartheta^0 & \text{in } \mathcal{F}, \end{cases} \quad (3.26)$$

$$\begin{cases} \partial_{tt} \eta + \Delta_s^2 \eta - \Delta_s \partial_t \eta = h & \text{in } (0, T) \times \mathcal{S}, \\ \eta = \nabla \eta \cdot n_S = 0 & \text{on } (0, T) \times \partial \mathcal{S}, \\ \eta(0, \cdot) = \hat{\eta}_1^0, \quad \partial_t \eta(0, \cdot) = \eta_2^0 & \text{in } \mathcal{S}, \end{cases} \quad (3.27)$$

where $\mathbb{A}^0, \mathbb{B}^0, \delta^0$ are defined in (3.6) and where $\mathcal{L}$ is defined by (3.12). Note that we also modify the initial conditions in the above system with respect to (3.7)–(3.10) since ρ^0 and η_1^0 already appear in the coefficients of (3.24)–(3.27). In the next section, we will take

$$\hat{\rho}^0 = \rho^0, \quad \hat{\eta}_1^0 = \eta_1^0$$

but here we do not assume the above relation. In particular, we assume that ρ^0 satisfies the second condition of (3.19) and that η_1^0 satisfies (1.18) but we do not impose these hypotheses on $\hat{\rho}^0$ and on $\hat{\eta}_1^0$.

We recall that (p, q) satisfies (1.11) and to simplify, we assume throughout this section that

$$T \in (0, 1].$$

This condition is only used to avoid the dependence in time of the constants in the estimates of this section.

We consider the subset of initial conditions

$$\begin{aligned} \mathcal{I}_{p,q} = \left\{ [\hat{\rho}^0, v^0, \vartheta^0, \hat{\eta}_1^0, \eta_2^0]^\top \in W^{1,q}(\mathcal{F}) \times B_{q,p}^{2(1-1/p)}(\mathcal{F})^3 \times B_{q,p}^{2(1-1/p)}(\mathcal{F}) \times B_{q,p}^{2(2-1/p)}(\mathcal{S}) \times B_{q,p}^{2(1-1/p)}(\mathcal{S}), \right. \\ \left. v^0 = 0 \text{ on } \Gamma_0, \quad v^0 = \eta_2^0 e_3 \text{ on } \Gamma_S, \quad \hat{\eta}_1^0 = \frac{\partial \hat{\eta}_1^0}{\partial n_S} = \eta_2^0 = 0 \text{ on } \partial \mathcal{S}, \right. \\ \left. \frac{\partial \eta_2^0}{\partial n_S} = 0 \text{ on } \partial \mathcal{S} \quad \text{and} \quad \mathbb{A}^0 \nabla \vartheta^0 \cdot n = 0 \text{ on } \partial \mathcal{F} \quad \text{if} \quad \frac{1}{p} + \frac{1}{2q} < \frac{1}{2} \right\}, \quad (3.28) \end{aligned}$$

endowed with the norm

$$\begin{aligned} \left\| [\hat{\rho}^0, v^0, \vartheta^0, \hat{\eta}_1^0, \eta_2^0]^\top \right\|_{\mathcal{I}_{p,q}} := & \|\hat{\rho}^0\|_{W^{1,q}(\mathcal{F})} + \|v^0\|_{B_{q,p}^{2(1-1/p)}(\mathcal{F})^3} + \|\vartheta^0\|_{B_{q,p}^{2(1-1/p)}(\mathcal{F})} \\ & + \|\hat{\eta}_1^0\|_{B_{q,p}^{2(2-1/p)}(\mathcal{S})} + \|\eta_2^0\|_{B_{q,p}^{2(1-1/p)}(\mathcal{S})}. \end{aligned}$$

We also consider the space $\mathcal{R}_{T,p,q}$ of the source terms in (3.24)–(3.27):

$$\begin{aligned} \mathcal{R}_{T,p,q} = \Big\{ [f_1, f_2, f_3, g, h]^\top ; \quad & f_1 \in L^p(0, T; W^{1,q}(\mathcal{F})), f_2 \in L^p(0, T; L^q(\mathcal{F}))^3, \\ & f_3 \in L^p(0, T; L^q(\mathcal{F})), g \in F_{p,q}^{(1-1/q)/2}(0, T; L^q(\partial\mathcal{F})) \cap L^p(0, T; W^{1-1/q,q}(\partial\mathcal{F})), \\ & h \in L^p(0, T; L^q(\mathcal{S})), \text{ with } g(0, \cdot) = 0 \text{ if } \frac{1}{p} + \frac{1}{2q} < \frac{1}{2} \Big\}, \end{aligned} \quad (3.29)$$

with

$$\begin{aligned} \|[f_1, f_2, f_3, g, h]^\top\|_{\mathcal{R}_{T,p,q}} &= \|f_1\|_{L^p(0,T;W^{1,q}(\mathcal{F}))} + \|f_2\|_{L^p(0,T;L^q(\mathcal{F}))^3} + \|f_3\|_{L^p(0,T;L^q(\mathcal{F}))} \\ &\quad + \|g\|_{F_{p,q}^{(1-1/q)/2}(0,T;L^q(\partial\mathcal{F})) \cap L^p(0,T;W^{1-1/q,q}(\partial\mathcal{F}))} + \|h\|_{L^p(0,T;L^q(\mathcal{S}))}. \end{aligned}$$

Finally, the space $\mathcal{W}_{T,p,q}$ of the solutions $[\rho, u, \vartheta, \eta]^\top$ of (3.24)–(3.27) is the Cartesian product:

$$\mathcal{W}_{T,p,q} = W^{1,p}(0, T; W^{1,q}(\mathcal{F})) \times W_{p,q}^{1,2}((0, T); \mathcal{F})^3 \times W_{p,q}^{1,2}((0, T); \mathcal{F}) \times W_{p,q}^{2,4}((0, T); \mathcal{S}), \quad (3.30)$$

with the norm

$$\|[\rho, u, \vartheta, \eta]^\top\|_{\mathcal{W}_{T,p,q}} := \|\rho\|_{W^{1,p}(0,T;W^{1,q}(\mathcal{F}))} + \|u\|_{W_{p,q}^{1,2}((0,T);\mathcal{F})^3} + \|\vartheta\|_{W_{p,q}^{1,2}((0,T);\mathcal{F})} + \|\eta\|_{W_{p,q}^{2,4}((0,T);\mathcal{S})}.$$

With the above notation, we can state the main result of this section:

Theorem 3.2. *Assume (1.11) (3.19), (3.20) and (1.18). Then for any*

$$[\hat{\rho}^0, v^0, \vartheta^0, \hat{\eta}_1^0, \eta_2^0]^\top \in \mathcal{I}_{p,q}, \quad [f_1, f_2, f_3, g, h]^\top \in \mathcal{R}_{T,p,q}, \quad (3.31)$$

the system (3.24)–(3.27) admits a unique solution $[\rho, v, \vartheta, \eta]^\top \in \mathcal{W}_{T,p,q}$ and there exists a constant $C > 0$ depending on p, q and independent of T such that

$$\|[\rho, v, \vartheta, \eta]^\top\|_{\mathcal{W}_{T,p,q}} \leq C \left(\left\| [\hat{\rho}^0, v^0, \vartheta^0, \hat{\eta}_1^0, \eta_2^0]^\top \right\|_{\mathcal{I}_{p,q}} + \left\| [f_1, f_2, f_3, g, h]^\top \right\|_{\mathcal{R}_{T,p,q}} \right). \quad (3.32)$$

In order to prove the above result, we notice that the system (3.24)–(3.27) can be solved in “cascades”. Systems (3.26) and (3.27) can be solved independently. With the solution of system (3.27) we can solve the system (3.25) and then (3.24).

We first need the following result on the coefficients appearing in the system (3.24)–(3.27):

Lemma 3.3. *Assume (1.11) (3.19), (3.20) and (1.18). Then $\mathbb{A}^0, \mathbb{B}^0, \delta^0$ defined in (3.6) satisfy*

$$\delta^0 > 0, \quad \mathbb{A}^0 = (\mathbb{A}^0)^\top, \quad \frac{1}{\delta^0} \in W^{1,q}(\mathcal{F}), \quad \mathbb{B}^0, \mathbb{A}^0 \in W^{1,q}(\mathcal{F})^9,$$

and there exists $c^0 > 0$ such that

$$\mathbb{A}^0 \geq c^0 \mathbb{I}_3 \quad \text{in } \overline{\mathcal{F}}.$$

Proof. The proof relies on the dependence of the solutions of (3.1) with respect to the initial conditions. Using that $\eta_1^0 \in W^{2,q}(\mathbb{R}^2)$ for $q > 3$ and Sobolev embedding, we have that $\Lambda \in C_b^1(\mathbb{R}^3)$. In particular, from standard results (see, for instance, [2, p.116]), we have that $\zeta \in C^1(\mathbb{R} \times \mathbb{R}^3)$ and by using the ordinary differential equation satisfied by the derivatives of ζ in space, we find that $X^0 \in W^{2,q}(\mathcal{F})^3$ and ∇X^0 is invertible. This yields the result. $\square$

We are now in a position to prove Theorem 3.2:

Proof of Theorem 3.2. The proof is divided in several steps devoted to the resolution of each system.

Step 1: we show here that (3.27) admits a unique solution $\eta \in W_{p,q}^{2,4}((0,T) \times \mathcal{S})$ and that there exists a constant C independent of T such that

$$\|\eta\|_{W_{p,q}^{2,4}((0,T);\mathcal{S})} + \|\partial_t \eta\|_{W_{p,q}^{1,2}((0,T);\mathcal{S})} \leq C \left(\|\hat{\eta}_1^0\|_{B_{q,p}^{2(2-1/p)}(\mathcal{S})} + \|\eta_2^0\|_{B_{q,p}^{2(1-1/p)}(\mathcal{S})} + \|h\|_{L^p(0,T;L^q(\mathcal{S}))} \right). \quad (3.33)$$

To prove this, we combine [14, Theorem 5.1] and [40, Theorem 4.2]. For the sake of clarity, we provide brief details about the proof. We first consider

$$\mathcal{X}_S := W_0^{2,q}(\mathcal{S}) \times L^q(\mathcal{S}), \quad (3.34)$$

and the operator A_S defined by

$$\mathcal{D}(A_S) = \left(W^{4,q}(\mathcal{S}) \cap W_0^{2,q}(\mathcal{S}) \right) \times W_0^{2,q}(\mathcal{S}), \quad A_S = \begin{bmatrix} 0 & \text{Id} \\ -\Delta^2 & \Delta \end{bmatrix}. \quad (3.35)$$

With the above notation, the system (3.27) can be written as

$$\frac{d}{dt} \begin{bmatrix} \eta \\ \partial_t \eta \end{bmatrix} = A_S \begin{bmatrix} \eta \\ \partial_t \eta \end{bmatrix} + \begin{bmatrix} 0 \\ h \end{bmatrix}, \quad \begin{bmatrix} \eta \\ \partial_t \eta \end{bmatrix} (0) = \begin{bmatrix} \hat{\eta}_1^0 \\ \eta_2^0 \end{bmatrix}.$$

Applying Theorem 5.1 in [14], we have that A_S is $\mathcal{R}$ -sectorial in $\mathcal{X}_S$ of angle $\beta_0 > \pi/2$ (see Section 2). Thus the operator A_S has maximal regularity L^p -regularity in $\mathcal{X}_S$ ([40, Theorem 4.2] or Corollary 2.4). More precisely, for every $h \in L^p(0,T;L^q(\mathcal{F}))$ and for every $(\hat{\eta}_1^0, \eta_2^0) \in (\mathcal{X}_S, \mathcal{D}(A_S))_{1-1/p,p}$, the system (3.27) admits a unique strong solution with

$$\eta \in L^p(0,T;W^{4,q}(\mathcal{S})) \cap W^{2,p}(0,T;L^q(\mathcal{S})).$$

In order to obtain the estimate (3.33) independent of T , we proceed as [24, Proposition 2.2].

Step 2: we show now that the system (3.25) admits a unique solution $v \in W_{p,q}^{1,2}((0,T);\mathcal{F})^3$ and that there exists a constant $C > 0$ depending only on the geometry such that

$$\|v\|_{W_{p,q}^{1,2}((0,T);\mathcal{F})^3} \leq C \left(\|\hat{\eta}_1^0\|_{B_{q,p}^{2(2-1/p)}(\mathcal{S})} + \|\eta_2^0\|_{B_{q,p}^{2(1-1/p)}(\mathcal{S})} + \|v^0\|_{B_{q,p}^{2(1-1/p)}(\mathcal{F})^3} + \|h\|_{L^p(0,T;L^q(\mathcal{S}))} + \|f_2\|_{L^p(0,T;L^q(\mathcal{F}))^3} \right). \quad (3.36)$$

To do this, we are going to apply [12, Theorem 2.3] and for this, we first reduce the problem to the case of homogeneous boundary conditions.

Using that $\mathcal{F}$ is a smooth domain, there exists an open bounded neighborhood $\tilde{\mathcal{S}}$ of $\bar{\mathcal{S}}$ in $\mathbb{R}^2$, $\tilde{\varepsilon} > 0$ and $\tilde{\eta} : \tilde{\mathcal{S}} \rightarrow \mathbb{R}$ smooth such that

$$\left(\tilde{\mathcal{S}} \times [-\tilde{\varepsilon}, \tilde{\varepsilon}] \right) \cap \partial\mathcal{F} = \left\{ (s, \tilde{\eta}(s)), s \in \tilde{\mathcal{S}} \right\}. \quad (3.37)$$

We consider $\chi \in C_c^\infty(\mathbb{R}^3)$ such that

$$\text{supp } \chi \subset \tilde{\mathcal{S}} \times [-\tilde{\varepsilon}, \tilde{\varepsilon}], \quad \chi \equiv 1 \quad \text{in } \mathcal{S} \times [-\tilde{\varepsilon}/2, \tilde{\varepsilon}/2].$$

Then we define

$$w(t, y_1, y_2, y_3) := \chi(y_1, y_2, y_3) \partial_t \eta(t, y_1, y_2) e_3 \quad ((t, y_1, y_2, y_3) \in (0, T) \times \mathbb{R}^3) \quad (3.38)$$

and we set $u = v - w$ so that u is the solution of

$$\begin{cases} \partial_t u - \mathcal{L}u = \hat{f}_2 := f_2 - \partial_t w - \mathcal{L}w & \text{in } (0, T) \times \mathcal{F}, \\ u = 0 & \text{on } (0, T) \times \partial\mathcal{F}, \\ u(0, \cdot) = u^0 := v^0 - w(0, \cdot) & \text{in } \mathcal{F}, \end{cases} \quad (3.39)$$

From Lemma 3.3 and (3.33), there exists a positive constant C independent of T such that

$$\begin{aligned} \|\widehat{f}_2\|_{L^p(0,T;L^q(\mathcal{F}))^3} &\leq C \left(\|\widehat{\eta}_1^0\|_{B_{q,p}^{2(2-1/p)}(\mathcal{S})} + \|\eta_2^0\|_{B_{q,p}^{2(1-1/p)}(\mathcal{S})} + \|h\|_{L^p(0,T;L^q(\mathcal{S}))} + \|f_2\|_{L^p(0,T;L^q(\mathcal{F}))^3} \right), \\ \|u^0\|_{B_{q,p}^{2(1-1/p)}(\mathcal{F})^3} &\leq \|v^0\|_{B_{q,p}^{2(1-1/p)}(\mathcal{F})^3} + \|\eta_2^0\|_{B_{q,p}^{2(1-1/p)}(\mathcal{S})}. \end{aligned}$$

Moreover, $u^0 = 0$ on $\partial\mathcal{F}$. To obtain the result it remains to show that for $u^0 \in B_{q,p}^{2(1-1/p)}(\mathcal{F})^3$ with $u^0 = 0$ on $\partial\mathcal{F}$ and for $\widehat{f}_2 \in L^p(0,T;L^q(\mathcal{F}))^3$, system (3.39) admits a unique strong solution in $W_{p,q}^{1,2}((0,T);\mathcal{F})^3$ with an estimate independent of T . In order to do this, we are going to apply [12, Theorem 2.3].

Let us denote by $\mathcal{L}_0(y, \xi)$ the principal symbol of the operator $\mathcal{L}$ defined by (3.12). Then we have

$$\mathcal{L}_0(\cdot, \xi) = \frac{\mu}{\rho^0 \delta^0} (\mathbb{A}^0 \xi \cdot \xi) I_3 + \frac{\mu + \alpha}{\rho^0 (\delta^0)^2} (\mathbb{B}^0 \xi) \otimes (\mathbb{B}^0 \xi).$$

In particular, $\mathcal{L}_0(\cdot, \xi)$ is symmetric and using (3.6) and (1.7), there exists c^0 such that

$$\mathcal{L}_0(y, \xi) a \cdot a \geq c^0 |a|^2 \quad (y \in \mathcal{F}, a, \xi \in \mathbb{R}^3, |\xi| = 1). \quad (3.40)$$

This shows condition **(E)** (ellipticity of the interior symbol) of [12].

Since we are in the case of the Dirichlet boundary conditions, (3.40) yields the Lopatinskii–Shapiro condition **(LS)**, see for instance, [34, Proposition 6.2.13 and Remark (i), p.270].

Finally, applying again Lemma 3.3 and using that $q > 3$, we can verify that **(SD1)** and **(SB1)** hold true. We can thus apply [12, Theorem 2.3] and deduce that the system (3.39) admits a unique solution $u \in W_{p,q}^{1,2}((0,T);\mathcal{F})^3$. This yields that the system (3.25) admits a unique solution $v \in W_{p,q}^{1,2}((0,T);\mathcal{F})^3$. In order to show that the estimate (3.36) holds with a constant independent of T , we can proceed as [24, Proposition 2.2].

Step 3: next we prove that the system (3.26) admits a unique strong solution $\vartheta \in W_{p,q}^{1,2}((0,T);\mathcal{F})$ and that there exists a constant $C > 0$, depending only on the geometry such that

$$\begin{aligned} \|\vartheta\|_{W_{p,q}^{1,2}((0,T);\mathcal{F})} &\leq C \left(\|\vartheta^0\|_{B_{q,p}^{2(1-1/p)}(\mathcal{F})} + \|f_3\|_{L^p(0,T;L^q(\mathcal{F}))} + \|g\|_{F_{p,q}^{(1-1/q)/2}(0,T;L^q(\partial\mathcal{F}))} \right. \\ &\quad \left. + \|g\|_{L^p(0,T;W^{1-1/q,q}(\partial\mathcal{F}))} \right). \quad (3.41) \end{aligned}$$

As for the previous step, we are going to apply [12, Theorem 2.3]. The principal symbol associated with the operator $\vartheta \mapsto -\frac{\kappa}{c_v \rho^0 \delta^0} \operatorname{div}(\mathbb{A}^0 \nabla \vartheta)$ is

$$a_0(\cdot, \xi) = \frac{\kappa}{c_v \rho^0 \delta^0} \mathbb{A}^0 \xi \cdot \xi$$

and from Lemma 3.3 it satisfies $a_0(\cdot, \xi) \geq c^1 > 0$ for ξ such that $|\xi| = 1$. This shows condition **(E)** (ellipticity of the interior symbol) of [12].

Due to Theorem 10.4 in [41, p.145], the above operator is properly elliptic and following Example 11.6 in [41, pp.160-161], we see that the Lopatinskii–Shapiro condition **(LS)** holds true.

Finally, applying again Lemma 3.3 and using that $q > 3$, we can verify that **(SD1)** and **(SB1)** hold true.

Thus all the conditions of [12, Theorem 2.3] are satisfied. Finally, to obtain the estimate (3.41) with constant independent of T we can proceed as [24, Proposition 2.2].

Step 4: it only remains to prove the estimate for ρ . It follows from $v \in W_{p,q}^{1,2}((0,T);\mathcal{F})^3$ and Lemma 3.3 that the system (3.24) admits a unique solution $\rho \in W^{1,p}(0,T;W^{1,q}(\mathcal{F}))$ and there exists

a constant C independent of T such that

$$\|\rho\|_{W^{1,p}(0,T;W^{1,q}(\mathcal{F}))} \leq C \left(\|v\|_{W^{1,2}_{p,q}((0,T);\mathcal{F})^3} + \|\hat{\rho}^0\|_{W^{1,q}(\mathcal{F})} + \|f_1\|_{L^p(0,T;W^{1,q}(\mathcal{F}))} \right). \quad (3.42)$$

Combining **Step 1** to **Step 4**, we deduce the result. $\square$

3.3. Proof of Theorem 3.1. Here, we show the local in time existence of solutions for (3.7)–(3.18). For this, we notice that a solution of (3.7)–(3.17) is a solution of (3.24)–(3.27) such that the source terms satisfy

$$[f_1, f_2, f_3, g, h]^\top = [F_1, F_2, F_3, G, H]^\top,$$

where F_1, F_2, F_3, G and H are given by (3.13)–(3.17). This suggests to prove Theorem 3.1 by showing that the following mapping admits a fixed point:

$$\Xi_{T,R} : \mathcal{B}_{T,R} \longrightarrow \mathcal{B}_{T,R}, \quad [f_1, f_2, f_3, g, h]^\top \longmapsto [F_1, F_2, F_3, G, H]^\top, \quad (3.43)$$

where

$$\mathcal{B}_{T,R} = \left\{ [f_1, f_2, f_3, g, h]^\top \in \mathcal{R}_{T,p,q} ; \|[f_1, f_2, f_3, g, h]^\top\|_{\mathcal{R}_{T,p,q}} \leq R \right\}$$

(recall that $\mathcal{R}_{T,p,q}$ is defined by (3.29)) and where $[\rho, v, \vartheta, \eta]^\top$ is the solution of (3.24)–(3.27) associated with $[f_1, f_2, f_3, g, h]^\top$ and with initial conditions $[\rho^0, v^0, \vartheta^0, \eta_1^0, \eta_2^0]^\top \in \mathcal{I}_{p,q}$. More precisely, we take R large enough so that

$$\|[\rho^0, v^0, \vartheta^0, \eta_1^0, \eta_2^0]^\top\|_{\mathcal{I}_{p,q}} \leq R, \quad (3.44)$$

and we assume (1.11) (3.19), (3.20) and (1.18) so that we can apply Theorem 3.2: the system (3.24)–(3.27) admits a unique solution $(\rho, v, \vartheta, \eta) \in \mathcal{W}_{T,p,q}$ and

$$\|[\rho, v, \vartheta, \eta]^\top\|_{\mathcal{W}_{T,p,q}} \leq C \left(\|[\rho^0, v^0, \vartheta^0, \eta_1^0, \eta_2^0]^\top\|_{\mathcal{I}_{p,q}} + \|[f_1, f_2, f_3, g, h]^\top\|_{\mathcal{R}_{T,p,q}} \right).$$

To prove Theorem 3.1, we need to show that, for T small enough, the mapping $\Xi_{T,R}$ is well-defined, that $\Xi_{T,R}(\mathcal{B}_{T,R}) \subset \mathcal{B}_{T,R}$ and $\Xi_{T,R}|_{\mathcal{B}_{T,R}}$ is a strict contraction.

In this proof, we write C_R for any positive constant of the form $C(1 + R^N)$ for $N \in \mathbb{N}$, with C a constant that only depends on the geometry and on the physical parameters, and in particular independent of T . In particular the above inequality can be written as

$$\|\rho\|_{W^{1,p}(0,T;W^{1,q}(\mathcal{F}))} + \|v\|_{W^{1,2}_{p,q}((0,T);\mathcal{F})^3} + \|\vartheta\|_{W^{1,2}_{p,q}((0,T);\mathcal{F})} + \|\eta\|_{W^{2,4}_{p,q}((0,T);\mathcal{S})} \leq C_R. \quad (3.45)$$

We are going to use several times that since $q > 3$, $W^{1,q}(\mathcal{F})$ is an algebra and $W^{1,q}(\mathcal{F}) \subset L^\infty(\mathcal{F})$. We also have that $W^{1-\frac{1}{q},q}(\partial\mathcal{F}) \subset L^\infty(\partial\mathcal{F})$.

We also recall the following elementary inequalities:

$$\|f\|_{L^p(0,T)} \leq T^{\frac{1}{p}-\frac{1}{r}} \|f\|_{L^r(0,T)} \quad (f \in L^r(0,T)) \quad \text{if } r > p, \quad (3.46)$$

$$\|f - f(0)\|_{L^\infty(0,T)} \leq T^{\frac{1}{p'}} \|f\|_{W^{1,p}(0,T)} \quad (f \in W^{1,p}(0,T)) \quad \text{if } \frac{1}{p} + \frac{1}{p'} = 1. \quad (3.47)$$

In particular, we deduce from (3.45) and the above inequality

$$\|\rho - \rho^0\|_{L^\infty(0,T;W^{1,q}(\mathcal{F}))} \leq C_R T^{\frac{1}{p'}}, \quad \|\rho\|_{L^\infty(0,T;W^{1,q}(\mathcal{F}))} \leq C_R. \quad (3.48)$$

The above estimate with (3.46) yields

$$\|\rho\|_{L^p(0,T;W^{1,q}(\mathcal{F}))} \leq C_R T^{\frac{1}{p}}. \quad (3.49)$$

Since $2 < p < \infty$, one has $B_{q,p}^{2(1-\frac{1}{p})}(\mathcal{F}) \hookrightarrow W^{1,q}(\mathcal{F})$. Therefore, using (3.45) and (1.9), we obtain

$$\|v\|_{L^\infty(0,T;W^{1,q}(\mathcal{F}))^3} + \|\vartheta\|_{L^\infty(0,T;W^{1,q}(\mathcal{F}))} \leq C_R. \quad (3.50)$$

Using (3.47) and (3.18), we deduce successively

$$\|X\|_{W^{1,p}(0,T;W^{2,q}(\mathcal{F}))^3} \leq C_R, \quad \|X - X^0\|_{L^\infty(0,T;W^{2,q}(\mathcal{F}))^3} \leq C_R T^{\frac{1}{p'}}, \quad \|X\|_{L^\infty(0,T;W^{2,q}(\mathcal{F}))^3} \leq C_R. \quad (3.51)$$

Since X^0 is a C^1 -diffeomorphism, we deduce from the above estimates that X is a C^1 -diffeomorphism for T small enough. Moreover, by combining the above estimates with Lemma 3.3 and with (3.5), we also deduce

$$\|\mathbb{B}_X\|_{W^{1,p}(0,T;W^{1,q}(\mathcal{F}))^9} \leq C_R, \quad \|\mathbb{B}_X - \mathbb{B}^0\|_{L^\infty(0,T;W^{1,q}(\mathcal{F}))^9} \leq C_R T^{\frac{1}{p'}}, \quad \|\mathbb{B}_X\|_{L^\infty(0,T;W^{1,q}(\mathcal{F}))^9} \leq C_R, \quad (3.52)$$

$$\|\delta_X\|_{W^{1,p}(0,T;W^{1,q}(\mathcal{F}))} \leq C_R, \quad \|\delta_X - \delta^0\|_{L^\infty(0,T;W^{1,q}(\mathcal{F}))} \leq C_R T^{\frac{1}{p'}}, \quad \|\delta_X\|_{L^\infty(0,T;W^{1,q}(\mathcal{F}))} \leq C_R, \quad (3.53)$$

and in particular, there exists c_0 depending on η_1^0 such that for T small enough,

$$\delta_X \geq c_0 > 0. \quad (3.54)$$

We thus deduce

$$\left\| \frac{1}{\delta_X} \right\|_{W^{1,p}(0,T;W^{1,q}(\mathcal{F}))} \leq C_R, \quad \left\| \frac{1}{\delta_X} - \frac{1}{\delta^0} \right\|_{L^\infty(0,T;W^{1,q}(\mathcal{F}))} \leq C_R T^{\frac{1}{p'}}, \quad \left\| \frac{1}{\delta_X} \right\|_{L^\infty(0,T;W^{1,q}(\mathcal{F}))} \leq C_R. \quad (3.55)$$

Using the above estimates and (3.5), we also obtain

$$\|\mathbb{A}_X\|_{W^{1,p}(0,T;W^{1,q}(\mathcal{F}))^9} \leq C_R, \quad \|\mathbb{A}_X - \mathbb{A}^0\|_{L^\infty(0,T;W^{1,q}(\mathcal{F}))^9} \leq C_R T^{\frac{1}{p'}}, \quad \|\mathbb{A}_X\|_{L^\infty(0,T;W^{1,q}(\mathcal{F}))^9} \leq C_R. \quad (3.56)$$

We are now in position to estimate the non linear terms in (3.13)-(3.17). From the above estimates, we deduce

$$\|F_1(\rho, v, \vartheta, \eta)\|_{L^p(0,T;W^{1,q}(\mathcal{F}))} + \|F_2(\rho, v, \vartheta, \eta)\|_{L^p(0,T;L^q(\mathcal{F}))^3} + \|F_3(\rho, v, \vartheta, \eta)\|_{L^p(0,T;L^q(\mathcal{F}))} \leq C_R T^{\frac{1}{p}}. \quad (3.57)$$

By using the trace theorems, we also have

$$\|G(\rho, v, \vartheta, \eta)\|_{L^p(0,T;W^{1-1/q,q}(\partial\mathcal{F}))} + \|H(\rho, v, \vartheta, \eta)\|_{L^p(0,T;L^q(\mathcal{S}))} \leq C_R T^{\frac{1}{p}}. \quad (3.58)$$

It only remains to estimate G given by (3.16) in $F_{p,q}^{(1-1/q)/2}(0,T;L^q(\partial\mathcal{F}))$. First, using [12, Proposition 6.4], since $\vartheta \in W_{p,q}^{1,2}((0,T);\mathcal{F})$, we have that

$$\forall i, j, \quad \frac{\partial \vartheta}{\partial y_j} n_i \in F_{p,q}^{(1-1/q)/2}(0,T;L^q(\partial\mathcal{F})), \quad \left\| \frac{\partial \vartheta}{\partial y_j} n_i \right\|_{F_{p,q}^{(1-1/q)/2}(0,T;L^q(\partial\mathcal{F}))} \leq C_R.$$

Then we apply the general result [24, Proposition 2.7] with $s = (1-\frac{1}{q})/2$, $U_1 = U_3 = L^q(\partial\mathcal{F})$, $U_2 = W^{1-\frac{1}{q},q}(\partial\mathcal{F})$. Note that since $2 < p < \infty$, we have the condition $s + \frac{1}{p} < 1$. From [24, Proposition 2.7], we deduce that for some positive constant δ ,

$$\begin{aligned} & \|(\mathbb{A}^0 - \mathbb{A}_X) \nabla \vartheta \cdot n\|_{F_{p,q}^{(1-1/q)/2}(0,T;L^q(\partial\mathcal{F}))} \\ & \leq C T^\delta \|\mathbb{A}^0 - \mathbb{A}_X\|_{W^{1,p}(0,T;W^{1-\frac{1}{q},q}(\partial\mathcal{F}))} \sum_{i,j} \left\| \frac{\partial \vartheta}{\partial y_j} n_i \right\|_{F_{p,q}^{(1-1/q)/2}(0,T;L^q(\partial\mathcal{F}))} \leq C_R T^\delta. \end{aligned} \quad (3.59)$$

Combining (3.57), (3.58), (3.59), we deduce

$$\|\Xi_{T,R}(f_1, f_2, f_3, g, h)\|_{\mathcal{R}_{T,p,q}} \leq C_R T^\delta \quad (3.60)$$

for some power $\delta > 0$. Thus for T small enough, $\Xi_{T,R}(\mathcal{B}_{T,R}) \subset \mathcal{B}_{T,R}$.

To show that $\Xi_{T,R}|_{\mathcal{B}_{T,R}}$ is a strict contraction, we proceed similarly: we consider

$$\left[f_1^{(i)}, f_2^{(i)}, f_3^{(i)}, g^{(i)}, h^{(i)} \right]^\top \in \mathcal{B}_{T,R}, \quad i = 1, 2$$

and we denote by $[\rho^{(i)}, v^{(i)}, \vartheta^{(i)}, \eta^{(i)}]^\top$ the solution of (3.24)–(3.27) associated with

$$\left[f_1^{(i)}, f_2^{(i)}, f_3^{(i)}, g^{(i)}, h^{(i)} \right]^\top \in \mathcal{R}_{T,p,q} \quad \text{and} \quad [\rho^0, v^0, \vartheta^0, \eta_1^0, \eta_2^0]^\top \in \mathcal{I}_{p,q}.$$

We also write

$$\begin{aligned} [f_1, f_2, f_3, g, h]^\top &= \left[f_1^{(1)}, f_2^{(1)}, f_3^{(1)}, g^{(1)}, h^{(1)} \right]^\top - \left[f_1^{(2)}, f_2^{(2)}, f_3^{(2)}, g^{(2)}, h^{(2)} \right]^\top, \\ [\rho, v, \vartheta, \eta]^\top &= \left[\rho^{(1)}, v^{(1)}, \vartheta^{(1)}, \eta^{(1)} \right]^\top - \left[\rho^{(2)}, v^{(2)}, \vartheta^{(2)}, \eta^{(2)} \right]^\top. \end{aligned}$$

We can apply Theorem 3.2 and deduce that

$$\begin{aligned} \|\rho\|_{W^{1,p}(0,T;W^{1,q}(\mathcal{F}))} + \|v\|_{W_{p,q}^{1,2}((0,T);\mathcal{F})^3} + \|\vartheta\|_{W_{p,q}^{1,2}((0,T);\mathcal{F})} + \|\eta\|_{W_{p,q}^{2,4}((0,T);\mathcal{S})} \\ \leq C \left\| [f_1, f_2, f_3, g, h]^\top \right\|_{\mathcal{R}_{T,p,q}}, \end{aligned} \quad (3.61)$$

and since the initial conditions of $[\rho, v, \vartheta, \eta]^\top$ are null, we can apply (3.47):

$$\|\rho\|_{L^\infty(0,T;W^{1,q}(\mathcal{F}))} \leq CT^{\frac{1}{p'}} \left\| [f_1, f_2, f_3, g, h]^\top \right\|_{\mathcal{R}_{T,p,q}}. \quad (3.62)$$

We deduce similarly that

$$\|X^{(1)} - X^{(2)}\|_{L^\infty(0,T;W^{2,q}(\mathcal{F}))} \leq CT^{\frac{1}{p'}} \left\| [f_1, f_2, f_3, g, h]^\top \right\|_{\mathcal{R}_{T,p,q}}, \quad (3.63)$$

and we obtain similar estimates for $\mathbb{B}_{X^{(1)}} - \mathbb{B}_{X^{(2)}}$, $\mathbb{A}_{X^{(1)}} - \mathbb{A}_{X^{(2)}}$, $\delta_{X^{(1)}} - \delta_{X^{(2)}}$. Proceeding as above, we deduce that F_1, F_2, F_3, G and H given by (3.13)–(3.17) satisfy

$$\begin{aligned} &\left\| F_1(\rho^{(1)}, v^{(1)}, \vartheta^{(1)}, \eta^{(1)}) - F_1(\rho^{(2)}, v^{(2)}, \vartheta^{(2)}, \eta^{(2)}) \right\|_{L^p(0,T;W^{1,q}(\mathcal{F}))} \\ &+ \left\| F_2(\rho^{(1)}, v^{(1)}, \vartheta^{(1)}, \eta^{(1)}) - F_2(\rho^{(2)}, v^{(2)}, \vartheta^{(2)}, \eta^{(2)}) \right\|_{L^p(0,T;L^q(\mathcal{F}))^3} \\ &+ \left\| F_3(\rho^{(1)}, v^{(1)}, \vartheta^{(1)}, \eta^{(1)}) - F_3(\rho^{(2)}, v^{(2)}, \vartheta^{(2)}, \eta^{(2)}) \right\|_{L^p(0,T;L^q(\mathcal{F}))} \\ &+ \left\| G(\rho^{(1)}, v^{(1)}, \vartheta^{(1)}, \eta^{(1)}) - G(\rho^{(2)}, v^{(2)}, \vartheta^{(2)}, \eta^{(2)}) \right\|_{L^p(0,T;W^{1-1/q,q}(\partial\mathcal{F})) \cap F_{p,q}^{(1-1/q)/2}(0,T;L^q(\partial\mathcal{F}))} \\ &+ \left\| H(\rho^{(1)}, v^{(1)}, \vartheta^{(1)}, \eta^{(1)}) - H(\rho^{(2)}, v^{(2)}, \vartheta^{(2)}, \eta^{(2)}) \right\|_{L^p(0,T;L^q(\mathcal{S}))} \leq C_R T^\delta \left\| [f_1, f_2, f_3, g, h]^\top \right\|_{\mathcal{R}_{T,p,q}} \end{aligned} \quad (3.64)$$

for some positive constant δ . Thus taking T small enough, we deduce that $\Xi_{T,R}|_{\mathcal{B}_{T,R}}$ is a strict contraction and this ends the proof of the theorem. $\square$

4. GLOBAL IN TIME EXISTENCE

In this section we prove Theorem 1.2.

4.1. Change of variables and Linearization. As in the first part of this work, in order to show global existence in time we use a change of variables to write the system (1.3)–(1.8) in the fixed spatial domain $\mathcal{F}$. We consider the same transformation as in Section 3.1, that is X is defined by (3.3). Note that (1.20) for R small enough yields condition (1.18). However, we modify (3.4) since we linearize here the system around the constant steady state $[\bar{\rho}, 0, \bar{\vartheta}, 0]^\top$, with $\bar{\rho}, \bar{\vartheta} \in \mathbb{R}_+^*$:

$$\rho(t, y) = \tilde{\rho}(t, X(t, y)) - \bar{\rho}, \quad v(t, y) = \tilde{v}(t, X(t, y)), \quad \vartheta(t, y) = \tilde{\vartheta}(t, X(t, y)) - \bar{\vartheta}, \quad (4.1)$$

for $(t, y) \in (0, T) \times \mathcal{F}$. In particular,

$$\tilde{\rho}(t, x) = \bar{\rho} + \rho(t, Y(t, x)), \quad \tilde{v}(t, x) = v(t, Y(t, x)), \quad \tilde{\vartheta}(t, x) = \bar{\vartheta} + \vartheta(t, Y(t, x)), \quad (4.2)$$

for $(t, x) \in (0, T) \times \mathcal{F}(\eta(t))$.

This change of variables transforms (1.3)–(1.8) into the following system for $[\rho, v, \vartheta, \eta]^\top$:

$$\begin{cases} \partial_t \rho + \bar{\rho} \operatorname{div} v = F_1(\rho, v, \vartheta, \eta) & \text{in } (0, \infty) \times \mathcal{F}, \\ \partial_t v - \frac{1}{\bar{\rho}} \operatorname{div} \mathbb{T}(\rho, v, \vartheta) = F_2(\rho, v, \vartheta, \eta) & \text{in } (0, \infty) \times \mathcal{F}, \\ \partial_t \vartheta - \bar{\kappa} \Delta \vartheta = F_3(\rho, v, \vartheta, \eta) & \text{in } (0, \infty) \times \mathcal{F}, \\ \partial_{tt} \eta + \Delta_s^2 \eta - \Delta_s \partial_t \eta = -\mathbb{T}(\rho, v, \vartheta) e_3 \cdot e_3 + H(\rho, v, \vartheta, \eta) & \text{in } (0, \infty) \times \mathcal{S}, \end{cases} \quad (4.3)$$

$$\begin{cases} v = 0 & \text{on } (0, \infty) \times \Gamma_F, \\ v = \partial_t \eta e_3 & \text{on } (0, \infty) \times \Gamma_S, \\ \frac{\partial \vartheta}{\partial n} = G(\rho, v, \vartheta, \eta) & \text{on } (0, \infty) \times \partial \mathcal{F}, \\ \eta = \nabla_s \eta \cdot n_S = 0 & \text{on } (0, \infty) \times \partial \mathcal{S}, \end{cases} \quad (4.4)$$

$$\begin{cases} \eta(0, \cdot) = \eta_1^0, & \partial \eta(0, \cdot) = \eta_2^0 & \text{in } \mathcal{S}, \\ \rho(0, \cdot) = \rho^0, & v(0, \cdot) = v^0, & \vartheta(0, \cdot) = \vartheta^0 & \text{in } \mathcal{F}, \end{cases} \quad (4.5)$$

where

$$\mathbb{T}(\rho, v, \vartheta) = 2\mu \mathbb{D}v + (\alpha \operatorname{div} v - R_0 \bar{\vartheta} \rho - R_0 \bar{\rho} \vartheta) I_3, \quad (4.6)$$

$$\bar{\kappa} = \frac{\kappa}{c_v \bar{\rho}} \quad (4.7)$$

$$\rho^0 = \tilde{\rho}^0 \circ X^0 - \bar{\rho}, \quad v^0 = \tilde{v}^0 \circ X^0, \quad \vartheta^0 = \tilde{\vartheta}^0 \circ X^0 - \bar{\vartheta}. \quad (4.8)$$

The nonlinear terms in (4.3)–(4.5) can be written as

$$F_1(\rho, v, \vartheta, \eta) = -\rho \operatorname{div} v - (\rho + \bar{\rho}) \left(\frac{1}{\delta_X} \mathbb{B}_X - I_3 \right) : \nabla v, \quad (4.9)$$

$$\begin{aligned} F_2(\rho, v, \vartheta, \eta) = & \frac{1}{\bar{\rho}} \left[-\bar{\rho}(\delta_X - 1) \partial_t v - \rho \delta_X \partial_t v + \mu \operatorname{div} (\nabla v (\mathbb{A}_X - I_3)) \right. \\ & \left. + (\mu + \alpha) \operatorname{div} \left(\frac{1}{\delta_X} \mathbb{B}_X (\nabla v)^\top \mathbb{B}_X - (\nabla v)^\top \right) + R_0 \mathbb{B}_X \nabla (\rho \vartheta) + R_0 (\mathbb{B}_X - I_3) (\bar{\rho} \nabla \vartheta + \bar{\vartheta} \nabla \rho) \right] \end{aligned} \quad (4.10)$$

$$\begin{aligned} F_3(\rho, v, \vartheta, \eta) = & \frac{1}{c_v \bar{\rho}} \left[-c_v \delta_X \rho \partial_t \vartheta - c_v \bar{\rho} (\delta_X - 1) \partial_t \vartheta - R_0 (\rho \vartheta + \bar{\rho} \vartheta + \bar{\vartheta} \rho) (\mathbb{B}_X : \nabla v) \right. \\ & \left. + \kappa \operatorname{div} \left((\mathbb{A}_X - I_3) \nabla \vartheta \right) + \frac{\alpha}{\delta_X} (\mathbb{B}_X : \nabla v)^2 + \frac{2\mu}{\delta_X} \left| \nabla v \mathbb{B}_X^\top + \mathbb{B}_X \nabla v^\top \right|^2 \right], \end{aligned} \quad (4.11)$$

$$G(\rho, v, \vartheta, \eta) = (I_3 - \mathbb{A}_X) \nabla \vartheta \cdot n, \quad (4.12)$$

$$H(\rho, v, \vartheta, \eta) = -\mu \left[\frac{1}{\delta_X} \left(\nabla v \mathbb{B}_X^\top + \mathbb{B}_X \nabla v^\top \right) \begin{bmatrix} -\nabla_s \eta \\ 1 \end{bmatrix} - 2\mu \mathbb{D}(v) e_3 \right] \cdot e_3 \\ - \alpha \left(\frac{1}{\delta_X} \mathbb{B}_X - I_3 \right) : \nabla v + R_0 \rho \vartheta, \quad (4.13)$$

where $\mathbb{A}_X$, $\mathbb{B}_X$ and δ_X are defined in (3.5). The hypotheses (1.12)–(1.15) on the initial conditions are transformed into (3.20)–(3.23) and

$$\rho^0 \in W^{1,q}(\mathcal{F}), \quad \min_{\overline{\mathcal{F}}} \rho^0 + \bar{\rho} > 0, \quad (4.14)$$

Using the above change of variables, Theorem 1.2 can be reformulated as

Theorem 4.1. *Assume (p, q) satisfies (1.11) and assume that $\bar{\rho}$ and $\bar{\vartheta}$ are two given positive constants such that (1.19) holds. Then there exist $\beta > 0$ and $R > 0$ such that, for any $[\rho^0, v^0, \vartheta^0, \eta_1^0, \eta_2^0]^\top$ satisfying (1.18), (4.14), (3.20)–(3.23) and*

$$\|\rho^0\|_{W^{1,q}(\mathcal{F})} + \|v^0\|_{B_{q,p}^{2(1-1/p)}(\mathcal{F})^3} + \|\vartheta^0\|_{B_{q,p}^{2(1-1/p)}(\mathcal{F})} + \|\eta_1^0\|_{B_{q,p}^{2(2-1/p)}(\mathcal{S})} + \|\eta_2^0\|_{B_{q,p}^{2(1-1/p)}(\mathcal{S})} \leq R,$$

the system (4.3)–(4.13) admits a unique strong solution $[\rho, v, \vartheta, \eta]^\top$ in the class of functions satisfying

$$\rho \in C_b^0([0, \infty); W^{1,q}(\mathcal{F})), \quad \nabla \rho \in W_\beta^{1,p}(0, \infty; L^q(\mathcal{F})), \quad \partial_t \rho \in L_\beta^p(0, \infty; W^{1,q}(\mathcal{F})), \quad (4.15)$$

$$\vartheta \in C_b^0([0, \infty); B_{q,p}^{2(1-1/p)}(\mathcal{F})), \quad \nabla \vartheta \in L_\beta^p(0, \infty; W^{1,q}(\mathcal{F})), \quad \partial_t \vartheta \in L_\beta^p(0, \infty; L^q(\mathcal{F})), \quad (4.16)$$

$$v \in W_{p,q,\beta}^{1,2}((0, \infty); \mathcal{F})^3, \quad \partial_t v \in W_{p,q,\beta}^{1,2}((0, \infty); \mathcal{S}), \quad (4.17)$$

$$\eta \in C_b^0([0, \infty); B_{q,p}^{2(2-1/p)}(\mathcal{S})), \quad \eta \in L_\beta^p(0, \infty; W^{4,q}(\mathcal{S})) + L^\infty(0, \infty; W^{4,q}(\mathcal{S})). \quad (4.18)$$

Moreover,

$$\min_{[0, \infty) \times \overline{\mathcal{F}}} \rho + \bar{\rho} > 0, \quad \Gamma_0 \cap \Gamma_S(\eta(t)) = \emptyset \quad (t \in [0, \infty)),$$

and for all $t \in [0, \infty)$, $X(t, \cdot) : \mathcal{F} \rightarrow \mathcal{F}(\eta(t))$ is a C^1 -diffeomorphism.

The proof of Theorem 4.1 relies on the Banach fixed point theorem and on the maximal $L^p - L^q$ regularity of a linearized system over the time interval $(0, \infty)$. In order to introduce the linearized system associated with (4.3)–(4.13), we introduce the following operator $\mathcal{T} : W_0^{2,q}(\mathcal{S}) \rightarrow W^{2,q}(\partial\mathcal{F})^3$ defined by

$$(\mathcal{T}\eta)(y) = \begin{cases} \eta(s)e_3 & \text{if } y = (s, 0) \in \Gamma_S, \\ 0 & \text{if } y \in \Gamma_0. \end{cases} \quad (4.19)$$

We also write $\eta_1 = \eta$ and $\eta_2 = \partial_t \eta$ and we consider the following system where we have replaced in (4.3)–(4.8), the nonlinearities F_1, F_2, F_3, G, H by given source terms f_1, f_2, f_3, g, h :

$$\begin{cases} \partial_t \rho + \bar{\rho} \operatorname{div} v = f_1 & \text{in } (0, \infty) \times \mathcal{F}, \\ \partial_t v - \frac{1}{\bar{\rho}} \operatorname{div} \overline{\mathbb{T}}(\rho, v, \vartheta) = f_2 & \text{in } (0, \infty) \times \mathcal{F}, \\ \partial_t \vartheta - \bar{\kappa} \Delta \vartheta = f_3 & \text{in } (0, \infty) \times \mathcal{F}, \\ \partial_t \eta_1 - \eta_2 = 0 & \text{in } (0, \infty) \times \mathcal{F}, \\ \partial_t \eta_2 + \Delta_s^2 \eta_1 - \Delta_s \eta_2 = -\overline{\mathbb{T}}(\rho, v, \vartheta) e_3 \cdot e_3 + h & \text{in } (0, \infty) \times \mathcal{S}, \end{cases} \quad (4.20)$$

$$\begin{cases} v = \mathcal{T}\eta_2 & \text{on } (0, \infty) \times \partial\mathcal{F}, \\ \frac{\partial \vartheta}{\partial n} = g & \text{on } (0, \infty) \times \partial\mathcal{F}, \\ \eta_1 = \nabla_s \eta_1 \cdot n_S = 0 & \text{on } (0, \infty) \times \partial\mathcal{S}, \end{cases} \quad (4.21)$$

$$\begin{cases} \eta_1(0, \cdot) = \eta_1^0, & \eta_2(0, \cdot) = \eta_2^0 & \text{in } \mathcal{S}, \\ \rho(0, \cdot) = \rho^0, & v(0, \cdot) = v^0, & \vartheta(0, \cdot) = \vartheta^0 & \text{in } \mathcal{F}, \end{cases} \quad (4.22)$$

Our aim is to show that the linearized operator associated to the above linear system is $\mathcal{R}$ -sectorial in a suitable function space.

4.2. The fluid-structure operator. Here we introduce the operator associated to the linear system (4.20)–(4.22). To this aim, we first define

$$\mathcal{D}(A_v) = \left\{ v \in W^{2,q}(\mathcal{F})^3 ; v = 0 \text{ on } \partial\mathcal{F} \right\}, \quad A_v = \frac{\mu}{\bar{\rho}} \Delta + \frac{\alpha + \mu}{\bar{\rho}} \nabla \operatorname{div}, \quad (4.23)$$

and

$$\mathcal{D}(A_\vartheta) = \left\{ \vartheta \in W^{2,q}(\mathcal{F}) ; \frac{\partial \vartheta}{\partial n} = 0 \text{ on } \partial\mathcal{F} \right\}, \quad A_\vartheta = \bar{\kappa} \Delta. \quad (4.24)$$

From [35, Theorem 1.4], A_v is an isomorphism from $\mathcal{D}(A_v)$ onto $L^q(\mathcal{F})^3$ for any $q \in (1, \infty)$. Using trace properties, this allows us to introduce the operator

$$D_v \in \mathcal{L}(W_0^{2,q}(\mathcal{S}); W^{2,q}(\mathcal{F})^3), \quad (4.25)$$

where $w = D_v g$ is the solution to the system

$$\begin{cases} -\frac{\mu}{\bar{\rho}} \Delta w - \frac{\alpha + \mu}{\bar{\rho}} \nabla(\operatorname{div} w) = 0 & \text{in } \mathcal{F}, \\ w = \mathcal{T}g & \text{on } \partial\mathcal{F}. \end{cases} \quad (4.26)$$

By a standard transposition method, the operator D_v can be extended as a bounded operator from $L^q(\mathcal{S})$ to $L^q(\mathcal{F})^3$.

Using the above definitions and recalling the definitions (3.34), (3.35) of A_S and $\mathcal{X}_S$, we can write (4.20)–(4.22) as follows (in the case $g = 0$):

$$\frac{d}{dt} \begin{bmatrix} \rho \\ v \\ \vartheta \\ \eta_1 \\ \eta_2 \end{bmatrix} = \mathcal{A}_{FS} \begin{bmatrix} \rho \\ v \\ \vartheta \\ \eta_1 \\ \eta_2 \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ 0 \\ h \end{bmatrix}, \quad \begin{bmatrix} \rho \\ v \\ \vartheta \\ \eta_1 \\ \eta_2 \end{bmatrix}(0) = \begin{bmatrix} \rho^0 \\ v^0 \\ \vartheta^0 \\ \eta_1^0 \\ \eta_2^0 \end{bmatrix}, \quad (4.27)$$

where $\mathcal{A}_{FS} : \mathcal{D}(\mathcal{A}_{FS}) \rightarrow \mathcal{X}$ is defined by

$$\mathcal{X} = W^{1,q}(\mathcal{F}) \times L^q(\mathcal{F})^3 \times L^q(\mathcal{F}) \times W_0^{2,q}(\mathcal{S}) \times L^q(\mathcal{S}), \quad (4.28)$$

$$\mathcal{D}(\mathcal{A}_{FS}) = \left\{ [\rho, v, \vartheta, \eta_1, \eta_2]^\top \in W^{1,q}(\mathcal{F}) \times W^{2,q}(\mathcal{F})^3 \times \mathcal{D}(A_\vartheta) \times \mathcal{D}(A_S) ; v - D_v \eta_2 \in \mathcal{D}(A_v) \right\}, \quad (4.29)$$

and

$$\mathcal{A}_{FS} = \mathcal{A}_{FS}^0 + \mathcal{B}_{FS},$$

with

$$\mathcal{A}_{FS}^0 \begin{bmatrix} \rho \\ v \\ \vartheta \\ \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} -\bar{\rho} \operatorname{div} v \\ A_v(v - D_v \eta_2) \\ A_\vartheta \vartheta \\ \eta_2 \\ -\Delta_s^2 \eta_1 + \Delta_s \eta_2 \end{bmatrix} \quad \text{and} \quad \mathcal{B}_{FS} \begin{bmatrix} \rho \\ v \\ \vartheta \\ \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{R_0 \bar{\vartheta}}{\bar{\rho}} \nabla \rho - R_0 \nabla \vartheta \\ 0 \\ 0 \\ -\bar{\mathbb{T}}(\rho, v, \vartheta) e_3 \cdot e_3 \end{bmatrix}. \quad (4.30)$$

We recall that the definition of a $\mathcal{R}$ -sectorial operator is given in Definition 2.2. We now prove the following theorem :

Theorem 4.2. *Let $1 < q < \infty$. Then there exists $\gamma > 0$ such that $\mathcal{A}_{FS} - \gamma$ is an $\mathcal{R}$ -sectorial operator in $\mathcal{X}$ of angle $\beta > \pi/2$.*

Proof. In order to prove the theorem, we first combine [16, Theorem 2.5], [11, Theorem 8.2] and [14, Theorem 5.1]: there exist $\gamma > 0$ and $\beta > \pi/2$ such that the operators $A_v - \gamma$, $A_\vartheta - \gamma$ and $A_S - \gamma$ are $\mathcal{R}$ -sectorial operators of angle β .

Second, standard calculation shows that for $\lambda \in \gamma + \Sigma_\beta$ (see (2.2)),

$$\lambda(\lambda I - \mathcal{A}_{FS}^0)^{-1} = \begin{bmatrix} \text{Id} & -\bar{\rho} \operatorname{div}(\lambda I - A_v)^{-1} & 0 & \bar{\rho} \operatorname{div} A_v(\lambda I - A_v)^{-1} \widetilde{D}_v(\lambda I - A_S)^{-1} \\ 0 & \lambda(\lambda I - A_v)^{-1} & 0 & -A_v(\lambda I - A_v)^{-1} \widetilde{D}_v \lambda(\lambda I - A_S)^{-1} \\ 0 & 0 & \lambda(\lambda I - A_\vartheta)^{-1} & 0 \\ 0 & 0 & 0 & \lambda(\lambda I - A_S)^{-1} \end{bmatrix},$$

where $\widetilde{D}_v[\eta_1, \eta_2]^\top = D_v \eta_2$. Using the properties of $\mathcal{R}$ -boundedness recalled in Section 2, we deduce that $\mathcal{A}_{FS}^0 - \gamma$ is $\mathcal{R}$ -sectorial operator in $\mathcal{X}$ of angle β . Note that in instance, we can write

$$\operatorname{div} A_v(\lambda I - A_v)^{-1} \widetilde{D}_v(\lambda I - A_S)^{-1} = -\operatorname{div} \widetilde{D}_v(\lambda I - A_S)^{-1} + \operatorname{div}(\lambda I - A_v)^{-1} \widetilde{D}_v \lambda(\lambda I - A_S)^{-1}$$

and then use that $D_v \in \mathcal{L}(W_0^{2,q}(\mathcal{S}); W^{2,q}(\mathcal{F})^3) \cap \mathcal{L}(L_0^q(\mathcal{S}); L^q(\mathcal{F})^3)$.

Next, using trace results, for $s \in (1/q, 1)$ there exists a constant C such that

$$\left\| \mathcal{B}_{FS}[\rho, v, \vartheta, \eta_1, \eta_2]^\top \right\|_{\mathcal{X}} \leq C \left(\|\rho\|_{W^{1,q}(\mathcal{F})} + \|v\|_{W^{1+s,q}(\mathcal{F})^3} + \|\vartheta\|_{W^{1+s,q}(\mathcal{F})} \right) \quad [\rho, v, \vartheta, \eta_1, \eta_2]^\top \in \mathcal{D}(\mathcal{A}_{FS}).$$

Since the embedding $W^{1+s,q}(\mathcal{F}) \hookrightarrow W^{2,q}(\mathcal{F})$ is compact for $s \in (1/q, 1)$, for any $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that

$$\left\| \mathcal{B}_{FS}[\rho, v, \vartheta, \eta_1, \eta_2]^\top \right\|_{\mathcal{X}} \leq \varepsilon \left\| \mathcal{A}_{FS}^0[\rho, v, \vartheta, \eta_1, \eta_2]^\top \right\|_{\mathcal{X}} + C(\varepsilon) \left\| [\rho, v, \vartheta, \eta_1, \eta_2]^\top \right\|_{\mathcal{X}}. \quad (4.31)$$

Finally using Proposition 2.5 we conclude the proof of the theorem. $\square$

4.3. Exponential stability of the fluid-structure semigroup. The aim of this subsection is to show that the operator $\mathcal{A}_{FS}$ generates an analytic semigroup of negative type in the following subspace of $\mathcal{X}$:

$$\mathcal{X}_m = \left\{ [f_1, f_2, f_3, h_1, h_2]^\top \in \mathcal{X} ; \int_{\mathcal{F}} f_1 dy + \bar{\rho} \int_{\mathcal{S}} h_1 ds = 0, \int_{\mathcal{F}} f_3 dy = 0 \right\}. \quad (4.32)$$

We can verify that $\mathcal{X}_m$ is invariant under $(e^{t\mathcal{A}_{FS}})_{t \geq 0}$. Therefore we can consider the restriction of $\mathcal{A}_{FS}$ to the domain $\mathcal{D}(\mathcal{A}_{FS}) \cap \mathcal{X}_m$ ([39, Definition 2.4.1]). For this operator, we have the following result:

Theorem 4.3. *Let $1 < q < \infty$. The part of $\mathcal{A}_{FS}$ in $\mathcal{X}_m$ generates an exponentially stable semigroup $(e^{t\mathcal{A}_{FS}})_{t \geq 0}$ on $\mathcal{X}_m$: there exists constants $C > 0$ and $\beta_0 > 0$ such that*

$$\left\| e^{t\mathcal{A}_{FS}}[\rho^0, v^0, \vartheta^0, \eta_1^0, \eta_2^0]^\top \right\|_{\mathcal{X}} \leq C e^{-\beta_0 t} \left\| [\rho^0, v^0, \vartheta^0, \eta_1^0, \eta_2^0]^\top \right\|_{\mathcal{X}}, \quad (t \geq 0), \quad (4.33)$$

for all $[\rho^0, v^0, \vartheta^0, \eta_1^0, \eta_2^0]^\top \in \mathcal{X}_m$.

To show the above theorem, it sufficient to show that $\mathbb{C}^+ \subset \rho(\mathcal{A}_{FS}|\mathcal{D}(\mathcal{A}_{FS}) \cap \mathcal{X}_m)$. We thus consider the following resolvent problem

$$\begin{cases} \lambda \rho + \bar{\rho} \operatorname{div} v = f_1 & \text{in } \mathcal{F}, \\ \lambda v - \frac{1}{\bar{\rho}} \operatorname{div} \bar{\mathbb{T}}(\rho, v, \vartheta) = f_2 & \text{in } \mathcal{F}, \\ \lambda \vartheta - \bar{\kappa} \Delta \vartheta = f_3 & \text{in } \mathcal{F}, \\ v = \mathcal{T} \eta_2, \quad \frac{\partial \vartheta}{\partial n} = 0 & \text{on } \partial \mathcal{F}, \\ \lambda \eta_1 - \eta_2 = h_1 & \text{in } \mathcal{S}, \\ \lambda \eta_2 + \Delta_s^2 \eta_1 - \Delta_s \eta_2 = -\bar{\mathbb{T}}(\rho, v, \vartheta) e_3 \cdot e_3 + h_2 & \text{in } \mathcal{S}, \\ \eta_1 = \nabla_s \eta_1 \cdot n_S = \eta_2 = 0 & \text{on } \partial \mathcal{S}. \end{cases} \quad (4.34)$$

Remark 4.4. If $\lambda = 0$, integrating the first and third equation of (4.34) and using the boundary conditions of v and ϑ we obtain

$$\int_{\mathcal{F}} f_1 dy + \bar{\rho} \int_{\mathcal{S}} h_1 ds = 0 \text{ and } \int_{\mathcal{F}} f_3 dy = 0.$$

Therefore, in order to study exponential stability of the semigroup it is necessary to consider the space $\mathcal{X}_m$ instead of $\mathcal{X}$.

Proof. Assume $\lambda \in \mathbb{C}^+$ and $[f_1, f_2, f_3, h_1, h_2]^\top \in \mathcal{X}_m$. We need to show that the system (4.34) admits a unique solution $[\rho, v, \vartheta, \eta_1, \eta_2]^\top \in \mathcal{D}(\mathcal{A}_{FS}) \cap \mathcal{X}_m$ together with an estimate

$$\left\| [\rho, v, \vartheta, \eta_1, \eta_2]^\top \right\|_{\mathcal{D}(\mathcal{A}_{FS})} \leq C \left\| [f_1, f_2, f_3, h_1, h_2]^\top \right\|_{\mathcal{X}}.$$

The proof is divided into several parts.

Step 1: Uniqueness. Let us assume that $[\rho, v, \vartheta, \eta_1, \eta_2]^\top \in \mathcal{D}(\mathcal{A}_{FS}) \cap \mathcal{X}_m$ solves the system (4.34) with $[f_1, f_2, f_3, h_1, h_2]^\top = 0$. We notice that

$$[\rho, v, \vartheta, \eta_1, \eta_2]^\top \in W^{1,2}(\mathcal{F}) \times W^{2,2}(\mathcal{F})^3 \times W^{2,2}(\mathcal{F}) \times W^{4,2}(\mathcal{S}) \times W^{2,2}(\mathcal{S}). \quad (4.35)$$

If $q \geq 2$ then it is a consequence of Hölder's inequality. Else, $1 < q < 2$ and we take $\lambda_0 \in \rho(\mathcal{A}_{FS})$ to rewrite (4.34) as

$$(\lambda_0 - \mathcal{A}_{FS}) [\rho, v, \vartheta, \eta_1, \eta_2]^\top = (\lambda_0 - \lambda) [\rho, v, \vartheta, \eta_1, \eta_2]^\top.$$

Since $W^{2,q}(\mathcal{F}) \hookrightarrow L^2(\mathcal{F})$ and $W^{2,q}(\mathcal{S}) \hookrightarrow L^2(\mathcal{S})$, we deduce (4.35) from the invertibility of the operator $(\lambda_0 - \mathcal{A}_{FS})$.

Multiplying (4.34)₃ by ϑ , we obtain after integration by parts

$$\lambda \int_{\mathcal{F}} |\vartheta|^2 dy + \bar{\kappa} \int_{\mathcal{F}} |\nabla \vartheta|^2 dy = 0.$$

Since $\operatorname{Re} \lambda \geq 0$ and $\int_{\mathcal{F}} \vartheta dy = 0$, we obtain $\vartheta = 0$.

Next, multiplying (4.34)₂ by v , (4.34)₆ by η_2 , after integration by parts and taking the real part, we deduce

$$\begin{aligned} \frac{R_0 \bar{\vartheta}}{\bar{\rho}} (\operatorname{Re} \lambda) \int_{\mathcal{F}} |\rho|^2 dy + \bar{\rho} (\operatorname{Re} \lambda) \int_{\mathcal{F}} |v|^2 dy + 2\mu \int_{\mathcal{F}} |\mathbb{D}v|^2 dy + \alpha \int_{\mathcal{F}} (\operatorname{div} v)^2 dy \\ + \operatorname{Re} \lambda \int_{\mathcal{S}} |\eta_2|^2 ds + \operatorname{Re} \lambda \int_{\mathcal{S}} |\Delta_s \eta_1|^2 + \int_{\mathcal{S}} |\nabla_s \eta_2|^2 = 0. \end{aligned}$$

Since $\operatorname{Re} \lambda \geq 0$, using (1.7) and using the boundary conditions we obtain $v = \eta_2 = 0$ and that ρ is a constant. Using that $[\rho, v, \vartheta, \eta_1, \eta_2]^\top \in \mathcal{X}_m$ we deduce that η_1 solves

$$\begin{cases} \Delta_s^2 \eta_1 + \frac{R_0 \bar{\vartheta} \bar{\rho}}{|\mathcal{F}|} \int_S \eta_1 ds = 0 & \text{in } \mathcal{S}, \\ \eta_1 = \nabla_s \eta_1 \cdot n_S = 0 & \text{on } \partial \mathcal{S}. \end{cases} \quad (4.36)$$

Multiplying the first equation of the above system by η_1 and integrating by parts, we deduce that $\eta_1 = 0$ and that $\rho = 0$.

Step 2. Existence for $\lambda = 0$. We consider the system (4.34) with $\lambda = 0$. It can be written as follows

$$\begin{aligned} \eta_2 &= -h_1 \text{ in } \mathcal{S}, \\ -\bar{\kappa} \Delta \vartheta &= f_3 \text{ in } \mathcal{F}, \quad \frac{\partial \vartheta}{\partial n} = 0 \text{ on } \partial \mathcal{F}, \quad \int_{\mathcal{F}} \vartheta dy = 0, \\ \begin{cases} -\mu \Delta v + R_0 \bar{\vartheta} \nabla \rho = \bar{\rho} f_2 + \frac{\alpha + \mu}{\bar{\rho}} \nabla f_1 - R_0 \bar{\rho} \nabla \vartheta & \text{in } \mathcal{F}, \\ \operatorname{div} v = \frac{1}{\bar{\rho}} f_1 & \text{in } \mathcal{F}, \\ v = -\mathcal{T} h_1 & \text{in } \partial \mathcal{F}, \end{cases} \end{aligned} \quad (4.37)$$

$$\begin{cases} \Delta_s^2 \eta_1 = -\bar{\mathbb{T}}(\rho, v, \vartheta) e_3 \cdot e_3 - \Delta_s h_1 + h_2 & \text{in } \mathcal{S}, \\ \eta_1 = \nabla_s \eta_1 \cdot n_S = 0 & \text{on } \partial \mathcal{S}. \end{cases} \quad (4.38)$$

$$\int_{\mathcal{F}} \rho dy + \bar{\rho} \int_S \eta_1 ds = 0, \quad (4.39)$$

We can solve the two first equations and obtain the existence and uniqueness of $\vartheta \in W^{2,q}(\mathcal{F})$ and $\eta_2 \in W_0^{2,q}(\mathcal{S})$ and we have the following estimate

$$\|\vartheta\|_{W^{2,q}(\mathcal{F})} \leq C \|f_3\|_{L^q(\mathcal{F})}, \quad \|\eta_2\|_{W^{2,q}(\mathcal{S})} = \|h_1\|_{W^{2,q}(\mathcal{S})}.$$

Using that $[f_1, f_2, f_3, h_1, h_2]^\top \in \mathcal{X}_m$, we can solve (4.37) (see, for instance [36, Proposition 2.3, p.35]) and we obtain the existence and uniqueness of $(\rho, v) \in (W^{1,q}(\mathcal{F})/\mathbb{R}) \times W^{2,q}(\mathcal{F})^3$ with the following estimate

$$\|v\|_{W^{2,q}(\mathcal{F})^3} + \|\rho\|_{W^{1,q}(\mathcal{F})/\mathbb{R}} \leq \left(\|f_1\|_{W^{1,q}(\mathcal{F})} + \|f_2\|_{L^q(\mathcal{F})^3} + \|f_3\|_{L^q(\mathcal{F})} + \|h_1\|_{W_0^{2,q}(\mathcal{S})} \right).$$

Then we decompose $\rho = \rho_m + \rho_{\text{avg}}$, with

$$\rho_{\text{avg}} = \frac{1}{|\mathcal{F}|} \int_{\mathcal{F}} \rho dy = -\frac{\bar{\rho}}{|\mathcal{F}|} \int_S \eta_1 ds$$

and we can rewrite (4.38) as

$$\begin{cases} \Delta_s^2 \eta_1 + \frac{R_0 \bar{\vartheta} \bar{\rho}}{|\mathcal{F}|} \int_S \eta_1 ds = -\bar{\mathbb{T}}(\rho_m, v, \vartheta) e_3 \cdot e_3 - \Delta_s h_1 + h_2 & \text{in } \mathcal{S}, \\ \eta_1 = \nabla_s \eta_1 \cdot n_S = 0 & \text{on } \partial \mathcal{S}. \end{cases} \quad (4.40)$$

Using the Fredholm alternative, the above system admits a unique solution $\eta_1 \in W^{4,q}(\mathcal{S})$ and

$$\|\eta_1\|_{W^{4,q}(\mathcal{S})} \leq C \left\| [f_1, f_2, f_3, h_1, h_2]^\top \right\|_{\mathcal{X}}.$$

Step 3. Existence for $\lambda \in \mathbb{C}^+, \lambda \neq 0$. By setting $\rho = \frac{1}{\lambda}(f_1 - \bar{\rho} \operatorname{div} v)$, the system (4.34) can be rewritten as

$$\begin{cases} \lambda v - \frac{1}{\bar{\rho}} \operatorname{div} \widehat{\mathbb{T}}_\lambda(v, \vartheta) = \widehat{f}_2 & \text{in } \mathcal{F}, \\ \lambda \vartheta - \bar{\kappa} \Delta \vartheta = f_3 & \text{in } \mathcal{F}, \\ v = \mathcal{T} \eta_2, \quad \frac{\partial \vartheta}{\partial n} = 0 & \text{on } \partial \mathcal{F}, \\ \lambda \eta_1 - \eta_2 = h_1 & \text{in } \mathcal{S}, \\ \lambda \eta_2 + \Delta_s^2 \eta_1 - \Delta_s \eta_2 = -\widehat{\mathbb{T}}_\lambda(v, \vartheta) e_3 \cdot e_3 + \widehat{h}_2 & \text{in } \mathcal{S}, \\ \eta_1 = \nabla_s \eta_1 \cdot n_S = \eta_2 = 0 & \text{on } \partial \mathcal{S}. \end{cases} \quad (4.41)$$

where

$$\begin{aligned} \widehat{\mathbb{T}}_\lambda(v, \vartheta) &= 2\mu \mathbb{D}(v) + \left(\left(\alpha + \frac{R_0 \bar{\rho} \bar{\vartheta}}{\lambda} \right) \operatorname{div} v - R_0 \bar{\rho} \bar{\vartheta} \right) I_3, \\ \widehat{f}_2 &= f_2 - \frac{R_0 \bar{\vartheta}}{\lambda \bar{\rho}} \nabla f_1, \quad \widehat{h}_2 = h_2 + \frac{R_0 \bar{\vartheta}}{\lambda} f_1|_S. \end{aligned}$$

Let us set $\widehat{\mathcal{X}} = L^q(\mathcal{F})^3 \times L^q(\mathcal{F}) \times W_0^{2,q}(\mathcal{S}) \times L^q(\mathcal{S})$. We define (see (4.23))

$$\mathcal{D}(A_{v,\lambda}) = \mathcal{D}(A_v), \quad A_{v,\lambda} = \frac{\mu}{\bar{\rho}} \Delta + \left(\frac{\alpha + \mu}{\bar{\rho}} + \frac{R_0 \bar{\rho} \bar{\vartheta}}{\lambda} \right) \nabla \operatorname{div}.$$

In view of [35, Theorem 1.4] and of the Fredholm theorem, for each λ with $\operatorname{Re} \lambda \geq 0$, $A_{v,\lambda}$ is an isomorphism from $\mathcal{D}(A_{v,\lambda})$ onto $L^q(\mathcal{F})^3$ for any $q \in (1, \infty)$. Let $D_{v,\lambda} \in \mathcal{L}(W_0^{2,q}(\mathcal{S}), W^{2,q}(\mathcal{F})^3)$ defined by $D_{v,\lambda} g = w$, where w is the solution to the problem

$$\begin{cases} -\frac{\mu}{\bar{\rho}} \Delta w - \left(\frac{\alpha + \mu}{\bar{\rho}} + \frac{R_0 \bar{\rho} \bar{\vartheta}}{\lambda} \right) \nabla(\operatorname{div} w) = 0 & \text{in } \mathcal{F}, \\ w = \mathcal{T} g & \text{on } \partial \mathcal{F}. \end{cases}$$

We introduce the unbounded operator $\mathcal{A}_\lambda : \mathcal{D}(\mathcal{A}_\lambda) \rightarrow \widehat{\mathcal{X}}$ defined by

$$\mathcal{D}(\mathcal{A}_\lambda) = \left\{ [v, \vartheta, \eta_1, \eta_2]^\top \in W^{2,q}(\mathcal{F})^3 \times \mathcal{D}(A_\vartheta) \times \mathcal{D}(A_S) ; v - D_{v,\lambda} \eta_2 \in \mathcal{D}(A_{v,\lambda}) \right\},$$

and

$$\mathcal{A}_\lambda \begin{bmatrix} v \\ \vartheta \\ \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} A_{v,\lambda}(v - D_{v,\lambda} \eta_2) - R_0 \nabla \vartheta \\ A_\vartheta \vartheta \\ \eta_2 \\ -\Delta_s^2 \eta_1 + \Delta_s \eta_2 - \widehat{\mathbb{T}}_\lambda(v, \vartheta) e_3 \cdot e_3 \end{bmatrix}.$$

With the above notations, the system (4.41) can be written as

$$(\lambda I - \mathcal{A}_\lambda)[v, \vartheta, \eta_1, \eta_2]^\top = [\widehat{f}_2, f_3, h_1, \widehat{h}_2]^\top. \quad (4.42)$$

Proceeding as in the proof of Theorem 4.2, one can show the existence of $\tilde{\lambda} \in \rho(\mathcal{A}_\lambda)$. Using that $\mathcal{A}_\lambda$ has compact resolvent and the Fredholm alternative theorem, the existence and uniqueness of a solution to the system (4.41) are equivalent. Let us consider a solution of (4.41) with $[\widehat{f}_2, f_3, h_1, \widehat{h}_2]^\top = 0$. As in Step 1, we can deduce that

$$[v, \vartheta, \eta_1, \eta_2]^\top \in W^{2,2}(\mathcal{F})^3 \times W^{2,2}(\mathcal{F}) \times W^{4,2}(\mathcal{S}) \times W^{2,2}(\mathcal{S}). \quad (4.43)$$

Then $\vartheta = 0$ and multiplying (4.41) by v and by η_2 , we deduce as in Step 1 that

$$[v, \vartheta, \eta_1, \eta_2]^\top = 0.$$

This completes the proof of the proposition. $\square$

4.4. Maximal L^p - L^q regularity of the linear system. Assume

$$p, q \in (1, \infty), \quad \frac{1}{p} + \frac{1}{2q} \neq 1, \quad \frac{1}{p} + \frac{1}{2q} \neq \frac{1}{2}. \quad (4.44)$$

Note that (1.11) implies (4.44). In order to show the maximal L^p - L^q regularity of the system (4.20)–(4.22), we first introduce the following decomposition: for any $f \in L^1(\mathcal{F})$,

$$f = f_m + f_{\text{avg}}, \quad \text{with} \quad \int_{\mathcal{F}} f_m \, dy = 0, \quad f_{\text{avg}} = |\mathcal{F}|^{-1} \int_{\mathcal{F}} f(y) \, dy. \quad (4.45)$$

We use the same decomposition and the same notation for $L^1(\partial\mathcal{F})$ and $L^1(\mathcal{S})$.

Let us recall some standard results on the heat equation and on the linearized compressible Navier-Stokes system:

Lemma 4.5. *There exists $\beta_1 > 0$ such that, for any $\beta \in (0, \beta_1)$ and for any $\eta_{2,\dagger} \in W_{p,q,\beta}^{2,4}((0, \infty); \mathcal{S})$ with*

$$\eta_{2,\dagger}(0, \cdot) \equiv 0,$$

the following linear system

$$\begin{cases} \partial_t \rho_{\dagger} + \bar{\rho} \operatorname{div} v_{\dagger} = 0 & \text{in } (0, \infty) \times \mathcal{F}, \\ \partial_t v_{\dagger} - \frac{1}{\bar{\rho}} \operatorname{div} \bar{\mathbb{T}}(\rho_{\dagger}, v_{\dagger}, 0) = 0 & \text{in } (0, \infty) \times \mathcal{F}, \\ v_{\dagger} = \mathcal{T} \eta_{2,\dagger} & \text{on } (0, \infty) \times \partial\mathcal{F}, \\ \rho_{\dagger}(0, \cdot) = 0, \quad v_{\dagger}(0, \cdot) = 0 & \text{in } \mathcal{F}. \end{cases} \quad (4.46)$$

admits a unique solution

$$\rho_{\dagger} = \rho_{\dagger,m} + \rho_{\dagger,\text{avg}}, \quad \rho_{\dagger,m} \in W_{\beta}^{1,p}(0, \infty; W^{1,q}(\mathcal{F})), \quad \partial_t \rho_{\dagger,\text{avg}} \in L_{\beta}^p(0, \infty), \quad (4.47)$$

$$v_{\dagger} \in W_{p,q,\beta}^{1,2}((0, \infty) \times \mathcal{F}). \quad (4.48)$$

Moreover, the following estimate holds

$$\begin{aligned} \|\rho_{\dagger,m}\|_{W_{\beta}^{1,p}(0,\infty;W^{1,q}(\mathcal{F}))} + \|\rho_{\dagger,\text{avg}}\|_{L^{\infty}(0,\infty)} + \|\partial_t \rho_{\dagger,\text{avg}}\|_{L_{\beta}^p(0,\infty)} \\ + \|v_{\dagger}\|_{W_{p,q,\beta}^{1,2}((0,\infty)\times\mathcal{F})} \leq C \|\eta_{2,\dagger}\|_{W_{p,q,\beta}^{2,4}((0,\infty)\times\mathcal{S})}. \end{aligned} \quad (4.49)$$

Proof. Let χ be the cut-off function defined in (3.38) and we define

$$w_{\dagger}(t, y_1, y_2, y_3) := \chi(y_1, y_2, y_3) \eta_{2,\dagger}(t, y_1, y_2) e_3 \quad (t, y) \in (0, \infty) \times \bar{\mathcal{F}}.$$

Let us set $u_{\dagger} = v_{\dagger} - w_{\dagger}$. Then $(\rho_{\dagger}, u_{\dagger})$ solves

$$\begin{cases} \partial_t \rho_{\dagger} + \bar{\rho} \operatorname{div} u_{\dagger} = f_{1,\dagger} & \text{in } (0, \infty) \times \mathcal{F}, \\ \partial_t u_{\dagger} - \frac{1}{\bar{\rho}} \operatorname{div} \bar{\mathbb{T}}(\rho_{\dagger}, u_{\dagger}, 0) = f_{2,\dagger} & \text{in } (0, \infty) \times \mathcal{F}, \\ u_{\dagger} = 0 & \text{on } (0, \infty) \times \partial\mathcal{F}, \\ \rho_{\dagger}(0, \cdot) = 0, \quad u_{\dagger}(0, \cdot) = 0 & \text{in } \mathcal{F}, \end{cases} \quad (4.50)$$

where

$$f_{1,\dagger} = -\bar{\rho} \operatorname{div} w_{\dagger}, \quad f_{2,\dagger} = -\partial_t w_{\dagger} - \frac{1}{\bar{\rho}} \operatorname{div} \bar{\mathbb{T}}(0, w_{\dagger}, 0).$$

It is easy to see that

$$\|f_{1,\dagger}\|_{L_\beta^p(0,\infty;W^{1,q}(\mathcal{F}))} + \|f_2\|_{L_\beta^p(0,\infty;L^q(\mathcal{F}))} \leq C \|\eta_{2,\dagger}\|_{W_{p,q,\beta}^{2,4}((0,\infty)\times\mathcal{S})},$$

for any $\beta > 0$. We look for a solution to the system (4.50) of the form $\rho_\dagger = \rho_{\dagger,m} + \rho_{\dagger,\text{avg}}$, where $(\rho_{\dagger,m}, u_\dagger)$ solves the system (4.50) with $f_{1,\dagger}$ replaced by $f_{1,\dagger,m}$ and $\rho_{\dagger,\text{avg}} = \int_0^t f_{1,\dagger,\text{avg}}(s) \, ds$. By [16, Theorem 2.9], there exists $\beta_1 > 0$ such that for any $\beta \in (0, \beta_1)$, $(f_{1,\dagger,m}, f_{2,\dagger}) \in L_\beta^p(0, \infty; W^{1,q}(\mathcal{F})) \times L_\beta^p(0, \infty; L^q(\mathcal{F}))$, we have

$$\|\rho_{\dagger,m}\|_{W_\beta^{1,p}(0,\infty;W^{1,q}(\mathcal{F}))} + \|v_\dagger\|_{W_{p,q,\beta}^{1,2}((0,\infty)\times\mathcal{F})} \leq C \|f_{1,\dagger}\|_{L_\beta^p(0,\infty;W^{1,q}(\mathcal{F}))} + \|f_2\|_{L_\beta^p(0,\infty;L^q(\mathcal{F}))}.$$

Combining the above estimates we obtain the conclusion of the lemma. $\square$

Combining Step 3 of the proof of Theorem 3.2 and [12, Proposition 6.4], we deduce the following result:

Lemma 4.6. *Assume $\beta > 0$. There exists $\gamma_1 > 0$ such that for any*

$$\vartheta^0 \in B_{q,p}^{2(1-1/p)}(\mathcal{F}), \quad f_3 \in L_\beta^p(0, \infty; L^q(\mathcal{F})),$$

$$g \in F_{p,q,\beta}^{(1-1/q)/2}(0, \infty; L^q(\partial\mathcal{F})) \cap L_\beta^p(0, \infty; W^{1-1/q,q}(\partial\mathcal{F}))$$

with

$$\frac{\partial \vartheta^0}{\partial n} = g(0, \cdot) \quad \text{on } \partial\mathcal{F},$$

the following heat equation

$$\begin{cases} \partial_t \vartheta_\# + \gamma_1 \vartheta_\# - \bar{\kappa} \Delta \vartheta_\# = f_3 & \text{in } (0, \infty) \times \mathcal{F}, \\ \frac{\partial \vartheta_\#}{\partial n} = g & \text{on } (0, \infty) \times \partial\mathcal{F}, \\ \vartheta_\#(0, \cdot) = \vartheta^0 & \text{in } \mathcal{F}. \end{cases} \quad (4.51)$$

admits a unique solution $\vartheta_\# \in W_{p,q,\beta}^{1,2}((0, \infty); \mathcal{F})$. Moreover, we have the following estimate

$$\|\vartheta_\#\|_{W_{p,q,\beta}^{1,2}((0,\infty);\mathcal{F})} \leq C \left(\|\vartheta^0\|_{B_{q,p}^{2(1-1/p)}(\mathcal{F})} + \|f_3\|_{L_\beta^p(0,\infty;L^q(\mathcal{F}))} + \|g\|_{F_{p,q,\beta}^{(1-1/q)/2}(0,\infty;L^q(\partial\mathcal{F}))} + \|g\|_{L_\beta^p(0,\infty;W^{1-1/q,q}(\partial\mathcal{F}))} \right). \quad (4.52)$$

We consider the subset of initial conditions

$$\begin{aligned} \mathcal{J}_{p,q} := \left\{ [\rho^0, v^0, \vartheta^0, \eta_1^0, \eta_2^0]^\top \in W^{1,q}(\mathcal{F}) \times B_{q,p}^{2(1-1/p)}(\mathcal{F})^3 \times B_{q,p}^{2(1-1/p)}(\mathcal{F}) \times B_{q,p}^{2(2-1/p)}(\mathcal{S}) \times B_{q,p}^{2(1-1/p)}(\mathcal{S}) \right. \\ \eta_1^0 = \nabla_s \eta_1^0 \cdot n_S = 0 \quad \text{on } \mathcal{S}, \\ v^0 = \mathcal{T} \eta_2^0 \text{ on } \partial\mathcal{F} \quad \text{and} \quad \eta_2^0 = 0 \text{ on } \partial\mathcal{S} \quad \text{if} \quad \frac{1}{p} + \frac{1}{2q} < 1, \\ \left. \nabla_s \eta_2^0 \cdot n_S = 0 \text{ on } \partial\mathcal{S} \quad \text{if} \quad \frac{1}{p} + \frac{1}{2q} < \frac{1}{2} \right\} \quad (4.53) \end{aligned}$$

with

$$\begin{aligned} \left\| [\rho^0, v^0, \vartheta^0, \eta_1^0, \eta_2^0]^\top \right\|_{\mathcal{J}_{p,q}} &:= \|\rho^0\|_{W^{1,q}(\mathcal{F})} + \|v^0\|_{B_{q,p}^{2(1-1/p)}(\mathcal{F})^3} + \|\vartheta^0\|_{B_{q,p}^{2(1-1/p)}(\mathcal{F})} \\ &\quad + \|\eta_1^0\|_{B_{q,p}^{2(2-1/p)}(\mathcal{S})} + \|\eta_2^0\|_{B_{q,p}^{2(1-1/p)}(\mathcal{S})}. \end{aligned}$$

We also consider the following subset for the source terms:

$$\begin{aligned} \mathcal{R}_{p,q,\beta}^{cc} &= \left\{ [f_1, f_2, f_3, g, \tilde{h}, \hat{h}]^\top ; f_1 \in L_\beta^p(0, \infty, W^{1,q}(\mathcal{F})), f_2 \in L_\beta^p(0, \infty; L^q(\mathcal{F}))^3, \right. \\ &\quad f_3 \in L_\infty^p(0, \infty; L^q(\mathcal{F})), g \in F_{p,q,\beta}^{(1-1/q)/2}(0, \infty; L^q(\partial\mathcal{F})) \cap L_\beta^p(0, \infty; W^{1-1/q,q}(\partial\mathcal{F})), \\ &\quad \hat{h} \in L^\infty(0, \infty), \quad \partial_t \hat{h} \in L_\beta^p(0, \infty), \quad \tilde{h} \in L_\beta^p(0, \infty; L^q(\mathcal{S})), \\ &\quad \left. \text{with } g(0, \cdot) = \frac{\partial \vartheta_0}{\partial n} \text{ if } \frac{1}{p} + \frac{1}{2q} < \frac{1}{2} \right\}, \quad (4.54) \end{aligned}$$

with

$$\begin{aligned} \|[f_1, f_2, f_3, g, \tilde{h}, \hat{h}]^\top\|_{\mathcal{R}_{p,q,\beta}^{cc}} &= \|f_1\|_{L_\beta^p(0, \infty; W^{1,q}(\mathcal{F}))} + \|f_2\|_{L_\beta^p(0, \infty; L^q(\mathcal{F}))^3} + \|f_3\|_{L_\beta^p(0, \infty; L^q(\mathcal{F}))} \\ &\quad + \|g\|_{F_{p,q,\beta}^{(1-1/q)/2}(0, \infty; L^q(\partial\mathcal{F})) \cap L_\beta^p(0, \infty; W^{1-1/q,q}(\partial\mathcal{F}))} \\ &\quad + \|\tilde{h}\|_{L_\beta^p(0, \infty; L^q(\mathcal{S}))} + \|\hat{h}\|_{L^\infty(0, \infty)} + \|\partial_t \hat{h}\|_{L_\beta^p(0, \infty)}. \end{aligned}$$

We take $\beta = \min(\beta_0, \beta_1) > 0$ where β_0 is the constant in Theorem 4.3 and where β_1 is the constant in Lemma 4.5. We decompose the solution of the system (4.20)–(4.22) as follows

$$\rho = \rho_b + \rho_\diamond + \rho_\dagger, \quad v = v_\diamond + v_\dagger, \quad \vartheta = \vartheta_\diamond + \vartheta_\# + \vartheta_b, \quad \eta_1 = \eta_{1,\diamond} + \eta_{1,\dagger}, \quad \eta_2 = \eta_{2,\diamond} + \eta_{2,\dagger}, \quad (4.55)$$

where $\vartheta_\#$ is the solution of (4.51) given by Lemma 4.6, where

$$\vartheta_b(t) := \int_0^t \gamma_1 \vartheta_{\#, \text{avg}}(r) \, dr, \quad \rho_b(t) = \frac{1}{|\mathcal{F}|} \left(\int_{\mathcal{F}} \rho^0 \, dy + \bar{\rho} \int_{\mathcal{S}} \eta_1^0 \, ds \right) + \int_0^t f_{1, \text{avg}}(r) \, dr. \quad (4.56)$$

where $[\rho_\diamond, v_\diamond, \vartheta_\diamond, \eta_{1,\diamond}, \eta_{2,\diamond}]^\top$ is solution of the following system

$$\begin{cases} \partial_t \rho_\diamond + \bar{\rho} \operatorname{div} v_\diamond = f_{1,m} & \text{in } (0, \infty) \times \mathcal{F}, \\ \partial_t v_\diamond - \frac{1}{\bar{\rho}} \operatorname{div} \bar{\mathbb{T}}(\rho_\diamond, v_\diamond, \vartheta_\diamond) = f_2 - R_0 \nabla \vartheta_\# & \text{in } (0, \infty) \times \mathcal{F}, \\ \partial_t \vartheta_\diamond - \bar{\kappa} \Delta \vartheta_\diamond = \gamma_1 \vartheta_{\#, m} & \text{in } (0, \infty) \times \mathcal{F}, \\ \partial_t \eta_{1,\diamond} - \eta_{2,\diamond} = 0 & \text{in } (0, \infty) \times \mathcal{F}, \\ \partial_t \eta_{2,\diamond} + \Delta_s^2 \eta_{1,\diamond} - \Delta_s \eta_{2,\diamond} = -\bar{\mathbb{T}}(\rho_\diamond, v_\diamond, \vartheta_\diamond) e_3 \cdot e_3 + \tilde{h} + R_0 \bar{\rho} \vartheta_{\#|_{\mathcal{S}}} & \text{in } (0, \infty) \times \mathcal{S}, \end{cases} \quad (4.57)$$

$$\begin{cases} v_\diamond = \mathcal{T} \eta_{2,\diamond} & \text{on } (0, \infty) \times \partial\mathcal{F}, \\ \frac{\partial \vartheta_\diamond}{\partial n} = 0 & \text{on } (0, \infty) \times \partial\mathcal{F}, \\ \eta_{1,\diamond} = \nabla_s \eta_{1,\diamond} \cdot n_{\mathcal{S}} = 0 & \text{on } (0, \infty) \times \partial\mathcal{S}, \end{cases} \quad (4.58)$$

$$\begin{cases} \eta_{1,\diamond}(0, \cdot) = \eta_1^0, \quad \eta_{2,\diamond}(0, \cdot) = \eta_2^0 & \text{in } \mathcal{S}, \\ \rho_\diamond(0, \cdot) = \rho^0 - \rho_b(0), \quad v_\diamond(0, \cdot) = v^0, \quad \vartheta_\diamond(0, \cdot) = 0 & \text{in } \mathcal{F}. \end{cases} \quad (4.59)$$

and where $[\rho_{\dagger}, v_{\dagger}, \vartheta_{\dagger}, \eta_{1,\dagger}, \eta_{2,\dagger}]^{\top}$ is solution of the following system

$$\begin{cases} \partial_t \rho_{\dagger} + \bar{\rho} \operatorname{div} v_{\dagger} = 0 & \text{in } (0, \infty) \times \mathcal{F}, \\ \partial_t v_{\dagger} - \frac{1}{\bar{\rho}} \operatorname{div} \bar{\mathbb{T}}(\rho_{\dagger}, v_{\dagger}, \vartheta_{\dagger}) = 0 & \text{in } (0, \infty) \times \mathcal{F}, \\ \partial_t \vartheta_{\dagger} - \bar{\kappa} \Delta \vartheta_{\dagger} = 0 & \text{in } (0, \infty) \times \mathcal{F}, \\ \partial_t \eta_{1,\dagger} - \eta_{2,\dagger} = 0 & \text{in } (0, \infty) \times \mathcal{F}, \\ \partial_t \eta_{2,\dagger} + \Delta_s^2 \eta_{1,\dagger} - \Delta_s \eta_{2,\dagger} = -\bar{\mathbb{T}}(\rho_{\dagger}, v_{\dagger}, \vartheta_{\dagger}) e_3 \cdot e_3 + \hat{h} + R_0 \bar{\vartheta} \rho_b + R_0 \bar{\rho} \vartheta_b & \text{in } (0, \infty) \times \mathcal{S}, \end{cases} \quad (4.60)$$

$$\begin{cases} v_{\dagger} = \mathcal{T} \eta_{2,\dagger} & \text{on } (0, \infty) \times \partial \mathcal{F}, \\ \frac{\partial \vartheta_{\dagger}}{\partial n} = 0 & \text{on } (0, \infty) \times \partial \mathcal{F}, \\ \eta_{1,\dagger} = \nabla_s \eta_{1,\dagger} \cdot n_S = 0 & \text{on } (0, \infty) \times \partial \mathcal{S}, \end{cases} \quad (4.61)$$

$$\begin{cases} \eta_{1,\dagger}(0, \cdot) = 0, & \eta_{2,\dagger}(0, \cdot) = 0 & \text{in } \mathcal{S}, \\ \rho_{\dagger}(0, \cdot) = 0, & v_{\dagger}(0, \cdot) = 0, & \vartheta_{\dagger}(0, \cdot) = 0 & \text{in } \mathcal{F}. \end{cases} \quad (4.62)$$

Let us show that the decomposition (4.55) is valid. First, we can check that

$$\vartheta_b, \rho_b \in C_b^0([0, \infty)), \quad \partial_t \vartheta_b, \partial_t \rho_b \in L_{\beta}^p(0, \infty).$$

Second, for the system (4.57)–(4.59), we note that from (4.32) and (4.53)

$$\left[f_{1,m}, f_2 - R_0 \nabla \vartheta_{\sharp}, \gamma_1 \vartheta_{\sharp,m}, 0, \tilde{h} + R_0 \bar{\rho} \vartheta_{\sharp|S} \right]^{\top} \in L_{\beta}^p(0, \infty; \mathcal{X}_m),$$

$$\left[\rho^0 - \rho_b(0), v^0, 0, \eta_1^0, \eta_2^0 \right]^{\top} \in (\mathcal{X}_m, \mathcal{D}(\mathcal{A}_{FS}))_{1-1/p, p}.$$

From Theorem 4.2 and Theorem 4.3 we know that $\mathcal{A}_{FS} + \beta I$ is a $\mathcal{R}$ -sectorial operator on $\mathcal{X}_m$ and generates an analytic exponential stable semigroup on $\mathcal{X}_m$. Therefore, by Corollary 2.4, the system (4.57)–(4.59) admits a unique solution

$$\left[\rho_{\diamond}, v_{\diamond}, \vartheta_{\diamond}, \eta_{1,\diamond}, \eta_{2,\diamond} \right]^{\top} \in L_{\beta}^p(0, \infty; \mathcal{D}(\mathcal{A}_{FS}) \cap \mathcal{X}_m) \cap W_{\beta}^{1,p}(0, \infty; \mathcal{X}_m). \quad (4.63)$$

Finally, let us consider the system (4.60)–(4.62). Note that $\vartheta_{\dagger} \equiv 0$. Moreover

$$\frac{d}{dt} \begin{bmatrix} \partial_t \rho_{\dagger} \\ \partial_t v_{\dagger} \\ \partial_t \vartheta_{\dagger} \\ \partial_t \eta_{1,\dagger} \\ \partial_t \eta_{2,\dagger} \end{bmatrix} = \mathcal{A}_{FS} \begin{bmatrix} \partial_t \rho_{\dagger} \\ \partial_t v_{\dagger} \\ \partial_t \vartheta_{\dagger} \\ \partial_t \eta_{1,\dagger} \\ \partial_t \eta_{2,\dagger} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \partial_t \hat{h} + R_0 \bar{\vartheta} f_{1,\text{avg}} + R_0 \bar{\rho} \gamma_1 \vartheta_{\sharp,\text{avg}} \end{bmatrix}, \quad \begin{bmatrix} \partial_t \rho_{\dagger} \\ \partial_t v_{\dagger} \\ \partial_t \vartheta_{\dagger} \\ \partial_t \eta_{1,\dagger} \\ \partial_t \eta_{2,\dagger} \end{bmatrix}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \quad (4.64)$$

Using that $\partial_t \hat{h} + R_0 \bar{\vartheta} f_{1,\text{avg}} + R_0 \bar{\rho} \gamma_1 \vartheta_{\sharp,\text{avg}} \in L_{\beta}^p(0, \infty)$, and combining as above Theorem 4.2, Theorem 4.3 and Corollary 2.4, we infer that

$$\left[\partial_t \rho_{\dagger}, \partial_t v_{\dagger}, \partial_t \vartheta_{\dagger}, \partial_t \eta_{1,\dagger}, \partial_t \eta_{2,\dagger} \right]^{\top} \in L_{\beta}^p(0, \infty; \mathcal{D}(\mathcal{A}_{FS}) \cap \mathcal{X}_m) \cap W_{\beta}^{1,p}(0, \infty; \mathcal{X}_m). \quad (4.65)$$

In particular,

$$\eta_{2,\dagger} \in W_{p,q,\beta}^{2,4}((0, \infty); \mathcal{S}). \quad (4.66)$$

Then, we use Lemma 4.5 to deduce $(\rho_{\dagger}, v_{\dagger})$ satisfies (4.47)–(4.48).

Let us also write

$$\tilde{\rho} = \rho_{\diamond} + \rho_{\dagger,m}, \quad \hat{\rho} = \rho_b + \rho_{\dagger,\text{avg}}, \quad \tilde{\vartheta} = \vartheta_{\diamond} + \vartheta_{\sharp}, \quad \hat{\vartheta} = \vartheta_b, \quad (4.67)$$

so that

$$\rho = \tilde{\rho} + \hat{\rho}, \quad \vartheta = \tilde{\vartheta} + \hat{\vartheta}. \quad (4.68)$$

Gathering the above properties, we have obtained the following theorem:

Theorem 4.7. *Assume (4.44). There exists $\beta > 0$ such that for any*

$$[\rho^0, v^0, \vartheta^0, \eta_1^0, \eta_2^0]^\top \in \mathcal{J}_{p,q}, \quad [f_1, f_2, f_3, g, \tilde{h}, \hat{h}]^\top \in \mathcal{R}_{p,q,\beta}^{cc}, \quad h = \tilde{h} + \hat{h},$$

the system (4.20)–(4.22) admits a unique solution satisfying (4.15)–(4.18) and

$$\begin{aligned} & \|\rho\|_{L^\infty(0,\infty;W^{1,q}(\mathcal{F}))} + \|\nabla \rho\|_{W_\beta^{1,p}(0,\infty;L^q(\mathcal{F}))^3} + \|\partial_t \rho\|_{L_\beta^p(0,\infty;W^{1,q}(\mathcal{F}))} + \|v\|_{W_{p,q,\beta}^{1,2}((0,\infty);\mathcal{F})^3} \\ & + \|\vartheta\|_{L^\infty(0,\infty;B_{q,p}^{2(1-1/p)}(\mathcal{F}))} + \|\nabla \vartheta\|_{L_\beta^p(0,\infty;W^{1,q}(\mathcal{F}))^3} + \|\partial_t \vartheta\|_{L_\beta^p(0,\infty;L^q(\mathcal{F}))} \\ & + \|\eta_1\|_{L^\infty(0,\infty;B_{q,p}^{2(2-1/p)}(\mathcal{S}))} + \|\eta_2\|_{W_{p,q,\beta}^{1,2}((0,\infty);\mathcal{S})} \\ & \leq C_L \left(\left\| [\rho^0, v^0, \vartheta^0, \eta_1^0, \eta_2^0]^\top \right\|_{\mathcal{J}_{p,q}} + \left\| [f_1, f_2, f_3, g, \tilde{h}, \hat{h}]^\top \right\|_{\mathcal{R}_{p,q,\beta}^{cc}} \right). \end{aligned} \quad (4.69)$$

Moreover, we can decompose the solution as (4.67)–(4.68), with

$$\tilde{\rho} \in W_\beta^{1,p}(0,\infty;W^{1,q}(\mathcal{F})), \quad \tilde{\vartheta} \in W_{p,q,\beta}^{1,2}((0,\infty);\mathcal{F}), \quad [\hat{\rho}, \hat{\vartheta}]^\top \in L^\infty(0,\infty)^2, \quad [\partial_t \hat{\rho}, \partial_t \hat{\vartheta}]^\top \in L_\beta^p(0,\infty)^2, \quad (4.70)$$

and

$$\begin{aligned} & \|\tilde{\rho}\|_{W_\beta^{1,p}(0,\infty;W^{1,q}(\mathcal{F}))} + \|\tilde{\vartheta}\|_{W_{p,q,\beta}^{1,2}((0,\infty);\mathcal{F})} + \left\| [\hat{\rho}, \hat{\vartheta}]^\top \right\|_{L^\infty(0,\infty)^2} + \left\| [\partial_t \hat{\rho}, \partial_t \hat{\vartheta}]^\top \right\|_{L_\beta^p(0,\infty)^2} \\ & \leq C_L \left(\left\| [\rho^0, v^0, \vartheta^0, \eta_1^0, \eta_2^0]^\top \right\|_{\mathcal{J}_{p,q}} + \left\| [f_1, f_2, f_3, g, \tilde{h}, \hat{h}]^\top \right\|_{\mathcal{R}_{p,q,\beta}^{cc}} \right). \end{aligned} \quad (4.71)$$

4.5. Proof of Theorem 4.1. In this subsection, we prove Theorem 4.1 (or equivalently Theorem 1.2): we show the existence and uniqueness of global in time solutions for the system (4.3)–(4.13) under a smallness assumption on the initial data.

Let us assume the hypotheses of Theorem 4.1, with β given by Theorem 4.7. Assume

$$[\rho^0, v^0, \vartheta^0, \eta_1^0, \eta_2^0]^\top \in \mathcal{J}_{p,q},$$

where $\mathcal{J}_{p,q}$ is defined by (4.53). For $R > 0$, we define $\mathcal{B}_R$ as follows

$$\mathcal{B}_R = \left\{ [f_1, f_2, f_3, g, \tilde{h}, \hat{h}]^\top \in \mathcal{R}_{p,q,\beta}^{cc} ; \left\| [f_1, f_2, f_3, g, \tilde{h}, \hat{h}]^\top \right\|_{\mathcal{R}_{p,q,\beta}^{cc}} \leq R \right\}, \quad (4.72)$$

where $\mathcal{R}_{p,q,\beta}^{cc}$ is defined by (4.54). By using Lemma 4.6 with $f_3 = 0$ and $g = 0$, we see that there exists a constant $C > 0$ independent of R such that if

$$\left\| [\rho^0, v^0, \vartheta^0, \eta_1^0, \eta_2^0]^\top \right\|_{\mathcal{J}_{p,q}} \leq CR, \quad (4.73)$$

then $\mathcal{B}_R$ is a nonempty closed subset of the Banach space

$$\begin{aligned} \mathcal{R}_{p,q,\beta} = & \left\{ [f_1, f_2, f_3, g, \tilde{h}, \hat{h}]^\top ; f_1 \in L_\beta^p(0,\infty;W^{1,q}(\mathcal{F})), \quad f_2 \in L_\beta^p(0,\infty;L^q(\mathcal{F}))^3, \right. \\ & f_3 \in L_\infty^p(0,\infty;L^q(\mathcal{F})), \quad g \in F_{p,q,\beta}^{(1-1/q)/2}(0,\infty;L^q(\partial\mathcal{F})) \cap L_\beta^p(0,\infty;W^{1-1/q,q}(\partial\mathcal{F})), \\ & \left. \hat{h} \in L^\infty(0,\infty), \quad \partial_t \hat{h} \in L_\beta^p(0,\infty), \quad \tilde{h} \in L_\beta^p(0,\infty;L^q(\mathcal{S})) \right\}. \end{aligned} \quad (4.74)$$

We define the map

$$\Xi_R : \mathcal{B}_R \longrightarrow \mathcal{B}_R, \quad [f_1, f_2, f_3, g, \tilde{h}, \hat{h}]^\top \longmapsto [F_1, F_2, F_3, G, \tilde{H}, \hat{H}]^\top, \quad (4.75)$$

where $[\rho, v, \vartheta, \eta, \partial_t \eta]^\top$ is the solution to the system (4.20)–(4.22) associated with $[f_1, f_2, f_3, g, h]^\top$ and $[\rho^0, v^0, \vartheta^0, \eta_1^0, \eta_2^0]^\top$, (see Theorem 4.7), where F_1, F_2, F_3 and G are given by (4.9)–(4.12) and where

$$\begin{aligned} \tilde{H} = -\mu \left[\frac{1}{\delta_X} \left(\nabla v \mathbb{B}_X^\top + \mathbb{B}_X \nabla v^\top \right) \begin{bmatrix} -\nabla_s \eta \\ 1 \end{bmatrix} - 2\mu \mathbb{D}(v) e_3 \right] \cdot e_3 - \alpha \left(\frac{1}{\delta_X} \mathbb{B}_X - I_3 \right) : \nabla v \\ + R_0 \tilde{\rho} \tilde{\vartheta} + R_0 \tilde{\rho} \hat{\vartheta} + R_0 \hat{\rho} \tilde{\vartheta}, \end{aligned} \quad (4.76)$$

and

$$\hat{H} = R_0 \hat{\rho} \hat{\vartheta}. \quad (4.77)$$

In the above definitions, we have used the decomposition of ρ and ϑ given by (4.67)–(4.68). We can check that H defined by (4.13) satisfies

$$H = \tilde{H} + \hat{H}.$$

In order to prove Theorem 4.1, it is enough to show that the mapping Ξ_R is well defined, $\Xi(\mathcal{B}_R) \subset \mathcal{B}_R$ and $\Xi|_{\mathcal{B}_R}$ is a strict contraction, for R small enough.

Throughout this subsection, C will be a positive constant depending on p, q and β but independent of R , which may change from line to line. To simplify the computations, we assume that $R \in (0, 1)$.

Since $2 < p < \infty$ and $3 < q < \infty$, one has (see, for instance [38, (7), p. 196])

$$B_{q,p}^{2(1-1/p)}(\mathcal{F}) \hookrightarrow W^{1,q}(\mathcal{F}) \hookrightarrow L^\infty(\mathcal{F}).$$

Therefore, from Theorem 4.7, we obtain

$$\begin{aligned} \|v\|_{L_\beta^\infty(0,\infty;W^{1,q}(\mathcal{F}))^3} + \|\vartheta\|_{L^\infty(0,\infty;W^{1,q}(\mathcal{F}))} + \|\nabla \vartheta\|_{L_\beta^p(0,\infty;L^\infty(\mathcal{F}))^3} \\ + \|\eta\|_{L^\infty(0,\infty;W^{3,q}(\mathcal{S}))} + \|\eta\|_{L^\infty(0,\infty;C^2(\bar{\mathcal{S}}))} \leq CR. \end{aligned} \quad (4.78)$$

From the definition of X^0 from (3.2) and from (4.73) we deduce that

$$\|\nabla X^0 - I_3\|_{W^{2,q}(\mathcal{F})^9} \leq CR.$$

Using the above estimate and the definition of X (see (3.18)) it follows that

$$\|\nabla X - I_3\|_{L^\infty(0,\infty;W^{1,q}(\mathcal{F}))^9} \leq \|\nabla X^0 - I_3\|_{W^{2,q}(\mathcal{F})^9} + C \|\nabla v\|_{L_\beta^p(0,\infty;W^{1,q}(\mathcal{F}))^9} \leq CR. \quad (4.79)$$

In particular, by choosing R sufficiently small, we have

$$\|\nabla X - I_3\|_{L^\infty((0,\infty) \times \mathcal{F})^9} \leq \frac{1}{2}.$$

Thus X is a C^1 -diffeomorphism for R small enough. Moreover, by combining the above estimates with (3.5) and using that $\partial_t X = v$, we also deduce

$$\|\mathbb{B}_X - I_3\|_{L^\infty(0,\infty;W^{1,q}(\mathcal{F}))^9} \leq CR, \quad \|\partial_t \mathbb{B}_X\|_{L^p(0,\infty;W^{1,q}(\mathcal{F}))^9} \leq CR, \quad \|\mathbb{B}_X\|_{L^\infty(0,\infty;W^{1,q}(\mathcal{F}))^9} \leq C, \quad (4.80)$$

$$\|\delta_X - 1\|_{L^\infty(0,\infty;W^{1,q}(\mathcal{F}))} \leq CR, \quad \|\partial_t \delta_X\|_{L^p(0,\infty;W^{1,q}(\mathcal{F}))} \leq CR, \quad \|\delta_X\|_{L^\infty(0,\infty;W^{1,q}(\mathcal{F}))} \leq C. \quad (4.81)$$

Consequently, for R small enough

$$\delta_X \geq \frac{1}{2} \quad \text{for all } (t, y) \in (0, \infty) \times \mathcal{F}. \quad (4.82)$$

We thus deduce

$$\left\| \frac{1}{\delta_X} - 1 \right\|_{L^\infty(0,\infty;W^{1,q}(\mathcal{F}))} \leq CR, \quad \left\| \partial_t \left(\frac{1}{\delta_X} \right) \right\|_{L^p(0,\infty;W^{1,q}(\mathcal{F}))} \leq CR, \quad \left\| \frac{1}{\delta_X} \right\|_{L^\infty(0,\infty;W^{1,q}(\mathcal{F}))} \leq C. \quad (4.83)$$

Using the above estimates and (3.5), we also obtain

$$\|\mathbb{A}_X - I_3\|_{L^\infty(0,\infty;W^{1,q}(\mathcal{F}))^9} \leq CR, \quad \|\mathbb{A}_X\|_{L^\infty(0,\infty;W^{1,q}(\mathcal{F}))^9} \leq C, \quad (4.84)$$

$$\|\partial_t \mathbb{A}_X\|_{L^p(0,\infty;W^{1,q}(\mathcal{F}))^9} \leq CR, \quad \|\mathbb{A}_X - I_3\|_{C^{1/p'}([0,\infty);W^{1,q}(\mathcal{F}))^9} \leq CR. \quad (4.85)$$

For more details about the proof of the above estimates, we refer to [24, Lemma 3.19].

From (4.12) and (3.23), we notice that

$$G(\rho, v, \vartheta, \eta)|_{t=0} = (I_3 - \mathbb{A}^0) \nabla \vartheta^0 \cdot n = \frac{\partial \vartheta^0}{\partial n} \quad \text{on } \partial \mathcal{F}. \quad (4.86)$$

Using the above estimates we deduce that F_1, F_2, F_3, G and $\tilde{H}, \hat{H}$ defined by (4.9)–(4.12) and (4.76)–(4.77) satisfy the estimate

$$\left\| [F_1, F_2, F_3, G, \tilde{H}, \hat{H}]^\top \right\|_{\mathcal{R}_{p,q,\beta}^{cc}} \leq CR^2. \quad (4.87)$$

To details on the proof of (4.87) can be found in [24, Proposition 3.20]. This shows that $\Xi(\mathcal{B}_R) \subset \mathcal{B}_R$ for R small enough.

To show that $\Xi_R|_{\mathcal{B}_R}$ is a strict contraction, we proceed similarly: we consider

$$\left[f_1^{(i)}, f_2^{(i)}, f_3^{(i)}, g^{(i)}, \tilde{h}^{(i)}, \hat{h}^{(i)} \right]^\top \in \mathcal{B}_R, \quad i = 1, 2$$

and we denote by $[\rho^{(i)}, v^{(i)}, \vartheta^{(i)}, \eta^{(i)}, \partial_t \eta^{(i)}]^\top$ the solutions to the system (4.20)–(4.20) associated with $[f_1^{(i)}, f_2^{(i)}, f_3^{(i)}, g^{(i)}, \tilde{h}^{(i)}, \hat{h}^{(i)}]^\top$ and $[\rho^0, v^0, \vartheta^0, \eta_1^0, \eta_2^0]^\top$ (see Theorem 4.7). We can thus define

$$\left[F_1^{(i)}, F_2^{(i)}, F_3^{(i)}, G^{(i)}, \tilde{H}^{(i)}, \hat{H}^{(i)} \right]^\top := \Xi_R \left(\left[f_1^{(i)}, f_2^{(i)}, f_3^{(i)}, g^{(i)}, \tilde{h}^{(i)}, \hat{h}^{(i)} \right]^\top \right).$$

We also write

$$[f_1, f_2, f_3, g, \tilde{h}, \hat{h}]^\top = [f_1^{(1)}, f_2^{(1)}, f_3^{(1)}, g^{(1)}, \tilde{h}^{(1)}, \hat{h}^{(1)}]^\top - [f_1^{(2)}, f_2^{(2)}, f_3^{(2)}, g^{(2)}, \tilde{h}^{(2)}, \hat{h}^{(2)}]^\top,$$

$$[\rho, v, \vartheta, \eta]^\top = [\rho^{(1)}, v^{(1)}, \vartheta^{(1)}, \eta^{(1)}]^\top - [\rho^{(2)}, v^{(2)}, \vartheta^{(2)}, \eta^{(2)}]^\top,$$

$$[\tilde{\rho}, \tilde{\vartheta}]^\top = [\tilde{\rho}^{(1)}, \tilde{\vartheta}^{(1)}]^\top - [\tilde{\rho}^{(2)}, \tilde{\vartheta}^{(2)}]^\top, \quad [\hat{\rho}, \hat{\vartheta}]^\top = [\hat{\rho}^{(1)}, \hat{\vartheta}^{(1)}]^\top - [\hat{\rho}^{(2)}, \hat{\vartheta}^{(2)}]^\top.$$

Therefore, from Theorem 4.7, we obtain

$$\begin{aligned} & \|v\|_{L_\beta^\infty(0,\infty;W^{1,q}(\mathcal{F}))^3} + \|\vartheta\|_{L^\infty(0,\infty;W^{1,q}(\mathcal{F}))} + \|\nabla \vartheta\|_{L_\beta^p(0,\infty;L^\infty(\mathcal{F}))^3} \\ & + \|\eta\|_{L^\infty(0,\infty;W^{3,q}(\mathcal{S}))} + \|\eta\|_{L^\infty(0,\infty;C^2(\overline{\mathcal{S}}))} \leq C \left\| [f_1, f_2, f_3, g, \tilde{h}, \hat{h}]^\top \right\|_{\mathcal{R}_{p,q,\beta}^{cc}} \end{aligned} \quad (4.88)$$

and

$$\begin{aligned} & \|\tilde{\rho}\|_{W_\beta^{1,p}(0,\infty;W^{1,q}(\mathcal{F}))} + \|\tilde{\vartheta}\|_{W_{p,q,\beta}^{1,2}((0,\infty);\mathcal{F})} + \left\| [\hat{\rho}, \hat{\vartheta}]^\top \right\|_{L^\infty(0,\infty)^2} + \left\| [\partial_t \hat{\rho}, \partial_t \hat{\vartheta}]^\top \right\|_{L_\beta^p(0,\infty)^2} \\ & \leq C \left(\left\| [f_1, f_2, f_3, g, \tilde{h}, \hat{h}]^\top \right\|_{\mathcal{R}_{p,q,\beta}^{cc}} \right). \end{aligned} \quad (4.89)$$

In particular, from (3.18),

$$\left\| \nabla X^{(1)} - \nabla X^{(2)} \right\|_{L^\infty(0,\infty;W^{1,q}(\mathcal{F}))^9} \leq C \left\| [f_1, f_2, f_3, g, \tilde{h}, \hat{h}]^\top \right\|_{\mathcal{R}_{p,q,\beta}^{cc}}. \quad (4.90)$$

By combining the above estimates with (3.5) and with (4.79), we deduce

$$\begin{aligned}
& \|\mathbb{B}_{X^{(1)}} - \mathbb{B}_{X^{(2)}}\|_{L^\infty(0,\infty;W^{1,q}(\mathcal{F}))^9} + \|\partial_t \mathbb{B}_{X^{(1)}} - \partial_t \mathbb{B}_{X^{(2)}}\|_{L^p(0,\infty;W^{1,q}(\mathcal{F}))^9} \\
& \quad + \|\delta_{X^{(1)}} - \delta_{X^{(2)}}\|_{L^\infty(0,\infty;W^{1,q}(\mathcal{F}))} + \|\partial_t \delta_{X^{(1)}} - \partial_t \delta_{X^{(2)}}\|_{L^p(0,\infty;W^{1,q}(\mathcal{F}))} \\
& \quad + \left\| \frac{1}{\delta_{X^{(1)}}} - \frac{1}{\delta_{X^{(2)}}} \right\|_{L^\infty(0,\infty;W^{1,q}(\mathcal{F}))} + \left\| \partial_t \left(\frac{1}{\delta_{X^{(1)}}} \right) - \partial_t \left(\frac{1}{\delta_{X^{(2)}}} \right) \right\|_{L^p(0,\infty;W^{1,q}(\mathcal{F}))} \\
& \quad + \|\mathbb{A}_{X^{(1)}} - \mathbb{A}_{X^{(2)}}\|_{L^\infty(0,\infty;W^{1,q}(\mathcal{F}))^9} + \|\partial_t \mathbb{A}_{X^{(1)}} - \partial_t \mathbb{A}_{X^{(2)}}\|_{L^p(0,\infty;W^{1,q}(\mathcal{F}))^9} \\
& \quad + \|\partial_t \mathbb{A}_{X^{(1)}} - \partial_t \mathbb{A}_{X^{(2)}}\|_{C^{1/p'}([0,\infty);W^{1,q}(\mathcal{F}))^9} \\
& \leq CR \left\| [f_1, f_2, f_3, g, \tilde{h}, \hat{h}]^\top \right\|_{\mathcal{R}_{p,q,\beta}^{cc}}. \quad (4.91)
\end{aligned}$$

Using the above estimates we deduce that F_1, F_2, F_3, G and $\tilde{H}, \hat{H}$ defined by (4.9)–(4.12), (4.76), (4.77) satisfy the estimate

$$\begin{aligned}
& \left\| F_1^{(1)} - F_1^{(2)} \right\|_{L_\beta^p(0,\infty;W^{1,q}(\mathcal{F}))} + \left\| F_2^{(1)} - F_2^{(2)} \right\|_{L_\beta^p(0,\infty;L^q(\mathcal{F}))^3} + \left\| F_3^{(1)} - F_3^{(2)} \right\|_{L_\beta^p(0,\infty;L^q(\mathcal{F}))} \\
& \quad + \left\| G_1^{(1)} - G_1^{(2)} \right\|_{F_{p,q,\beta}^{(1-1/q)/2}(0,\infty;L^q(\partial\mathcal{F})) \cap L_\beta^p(0,\infty;W^{1-1/q,q}(\partial\mathcal{F}))} + \left\| \tilde{H}^{(1)} - \tilde{H}^{(2)} \right\|_{L_\beta^p(0,\infty;L^q(S))} \\
& \quad + \left\| \hat{H}^{(1)} - \hat{H}^{(2)} \right\|_{L^\infty(0,\infty)} + \left\| \partial_t \hat{H}^{(1)} - \partial_t \hat{H}^{(2)} \right\|_{L_\beta^p(0,\infty)} \\
& \leq CR \left\| [f_1, f_2, f_3, g, h]^\top \right\|_{\mathcal{R}_{p,q,\beta}^{cc}}. \quad (4.92)
\end{aligned}$$

This shows that $\Xi|_{\mathcal{B}_R}$ is a strict contraction, for R small enough. This completes the proof of Theorem 4.1 and Theorem 1.2. $\square$

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