Ordered models for concept representation

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Abstract

Basic notions linked with concept theory can be accounted for by partial order relations. These orders translate the fact that, for an agent, an object may be seen as a better or a more typical exemplar of a concept than an other. They adequately model notions linked with categorial membership, typicality and resemblance, without any of the drawbacks that are classically encountered in conjunction theory. An interesting consequence of such a concept representation is the possibility of using the tools of non-monotonic logic to address some well-known problems of cognitive psychology. Thus, conceptual entailment and concept induction can be reexamined in the framework of preferential inference relations. This leads to a rigorous definition of the basic notions used in the study of category-based induction.

Keywords categorization, prototype theory, categorial membership, typicality, inference relations, nonmonotonic logic, category-based induction, resemblance theory, deep learning.

Introduction

This paper aims at providing a mathematical formalism for concept representation. Its starting point is the *binary model* proposed by Osherson and Smith [26], in which the notions of categorial membership and typicality receive independent treatments. Order relations can be defined from the two fundamental sets that, in this model, govern concept analysis: the *defining* feature set and the *characteristic* one. As we shall see, such orders reveal themselves to be an adequate tool to address the classical problems encountered in this domain.

Concept analysis, with which this paper is concerned, is an important domain of cognitive psychology. The results we present here follow in a long line of uninterrupted theoretical and experimental studies that go back to the beginning of the seventies. They differ from the ones obtained in [11], [12] and [13] in several points. First, the domain of this study has been noticeably enlarged. It now covers most familiar concepts and is no longer restricted to the family of concepts recursively obtained from sharp ones, as was previously the case. Next, an operational distinction has been drawn between *concepts* and *features*. Such a distinction renders possible the construction of three different orders associated with a given concept: one, based on the defining feature set, that compares the *categorial membership* of two items, the second, based on the characteristic set, that compares the *typicality* of two exemplars, and the last one, built on the union of the two preceding ones, that can be used to evaluate *resemblance* between concepts. With the help of these orders, a refinement of the notion of subconcept has also been operated, providing an interesting solution to the problem of within-category induction.

We have to emphasize that the model we propose is not universal. It only applies to the subfamily of concepts for which categorial membership and typicality can be defined through auxiliary sets of features. This model is not objective either, in the sense that the concepts on which is based our study and their attached feature sets are those that a given (human) agent considers at a given time. Finally, the reader has to be aware that, at this stage, this model is to be considered as a simple working hypothesis, since no experiment has yet tested its validity.

Plan of this paper

In order to make this paper self-contained, we revisit in the first part the fundamental notions on which is built concept theory. We first recall the basic definitions that are linked with the notion of concept. After characterizing the family of concepts that constitutes our domain of study, we propose an ordered model to evaluate categorial membership for simple or composed concepts (Sections 2 and 3). A similar construction is introduced in Section

4 to account for the notion of typicality. There we characterize the family of subconcepts that can be seen as a determination of a given concept. Section 5 deals with resemblance theory: it introduces a new way of computing a resemblance degree between concepts.

The second part of this paper is devoted to concept entailment. In section 6 we present a logical model that parallels that used by Kraus, Lehmann and Magidor to represent preferential inference relations ([22]). This, as we show in Section 7, provides a particularly well adapted framework to revisit some classical problems in category-based induction. Section 8 is a conclusion.

The proofs of the propositions are given in the Appendix.

Part I
Representing the basic notions of concept analysis

1 Concepts, features and objects

We denote by $\mathcal{O}$ the set of all objects, real or imaginary, that a human agent has at her disposal at a given time. Together with this set, we suppose given a set of concepts $\mathcal{F}$ that reflects the agent's knowledge of her environment, and on which she builds her reasoning process. For simplification, we shall adopt the original presentation of Frege [9] who assimilated concepts with one-place predicates. In this perspective, concepts will be generally introduced through the auxiliary *to-be* followed by a noun: *to-be-a-bird, to-be-a-vector-space, to-be-a-democracy.*

Objects are entities that are identified as such by some agent. The process of identification may consist in recognizing an object as falling under one or several concepts that are part of the agent's knowledge. In this case, concepts appear to be prior to objects, and this conforms with Quine's famous *No entity without identity* [28]. A concept may also arise from the observation of an object. This is for example the case for some nominal or scientific concepts. Thus, the concept *to-be-a-group* stemmed from the study by Galois of an object, which was the set of permutations of the roots of an algebraic equation... In any case, whatever process has been used by an agent to recognize and name the objects of her environment, the starting point of this study is the existence in her mind of these two sets $\mathcal{O}$ and $\mathcal{F}$, which are related by a *categorial membership* relation. The agent will consider that an object x *falls* under a concept α, or that α *applies* to x, if this relation is satisfied between x and α. For example, we may say that 'Beethoven's opus 111 falls under the concept *to-be-a-piano-sonata*', or that 'The dog Tobby does not fall under the concept *to-be-a cat*'.

An object that falls under a concept is said to be an *exemplar* or an *instance* of this concept. The set $Ext\,\alpha$ of exemplars of a concept α forms its associated *category* or *extension.*

Restricting if necessary our field of study, we shall suppose that the categorial membership relation is complete. This means that, given any object of her environment, the agent is able to decide whether or not this object falls under a given concept.

The notion of *feature* differs from that of *concept*. Formally, features may be introduced through a verb (e.g. *to-fly*), through the auxiliary *to-have* followed by a noun (e.g. *to-have-a-beak)*, or through the auxiliary *to-be* followed by an adjective (e.g. *to-be-tall*). Features, like concepts, apply to the objects at hand but, contrary to the concepts that an agent may use, they are context-sensitive: they borrow part of their significance from the concept to which they are attached[1]. Properties like *to-be-tall*, *to-be-rich* or *to-be-red* take their full meaning only in a given context, that is when qualifying a well-defined entity. Even simple verbal forms like *to-fly*, *to-run*, *to-live-in-water*, *to-be-made-of-metal* need a principal referent concept to fully seize the strength with which they apply to different items. The concept to which a feature applies may be seen itself as a contextual determination of this feature. To summarize, the meaning of a feature depends on the context where this feature is used, contrary to the meaning of a concept, which, for a given agent, exists by itself.

It does not seem at this stage that any formalism can fully account for the difference between features and concepts. It is true that in Description Logics, a different treatment is applied for one and two-place predicates: binary predicates characterize indeed the *roles* of the language, which are used to express relationship between the concepts [23]. In such a framework, *to-be-a-tree* will be a concept, expressible by a single symbol A, but *to-have-green-leaves* is a role, expressed by a formula of the type '$\forall$ hasLeaves.Green'. However, no difference is made in Description Logics between the unary predicates that translate a notion of concept and those that translate a notion of feature.

In spite of these distinctions, features may be assimilated with concepts when their meaning is well determined, that is when they intervene in a given context. For instance, referring to men, the feature *to-be-short* is clear enough, even though it is a vague concept. This identification between concepts and unambiguous features renders possible a simplified universe of discourse.

In the classical theory of categorization, it is through its features that a

[1]The concepts that an agent has in mind are themselves context-sensitive, but at a different level: they indeed depend on the intellectual and social environment of this agent. This sensitivity may be ignored when working, as we do, on a single agent's *Weltanschauung*.

concept is defined or characterized. Vertebrates that have beaks and feathers will be labelled *birds*; a *car* may be defined as a road vehicle powered by an engine able to carry a small number of people; *democracy* is a system of government by all the eligible members of a state. These features may include the potential actions that are expected from instances of the concept and that our perception is aware of [24]. To quote E. Gregoromichelaki [15]: '*Perception of an entity will then be constituted by the set of expectations concerning the possible actions enabled by it (its* affordance*), rather than its association with a mental symbol and stored propositional knowledge'.* Thus, *to-drive-in-nails* may be listed among the defining features of the concept *to-be-a-hammer*.

In this perspective, where each concept is endowed with a set of attributes, categorization relative to a concept boils down to categorization relative to its defining features. On this *attributional view* rests the *binary model* ([32] and [31]), in which two auxiliary sets are attached to a concept. On the one hand, the *defining feature set* provides the conditions that an object should satisfy in order to fall under a given concept; on the other hand, the *characteristic set* lists the features that an object should have to be qualified as a *typical* instance of this concept. Given for instance the concept *to-be-a-fruit*, an agent may take as defining feature set the set consisting of the two elements *to-be-a-vegetable* and *to-have-seeds*, while the characteristic set would include features like *to-grow-on-trees*, *to-be-sweet*, *to-be-raw-edible*, *to-yield-juice*.

The attributional theory was thereafter rejected by most researchers, as it appeared that concepts defined by a conjunction of features form an exceptional subclass. Fodor for instance [7] argued that there exists practically no examples of successful definitions around. Without being so radical, it is clear that a great deal of concepts are deprived of any set of defining features: what list of attributes could be attached for concepts like *to-be-a-game*, *to-be-a-lie*, or *to-be-a-heap* ? However, in spite of this fact, it appears that the attributional view is justified for certain well-defined families of concepts: such is for instance the case for most *nominal* concepts, i.e. concepts that are conventionally defined, like *to-be-a-mammal*, *to-be-a-theft* or *to-be-a-refugee*. In particular, this remains true for scientific concepts, like *to-be-a-vortex* or *to-be-a-square*. Furthermore, it may happen that a concept, first grasped through its exemplars, is thereafter sharpened with the help of a set of defining features, as happened to the concept *to-be-a-bird*, a natural kind concept that was revisited by naturalists and turned into the pseudo-nominal *to-have-feathers* + *to-have-a-beak* + *to-have-wings*. Note also that even for non-definable concepts, class membership may still depend on auxiliary features: this occurs when categorial membership is induced through resemblance to a prototypical exemplar. Then, it is the features of this prototype that play the role of defining features.

These considerations show that the family of concepts whose categorial membership and typicality may be defined through their corresponding set of features is large enough to deserve a treatment of its own. Since this class of concepts can be studied with the help of quite elementary mathematical tools, there is no reason not to devote to them a special study. Such is the aim of the present paper.

2 Categorial membership

An attributional concept being determined by its features, its categorization process depends on the way these features apply to the different objects that are part of the universe of discourse. As a first approximation, one may use two-valued functions to measure the applicability of a feature to an object, and decide that a concept applies to an object if and only if all its defining features apply to this object. Such an attitude, however, would reduce categorial membership to an all-or-none matter, ignoring the fact that an object may be *closer* than another to falling under a concept, even if this concept does not fully apply to it. This observation drove Eleanor Rosch [29] and her successors to the conclusion that *membership is a matter of degree*. In fact, even this assertion is disputable, as, in many cases, an agent will be unable to assign a precise membership degree to a given item: the Neolithic man was able to categorize and compare items without knowing anything about degrees or scales, and the same is clearly true for young children.

At a basic level, one may consider that membership relative to a concept boils down to a comparison relation between the objects of the agent's universe. Thus, without trying to be more precise, and while unable to assign a membership degree to the items at hand, an agent may judge that an arquebus is definitely less *a-weapon-of-mass-destruction* than a machine-gun. This means that, for an agent, a concept may be grasped through the *membership order* that this concept induces among the objects of the universe. The process of categorization then consists in comparing items and evaluating their relative membership to the target category. Note that, in some cases, this membership order may generate a corresponding degree of membership. For example, the agent could consider a chain of objects that includes *machine-gun*, dispose these objects on a [0,1] scale, and thereafter deduce from this its respective membership degree. Thus, considering the chain

$$\textit{bludgeon} \leq \textit{sword} \leq \textit{crossbow} \leq \textit{arquebus} \leq \textit{gun} \leq \textit{machine-gun} \leq$$
$$\textit{flamethrower} \leq \textit{conventional bomb} \leq \textit{scud} \leq \textit{atomic bomb}$$

the agent may subsequently decide that a machine-gun is a WMD with degree 0.6. Nevertheless - and this is the hypothesis on which our whole study is based - the point is that, in most cases, numerical values are attributed only after a pre-recognized order has been set among the different items. An example of such a construction will be examined in section 2.6.

Coming back to the notion of membership order, one may ask what specific properties should such a relation enjoy. Clearly, this kind of order cannot be arbitrary. To correctly model membership relative to a concept, an ordering relation among the objects has to satisfy at least two properties. First, its set of maximal elements should agree with the *extension* of the concept. A concept should *maximally* apply to all its exemplars and only to them. Next, the membership order should be *finite*, in the sense that any strictly increasing chain of elements should be of bounded finite length. This does not mean that the agent has only a finite number of objects at her disposal. On the contrary, we are dealing with real or imaginary items, and the set $\mathcal{O}$ is therefore supposed to be infinite. The finiteness condition simply means that, relative to categorization, only a finite number of classes are to be considered: in the WMD example, any element of $\mathcal{O}$ either shares one of the ten positions on the WMD chain, or is incomparable with the elements of that chain.

A first way of building such a membership order could consist in simply counting the features: an object x would be considered as falling less under a concept α than an object y if the number of α-defining features that apply to x is less than the number of these features that apply to y. Alternatively, one could consider that x falls less under α than y iff the set of defining features that apply to x is a subset of the set of defining features that apply to y. These elementary approaches however suffer from two drawbacks: first, they suppose that the applicability of features to objects is an all-or-none matter - a simplification that is supported by no rationale: clearly there exist degrees in the way features apply to objects. For instance, the degree with which the feature *to-fly*, taken in the context of *to-be-a-bird*, applies to the fluttering of a hen is not the same as the degree with which it applies to the flight of an eagle. Secondly, these methods treat all the defining features of a concept at the same level, ignoring the fact that, for an agent, some features may appear to be more important than others. Thus, after having associated with the concept *to-be-a-bird* the set of features $\{$*to-be-a-vertebrate, to-be-oviparous, to-have-feathers, to-have-a-beak, to-have-wings*$\}$, an agent may consider that *having wings* is a more important feature for birdhood than *having a beak*. From this point of view, a bat will be endowed with more birdhood than a tortoise.

2.1 Applicability functions and applicability orders induced by concept features

The strength with which a feature f of a concept α applies to an object is usually measured in a given ontology through a *percentage* or an *applicability degree function* δ_f^α that takes its values in the unit interval. Concerning the range of this function, it is important to observe that it can be most often circumscribed to a *finite* subset of $[0, 1]$: this is clearly true for fuzzy features like *to-be-tall*, *to-be-rich* or *to-be-warm*, since the measure of their applicability is always approximative (to an inch, a cent or a degree). On the Brittany coast of France, for instance, the set of water temperatures t in July ranges from 15 to to 25 Celsius degrees, thus covering 11 possible (integral) values. In this context, the function associated with the feature *warm* may be given by $\delta_f^\alpha = t/10 - 1.5$. The finiteness of the range of δ_f^α is even more obvious in the general process of categorization: ranking the objects according to a feature of a given concept only yields a small number of equivalence classes. Recall indeed that we are dealing with a *phenomenal* representation of cognitive structures (see [14] for the difference between *scientific* and *phenomenal* representations). For instance, to determine to which extent a flower may be considered as a *poppy*, one roughly evaluates its redness, its shape and the size of its petals. Concerning the redness, comparison with other objects shows that, even though it is theoretically possible to list hundreds of different red shades, only a finite number of discernible non-equivalent reds separate the color of that particular flower from that of an ideal poppy. The same observation can be made concerning the other features that define or describe poppies, like the shape and the size of the petals.

For this reason, we consider that there is no loss of generality to restrict the study of this paper to attributional concepts whose features can be weighed on a finite scale. Given an attributional concept α, the way any of its defining features f applies to an item is therefore accounted for by an applicability degree function δ_f^α that takes only a finite number of values in the unit interval. Equivalently, we may say that f, as an α-feature, generates an *applicability order* $\preceq_f^\alpha$ defined for any couple of objects (x, y) by:

$$x \preceq_f^\alpha y \text{ iff } \delta_f^\alpha(x) \leq \delta_f^\alpha(y).$$

The relation thus defined is a total pre-order that has only finitely many equivalence classes : there exists only a finite number of intermediate states between an object x totally deprived of f and an object y to which the feature f fully applies.

Apart from the finiteness condition, any attributional concept will be required to satisfy an *agreement condition* that guarantees the existence of at least one object to which simultaneously apply all its defining features. This simply means that every considered concept has a non-empty extension: there exist objects to which this concept applies.

Definition 1 *An* attributional *concept α is a concept for which there exists a non-empty finite set of defining features $\Delta(\alpha)$ that satisfy the two properties:*

1. for every defining feature f, the corresponding applicability function δ_f^α takes a finite number of values.
2. There exists at least one item z such that $\delta_f^\alpha(z) = 1$ for all defining features f.

In the agent's mind, the set $\Delta(\alpha)$ is constituted by a choice of attributes that appear to be *essential* for the concept realization: the objects to which α applies are exactly those that possess all these attributes.

It will be useful to assimilate a well determined feature f with an attributional concept whose set of defining features is equal to $\{f\}$.

Recall that the extension $Ext\,\alpha$ of the concept α is the set of all objects that fall under this concept. If α is an attributional concept, we have therefore $Ext\,\alpha = \{x \in \mathcal{O} \,|\, \delta_f^\alpha(x) = 1 \;\forall f \in \Delta(\alpha)\}$.

In order to lighten the notation, the superscript of the applicability orders and degrees will be omitted, so that $\preceq_f$ and δ_f stand for $\preceq_f^\alpha$ and δ_f^α. However, it is necessary to keep in mind that the applicability order or the applicability function of a feature is always defined relative to the concept it qualifies.

2.2 Salience in the defining feature set

Let α be an attributional concept and $\Delta(\alpha)$ the defining feature set that an agent associates with it. As we noted, some defining features may be considered as more important than others, or more salient to the eyes of this agent. Thus, it is not only the number of features applying to an object, but also their relative importance that have to be considered when evaluating its categorial membership.

This can be easily done when salience is accounted for by a ranking function s that associates with each feature f its salience degree $s(f)$. In this case for instance, following Hampton [17], or [19], we can introduce for every object x a membership degree $\delta_\alpha(x)$ defined by $\delta_\alpha = \frac{1}{N(\alpha)} \sum_{f \in \Delta(\alpha)} s(f)\delta_f(x)$, with $N(\alpha) = |\Delta(\alpha)| \sum_{f \in \Delta(\alpha)} s(f)$. The associated membership order is then given by $x \leq_\alpha y$ iff $\delta_\alpha(x) \leq \delta_\alpha(y)$.

Other systems exist, and the literature on the subject - multiple criteria decision making - is abundant: see for instance [27], [33] or [35]. The point, however, is that, in general, the agent is not supposed to endow each defining feature with a specific rank of importance. Again, for an agent who doesn't know about numbers and scales for instance, comparing the relative salience of two features may be simpler than attributing a precise salience degree to each of them: salience is then simply viewed as a binary relation of comparison between features.

If we want to model in all its generality an agent's representation of categorial membership, we have therefore to start from the very basic case where the agent has equipped the set of defining features with a (possibly empty) strict partial order $>_{\Delta(\alpha)}$.

2.3 The membership order

In order to directly define a categorial membership order from the relation $>_{\Delta(\alpha)}$, we may proceed as follows:

Let α be an attributional concept, and $\Delta(\alpha)$ its defining set equipped with the salience order $>_{\Delta(\alpha)}$. We consider that an object y is at least as close to be an instance of α as an object x if, for each defining feature that applies more to x than to y, there exists a more salient feature that applies more to y than to x. We set therefore:

(1) $\quad x \preceq_\alpha y$ iff $\forall f \in \Delta(\alpha) \,|\, y \prec_f x$, $\exists g \in \Delta(\alpha)$, $g >_{\Delta(\alpha)} f \,|\, x \prec_g y$.

We shall say in this case that the concept α applies at least as much to y as to x, or that y is at least as close to be an α as x. This exactly means that each defining feature that applies more to x than to y is dominated by a defining feature that applies more to y than to x.

Before analyzing the positive and negative aspects of this construction, let us check that the relation $\preceq_\alpha$ is indeed a weak order.

Proposition 1 *The relation $\preceq_\alpha$ is a partial weak order on $\mathcal{O}$.*

(We recall that the proofs are given in the Appendix.)
We shall refer to $\preceq_\alpha$ as the order induced by $\Delta(\alpha)$. The strict partial order associated with $\preceq_\alpha$ is easily seen to be given by:

(2) $$x \prec_\alpha y \text{ iff } x \preceq_\alpha y \text{ and } \exists f \in \Delta(\alpha) \text{ such that } x \prec_f y.$$

We can now check that $\preceq_\alpha$ correctly models categorial membership:

Proposition 2 *An object is $\preceq_\alpha$-maximal in $\mathcal{O}$ if and only if it is an element of $Ext\,\alpha$. One has $z \prec_\alpha x$ for any $x \in Ext\,\alpha$ and $z \notin Ext\,\alpha$. Furthermore, any increasing $\prec_\alpha$-chain is of finite bounded length.*

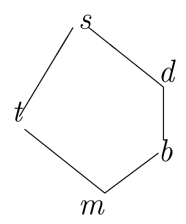
Let us evaluate an example of membership order induced by the concept *to-be-a-bird*:

Example 1 *Let α be the concept* to-be-a-bird*, and suppose that, from the point of view of a particular agent, its defining feature set in the context of* living beings *is the set* {to-have-two-legs, to-lay-eggs, to-have-a-beak, to-have-wings}*, equipped the salience order:* to-have-a-beak $>_s$ to-lay-eggs $>_s$ to-have-two-legs*, and* to-have wings $>_s$ to-lay-eggs$>_s$ to-have-two-legs. *Suppose for the sake of simplicity that, in the agent's mind, membership to any of these features is a two-valued function. Let s, m, t, b and d respectively stand for a sparrow, a mouse, a tortoise, a bat and a dragonfly. Then the induced membership order is determined by the following array :*

	$two-legs$	$lay-eggs$	$beak$	$wings$
sparrow	$\star$	$\star$	$\star$	$\star$
mouse				
tortoise		$\star$	$\star$	
bat	$\star$			$\star$
dragonfly		$\star$		$\star$

One readily checks that $d \prec_\alpha s$, $m \prec_\alpha t$, *and* $m \prec_\alpha b$. *Note that one has* $b \preceq_\alpha d$, *since the concept* to-have-two-legs *under which the bat falls, contrary to the dragonfly, is dominated by the concept* to-lay-eggs *that applies to the dragonfly and not to the bat. On the other hand, one does not have* $d \preceq_\alpha b$, *as nothing compensates the fact that the dragonfly lays eggs and the bat does not. This yields* $b \prec_\alpha d$. *Note also that the tortoise and the bat are incomparable: one has neither* $b \preceq_\alpha t$, *nor* $t \preceq_\alpha b$.

The strict α*-membership order therefore reads:*



Remark 1 *The notion of* extension *is not sufficient to characterize a concept. In the preceding example for instance, a different agent may agree on the defining feature set of the* bird-concept, *while disagreeing on the associated salience order. Thus, the concept* to-be-a-bird *would differ from one agent to the other, although its exemplars would be the same.*

This suggests that the original binary model needs to be completed: not only the defining and characteristic feature sets have to be taken into account, but also the salience orders provided therein. Thus, strictly speaking, a 'personal' (as opposed to 'universal') attributional concept α *should be defined by a quadruplet* $(\Delta(\alpha), >_{\Delta(\alpha)}, \chi(\alpha), >_{\chi(\alpha)})$, *where* $\Delta(\alpha)$ *and* $\chi(\alpha)$ *are finite sets of features respectively equipped with the strict partial orders* $>_{\Delta(\alpha)}$ *and* $>_{\chi(\alpha)}$.

Coming back to the construction of the order proposed by formula (1), we have now to mention two negative aspects.

The first one occurs when the salience order on $\Delta(\alpha)$ is empty. Then the membership order $\preceq_\alpha$ boils down to the intersection of the $\prec_{f_i}$'s: one has $x \preceq_\alpha y$ iff $\delta_f(x) \leq \delta_f(y) \, \forall f \in \Delta(\alpha)$. This is a very strict condition that only few pairs of objects will satisfy. Furthermore, it leads to counterintuitive results: for instance, an agent may consider that an a tortoise, that shares with birds the features of *having-a-beak* and *laying-eggs* has more birdhood than a monkey, to which only applies the single feature *to-have-two-legs*.

For this reason, it seems preferable to require that, in the absence of any salience order, $\preceq_\alpha$ is the order:

$$x \preceq_\alpha y \text{ iff } \sum_{f\in\Delta(\alpha)} \delta_f(x) \leq \sum_{f\in\Delta(\alpha)} \delta_f(y). \tag{3}$$

The second drawback of $\preceq_\alpha$ is that a single feature dictates α-membership whenever it is more salient than the others. That is, if such a feature applies more to y than to x, y will be considered as closer than x to α, even if all other features apply more to x than to y. Note that this phenomenon can be also encountered in the orders defined through a membership degree $\delta_\alpha = \frac{1}{N(\alpha)} \sum_{f\in\Delta(\alpha)} s(f)\delta_f(x)$, as defined in section 2.2. Indeed, suppose that for a particular feature f_0, one has $s(f_0) > \sum_{f\neq f_0} s(f)$. Suppose also that f_0 applies to an object x but not to an object y, but that all other features apply to y and not to x. Then the α-membership degree of x will be greater than that of y, although no feature, except one, applies to x, while all but one apply to y.

It is doubtful that, for an agent, the defining feature set attached to a concept includes more than a handful of features - those found, for instance, in a dictionary. It follows that, in the phenomenal framework in which we circumscribe this study, the hypothesis of a single feature overruling dozens of others appears to be a theoretical one.

Henceforth, we shall suppose that the membership order induced by a defining feature set is evaluated by the relation of weak partial order given by equation (1), which will be denoted by $\preceq^\mu_\alpha$ or $\preceq^\mu_{\Delta(\alpha)}$. This will be considered as *the principal case*. Some results established in this paper remain valid in the special case where, the salience order on $\Delta(\alpha)$ being empty, the membership order is given by (3): these will be marked with an asterisk (*). [2]

2.4 Subconcepts

Let α be an attributional concept. Following the usual definition, we shall say that β is a *subconcept* of α if $Ext\,\beta \subseteq Ext\,\alpha$. This is equivalent to saying that every defining feature of α applies to the instances of β. An interesting class of subconcepts is given by the following lemma:

[2] An attempt to an alternative solution was proposed in [13]. The resulting membership order appears to be free from the dictatorship phenomenon encountered in $\preceq^\mu_\alpha$, but reveals itself to be hard to implement, even in the simplest cases.

Lemma 1 *Every attributional concept β such that $\preceq^{\mu}_{\beta} \subseteq \preceq^{\mu}_{\alpha}$ is a subconcept of α.(*)*

This leads to the following definition:

Definition 2 *A* regular *subconcept of α is an attributional concept β such that $\preceq^{\mu}_{\beta} \subseteq \preceq^{\mu}_{\alpha}$.*

Such regular subconcepts may be found by adding to $\Delta(\alpha)$ some defining features considered as less salient than the defining features of α. In this sense, a regular subconcept β does not show particularly original features compared to those of α. For instance, for an agent who only knows that the macaw is a kind of parrot, *to-be-a-macaw* will be considered as a regular subconcept of *to-be-a-parrot*. On the contrary, *to-be-a-parrot* will not be considered as a regular subconcept of *to-be-a-bird*: no agent indeed would think that *to-talk* or *to-have-scarlet-feathers* is a less salient feature of parrots than *to-have-two-legs*.

Definition 3 *Two attributional concepts α and β are* similar *if equality holds between their membership orders.*

Denoting by $\sim$ the relation of similarity, we have therefore $\alpha \sim \beta$ iff $\preceq^{\mu}_{\beta} = \preceq^{\mu}_{\alpha}$. An example of non-trivial similarity will be given in section 3.2.

2.5 Essence and extension

Definition 4 *Let α be an attributional concept and $Ext\,\alpha$ its extension. The* essence *$Ess\,\alpha$ of α is the set of all concepts β that apply to every element of $Ext\,\alpha$: $Ess\,\alpha = \{\beta \in \mathcal{F}; Ext\,\alpha \subseteq Ext\,\beta\}$.*

One may consider the elements of this set as the *essential attributes* of the concept. Since $\alpha \in Ess\,\alpha$, we have $Ext\,\alpha = \bigcap(Ext\,k)_{k \in Ess\,\alpha}$, showing that the extension of a concept can be retrieved from its essence. In the language of Formal Concept Analysis, the pair $(Ext\,\alpha, Ess\,\alpha)$ can be considered as a formal concept in the context $(\mathcal{O}, \mathcal{F}, I)$, where $\mathcal{O}$ is the set of objects, $\mathcal{F}$ the set of concepts and $(x,\,\alpha) \in I$ iff x falls under α.

The essence of a concept, as defined above, corresponds to the classical one (see [10]), provided one remembers that the universe of discourse includes, with real objects, *imaginary* ones - that is objects that the agent my think of: eg. a unicorn, a pink elephant, a flying cow... Restricting the set $\mathcal{O}$ to real objects would indeed lead to counterintuitive results: consider for instance the concept *to-be-the-US-president*, and suppose that it is known that all past and present US presidents were golf players. Then the attribute *to-be-a-golf-player* applies to all instances of *to-be-the-US-president*; if only real existing objects were considered, the feature *to-be-a-golf-player* would become part of the *essence* of *to-be-the-US-president*...! Enlarging the set of real objects to imaginary ones avoids this drawback, because we can imagine a US president that does not play golf.

Remark 2 *The essence of a concept gathers two different families of concepts or attributes. On the one hand, we find the* specific *attributes that are attached to this concept and help distinguishing it from neighboring concepts. In the example of* to-be-the-US-president*, such is for instance the case of the concepts* to-sleep-in-the White-House*,* to-convene-Congress*,* to-command-the-US-armed-forces*. In the current literature, this set is usually referred to as the 'Intension' of the concept. On the other hand, we have the* generic *attributes, which the concept has inherited from some super-concept: for instance, the generic attributes of the concept* to-be-US-president *include the features* to-be-mortal*,* to-be-a-vertebrate *or* to-have-a-heart*, which are part of the essence of* to-be-a-man*. The distinction between specific and non specific features plays an important role in category-based induction. It will be more precisely studied in section 7.2.2.*

2.6 Membership distance and membership degree

Since $\preceq^{\mu}_{\alpha}$ is generally not a ranked order, it cannot be faithfully translated by a numerical function It is however possible to approximate the notion of membership degree by considering increasing chains of objects, similarly to what was evoked in the WMD example. The following construction may be used to evaluate how far an object x stands from falling under a concept α.

Consider the *maximal length* of a chain $x \prec^{\mu}_{\alpha} x_1 \prec^{\mu}_{\alpha} \ldots \prec^{\mu}_{\alpha} x_n$ with last term $x_n \in Ext\,\alpha$. This length measures the distance that separates x from $Ext\,\alpha$. It will be referred to as the *membership distance* of x, and denoted by $\mu_{\alpha}(x)$.

Let us denote by N_{α} the length of a maximal $\prec^{\mu}_{\alpha}$-chain in $\mathcal{O}$. *The membership degree* δ^{μ}_{α} can be then defined, for all objects x, by $\delta^{\mu}_{\alpha}(x) = 1 - \frac{\mu_{\alpha}(x)}{N_{\alpha}}$. Hence, $\delta^{\mu}_{\alpha}(x) = 1$ if and only $x \in Ext\,\alpha$, and $\delta^{\mu}_{\alpha}(x) = 0$ if and only if x is maximally distant from $Ext\,\alpha$. Note that $\delta^\mu_\alpha(x) < \delta^\mu_\alpha(y)$ whenever $x \prec^\mu_\alpha y$.

Example 2 *In Example 1, the membership order yields $\mu_\alpha(t) = 1$ provided that, for the agent, there exists no animal z such that $t \prec^\mu_\alpha z \prec^\mu_\alpha s$. Similarly $\mu_\alpha(d) = 1$ and $\mu_\alpha(b) = 2$. Concerning the mouse, the agent may consider that the chain $m \prec^\mu_\alpha k \prec^\mu_\alpha b \prec^\mu_\alpha d \prec^\mu_\alpha s$ is a maximal one, where k stands for* a monkey*, so that $\mu_\alpha(m) = 4$. The maximal length of a chain is* 4*, and the membership degrees are $\delta^\mu_\alpha(m) = 0$, $\delta^\mu_\alpha(b) = 1/2$, $\delta^\mu_\alpha(t) = \delta^\mu_\alpha(d) = 3/4)$ and $\delta^\mu_\alpha(s) = 1$.*

3 The case of compound concepts

Elementary concepts may aggregate in different ways to give birth to compound ones. The simplest one seems to be the ordinary *conjunction*, &, which corresponds to a simple juxtaposition of terms, as in (*to-be-a-woman*)&(*to-be-a-physician*) or (*to-be-a-car*)&(*to-be-a-home*). For such an operation to be meaningful, it is clearly necessary that the extensions of the components have non-empty intersection. The result should yield a concept where both components have same importance, so that the connective & should be a symmetric operator. For different reasons however, examples are rare where $\alpha\&\beta$ and $\beta\&\alpha$ exactly convey the same notion. The order in which components are articulated impacts the meaning one gives to the conjunction. (*To-be-a-woman*)&(*to-be-a-doctor*) does not bear the same signification as (*To-be-a-doctor*)&(*to-be-a-woman*). For this reason, we shall turn to a more sophisticated operator. This connective was introduced in [11] to account for the *determination* of a concept by another.

3.1 The determination connective

It is possible to determine a concept α by another concept or by a feature β whose meaning is unambiguous. We obtain in this way a compound concept, denoted by $\beta \star \alpha$. This determination is most often realized by the combination of an adjective, or a verb in the participle form, with a noun, like in the compositions *to-be-a-carnivorous-animal, to-be-a-flying-bird, to-be-a-french-student, to-be-a-red-apple.* The determination can also take the form of a noun-noun combination, like in *to-be-a-pet-fish, to-be-a-barnyard-bird*, and, more generally, of a relative clause that will be globally encapsulated by the concept β (e.g. *to-be-an-American*-who-lives-in-Paris). Typically, the determiner β becomes a simple feature of the compound concept $\beta \star \alpha$. Its role can be considered as secondary, compared with that played by the principal concept α: *to-be-red* is a feature of the composed concept *to-be-a-red-car*, and *to-be-a-woman* becomes a feature of a *to-be-physician-that-is-a-woman.* Unlike conjunction, concept determination is not commutative : to take a well-known example, the concept of *games-that-are-sports* differs from the concept of *sports-that-are-games.*

The idea of introducing a determination connective in the logic of concepts goes back to J.P. Desclés - see for instance [4]. In his Object Determination Logic, Desclés defined a partial operator from $\mathcal{F}$ x $\mathcal{O}$ into $\mathcal{O}$, that associates to a concept f and an object x, the 'more determined' object $\delta f(x)$. A-red-car would result for instance of the determination of an object a-car by the concept *to-be-red.* In our framework, it seems simpler to directly work in the set of concepts, as we consider that an 'undetermined' object should be assimilated with a concept.

As a last remark, it is important to keep in mind that only the *intersective* conceptual combinations are accounted for: we consider the determination of α by β in the only case where $Ext\,\alpha \cap Ext\,\beta \neq \emptyset$. (see [21] for the distinction between intersective and non-intersective determiner). This shows that the determination connective $\star$ is only a *partial* operator: given arbitrary α and β, it may be meaningless to build the concept $\beta \star \alpha$. For instance, there is no sense in talking of a *sailing-number* or a *wooden-salience.* Such pseudo-concepts correspond to nothing, and no object, real or fictitious, can be thought of falling under them, contrary to imaginary concepts like a *pink-elephant*, a *striped-apple* or a *flying-cow*: these latter definitely have a non-empty extension, because we can imagine a pink elephant, a striped apple or a flying cow.

Our aim is now to find a model that describes a precise type of concept determination, in which the determiner plays only a secondary role. Other types of connective exist, where the principal role is attributed to the determiner, like in the concept *to-be-a-Picasso-painting.* These will not be studied in this paper.

3.2 Membership order for compound concepts

Let α be an attributional concept and β an attributional concept or a feature whose applicability has been defined in the context of α. If the determination of α by β is meaningful, that is if $Ext\,\alpha \cap Ext\,\beta \neq \emptyset$, it is not difficult to define a membership order on the concept $\beta \star \alpha$ that gives pre-eminence to α over β. This can be done by setting:

(4) $$x \preceq^{\mu}_{\beta\star\alpha} y \text{ if } x \preceq^{\mu}_{\alpha} y \text{ and either } x \prec^{\mu}_{\alpha} y, \text{ or } x \preceq^{\mu}_{\beta} y.$$

In this framework, the concept *to-be-a-flying-bird* will be considered as applying more to a penguin than to a bat: indeed, the principal concept is that of *being-a-bird*, while *to-fly* appears as a simple feature, less important than the concept it determines.

The relation $\preceq^{\mu}_{\beta\star\alpha}$ is reflexive and transitive. Since it is clearly a subrelation of $\preceq^{\mu}_{\alpha}$, it makes $\beta\star\alpha$ a regular subconcept of α in the sense of Definition 2. Its associated strict partial order reads:

(5) $$x \prec^{\mu}_{\beta\star\alpha} y \text{ if and only if either } x \prec^{\mu}_{\alpha} y, \text{ or } x \preceq^{\mu}_{\beta\star\alpha} y \text{ and } x \prec^{\mu}_{\beta} y.$$

It is interesting to observe that the compound order $\preceq^{\mu}_{\beta\star\alpha}$ can be directly recovered from a fictitious set of defining features attached to the concept $\beta \star \alpha$:

Theorem 1 (*) *$\beta \star \alpha$ is similar to an attributional concept whose set of features $\Delta(\beta \star \alpha)$ is equal in the principal case to $\Delta(\alpha) \cup \Delta(\beta)$.*

It follows from this result that the set of attributional concepts is stable under concept determination, so that concept determination can be iterated. This operation satisfies idempotence and associativity:

Proposition 3 (*) *Let α, β and γ be three attributional concepts such that $Ext\,\alpha \cap Ext\,\beta \cap Ext\,\gamma \neq \emptyset$. Then*

- $\alpha \star \alpha \sim \alpha$.
- $\gamma \star (\beta \star \alpha) \sim (\gamma \star \beta) \star \alpha$.

As can be expected, the determination of a concept by one of its own defining feature leaves unchanged its induced membership order:

Proposition 4 *One has $f \star \alpha \sim \alpha$ for every $f \in \Delta(\alpha)$.*

The family of features f that satisfy $f \star \alpha \sim \alpha$ therefore includes the defining features of α. In fact, we can consider any element of this family as a 'generalized defining feature', as can be seen from the following proposition.

Proposition 5 *Let f be a feature such that $f \star \alpha \sim \alpha$. Then f can be added to the set $\Delta(\alpha)$ without changing the order $\preceq^{\mu}_{\alpha}$.*

By the above proposition, a generalized feature may be seen as a feature that the agent could have included in her defining set, although considering it as less salient than the other ones. For instance, if an agent thinks that, as far as class-membership is concerned, there is no difference between the concept *to-be-a-bird* and the concept *to-be-a-bird-that-has-feathers*, this agent will consider that the feature *to-have feathers*, taken in the context of being a bird, may be part of *the defining feature set* given in Example 1, this feature being less salient than *to-have-two-legs*.

We now turn to the *extension* of a compound concept. By Theorem 1, this notion can be directly defined similarly to what was done for simple concepts (see paragraph 2.1).

Proposition 6 (*) *The extension of $(\beta \star \alpha)$ is the set of $\preceq^{\mu}_{\beta\star\alpha}$-maximal elements of $\mathcal{O}$. It satisfies $Ext\,(\beta \star \alpha) = Ext\,\alpha \cap Ext\,\beta$.*

This proposition shows that full categorial membership is *compositional*: an object falls under a determined concept if and only if it falls under the concept and under its determiner. The category of *red-cars* exactly covers all the items that are red and that are cars. It is however important to observe that although compositionality holds for full membership, it does not hold for relative membership: we do not have $\preceq^{\mu}_{\beta\star\alpha} = \preceq_{\alpha} \cap \preceq_{\beta}$. An object y may fall more than an object x under $\beta \star \alpha$, while falling less than x under β: the concept *to-be-a-flying-bird* for instance applies more to a penguin than to a bat, although *to-fly* applies more to a bat than to a penguin.

Remark 3 *We can see from (5) or from Theorem 1 that any strictly increasing $\prec^{\mu}_{\beta\star\alpha}$-chain is of finite bounded length. As was done for simple concepts, a membership distance can be defined for compound concepts of the form $\beta \star \alpha$: for any object x, let $\mu_{\beta\star\alpha}(x)$ denote the length of a maximal $\prec^{\mu}_{\beta\star\alpha}$-chain starting from x and ending in $Ext\,(\beta \star \alpha)$. Note that $\mu_{\alpha}(x) \leq \mu_{\beta\star\alpha}(x)$. If $N_{\beta\star\alpha}$ is the length of a maximal $\prec^{\mu}_{\beta\star\alpha}$-chain in $\mathcal{O}$, the $(\beta \star \alpha)$-membership degree of x may be then defined by $\delta^{\mu}_{\beta\star\alpha}(x) = 1 - \frac{\mu_{\beta\star\alpha}(x)}{N_{\beta\star\alpha}}$.*

Given an attributional concept α, one may ask which concepts can be characterized as a determination of α. The answer is given by the following proposition:

Proposition 7 (*) *Let α and γ be two attributional concepts. Then the following conditions are equivalent:*

1. $\gamma \sim (\gamma \star \alpha)$
2. *There exists a concept β such that $\gamma \sim (\beta \star \alpha)$*
3. *γ is a regular subconcept of α such that $\prec_\alpha^\mu \subseteq \prec_\gamma^\mu$.*

3.3 The conjunction effect

The *conjunction effect* or *guppy effect* was observed in 1981 by Osherson and Smith [25]; it has been thereafter at the center of numerous research and experiments (see in particular [16], [21] or more recently [1] and [8]); it can be described by the fact that an item may be found to be more strongly a member of the composition of two concepts than a member of one of them. Thus, a cuckoo was found to be more strongly a member of the composed concept *(to-be-a-pet-bird)* than a member of the concept *to-be-a-pet* on its own, and a guppy was considered more a member of the concept *(to-be-a-pet-fish)* than a member of the concept *to-be-a-fish.* This appears to be paradoxical, since any item falling under a composition $\beta \star \alpha$ must already fall under each of its components.

A similar effect, the *Linda paradox*, was observed in [34]: subjects were told about a woman, Linda, who was involved in liberal politics in college. Some subjects were then asked to rate the probability that Linda has become a *bank teller,* other subjects were asked to rate the probability that she became a *feminist bank teller.* The result showed that it was judged more probable that Linda became *a feminist bank teller.* This again seems in contradiction with classical logic and probability theory, since being a feminist bank teller necessarily implies being a bank teller, so that the probability of the first event should not exceed that of the second event. However, and in spite of '*desperate manipulations designed to induce subjects to obey the conjunction rule*', the result of the experiments all concluded in the sense of a so-called 'violation' of the conjunction rule.

These two 'paradoxes' seem to have promoted the introduction in concept theories of the quantum mechanics formalism: thus, Franco writes in [8] that

‘*Quantum mechanics, for its counterintuitive predictions, seems to provide a good formalism to describe puzzling effects of contextuality*’. Similarly, Aerts [1] pleads for adopting in cognition theory the attitude of theoretical physicists for whom ‘*data showing deviations from set theoretic rules are a major indication of the presence of quantum structure*’; he thus devotes special attention to the example of the guppy effect as ‘*none of the currently existing concept theories provides a satisfactory description and/or explanation of such effect for concept combinations.*’

We do not take sides on the question whether the rather complex and artificial formalism of quantum mechanics is or is not suitable to model cognitive theories; however, we depart from the quoted authors in that we think that the guppy effect and the conjunction fallacy can be simply described and explained through a classical formalism, using for instance the tools which we developed in the preceding sections. This can be seen in the following example.

Example 3 *Let us take again the degree computations of example 2. The chain* $m \prec^{\mu}_{\alpha} k \prec^{\mu}_{\alpha} b \prec^{\mu}_{\alpha} d \prec^{\mu}_{\alpha} s$ *being supposed to be maximal, one has* $\delta^{\mu}_{\alpha}(k) = 1/4$*. If* β *is the feature* to-be-black*, equipped with a two-valued membership degree function, one may think of a maximal* $\beta \star \alpha$*-chain like*

$$m \prec^{\mu}_{\beta\star\alpha} m' \prec^{\mu}_{\beta\star\alpha} k' \prec^{\mu}_{\beta\star\alpha} k \prec^{\mu}_{\beta\star\alpha} b \prec^{\mu}_{\beta\star\alpha} b' \prec^{\mu}_{\beta\star\alpha} d \prec^{\mu}_{\beta\star\alpha} d' \prec^{\mu}_{\beta\star\alpha} s \prec^{\mu}_{\beta\star\alpha} r,$$

with m*: a white mouse;* m'*: a black cat;* k'*: a (red-brown) kangaroo;* k*: a black macaque;* b*: a brown bat;* b'*: a black bat;* d*: a blue dragonfly;* d'*: a black fly;* s*: a sparrow;* r*: a raven. This leads to* $N_{\beta\star\alpha} = 9$*, and* $\delta^{\mu}_{\beta\star\alpha}(k) = 1/3$*, showing that, in this model, the macaque is closer to the concept* to-be-a-black-bird *than to the concept to-be-a-bird.*

3.3.1 Objections to the compositional theory

Several arguments have been developed against a theory of compositionality in which membership relative to a compound concept simply boils down to membership relative to each of its constituents. It has been objected for instance ([16] and [18]) that agents make a marked difference between, for instance, the concept $s \star g$ (*being-a-game-that-is-a-sport* and the concept $g \star s$ (*being-a-sport-that-is-a-game*): consequently, these concepts should not share the same exemplars... However, as we underlined (see remark 1) two concepts may be different while sharing the same extension. We agree that the concepts $s \star g$ and $g \star s$ are different, but this does not at all preclude their extensions of being the same. Our opinion is that an item which is considered as definitely *a sport that is a game* must be both a sport and a game. But this will not remain true for an object that is simply 'close' to being *a sport that is a game*.

Concerning the composition of extensions, our model does not seem to conform with Hampton's conclusions ([16] or [18]) that *a large number of items ...(are) more often judged to belong in a conjunction such as* school furniture *or* protective clothing *than in the categories from which these concepts are supposedly drawn, namely* furniture *and* clothing*... Thus, items belonged in a conjunction that did not belong in one of its constituent concepts.*

This seems to stand in contradiction with the results of Proposition 5. But if we look closely at the different experimental results displayed in Hampton's paper, we note that they are not really explicit as long as *full membership* is concerned. Let us recall indeed that membership, in Hampton's experiments, was rated by a positive number from 1 to 3 'to indicate degree of typicality', while non-membership was rated by a negative number from -1 to -3 'to indicate relatedness as a non member'. In other words, Hampton treated on the same level categorial membership and typicality. It turns out that in the case of proper concept determination (thus excluding the cases of *kitchen furniture, sport vehicle* and *protective clothing*), the results, displayed pp. 22-23 of Hampton's paper show that when an item's membership relative to one of the components is less than or equal to -1, its membership relative to the conjunction is always strictly less than 1. This means that an item that is known not to fall under one of the components of a conjunction will never be considered as (fully) falling under this conjunction.

To be more specific, let us return to the *blackboard* example. Experiments show that people consider such an item as a clear exemplar of the concept *to-be-school-furniture*, although not an exemplar of the unmodified *to-be-furniture*. Therefore, the extension of the compound concept is not equal to the intersection of the extensions of its components, in apparent contradiction again with Proposition 4.

From our point of view, however, we do not consider this fact as contradicting our model, because we claim that *to-be-school-furniture* is *not* a determination of the concept *to-be-furniture*. Indeed, by default, *to-be-furniture* refers to *to-be-home-furniture*. This can be seen from the definitions found in the dictionaries. In the Collins English Dictionary, for instance, furniture *consists of large objects such as tables, chairs, or beds that* *are used in a room for sitting or lying or for putting things on or in.* In the Oxford Lexico, after giving for principal definition *the movable articles that are used to make a room or building suitable for living or working in, such as tables, chairs, or desks*, a second meaning is proposed that reads: *usually with adjective or noun modifier, the small accessories or fittings that are required for a particular task or function.*

From this follows that the word *furniture*, when not preceded by a noun modifier, is principally used in the sense of *home-furniture*. Since school-furniture is *not* home-furniture that can be found in school, the corresponding composed concept is not the result of a determination of *home-furniture* by the relative clause *used-in-school*. The intersection of the extensions of these two concepts is empty, and this prohibits their composition through determination, as we underlined in section 3.2.

It is interesting to note that the compound concept *to-be-school-furniture* cannot be translated in French through a determination or a modification of the words *meubles* or *ameublement*, two expressions that are are used to designate usual (home) furniture. The French word that translates the concept of furniture dedicated to some specific use is *mobilier*: one speaks then of *mobilier d'école* (school-furniture), *mobilier urbain* (urban-furniture), or *mobilier de bureau* (office-furniture). The compound concept obtained in this way is fully compositional.

4 Typicality

Among the exemplars of a concept, some objects can be considered as *typically representing* this concept. For a European agent for instance, a robin, a blackbird or a sparrow *typically* represents the concept of *to-be-a-bird*, contrary to a penguin or an ostrich, that would be qualified as *atypical*. Indeed sparrows, blackbirds and robins fly, sing, and live in the trees, all attributes that are usually expected from birds, and that do not possess penguins or ostriches. The typical exemplars of a concept differ from the non-typical ones in that they fall under a set of particular features that naturally come to mind when evoking the concept. These features are not part of the essence of the concept, but they are 'by default' supposed to apply to its non-exceptional instances. In the agent's mind, they form a *characteristic set*, that will play for typicality a role that is analogous to that of the defining feature set in categorial membership.

4.1 Typical exemplars, essence and intension

As was recalled in section 1, in the binary model, for some concepts, typicality is fully accounted for by their associated characteristic feature set. Using the definition of attributional concepts given in Definition 1, we can parallel the constructions operated in section 2, replacing the notion of categorial membership by that of typicality. For this purpose, we shall consider the following subfamily of attributional concepts.

Definition 5 *A* featured *concept α is an attributional concept for which there exists a non-empty finite set of characteristic features $\chi(\alpha)$ that satisfy the two properties:*

1. *For every characteristic feature f the corresponding applicability function δ_f takes a finite number of values.*
2. *There exists at least one exemplar z of α such that $\delta_f(z) = 1$ for any characteristic feature f.*

This definition ensures that $\{x \in Ext\,\alpha \mid \delta_f^\alpha(x) = 1\ \forall f \in \chi(\alpha)\}$ is a non-empty set. We shall denote this set by $Typ\,\alpha$, and refer to its elements as the *typical exemplars* of α.

As was done for attributional concepts (see section 2.1 *in fine*), it will be useful to assimilate a well determined feature f with a featured concept whose characteristic set is equal to $\{f\}$.

Definition 6 *The* intension $Int\,\alpha$ *of a featured concept α is the set of all attributes that apply to every element of $Typ\,\alpha$. Its elements will be called the* typical attributes*, or the* typical features *of α.*

We have therefore
$Int\,\alpha = \{f \mid \delta_f^\alpha(x) = 1\ \forall x \in Typ\,\alpha\}$, and
$Typ\,\alpha = Ext\,\alpha \bigcap_{k\in Int\,\alpha}(Ext\,k) = \bigcap_{f\in(\Delta(\alpha)\cup\chi(\alpha))} Ext\,f.$

Clearly, $Ess\,\alpha$ is a subset of $Int\,\alpha$. Note that the set $Int\,\alpha \setminus Ess\,\alpha$ is usually larger than the characteristic set: this latter only gathers the features that, for an agent, isolate the typical exemplars from the remaining ones. Thinking of *dogs,* for instance, an agent may consider the set $\{$*to-bark, to-gnaw-at-bones, to-be-a-cat-enemy*$\}$ sufficient to characterize typical dogs, neglecting attributes like *to-get-fleas* or *to-run-on-to-sticks*.

As was the case for the defining feature set, some characteristic features may be seen by the agent as more important than others. Consequently, we shall suppose that the characteristic set is equipped with a *salience order* which we denote by $>_{\chi(\alpha)}$.

4.2 The typicality order

Like membership, typicality is generally not an all-or-none matter. Given a featured concept α, the best tool to evaluate the relative typicality of its exemplars is the construction, on the set $Ext\,\alpha$, of an ordering stemming from its characteristic set. For this purpose, we can adapt the solutions proposed in section 2.3. The typicality order will be defined on the subset $Ext\,\alpha$ of $\mathcal{O}$, and the role played by the defining features will be replaced by that of the characteristic ones. Given two exemplars x and y of α, we shall therefore consider that y is at least as typical as x iff, for each characteristic feature f that applies more to y than to x, there exists a more salient characteristic feature g that applies more to y than to x:

$$x \preceq_\alpha^\tau y \text{ iff } \forall f \in \chi(\alpha) \mid y \prec_f x,\ \exists g \in \chi(\alpha),\ g >_{\chi(\alpha)} f : x \prec_g y. \tag{6}$$

Again, we shall mark with an asterisk the results established for the order $\preceq_\alpha^\tau$ that remain valid in the exceptional case where, the salience order on $\chi(\alpha)$ being empty, the typicality order on $Ext\,\alpha$ is given by

$$x \preceq_\alpha^\tau y \text{ iff } \sum_{f\in\chi(\alpha)} \delta_f(x) \leq \sum_{f\in\chi(\alpha)} \delta_f(y). \tag{7}$$

We have the analogue of Proposition 2:

Proposition 8 (*) *The relation $\preceq_\alpha^\tau$ is a partial weak order on $Ext\,\alpha$. Its maximal elements are the typical instances of α. One has $z \prec_\alpha^\tau x$ for any $x \in Typ\,\alpha$ and $z \notin Typ\,\alpha$.*

Membership and typicality orders fully determine a concept, as they describe how the universe of discourse is structured relative to this concept. This leads to the definition of *equivalent* concepts:

Definition 7 *Two featured concepts α and β are said to be* equivalent*, written $\alpha \equiv \beta$, if $\preceq^{\mu}_{\alpha} = \preceq^{\mu}_{\beta}$ and $\preceq^{\tau}_{\alpha} = \preceq^{\tau}_{\beta}$.*

This notion of equivalence is stronger than that considered in [11], where equivalence only required the double equality $Ext\,\alpha = Ext\,\beta$ and $Typ\,\alpha = Typ\,\beta$. Indeed, we may have these equalities without having $\alpha \equiv \beta$: such will be the case for example whenever $\Delta(\alpha) = \Delta(\beta)$ and $\chi(\alpha) = \chi(\beta)$ but $>_{\Delta(\alpha)} \neq >_{\Delta(\beta)}$ or $>_{\chi}(\alpha) \neq >_{\chi(\beta)}$.

Remark 4 *As was done in section 2.6, it is possible to build from the relation $\preceq^{\mu}_{\alpha}$ a typicality distance τ_{α} and a typicality degree δ^{τ}_{α} defined on the set $Ext\,\alpha$. This construction exactly parallels that proposed for membership distance and membership degree.*

4.3 Typicality for compound concepts

Let $\beta\star\alpha$ be the determination of α by a concept β. Similarly to what was done in section 3.2, we define the typicality order $\preceq^{\tau}_{\beta\star\alpha}$ on the set $Ext\,\alpha \cap Ext\,\beta$ by:

(8) $$x \preceq^{\tau}_{\beta\star\alpha} y \text{ if } x \preceq^{\tau}_{\alpha} y \text{ and either } x \prec^{\tau}_{\alpha} y, \text{ or } x \preceq^{\tau}_{\beta} y.$$

This relation is reflexive and transitive. The associated strict partial order reads

(9) $$x \prec^{\tau}_{\beta\star\alpha} y \text{ iff } x \preceq^{\mu}_{\alpha} y, \text{ and either } x \prec^{\mu}_{\alpha} y \text{ or } x \prec^{\mu}_{\beta} y.$$

As was the case for membership, the typicality order for a composed concept gives preeminence to the original concept: a typical exemplar of $\beta\star\alpha$ has maximal α-typicality among the elements of $Ext\,(\beta \star \alpha)$. Such a representation however leads to seemingly paradoxical results. It has been objected for instance (Hampton, *personal communication*) that this construction makes a gull a more typical exemplar of *Antarctic-bird* than a penguin, while, for most people, the penguin appears to be *the* typical Antarctic bird. The explanation, in our opinion, is that one confuses a *typical Antarctic bird* with a *bird that typically lives in Antarctic*, this latter being usually interpreted as a *bird that mainly lives in Antarctic*. Such is indeed the case for the penguin, which, contrary to the gull, is an Antarctic *endemic* bird. Moreover, the penguin's features are so different from those of a familiar European bird that they may appear as 'typical' of this atypical species.

Typicality, as we see, is often understood as a differentiation tool: the most typical exemplars of an atypical subcategory tend to be chosen among the less typical exemplars of this category... Such an interpretation however does not correspond to the usual definition of typicality by means of characteristic features. Indeed, it covers a different notion, that accounts for *representativeness* rather than *typicality.* To precisely grasp the difference, it is necessary to introduce a *representativeness* order inside a subcategory: for example, given two exemplars x, y of a subconcept β of α, one may say that, relatively to α, *y is a better representative of β than x* if y is at the same time more β-typical and less α-typical than x. It is this order relation, different from typicality, that would render the penguin more representative as an Antarctic bird than the skua or the petrel.

Using the notion of *equivalent* concepts yields some elementary properties of the determination connective.

Proposition 9 (*) *Let α, β, γ be three arbitrary featured concepts such that $Ext\,\alpha \cap Ext\,\beta \cap Ext\,\gamma \neq \emptyset$. Then*

- $\alpha \star \alpha \equiv \alpha$.
- $\gamma \star (\beta \star \alpha) \equiv (\gamma \star \beta) \star \alpha$.
- *If* $\alpha \equiv \beta$, *then* $\gamma \star \alpha \equiv \gamma \star \beta$

4.4 The typical instances of a compound concept

Given two featured concepts α and β, it follows from the definition of $\preceq^{\tau}_{\beta\star\alpha}$ and the finiteness of $\chi(\alpha)$ and $\chi(\beta)$ that the set of $\preceq^{\tau}_{\beta\star\alpha}$-maximal elements of $Ext\,(\beta \star \alpha)$ is not empty. This set will be denoted by $Typ(\beta \star \alpha)$ and its elements referred to as the *typical instances* of $\beta \star \alpha$.

Contrary to what was the case for membership (see Proposition 1 and Proposition 6), this set cannot be defined as the set of instances of $(\beta \star \alpha)$ that fall under the elements of $\chi(\alpha) \cup \chi(\beta)$. Indeed, nothing guarantees that this set is not empty. Typical *walking-birds* do not fall under the feature *to-fly.* In fact, we do not even have for typicality a result similar to that of Proposition 2: an element of $Typ(\beta\star\alpha)$ is not necessarily more $(\beta\star\alpha)$-typical than an arbitrary element of $Ext\,(\beta \star \alpha) \setminus Typ(\beta \star \alpha)$, as can be seen from the following example.

Example 4 *Let α denote the concept* to-be-a-student*, and β the concept* to-be-married. *We consider the case where, for an agent, the corresponding characteristic sets are $\chi(\alpha) = \{k\}$, with $k =$* to-live-with-one's-parents*, and $\chi(\beta) = \{f, g, h\}$, with $f =$* to-be-more-than-thirty-years-old*, $g =$* to-have-children*, and $h =$* to-have-a-job. *For simplicity, the applicability functions associated with these features are supposed to be two-valued functions.*

Suppose that the chosen agent cannot think of an exemplar of $\beta \star \alpha$ to which f, g and k would apply: for this agent, it is unconceivable that a student over thirty, married, with children and having a job would live with his or her parents. Then, from her point of view, typicality among married students can be described as follows :

1. *If the salience order on $\chi(\beta)$ is empty, the typical instances of $\beta \star \alpha$ are the exemplars that fall under g, h, k or under f, h, k. We have $z \prec^{\tau}_{\beta\star\alpha} x$ for any $x \in Typ(\beta \star \alpha)$ and $z \in Ext\,(\beta \star \alpha) \setminus Typ(\beta \star \alpha)$.*

2. *If $f >_{\chi(\beta)} g$, the typical instances of $\beta \star \alpha$ are the exemplars of $\beta \star \alpha$ that fall under f, h, k. We still have $z \prec^{\tau}_{\beta\star\alpha} x$ for any $x \in Typ(\beta \star \alpha)$ and $z \in Ext\,(\beta \star \alpha) \setminus Typ(\beta \star \alpha)$.*

3. *If $g >_{\chi(\beta)} f$, the typical instances of $\beta \star \alpha$ are the exemplars of $\beta \star \alpha$ that fall under g, h, k. We have again $z \prec^{\tau}_{\beta\star\alpha} x$ for any $x \in Typ(\beta\star\alpha)$ and $z \in Ext\,(\beta \star \alpha) \setminus Typ(\beta \star \alpha)$*

4. *If the salience order is given by $>_{\chi(\beta)}= \{(h, g)\}$, or $>_{\chi(\beta)}= \{(h, f)\}$, or $>_{\chi(\beta)}= \{(h, g); (h, f)\}$, we have once more the same result as in 1: the typical instances of $\beta \star \alpha$ are the exemplars that fall under g, h, k or under f, h, k, and one has $z \prec^{\tau}_{\beta\star\alpha} x$ for any $x \in Typ(\beta \star \alpha)$ and $z \in Ext\,(\beta \star \alpha) \setminus Typ(\beta \star \alpha)$.*

5. *If finally the salience order is given by $>_{\chi(\beta)}= \{(g, h)\}$, $>_{\chi(\beta)}= \{(f, h)\}$, or $>_{\chi(\beta)}= \{(g, h); (f, h)\}$, the typical instances of of $\beta\star\alpha$ are again the exemplars of $\beta \star \alpha$ that fall under g, h, k or f, h, k. However, in this particular case, there exist elements z of $Ext\,(\beta \star \alpha) \setminus Typ(\beta \star \alpha))$ that are not less $(\beta \star \alpha)$-typical than some typical exemplars of $\beta \star \alpha$. For example, suppose that $>_{\chi(\beta)}= \{(g, h)\}$, that z is an element of $Ext\,(\beta \star \alpha)$ that falls under f and k but not under h, and that x is any*

element of $Ext\,(\beta \star \alpha)$ that falls under g, h, and k. Then x is $(\beta \star \alpha)$-typical, z is not $(\beta \star \alpha)$-typical, but we no longer have $z \prec^{\tau}_{beta\star\alpha} x$: for an agent who considers that, in the context of being married, to-have-a-job *is less salient than* to-have-children*, a married thirty-something student who has no children, no job and who lives at his or her parents, is not less typical than a young student, married, with children who has a job and lives at his or her parents.*

The typical instances of $\beta \star \alpha$ however dominate the exemplars of α, provided there exist some exemplars of β that can be considered as typical instances of α:

Proposition 10 (*) *$Typ(\beta \star \alpha) \subseteq Typ\,\alpha$ whenever $Typ\,\alpha \cap Ext\,\beta \neq \emptyset$. In this case one has $z \prec^{\tau}_{\alpha} x$ for any $x \in Typ(\beta\star\alpha)$ and $z \in Ext\,(\beta\star\alpha) \setminus Typ(\alpha)$*

Contrary to membership, full typicality is *not* compositional. In general, the typical instances of $\beta\star\alpha$ cannot be retrieved from the typical instances of α and β. The set $Typ\,\alpha \cap Typ\,\beta$ may well be empty, for instance, as in the example (*to-be-an-ostrich*)$\star$(*to-be-a-bird*). That being noted, compositionality can be retrieved in a particular case:

Theorem 2 (*) *The equality $Typ\,(\beta\star\alpha) = Typ\,\beta \cap Typ\,\alpha$ holds if and only if $Typ\,\beta \cap Typ\,\alpha \neq \emptyset$. When this is the case, the concept $(\beta\star\alpha)$ is equivalent to a featured concept whose characteristic set $\chi(\alpha\star\beta)$ is equal, in the principal case, to $\chi(\alpha) \cup \chi(\beta)$.*

Thus, typical black olives are typical olives that are typically black.

We finish this section with a characterization of the subconcepts of α that can be be seen as a determination of α.

Proposition 11 (*) *Given two featured concepts α and γ, the following conditions are equivalent:*

1. *$\gamma \equiv (\gamma \star \alpha)$.*
2. *There exists a concept β such that $\gamma \equiv (\beta \star \alpha)$.*

5 On resemblance

The notion of resemblance is important in the categorization process. It has been at the center of numerous theoretical and experimental studies. Researchers have tried to clarify and study this notion, by defining a resemblance degree between two items, by determining the link between resemblance, membership and typicality (see [29], [34], [36]), or by using a geometrical interpretation of the notion of similarity ([14] or [2]).

To investigate this notion, we need to circumscribe our study. Resemblance may be considered as a binary relation between objects (*Henry resembles his brother*), or as a relation between concepts (*the wolf resembles a dog*), or as a relation between an object and a concept (*this picture resembles a Picasso*). We have therefore to specify the type of resemblance we are considering. In this paper, we shall only deal with the two latter notions.

5.1 Resemblance between objects and concepts

What does it mean to say that *Toby resembles a wolf* ? At first glance, one may be tempted to say that *Toby is close to being a wolf*, so that resemblance between an object and a concept would be directly linked with categorial membership. However, it appears that resemblance relative to a concept is first perceived as resemblance with the *typical instances* of this concept. When we say that a particular piece of music *resembles Beethoven's work*, the *Beethoven* we refer to is not the 'young Beethoven', whose compositions still reflect Haydn influence, but the later Beethoven of the second or third period. To take another example, if someone says that 'Peter's bedroom looks like a boat cabin', he clearly refers to a typical boat cabin, excluding for instance a destroyer's cabin.

It follows that, when an agent asserts that a particular item x resembles a concept α, one may infer, first, that x is not known by the agent to be an instance of α, and, secondly that, for this agent, x resembles a *typical* exemplar of this concept, sharing with it a certain amount of *typical attributes*. Thus, looking at a bat, one may say it resembles a bird, just because it has wings, flies, and has the size or the shape of a bird. Conversely, an animal may be declared *not* to resemble a bird if it does not resemble a *typical* bird, even though this animal is known to be a bird. For instance, looking at a penguin, an assertion like 'this animal does not resemble a bird' is perfectly justified. Resemblance first deals with the typical attributes of a concept.

The notion of resemblance in itself is difficult to analyze and model. As is the case for membership and typicality, it is more through a relation of comparison that we can address this problem. We will therefore be interested in interpreting assertions of the form: *the object y resembles more the concept α than the object x.*

Let $\Pi(\alpha)$ gather the defining and the characteristic features of α, that is $\Pi(\alpha) = \Delta(\alpha) \cup \chi(\alpha)$. This set is referred to as the *stereotypical* set of α (see [3], [6], or [20])). In view of what we said, the simplest way to grasp resemblance is to define on $\Pi(\alpha)$ a salience order stemming from the salience orders of $\Delta(\alpha)$ and $\chi(\alpha)$, and build thereafter a weak order relation on $\mathcal{O}$, similar to those proposed in section 2.3. However, the preeminence this kind of order gives to hierarchy over number, together with the fact that $\Pi(\alpha)$ includes more elements than $\Delta(\alpha)$ and $\chi(\alpha)$, may pose a problem. For instance, if α is the concept *to-be-a-bird*, and if *to-fly* has maximal salience in $\Pi(\alpha)$, bird-resemblance will principally rest on the ability of an item to fly; consequently, bats will be more bird-resemblant than kiwis, although *having feathers*, *singing* and *building nests*, taken together, should, at least, compensate the fact that kiwis do not fly.

An interesting alternative,is to use a salience degree directly built from the salience orders on the defining or the characteristic feature set. :

Definition 8 *Given a featured concept α, let $>_{\Pi(\alpha)}$ be the strict order on the set $\Pi(\alpha) = \Delta(\alpha) \cup \chi(\alpha)$ that extends the salience orders $>_{\chi(\alpha)}$ on $\chi(\alpha)$ and $>_{\Delta(\alpha)}$ on $\Delta(\alpha) \setminus \chi(\alpha)$, and satisfies $f >_{\Pi(\alpha)} g$ for all $f \in \chi(\alpha)$ and $g \in \Delta(\alpha) \setminus \chi(\alpha)$. Then the salience degree $s_\alpha(f)$ of an element $f \in \Pi(\alpha)$ is defined by: $s_\alpha(f) = 1 + |\{h \in \Pi(\alpha); f >_{\Pi(\alpha)} h\}|$.*

Using the f-applicability function $\delta_f(x)$ defined in 2.1 now allows measuring the resemblance between an object and a concept:

Definition 9 *The degree to which an object x resembles a concept α is the number $\delta^\rho_\alpha(x) = \frac{\Sigma_{f\in\ \Pi(\alpha)} s_\alpha(f)\delta_f(x)}{\Sigma_{f\in\Pi(\alpha)} s_\alpha(f)}$.*

In the trivial case where the defining and characteristic attributes are all measured through two-valued applicability functions, and no salience has been defined on the sets $\Delta(\alpha)$ and $\chi(\alpha)$, we have

$$\delta^\rho_\alpha(x) = \frac{(|\Delta(\alpha)| + 1)r + s}{(|\Delta(\alpha)| + 1)(|\chi(\alpha)|) + |\Delta(\alpha)|)} \tag{10}$$

where r denotes the number of characteristic features that apply to x, and s the number of defining features that apply to x. Note that resemblance to a concept is *not* preserved through equivalence: we may have $\alpha \equiv \beta$ while $\delta^{\rho}_{\alpha}(x) \neq \delta^{\rho}_{\beta}(x)$.

In the general case, the elements of $\mathcal{O}$ that are maximally resemblant to a concept α are its *typical* instances. Their resemblance degree is equal to 1. On the contrary, simple categorial membership cannot be directly retrieved through resemblance. This comes from the fact that, in evaluating resemblance, one compares objects with the *typical instances* of a concept, rather than with arbitrary exemplars of this concept.

Example 5 *Suppose that the defining feature set associated with* to-be-a-bird *is* $\Delta(\alpha)$ ={to-have-two-legs, to-lay-eggs, to-have-a-beak, to-have-wings} *with salience order:* to-have-a-beak $>_{\Delta(\alpha)}$ to-lay-eggs $>_{\Delta(\alpha)}$ to-have-two-legs, *and* to-have-wings $>_{\Delta(\alpha)}$ to-lay-eggs $>_{\Delta(\alpha)}$ to-have-two-legs. *Suppose also that the corresponding characteristic set consists of the three features* to-build-nests, to-sing, *and* to-fly, *equipped with the order :* to-fly$>_{\chi(\alpha)}$to-build-nests *and* to-fly$>_{\chi(\alpha)}$to-sing. *Computing the bird-resemblance degrees of a bat* b, *a penguin* p *and a kiwi* k *yields* $\delta^{\rho}_{\alpha}(b) = 11/26$, $\delta^{\rho}_{\alpha}(p) = 9/26$ *and* $\delta^{\rho}_{\alpha}(k) = 19/26$. *The bat is less bird-resemblant than the kiwi, but more bird-resemblant than the penguin.*

The following result shows that, given a concept, there exists an associated *resemblance threshold*, beyond which objects are endowed to some degree with the characteristic attributes of this concept:

Proposition 12 *Set* $\epsilon(\alpha) = \frac{|\Delta(\alpha)|}{\Sigma_{f\in\Pi(\alpha)} s_{\alpha}(f)}$, *and let x be an object such that* $\delta^{\rho}_{\alpha}(x) \geq 1 - \epsilon(\alpha)$. *Then all the characteristic features of α apply at least partially to x.*

If x is sufficiently α-resemblant, we have therefore $\delta_f(x) > 0 \; \forall f \in \chi(\alpha)$. This result is particularly interesting in the case where the applicability of the characteristic features of α is measured by two-valued functions. Then *any object sufficiently close to α falls under its characteristic features.* However, this does not mean that such an object is a typical instance of α: nothing indeed guarantees that x falls under α. But it may happen that, for an agent, the typical instances of α only apply to the typical exemplars of α, that is $Typ\,\alpha = \bigcap_{f\in Int\,\alpha} Ext\,f$. In this case, x has an α-resemblance degree greater than $1 - \epsilon(\alpha)$ if and only if $x \in Typ\,\alpha$.

Example 6 *Let* $\alpha =$ to-be-a-bird *with the stereotypical set* $\Pi(\alpha)$ *considered in Example 5, all features being evaluated through two-valued functions. Then one has* $\epsilon(\alpha) = 2/13$*. If an object* x *has a* birdhood *resemblance degree greater than, say, 85%,* x *will* fly, sing *and* build nests*. For any agent for whom only* birds *can at the same time* fly, sing *and* build nests, x *will necessarily appear to be a typical bird.*

5.2 Resemblance between concepts

Similarly to what happens when an object is compared with a concept, one may interpret concept resemblance as resemblance between their typical instances: judging for instance that 'Braque resembles Picasso' amounts to saying that a typical painting of Braque resembles a Picasso. In other words, we say that *a concept* α *resembles a concept* β whenever *every typical instance* of α resembles the concept β in the sense of the preceding paragraph.

This observation can be used to define a resemblance degree between concepts:

Definition 10 *The* α-resemblance degree *of a concept* β *is the number* $\delta^\rho_\alpha(\beta) = Min_{x \in Typ\,\beta} \delta^\rho_\alpha(x)$.

The resemblance of β to α is therefore measured by taking among the typical elements of β those that are the least α-resemblant. Note that this notion of resemblance between concepts is not symmetric.

Remark 5 *In distributional semantics, the resemblance degree of two items is defined by comparing their linguistic context and evaluating their distance in the underlying 'semantic space'. This latter consists in a vector space with several hundred thousands dimensions, each of which is associated with a context word (see for instance [2] or [5]). Distance in a topological vector space with 'qualitative dimensions' is also the tool that Gärdenfors uses in [14] to evaluate similarity. By opposition to the sophisticated technics developed in these two examples, the subjective notion of resemblance we define here only requires some elementary calculations. These will not provide information about the similarity of two items, but they will enable us to evaluate to what degree a given agent may consider that an item resembles another.*

From Definition 9 and 10, we see that the maximally α-resemblant concepts β are those for which $Typ\,\beta \subseteq Typ\,\alpha$. This justifies the following distinction:

Definition 11 *A subconcept β of α is* smooth *if $Typ\,\beta \subseteq Typ\,\alpha$.*

If β is a smooth subconcept of α, it has maximal resemblance to it. But maximal resemblance may also be found among concepts that are not subconcepts of α. For instance, although not all *Caravaggio paintings* are *portraits*, we can consider that *Caravaggio paintings* maximally resemble *portraits*, because typical paintings from Caravaggio are typical portraits.

5.2.1 The case of compound concepts

We now extend the notion of resemblance between an object and a simple concept to compound concepts. Let $\beta\star\alpha$ be a determination of α by a concept β such that $Typ\,\alpha \cap Typ\,\beta \neq \emptyset$. Then we know by Theorem 2 that $\beta \star \alpha$ is equivalent to a featured concept for which the defining and characteristic sets are $\Delta(\beta \star \alpha) = \Delta(\alpha) \cup \Delta(\beta)$ and $\chi(\beta \star \alpha) = \chi(\alpha) \cup \chi(\beta)$. Let $\Pi(\beta \star \alpha)$ denote the set $\Delta(\beta \star \alpha) \cup \chi(\beta \star \alpha)$ with the salience order that extends the salience orders $>_{\chi(\beta\star\alpha)}$ on $\chi(\beta \star \alpha)$ and $>_{\Delta(\beta\star\alpha)}$ on $\Delta(\beta \star \alpha) \setminus \chi(\beta \star \alpha)$, and satisfies moreover $f >_{\Pi(\beta\star\alpha)} g$ for all $f \in \chi(\beta\star\alpha)$ and $g \in \Delta(\beta\star\alpha)\setminus\chi(\beta\star\alpha)$.

Then the salience degree $s_{\beta\star\alpha}(f)$ of an element f of $\Pi(\beta \star \alpha)$ is defined by: $s_{\beta\star\alpha}(f) = 1 + |\{h \in \Pi(\beta \star \alpha); f >_{\Pi(\beta\star\alpha)} h\}|$.

Definition 12 *The $(\beta \star \alpha)$-resemblance degree of an object x is the number* $\delta^{\rho}_{\beta\star\alpha}(x) = \frac{\Sigma_{f\in\ \Pi(\beta\star\alpha)} s_{\beta\star\alpha}(f)\delta_f(x)}{\Sigma_{f\in\Pi(\beta\star\alpha)} s_{\beta\star\alpha}}$.

In the case where x is an element of $Typ\,\alpha$, its $\beta \star \alpha$-resemblance degree cannot be arbitrary small:

Proposition 13 *One has $\delta^{\rho}_{\beta\star\alpha}(x) \geq \frac{\Sigma_{f\in\Pi(\alpha)} s_{\alpha}(f)}{\Sigma_{f\in\Pi(\beta\star\alpha)} s_{\beta\star\alpha}(f)}$ whenever $x \in Typ\,\alpha$.*

The $(\beta \star \alpha)$-resemblance degree of a concept γ can be now defined similarly to simple concepts:

Definition 13 *The $(\beta \star \alpha)$-*resemblance degree *of a concept γ is the number* $\delta^{\rho}_{\beta\star\alpha}(\gamma) = Min_{x\in Typ\,\gamma}\delta^{\rho}_{\beta\star\alpha}(x)$.

We can now more accurately address the problem of resemblance between concepts. Observe indeed that comparison between two concepts mainly occurs when these concepts are considered as subconcepts of the same concept. One compares apples to pears, considered as fruits, or dogs to cats, considered as pets, but (generally) not apples to cats... Resemblance between concepts therefore principally concerns resemblance between two subconcepts of a given concept.

As an immediate consequence of proposition 13, we obtain a threshold of minimal resemblance:

Proposition 14 *Suppose that $Typ\,(\beta \star \alpha) = Typ\,\beta \cap Typ\,\alpha$. Then, for any smooth subconcept γ of α, one has $\delta^{\rho}_{\beta\star\alpha}(\gamma) \geq \frac{\Sigma_{f \in \Pi(\alpha)} s_{\alpha}(f)}{\Sigma_{f \in \Pi(\beta\star\alpha)} s_{\beta\star\alpha}(f)}$.*

We shall come back to the notion of resemblance in the last section of this paper to address the problem of induction through resemblance.

Now that the basic notions of concept analysis have been discussed, we can turn to the *dynamics* of concepts, and study the main problems linked with concept entailment.

Part II
Concept entailment

6 Conceptual inference

In the section, we shall reinterpret the notions of *essential* and *typical* attribute defined in section 4.1, and integrate it within the framework of non monotonic inference relations.

6.1 Necessary induction

Consider first a concept α, and let β be an element of $Ess\,\alpha$. Recall (see section 4) that, by definition, β applies to all exemplars of α. Thus, α cannot be 'true' on an object without β being 'true' at the same time. In other words, we can consider β as a *necessary* consequence of α. For instance, each of the concepts *to-have-a-beak*, *to-have-feathers* or *to-be-oviparous* may be considered as a *necessary consequence* of the concept *to-be-a-bird*: if something is a bird, then it is necessarily oviparous, and it has necessarily a beak and feathers.

By analogy with the semantics of propositional calculus, one may translate the binary relation $\beta \in Ess\,\alpha$ into the form of a *consequence* relation, written $\alpha \vdash \beta$, which we interpret as 'if α, then necessarily β'.

Proposition 15 *Set $\alpha \vdash \beta$ iff $\beta \in Ess\,\alpha$. The relation $\vdash$ thus defined is* reflexive *and* transitive*. Furthermore, it is* monotonic*, in the sense that $\alpha \vdash \beta$ implies $\gamma \star \alpha \vdash \beta$ and $\alpha \star \kappa \vdash \beta$ for all concepts γ and κ for which these expressions are meaningful.*

Using logical calculus notation, the properties of Proposition 15 can be written:

- $\frac{}{\alpha \vdash \alpha}$ ***(Reflexivity)***
- $\frac{\alpha \vdash \beta}{\gamma \star \alpha \vdash \beta,\, \alpha \star \kappa \vdash \beta}$ ***(Monotonicity)***

We can add three other properties, the proof of which is straightforward

- $\frac{\alpha\equiv\alpha', \beta\equiv\beta', \alpha\vdash\beta}{\alpha'\vdash\beta'}$ $\big($***Logical Equivalence***$\big)$

- $\frac{\alpha\vdash\beta, \alpha\vdash\gamma}{\alpha\vdash\beta\star\gamma}$ $\big($***And***$\big)$

- $\frac{\beta\star\alpha\vdash\gamma, \alpha\vdash\beta}{\alpha\vdash\gamma}$$\big($***Cut)***.

These properties do not put a special light on concept theory. Their interpretation is obvious. For instance, the application of ***(Cut)*** to the concepts α = *to-be-a-bird*, β = *to-be-oviparous* and γ = *to-build-nests* yields the scheme of reasoning: *since birds are oviparous and oviparous birds build nests, birds build nests...* In fact, the real interest of the above proposition is to set a new perspective for the study of concept analysis. It shows indeed that $\vdash$ behaves like a *monotonic consequence relation*, and that the set of featured concepts in which it operates plays a role similar to that of a *propositional language* equipped with an equivalence relation $\equiv$ between propositions and a (partial) connective $\star$. Thus, using the tools of classical and non-classical logic, it will be possible to obtain interesting results on concept entailment. Note however that the language is deprived of *negation* since the negation of a featured concept is generally not a concept, unless it has a positive equivalent. Out of any context, what meaning can we give, for example, to the negation of *to-be-a-dog*? As observed by Hampton [18], it is true that within a conjunctive phrase, a negation may make sense. In this case, the principal term of the conjunction provides a context that circumscribes the possible meanings, and Hampton rightly writes that '*Games-which-are-not-sports* is a concept for which subjects can sensibly judge the membership and typicality of items'. However, the complex construction necessary to define, characterize and study negated concepts in the general case goes far beyond the limit of this work. The restricted language we are developing on is sufficient to obtain a new and fruitful insight on the classical problems of concept entailment.

As we shall see now, the interpretation of typicality in the framework of consequence relations can be extended to a weak form of inference.

6.2 Typical induction

The typical instances of a concept are those that best represent it. In this sense, we may consider that *non-typicality is exceptional* among the instances of a concept α. This amounts to supposing that *by default*, any arbitrary exemplar of α is typical: thinking of a bird, for instance, an agent will most often think of a flying bird, and the images that are given on the web for *birds* all show flying birds.

This being noticed, let f be a typical attribute of α, that is an element of $Int\,\alpha$. By definition (see section 4.1), f applies to every typical instance of α, and f therefore appears to be *typically inferred* from α: for instance, we consider that the feature *to-fly* is typically inferred from the concept *to-be-a-bird*, because, typically, birds fly. Equivalently, we could say that typically, birds are flying birds, showing the existence of an inference relation between the concepts α = *to-be-bird* and β = *to-be-a-flying-bird*. This yields the following definition:

Definition 14 *Given two concepts α and β, we say that α* typically induces *β, or that β is a* typical consequence *of α, if $Typ\,\alpha \subseteq Ext\,\beta$. This will be denoted by $\alpha \mid\!\sim \beta$.*

In the particular case where β is a feature f, the relation $\alpha \mid\!\sim \beta$ exactly translates the fact that $\delta_f(x) = 1\ \forall x \in Typ\,\alpha$, that is that f is a typical attribute of α.

It is clear that the relation $\mid\!\sim$ is *not* monotonic: for instance, if α is the concept *to-be-a-bird*, β the concept *to-fly* and γ the concept *to-be-an-ostrich*, we have $\alpha \mid\!\sim \beta$ because, typically, birds fly, but not $(\gamma \star \alpha) \mid\!\sim \beta$, as typical ostriches don't fly. However, relative to the connective $\star$, the relation $\mid\!\sim$ enjoys several properties that are formally similar to those studied by Kraus, Lehmann and Magidor in their seminal paper [22]. More precisely we have the following result:

Proposition 16 *Denote by $\mid\!\sim$ the relation: $\alpha \mid\!\sim \beta$ iff $Typ\,\alpha \subseteq Ext\,\beta$. Let α, β and γ be three featured concepts such that $Ext\,\alpha \cap Ext\,\beta \cap Ext\,\gamma \neq \emptyset$. Then the following properties hold:*

- $\dfrac{\alpha \vdash \beta}{\alpha \mid\!\sim \beta}$ ***(Supraclassicality)***
- $\dfrac{\alpha \equiv \alpha', \beta \equiv \beta', \alpha \mid\!\sim \beta}{\alpha' \mid\!\sim \beta'}$ ***(Logical Equivalence)***
- $\dfrac{\alpha \mid\!\sim \beta, \beta \vdash \gamma}{\alpha \mid\!\sim \gamma}$ ***(Right weakening)***

- $\dfrac{\alpha|\!\sim\beta, \alpha|\!\sim\gamma}{\alpha|\!\sim\beta\star\gamma}$ ***(And)***
- $\dfrac{\alpha|\!\sim\beta, \alpha|\!\sim\gamma}{\beta\star\alpha|\!\sim\gamma}$ ***(Cautious Monotonicity)****.*

The first four properties directly follow from the definition of the typical inference relation $|\!\sim$. Cautious Monotonicity is a rule close to monotonicity: it models reasonings like: *typical birds fly and typical birds sing, therefore typical flying birds sing.*

The above proposition shows that typical inference is close to a *cumulative* inference relation (see [22] for a study of these relations). However, the important property of **Cut** mentioned in the preceding paragraph, which is satisfied by cumulative relations, is *not* satisfied by typical induction:

Example 7 *Let α and β be respectively the concepts* to-be-a-seed-eater *and* to-be-a-bird*, and take for γ the feature* to-fly*. Suppose that for an agent, the characteristic set of α is* {to-have-a-beak, to-have-wings, to-be-oviparous} *and the characteristic set of β is* {to-fly, to-sing, to-live-in-the-trees}*. Then $Typ\,\alpha$ is the set of seed-eaters that are birds, and $Typ\,\beta$ is the set of typical birds. We have $Typ\,(\beta\star\alpha) = Typ\,\alpha \cap Typ\,\beta$ (see Proposition 7). This set consists of all typical birds that eat seeds. For this agent therefore, $\beta\star\alpha|\!\sim\gamma$ and $\alpha|\!\sim\beta$, but not $\alpha|\!\sim\gamma$:* seed-eaters that are birds generally fly, and birds generally fly, but seed-eaters generally don't fly.

The fact that *Cut* is generally not satisfied by typical inference shows that the relation $|\!\sim$ admits no model analogous to the KLM cumulative models. In particular, there exists no possible representation of $|\!\sim$ through order relations in $\mathcal{O}$. Of course, the conditional $\alpha|\!\sim\beta$ holds if and only if the $\prec^{\tau}_{\alpha}$-maximal objects falling under α also fall under β. However, the chosen order $\prec^{\tau}_{\alpha}$ is here a local one, and depends on the antecedent α of the conditional $\alpha|\!\sim\beta$.

It will be shown in section 7.2.1 that a slightly weaker form of *Cut* still remains valid in our model.

7 Concept-based induction

Concept-based induction is the process through which, from a given concept, one infers information relative to another concept. Such an inference is based on a link between the source and the target concept. For instance, one may ask which attributes of a concept remain true for a given subconcept or, conversely, which attributes of a subconcept can be raised to the whole original one. One may also be interested in resemblance between concepts, and look for the attributes that can be transposed from a concept to another one that is considered as 'sufficiently resemblant'. These three different kinds of *category-based* induction will be successively examined.

7.1 Within-category induction

While the *essential* attributes (see section 2.5) of a concept apply to the exemplars of any of its subconcepts, this is no longer true for its *typical* attributes. Typical birds fly, but typical ostriches don't. In spite of this, typical ostriches inherit some other bird typical attributes, like that of having feathers with remiges. This poses the problem of determining *which attributes of a concept remain valid for a given subconcept.* To take a well-known example (see [3]), knowing that ducks have webbed feet, can we deduce that the same is true for *quacking* ducks ? Can we deduce that the same is true for *non-quacking* ducks ?

This *inheritance problem* is of particular interest in the case of regular subconcepts obtained through concept determination. To quote J.A.Hampton [20] '*Given that an attribute is not universally true of a concept (...), how should one determine whether the predicate should also be considered generically true of the complex concept formed when an adjectival or nominal determiner is applied to the noun ?*'.

In the particular case where the determiner β is itself a typical consequence of α, the rule of Cautious Monotonicity guarantees that inheritance holds for the composed regular subconcept $\beta \star \alpha$: any typical instance γ of α becomes a typical instance of $\beta \star \alpha$. Thus, *flying-birds inherit all the typical properties of birds.*

In the general case, however, the answer depends on the nature of the determiner, as will be seen in the next sections

7.1.1 Smooth subconcepts and non-exceptional determiners

Let us first distinguish between two different forms of concept determination:

Definition 15 *A concept β is* exceptional *for α if $Typ\,\alpha \cap Ext\,\beta = \emptyset$.*

β is therefore *exceptional* for α if no typical instance of α falls under β. For instance, taking for characteristic set of the concept *to-be-a-bird* the set $\{$*to-fly, to-leave-in-trees*$\}$, we can say that *to-talk* is *non-exceptional for birds* because there exist typical birds that talk.

When β is used to determine or modify α, we shall speak of an exceptional or a non-exceptional determiner. This notion may be seen as a formal definition of the *compatible determiner* introduced by [30]. It is tightly linked with that of smoothness (see Definition 11), since it readily follows from Proposition 10 that $(\beta \star \alpha)$ *is a smooth subconcept of* α *if and only if* β *is non-exceptional for* α.

We can interpret the notion of exceptional determiners in the framework of inference relations. To say that β is exceptional for α means that no typical instance of α falls under β, or that *typically,* α*'s are not* β*'s.* Although our language is deprived of negation, such a sentence can still be translated by means of a new notation:

Notation *We write '*$\alpha {\mid\!\sim} \neg\beta$*' if* β *is exceptional for* α*, and '*$\alpha {\mid\!\not\sim} \neg\beta$*' if* β *is not exceptional for* α*.*

Using this notation, it is possible to retrieve a well-known property that characterizes *rational* inference relations:

Proposition 17 *The following rule is valid in the proposed model:*

- $\frac{\alpha {\mid\!\sim} \gamma,\, \alpha {\mid\!\not\sim} \neg\beta}{\beta\star\alpha {\mid\!\sim} \gamma}$ ***(Rational Monotony)***

Proposition 17 provides a first answer to the within-category induction problem: *knowing that there exist typical birds that are white, and knowing that birds generally sing, we can conclude that white birds generally sing.*

7.1.2 The case of exceptional determiners

When β is exceptional for α, $(\beta \star \alpha)$ is no longer smooth in α and the full inheritance property does not apply anymore: *non-flying-birds* do not inherit the *birdhood* typical property of flying. The inheritance problem then amounts to determining *which attributes of* α *are preserved in the subconcept* $(\beta \star \alpha)$.

Since β is exceptional for α, we have $\emptyset = Ext\,\beta \cap Typ\,\alpha$, and thus (see paragraph 4.1) $Ext\,\beta \cap Ext\,\alpha \bigcap_{f\in\chi(\alpha)} Ext\,f = \emptyset$.

We shall examine the case where one single characteristic attribute k is responsible for the α-exceptionality of β:

Proposition 18 *Suppose that for a characteristic feature k of α, one has $Ext\,\beta \cap Ext\,\alpha \cap Ext\,k = \emptyset$, but $Ext\,\beta \cap Ext\,\alpha \cap (\bigcap(Ext\,f)_{f \in \chi(\alpha)-\{k\}}) \neq \emptyset$. Suppose also that the applicability function δ_k induced by k is a two-valued function. Then any characteristic feature of α different from k remains a typical attribute of $(\beta \star \alpha)$.*

By this result, we see that inheritance to an exceptional subcategory still holds for the characteristic features that do not directly contradict the determiner. If α is the concept *to-be-a-bird* for instance, with characteristic feature set $\chi(\alpha) = \{$*to-fly, to-build-nests, to-sing, to-eat-seeds*$\}$, we can conclude from the above proposition that,*typically, walking birds sing, walking birds build nests and walking birds eat seeds.*

7.2 Over-category induction

Over-category induction deals with the problem of extending to a whole category an attribute that is known to hold for a subcategory. This problem may be seen as the converse of within-category induction. However, it slightly differs in that one has to distinguish between *essential* and *non-essential* attributes. We may therefore consider the four problems of raising an *essential* or a *typical* attribute of a subcategory into an *essential* or a *typical* attribute to the whole category.

Clearly, extending a typical attribute γ into an essential attribute is possible only in the case where γ itself is an essential attribute. Furthermore, the problem of extending an essential attribute of a subcategory to an essential attribute of the category may be easily solved, at least for determined concepts, by the property ***Cut*** which is satisfied by $\vdash$. We are therefore left with the problem of raising an essential or a typical attribute into a typical one.

Let us first determine what essential attributes of a subcategory typically apply to the whole category. For instance, knowing that ducks fly and have webbed feet, which of these attributes can be extended to birds ?

7.2.1 Over-category induction from typically determined concepts

Extending an essential attribute of a subconcept into a typical attribute of the concept is generally impossible: *to-walk* is an essential attribute of *walking-birds*, although birds generally don't walk. However, a positive answer can be given when the subconcept is obtained through a determination of the initial one by one of its typical consequences : if β is a typical consequence of α, every essential attribute of $\beta \star \alpha$ becomes a typical attribute of α. More precisely, we can observe that a weak version of **Cut** is valid in our model:

Proposition 19 *The following rule is valid:*

- $\frac{\beta\star\alpha\vdash\gamma,\, \alpha\mid\!\sim\beta}{\alpha\mid\!\sim\gamma}$ ***(Cautious Cut)***

Thus, *from the fact that birds generally fly and that all flying-birds have feathers with remiges, we can conclude that birds have generally feathers with remiges.*

Concerning finally the possibility of extending a typical consequence from a typically determined concept, note that this is exactly the property of **Cut**, which is not satisfied by $\mid\!\sim$, as we observed before. However, a solution exists in a particular case:

Proposition 20 *The rule **Cut** holds for any concept determined by a typical attribute:*

- $\frac{\beta\star\alpha\mid\!\sim\gamma,\ \beta\in Int\,\alpha}{\alpha\mid\!\sim\gamma}$

By this property, *knowing that flying-birds are generally small and that birds generally fly, we can conclude that birds are generally small.*

We shall be interested now in a particular family of subconcepts, which we will refer to as *typical*. For this purpose, let us consider once more the concept *to-be-a-bird*. When an agent says that the robin is a *typical* exemplar of this concept, this assertion rests on the fact that robins inherit all the attributes that, from her point of view, a typical bird should have - *to fly, to sing, to live in the trees,* etc. This inheritance property then extends from individual items to the whole *category* of robins, thus becoming an *essential* attribute of the concept *to-be-a-robin*. These considerations lead to the following definition:

Definition 16 *A subconcept β of α such that $Ext\,\beta \subseteq Typ\,\alpha$ will be called a* typical *subconcept of α.*

Note that a typical subconcept of α is smooth in α (see Definition 11). As we shall see now, typical subconcepts play an important role in over-category induction.

7.2.2 Over-category induction from typical subconcepts

During some experiments conducted on category-based induction [3], it appeared that the predicates that were the best candidates for induction from a subcategory to a mother category were the so-called *blank* predicates: these predicates did not bear any special meaning, they were liable to apply to any category that was considered close enough to the tested one. On the contrary, features that were closely related or specific to the source category had to be discarded.

This shows the need of distinguishing between *specific* and *non-specific* attributes.

Definition 17 *Let β be a typical subconcept of α. An essential attribute of β is* non-specific *for β if it is shared by all the typical subconcepts of α.*

For instance, *to-fly* is a non specific attribute of *robin*, considered as a subcategory of *birds*. Indeed, *to-fly* is a typical attribute of any typical subconcept of *birds*.

The following result now provides an answer to over-category induction from typical subconcepts:

Proposition 21 *Let β be a typical subconcept of α, and κ an essential attribute of β. Then κ is a typical attribute of α if and only if κ is non-specific to β.*

By the above proposition, we see that *to-have-webbed-feet can be raised from the category of ducks to the category of aquatic birds, but not to the whole category of birds.*

After having studied the problem of in and over-category induction, we now turn to the last problem of concept-based induction, which is that of inducing a property from a category to another that sufficiently resembles the first one.

7.3 Induction through resemblance

When a concept β is sufficiently close to a concept α, one expects that some attributes of α also hold for β: for example, some features should transpose from bats to birds, from horses to cows, or from camels to dromedaries. Ideally, induction through resemblance should be encapsulated by a deduction scheme of the form $\frac{\alpha|\!\sim\gamma\ \beta\rhd\alpha}{\beta|\!\sim\gamma}$, where $\rhd$ would translate a certain kind of resemblance between concepts.

Recall (see definition 10) that the α-resemblance degree of a concept β is given by: $\delta^{\rho}_{\alpha}(\beta) = Min_{x\in Typ\,\beta}\,\delta^{\rho}_{\alpha}(x)$, where $\delta^{\rho}_{\alpha}(x) = \frac{\Sigma_{f\in\ \Pi(\alpha)}s_f\delta_f(x)}{\Sigma_{f\in\Pi(\alpha)}s_f}$ and $s_f = 1 + |\{h \in \Pi(\alpha); f >_{\Pi(\alpha)} h\}|$. We noted then that the maximally α-resemblant concepts β are those for which $Typ\,\beta \subseteq Typ\,\alpha$. From this readily follows the

Proposition 22 $\frac{\alpha|\!\sim\gamma,\ \delta^{\rho}_{\alpha}(\beta)=1}{\beta|\!\sim\gamma}$.

Thus, if β is maximally resemblant to α, it shares with it all its typical attributes.

In the case where β is only supposed to be 'sufficiently α-resemblant', we still have an interesting result that can be easily deduced from Proposition 12. It asserts that a non exceptional determiner h of α becomes a typical attribute of any concept β that sufficiently resembles $(h\star\alpha)$.

Proposition 23 *Let $h\star\alpha$ be the determination of a featured concept α by a feature h whose applicability is given by a two-valued function. Let $\epsilon(h\star\alpha)$ denote the number $\frac{|\Delta(h\star\alpha)|}{\Sigma_{f\in\Pi(h\star\alpha)}s_{h\star\alpha}(f)}$. Then the following rule is valid:*

- $\frac{\delta^{\rho}_{h\star\alpha}(\beta)\geq 1-\epsilon(h\star\alpha),\ \alpha|\!\not\sim\neg h}{\beta|\!\sim h}$

For instance, this result shows that *a fruit that resembles a red apple is very likely to be red itself . . .*

It is interesting to observe that the principal role in this situation is devoted to the determiner h and not to the principal concept α: the proposition says nothing about the link, resemblance or induction, that may exist between β and α.

8 Conclusion

We showed in this paper that a model for concept representation can be built from simple order relations. This model renders possible a mathematical definition of several notions that are at the heart of much research in cognitive psychology. At the same time, it renders possible the formalization of problems linked with concept entailment, for which it proposes an original and efficient answer. The results obtained in this framework however rest on a hypothesis, which is that agents grasp some of their concepts through the features they attach to them. If this assumption is founded, we may well ask how agents gather and bring these features together to build their knowledge of concepts. We may even wonder to what degree one could anticipate part of their cognitive behavior, knowing their knowledge bases. Conversely, observing the significance that a concept may bear for an agent, it could be interesting to deduce the nature and the role played by the features from which it stems. In this regard, ordered or logical models seem to provide an economical as well as a powerful tool for the study of categorization. They constitute an interesting alternative to more sophisticated theories, as they are sufficient to explain and foresee classical problems linked with concept combination, prototype theory and category-based induction.

The binary model which we used as a basis of our investigation provides a simple framework in which both the statics and the dynamics of concept analysis can be studied. Nevertheless, it is probable that similar results could be obtained using a unitary model, equipped with a single order that would altogether account for categorial membership, typicality and resemblance.

As a last point, we have to underline that the validity of this theoretical study depends on a confirmation by future experimentation. In this perspective, the distinctions we introduced between the different kinds of determiners in concept composition may be of some use. We also think that the introduction and the formalization of fundamental notions, like those of *smooth* or *typical* subconcept, remain of significant interest, independently of the value of the proposed model.

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A Proofs

Proposition 1 *The relation $\preceq_\alpha$ is a partial weak order on $\mathcal{O}$.*

Proof: Reflexivity is immediate, and we have only to prove transitivity.

Let therefore x, y and z be three items such that $x \preceq_\alpha y$ and $y \preceq_\alpha z$; suppose that there exists $f \in \Delta(\alpha)$ such that $z \prec_f x$. We have to show that there exists a feature $g \in \Delta(\alpha)$, g more salient than f, such that $x \prec_g z$. We consider two cases:

- Suppose first that $x \preceq_f y$. Then we have $z \prec_f y$, and there exists a feature $k \in \Delta(\alpha)$, k more salient than f, such that $y \prec_k z$. We can suppose that k is maximally salient in $\Delta(\alpha)$ for this property (recall that $\Delta(\alpha)$ is finite). If $x \preceq_k y$, we get $x \prec_k z$ and we are done. Otherwise, because of the connectedness of $\preceq_k$, we have $y \prec_k x$. Since we supposed $x \preceq_\alpha y$, this implies that there exists a concept $g \in \Delta(\alpha)$, g more salient than k, such that $x \prec_g y$. We cannot have $z \prec_g y$ otherwise there would exist $h \in \Delta(\alpha)$, h more salient than g, such that $y \prec_h z$, contradicting the choice of k. We have therefore $y \preceq_g z$, hence $x \prec_g z$ as desired.

- Suppose now that $y \prec_f x$. Then there exists $k \in \Delta(\alpha)$, k more salient than f such that $x \prec_k y$, and we can again suppose k maximally salient for this property. If $y \preceq_k z$, we get $x \prec_k z$ and we are through. Otherwise, we have $z \prec_k y$ and there exists g more salient than k such that $y \prec_g z$. Let us show that $x \preceq_g y$: if this were not the case, we would have $y \prec_g x$, so that there would exist h more salient than g such that $x \prec_h y$. But then h would be more salient than k, which is impossible. We have therefore $x \preceq_g y$, hence $x \prec_g z$, which completes the proof. ■

Proposition 2 *An object is $\preceq_\alpha$-maximal in $\mathcal{O}$ if and only if it is an element of $Ext\,\alpha$. One has $z \prec_\alpha x$ for any $x \in Ext\,\alpha$ and $z \notin Ext\,\alpha$. Furthermore, any increasing $\prec_\alpha$-chain is of finite bounded length.*

Proof: For the first part of the proposition, suppose that z is not in the extension of α. By definition, there exists a defining feature f such that $\delta_f(z) < 1$. Let x be an element such that $\delta_{f_i}(x)$ =1 for all i. We have then clearly $z \prec_\alpha x$, showing that z is not $\preceq_\alpha$-maximal in $\mathcal{O}$.

The second part of the proposition is a direct consequence of Definition 1: let indeed f_i, $1 \leq i \leq k$, be a defining feature of α, and n_i the number of values taken by δ_{f_i}. Then f_i appears as a ring of a strictly increasing chain at most δ_{f_i} times. It follows that the length of such a chain is bounded by $\sum_{i=1}^{k} n_i$. ■

Lemma 1 *Every attributional concept β such that $\preceq_\beta^\mu \subseteq \preceq_\alpha^\mu$ is a subconcept of α.*

Proof: We have to prove that $Ext\,\beta \subseteq Ext\,\alpha$. But if $z \in Ext\,\beta$, we have $x \preceq_\beta z$ for all objects $x \in \mathcal{O}$, and therefore $x \preceq_\alpha z$ for all objects $x \in \mathcal{O}$, showing that z is $\preceq_\alpha$-maximal. It follows that $z \in Ext\,\alpha$. ■

Theorem 1 *$\beta \star \alpha$ is similar to an attributional concept whose set of features $\Delta(\beta \star \alpha)$ is equal in the principal case to $\Delta(\alpha) \cup \Delta(\beta)$.*

Proof: In order to prove the theorem in the principal case, we need a technical lemma:

Lemma Given two objects x and y, one has $x \preceq_\alpha^\mu y$ and $y \preceq_\alpha^\mu x$ iff $\forall f \in \Delta(\alpha), \delta_f(x) = \delta_f(y.)$

Proof of the lemma It is clear that $x \preceq_\alpha^\mu y$ and $y \preceq_\alpha^\mu x$ whenever $\delta_f(x) = \delta_f(y) \forall f \in \Delta(\alpha)$. Conversely, suppose that $x \preceq_\alpha^\mu y$ and $y \preceq_\alpha^\mu x$ and suppose that $\delta_f(x) \neq \delta_f(y)$ for some element $f \in \Delta(\alpha)$. We can choose f of maximal salience for that property. We have for instance $\delta_f(x) < \delta_f(y)$. Since $y \preceq_\alpha^\mu x$, there would exist $g \in \Delta(\alpha)$, g more salient than f, such that $\delta_g(y) < \delta_g(x$, contradicting the choice of f. ■

Let us now prove the theorem in the principal case where the membership order is given by Equation (1).

Consider the set $\Delta(\beta\star\alpha) = \Delta(\alpha)\cup\Delta(\beta)$ equipped with the salience order $>_{\Delta(\beta\star\alpha)}$ that extends those of $\Delta(\alpha)$ and $\Delta(\beta) \setminus \Delta(\alpha)$ and satisfies moreover

$g >_{\Delta(\beta\star\alpha)} f$ for all $g \in \Delta(\alpha)$ and $f \in \Delta(\beta) \setminus \Delta(\alpha)$.

Denote by $\preceq^{\mu}_{\Delta(\beta\star\alpha)}$ the membership order induced by the concept $\beta \star \alpha$ equipped with the defining feature set equal to $\Delta(\beta \star \alpha)$. We have to prove that $\preceq^{\mu}_{\Delta(\beta\star\alpha)}$ agrees with $\preceq^{\mu}_{\beta\star\alpha}$.

- We first show that $\preceq^{\mu}_{\beta\star\alpha} \subseteq \preceq^{\mu}_{\Delta(\beta\star\alpha)}$. Suppose that we have $x \preceq^{\mu}_{\beta\star\alpha} y$, that is $x \preceq^{\mu}_{\alpha} y$ and either $x \prec^{\mu}_{\alpha} y$, or $x \preceq^{\mu}_{\beta} y$. Let f be an element of $\Delta(\beta\star\alpha)$ such that $y \prec_f x$. We have to prove that $\exists g \in \Delta(\beta \star \alpha), g >_{\Delta(\beta\star\alpha)} f$, such that $x \prec_g y$.

  - If $f \in \Delta(\alpha)$, the inequality $x \preceq^{\mu}_{\alpha} y$, implies the existence of $g \in \Delta(\alpha)$, $g >_{\Delta(\alpha)} f$, such that $x \prec_g y$. We have therefore $g \in \Delta(\beta \star \alpha)$ and $g >_{\Delta(\beta\star\alpha)} f$, as desired.

  -If $f \in \Delta(\beta) \setminus \Delta(\alpha)$, the inequality $x \preceq^{\mu}_{\beta\star\alpha} y$ implies either $x \prec^{\mu}_{\alpha} y$ or $x \preceq^{\mu}_{\beta} y$. In the first case, it follows from equation (2) that $\exists g \in \Delta(\alpha)$ such that $x \prec_g y$, and we conclude as above. For the second case, the inequalities $y \prec_f x$ and $x \preceq^{\mu}_{\beta} y$ show that there exists $g \in \Delta(\beta)$, $g >_{\Delta(\beta)} f$ such that $x \prec_g y$. This again implies $g >_{\Delta(\beta\star\alpha)} f$.

- Conversely, let us prove that $\preceq^{\mu}_{\Delta(\beta\star\alpha)} \subseteq \preceq^{\mu}_{\beta\star\alpha}$. Suppose that $x \preceq^{\mu}_{\Delta(\beta\star\alpha)} y$. To prove that $x \preceq^{\mu}_{\beta\star\alpha} y$, let us first check that $x \preceq^{\mu}_{\alpha} y$. If $f \in \Delta(\alpha)$ is such that $y <_f x$, we have $f \in \Delta(\beta\star\alpha)$. Since $x \preceq^{\mu}_{\Delta(\beta\star\alpha)} y$, there exists $g \in \Delta(\beta \star \alpha)$, $g >_{\Delta(\beta\star\alpha)} f$, such that $x <_g y$. By the salience order on $\Delta(\beta\star\alpha)$, this necessarily implies $g \in \Delta(\alpha)$ and $g >_{\Delta(\alpha)} f$. This proves that $x \preceq^{\mu}_{\alpha} y$, as desired.

  It remains to prove that we have either $x \prec^{\mu}_{\alpha} y$, or $x \preceq^{\mu}_{\beta} y$.

  Suppose that we do not have $x \prec^{\mu}_{\alpha} y$, so that $x \preceq^{\mu}_{\alpha} y$ and $y \preceq^{\mu}_{\alpha} x$. To show that $x \preceq^{\mu}_{\beta} y$, let $f \in \Delta(\beta)$ such that $y <_f x$. Since $x \preceq^{\mu}_{\Delta(\beta\star\alpha)} y$, there exists $g \in \Delta(\beta \star \alpha)$, $g >_{\Delta(\beta\star\alpha)} f$ such that $x \prec_g y$. By the above lemma, we cannot have $g \in \Delta(\alpha)$. It follows that $g \in \Delta(\beta) \setminus \Delta(\alpha)$, so that $g >_{\beta} f$, completing the proof in the principal case.

We now prove Theorem 1 when the salience order on $\Delta(\alpha)$ is empty and the order is given by equation (3). Defining the α-membership degree of an object x by $\delta_\alpha(x) = \frac{1}{|\Delta(\alpha)|}\sum_{f\in\Delta(\alpha)} \delta_f(x)$, we have then $x \preceq^{\mu}_{\alpha} y$ iff $\delta_\alpha(x) \leq \delta_\alpha(y)$. It is easily seen that δ_α takes only a finite number of values in the unit interval. In this sense, α may be considered as a feature. This observation leads to the following construction:

Consider the set $\Delta(\beta \star \alpha) = \{\alpha\} \cup \Delta(\beta)$ equipped with the salience order $>_{\Delta(\beta\star\alpha)}$ that extends that of $\Delta(\beta)$ and satisfies moreover $\alpha >_{\Delta(\beta\star\alpha)} f$ for all $f \in \Delta(\beta) \setminus \alpha$. Denote by $\preceq^{\mu}_{\Delta(\beta\star\alpha)}$ the membership order induced by the concept $\beta \star \alpha$ with associated defining set $\Delta(\beta \star \alpha)$. We claim that $\preceq^{\mu}_{\Delta(\beta\star\alpha)} = \preceq^{\mu}_{\beta\star\alpha}$.

- We first prove that $\preceq^{\mu}_{\Delta(\beta\star\alpha)} \subseteq \preceq^{\mu}_{\beta\star\alpha}$. Suppose that $x \preceq^{\mu}_{\Delta(\beta\star\alpha)} y$. Note first that $x \preceq^{\mu}_{\alpha} y$: indeed, if this were not the case, we would have $y \prec^{\mu}_{\alpha} x$ because of the connexity of $\preceq^{\mu}_{\alpha}$. By definition of the order $\preceq^{\mu}_{\Delta(\beta\star\alpha)}$, this would imply the existence of a feature f of $\Delta(\beta\star\alpha)$ more salient than α, which is impossible.

  It remains to check that $x \preceq^{\mu}_{\beta} y$ if we do not have $x \prec^{\mu}_{\alpha} y$. Let f be an element of $\Delta(\beta)$ such that $y \prec_f x$. Since we supposed $x \preceq^{\mu}_{\Delta(\beta\star\alpha)} y$, we see that there exists an element g of $\{\alpha\} \cup \Delta(\beta)$, more salient than f, such that $x \prec_g y$. Since we do not have $x \prec^{\mu}_{\alpha} y$, we have $g \in \Delta(\beta)$, and we have shown that $x \preceq^{\mu}_{\beta} y$.

- Conversely, to prove that $\preceq^{\mu}_{\beta\star\alpha} \subseteq \preceq^{\mu}_{\Delta(\beta\star\alpha)}$, suppose that $x \preceq^{\mu}_{\beta\star\alpha} y$ so that $x \preceq^{\mu}_{\alpha} y$ and either $x \prec^{\mu}_{\alpha} y$ or $x \preceq^{\mu}_{\beta} y$. Let us check that $x \preceq^{\mu}_{\Delta(\beta\star\alpha)} y$. If $f \in \Delta(\beta \star \alpha)$ is such that $y \prec_f x$, we have necessarily $f \in \Delta(\beta) \setminus \{\alpha\}$. If $x \prec^{\mu}_{\alpha} y$, we are done, since $\alpha >_{\Delta(\beta\star\alpha)} f$. If $x \preceq^{\mu}_{\beta} y$, there exists $g \in \Delta(\beta)$, $g >_{\Delta(\beta\star\alpha)f}$ such that $x \prec_g y$ as desired. ■

Proposition 3 *Let α, β, and γ be three attributional concepts such that $Ext\,\alpha \cap Ext\,\beta \cap Ext\,\gamma \neq \emptyset$. Then*

- $\alpha \star \alpha \sim \alpha$
- $\gamma \star (\beta \star \alpha) \sim (\gamma \star \beta) \star \alpha$.

Proof:

- It follows from the definition of $\preceq^{\mu}_{\beta\star\alpha}$, that one has $\preceq^{\mu}_{\alpha\star\alpha} = \preceq^{\mu}_{\alpha}$, so that $\alpha \star \alpha \sim \alpha$.

- A straightforward computation shows that, given two elements x and y of $\mathcal{O}$, one has $x \preceq^{\mu}_{\gamma\star(\beta\star\alpha)} y$ if and only if $x \prec^{\mu}_{\alpha} y$, or $x \preceq^{\mu}_{\alpha} y$, $x \preceq^{\mu}_{\beta} y$ and either $x \prec^{\mu}_{\beta} y$ or $x \preceq^{\mu}_{\gamma} y$. This condition is readily seen to be equivalent to $x \preceq^{\mu}_{(\gamma\star\beta)\star\alpha} y$, showing that $\preceq^{\mu}_{\gamma\star(\beta\star\alpha)} = \preceq^{\mu}_{(\gamma\star\beta)\star\alpha}$. ■

Proposition 4 *One has $f \star \alpha \sim \alpha$ for every $f \in \Delta(\alpha)$.*

Proof: We have only to check that $\preceq^{\mu}_{\alpha} \subseteq \preceq^{\mu}_{f\star\alpha}$. Suppose therefore we have $x \preceq^{\mu}_{\alpha} y$ and that we have not $x \preceq_f y$. Then one has $y \prec_f x$. Since $f \in \Delta(\alpha)$, this implies that there exists $g \in \Delta(\alpha)$ such that $x \prec_g y$. By equation (2), it follows that $x \prec^{\mu}_{\alpha} y$. ■

Proposition 5 *Let f be a feature such that $f \star \alpha \sim \alpha$. Then f can be added to the set $\Delta(\alpha)$ without changing the order $\preceq^{\mu}_{\alpha}$.*

Proof: We can suppose $f \notin \Delta(\alpha)$. Define the set $\widetilde{\Delta}(\alpha) = \Delta(\alpha) \cup \{f\}$ equipped with the salience order $>_{\widetilde{\Delta}(\alpha)}$ that agrees with $>_{\Delta(\alpha)}$ on $\Delta(\alpha)$ and satisfies moreover $g >_{\widetilde{\Delta}(\alpha)} f$ for all $g \in \Delta(\alpha)$. Let $\preceq_{\widetilde{\Delta}(\alpha)}$ be the membership order induced by the set of features $\widetilde{\Delta}(\alpha)$. We claim that $\preceq_{\widetilde{\Delta}(\alpha)} = \preceq^{\mu}_{\alpha}$.

We have readily $\preceq_{\widetilde{\Delta}(\alpha)} \subseteq \preceq^{\mu}_{\alpha}$. For the converse inclusion, suppose that $x \preceq^{\mu}_{\alpha} y$. Let $g \in \widetilde{\Delta}(\alpha)$ be such that $y \prec_g x$. We have to show that there exists $h \in \widetilde{\Delta}(\alpha)$, $h >_{\widetilde{\Delta}(\alpha)} g$ such that $x \prec_h y$.

- If $g \in \Delta(\alpha)$, the hypothesis implies that there is an element $h \in \Delta(\alpha)$, h more salient than g such that $x \prec_h y$. By the construction of $>_{\widetilde{\Delta}(\alpha)}$, this implies that $h >_{\widetilde{\Delta}(\alpha)} g$.

- If $g = f$, we have $y \prec_f x$. From our hypothesis, since we supposed $x \preceq^{\mu}_{f\star\alpha} y$, we must have $x \prec^{\mu}_{\alpha} y$. This implies by equation (2) that there exits a defining feature h such that $x \prec_h y$. By construction, we have then $h >_{\widetilde{\Delta}(\alpha)} g$.

We have therefore proven the equality $\preceq_{\widetilde{\Delta}(\alpha)} = \preceq^{\mu}_{\alpha}$, showing that the membership order $\preceq^{\mu}_{\alpha}$ remains unchanged if f is added to the set of defining features. ■

Proposition 6 *The extension of $(\beta\star\alpha)$ is the set of $\preceq^{\mu}_{\beta\star\alpha}$-maximal elements of $\mathcal{O}$. It satisfies $Ext\,(\beta\star\alpha) = Ext\,\alpha \cap Ext\,\beta$.*

Proof: We have $x \in Ext\,(\beta \star \alpha)$ iff $\delta^\alpha_\phi(x) = 1\,\forall \phi \in \Delta(\beta \star \alpha)$, that is iff $\delta^\alpha_f(x) = 1\,\forall f \in \Delta(\alpha) \cup \Delta(\beta)$. This shows that $Ext\,(\beta \star \alpha) = Ext\,\alpha \cap Ext\,\beta$.

To prove the second part of the proposition, let $x \in Ext\,\alpha \cap Ext\,\beta$. Then x is $\prec^\mu_\alpha$ and $\prec^\mu_\beta$-maximal in $\mathcal{O}$. If x were not $\preceq^\mu_{\beta\star\alpha}$-maximal, there would exist an object y such that $x \prec^\mu_{\beta\star\alpha} y$. Therefore, we would have either $x \prec^\mu_\alpha y$ or $x \prec^\mu_\beta y$, a contradiction.

Conversely suppose that x is $\preceq^\mu_{\beta\star\alpha}$-maximal. Then x is clearly $\prec^\mu_\alpha$-maximal and therefore an element of $Ext\,\alpha$. If x were not $\prec^\mu_\beta$-maximal, we would have $x \prec^\mu_{\beta\star\alpha} z$ for any element z of the (non-empty) set $Ext\,\alpha \cap Ext\,\beta$, contradicting the choice of x. ■

Proposition 7 *Let α and γ be two attributional concepts. Then the following conditions are equivalent:*

1. $\gamma \sim (\gamma \star \alpha)$
2. *There exists a concept β such that $\gamma \sim (\beta \star \alpha)$*
3. *γ is a regular subconcept of α such that $\prec^\mu_\alpha \subseteq \prec^\mu_\gamma$.*

Proof: We have clearly 1) $\Rightarrow$ 2). To prove that 2) $\Rightarrow$ 3), note first that if $\gamma \sim (\beta \star \alpha)$, then $\preceq^\mu_\gamma = \preceq^\mu_{\beta\star\alpha} \subseteq \preceq^\mu_\alpha$, showing that γ is a regular subconcept of α. It remains to prove that $\prec^\mu_\alpha \subseteq \prec^\mu_\gamma$. Let x and y be two objects such that $x \prec^\mu_\alpha y$. Il follows from the definition of $\preceq^\mu_{\beta\star\alpha}$ that $x \preceq^\mu_{\beta\star\alpha} y$, that is $x \preceq^\mu_\gamma y$. We cannot have $y \preceq^\mu_\gamma x$ because this would imply $y \preceq^\mu_\alpha x$, contradicting the choice of x and y. We have therefore $x \prec^\mu_\gamma y$, as desired.

To prove that 3) $\Rightarrow$ 1), suppose that $\preceq^\mu_\gamma \subseteq \preceq^\mu_\alpha$ and $\prec^\mu_\alpha \subseteq \prec^\mu_\gamma$. We have to show that $\preceq^\mu_\gamma = \preceq^\mu_{\gamma\star\alpha}$. If one has $x \preceq^\mu_\gamma y$ for two objects x and y, the hypothesis implies that $x \preceq^\mu_\alpha y$. This together with the fact that $x \preceq^\mu_\gamma y$ implies $x \preceq^\mu_{\gamma\star\alpha} y$, so that $\preceq^\mu_\gamma \subseteq \preceq^\mu_{\gamma\star\alpha}$.

Conversely, suppose that $x \preceq^\mu_{\gamma\star\alpha} y$. This implies $x \preceq^\mu_\alpha y$, and either $x \prec_\alpha y$ - and therefore $x \prec^\mu_\gamma y$ by 3) - or $x \preceq^\mu_\gamma y$. In any case, we have $x \preceq^\mu_\gamma y$, and the proof of the proposition is completed. ■

Proposition 8 *The relation $\preceq^\tau_\alpha$ is a partial weak order on $Ext\,\alpha$. Its maximal elements are the typical instances of α. One has $z \prec^\tau_\alpha x$ for any $x \in Typ\,\alpha$ and $z \notin Typ\,\alpha$.*

Proof: Similar to the proof of Propositions 1 and 2. ■

Proposition 9 Let α, β, γ be three arbitrary featured concepts such that $Ext\,\alpha \cap Ext\,\beta \cap Ext\,\gamma \neq \emptyset$. Then

- $\alpha \star \alpha \equiv \alpha$.
- $\gamma \star (\beta \star \alpha) \equiv (\gamma \star \beta) \star \alpha$.
- If $\alpha \equiv \beta$, then $\gamma \star \alpha \equiv \gamma \star \beta$

Proof:

- We already observed (proposition 3) that $\preceq^{\mu}_{\alpha\star\alpha} = \preceq^{\mu}_{\alpha}$. Similarly, we have $\preceq^{\tau}_{\alpha\star\alpha} = \preceq^{\tau}_{\alpha}$, which proves the result.
- Since, by Proposition 3 we know that $\gamma\star(\beta\star\alpha) \sim (\gamma\star\beta)\star\alpha$, the second part of Proposition 9 amounts to proving that $\preceq^{\tau}_{\gamma\star(\beta\star\alpha)} = \preceq^{\tau}_{(\gamma\star\beta)\star\alpha)}$. A straightforward computation shows that, given two elements x and y of $Ext\,\alpha \cap Ext\,\beta \cap Ext\,\gamma$ one has $x \preceq^{\tau}_{\gamma\star(\beta\star\alpha)} y$ if and only if $x \prec^{\tau}_{\alpha} y$ or $x \preceq^{\tau}_{\alpha} y$, $x \preceq^{\tau}_{\beta} y$ and either $x \prec^{\tau}_{\beta} y$ or $x \preceq^{\tau}_{\gamma} y$. This condition is readily seen to be equivalent to $x \preceq^{\tau}_{(\gamma\star\beta)\star\alpha)} y$.
- The last part of the proposition is a direct consequence of the definition of $\preceq^{\mu}_{\gamma\star\alpha}$ and $\preceq^{\tau}_{\gamma\star\alpha}$. ■

Proposition 10 $Typ(\beta \star \alpha) \subseteq Typ\,\alpha$ whenever $Typ\,\alpha \cap Ext\,\beta \neq \emptyset$. In this case one has $z \prec^{\tau}_{\alpha} x$ for any $x \in Typ(\beta \star \alpha)$ and $z \in Ext\,(\beta \star \alpha) \setminus Typ(\alpha)$.

Proof: Let x be an element of $Typ(\beta \star \alpha)$. Note that $x \in Ext\,\alpha \cap Ext\,\beta$. Suppose $x \notin Typ\,\alpha$. By hypothesis, there exists an object $t \in Typ\,\alpha \cap Ext\,\beta$, and it follows from Proposition 8 that one has $x \prec^{\tau}_{\alpha} t$. But this readily implies $x \prec^{\tau}_{\beta\star\alpha} t$, contradicting the choice of x.

We have therefore $x \in Typ\,\alpha$. Since $z \notin Typ\,\alpha$, this implies, again by Proposition 8, $z \prec^{\tau}_{\alpha} x$, whence $z \prec^{\tau}_{\beta\star\alpha} x$, as desired. ■

Theorem 2 *The equality $Typ\,(\beta \star \alpha) = Typ\,\beta \cap Typ\,\alpha$ holds if and only if $Typ\,\beta \cap Typ\,\alpha \neq \emptyset$. When this is the case, the concept $\beta \star \alpha$ is equivalent to* *a featured concept whose characteristic set is equal, in the principal case, to* $\chi(\alpha) \cup \chi(\beta)$.

Proof: Since $Typ\,(\beta \star \alpha) \neq \emptyset$, the first part of the proposition amounts to proving that $Typ\,\beta \cap Typ\,\alpha = Typ\,(\beta \star \alpha)$ whenever $Typ\,\beta \cap Typ\,\alpha \neq \emptyset$.

- Let $x \in Typ\,\beta \cap Typ\,\alpha$. We have to show that x is $\preceq^{\tau}_{\beta\star\alpha}$-maximal. If this were not the case, we would have $x \prec^{\tau}_{\beta\star\alpha} z$ or for some element z of $Ext\,\alpha \cap Ext\,\beta$, that is $x \prec^{\tau}_{\alpha} z$ or $x \prec^{\tau}_{\beta} z$, contradicting the choice of x.

- Conversely let y be an element of $Typ\,(\beta \star \alpha)$, that is a $\prec^{\tau}_{\beta\star\alpha}$-maximal element of $Ext\,\alpha \cap Ext\,\beta$. By hypothesis, there exists an element z of $Typ\,\beta \cap Typ\,\alpha$. We have therefore $\delta_f(z) = 1\ \forall f \in \chi(\alpha) \cup \chi(\beta)$. If y were not an element of $Typ\,\beta \cap Typ\,\alpha$, we would have $\delta_f(y) < 1$ for some $f \in \chi(\alpha) \cup \chi(\beta)$. This would imply $y \prec^{\tau}_{\alpha} z$ or $y \prec^{\tau}_{\beta} z$, contradicting the $\prec^{\tau}_{\beta\star\alpha}$-maximality of y.

To prove that $\beta\star\alpha$ is equivalent to a featured concept, we distinguish between the principal case and the exceptional one, and accordingly define the set $\chi(\beta\star\alpha)$ in a way similar to the construction of the sets $\Delta(\beta\star\alpha)$ in the proof of Theorem 1. The proof that $\preceq^{\tau}_{\chi(\beta\star\alpha)} = \preceq^{\tau}_{\beta\star\alpha}$ then exactly parallels that of Theorem 1. The fact that $Typ\,\beta \cap Typ\,\alpha \neq \emptyset$ ensures that condition 2 of Definition 5 is satisfied, so that the concept $\beta\star\alpha$ equipped with the defining set $\Delta(\beta\star\alpha)$ and the characteristic set $\chi(\beta\star\alpha)$ is a featured concept. ∎

Proposition 11 *Given two featured concepts* α *and* γ*, the following conditions are equivalent:*

1. $\gamma \equiv (\gamma \star \alpha)$.
2. *There exists a concept* β *such that* $\gamma \equiv (\beta \star \alpha)$.

Proof: Suppose that 2 holds, so that we have $\gamma \equiv (\beta \star \alpha)$ for some concept β. By proposition 9, we get $\gamma \star \alpha \equiv \beta \star (\alpha \star \alpha) \equiv (\beta \star \alpha) \equiv \gamma$, which proves the proposition. ∎

Proposition 12 *Set* $\epsilon(\alpha) = \frac{|\Delta(\alpha)|}{\Sigma_{f\in\Pi(\alpha)} s_\alpha(f)}$*, and let* x *be an object such that*

$\delta^\rho_\alpha(x) \geq 1 - \epsilon(\alpha)$. *Then all the characteristic features of α apply at least partially to x.*

Proof: Let h be an element of $\chi(\alpha)$. By construction, we have $0 < s\alpha(h) - |\Delta(\alpha)|$, that is $\Sigma_{f\in\Pi(\alpha) f\neq h} s_\alpha(f) < \Sigma_{f\in\Pi(\alpha)} s_\alpha(f) - |\Delta(\alpha)|$. If $\delta_h(x) = 0$, this would imply $\frac{\Sigma_{f\in\ \Pi(\alpha)} s_f \delta_f(x)}{\Sigma_{f\in\Pi(\alpha)} s_\alpha(f)} < 1 - \frac{|\Delta(\alpha)|}{\Sigma_{f\in\Pi(\alpha)} s_\alpha(f)}$, and therefore $\delta^\rho_\alpha(x) < 1 - \epsilon(\alpha)$, contradicting the choice of x. ■

Proposition 13 *One has* $\delta^\rho_{\beta\star\alpha}(x) \geq \frac{\Sigma_{f\in\Pi(\alpha)} s_\alpha(f)}{\Sigma_{f\in\Pi(\beta\star\alpha)} s_{(\beta\star\alpha)}(f)}$ *whenever* $x \in Typ\,\alpha$.

Proof: We know from the proof of Theorem 1 and Theorem 2 that the salience orders on $\Delta(\beta\star\alpha)$ and $\chi(\beta\star\alpha)$ extend the salience orders on $\Delta(\alpha)$ and $\chi(\alpha)$. By definition of the salience degrees s_α and $s_{(\beta\star\alpha)}$, this implies $s_{(\beta\star\alpha)}(f) \geq s_\alpha(f)\,\forall f \in \Delta(\alpha)$. Similarly, one has $s_\alpha(f) \leq s_{(\beta\star\alpha)}(f)$ for all elements of $\chi(\alpha)$. If $x \in Typ\,\alpha$, all defining and characteristic features of α apply to x, and we have therefore $\Sigma_{f\in\ \Pi(\beta\star\alpha)} s_{\beta\star\alpha}(f)\delta_f(x) \geq \Sigma_{f\in\Pi(\alpha)} s_\alpha(f)$. ■

Proposition 14 *Suppose that* $Typ\,(\beta\star\alpha) = Typ\,\beta \cap Typ\,\alpha$. *Then, for any smooth subconcept* γ *of* α, *one has* $\delta^\rho_{\beta\star\alpha}(\gamma) \geq \frac{\Sigma_{f\in\Pi(\alpha)} s_\alpha(f)}{\Sigma_{f\in\Pi(\beta\star\alpha)} s_{\beta\star\alpha}(f)}$

Proof: Immediate from Proposition 13. ■

Proposition 15 *Set* $\alpha \vdash \beta$ *iff* $\beta \in Ess\,\alpha$. *The relation* $\vdash$ *thus defined is* reflexive *and* transitive. *Furthermore, it is* monotonic, *in the sense that* $\alpha \vdash \beta$ *implies* $\gamma\star\alpha \vdash \beta$ *and* $\alpha\star\kappa \vdash \beta$ *for all concepts* γ *and* κ *for which these expressions are meaningful.*

Proof: Straightforward, since $\alpha \vdash \beta \Leftrightarrow \beta \in Ess\,\alpha \Leftrightarrow Ext\,\alpha \subseteq Ext\,\beta$. ■

Proposition 16 *Denote by* $\mid\!\sim$ *the relation:* $\alpha \mid\!\sim \beta$ *iff* $Typ\,\alpha \subseteq Ext\,\beta$. *Let* α, β *and* γ *be three featured concepts such that* $Ext\,\alpha \cap Ext\,\beta \cap Ext\,\gamma \neq \emptyset$. *Then the following properties hold:*

- $\frac{\alpha\vdash\beta}{\alpha\mid\!\sim\beta}$ ***(Supraclassicality)***

- $\frac{\alpha \equiv \alpha', \beta \equiv \beta', \alpha \mid\!\sim \beta}{\alpha' \mid\!\sim \beta'}$ ***(Logical Equivalence)***

- $\frac{\alpha \equiv \beta, \alpha \mid\!\sim \gamma}{\beta \mid\!\sim \gamma}$ ***(Left Logical Equivalence)***

- $\frac{\alpha \mid\!\sim \beta, \beta \vdash \gamma}{\alpha \mid\!\sim \gamma}$ ***(Right weakening)***

- $\frac{\alpha \mid\!\sim \beta, \alpha \mid\!\sim \gamma}{\alpha \mid\!\sim \beta \star \gamma}$ ***(And)***

- $\frac{\alpha \mid\!\sim \beta, \alpha \mid\!\sim \gamma}{\beta \star \alpha \mid\!\sim \gamma}$ ***(Cautious Monotonicity)***.

Proof: To prove Cautious Monotonicity, note that the hypothesis $\alpha \mid\!\sim \beta$ implies $Typ\,\alpha \subseteq Ext\,\beta$, so that $Typ\,\alpha \cap Ext\,\beta \neq \emptyset$. By proposition 10, we have then $Typ\,(\beta \star \alpha) \subseteq Typ\,\alpha$. Since, by hypothesis, $Typ\,\alpha \subseteq Ext\,\gamma$, the conclusion follows. ■

Proposition 17 *The following rule is valid in the proposed model:*

- $\frac{\alpha \mid\!\sim \gamma, \alpha \not\mid\!\sim \neg\beta}{\beta \star \alpha \mid\!\sim \gamma}$ ***(Rational Monotony)***

Proof: Since β is not exceptional for α, we have $Typ\,\alpha \cap Ext\,\beta \neq \emptyset$, and we conclude again with proposition 10. ■

Proposition 18 *Suppose that for a characteristic feature k of α, one has $Ext\,\beta \cap Ext\,\alpha \cap Ext\,k = \emptyset$, but $Ext\,\beta \cap Ext\,\alpha \cap (\bigcap (Ext\,f)_{f \in \chi(\alpha) - \{k\}}) \neq \emptyset$. Suppose also that the applicability function δ_k, induced by k is a two-valued function. Then any characteristic feature of α different from k remains a typical attribute of $\beta \star \alpha$.*

Proof: Let $f \in \chi(\alpha)$, $f \neq k$. We have to prove that $Typ\,(\beta \star \alpha) \subseteq Ext\,f$. If this were not the case, there would exist an element $x \in Typ\,(\beta \star \alpha)$ such that $\delta_f(x) < 1$. Note that the hypothesis implies that $\delta_k(x) = 0$. For any element y of $Ext\,\beta \cap Ext\,\alpha \cap (\bigcap (Ext\,g)_{g \in \chi(\alpha) - \{k\}})$, we would then have $x \prec^{\tau}_{\alpha} y$, and therefore $x \prec^{\tau}_{\beta \star \alpha} y$, contradicting the $\preceq^{\tau}_{\beta \star \alpha}$-maximality of x in $Ext\,(\beta \star \alpha)$. ■

Proposition 19 *The following rule is valid in the proposed model:*

- $\frac{\beta\star\alpha\vdash\gamma,\, \alpha|\!\sim\beta}{\alpha|\!\sim\gamma}$ ***(Cautious Cut)***

Proof: Straightforward, since one has $Typ\,\alpha \subseteq Ext\,\beta$, and therefore $Typ\,\alpha \subseteq Ext\,(\beta \star \alpha)$. ■

Proposition 20 *The rule **Cut** holds for any concept determined by one of its typical attributes:*

- $\frac{\beta\in Int(\alpha),\ \beta\star\alpha|\!\sim\gamma}{\alpha|\!\sim\gamma}$

Proof: If $\beta \in Int\,\alpha$, all typical instances of α fall under β, so that $Typ\,\alpha \subseteq Ext\,\beta$. Since β is a feature, one has $\chi(\beta) = \Delta(\beta) = \{\beta\}$, and therefore $Typ\,\beta = Ext\,\beta$. ByTheorem 2, this yields $Typ(\beta \star \alpha) = Typ\,\alpha$, whence the result.

■

Proposition 21 *Let β be a typical subconcept of α, and κ an essential attribute of β. Then κ is a typical attribute of α if and only if κ is non-specific to β*

Proof: Suppose first that κ is an typical attribute of α. For any typical subconcept γ of α, we have $Ext\,\gamma \subseteq Typ\,\alpha$, showing that κ is an *essential* attribute of γ, so that κ is non-specific to β.

Conversely, suppose that κ is a non-specific attribute of β. Denote by $\chi_1, \chi_2, \ldots \chi_n$ the characteristic attributes of α, and let γ be the 'pseudo-concept' $\gamma = \chi_1 \star \chi_2 \star \ldots \star \chi_n \star \alpha$. Note that $Ext\,\gamma = Typ\,\alpha$, showing that γ is a typical subconcept of α. It follows that κ applies to $Ext\,\gamma$, hence to $Typ\,\alpha$. This proves that κ is a typical attribute of α.

We may see the concept γ thus defined as the concept *to-be-a-typical-α*, paralleling that of *typical object* introduced by Desclés [4].

Proposition 22 $\frac{\alpha|\!\sim\gamma,\ \delta^{\rho}_{\alpha}(\beta)=1}{\beta|\!\sim\gamma}$

Proof: Immediate.

Proposition 23 *Let $h \star \alpha$ the determination of a featured concept α by* *a feature whose applicability is given by a two-valued function. Let* $\epsilon(h \star \alpha)$ *be the number* $\frac{|\Delta(h\star\alpha)|}{\Sigma_{f\in\Pi(h\star\alpha)} s_{h\star\alpha}(f)}$. *Then the following rule is valid:*

- $\dfrac{\delta^{\rho}_{h\star\alpha}(\beta)\geq 1-\epsilon(h\star\alpha),\ \alpha\not\mid\!\sim\neg h}{\beta\mid\!\sim h}$

Proof: Note that the hypothesis $\alpha\not\mid\!\sim\neg h$ ensures that $Typ\,\alpha \cap Ext\,h \neq \emptyset$. Note also that, since h is a feature, we have $\Delta(h) = \chi(h) = \{h\}$, and consequently $Typ\,h = Ext\,h$, showing that $Typ\,\alpha \cap Typ\,h \neq \emptyset$. The notion of $h \star \alpha$-resemblance is therefore well defined, and we can apply the results of section 5.2.1, considering $h \star \alpha$ as a featured concept with characteristic set $\chi(h \star \alpha) = \{h\} \cup \chi(\alpha)$.

Let now x be an element of $Typ\,\beta$. Supposing that $\delta^{\rho}_{h\star\alpha}(\beta) \geq 1 - \epsilon(h \star \alpha)$, we have $\delta^{\rho}_{h\star\alpha}(x) \geq 1 - \epsilon(h \star \alpha)$. Since h is a characteristic feature of $h \star \alpha$, we know by proposition 12 that $x \in Ext\,h$. This being true for all elements of $Typ\,(\beta)$, we see that $Typ\,(\beta) \subseteq Ext\,h$, and we have therefore $\beta\mid\!\sim h$ as desired. ■

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