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Abstract

In this paper, we study an inverse Steklov problem in a class of n-dimensional manifolds having the topology of a hollow sphere and equipped with a warped product metric. Precisely, we aim at studying the continuous dependence of the warping function defining the warped product with respect to the Steklov spectrum. We first show that the knowledge of the Steklov spectrum up to an exponential decreasing error is enough to determine uniquely the warping function in a neighbourhood of the boundary. Second, when the warping functions are symmetric with respect to $1/2$, we prove a log-type stability estimate in the inverse Steklov problem. As a last result, we prove a log-type stability estimate for the corresponding Calderón problem.

Keywords. Inverse Calderón problem, Steklov spectrum, Weyl-Titchmarsh functions, Nevanlinna theorem, Müntz-Jackson's theorem.

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1 Introduction

1.1 Framework

For $n \geq 2$, let us consider a class of n-dimensional manifolds $M = [0,1] \times \mathbb{S}^{n-1}$ equipped with a warped product metric

$$g = f(x)(dx^2 + g_{\mathbb{S}})$$

where $g_{\mathbb{S}}$ denotes the usual metric on $\mathbb{S}^{n-1}$ induced by the euclidean metric on $\mathbb{R}^n$ and f is a smooth and positive function on $[0,1]$. Let ψ belong to $H^{1/2}(\partial M)$ and ω be in $\mathbb{R}$.

The Dirichlet problem is the following elliptic equation with boundary condition

$$\begin{cases} -\Delta_g u = \omega u \text{ in } M \\ u = \psi \text{ on } \partial M, \end{cases} \tag{1}$$

where, in a local coordinate system $(x_i)_{i=1,\dots,n}$, and setting $|g| = \det(g_{ij})$ and $(g^{ij}) = (g_{ij})^{-1}$, the Laplacian operator $-\Delta_g$ has the expression

$$-\Delta_g = -\sum_{1\leq i,j\leq n} \frac{1}{\sqrt{|g|}} \partial_i \big(\sqrt{|g|} g^{ij} \partial_j \big).$$

If ω does not belong to the Dirichlet spectrum of $-\Delta_g$, the equation (1) has a unique solution in $H^1(M)$ (see [11, 14]), so we can define the so-called *Dirichlet-to-Neumann* operator ("DN map") $\Lambda_g(\omega)$ as :

$$\begin{array}{ccccc} \Lambda_g(\omega) & : & H^{1/2}(\partial M) & \to & H^{-1/2}(\partial M) \\ & & \psi & \mapsto & \frac{\partial u}{\partial \nu}|_{\partial M} \end{array}$$

where ν is the unit outer normal vector on ∂M. The previous definition has to be understood in the weak sense by:

$$\forall (\psi,\phi) \in H^{1/2}(\partial M)^2 \ : \ \langle \Lambda_g(\omega)\psi, \phi\rangle = \int_M \langle du, dv\rangle_g, \ \mathrm{dVol}_g + \omega \int_M uv \, \mathrm{dVol}_g. \tag{2}$$

where v is any element of $H^1(M)$ such that $v|_{\partial M} = \phi$, $\langle .,. \rangle$ is the standard L^2 duality pairing between $H^{1/2}(\partial M)$ and its dual, and dVol_g is the volume form induced by g on M.

The DN map $\Lambda_g(\omega)$ is a self-adjoint pseudodifferential operator of order one on $L^2(\partial M)$. Then, it has a real and discrete spectrum accumulating at infinity. We shall denote the Steklov eigenvalues counted with multiplicity by

$$\sigma(\Lambda_g(\omega)) = \{0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k \to +\infty\},$$

usually called the *Steklov spectrum* (see [8], p.2 or [7] for details).

The inverse Steklov problem adresses the question whether the knowledge of the Steklov spectrum is enough to recover the metric g. Precisely:

If $\sigma\big(\Lambda_g(\omega)\big) = \sigma\big(\Lambda_{\tilde{g}}(\omega)\big)$, *is it true that* $g = \tilde{g}$ *?*

It is known that the answer is negative because of some gauge invariances in the Steklov problem. These gauge invariances are (see [8]):

1) Invariance under pullback of the metric by the diffeomorphisms of M :

$$\forall \psi \in \mathrm{Diff}(M), \quad \Lambda_{\psi^* g}(\lambda) = \varphi^* \circ \Lambda_g(\lambda) \circ \varphi^{*-1}.$$

where $\varphi := \psi|_{\partial M}$ and where $\varphi^* : C^\infty(\partial M) \to C^\infty(\partial M)$ is the application defined by $\varphi^* h := h \circ \varphi$.

2) In dimension $n = 2$ and for $\omega = 0$, there is one additional gauge invariance. Indeed, thanks to the conformal invariance of the Laplacian, for every smooth function $c > 0$, we have

$$\Delta_{cg} = \frac{1}{c} \Delta_g.$$

Consequently, the solutions of the Dirichlet problem (1) associated to the metrics g and cg are the same when $\omega = 0$. Moreover, if we assume that $c \equiv 1$ on the boundary, the unit outer normal vectors on ∂M are also the same for both metrics. Therefore,

$$\Lambda_{cg}(0) = \Lambda_g(0).$$

In our particular model, we have shown in [6] that the only gauge invariance is given by the involution $\eta : x \mapsto 1 - x$ when $n = 2$ and also when $n \geq 3$ under some technical estimates on the conformal factor f on the boundary. Precisely, we proved:

Theorem 1.1. *Let $M = [0, 1] \times \mathbb{S}^{n-1}$ be a smooth Riemannian manifold equipped with the metrics*

$$g = f(x)(dx^2 + g_{\mathbb{S}}), \quad \tilde{g} = \tilde{f}(x)(dx^2 + g_{\mathbb{S}}),$$

and let ω be a frequency not belonging to the Dirichlet spectrum of $-\Delta_g$ and $-\Delta_{\tilde{g}}$ on M. Then,

1. *For $n = 2$ and $\omega \neq 0$,*

$$\big(\sigma(\Lambda_g(\omega)) = \sigma(\Lambda_{\tilde{g}}(\omega))\big) \Leftrightarrow \big(f = \tilde{f} \quad \text{or} \quad f = \tilde{f} \circ \eta\big)$$

   where $\eta(x) = 1 - x$ for all $x \in [0, 1]$.

2. *For $n \geq 3$, and if moreover*

$$f, \tilde{f} \in \mathcal{C}_b := \left\{ f \in C^\infty([0, 1]), \ \left| \frac{f'(k)}{f(k)} \right| \leq \frac{1}{n-2}, \ k = 0 \text{ and } 1 \right\},$$

$$\big(\sigma(\Lambda_g(\omega)) = \sigma(\Lambda_{\tilde{g}}(\omega))\big) \Leftrightarrow \big(f = \tilde{f} \quad \text{or} \quad f = \tilde{f} \circ \eta\big)$$

Remark 1. In Theorem 1.1, there is no need to assume that $\omega \neq 0$ when $n \geq 3$ whereas this condition is necessary in dimension 2, due to the gauge invariance by a conformal factor.

In this paper, we will show two additional results on the Steklov inverse problem, that follow and precise the question of uniqueness. Namely, we will prove some *local uniqueness* and *stability* results. Before stating our results, recall that the boundary ∂M of M has two connected components. If we denote $-\Delta_{g_{\mathbb{S}}}$ the Laplace-Beltrami operator on $(\mathbb{S}^{n-1}, g_{\mathbb{S}})$ and

$$\sigma(-\Delta_{g_{\mathbb{S}}}) := \{0 = \mu_0 < \mu_1 \leq \mu_2 \leq ... \leq \mu_m \leq ... \to +\infty\}$$

the sequence of the eigenvalues of $-\Delta_{g_{\mathbb{S}}}$, counted with multiplicity, one can show that the spectrum of $\Lambda_g(\omega)$ is made of two sets of eigenvalues $\{\lambda^-(\mu_m)\}$ and $\{\lambda^+(\mu_m)\}$ whose asymptotics are given later in Lemma 2.1.

1.2 Closeness of two Steklov spectra

Let us define first what is the *closeness* between two spectra $\sigma\big(\Lambda_g(\omega)\big)$ and $\sigma\big(\Lambda_{\tilde{g}}(\omega)\big)$.

Definition 1.2. Let $(\varepsilon_m)_m$ a sequence of positive numbers. We say that $\sigma\big(\Lambda_g(\omega)\big)$ is close to $\sigma\big(\Lambda_{\tilde{g}}(\omega)\big)$ up to the sequence $(\varepsilon_m)_m$ if, for every $\lambda^\pm(\mu_m) \in \sigma\big(\Lambda_g(\omega)\big)$:

- there is $\tilde{\lambda}^\pm$ in $\sigma\big(\Lambda_{\tilde{g}}(\omega)\big)$ such that $|\lambda^\pm(\mu_m) - \tilde{\lambda}^\pm| \leq \varepsilon_m$.

- $\mathrm{Card}\big\{\lambda^{\pm} \in \sigma\big(\Lambda_g(\omega)\big),\ |\lambda^{\pm}(\mu_m) - \lambda^{\pm}| \leq \varepsilon_m\big\} = \mathrm{Card}\big\{\tilde{\lambda}^{\pm} \in \sigma\big(\Lambda_{\tilde{g}}(\omega)\big),\ |\lambda^{\pm}(\mu_m) - \tilde{\lambda}^{\pm}| \leq \varepsilon_m\big\}$.

where Card A is the cardinal of the set A. We denote it

$$\sigma\big(\Lambda_g(\omega)\big) \underset{(\varepsilon_m)}{\lesssim} \sigma\big(\Lambda_{\tilde{g}}(\omega)\big).$$

Remark 2. The second point of Definition 1.2 amounts to taking into account the multiplicity of the eigenvalues.

Definition 1.3. We say that $\sigma\big(\Lambda_g(\omega)\big)$ and $\sigma\big(\Lambda_{\tilde{g}}(\omega)\big)$ are close up to $(\varepsilon_m)_m$ if

$$\sigma\big(\Lambda_g(\omega)\big) \underset{(\varepsilon_m)}{\lesssim} \sigma\big(\Lambda_{\tilde{g}}(\omega)\big) \quad \text{and} \quad \sigma\big(\Lambda_{\tilde{g}}(\omega)\big) \underset{(\varepsilon_m)}{\lesssim} \sigma\big(\Lambda_g(\omega)\big).$$

We denote it $\sigma\big(\Lambda_g(\omega)\big) \underset{(\varepsilon_m)}{\asymp} \sigma\big(\Lambda_{\tilde{g}}(\omega)\big)$.

Constant sequence : if (ε_m) is a constant sequence such that, for all m, $\varepsilon_m = \varepsilon$, we just denote

$$\sigma\big(\Lambda_g(\omega)\big) \underset{\varepsilon}{\asymp} \sigma\big(\Lambda_{\tilde{g}}(\omega)\big).$$

Definition 1.4. If A and $\tilde{A}$ are any finite subset of $\mathbb{R}$, we will denote $A \underset{\varepsilon}{\lesssim} \tilde{A}$ if, for every $a \in A$

- There is $\tilde{a} \in \tilde{A}$ such that $|a - \tilde{a}| \leq \varepsilon$,
- $\mathrm{Card}\big\{\lambda \in A,\ \ |\lambda - a| \leq \varepsilon\big\} = \mathrm{Card}\big\{\tilde{\lambda} \in \tilde{A},\ \ |a - \tilde{\lambda}| \leq \varepsilon\big\}$.

We denote $A \underset{\varepsilon}{\asymp} \tilde{A}$ if $A \underset{\varepsilon}{\lesssim} \tilde{A}$ and $\tilde{A} \underset{\varepsilon}{\lesssim} A$.

This work is based on ideas developped by Daudé, Kamran and Nicoleau in [5]. However, due to the specific structure of our model that possesses a *disconnected* boundary (contrary to the model studied in [5]), some new difficulties arise.

Local uniqueness. We would like to answer the following question : if the data of the Steklov spectrum is known up to some exponentially decreasing sequence, is it possible to recover the conformal factor f in the neigbourhood of the boundary (or one of its component) up to a natural gauge invariance ? The main difficulty that appears here is due to the presence of two sets of eigenvalues, in each spectrum $\sigma\big(\Lambda_g(\omega)\big)$ and $\sigma\big(\Lambda_{\tilde{g}}(\omega)\big)$, instead of one as in [5]. With the previous definitions of closeness, it is not clear that we can get, for example, this kind of implication :

$$\Big(\ \sigma\big(\Lambda_g(\omega)\big) \text{ and } \sigma\big(\Lambda_{\tilde{g}}(\omega)\big) \text{ close }\Big) \Rightarrow \Big(\ \lambda^-(\mu_m) \text{ and } \tilde{\lambda}^-(\mu_m) \text{ close for all } m \in \mathbb{N}\ \Big)$$
$$\text{or } \Big(\ \lambda^-(\mu_m) \text{ and } \tilde{\lambda}^+(\mu_m) \text{ close for all } m \in \mathbb{N}\ \Big).$$

In order to overcome this problem, we will need to do some hypotheses on the warping function f to introduce a kind of asymmetry on the metric on each component. In that way, the previous implication will be true by replacing $\mathbb{N}$ with an infinite subset $\mathcal{L} \subset \mathbb{N}$ that satisfies, for m large enough, $\mathcal{L} \cap \{m, m+1\} \neq \emptyset$. In other word, the frequency of integers belonging to $\mathcal{L}$ will be greater than $1/2$.

Stability. As regards the problem of stability, if the Steklov eigenvalues are known up to a positive, fixed and small ε, is it possible to find an approximation of the conformal factor f depending on ε ? Thanks to Theorem 1.1, we know that there is no uniqueness in the problem of recovering f from $\sigma\big(\Lambda_g(\omega)\big)$. This seems to be a serious obstruction to establish any result of stability in a general framework. Indeed, the uniqueness problem solved in Theorem 1.1 is quite rigid (as well as the local uniqueness result) and is based on analyticity results that can no longer be used here. On the contrary, the condition

$$\sigma\big(\Lambda_g(\omega)\big) \underset{\varepsilon}{\asymp} \sigma\big(\Lambda_{\tilde{g}}(\omega)\big)$$

seems much less restrictive than an equality, and the non-uniqueness makes this new problem quite difficult to tackle. From Theorem 1.1, we see that the only way to get a strict uniqueness result is to assume that f is symmetric with respect to $1/2$. This natural - albeit restrictive - condition will be made on f in Section 4 devoted to the stability result.

1.3 The main results

Definition 1.5. The class of functions $\mathcal{D}_b$ is defined by

$$\mathcal{D}_b = \{h \in C^\infty([0,1]) \;\big|\; \exists k \in \mathbb{N},\; h^{(k)}(0) \neq (-1)^k h^{(k)}(1)\}.$$

Definition 1.6. The potential associated to the conformal factor f is the function q_f defined on $[0,1]$ by $q_f = \dfrac{(f^{\frac{n-2}{4}})''}{f^{\frac{n-2}{4}}} - \omega f$.

The potential q_f naturally appears when we solve the problem (1) by separating the variables in order to get an infinite system of ODE. We have at last to precise the following notation that will appear in the statement of Theorem 1.7.

Notation: Let x_0 be in $\mathbb{R}$ and g be a real function such that $\lim\limits_{x\to x_0} g(x) = 0$.

We say that $f(x) \underset{x_0}{=} \tilde{O}\big(g(x)\big)$ if

$$\forall \varepsilon > 0 \quad \lim_{x \to x_0} \frac{|f(x)|}{|g(x)|^{1-\varepsilon}} = 0.$$

Here is our local uniqueness result.

Theorem 1.7. *Let $M = [0,1] \times \mathbb{S}^{n-1}$ be a smooth Riemannian manifold equipped with the metrics*

$$g = f(x)(dx^2 + g_{\mathbb{S}}), \quad \tilde{g} = \tilde{f}(x)(dx^2 + g_{\mathbb{S}}),$$

and let ω be a frequency not belonging to the Dirichlet spectrum of $-\Delta_g$ or $-\Delta_{\tilde{g}}$ on M. Let $a \in]0,1[$ and $\mathcal{E}$ be the set of all the positive sequences $(\varepsilon_m)_m$ satisfying

$$\varepsilon_m = \tilde{O}\big(e^{-2a\sqrt{\mu_m}}\big),$$

In order to simplify the statements of the results, let us denote the propositions:

- $*$ (P_1) : $f = \tilde{f}$ on $[0,a]$
- $*$ (P_2) : $f = \tilde{f} \circ \eta$ on $[0,a]$
- $*$ (P_3): $f = \tilde{f}$ on $[1-a,1]$
- $*$ (P_4) : $f = \tilde{f} \circ \eta$ on $[1-a,1]$

where $\eta(x) = 1 - x$ for all $x \in [0,1]$.

Assume that f and $\tilde{f}$ belong to $C^\infty([0,1]) \cap \mathcal{C}_b$ where $\mathcal{C}_b$ is defined as

$$\mathcal{C}_b = \left\{ \bullet \; \left| \frac{f'(k)}{f(k)} \right| \leq \frac{1}{n-2},\; k \in \{0,1\}, \qquad \bullet \; q_f \in \mathcal{D}_b \right\}.$$

Then :

- *For $n = 2$ and $\omega \neq 0$ or $n \geq 3$:*

$$\left(\exists\, (\varepsilon_m) \in \mathcal{E},\; \sigma(\Lambda_g(\omega)) \underset{(\varepsilon_m)}{\asymp} \sigma(\Lambda_{\tilde{g}}(\omega)) \right) \Rightarrow (P_1) \text{ or } (P_2) \text{ or } (P_3) \text{ or } (P_4),$$

Remark 3. When $n = 2$, the condition $\left| \dfrac{f'(k)}{f(k)} \right| \leq \dfrac{1}{n-2}$ is always satisfied.

Remark 4. The converse is not true if $f(0) \neq f(1)$. If one of the (P_i) is satisfied, it cannot imply more than the closeness of one of the subsequence $\big(\lambda^-(\mu_m)\big)$ or $\big(\lambda^+(\mu_m)\big)$ with $\big(\tilde{\lambda}^-(\mu_m)\big)$ or $\big(\tilde{\lambda}^+(\mu_m)\big)$.

Special case *: when $f(0) = f(1)$, we have the following equivalence :*

$$\Big(\exists\, (\varepsilon_m) \in \mathcal{E},\ \sigma(\Lambda_g(\omega)) \underset{(\varepsilon_m)}{\asymp} \sigma(\Lambda_{\tilde{g}}(\omega))\Big) \Leftrightarrow \Big((P_1) \text{ and } (P_3)\Big) \text{ or } \Big((P_2) \text{ and } (P_4)\Big).$$

Let us also give our stability result. It requires to assume that, for some $A > 0$, the unknown conformal factor belongs to the set of A-admissible functions that we define now.

Definition 1.8. Let $A > 0$. The set of A-admissible functions is defined as :

$$\mathcal{C}(A) = \left\{ \bullet\ f \in C^2([0,1]) \qquad \bullet\ \forall k \in [\![0,2]\!],\ \|f^{(k)}\|_\infty + \left\|\frac{1}{f}\right\|_\infty \leq A \right\}$$

Our stability result for the Steklov problem is the following:

Theorem 1.9. *Let $M = [0,1] \times \mathbb{S}^{n-1}$ be a smooth Riemanniann manifold equipped with the metrics*

$$g = f(x)(dx^2 + g_{\mathbb{S}}), \quad \tilde{g} = \tilde{f}(x)(dx^2 + g_{\mathbb{S}}),$$

with f, $\tilde{f}$ positive on $[0,1]$ and symmetric with respect to $x = 1/2$.

Let $A > 0$ be fixed and ω be a frequency not belonging to the Dirichlet spectrum of $-\Delta_g$ and $-\Delta_{\tilde{g}}$ on M. Then, for $n \geq 2$, for a sufficiently small $\varepsilon > 0$ and under the assumption

$$f, \tilde{f} \in \mathcal{C}(A)$$

we have the implication :

$$\sigma(\Lambda_g(\omega)) \underset{\varepsilon}{\asymp} \sigma(\Lambda_{\tilde{g}}(\omega)) \Rightarrow \left\|q_f - \tilde{q}_f\right\|_2 \leq C_A \frac{1}{\ln\left(\frac{1}{\varepsilon}\right)}$$

where C_A is a constant that only depends on A.

As a by-product, we get two corollaries :

Corollary 1.10. *Using the same notations and assumptions as in Theorem 1.9, for all $0 \leq s \leq 2$, we have*

$$\sigma(\Lambda_g(\omega)) \underset{\varepsilon}{\asymp} \sigma(\Lambda_{\tilde{g}}(\omega)) \Rightarrow \left\|q_f - \tilde{q}_f\right\|_{H^s(0,1)} \leq C_A \frac{1}{\ln\left(\frac{1}{\varepsilon}\right)^\theta}$$

where $\theta = (2-s)/2$ and C_A is a constant that only depends on A. In particular, from the Sobolev embedding, one gets

$$\sigma(\Lambda_g(\omega)) \underset{\varepsilon}{\asymp} \sigma(\Lambda_{\tilde{g}}(\omega)) \Rightarrow \left\|q_f - \tilde{q}_f\right\|_\infty \leq C_A \sqrt{\frac{1}{\ln\left(\frac{1}{\varepsilon}\right)}}$$

Corollary 1.11. *Using the same notations and assumptions as in Theorem 1.9, if moreover $\omega = 0$ and $n \geq 3$, one has*

$$\sigma(\Lambda_g(\omega)) \underset{\varepsilon}{\asymp} \sigma(\Lambda_{\tilde{g}}(\omega)) \Rightarrow \left\|f - \tilde{f}\right\|_\infty \leq C_A \frac{1}{\ln\left(\frac{1}{\varepsilon}\right)}$$

where C_A is a constant that only depends on A.

The stability in the inverse Calderón problem somehow precedes the inverse Steklov problem, so we say few words about it. Let $\mathcal{B}(H^{1/2}(\partial M))$ be the set of bounded operators from $H^{1/2}(\partial M)$ to $H^{1/2}(\partial M)$ equipped with the norm

$$\|F\|_* = \sup_{\psi \in H^{1/2}(\partial M)\setminus\{0\}} \frac{\|F\psi\|_{H^{1/2}}}{\|\psi\|_{H^{1/2}}}.$$

In Lemma 5.1 (Section 5) we show the equivalence

$$\Lambda_g(\omega) - \Lambda_{\tilde{g}}(\omega) \in \mathcal{B}(H^{1/2}(\partial M)) \Leftrightarrow \begin{cases} f(0) = \tilde{f}(0) \\ f(1) = \tilde{f}(1) \end{cases}$$

and prove the following stability result for the Calderón problem. We draw the reader's attention to the fact that the symmetry hypothesis no longer occurs here since the strict uniqueness is true (see [4]). However, it is replaced by a technical assumption on the mean of the difference of the potentials.

Theorem 1.12. *Let $M = [0,1] \times \mathbb{S}^{n-1}$ be a smooth Riemanniann manifold equipped with the metrics*

$$g = f(x)(dx^2 + g_{\mathbb{S}}), \quad \tilde{g} = \tilde{f}(x)(dx^2 + g_{\mathbb{S}}),$$

with f and $\tilde{f}$ positive on $[0,1]$.

Let $A > 0$ be fixed and ω be a frequency not belonging to the Dirichlet spectrum of $-\Delta_g$ and $-\Delta_{\tilde{g}}$ on M. Let $n \geq 2$, $\varepsilon > 0$ and assume that

- $f(0) = \tilde{f}(0)$ *and* $f(1) = \tilde{f}(1)$,
- $f, \tilde{f} \in \mathcal{C}(A)$,
- $\left| \int_0^1 (q_f - \tilde{q}_f) \right| + \|\Lambda_g(\omega) - \Lambda_{\tilde{g}}(\omega)\|_* \leq \varepsilon.$

Then :

$$\|q_f - \tilde{q}_f\|_2 \leq C_A \frac{1}{\ln\left(\frac{1}{\varepsilon}\right)},$$

where C_A is a constant that only depends on A.

Corollary 1.13. *Using the same notations and assumptions as in Theorem 1.12, for all $0 \leq s \leq 2$, we obtain also*

$$\|q_f - \tilde{q}_f\|_{H^s(0,1)} \leq C_A \frac{1}{\ln\left(\frac{1}{\varepsilon}\right)^\theta}$$

where $\theta = (2-s)/2$ and C_A is a constant that only depends on A. In particular, from the Sobolev embedding, one gets

$$\|q_f - \tilde{q}_f\|_\infty \leq C_A \sqrt{\frac{1}{\ln\left(\frac{1}{\varepsilon}\right)}}$$

Corollary 1.14. *Using the same notations and assumptions as in Theorem 1.12, if moreover $\omega = 0$ and $n \geq 3$, one has the stronger conclusion:*

$$\|f - \tilde{f}\|_\infty \leq C_A \frac{1}{\ln\left(\frac{1}{\varepsilon}\right)}$$

where C_A is a constant that only depends on A.

2 Asymptotics of the Steklov spectrum

The proof of both theorems is based on the separation of variables that leads to reformulating the Dirichlet problem in terms of boundary value problems for ordinary differential equations. All the details can be found in [6, 4] but we outline the main points for the sake of completeness.

2.1 From PDE to ODE using separation of variables

The equation

$$\begin{cases} -\Delta_g u = \omega u \text{ in } M \\ u = \psi \text{ on } \partial M \end{cases} \tag{3}$$

can be reduced to a countable system of Sturm Liouville boundary value problems on $[0,1]$. Indeed, the boundary ∂M of the manifold M has two distinct connected components

$$\Gamma_0 = \{0\} \times \mathbb{S}^{n-1} \text{ and } \Gamma_1 = \{1\} \times \mathbb{S}^{n-1},$$

so we can decompose $H^1(\partial M)$ as the direct sum :

$$H^{1/2}(\partial M) = H^{1/2}(\Gamma_0) \bigoplus H^{1/2}(\Gamma_1).$$

Each element ψ of $H^{1/2}(\partial M)$ can be written as

$$\psi = \begin{pmatrix} \psi^0 \\ \\ \psi^1 \end{pmatrix}, \qquad \psi^0 \in H^{1/2}(\Gamma_0) \text{ and } \psi^1 \in H^{1/2}(\Gamma_1).$$

Using separation of variables, one can write the solution of (1) as

$$u(x,y) = \sum_{m=0}^{+\infty} u_m(x) Y_m(y),$$

and ψ^0 and ψ^1 as

$$\psi^0 = \sum_{m\in\mathbb{N}} \psi_m^0 Y_m, \qquad \psi^1 = \sum_{m\in\mathbb{N}} \psi_m^1 Y_m,$$

where (Y_m) represents an orthonormal basis of eigenfunctions of $-\Delta_{\mathbb{S}}$ associated to the sequence of its eigenvalues counted with multiplicity

$$\sigma\big(\Delta_{\mathbb{S}}\big) = \{0 = \mu_0 \leq \mu_1 ... \leq \mu_m \to +\infty\}.$$

By setting

$$v(x,y) = f^{\frac{n-2}{4}} u(x,y) = \sum_{m=0}^{+\infty} v_m(x) Y_m(y), \quad x \in [0,1], \;\; y \in \mathbb{S}^{n-1},$$

one can show the equivalence (cf [6]) :

$$u \text{ solves (1)} \Leftrightarrow \forall m \in \mathbb{N}, \; \begin{cases} -v_m''(x) + q_f(x) v_m(x) = -\mu_m v_m(x), \quad \forall x \in]0,1[\\ v_m(0) = f^{\frac{n-2}{4}}(0)\psi_m^0, \;\; v_m(1) = f^{\frac{n-2}{4}}(1)\psi_m^1, \end{cases} \tag{4}$$

with $q_f = \dfrac{(f^{\frac{n-2}{4}})''}{f^{\frac{n-2}{4}}} - \omega f$ (the dependence in f will be omitted in the following and we will just write q instead of q_f).

We thus are brought back to a countable system of 1D Schrödinger equations whose potential does not depend on $m \in \mathbb{N}$. Thanks to the Weyl-Titchmarsh theory, we are able to give a nice representation of the DN map that involves the Weyl-Titchmarsh functions of (4) evaluated at the sequence (μ_m).

2.2 Diagonalization of the DN map

From the equation on $[0,1]$

$$-u'' + qu = -zu, \quad z \in \mathbb{C}. \tag{5}$$

one can define two fundamental systems of solutions of (5), $\{c_0, s_0\}$ and $\{c_1, s_1\}$, whose initial Cauchy conditions satisfy

$$\begin{cases} c_0(0,z) = 1, \ c_0'(0,z) = 0 \\ c_1(1,z) = 1, \ c_1'(1,z) = 0 \end{cases} \quad \text{and} \quad \begin{cases} s_0(0,z) = 0, \ s_0'(0,z) = 1 \\ s_1(1,z) = 0, \ s_1'(1,z) = 1. \end{cases} \tag{6}$$

We shall add the subscript $\tilde{}$ to all the quantities referring to $\tilde{q}$.

The characteristic function $\Delta(z)$ associated to the equation (5) is defined by the Wronskian

$$\Delta(z) = W(s_0, s_1) := s_0 s_1' - s_0' s_1. \tag{7}$$

Furthermore, there are two (uniqueLy defined) Weyl-solutions ψ and ϕ of (5) having the form :

$$\psi(x) = c_0(x) + M(z)s_0(x), \quad \phi(x) = c_1(x) - N(z)s_1(x)$$

with Dirichlet boundary conditions at $x = 1$ and $x = 0$ respectively. The meromorphic functions M and N are called the *Weyl-Titchmarsh functions.* Denoting

$$D(z) := W(c_0, s_1), \qquad E(z) := W(c_1, s_0)$$

an easy calculation leads to

$$M(z) = -\frac{D(z)}{\Delta(z)}, \qquad N(z) = -\frac{E(z)}{\Delta(z)}.$$

Remark 5. The function N has the same role as M for the potential $q(1-x)$, *i.e* :

$$N(z,q) = M(z, q \circ \eta)$$

where, for all $x \in [0,1]$, $\eta(x) = 1 - x$.

Those meromorphic functions naturally appear in the expression of the DN map $\Lambda_g(\omega)$ in a specific basis of $H^{1/2}(\Gamma_0) \oplus H^{1/2}(\Gamma_1)$. More precisely, in the basis $\mathscr{B} = \left(\{e_m^1, e_m^2\}\right)_{m \geq 0}$ where e_m^1 and e_m^2 are defined as :

$$e_m^1 = (Y_m, 0) \qquad e_m^2 = (0, Y_m)$$

one can prove the that the operator $\Lambda_g(\omega)$ can be bloc-diagonalized :

$$[\Lambda_g]_{\mathscr{B}} = \begin{pmatrix} \Lambda_g^1(\omega) & 0 & 0 & \cdots \\ 0 & \Lambda_g^2(\omega) & 0 & \cdots \\ 0 & 0 & \Lambda_g^3(\omega) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

with, for every $m \in \mathbb{N}$ and setting $h = f^{n-2}$ (cf [4]):

$$\Lambda_g^m(\omega) = \begin{pmatrix} -\frac{M(\mu_m)}{\sqrt{f(0)}} + \frac{1}{4\sqrt{f(0)}}\frac{h'(0)}{h(0)} & -\frac{1}{\sqrt{f(0)}}\frac{h^{1/4}(1)}{h^{1/4}(0)}\frac{1}{\Delta(\mu_m)} \\ -\frac{1}{\sqrt{f(1)}}\frac{h^{1/4}(0)}{h^{1/4}(1)}\frac{1}{\Delta(\mu_m)} & -\frac{N(\mu_m)}{\sqrt{f(1)}} - \frac{1}{4\sqrt{f(1)}}\frac{h'(1)}{h(1)} \end{pmatrix}.$$

2.3 Asymptotics of the eigenvalues

It is then possible, with this representation of $\Lambda_g(\omega)$, to get the following precise asymptotics of the eigenvalues $\lambda^{\pm}(\mu_m)$ of $\Lambda_g(\omega)$:

Lemma 2.1. *When q belongs to $\mathcal{D}_b$, $\Lambda_g^m(\omega)$ has two eigenvalues $\lambda^-(\mu_m)$ and $\lambda^+(\mu_m)$ whose asymptotics are given by :*

$$\begin{cases} \lambda^-(\mu_m) = -\dfrac{N(\mu_m)}{\sqrt{f(1)}} - \dfrac{(\ln h)'(1)}{4\sqrt{f(1)}} + \tilde{O}\Big(e^{-2\sqrt{\mu_m}}\Big) \\ \lambda^+(\mu_m) = -\dfrac{M(\mu_m)}{\sqrt{f(0)}} + \dfrac{(\ln h)'(0)}{4\sqrt{f(0)}} + \tilde{O}\Big(e^{-2\sqrt{\mu_m}}\Big). \end{cases}$$

Proof. The characteristic polynomial $\chi_m(X)$ of $\Lambda_g^m(\omega)$ is

$$\chi_m(X) = X^2 - \mathrm{Tr}(\Lambda_g^m(\omega))X + \det(\Lambda_g^m(\omega)).$$

To simplify the notations, we set

$$C_0 = \frac{\ln(h)'(0)}{4\sqrt{f(0)}}, \qquad C_1 = \frac{\ln(h)'(1)}{4\sqrt{f(1)}}.$$

Thanks to the matrix representation of $\Lambda_g(\omega)$, we see that $\mathrm{Tr}(\Lambda_g^m(\omega))$ and $\det(\Lambda_g^m(\omega))$ satisfy the equalities:

$$\begin{cases} \mathrm{Tr}(\Lambda_g^m(\omega)) = -\dfrac{M(\mu_m)}{\sqrt{f(0)}} - \dfrac{N(\mu_m)}{\sqrt{f(1)}} + C_0 - C_1. \\ \det(\Lambda_g^m(\omega)) = \Big(-\dfrac{M(\mu_m)}{\sqrt{f(0)}} + C_0\Big)\Big(-\dfrac{N(\mu_m)}{\sqrt{f(1)}} - C_1\Big) + O(\mu_m e^{-2\sqrt{\mu_m}}), \qquad m \to +\infty \end{cases}$$

The asymptotics of the discriminant δ_m of $\chi_m(X)$ is thus :

$$\begin{aligned} \delta_m &= \Big(-\frac{M(\mu_m)}{\sqrt{f(0)}} + C_0 - \frac{N(\mu_m)}{\sqrt{f(1)}} - C_1\Big)^2 - 4\Big(-\frac{M(\mu_m)}{\sqrt{f(0)}} + C_0\Big)\Big(-\frac{N(\mu_m)}{\sqrt{f(1)}} - C_1\Big) \\ &\qquad + O(\mu_m e^{-2\sqrt{\mu_m}}). \\ &= \Big(-\frac{M(\mu_m)}{\sqrt{f(0)}} + C_0 + \frac{N(\mu_m)}{\sqrt{f(1)}} + C_1\Big)^2 + O(\mu_m e^{-2\sqrt{\mu_m}}). \end{aligned}$$

Now, let us recall the result obtained by Simon in [13] :

Theorem 2.2. *$M(z^2)$ has the following asymptotic expansion :*

$$\forall B \in \mathbb{N}, \ -M(z^2) \underset{z\to\infty}{=} z + \sum_{j=0}^{B} \frac{\beta_j(0)}{z^{j+1}} + o\Big(\frac{1}{z^{B+1}}\Big)$$

where, for every $x \in [0,1]$, $\beta_j(x)$ is defined by : $\begin{cases} \beta_0(x) = \dfrac{1}{2}q(x) \\ \beta_{j+1}(x) = \dfrac{1}{2}\beta_j'(x) + \dfrac{1}{2}\displaystyle\sum_{l=0}^{j} \beta_l(x)\beta_{j-l}(x). \end{cases}$

From Remark 5, by symmetry, one has immediately:

Corollary 2.3. *$N(z^2)$ has the following asymptotic expansion :*

$$\forall B \in \mathbb{N}, \ -N(z^2) \underset{z\to\infty}{=} z + \sum_{j=0}^{B} \frac{\gamma_j(0)}{z^{j+1}} + o\Big(\frac{1}{z^{B+1}}\Big)$$

where, for all $x \in [0,1]$, $\gamma_j(x)$ is defined by : $\begin{cases} \gamma_0(x) = \frac{1}{2}q(1-x) \\ \gamma_{j+1}(x) = \frac{1}{2}\gamma_j'(x) + \frac{1}{2}\sum_{l=0}^{j}\gamma_l(x)\gamma_{j-l}(x). \end{cases}$

If $f(0) \neq f(1)$, we deduce from Theorem 2.2 and Corollary 2.3:

$$-\frac{M(\mu_m)}{\sqrt{f(0)}} + \frac{N(\mu_m)}{\sqrt{f(1)}} = \underbrace{\left(\frac{1}{\sqrt{f(0)}} - \frac{1}{\sqrt{f(1)}}\right)}_{\neq 0} \sqrt{\mu_m} + O\Big(\frac{1}{\sqrt{\mu_m}}\Big).$$

If $f(0) = f(1)$, we will need the elementary general following lemma:

Lemma 2.4. *We have the equivalence :*

$$q^{(k)}(0) = (-1)^k q^{(k)}(1), \ \forall k \in \mathbb{N} \quad \Leftrightarrow \quad \beta_k(0) = \gamma_k(0), \ \forall k \in \mathbb{N}.$$

Proof.

Let us prove by induction that, for every $j \in \mathbb{N}$, there exists $P_j \in \mathbb{R}[X_1, ..., X_j]$ such that

$$\begin{cases} \beta_j(x) = \frac{1}{2^{j+1}} q^{(j)}(x) + P_j(q, q', ..., q^{(j-1)})(x) \\ \gamma_j(x) = \frac{1}{2^{j+1}} (q\circ\eta)^{(j)}(x) + P_j(q\circ\eta, (q\circ\eta)', ..., (q\circ\eta)^{(j-1)})(x) \end{cases} \tag{8}$$

where $\eta(x) = 1 - x$.

- $\beta_0(x) = \frac{1}{2}q(x)$ and $\gamma_0(x) = \frac{1}{2}q(1-x)$, so the result holds with $P_0(X) = 0$.
- Let $j \in \mathbb{N}$ and assume that

$$\begin{cases} \beta_k(x) = \frac{1}{2^{k+1}} q^{(k)}(x) + P_k(q, q', ..., q^{(k-1)})(x) \\ \gamma_k(x) = \frac{1}{2^{k+1}} (q\circ\eta)^{(k)}(x) + P_k(q\circ\eta, (q\circ\eta)', ..., (q\circ\eta)^{(k-1)})(x), \end{cases}$$

  for every $0 \leq k \leq j$. Then :

$$\begin{aligned} \beta_{j+1}(x) &= \frac{1}{2}\beta_j'(x) + \frac{1}{2}\sum_{l=0}^{j}\beta_l(x)\beta_{j-l}(x) \\ &= \frac{1}{2^{j+2}} q^{(j+1)}(x) + P_{j+1}(q, q', ..., q^{(j)})(x), \end{aligned}$$

  where we have set $P_{j+1}(q, q', ..., q^{(j)}) = \frac{1}{2}\big[P_j(q, q', ..., q^{(j-1)})\big]' + \frac{1}{2}\sum_{l=0}^{j}\beta_l(x)\beta_{j-l}(x)$.

  In the same way, one also has

$$\begin{aligned} \gamma_{j+1}(x) &= \frac{1}{2}\gamma_j'(x) + \frac{1}{2}\sum_{l=0}^{j}\gamma_l(x)\gamma_{j-l}(x) \\ &= \frac{1}{2^{j+2}} (q\circ\eta)^{(j+1)}(x) + P_{j+1}(q\circ\eta, (q\circ\eta)', ..., (q\circ\eta)^{(j)})(x). \end{aligned}$$

- Hence, we get the result by induction.

We are now able to prove the equivalence.

($\Rightarrow$) If $q^{(j)}(0) = (-1)^j q^{(j)}(1)$ for every $j \in \mathbb{N}$ then one has, for every $k \in \mathbb{N}$ and every $P \in \mathbb{R}[X_1, ..., X_k]$:

$$P(q, q', ..., q^{(k-1)})(0) = P(q \circ \eta, (q \circ \eta)', ..., (q \circ \eta)^{(k-1)})(0),$$

so, thanks to (8):

$$\beta_j(0) = \gamma_j(0) \text{ for every } j \in \mathbb{N}.$$

($\Leftarrow$) Conversely, assume that there is $j \in \mathbb{N}$ such that $q^{(j)}(0) \neq (-1)^j q^{(j)}(1)$ and set $k = \min\{j \in \mathbb{N},\ q^{(j)}(0) \neq (-1)^j q^{(j)}(1)\}$. As previously, for every $P \in \mathbb{R}[X_1, ..., X_k]$:

$$P(q, q', ..., q^{(k-1)})(0) = P(q \circ \eta, (q \circ \eta)', ..., (q \circ \eta)^{(k-1)})(0).$$

Hence :

$$\begin{aligned}
\beta_k(0) \neq \gamma_k(0) \Leftrightarrow \frac{1}{2^{k+1}} q^{(k)}(0) + P_k(q, ..., q^{(k-1)})(0) \neq \frac{1}{2^{k+1}} &(q \circ \eta)^{(k)}(0) \\
&+ P_k\big((q \circ \eta), ..., (q \circ \eta)^{(k-1)}\big)(0) \\
\Leftrightarrow \frac{1}{2^{k+1}} q^{(k)}(0) \neq \frac{1}{2^{k+1}} (q \circ \eta)^{(k)}(0)& \\
\Leftrightarrow q^{(k)}(0) \neq (-1)^k q^{(k)}(1),&
\end{aligned}$$

and that is true by definition of k. □

As we have assumed that q belongs to $\mathcal{D}_b$, by setting $k = \min\{j \in \mathbb{N},\ q^{(j)}(0) \neq (-1)^j q^{(j)}(1)\}$, we get, thanks to (2.2), (2.3) and Lemma 2.4:

$$-\frac{M(\mu_m)}{\sqrt{f(0)}} + \frac{N(\mu_m)}{\sqrt{f(1)}} = \underbrace{\left(\frac{\beta_k(0) - \gamma_k(0)}{\sqrt{f(0)}}\right)}_{\neq 0} \frac{1}{(\sqrt{\mu_m})^{k+1}} + O\left(\frac{1}{(\sqrt{\mu_m})^{k+2}}\right).$$

In both cases, there is $A \in \mathbb{R}\backslash\{0\}$ and $k \in \mathbb{Z}$ such that

$$-\frac{M(\mu_m)}{\sqrt{f(0)}} + \frac{N(\mu_m)}{\sqrt{f(1)}} = A(\sqrt{\mu_m})^k + o\big((\sqrt{\mu_m})^k\big) \tag{9}$$

Thus, recalling that

$$\delta = \left(-\frac{M(\mu_m)}{\sqrt{f(0)}} + C_0 + \frac{N(\mu_m)}{\sqrt{f(1)}} + C_1\right)^2 + O(\mu_m e^{-2\sqrt{\mu_m}}),$$

we obtain, as A is not 0:

$$\begin{aligned}
\sqrt{\delta} &= \left[\left(-\frac{M(\mu_m)}{\sqrt{f(0)}} + C_0 + \frac{N(\mu_m)}{\sqrt{f(1)}} + C_1\right)^2 + O(\mu_m e^{-2\sqrt{\mu_m}})\right]^{\frac{1}{2}} \\
&= \left[\left(A(\sqrt{\mu_m})^k + C_0 + C_1 + o\big((\sqrt{\mu_m})^k\big)\right)^2 + O(\mu_m e^{-2\sqrt{\mu_m}})\right]^{\frac{1}{2}} \\
&= \left|A(\sqrt{\mu_m})^k + C_0 + C_1 + o\big((\sqrt{\mu_m})^k\big)\right| \left[1 + O\left(\big(\sqrt{\mu_m}\big)^{-2k+2} e^{-2\sqrt{\mu_m}}\right)\right]^{\frac{1}{2}} \\
&= \left|\frac{N(\mu_m)}{\sqrt{f(1)}} - \frac{M(\mu_m)}{\sqrt{f(0)}} + C_0 + C_1\right| \left[1 + O\left(\big(\sqrt{\mu_m}\big)^{-2k+2} e^{-2\sqrt{\mu_m}}\right)\right] \\
&= \left|\frac{N(\mu_m)}{\sqrt{f(1)}} - \frac{M(\mu_m)}{\sqrt{f(0)}} + C_0 + C_1\right| + \tilde{O}\big(e^{-2\sqrt{\mu_m}}\big).
\end{aligned}$$

Therefore, the two eigenvalues $\lambda^{\pm}(\mu_m)$ of $\Lambda_g^m(\omega)$ satisfy the asymptotics equalities

$$\begin{cases} \lambda^{-}(\mu_m) = -\dfrac{N(\mu_m)}{\sqrt{f(1)}} - \dfrac{\ln(h)'(1)}{4\sqrt{f(1)}} + \tilde{O}\big(e^{-2\sqrt{\mu_m}}\big) \\ \lambda^{+}(\mu_m) = -\dfrac{M(\mu_m)}{\sqrt{f(0)}} + \dfrac{\ln(h)'(0)}{4\sqrt{f(0)}} + \tilde{O}\big(e^{-2\sqrt{\mu_m}}\big). \end{cases}$$

□

In fact, the eigenvalues μ_m being counted with multiplicity, the asymptotics of Lemma 2.1 won't be sufficiently precise for our purpose. Indeed, by Theorem 2.2 and its Corollary, the Weyl-Titchmarsh functions always satisfy

$$\begin{cases} -N(z^2) = z + O\Big(\dfrac{1}{z}\Big) \\ -M(z^2) = z + O\Big(\dfrac{1}{z}\Big). \end{cases}$$

So, using the Weyl law, one can prove immediately that

$$\begin{cases} \lambda^{-}(\mu_m) = \dfrac{\sqrt{\mu_m}}{\sqrt{f(1)}} - \dfrac{(\ln h)'(1)}{4\sqrt{f(1)}} + O\Big(\dfrac{1}{\sqrt{\mu_m}}\Big) = C_1 m^{\frac{1}{n-1}} + O(1) \\ \lambda^{+}(\mu_m) = \dfrac{\sqrt{\mu_m}}{\sqrt{f(0)}} + \dfrac{(\ln h)'(0)}{4\sqrt{f(0)}} + O\Big(\dfrac{1}{\sqrt{\mu_m}}\Big) = C_0 m^{\frac{1}{n-1}} + O(1) \end{cases} \tag{10}$$

with $C_0, C_1 > 0$. In order to have a much more precise asymptotic expansion in m, let us introduce the set

$$\Sigma\big(\Lambda_g(\omega)\big) = \{\lambda^{\pm}(\kappa_m),\ m \in \mathbb{N}\} \tag{11}$$

where the κ_m are the eigenvalues of $-\Delta_{\mathbb{S}}$ counted *without multiplicity*. We have an explicit formula for κ_m (cf [12]) given by

$$\kappa_m = m(m+n-2).$$

From now on, we will always use the asymptotics of Lemma 2.1 by replacing μ_m by κ_m. Of course, one can also define the closeness between $\Sigma\big(\Lambda_g(\omega)\big)$ and $\Sigma\big(\Lambda_{\tilde{g}}(\omega)\big)$ up to a sequence (ε_m) by replacing μ_m by κ_m in Definitions 1.2 and 1.3.

3 A local uniqueness result

Now, let us give the proof of Theorem 1.7.

Proof. Let (ε_m) be a sequence such that $\varepsilon_m = \tilde{O}(e^{-2a\sqrt{\mu_m}})$ and $\sigma(\Lambda_g(\omega)) \underset{(\varepsilon_m)}{\asymp} \sigma(\Lambda_{\tilde{g}}(\omega))$.
Then, there is a subsequence of (ε_m), that we will still denote (ε_m) which satisfies the estimate

$$\varepsilon_m = \tilde{O}(e^{-2a\sqrt{\kappa_m}})$$

and the relation

$$\Sigma(\Lambda_g(\omega)) \underset{(\varepsilon_m)}{\asymp} \Sigma(\Lambda_{\tilde{g}}(\omega)).$$

Lemma 3.1. *Under the hypothesis* $\Sigma(\Lambda_g(\omega)) \underset{(\varepsilon_m)}{\asymp} \Sigma(\Lambda_{\tilde{g}}(\omega))$, *we have the alternative :*

$$\begin{cases} f(0) = \tilde{f}(0) \\ f(1) = \tilde{f}(1) \end{cases} \quad \textit{or} \quad \begin{cases} f(0) = \tilde{f}(1) \\ f(1) = \tilde{f}(0). \end{cases}$$

Proof.

- We first show the equality

$$\sqrt{f(0)} + \sqrt{f(1)} = \sqrt{\tilde{f}(0)} + \sqrt{\tilde{f}(1)} \tag{12}$$

As $\sqrt{\kappa_m} = m + \dfrac{n-2}{2} + O\left(\dfrac{1}{m}\right)$, we get from Lemma 2.1 the following asymptotics

$$\begin{cases} \lambda^-(\kappa_m) = \dfrac{m}{\sqrt{f(1)}} + \dfrac{n-2}{2\sqrt{f(1)}} - \dfrac{(\ln h)'(1)}{4\sqrt{f(1)}} + O\left(\dfrac{1}{m}\right) \\ \lambda^+(\kappa_m) = \dfrac{m}{\sqrt{f(0)}} + \dfrac{n-2}{2\sqrt{f(0)}} + \dfrac{(\ln h)'(0)}{4\sqrt{f(0)}} + O\left(\dfrac{1}{m}\right) \end{cases} \tag{13}$$

Let $L > 0$. The sequences $(\lambda^\pm(\kappa_m))$ are asymptotically in arithmetic progression. Combined with the relation

$$\Sigma\big(\Lambda_g(\omega)\big) \underset{(\varepsilon_m)}{\asymp} \Sigma\big(\Lambda_{\tilde{g}}(\omega)\big),$$

this leads to the equality (when $L \to +\infty$)

$$\begin{aligned} &\mathrm{Card}\big\{m \in \mathbb{N},\ \lambda^-(\kappa_m) \le L\big\} \quad + \quad \mathrm{Card}\big\{m \in \mathbb{N},\ \lambda^+(\kappa_m) \le L\big\} \\ &\qquad = \quad \mathrm{Card}\big\{m \in \mathbb{N},\ \tilde{\lambda}^-(\kappa_m) \le L\big\} \quad + \quad \mathrm{Card}\big\{m \in \mathbb{N},\ \tilde{\lambda}^+(\kappa_m) \le L\big\} + O(1). \end{aligned} \tag{14}$$

Thanks to the asymptotics (13), we deduce that :

$$\begin{aligned} &\mathrm{Card}\big\{m \in \mathbb{N},\ m \le \sqrt{f(1)}L\big\} \quad + \quad \mathrm{Card}\big\{m \in \mathbb{N},\ m \le \sqrt{f(0)}L\big\} \\ &\qquad = \quad \mathrm{Card}\big\{m \in \mathbb{N},\ m \le \sqrt{\tilde{f}(1)}L\big\} \quad + \quad \mathrm{Card}\big\{m \in \mathbb{N},\ m \le \sqrt{\tilde{f}(0)}L\big\} + O(1), \end{aligned}$$

and then that :

$$\sqrt{f(1)}L + \sqrt{f(0)}L = \sqrt{\tilde{f}(1)}L + \sqrt{\tilde{f}(0)}L + O(1), \qquad L \to +\infty.$$

As L is any positive number, this proves (12).

- Now, we have to show : $f(0) \in \{\tilde{f}(0), \tilde{f}(1)\}$.

Assume that it is not true, for example

$$f(0) < \min\{f\tilde{(0)}, \tilde{f}(1)\}. \tag{15}$$

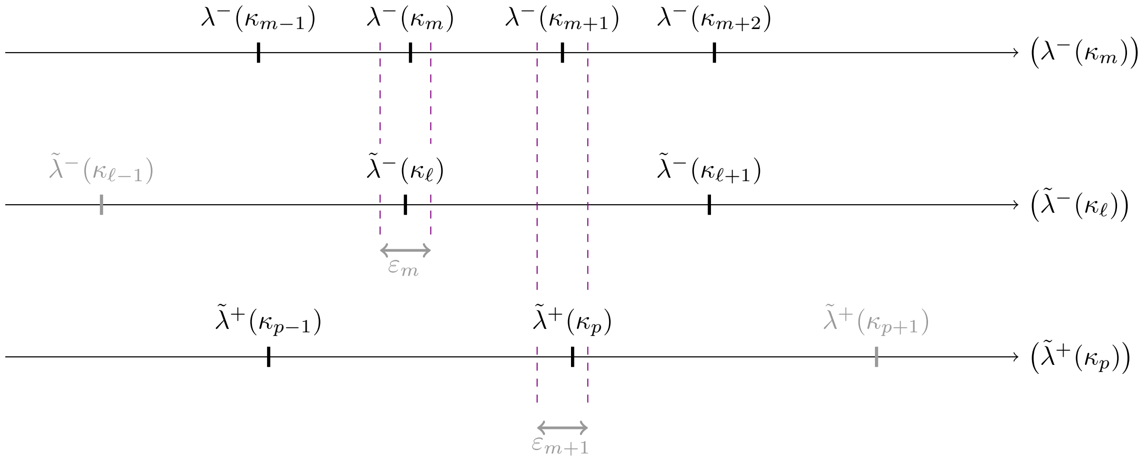
Then (12) implies

$$f(1) > \max\{\tilde{f}(0), \tilde{f}(1)\}. \tag{16}$$

Let m be in $\mathbb{N}$. There is $\ell \in \mathbb{N}$ such that

$$\lambda^-(\kappa_m) - \tilde{\lambda}^-(\kappa_\ell) = O(\varepsilon_m), \tag{17}$$

with $|O(\varepsilon_m)| \le \varepsilon_m$. From the assumption $\sigma\big(\Lambda_g(\omega)\big) \underset{(\varepsilon_m)}{\asymp} \sigma\big(\Lambda_{\tilde{g}}(\omega)\big)$, each of the eigenvalues $\lambda^-(\kappa_{m-1})$, $\lambda^-(\kappa_{m+1})$ and $\lambda^-(\kappa_{m+2})$ is also close to an element of $\sigma\big(\Lambda_{\tilde{g}}(\omega)\big)$. If m is large enough, the situation is necessary the following:



Indeed, since $f(1) > \tilde f(1)$, for m large enough, from (13) and (17), we have

$$\begin{aligned}\lambda^-(\kappa_{m+1}) &= \lambda^-(\kappa_m) + \frac{1}{\sqrt{f(1)}} + o(1)\\ &= \tilde\lambda^-(\kappa_\ell) + O(\varepsilon_m) + \frac{1}{\sqrt{f(1)}} + o(1)\\ &= \tilde\lambda^-(\kappa_{\ell+1}) + \underbrace{\frac{1}{\sqrt{f(1)}} - \frac{1}{\sqrt{\tilde f(1)}}}_{<0} + O(\varepsilon_m) + o(1)\end{aligned}$$

Let us chose m large enough such that

$$\begin{cases}\dfrac{1}{\sqrt{f(1)}} - \dfrac{1}{\sqrt{\tilde f(1)}} + O(\varepsilon_m) + o(1) < \varepsilon_{m+1}\\ O(\varepsilon_m) + \dfrac{1}{\sqrt{f(1)}} + o(1) > \varepsilon_{m+1}\end{cases}$$

Then:

$$\tilde\lambda^-(\kappa_\ell) + \varepsilon_{m+1} < \lambda^-(\kappa_{m+1}) < \tilde\lambda^-(\kappa_{\ell+1}) - \varepsilon_{m+1}$$

so, as $\big(\tilde\lambda^-(\kappa_\ell)\big)$ is a strictly increasing sequence, for m large enough $\lambda^-(\kappa_{m+1})$ is not ε_{m+1}-close to any element of $\big(\tilde\lambda^-(\kappa_\ell)\big)$: there is thus $p \in \mathbb{N}$ such that

$$\lambda^-(\kappa_{m+1}) - \tilde\lambda^+(\kappa_p) = O(\varepsilon_{m+1}), \tag{18}$$

with $|O(\varepsilon_{m+1})| \le \varepsilon_{m+1}$.

For the same reasons, we get also $\tilde\lambda^-(\kappa_{\ell-1}) + \varepsilon_{m-1} < \lambda^-(\kappa_{m-1}) < \tilde\lambda^-(\kappa_\ell) - \varepsilon_{m-1}$ and one deduces that (the previous picture helps to visualize it)

$$\lambda^-(\kappa_{m-1}) - \tilde\lambda^+(\kappa_{p-1}) = O(\varepsilon_{m-1}), \tag{19}$$

with $|O(\varepsilon_{m-1})| \le \varepsilon_{m-1}$. Since we have $f(1) > \tilde f(0)$, we get also

$$\tilde\lambda^+(\kappa_p) + \varepsilon_{m+2} < \lambda^-(\kappa_{m+2}) < \tilde\lambda^+(\kappa_{p+1}) - \varepsilon_{m+2}.$$

Consequently

$$\lambda^-(\kappa_{m+2}) - \tilde\lambda^-(\kappa_{\ell+1}) = O(\varepsilon_{m+2}) \tag{20}$$

with $|O(\varepsilon_{m+2})| \le \varepsilon_{m+2}$.

Then, by (17), (18), (19) and (20), we have for m large enough :

$$\begin{cases} \lambda^-(\kappa_{m+1}) - \lambda^-(\kappa_{m-1}) = \tilde{\lambda}^+(\kappa_p) - \tilde{\lambda}^+(\kappa_{p-1}) + O(\varepsilon_{m+1}) - O(\varepsilon_{m-1}) \\ \lambda^-(\kappa_{m+2}) - \lambda^-(\kappa_m) = \tilde{\lambda}^-(\kappa_{\ell+1}) - \tilde{\lambda}^-(\kappa_\ell) + O(\varepsilon_{m+2}) - O(\varepsilon_m). \end{cases}$$

In particular:

$$\begin{cases} \dfrac{2}{\sqrt{f(1)}} = \dfrac{1}{\sqrt{\tilde{f}(0)}} + o(1) \\ \dfrac{2}{\sqrt{f(1)}} = \dfrac{1}{\sqrt{\tilde{f}(1)}} + o(1), \end{cases}$$

and so, taking $m \to +\infty$, one deduces

$$2\sqrt{\tilde{f}(0)} = \sqrt{f(1)} \quad \text{and} \quad 2\sqrt{\tilde{f}(1)} = \sqrt{f(1)}.$$

As $\sqrt{f(0)} + \sqrt{f(1)} = \sqrt{\tilde{f}(0)} + \sqrt{\tilde{f}(1)}$, we get

$$\begin{aligned} 2\sqrt{f(0)} = 2\Big(\sqrt{\tilde{f}(0)} + \sqrt{\tilde{f}(1)} - \sqrt{f(1)}\Big) &= \Big(2\sqrt{\tilde{f}(0)} - \sqrt{f(1)}\Big) + \Big(2\sqrt{\tilde{f}(1)} - \sqrt{f(1)}\Big) \\ &= 0. \end{aligned}$$

and we get a contradiction as $f(0) > 0$.

Hence $f(0) \in \{\tilde{f}(0), \tilde{f}(1)\}$. The equality (12) gives the conclusion. □

Assume from now that $f(0) = \tilde{f}(0)$ and $f(1) = \tilde{f}(1)$. The other case is obtained by substituting the roles of $\tilde{\lambda}^-(\kappa_m)$ and $\tilde{\lambda}^+(\kappa_m)$.

3.1 The case $\mathbf{f(0) \neq f(1)}$

Without loss of generality, we assume that $f(0) < f(1)$. The following lemma focuses on the sequence $\big(\lambda^-(\kappa_p)\big)$ since this is the sequence that grows slower. If we had treated the case $f(0) > f(1)$, the sequence considered in this section would have been $\big(\lambda^+(\kappa_p)\big)$.

Lemma 3.2. *Assume that $f(0) < f(1)$. There is an infinite subset $\mathcal{L}$ of $\mathbb{N}$ such that*

- $\lambda^-(\kappa_p) - \tilde{\lambda}^-(\kappa_p) = \tilde{O}(e^{-2a\kappa_p}), \quad p \in \mathcal{L}.$
- *For all m in $\mathbb{N}$ large enough, $\{m, m+1\} \cap \mathcal{L} \neq \emptyset$.*

Proof. Let us denote U the subset of $\{\lambda^-(\kappa_m),\ m \in \mathbb{N}\}$ such that :

$$U \underset{(\varepsilon_m)}{\subsetneq} \{\tilde{\lambda}^+(\kappa_m),\ m \in \mathbb{N}\}$$

Case 1 : U is finite. Then there is $m_0 \in \mathbb{N}$ such that :

$$\forall m \geq m_0, \quad \exists p \in \mathbb{N},\ |\lambda^-(\kappa_m) - \tilde{\lambda}^-(\kappa_p)| \leq \varepsilon_m$$

This can be written as :

$$\lambda^-(\kappa_m) - \tilde{\lambda}^-(\kappa_p) = O(\varepsilon_m)$$

with $|O(\varepsilon_m)| \leq \varepsilon_m$. By replacing the eigenvalues by their asymptotics (13) in the previous equality, one finds, as $f(1) = \tilde{f}(1)$:

$$\frac{m}{\sqrt{f(1)}} + \frac{n-2}{2\sqrt{f(1)}} - \frac{\ln(h)'(1)}{4\sqrt{f(1)}} = \frac{p}{\sqrt{f(1)}} + \frac{n-2}{2\sqrt{f(1)}} - \frac{\ln(\tilde{h})'(1)}{4\sqrt{f(1)}} + O(\varepsilon_m)$$

- If $n = 2$ then $h = f^{n-2}$ is a constant. One has $\dfrac{m}{\sqrt{f(1)}} = \dfrac{p}{\sqrt{f(1)}} + O(\varepsilon_m)$. Hence, as m and p are integers, if m is large enough, we have $m = p$.

- If $n \geq 3$, then $m - p = \dfrac{\ln(h)'(1)}{4} - \dfrac{\ln(\tilde{h})'(1)}{4} + O(\varepsilon_m)$. By hypothesis :
$$\left|\frac{\ln(h)'(1)}{4} - \frac{\ln(\tilde{h})'(1)}{4}\right| = \frac{n-2}{4}\left|\frac{f'(1)}{f(1)} - \frac{\tilde{f}'(1)}{f(1)}\right| \leq \frac{n-2}{4} \times \frac{2}{n-2} = \frac{1}{2}.$$
Hence, $m = p$ for m, p greater than some integer m_0.

The set $\mathcal{L} = \{m \in \mathbb{N},\ m \geq m_0\}$ satisfies the properties of Lemma 3.2.

Case 2 : U is infinite. Then, there exists $\varphi, \psi : \mathbb{N} \to \mathbb{N}$ two strictly increasing functions such that :
$$\lambda^-(\kappa_{\psi(m)}) - \tilde{\lambda}^+(\kappa_{\varphi(m)}) = O(\varepsilon_{\psi(m)}), \tag{21}$$
with $|O(\varepsilon_{\psi(m)})| \leq \varepsilon_{\psi(m)}$.

Remark 6. φ and ψ are built in such a way that an integer $m \in \mathbb{N}$ that is not in the range of ψ (respectively not in the range of φ) satisfies $|\lambda^-(\kappa_m) - \tilde{\lambda}^-(\kappa_n)| < \varepsilon_m$ for some $n \in \mathbb{N}$ (respectively $|\lambda^+(\kappa_n) - \tilde{\lambda}^+(\kappa_m)| < \varepsilon_n$ for some $n \in \mathbb{N}$).

By replacing $\lambda^+(\kappa_{\varphi(m)})$ and $\tilde{\lambda}^-(\kappa_{\psi(m)})$ with their asymptotics in the equality (21), one has :
$$\frac{\varphi(m)}{\sqrt{f(0)}} = \frac{\psi(m)}{\sqrt{f(1)}} + C + O(\varepsilon_{\psi(m)}) + O\left(\frac{1}{\varphi(m)}\right) \tag{22}$$
where $C = -\dfrac{\ln(h)'(1)}{4\sqrt{f(1)}} - \dfrac{\ln(h)'(0)}{4\sqrt{f(0)}} + \dfrac{n-2}{2\sqrt{f(1)}} - \dfrac{n-2}{2\sqrt{f(0)}}$.

Lemma 3.3. *There is an integer* $m_0 \in \mathbb{N}$ *such that* $\big(m \geq m_0 \Rightarrow \psi(m+1) \geq \psi(m) + 2\big)$.

Proof. Set $B = \dfrac{\sqrt{f(1)}}{\sqrt{f(0)}} > 1$ and $C' = -\sqrt{f(1)}C$. From (22), one gets :
$$\psi(m) = B\varphi(m) + C' + o(1).$$
Assume $\psi(m+1) = \psi(m) + 1$. Then :
$$\begin{aligned}\psi(m) + 1 = \psi(m+1) &= B\varphi(m+1) + C' + o(1)\\ &\geq B(\varphi(m) + 1) + C' + o(1)\\ &= B\varphi(m) + C' + B + o(1)\\ &= \psi(m) + B + o(1).\end{aligned}$$
Thus, we find :
$$1 \geq B + o(1)$$
which is false if $m \geq m_0$ for some $m_0 \in \mathbb{N}$. □

Consequently, the range of ψ does not contain two consecutive integers. Let us set :
$$\mathcal{L} = \{m \in \mathbb{N} \,\big|\, m \geq m_0,\ m \notin \mathrm{range}(\psi)\}.$$
Then $\mathcal{L}$ satisfies the condition $\mathcal{L} \cap \{m, m+1\} \neq \emptyset$ for any $m \geq m_0$. Moreover, for any $m \in \mathcal{L}$, there is $\ell \in \mathbb{N}$ such that $|\lambda^-(\kappa_m) - \tilde{\lambda}^-(\kappa_\ell)| \leq \varepsilon_m$. One deduces, as previously, $m = \ell$. □

Remark 7. Lemma 3.2 and asymptotics (13) imply in particular

$$\frac{(\ln h)'(1)}{4\sqrt{f(1)}} = \frac{(\ln \tilde{h})'(1)}{4\sqrt{\tilde{f}(1)}}.$$

Now, let us recall an asymptotic integral representation of the Weyl-Titschmarsh function $N(z^2)$ obtained by Simon in [13] (Theorem 3.1) :

Theorem 3.4. *For every $0 < a < 1$, there is $A \in L^1([0, a])$ such that*

$$N(z^2) = -z - \int_0^a A(x)e^{-2xz}dx + \tilde{O}(e^{-2az}), \qquad z \to +\infty. \tag{23}$$

From the asymptotic of $\lambda^-(\kappa_m)$ obtained in Lemma 2.1, we have

$$\lambda^-(\kappa_m) = -\frac{N(\kappa_m)}{\sqrt{f(1)}} - \frac{(\ln h)'(1)}{4\sqrt{f(1)}} + \tilde{O}\left(e^{-2\sqrt{\kappa_m}}\right).$$

Hence

$$\begin{aligned}\lambda^-(\kappa_m) - &\tilde{\lambda}^-(\kappa_m) = \tilde{O}(e^{-2a\sqrt{\kappa_m}}) \\ &\Rightarrow -\frac{N(\kappa_m)}{\sqrt{f(1)}} - \frac{(\ln h)'(1)}{4\sqrt{f(1)}} = -\frac{\tilde{N}(\kappa_m)}{\sqrt{\tilde{f}(1)}} - \frac{(\ln \tilde{h})'(1)}{4\sqrt{\tilde{f}(1)}} + \tilde{O}(e^{-2a\sqrt{\kappa_m}}) + \tilde{O}\left(e^{-2\sqrt{\kappa_m}}\right)\end{aligned}$$

The equalities $f(1) = \tilde{f}(1)$ and $\dfrac{(\ln h)'(1)}{4\sqrt{f(1)}} = \dfrac{(\ln \tilde{h})'(1)}{4\sqrt{\tilde{f}(1)}}$ (see Remark 7) imply

$$\begin{aligned}\lambda^-(\kappa_m) - \tilde{\lambda}^-(\kappa_m) = \tilde{O}(e^{-2a\sqrt{\kappa_m}}) &\Rightarrow N(\kappa_m) = \tilde{N}(\kappa_m) + \tilde{O}(e^{-2a\sqrt{\kappa_m}}) \\ &\Rightarrow \int_0^a A(x)e^{-2x\sqrt{\kappa_m}}dx = \int_0^a \tilde{A}(x)e^{-2x\sqrt{\kappa_m}}dx + \tilde{O}(e^{-2a\sqrt{\kappa_m}}) \\ &\Rightarrow \int_0^a (A(x) - \tilde{A}(x))e^{-2x\sqrt{\kappa_m}}dx = \tilde{O}(e^{-2a\sqrt{\kappa_m}}).\end{aligned}$$

Let $\varepsilon > 0$ and set $F(z) = e^{2a(1-\varepsilon)z}\displaystyle\int_0^a (A(x) - \tilde{A}(x))e^{-2xz}dx$. The function F is entire and satisfies

$$\forall z \in \mathbb{C}, \quad \mathrm{Re}(z) > 0 \Rightarrow |F(z)| \leq \|A - \tilde{A}\|_1 e^{2a(1-\varepsilon)\mathrm{Re}(z)}$$

Let m be an integer large enough. From Lemma 3.2, we can find an integer p in $\{2m, 2m+1\}$ such that $|\lambda^-(\kappa_p) - \tilde{\lambda}^-(\kappa_p)| \leq \varepsilon_p$. We can thus build a sequence (u_m) by setting, for each m large enough, $u_m = \dfrac{\sqrt{\kappa_p}}{2}$. This sequence satisfies

$$u_m - m = O(1).$$

We set at last $G(z) = F(2z)$. Then $|G(z)| \leq \|A - \tilde{A}\|_1 e^{4a(1-\varepsilon)\mathrm{Re}(z)}$ and moreover :

$$G(u_m) = o(1)$$

Consequently (cf [2], Theorem 10.5.1, p.191), G is bounded on $\mathbb{R}_+$, and so is F :

$$\forall u \in \mathbb{R}_+, \quad \int_0^a (A(x) - \tilde{A}(x))e^{-2xu}dx = O\left(e^{-2a(1-\varepsilon)u}\right)$$

As this estimate is true for all $\varepsilon > 0$, we have :

$$\forall u \in \mathbb{R}_+, \quad \int_0^a (A(x) - \tilde{A}(x))e^{-2xu}dx = \tilde{O}\left(e^{-2au}\right),$$

hence ([13], Lemma A.2.1) $A = \tilde{A}$ on $[0, a]$. One deduces :

$$\forall t \in \mathbb{R}, \quad N(t^2) - \tilde{N}(t^2) = \tilde{O}(e^{-2at})$$

From Remark 5, N (resp. $\tilde{N}$) has the same role as M (resp. $\tilde{M}$) for the potential $x \mapsto q(1-x)$ (resp. $x \mapsto \tilde{q}(1-x)$). Now, it follows, from [13], Theorem A.1.1, that we get $q(1-x) = \tilde{q}(1-x)$ for all $x \in [0, a]$, *i.e*

$$\frac{(f^{n-2})''(x)}{f^{n-2}(x)} - \omega f(x) = \frac{(\tilde{f}^{n-2})''(x)}{\tilde{f}^{n-2}(x)} - \omega \tilde{f}(x) := r(x), \quad \forall x \in [1-a, 1].$$

The functions f and $\tilde{f}$ solve on $[1-a, 1]$ the same ODE

$$(y^{n-2})''(x) - \lambda y^{n-1}(x) = r(x) y^{n-2}(x)$$

Moreover $f(1) = \tilde{f}(1)$ and, thanks to the equality

$$\frac{(\ln h)'(1)}{4\sqrt{f(1)}} = \frac{(\ln \tilde{h})'(1)}{4\sqrt{\tilde{f}(1)}},$$

we also have $f'(1) = \tilde{f}'(1)$. Hence, the Cauchy-Lipschitz Theorem entails that $f = \tilde{f}$ on $[1-a, 1]$.

Remark 8. If we had assumed that $f(0) > f(1)$, we would have worked with $\big(\lambda^+(\kappa_p)\big)$ and $\big(\tilde{\lambda}^+(\kappa_p)\big)$, and found that $f = \tilde{f}$ on $[0, a]$.

3.2 The case $\mathbf{f(0) = f(1)}$

Assume, without loss of generality, that $f(0) = f(1) = 1$. From Lemma 2.1, the eigenvalues $\lambda^{\pm}(\kappa_m)$ satisfy the asymptotics :

$$\begin{cases} \lambda^-(\kappa_m) = -N(\kappa_m) - \dfrac{(\ln h)'(1)}{4} + \tilde{O}\Big(e^{-2\sqrt{\kappa_m}}\Big) \\ \lambda^+(\kappa_m) = -M(\kappa_m) + \dfrac{(\ln h)'(0)}{4} + +\tilde{O}\Big(e^{-2\sqrt{\kappa_m}}\Big). \end{cases}$$

Let us denote $C_0 = \dfrac{(\ln h)'(0)}{4}$, $C_1 = \dfrac{(\ln h)'(1)}{4}$, $\tilde{C}_0 = \dfrac{(\ln \tilde{h})'(0)}{4}$ and $\tilde{C}_1 = \dfrac{(\ln \tilde{h})'(1)}{4}$.
Using the asymptotics of M and N given in Theorem 2.2 and Corollary 2.3, and the explicit expression of $\kappa_m = m(m+n-2)$, we get

$$\begin{cases} \lambda^-(\kappa_m) = m + \dfrac{n-2}{2} - C_1 + O\Big(\dfrac{1}{m}\Big) \\ \lambda^+(\kappa_m) = m + \dfrac{n-2}{2} + C_0 + O\Big(\dfrac{1}{m}\Big). \end{cases} \qquad m \to +\infty. \tag{24}$$

Let us set also

$$V_m = \left\{\lambda^-(\kappa_m) - \frac{n-2}{2}\,,\, \lambda^+(\kappa_m) - \frac{n-2}{2}\right\} \text{ and } \tilde{V}_m = \left\{\tilde{\lambda}^-(\kappa_m) - \frac{n-2}{2}\,,\, \tilde{\lambda}^+(\kappa_m) - \frac{n-2}{2}\right\}.$$

As f and $\tilde{f}$ belong to $\mathcal{C}_b$, one has

$$|C_i| \leq \frac{1}{4}, \quad |\tilde{C}_i| \leq \frac{1}{4}, \quad i \in \{0, 1\}. \tag{25}$$

Hence, thanks to (24) and (25), we get for m large enough:

$$V_m, \tilde{V}_m \subset \left[m - \frac{1}{3}, m + \frac{1}{3}\right] \tag{26}$$

Of course, the assumption $\Sigma\big(\Lambda_g(\omega)\big) \underset{(\varepsilon_m)}{\asymp} \Sigma\big(\Lambda_{\tilde{g}}(\omega)\big)$ implies

$$-\frac{n-2}{2} + \Sigma\big(\Lambda_g(\omega)\big) \underset{(\varepsilon_m)}{\asymp} -\frac{n-2}{2} + \Sigma\big(\Lambda_{\tilde{g}}(\omega)\big) \tag{27}$$

From (26) and (27), for each m large enough, we have the alternative

$$\begin{cases} \lambda^-(\kappa_m) - \tilde{\lambda}^-(\kappa_m) = O\big(e^{-2a\sqrt{\kappa_m}}\big) \\ \lambda^+(\kappa_m) - \tilde{\lambda}^+(\kappa_m) = O\big(e^{-2a\sqrt{\kappa_m}}\big) \end{cases} \quad \text{or} \quad \begin{cases} \lambda^-(\kappa_m) - \tilde{\lambda}^+(\kappa_m) = O\big(e^{-2a\sqrt{\kappa_m}}\big) \\ \lambda^+(\kappa_m) - \tilde{\lambda}^-(\kappa_m) = O\big(e^{-2a\sqrt{\kappa_m}}\big) \end{cases} \tag{28}$$

There is thus an infinite set $\mathcal{S} \subset \mathbb{N}$ such that either

$$\begin{aligned} &\forall m \in \mathcal{S}, \quad \begin{cases} \lambda^-(\kappa_m) - \tilde{\lambda}^-(\kappa_m) = O\big(e^{-2a\sqrt{\kappa_m}}\big) \\ \lambda^+(\kappa_m) - \tilde{\lambda}^+(\kappa_m) = O\big(e^{-2a\sqrt{\kappa_m}}\big) \end{cases} \quad \text{or} \\ &\qquad\qquad \forall m \in \mathcal{S}, \quad \begin{cases} \lambda^-(\kappa_m) - \tilde{\lambda}^+(\kappa_m) = O\big(e^{-2a\sqrt{\kappa_m}}\big) \\ \lambda^+(\kappa_m) - \tilde{\lambda}^-(\kappa_m) = O\big(e^{-2a\sqrt{\kappa_m}}\big). \end{cases} \end{aligned} \tag{29}$$

Assume, for example, that the former is true. Then we have, using (24) :

$$C_1 = \tilde{C}_1 \quad \text{and} \quad C_0 = \tilde{C}_0. \tag{30}$$

Case 1 : $C_0 \neq -C_1$.
Let us denote

$$\delta = \frac{|C_0 + C_1|}{3} \in \left]0, \frac{1}{4}\right[.$$

For m large enough, we have, thanks to (24):

- $\lambda^-(\kappa_m) - \frac{n-2}{2}$ and $\tilde{\lambda}^-(\kappa_m) - \frac{n-2}{2}$ both belong to the interval $-C_1 + [m-\delta, m+\delta] := I_{m,1}$
- $\lambda^+(\kappa_m) - \frac{n-2}{2}$ and $\tilde{\lambda}^+(\kappa_m) - \frac{n-2}{2}$ both belong to the interval $C_0 + [m-\delta, m+\delta] := I_{m,0}$.

Moreover, as $C_0 \neq -C_1$, we have $d(I_{m,1}, I_{m,0}) \geq \delta$ for all m large enough, where $d(I,J) = \inf_{x\in I, y \in J} |x-y|$. We can therefore associate eigenvalues as follows :

$$\begin{cases} \lambda^-(\kappa_m) = \tilde{\lambda}^-(\kappa_m) + O(e^{-2a\sqrt{\kappa_m}}) \\ \lambda^+(\kappa_m) = \tilde{\lambda}^+(\kappa_m) + O(e^{-2a\sqrt{\kappa_m}}) \end{cases} \qquad m \to +\infty.$$

One shows, as in Section 3.1, that

$$\begin{cases} N(t^2) - \tilde{N}(t^2) = \tilde{O}(e^{-2at}) \\ M(t^2) - \tilde{M}(t^2) = \tilde{O}(e^{-2at}) \end{cases} \qquad t \to +\infty$$

and then, that

$$f(x) = \tilde{f}(x) \quad \forall x \in [1-a, 1] \quad \text{and} \quad f(x) = \tilde{f}(x) \quad \forall x \in [0, a].$$

Case 2 : $C_0 = -C_1$.

By hypothesis : $f, \tilde{f}$ belong to $\mathcal{C}_b$ so q belongs to $\mathcal{D}_b$. Thanks to Lemma 2.4, there is $j_0 \in \mathbb{N}$ such that $\beta_j(0) \neq \gamma_j(0)$. Let us set

$$j_0 = \min\{j \geq 2,\ \beta_j(0) \neq \gamma_j(0)\}.$$

The asymptotics given by Theorem 2.2 and Corollary (2.3) imply

$$\lambda^-(\kappa_m) - \lambda^+(\kappa_m) = \frac{\gamma_{j_0} - \beta_{j_0}}{m^{j_0}} + O\left(\frac{1}{m^{j_0+1}}\right).$$

Because of the relation $\Sigma(\Lambda_g(\omega)) \underset{(\varepsilon_m)}{\asymp} \Sigma(\Lambda_{\tilde{g}}(\omega))$, one can show that, for the same j_0:

$$\tilde{\lambda}^-(\kappa_m) - \tilde{\lambda}^+(\kappa_m) = \frac{\tilde{\gamma}_{j_0} - \tilde{\beta}_{j_0}}{m^{j_0}} + O\left(\frac{1}{m^{j_0+1}}\right)$$

It is then possible to order the eigenvalues $\lambda^-(\kappa_m)$ and $\lambda^+(\kappa_m)$ (also $\tilde{\lambda}^-$ and $\tilde{\lambda}^+$), and this order depends on the sign of $\gamma_{j_0} - \beta_{j_0}$ (resp. $\tilde{\gamma}_{j_0} - \tilde{\beta}_{j_0}$). If $\gamma_{j_0} - \beta_{j_0}$ and $\tilde{\gamma}_{j_0} - \tilde{\beta}_{j_0}$ have the same sign, we claim that

$$\begin{cases} \lambda^-(\kappa_m) = \tilde{\lambda}^-(\kappa_m) + \tilde{O}(e^{-2a\sqrt{\kappa_m}}) \\ \lambda^+(\kappa_m) = \tilde{\lambda}^+(\kappa_m) + \tilde{O}(e^{-2a\sqrt{\kappa_m}}). \end{cases} \tag{31}$$

Indeed, if not, from (28), there is an infinite subset $\mathcal{F} \subset \mathbb{N}$ such that :

$$\begin{cases} \lambda^-(\kappa_m) = \tilde{\lambda}^+(\kappa_m) + \tilde{O}(e^{-2a\sqrt{\kappa_m}}) \\ \lambda^+(\kappa_m) = \tilde{\lambda}^-(\kappa_m) + \tilde{O}(e^{-2a\sqrt{\kappa_m}}). \end{cases}, \qquad m \to +\infty,\ m \in \mathcal{F}.$$

Then $\lambda^-(\kappa_m) - \lambda^+(\kappa_m) = \tilde{\lambda}^+(\kappa_m) - \tilde{\lambda}^-(\kappa_m) + O(e^{-2a\sqrt{\kappa_m}})$, and letting m go to infinity :

$$\gamma_{j_0} - \beta_{j_0} = \tilde{\beta}_{j_0} - \tilde{\gamma}_{j_0}$$

and we have a contradiction. Using (31) and the same method as in Section 3.1, we find

$$\forall t \in \mathbb{R}_+, \quad M(t^2) - \tilde{M}(t^2) = \tilde{O}(e^{-2at}) \quad \text{and} \quad N(t^2) - \tilde{N}(t^2) = \tilde{O}(e^{-2at})$$

and at last

$$f = \tilde{f} \text{ on } [0, a] \quad \text{and} \quad f = \tilde{f} \text{ on } [1-a, 1].$$

If $\gamma_{j_0} - \beta_{j_0}$ and $\tilde{\gamma}_{j_0} - \tilde{\beta}_{j_0}$ have opposite sign, then :

$$\begin{cases} \lambda^-(\kappa_m) = \tilde{\lambda}^+(\kappa_m) + O(e^{-2a\sqrt{\kappa_m}}) \\ \lambda^+(\kappa_m) = \tilde{\lambda}^-(\kappa_m) + O(e^{-2a\sqrt{\kappa_m}}). \end{cases}$$

In this case, one can prove that

$$f = \tilde{f} \circ \eta \text{ on } [0, a] \quad \text{and} \quad f = \tilde{f} \circ \eta \text{ on } [1-a, 1].$$

3.3 Special case

When $f(0)=f(1)$, the direct implication has already been established. Now, we prove the converse in this case. Let $a\in]0,1[$ and assume that, for example:

$$f=\tilde{f} \text{ on } [0,a] \quad \text{and} \quad f=\tilde{f} \text{ on } [1-a,1]$$

In that case, $q=\tilde{q}$ on $[0,a]$ and $q\circ\eta=\tilde{q}\circ\eta$ on $[0,a]$. But thanks to Theorem 3.1 in [13], the potential q determines the function A that appears in the representation (3.4). Hence,

$$\begin{cases} M(z^2)-\tilde{M}(z^2)=\tilde{O}(e^{-2az}) \\ N(z^2)-\tilde{N}(z^2)=\tilde{O}(e^{-2az}) \end{cases}$$

The hypothesis (P_1) implies in particular that $f(0)=\tilde{f}(0)$ and $f'(0)=\tilde{f}'(0)$. From (P_3) we have also $f(1)=\tilde{f}(1)$ and $f'(1)=\tilde{f}'(1)$. Using the asymptotics given by Lemma 2.1, one deduces immediately that

$$\begin{cases} \lambda^+(\kappa_m)-\tilde{\lambda}^+(\kappa_m)=\tilde{O}(e^{-2a\sqrt{\kappa_m}}) \\ \lambda^-(\kappa_m)-\tilde{\lambda}^-(\kappa_m)=\tilde{O}(e^{-2a\sqrt{\kappa_m}}) \end{cases} \qquad m\to+\infty$$

which concludes the proof. □

Remark 9. We emphasize that if $f(0)=f(1)$ and $\frac{1}{2}\leq a<1$, then we have a global uniqueness result.

Corollary 3.5. *If f and $\tilde{f}$ are analytic functions on $[0,1]$ the previous local uniqueness result becomes a global uniqueness result without the additional constraint that $q,\tilde{q}\in\mathcal{D}_b$.*

Proof. In proving Theorem 1.7, we needed the hypothesis $q,\tilde{q}\in\mathcal{D}_b$ only in the case where $f(0)=f(1)$ and $C_0=-C_1$. In all other cases, without this hypothesis, one of the properties (P_1), (P_2), (P_3) or (P_4) was obtained and, as f and $\tilde{f}$ are assumed to be analytic, the corresponding equalities extend over $[0,1]$ by analytic continuation. Then, only this latter case remains to be dealt with. Let us assume, without loss of generality, that $f(0)=1$.

Subcase 1 : $q,\tilde{q}\in\mathcal{D}_b$ and the situation has already been studied.

Subcase 2 : q or $\tilde{q}$ does not belong to $\mathcal{D}_b$. Assume, for example, that $q\notin\mathcal{D}_b$. As f is analytic, so is q. The function $\varphi:[0,1]\mapsto\mathbb{R},\ x\mapsto q(x)-q(1-x)$ is also analytic and, as q is not in $\mathcal{D}_b$, satisfies :

$$\forall k\in\mathbb{N},\quad \varphi^{(k)}(0)=0.$$

By analytic continuation, φ vanishes everywhere on $[0,1]$, *i.e* $q=q\circ\eta$ on $[0,1]$. This is equivalent to $M=N$. The eigenvalues $\lambda^\pm(\kappa_m)$ then satisfy the asymptotics

$$\begin{cases} \lambda^-(\kappa_m)=M(\kappa_m)+C_0+\tilde{O}\left(e^{-2a\sqrt{\kappa_m}}\right) \\ \lambda^+(\kappa_m)=M(\kappa_m)+C_0+\tilde{O}\left(e^{-2a\sqrt{\kappa_m}}\right) \end{cases}$$

Using the assumption $\sigma(\Lambda_g(\omega)) \underset{(\varepsilon_m)}{\asymp} \sigma(\Lambda_{\tilde{g}}(\omega))$ and the same arguments as above, we prove that $C_0=\tilde{C}_0=-\tilde{C}_1$ and

$$\begin{cases} M(\kappa_m)=\tilde{M}(\kappa_m)+O\big(e^{-2a\sqrt{\kappa_m}}\big) \\ M(\kappa_m)=\tilde{N}(\kappa_m)+O\big(e^{-2a\sqrt{\kappa_m}}\big) \end{cases} \qquad m\to+\infty$$

One can show, as in the proof of Theorem 1.7, that :

$$M(t^2) - \tilde{M}(t^2) = O(e^{-2at}) \quad \text{and} \quad M(t^2) - \tilde{N}(t^2) = O(e^{-2at}).$$

Hence $f = \tilde{f}$ on $[0, a]$ and $f = \tilde{f} \circ \eta$ on $[0, a]$. By analytic continuation :

$$f = \tilde{f} = \tilde{f} \circ \eta \text{ sur } [0, 1].$$

□

The next section is devoted to the proof of Theorem 1.9.

4 Stability estimates for symmetric conformal factors

4.1 Discrete estimates on Weyl-Titchmarsh functions

Preliminary remarks:

1. Until the end of the paper, we will denote by C_A any constant depending only on A, even within the same calculation.
2. In this section, each factor f and $\tilde{f}$ is supposed so be symmetric. This simplifies many formula. However, in order to generalize our arguments as much as possible, we will use this property of symmetry only when it seems necessary and write the formulas in their generic forms. For example, we will distinguish M from N whereas those two functions are equal.

The goal of this subsection is to prove the following result which will be useful in Subsection 4.2.

Proposition 4.1. *Let $\varepsilon > 0$ small enough. Assume that $\sigma(\Lambda_g(\omega)) \underset{\varepsilon}{\asymp} \sigma(\Lambda_{\tilde{g}}(\omega))$. There is $C_A > 0$ and $m_0 \in \mathbb{N}$ (independant of ε) such that, for all $m \geq m_0$ and by setting $y_m = \sqrt{\kappa_m}$:*

$$\left| \left(M(\kappa_m) N(\kappa_m) - \frac{1}{\Delta^2(\kappa_m)} \right) - \left(\tilde{M}(\kappa_m) \tilde{N}(\kappa_m) - \frac{1}{\tilde{\Delta}^2(\kappa_m)} \right) \right| \leq C_A \varepsilon \times y_m$$

Proof. We first need the following result.

Lemma 4.2. *Under the hypothesis $\sigma(\Lambda_g(\omega)) \underset{\varepsilon}{\asymp} \sigma(\Lambda_{\tilde{g}}(\omega))$, and under the hypothesis that f and $\tilde{f}$ are symmetric, we have :*

$$f(0) = \tilde{f}(0).$$

Proof. Using the same argument as in the proof of Lemma 3.1, one proves the equality

$$\sqrt{f(1)} + \sqrt{f(0)} = \sqrt{\tilde{f}(1)} + \sqrt{\tilde{f}(0)}.$$

As f and $\tilde{f}$ are symmetric with respect to $1/2$, we have $f(0) = f(1)$ and $\tilde{f}(0) = \tilde{f}(1)$. Hence $2f(0) = 2\tilde{f}(0)$. This proves Lemma 4.2. □

Lemma 4.3. *For m large enough, we have :*

$$\{\lambda^-(\kappa_m), \lambda^+(\kappa_m)\} \underset{\varepsilon}{\asymp} \{\tilde{\lambda}^-(\kappa_m), \tilde{\lambda}^+(\kappa_m)\} \tag{32}$$

Proof. For every $m \in \mathbb{N}$, there p such that :

$$|\lambda^{\pm}(\kappa_m) - \tilde{\lambda}^{\pm}(\kappa_p| \leq \varepsilon \tag{33}$$

Let us denote

$$C_1 = \frac{1}{4\sqrt{f(1)}}\frac{h'(1)}{h(1)} \quad \text{et} \quad C_0 = \frac{1}{4\sqrt{f(0)}}\frac{h'(0)}{h(0)}.$$

Since f and $\tilde{f}$ are supposed symmetric, we have

$$C_0 = -C_1 \quad \text{and} \quad \tilde{C}_0 = -\tilde{C}_1$$

Thus, setting

$$C = C_0 - \tilde{C}_0$$

one has, from Lemma 2.1 :

$$\sqrt{f(0)}\Big(\lambda^{\pm}(\kappa_m) - \tilde{\lambda}^{\pm}(\kappa_p)\Big) = \big(m - p\big) + C + o(1).$$

Let $k = \lfloor C \rfloor$. Then

$$m - p + k = \sqrt{f(0)}\Big(\lambda^{\pm}(\kappa_m) - \tilde{\lambda}^{\pm}(\kappa_p)\Big) + \underbrace{k - C}_{\in]-1,0[} + o(1) \tag{34}$$

and, as $m - p + k$ is an integer, using (33), this leads, for m large enough and ε small enough, to

$$p = m + k.$$

Hence, for m large enough, (33) is equivalent to

$$\begin{cases} |\lambda^{-}(\kappa_m) - \tilde{\lambda}^{-}(\kappa_{m+k})| \leq \varepsilon \\ |\lambda^{+}(\kappa_m) - \tilde{\lambda}^{+}(\kappa_{m+k})| \leq \varepsilon \end{cases} \quad \text{or} \quad \begin{cases} |\lambda^{-}(\kappa_m) - \tilde{\lambda}^{+}(\kappa_{m+k})| \leq \varepsilon \\ |\lambda^{+}(\kappa_m) - \tilde{\lambda}^{-}(\kappa_{m+k})| \leq \varepsilon \end{cases}$$

The relation $\Sigma\big(\Lambda_g(\omega)\big) \underset{\varepsilon}{\asymp} \Sigma\big(\Lambda_{\tilde{g}}(\omega)\big)$ implies

$$2m = 2(m + k)$$

and then $k = 0$.

This means that, for m greater than some m_0 (that does not depend on ε) one has

$$\{\lambda^{-}(\kappa_m), \lambda^{+}(\kappa_m)\} \underset{\varepsilon}{\asymp} \{\tilde{\lambda}^{-}(\kappa_m), \tilde{\lambda}^{+}(\kappa_m)\} \tag{35}$$

□

Of course, Lemma 4.3 is still true by replacing κ_m by μ_m. For m large enough, we have :

$$\{\lambda^{-}(\mu_m), \lambda^{+}(\mu_m)\} \underset{\varepsilon}{\asymp} \{\tilde{\lambda}^{-}(\mu_m), \tilde{\lambda}^{+}(\mu_m)\} \tag{36}$$

Recall that

$$\Lambda_g^m(\omega) = \begin{pmatrix} -\frac{M(\mu_m)}{\sqrt{f(0)}} + C_0 & -\frac{1}{\sqrt{f(0)}}\frac{h^{1/4}(1)}{h^{1/4}(0)}\frac{1}{\Delta(\mu_m)} \\ -\frac{1}{\sqrt{f(1)}}\frac{h^{1/4}(0)}{h^{1/4}(1)}\frac{1}{\Delta(\mu_m)} & -\frac{N(\mu_m)}{\sqrt{f(1)}} - C_1 \end{pmatrix}$$

Hence

$$
\begin{aligned}
&\mathrm{Tr}\big(\Lambda_g^m(\omega)\big) - \mathrm{Tr}\big(\Lambda_{\tilde g}^m(\omega)\big)\\
&\qquad = \Big[-\frac{M(\mu_m)}{\sqrt{f(0)}} + C_0 - \frac{N(\mu_m)}{\sqrt{f(1)}} - C_1\Big] - \Big[-\frac{\tilde M(\mu_m)}{\sqrt{\tilde f(0)}} + \tilde C_0 - \frac{\tilde N(\mu_m)}{\sqrt{\tilde f(1)}} - \tilde C_1\Big]\\
&\qquad = -\frac{1}{\sqrt{f(0)}}\Big(M(\mu_m) - \tilde M(\mu_m)\Big) - \frac{1}{\sqrt{f(1)}}\Big(N(\mu_m) - \tilde N(\mu_m)\Big)\\
&\qquad\qquad + (\tilde C_0 - C_0) + (C_1 - \tilde C_1)
\end{aligned}
\tag{37}
$$

Thanks to (34), with $k = 0$ and $m - p = 0$, we have

$$
|C| = |\tilde C_0 - C_0| = |C_1 - \tilde C_1| \leq C_A \varepsilon. \tag{38}
$$

Hence, combining (36), (37) and (38), we get :

$$
\begin{aligned}
\left| \frac{1}{\sqrt{f(0)}}\Big(M(\mu_m) - \tilde M(\mu_m)\Big) + \frac{1}{\sqrt{f(1)}}\Big(N(\mu_m) - \tilde N(\mu_m)\Big)\right| &\leq \underbrace{\left|\mathrm{Tr}\big(\Lambda_g^m(\omega)\big) - \mathrm{Tr}\big(\Lambda_{\tilde g}^m(\omega)\big)\right|}_{\leq 2\varepsilon} + C_A\varepsilon\\
&\leq C_A \varepsilon.
\end{aligned}
$$

As f and $\tilde f$ are symmetric with respect to $1/2$, this leads to

$$
\Big| M(\mu_m) - \tilde M(\mu_m)\Big| \leq C_A \varepsilon
$$

We also have an estimate on the determinant. From (36), assume for example that

$$
\big|\lambda^+(\mu_m) - \tilde\lambda^+(\mu_m)\big| \leq \varepsilon \quad \text{and} \quad \big|\tilde\lambda^-(\mu_m) - \lambda^-(\mu_m)\big| \leq \varepsilon.
$$

Then :

$$
\begin{aligned}
\Big|\det(\Lambda_g^m(\omega)) - \det(\Lambda_{\tilde g}^m(\omega))\Big| &= \Big|\lambda^-(\mu_m)\lambda^+(\mu_m) - \tilde\lambda^-(\mu_m)\tilde\lambda^+(\mu_m)\Big|\\
&\leq \big|\lambda^-(\mu_m)\big|\big|\lambda^+(\mu_m) - \tilde\lambda^+(\mu_m)\big| + \big|\tilde\lambda^-(\mu_m) - \lambda^-(\mu_m)\big|\big|\tilde\lambda^+(\mu_m)\big|\\
&\leq C_A\varepsilon \times \sqrt{\mu_m}
\end{aligned}
$$

We write :

$$
\det(\Lambda_g^m(\lambda)) - \det(\Lambda_{\tilde g}^m(\lambda)) = \mathrm{I}(\mu_m) + \mathrm{II}(\mu_m) + \mathrm{III}(\mu_m) + \mathrm{IV}
$$

with

$$
\begin{aligned}
\mathrm{I}(\mu_m) &= \frac{1}{\sqrt{f(0)f(1)}}\Big[\Big(M(\mu_m)N(\mu_m) - \frac{1}{\Delta^2(\mu_m)}\Big) - \Big(\tilde M(\mu_m)\tilde N(\mu_m) - \frac{1}{\tilde\Delta^2(\mu_m)}\Big)\Big],\\
\mathrm{II}(\mu_m) &= \frac{1}{\sqrt{f(0)}}\Big[C_1(M(\mu_m) - \tilde M(\mu_m)) + (C_1 - \tilde C_1)\tilde M(\mu_m)\Big]\\
\mathrm{III}(\mu_m) &= \frac{1}{\sqrt{f(1)}}\Big[\tilde C_0(\tilde N(\mu_m) - N(\mu_m)) + (\tilde C_0 - C_0)N(\mu_m)\Big]
\end{aligned}
$$

and

$$
\mathrm{IV} = (\tilde C_0 - C_0)\tilde C_1 + C_0(\tilde C_1 - C_1)
$$

We have:

$$\begin{aligned}
|\mathrm{II}(\mu_m)| &\leq \frac{1}{\sqrt{f(0)}}|C_1||M(\mu_m)-\tilde{M}(\mu_m)| + \frac{1}{\sqrt{f(0)}}|C_1-\tilde{C}_1||\tilde{M}(\mu_m)| \\
&\leq C_A\varepsilon + C_A\varepsilon\sqrt{\mu_m} \\
&\leq C_A\varepsilon\sqrt{\mu_m}.
\end{aligned}$$

Similarly:

$$|\mathrm{II}(\mu_m)| \leq C_A\varepsilon\sqrt{\mu_m} \quad \text{and} \quad |\mathrm{IV}| \leq C_A\varepsilon.$$

Finally :

$$\begin{aligned}
|I(\mu_m)| &\leq \left|\det(\Lambda_g^m(\lambda)) - \det(\Lambda_{\tilde{g}}^m(\lambda))\right| + C_A\varepsilon\sqrt{\mu_m} \\
&\leq C_A\varepsilon\sqrt{\mu_m}
\end{aligned}$$

As this is true for μ_m, with $m \geq m_0$ with m_0 not depending on ε, this is also true for κ_m with $m \geq m_0$ (with m_0 different from the other one but still independent of ε). Hence, by setting $y_m = \sqrt{\kappa_m}$, we have proved that there exists $C_A > 0$ such that, for $m \geq m_0$:

$$\left|\left(M(\kappa_m)N(\kappa_m) - \frac{1}{\Delta^2(\kappa_m)}\right) - \left(\tilde{M}(\kappa_m)\tilde{N}(\kappa_m) - \frac{1}{\tilde{\Delta}^2(\kappa_m)}\right)\right| \leq C_A\varepsilon \times y_m$$

□

4.2 An integral estimate

In all this section, we will use the estimate of Proposition 4.1 in order to show that

$$\|q-\tilde{q}\|_{L^2(0,1)} \leq C_A\frac{1}{\ln\left(\frac{1}{\varepsilon}\right)}.$$

where q is the potential defined in (4).

Let us go back to the Sturm-Liouville equation

$$-u'' + qu = -zu, \quad z \in \mathbb{C} \tag{39}$$

and to the fundamental system of solutions $\{c_0, s_0\}$ and $\{c_1, s_1\}$ given by (6). We define ψ and ϕ as the two unique solutions of (39) that can be written as

$$\psi(x,z) = c_0(x,z) + M(z)s_0(x,z), \quad \phi(x,z) = c_1(x,z) - N(z)s_1(x,z), \tag{40}$$

with Dirichlet boundary conditions at $x=1$ and $x=0$ respectively.

Proposition 4.4. *We have the following relations*

$$\begin{aligned}
s_0(1,z) &= \Delta(z) & && s_1(0,z) &= -\Delta(z) \\
s_0'(1,z) &= -N(z)\Delta(z) & && s_1'(0,z) &= -M(z)\Delta(z) \\
c_0(1,z) &= -M(z)\Delta(z) & \text{and} && c_1(0,z) &= -N(z)\Delta(z) \\
c_0'(1,z) &= M(z)N(z)\Delta(z) - \frac{1}{\Delta(z)} & && c_1'(0,z) &= \frac{1}{\Delta(z)} - N(z)M(z)\Delta(z).
\end{aligned}$$

Proof. First of all, the equalities $s_0(1,z) = \Delta(z)$ and $s_1(0,z) = -\Delta(z)$ come from the relation (7). The set of solutions of (39) that satisfy $u(0,z) = 0$ is a one dimensional vector space. Therefore, there exists $A(z) \in \mathbb{C}$ such that :

$$\forall x \in [0,1], \quad s_0(x,z) = A(z)\phi(x,z) \tag{41}$$

The conditions on c_1 and s_1 at $x=1$ lead to the equality $A(z) = s_0(1,z) = \Delta(z)$. We get also, by differentiating (41):

$$s_0'(1,z) = A(z)\big(c_1'(1,z) - N(z)s_1'(1,z)\big) = -N(z)A(z) = -\Delta(z)N(z).$$

Analogously, there is $B(z) \in \mathbb{C}$ such that

$$s_1(x,z) = B(z)\psi(x,z)$$

and we show $B(z) = -\Delta(z)$. Hence $s_1'(1,z) = -\Delta(z)\psi'(1,z)$ and so $\psi'(1,z) = -\dfrac{1}{\Delta(z)}$.
By differentiating (41) and taking $x=1$, we get:

$$c_0'(1,z) + M(z)s_0'(1,z) = -\frac{1}{\Delta(z)}$$

and then

$$c_0'(1,z) = M(z)N(z)\Delta(z) - \frac{1}{\Delta(z)}.$$

This proves the equalities on c_0 and s_0. We proceed similarly to establish those on c_1 and s_1. $\square$

Thanks to those relations, we are now able to prove the following lemma:

Lemma 4.5. *Denote $\mathcal{P}$ the poles of N. For any $z \in \mathbb{C}\backslash\mathcal{P}$ we have the equality :*

$$\begin{aligned}\tilde{\Delta}(z)\Delta(z)\left(M(z)N(z) - \frac{1}{\Delta(z)^2}\right) - M(z)\tilde{N}(z)\Delta(z)\tilde{\Delta}(z) + 1& \\ = \int_0^1 \big(q(x) - \tilde{q}(x)\big)c_0(x,z)\tilde{s}_0(x,z)dx&\end{aligned} \tag{42}$$

Proof. Let us define $\theta : x \mapsto c_0(x,z)\tilde{s}_0'(x,z) - s_0'(x,z)\tilde{c}_0(x,z)$. Then :

$$\begin{aligned}\theta'(x) &= c_0(x,z)\tilde{s}_0''(x,z) + c_0'(x,z)\tilde{s}_0'(x,z) - c_0'(x,z)\tilde{s}_0'(x,z) - c_0''(x,z)\tilde{s}_0(x,z)\\ &= c_0(x,z)(\tilde{q}(x)\tilde{s}_0(x,z) + z\tilde{s}_0(x,z)) - (q(x)c_0(x,z) + zc_0(x,z))\tilde{s}_0(x,z)\\ &= \big(\tilde{q}(x) - q(x)\big)c_0(x,z)\tilde{s}_0(x,z)\end{aligned}$$

Hence, by integrating between 0 and 1 :

$$\theta(1) - \theta(0) = \int_0^1 \big(q(x) - \tilde{q}(x)\big)c_0(x,z)\tilde{s}_0(x,z)dx.$$

By replacing $c_0'(1,z)$, $\tilde{s}_0(1,z)$, $c_0(1,z)$ and $\tilde{s}_0'(1,z)$ by the expressions given in Proposition 4.4, we get the relation of Lemma 4.5. $\square$

By inverting the roles of q and $\tilde{q}$, we get

$$\begin{aligned}\Delta(z)\tilde{\Delta}(z)\left(\tilde{M}(z)\tilde{N}(z) - \frac{1}{\tilde{\Delta}(z)^2}\right) - \tilde{M}(z)N(z)\Delta(z)\tilde{\Delta}(z) + 1& \\ = \int_0^1 \big(\tilde{q}(x) - q(x)\big)\tilde{c}_0(x,z)s_0(x,z)dx&\end{aligned} \tag{43}$$

At last, from Remark 5, if we replace $q(x)$ and $\tilde{q}(x)$ by $q(1-x)$ and $\tilde{q}(1-x)$, then, the roles of M and N are inverted. Moreover, we remark that $c_1(1-x)$ and $-s_1(1-x)$ play the roles of $c_0(x)$ and $s_0(x)$ but for the potential $q(1-x)$, *i.e.* denoting $\eta(x) = 1-x$:

$$c_0(x,z,,q\circ\eta) = c_1(1-x,z,q) \quad \text{and} \quad s_0(x,z,,q\circ\eta) = -s_1(1-x,z,q)$$

Hence :

$$\Delta(z)\tilde{\Delta}(z)\Big(\tilde{M}(z)\tilde{N}(z)-\frac{1}{\tilde{\Delta}(z)^2}\Big)-\tilde{N}(z)M(z)\Delta(z)\tilde{\Delta}(z)+1 \\ =-\int_0^1\big(\tilde{q}(1-x)-q(1-x)\big)\tilde{c}_1(1-x,z)s_1(1-x,z)dx \tag{44}$$

As q is symmetric, we have $c_1(1-x)=c_0(x)$ and $s_1(1-x)=-s_0(x)$. The previous equality can be written

$$\Delta(z)\tilde{\Delta}(z)\Big(\tilde{M}(z)\tilde{N}(z)-\frac{1}{\tilde{\Delta}(z)^2}\Big)-\tilde{N}(z)M(z)\Delta(z)\tilde{\Delta}(z)+1 \\ =\int_0^1\big(\tilde{q}(x)-q(x)\big)\tilde{c}_0(x,z)s_0(x,z)dx \tag{45}$$

Hence, by substracting the relation of Lemma 4.5 from equality (45), we get

$$\Delta(x)\tilde{\Delta}(z)\Big[\Big(M(z)N(z)-\frac{1}{\Delta(z)^2}\Big)-\Big(\tilde{M}(z)\tilde{N}(z)-\frac{1}{\tilde{\Delta}(z)^2}\Big)\Big]=\int_0^1\big(q(x)-\tilde{q}(x)\big)c_0(x,z)\tilde{s}_0(x,z)dx \\ +\int_0^1\big(q(x)-\tilde{q}(x)\big)\tilde{c}_0(x,z)s_0(x,z)dx$$

Using Proposition 4.1, we have proved:

Proposition 4.6. *There is $m_0\in\mathbb{N}$ such that, for $m\geq m_0$:*

$$\Big|\int_0^1\big(q(x)-\tilde{q}(x)\big)\big[c_0(x,\kappa_m)\tilde{s}_0(x,\kappa_m)+\tilde{c}_0(x,\kappa_m)s_0(x,\kappa_m)\big]dx\Big|\leq C_A y_m|\Delta(\kappa_m)||\tilde{\Delta}(\kappa_m)|\varepsilon \tag{46}$$

4.3 Construction of an inverse integral operator

From now on, we set $L(x)=q(x)-\tilde{q}(x)$. We want to express the integrand in the left-hand-side of (46) in terms of an operator acting on L.

Proposition 4.7. *There is an operator $B:L^2([0,1])\to L^2([0,1])$ such that :*

1. *For all $m\in\mathbb{N}$,*

$$\int_0^1\big[c_0(x,\kappa_m)\tilde{s}_0(x,\kappa_m)+\tilde{c}_0(x,\kappa_m)s_0(x,\kappa_m)\big]L(x)dx=\frac{1}{y_m}\int_0^1\sinh(2\tau y_m)BL(\tau)d\tau.$$

2. *The function $\tau\mapsto BL(\tau)$ is C^1 on $[0,1]$ and BL and $(BL)'$ are uniformly bounded by a constant C_A.*

Proof. Let us extend on $[-1,0]$ q and $\tilde{q}$ into even functions. From [10] (page 9) we have the following integral representations of the functions c_0 and s_0:

$$s_0(x,-z^2)=\frac{\sin(zx)}{z}+\int_0^x H(x,t)\frac{\sin(zt)}{z}dt$$
$$c_0(x,-z^2)=\cos(zx)+\int_0^x P(x,t)\cos(zt)dt$$

where $H(x,t)$ and $P(x,t)$ can be written as

$$H(x,t)=K(x,t)-K(x,-t) \\ P(x,t)=K(x,t)+K(x,-t) \tag{47}$$

with K a C^1 function on $[-1,1]\times[-1,1]$ satisfying some good estimates. More precisely ([10], p.14), we have:

Theorem 4.8. *On $[-1,1]\times[-1,1]$, K satisfies the estimate*

$$|K(x,t)| \leq \frac{1}{2}w\left(\frac{x+t}{2}\right)\exp\left(\sigma_1(x)-\sigma_1\left(\frac{x+t}{2}\right)-\sigma_1\left(\frac{x-t}{2}\right)\right)$$

with $w(u)=\max_{0\leq\xi\leq u}\left|\int_0^\xi q(y)dy\right|$, $\qquad \sigma_0(x)=\int_0^x |q(t)|dt, \qquad \sigma_1(x)=\int_0^x \sigma_0(t)dt.$

We thus have the following estimate :

Proposition 4.9. *There is a constant $C_A>0$, which only depends on A, such that*

$$\|K\|_\infty + \left\|\frac{\partial K}{\partial x}\right\|_\infty + \left\|\frac{\partial K}{\partial t}\right\|_\infty \leq C_A.$$

Proof. Since $f\in C(A)$, the potential q is bounded by a constant that only depends on A, so are σ_0, σ_1, w and K. Denote $J(u,v)=K(u+v,u-v)$. Then J is uniformly bounded by C_A and moreover (cf [10], p. 14 and 16), one has the equalities :

$$\begin{cases} \dfrac{\partial J(u,v)}{\partial u} = \dfrac{1}{2}q(u)+\displaystyle\int_0^v q(u+\beta)J(u,\beta)d\beta \\ \dfrac{\partial J(u,v)}{\partial v} = \displaystyle\int_0^u q(v+\alpha)J(\alpha,v)d\beta \end{cases}$$

We deduce that the partial derivative of J are uniformly bounded by C_A. Returning to the (x,t) coordinates, the conclusion of Proposition 4.9 follows. $\square$

For $z=i\sqrt{\kappa_m}=:iy_m$, we have thus :

$$s_0(x,\kappa_m)=\frac{\sinh(y_m x)}{y_m}+\int_0^x H(x,t)\frac{\sinh(y_m t)}{y_m}dt$$
$$c_0(x,\kappa_m)=\cosh(y_m x)+\int_0^x H(x,t)\cosh(y_m t)dt$$

We will take advantage of this representation to write the estimates of Proposition 4.6 as an integral estimate.

We have

$$\begin{aligned}\int_0^1 L(x)s_0(x)\tilde{c}_0(x)dx &= \int_0^1 L(x)\left[\frac{\sinh(y_m x)}{y_m}+\int_0^x H(x,t)\frac{\sinh(y_m t)}{y_m}dt\right]\times \\ &\qquad\qquad \left[\cosh(y_m x)+\int_0^x \tilde{P}(x,u)\cosh(y_m u)du\right]dx \\ &= \mathrm{I}_0+\mathrm{II}_0+\mathrm{III}_0+\mathrm{IV}_0,\end{aligned}$$

with

- $\mathrm{I}_0 = \displaystyle\int_0^1 L(x)\frac{\sinh(y_m x)\cosh(y_m x)}{y_m}dx$
- $\mathrm{II}_0 = \displaystyle\int_0^1 L(x)\left[\int_0^x \tilde{P}(x,u)\frac{\sinh(y_m x)\cosh(y_m u)}{y_m}du\right]dx$
- $\mathrm{III}_0 = \displaystyle\int_0^1 L(x)\left[\int_0^x H(x,t)\frac{\sinh(y_m t)\cosh(y_m x)}{y_m}dt\right]dx$
- $\mathrm{IV}_0 = \displaystyle\int_0^1 L(x)\left[\int_0^x\int_0^x \tilde{P}(x,u)H(x,t)\frac{\sinh(y_m t)\cosh(y_m u)}{y_m}du\, dt\right]dx$

Let us compute those four quantities independently.

$$\mathrm{I}_0 = \int_0^1 L(x)\frac{\sinh(y_m x)\cosh(y_m x)}{y_m}dx = \frac{1}{2y_m}\int_0^1 \sinh(2xy_m)L(x)dx.$$

$$\begin{aligned}
\mathrm{II}_0 &= \int_0^1 L(x)\Big[\int_0^x \tilde{P}(x,u)\frac{\sinh(y_m x)\cosh(y_m u)}{y_m}du\Big]dx\\
&= \frac{1}{y_m}\int_0^1 L(x)\Big[\int_0^x \tilde{P}(x,u)\frac{\sinh(y_m(x+u))+\sinh(y_m(x-u))}{2}du\Big]dx\\
&= \frac{1}{2y_m}\int_0^1 L(x)\Big[\int_0^x \tilde{P}(x,u)\sinh(y_m(x+u))du + \int_0^x \tilde{P}(x,u)\sinh(y_m(x-u))du\Big]dx\\
&= \frac{1}{y_m}\int_0^1 L(x)\Big[\int_{\frac{x}{2}}^x \tilde{P}(x,2\tau-x)\sinh(2\tau y_m)d\tau + \int_0^{\frac{x}{2}} \tilde{P}(x,x-2\tau)\sinh(2\tau y_m)d\tau\Big]dx\\
&= \frac{1}{y_m}\int_0^1 \sinh(2\tau y_m)\Big[\int_\tau^{2\tau} \tilde{P}(x,2\tau-x)L(x)dx + \int_{2\tau}^1 \tilde{P}(x,x-2\tau)L(x)dx\Big]d\tau
\end{aligned}$$

But, for all (x,τ) in $\mathbb{R}^2$, we have $\tilde{P}(x,x-2\tau)=\tilde{P}(x,2\tau-x)$. Then

$$\mathrm{II}_0 = \frac{1}{y_m}\int_0^1 \sinh(2\tau y_m)\Big[\int_\tau^1 \tilde{P}(x,2\tau-x)L(x)dx\Big]d\tau$$

Let us compute III_0 :

$$\begin{aligned}
\mathrm{III}_0 &= \int_0^1 L(x)\Big[\int_0^x H(x,t)\frac{\sinh(y_m t)\cosh(y_m x)}{y_m}dt\Big]dx\\
&= \frac{1}{2y_m}\int_0^1 L(x)\Big[\int_0^x H(x,t)\sinh(y_m(t+x))dt + \int_0^x H(x,t)\sinh(y_m(t-x))dt\Big]dx\\
&= \frac{1}{y_m}\int_0^1 L(x)\Big[\int_{\frac{x}{2}}^x H(x,2\tau-x)\sinh(2\tau y_m)d\tau + \int_{-\frac{x}{2}}^0 H(x,2\tau+x)\sinh(2\tau y_m)d\tau\Big]dx
\end{aligned}$$

By changing τ in $-\tau$, we get

$$\begin{aligned}
\mathrm{III}_0 &= \frac{1}{y_m}\int_0^1 L(x)\Big[\int_{\frac{x}{2}}^x H(x,2\tau-x)\sinh(2\tau y_m)dt + \int_0^{\frac{x}{2}} H(x,-2\tau+x)\sinh(-2\tau y_m)dt\Big]dx\\
&= \frac{1}{y_m}\int_0^1 L(x)\Big[\int_{\frac{x}{2}}^x H(x,2\tau-x)\sinh(2\tau y_m)d\tau - \int_0^{\frac{x}{2}} H(x,-2\tau+x)\sinh(2\tau y_m)d\tau\Big]dx
\end{aligned}$$

As H is odd with respect to the second variable :

$$\begin{aligned}
\mathrm{III}_0 &= \frac{1}{y_m}\int_0^1 L(x)\Big[\int_0^x H(x,2\tau-x)\sinh(2\tau y_m)d\tau\Big]dx\\
&= \frac{1}{y_m}\int_0^1 \sinh(2\tau y_m)\Big[\int_\tau^1 H(x,2\tau-x)L(x)dx\Big]d\tau
\end{aligned}$$

At last :

$$\begin{aligned}
\mathrm{IV}_0 &= \frac{1}{2y_m}\int_0^1 L(x)\Big[\int_0^x\int_0^x \tilde{P}(x,u)H(x,t)\big(\sinh(y_m(t+u))+\sinh(y_m(t-u))\big)du\,dt\Big]dx\\
&= \mathrm{IV}_0(1)+\mathrm{IV}_0(2)
\end{aligned}$$

where

$$
\begin{aligned}
\mathrm{IV}_0(1) &= \frac{1}{2y_m}\int_0^1 L(x)\int_0^x\int_0^x \tilde{P}(x,u)H(x,t)\sinh(y_m(t+u))dudtdx \\
&= \frac{1}{2y_m}\int_0^1 L(x)\int_0^1 \mathbf{1}_{[0,x]}(t)\int_{\frac{t}{2}}^{\frac{x+t}{2}} 2\tilde{P}(x,2\tau-t)H(x,t)\sinh(2\tau y_m)d\tau dtdx \\
&= \frac{1}{y_m}\int_0^1 L(x)\int_0^1 \sinh(2\tau y_m)\mathbf{1}_{[0,x]}(\tau)\int_{2\tau-x}^{2\tau} \tilde{P}(x,2\tau-t)H(x,t)\mathbf{1}_{[0,x]}(t)dtd\tau dx \\
&= \frac{1}{y_m}\int_0^1 \sinh(2\tau y_m)\int_\tau^1 L(x)\int_{2\tau-x}^{2\tau} \tilde{P}(x,2\tau-t)H(x,t)\mathbf{1}_{[0,x]}(t)dtdxd\tau
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{IV}_0(2) &= \frac{1}{2y_m}\int_0^1 L(x)\int_0^x\int_0^x \tilde{P}(x,u)H(x,t)\sinh(y_m(t-u))du\,dtdx \\
&= \frac{1}{2y_m}\int_0^1 L(x)\int_0^1 \mathbf{1}_{[0,x]}(t)\int_{\frac{t-x}{2}}^{\frac{t}{2}} 2\tilde{P}(x,t-2\tau)H(x,t)\sinh(2\tau y_m)d\tau\,dtdx \\
&= \frac{1}{y_m}\int_0^1 L(x)\int_{-1}^1 \sinh(2\tau y_m)\mathbf{1}_{[-x,x]}(2\tau)\int_{2\tau}^{2\tau+x} \tilde{P}(x,t-2\tau)H(x,t)\mathbf{1}_{[0,x]}(t)\,dtd\tau dx \\
&= \frac{1}{y_m}\int_{-1}^1 \sinh(2\tau y_m)\int_{2|\tau|}^1 L(x)\int_{2\tau}^{2\tau+x} \tilde{P}(x,t-2\tau)H(x,t)\mathbf{1}_{[0,x]}(t)\,dtdxd\tau \\
&= \mathrm{IV}_0(2,1)+\mathrm{IV}_0(2,2)
\end{aligned}
$$

with

$$
\begin{aligned}
\mathrm{IV}_0(2,1) &= \frac{1}{y_m}\int_{-1}^0 \sinh(2\tau y_m)\int_{-2\tau}^1 L(x)\int_{2\tau}^{2\tau+x} \tilde{P}(x,t-2\tau)H(x,t)\mathbf{1}_{[0,x]}(t)\,dtdxd\tau \\
&= \frac{1}{y_m}\int_0^1 \sinh(2\tau y_m)\int_{2\tau}^1 L(x)\int_{-2\tau}^{-2\tau+x} \tilde{P}(x,t+2\tau)H(x,t)\mathbf{1}_{[0,x]}(t)\,dtdxd\tau \\
&= \frac{1}{y_m}\int_0^1 \sinh(2\tau y_m)\int_{2\tau}^1 L(x)\int_0^{-2\tau+x} \tilde{P}(x,t+2\tau)H(x,t)\,dtdxd\tau \\
&= \frac{1}{y_m}\int_0^1 \sinh(2\tau y_m)\int_{2\tau}^1 L(x)\int_{2\tau}^x \tilde{P}(x,t)H(x,t-2\tau)\,dtdxd\tau
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{IV}_0(2,2) &= \frac{1}{y_m}\int_0^1 \sinh(2\tau y_m)\int_{2\tau}^1 L(x)\int_{2\tau}^{2\tau+x} \tilde{P}(x,t-2\tau)H(x,t)\mathbf{1}_{[0,x]}(t)\,dtdxd\tau \\
&= \frac{1}{y_m}\int_0^1 \sinh(2\tau y_m)\int_{2\tau}^1 L(x)\int_{2\tau}^x \tilde{P}(x,t-2\tau)H(x,t)\,dtdxd\tau
\end{aligned}
$$

Finally :

$$
\int_0^1 s_0(x)\tilde{c}_0(x)L(x)dx = \frac{1}{y_m}\int_0^1 \sinh(2\tau y_m)QL(\tau)d\tau
$$

with

$$
\begin{aligned}
QL(\tau) = \frac{1}{2}L(\tau) &+ \int_\tau^1 \tilde{P}(x,2\tau-x)L(x)dx + \int_\tau^1 H(x,2\tau-x)L(x)dx \\
&+ \int_\tau^1 L(x)\int_{2\tau-x}^{2\tau} \tilde{P}(x,2\tau-t)H(x,t)\mathbf{1}_{[0,x]}(t)dtdx \\
&+ \int_{2\tau}^1 L(x)\int_{2\tau}^{x} \tilde{P}(x,t)H(x,t-2\tau)\,dtdx \\
&+ \int_{2\tau}^1 L(x)\int_{2\tau}^{x} \tilde{P}(x,t-2\tau)H(x,t)\,dtdx
\end{aligned}
$$

Similarly, inverting the $\tilde{}$, we construct as well an operator $R : L^2(0,1) \to L^2(0,1)$ such that

$$
\int_0^1 \tilde{s}_0(x)c_0(x)L(x)dx = \frac{1}{y_m}\int_0^1 \sinh(2\tau y_m)RL(\tau)d\tau
$$

with

$$
\begin{aligned}
RL(\tau) = \frac{1}{2}L(\tau) &+ \int_\tau^1 P(x,2\tau-x)L(x)dx + \int_\tau^1 \tilde{H}(x,2\tau-x)L(x)dx \\
&+ \int_\tau^1 L(x)\int_{2\tau-x}^{2\tau} P(x,2\tau-t)\tilde{H}(x,t)\mathbf{1}_{[0,x]}(t)dtdx \\
&+ \int_{2\tau}^1 L(x)\int_{2\tau}^{x} P(x,t)\tilde{H}(x,t-2\tau)\,dtdx \\
&+ \int_{2\tau}^1 L(x)\int_{2\tau}^{x} P(x,t-2\tau)\tilde{H}(x,t)\,dtdx
\end{aligned}
$$

Let us denote $B = Q + R$. Then

$$
\begin{aligned}
\int_0^1 \left[c_0(x,z)\tilde{s}_0(x,z) + \tilde{c}_0(x,z)s_0(x,z)\right]L(x)dx &= \frac{1}{y_m}\int_0^1 \sinh(2\tau y_m)(R+Q)L(\tau)d\tau \\
&= \frac{1}{y_m}\int_0^1 \sinh(2\tau y_m)BL(\tau)d\tau.
\end{aligned}
$$

Now, let us prove the second part of the proposition. As the conformal factors f and $\tilde{f}$ belong to $C(A)$, and thanks to Proposition 4.9, we know that H and $\tilde{H}$ are C^1 and uniformly bounded by a constant C_A (and also are their partial derivatives). Moreover, it is known that, for a function g that is C^1 on $[0,1]$, for any $a \in]0,1[$ the function G_a defined as $G_a(\tau) = \int_a^\tau g(\tau,x)dx$ is also C^1 and its derivative is

$$
G_a'(\tau) = \int_a^\tau \frac{\partial g}{\partial \tau}(\tau,x)dx + g(\tau,\tau).
$$

Hence BL and its derivative are also bounded by some constant C_A. □

Thus, we have obtained:

$$
\left|\frac{1}{y_m^2}\int_0^1 \sinh(2\tau y_m)BL(\tau)d\tau\right| \le C_A\varepsilon \times \Delta(\kappa_m)\tilde{\Delta}(\kappa_m) \tag{48}
$$

Moreover

$$
\begin{aligned}
y_m^2 e^{-2ym} \times \frac{1}{y_m^2}\int_0^1 \sinh(2\tau y_m)BL(\tau)d\tau &= \frac{1}{2}\left[e^{-2ym}\int_0^1 e^{2\tau y_m}BL(\tau)d\tau + e^{-2ym}\int_0^1 e^{-2\tau y_m}BL(\tau)d\tau\right] \\
&= \frac{1}{2}\left[\int_0^1 e^{2(\tau-1)y_m}BL(\tau)d\tau + \int_0^1 e^{-2(\tau+1)y_m}BL(\tau)d\tau\right] \\
&= \frac{1}{2}\left[\int_0^1 e^{-2\tau y_m}BL(1-\tau)d\tau + \int_1^2 e^{-2\tau y_m}BL(\tau-1)d\tau\right]
\end{aligned}
$$

and so, by multiplying (48) by $y_m^2 e^{-2y_m}$, one gets, for $m \geq m_0$:

$$\begin{aligned}\Big|\int_0^{+\infty} e^{-2\tau y_m}\Big(BL(1-\tau)\mathbf{1}_{[0,1]}(\tau)+BL(\tau-1)\mathbf{1}_{[1,2]}(\tau)\Big)d\tau\Big| &\leq C_A\varepsilon \times \big[y_m^2 e^{-2ym}\Delta(\kappa_m)\tilde{\Delta}(\kappa_m)\big]\\ &\leq C_A\varepsilon.\end{aligned}$$

4.4 A Müntz approximation theorem

4.4.1 A Hausdorf moment problem

Let us set

$$g(\tau)=BL(1-\tau)\mathbf{1}_{[0,1]}(\tau)+BL(\tau-1)\mathbf{1}_{[1,2]}(\tau)$$

The change of variable $t=e^{-\tau}$ leads to the estimates :

$$\forall m \geq m_0, \quad \Big|\int_0^1 t^{2y_m-1} g(-\ln(t))dt\Big| \leq C_A\varepsilon.$$

We recall that, for all $m \in \mathbb{N}$, we have set $y_m=\sqrt{\kappa_m}$, where $\kappa_m=m(m+n-2)$. Let us set $\alpha=2y_{m_0}-1$ and

$$\lambda_m := 2y_m-1-\alpha \tag{49}$$

Then, by denoting

$$h(t)=t^\alpha g(-\ln(t)),$$

we get:

$$\Big|\int_0^1 t^{\lambda_m} h(t)dt\Big| \leq C_A\varepsilon, \quad \forall m \in \mathbb{N}. \tag{50}$$

Thus, we would like now to answer the following question : does the approximate knowledge of the moments of h on the sequence $(\lambda_m)_{m\in\mathbb{N}}$ determine h up to a small error in L^2 norm ?

Let us fix $m \in \mathbb{N}$ (we will precise it later) and consider the finite real sequence :

$$\Lambda_m : 0=\lambda_0<\lambda_1<\ldots<\lambda_m.$$

Definition 4.10. The subspace of the Müntz polynomials of degree λ_m is defined as :

$$\mathcal{M}(\Lambda_m)=\{P:\ P(x)=\sum_{k=0}^m a_k x^{\lambda_k}\}.$$

Definition 4.11. The L^2-error of approximation from $\mathcal{M}(\Lambda_m)$ of a function $f \in L^2([0,1])$ is :

$$E_2(f,\Lambda_m)=\inf_{P\in\mathcal{M}(\Lambda_m)}\|f-P\|_2.$$

$E_2(h,\Lambda_m)$ appears in an estimate of $\|h\|_2$ given by Proposition 4.12. Thanks to the Gram-Schmidt process, we define the sequence of Müntz polynomials $\big(L_p(x)\big)$ as $L_0\equiv 1$ and, for $p\geq 1$:

$$L_p(x)=\sum_{j=0}^p C_{pj}x^{\lambda_j},$$

where :

$$C_{pj}=\sqrt{2\lambda_p+1}\frac{\prod_{r=0}^{p-1}(\lambda_j+\lambda_r+1)}{\prod_{r=0,r\neq j}^{p}(\lambda_j-\lambda_r)}.$$

Proposition 4.12. *Under the assumption (50), we have the following estimate : We have the following estimate :*

$$\|h\|_2^2 \leq C_A\varepsilon^2 \sum_{k=0}^{m} \left(\sum_{\ell=0}^{k} |C_{k\ell}| \right)^2 + E_2(h, \Lambda_m)^2.$$

Proof. Let us denote $\pi(h) = \sum_{k=0}^{m} \langle L_k, h\rangle L_k$ the orthogonal projection of h on $\mathcal{M}(\Lambda_m)$.

$$\begin{aligned} \|h\|_2^2 &= \|\pi(h)\|^2 + \|h - \pi(h)\|_2^2 \\ &= \sum_{k=0}^{m} \langle L_k, h\rangle^2 + E_2(\Lambda_m, h)^2. \end{aligned}$$

As

$$|\langle L_k, h\rangle| = \left| \sum_{\ell=0}^{k} C_{k\ell} \underbrace{\int_0^1 x^{\lambda_\ell} h(x)dx}_{\leq C_A\varepsilon} \right| \leq C_A\varepsilon \sum_{\ell=0}^{k} |C_{k\ell}|,$$

one gets

$$\|h\|_2^2 \leq C_A\varepsilon^2 \sum_{k=0}^{m} \left(\sum_{\ell=0}^{k} |C_{k\ell}| \right)^2 + E_2(\Lambda_m, h)^2.$$

□

We would like to find $m(\varepsilon) \in \mathbb{N}$ satisfying :

$$\lim_{\varepsilon \to 0} m(\varepsilon) = +\infty$$

and such that

$$\sum_{k=0}^{m(\varepsilon)} \left(\sum_{\ell=0}^{k} |C_{k\ell}| \right)^2 \leq \frac{1}{\varepsilon},$$

in order to obtain $\|h\|_2^2 \leq C_A\varepsilon + E_2(\Lambda_{m(\varepsilon)}, h)$.

Lemma 4.13.

1. *For all $m \in \mathbb{N}$, $\lambda_{m+1} - \lambda_m \geq 2$.*
2. *For all $m \in \mathbb{N}$, $\lambda_{m+1} - \lambda_m = 2 + O\left(\frac{1}{m}\right)$.*

Proof.

1. Let $m \in \mathbb{N}$ and set $a = n-2$. From (49) we have the equivalence $\lambda_{m+1} - \lambda_m \geq 2 \Leftrightarrow y_{m+1} - y_m \geq 1$, where $y_m = \sqrt{m^2 + am}$. For $m \in \mathbb{N}$, one has :

$$\begin{aligned} y_{m+1} - y_m \geq 1 &\Leftarrow \sqrt{(m+1)^2 + a(m+1)} - \sqrt{m^2 + am} \geq 1 \\ &\Leftrightarrow (m+1)^2 + a(m+1) - m^2 - am \geq \sqrt{(m+1)^2 + a(m+1)} + \sqrt{m^2 + am} \\ &\Leftrightarrow 2m + 1 + a \geq m + 1 + \frac{a}{2} - \frac{a}{8(m+1)} + m + \frac{a}{2} - \frac{a}{8m} + o\left(\frac{1}{m}\right) \\ &\Leftrightarrow \frac{a}{8(m+1)} \geq -\frac{a}{8m} + o\left(\frac{1}{m}\right), \end{aligned}$$

and that is true for m large enough. We assume, without loss of generality, that it is true for all $m \geq m_0$. Hence, for all $m \in \mathbb{N}$, $\lambda_{m+1} - \lambda_m \geq 2$.

2. Let $m \in \mathbb{N}$ and $u_m = \sqrt{\kappa_\ell}$ for some $\ell \in \mathbb{N}$. Then

$$y_{m+1} = \sqrt{\kappa_{m+1}} = \sqrt{\kappa_m} + 1 + O\left(\frac{1}{m}\right) = y_m + 1 + O\left(\frac{1}{m}\right),$$

so we have the result. $\square$

Hence, there is $C > 0$ such that, for all $m \in \mathbb{N}$, $\lambda_m \leq 2m + C$. By setting $M_1 = \max(2, 2C+1)$, one gets :

$$\prod_{r=0}^{p-1}(\lambda_j + \lambda_r + 1) \leq \prod_{r=0}^{p-1}(2j + 2r + 2C + 1) \leq M_1^p \prod_{r=0}^{p-1}(j + r + 1).$$

On the other hand, for all $m \in \mathbb{N}$, $\lambda_{m+1} - \lambda_m \geq 2$. Let $m \in \mathbb{N}$ and $(r, j) \in \mathbb{N}$ such that $0 \leq r, j \leq m$, $r \neq j$.

$$\begin{aligned}|\lambda_j - \lambda_r| &= |\lambda_j - \lambda_{j-1}| + |\lambda_{j-1} - \lambda_{j-2}| + ... + |\lambda_{r+1} - \lambda_r| \\ &\geq 2|j - r|.\end{aligned}$$

Consequently:

$$|\prod_{r=0, r\neq j}^{p}(\lambda_j - \lambda_r)| \geq 2^p \left|\prod_{r=0, r\neq j}^{p}(j - r)\right|$$

It follows that

$$\begin{aligned}|C_{pj}| &\leq \sqrt{4p + 2C + 1}\left(\frac{M_1}{2}\right)^p \frac{\prod_{r=0}^{p-1}|j + r + 1|}{\prod_{r=0, r\neq j}^{p}|j - r|} \\ &= \sqrt{4p + 2C + 1}\left(\frac{M_1}{2}\right)^p \frac{(j+1)...(j+p)}{j(j-1)...2 \times 1 \times 2 \times ...(p-j)} \\ &= \sqrt{4p + 2C + 1}\left(\frac{M_1}{2}\right)^p \frac{(j+p)!}{(j!)^2(p-j)!}\end{aligned}$$

The multinomial formula stipulates that for any real finite sequence $(x_0, ..., x_m)$ and any $n \in \mathbb{N}$:

$$\left(\sum_{k=0}^{m} x_k\right)^n = \sum_{k_1 + ... + k_m = n} \binom{n}{k_1, k_2, ..., k_m} x_1^{k_1} ... x_m^{k_m},$$

where $\displaystyle\binom{n}{k_1, k_2, ..., k_m} = \frac{n!}{k_1! k_2! ... k_m!}$.

As $j + j + (p - j) = j + p$, one deduces that :

$$\frac{(j+p)!}{(j)!(j)!(p-j)!} \leq (1 + 1 + 1)^{j+p} = 3^{j+p}$$

Hence (see [5] or [1], chapter 4, for similar computations) :

$$\begin{aligned}\varepsilon^2 \sum_{k=0}^{m}\left(\sum_{\ell=0}^{k}|C_{k\ell}|\right)^2 &\leq \varepsilon^2 \sum_{k=0}^{m}\left(\sum_{\ell=0}^{k}\sqrt{4k + 2C + 1}\left(\frac{M_1}{2}\right)^k 3^{k+\ell}\right)^2 \\ &= \varepsilon^2 \sum_{k=0}^{m}\left(\frac{3M_1}{2}\right)^{2k}(4k + 2C + 1)\left(\sum_{\ell=0}^{k}3^\ell\right)^2 \\ &\leq \varepsilon^2(4m + 2C + 1)\sum_{k=0}^{m}\left(\frac{3M_1}{2}\right)^{2k}\left(\sum_{\ell=0}^{k}3^\ell\right)^2 \\ &\leq \varepsilon^2(4m + 2C + 1)\sum_{k=0}^{m}\left(\frac{3M_1}{2}\right)^{2k}\frac{3}{2} \times 3^{2k} \\ &\leq \varepsilon^2 \times \frac{3}{2}(4m + 2C + 1)\sum_{k=0}^{m}\left(\frac{9M_1}{2}\right)^{2k} \\ &\leq \varepsilon^2 \times \frac{3}{2}(4m + 2C + 1)(m+1)\left(\frac{9M_1}{2}\right)^{2m} \\ &= \varepsilon^2 g(m)^2\end{aligned}$$

where $g(t) := \dfrac{3}{2}(4t+2C+1)(t+1)\left(\dfrac{9M_1}{2}\right)^{2t}$.

As g is a strictly increasing function on $\mathbb{R}_+$, we can set, for ε small enough, $m(\varepsilon) = E\left(g^{-1}\left(\dfrac{1}{\sqrt{\varepsilon}}\right)\right)$.
Thanks to this choice, we have

$$g\big(m(\varepsilon)\big) \leq \frac{1}{\sqrt{\varepsilon}},$$

so that

$$\varepsilon^2 \sum_{k=0}^{m(\varepsilon)} \left(\sum_{p=0}^{k} |C_{kp}|\right)^2 \leq \varepsilon.$$

Let us now estimate $E_2(\Lambda_m, h)$. To this end, we recall some definitions.

Definition 4.14. The index of approximation of Λ_m in $L^2([0,1])$ is :

$$\varepsilon_2(\Lambda_m) = \max_{y\geq 0}\left|\frac{B(1+iy)}{1+iy}\right|$$

where $B : \mathbb{C} \to \mathbb{C}$ is the Blaschke product defined as :

$$B(z) := B(z, \Lambda_m) = \prod_{k=0}^{m} \frac{z-\lambda_k-\frac{1}{2}}{z+\lambda_k+\frac{1}{2}}$$

We will take advantage of a much simpler expression of $\varepsilon_2\big(\Lambda_m\big)$, thanks to the following Theorem ([9], p.360):

Theorem 4.15. *Let $\Lambda_m : 0 = \lambda_0 < \lambda_1 < ... < \lambda_m$ be a finite sequence. Assume that $\lambda_{k+1} - \lambda_k \geq 2$ for $k \geq 0$. Then :*

$$\varepsilon_2\big(\Lambda_m\big) = \prod_{k=0}^{m} \frac{\lambda_k - \frac{1}{2}}{\lambda_k + \frac{3}{2}}$$

Definition 4.16. For a function $f \in L^2([0,1])$, its L^2-modulus of continuity $w(f,.) :]0,1[\to \mathbb{R}$ is defined as:

$$w(f,u) = \sup_{0\leq r\leq u}\left(\int_0^{1-r} |f(x+r)-f(x)|^2 dx\right)^{\frac{1}{2}}.$$

The introduction of the two previous concepts is motivated by the following result (cf [9], Theorem 2.7 p.352) :

Theorem 4.17. *Let $f \in L^2([0,1])$. Then there is an universal constant $C > 0$ such that :*

$$E_2(\Lambda_m) \leq C\omega(f, \varepsilon\big(\Lambda_m\big))$$

Lemma 4.18. *$w(h,u) \leq C_A u$, $\forall u \in [0, 1/e^2]$.*

Proof. We write $h(x)$ as the sum of two functions with disjoint support :

$$h = h_1 + h_2,$$

with :

- $h_1(t) = t^\alpha BL(-\ln(x)-1)\mathbf{1}_{[\frac{1}{e^2},\frac{1}{e}]}(t)$,
- $h_2(t) = t^\alpha BL(1+\ln(t))\mathbf{1}_{[\frac{1}{e},1[}(t)$.

Thanks to the second part of Proposition 4.7, the function BL is bounded by a constant C_A so, for $i \in [\![1,2]\!]$, each of the function h_i is bounded by some constant C_A depending on A. Moreover, BL is C^1 on $[\frac{1}{e^2}, \frac{1}{e}]$ and $[\frac{1}{e}, 1]$, and, for $i \in [\![1,2]\!]$, h_i' is bounded by a constant C_A. Let $x \in [0, 1/e^2]$, $r \in [0, x]$. We have :

$$\begin{aligned}
\int_0^{1-r} |h(t+r) - h(t)|^2 dt &= \int_{\frac{1}{e^2}}^{\frac{1}{e}-r^2} |h_1(t+r) - h_1(t)|^2 dt + \int_{\frac{1}{e}-r^2}^{\frac{1}{e}} |h_2(x+r) - h_1(t)|^2 dt \\
&+ \int_{\frac{1}{e}}^{1-r} |h_2(t+r) - h_2(t)|^2 dt \\
&\leq \left(\frac{1}{e} - \frac{1}{e^2} - r^2\right) \|h_1'\|_\infty^2 r^2 + r^2 \Big(\|h_2\|_\infty + \|h_1\|_\infty \Big)^2 \\
&+ \left(1 - \frac{1}{e} - r\right) \|h_2'\|_\infty^2 r^2 \\
&\leq C_A r^2.
\end{aligned}$$

Taking the square root and the supremum on r on each side, the result is proved. □

Lemma 4.19.

$$\varepsilon_2\big(\Lambda_m\big) = O\Big(\frac{1}{m}\Big), \qquad m \to +\infty.$$

Proof.

Using Theorem 4.15 and Lemma 4.13, the expression of $\varepsilon_2(\Lambda_m)$ defined above can be written as

$$\varepsilon_2(\Lambda_m) = \prod_{k=0}^m \frac{\lambda_k - \frac{1}{2}}{\lambda_k + \frac{3}{2}}.$$

Recall there exists $C > 0$ such that for all $m \in \mathbb{N}$, $\lambda_m \leq 4m + C$. Consequently, one has :

$$\begin{aligned}
\forall m \in \mathbb{N}, \quad \ln\left(\prod_{k=0}^m \frac{\lambda_k - \frac{1}{2}}{\lambda_k + \frac{3}{2}}\right) &= \ln\left(\prod_{k=0}^m \left(1 - \frac{2}{\lambda_k + \frac{3}{2}}\right)\right) \\
&= \sum_{k=0}^m \ln\left(1 - \frac{2}{\lambda_k + \frac{3}{2}}\right)\Bigg) \\
&\leq -2 \sum_{k=0}^m \frac{1}{\lambda_k + \frac{3}{2}} \\
&\leq -2 \sum_{k=0}^m \frac{1}{2k + C + \frac{3}{2}}.
\end{aligned}$$

But $-2 \sum_{k=0}^m \frac{1}{2k + C + \frac{3}{2}} \underset{m \to +\infty}{=} -\ln(m) + O(1)$. Hence

$$\varepsilon_2\big(\Lambda_m\big) = O\Big(\frac{1}{m}\Big).$$

□

Hence, as $\varepsilon_2(\Lambda_{m(\varepsilon)}) \in [0, 1/e^2]$ for ε small enough, we get thanks to Lemma 4.18 and Theorem 4.17:

$$E(h, \Lambda_{m(\varepsilon)})_2 \leq C_A \varepsilon_2(\Lambda_{m(\varepsilon)}).$$

To sum up, we have shown that :

$$\|h\|_2^2 \le C_A\Big(\varepsilon + \varepsilon_2(\Lambda_{m(\varepsilon)})^2\Big).$$

Now, we know that $\varepsilon_2(\Lambda_{m(\varepsilon)})^2 \le \dfrac{C_A}{m(\varepsilon)^2}$. By virtue of the double inequality

$$\frac{1}{\sqrt{\varepsilon}} + o(1) \le g\big(m(\varepsilon)\big) \le \frac{1}{\sqrt{\varepsilon}}$$

one has

$$\frac{1}{2}\ln\left(\frac{1}{\varepsilon}\right) \underset{\varepsilon\to 0}{\sim} \ln\Big(g\big(m(\varepsilon)\big)\Big) \underset{\varepsilon\to 0}{\sim} C_A m(\varepsilon).$$

Hence (for another $C_A > 0$) : $\dfrac{1}{m(\varepsilon)} \le \dfrac{C_A}{\ln\left(\frac{1}{\varepsilon}\right)}$. Consequently :

$$\|h\|_2^2 \le C_A \frac{1}{\ln\left(\frac{1}{\varepsilon}\right)^2}.$$

Since h_1 and h_2 have disjoint support, we have

$$\|h\|_2^2 = \|h_1\|_2^2 + \|h_2\|_2^2.$$

In particular

$$\|h_2\|_2^2 \le \|h\|_2^2$$

But as

$$\|h_2\|_2^2 = \int_{\frac{1}{e}}^{1} t^{2\alpha}\Big|BL\big(1+\ln(t)\big)\Big|^2 dt$$

we get

$$\int_{\frac{1}{e}}^{1} t^{2\alpha+1}\Big|BL\big(1+\ln(t)\big)\Big|^2 \frac{dt}{t} \le \frac{C_{A,a}}{\ln\left(\frac{1}{\varepsilon}\right)^2}$$

Hence, as we integrate over $\left[\dfrac{1}{e^1}, 1\right]$, the term $t^{2\alpha+1}$ is minorated by $(1/e)^{(2\alpha+1)}$. By returning to the τ coordinate, we obtain :

$$\|BL(1-\tau)\|_{L^2([0,1])} \le C_A \frac{1}{\ln\left(\frac{1}{\varepsilon}\right)},$$

and then

$$\|BL\|_{L^2([0,1])} \le C_A \frac{1}{\ln\left(\frac{1}{\varepsilon}\right)}.$$

4.4.2 Invertibility of the B operator

Now, we want to prove that $B : L^2(0,1) \to L^2(0,1)$ is invertible and that its inverse is bounded with respect to C_A. We can write :

$$B = I + C$$

where $Ch(\tau) = \displaystyle\int_\tau^1 H_1(x,\tau)h(x)dx$, with :

$$\begin{aligned}
H_1(x,\tau) &= \tilde{P}(x,2\tau-x) + H(x,2\tau-x) + \int_{2\tau-x}^{2\tau} \tilde{P}(x,2\tau-t)H(x,t)\mathbf{1}_{[0,x]}(t)dt \\
&+ \int_{2\tau}^{x} \tilde{P}(x,t)H(x,t-2\tau)\, dt\mathbf{1}_{[2\tau,1]}(x) + \int_{2\tau}^{x} \tilde{P}(x,t-2\tau)H(x,t)\, dt\mathbf{1}_{[2\tau,1]}(x) \\
&+ P(x,2\tau-x) + \tilde{H}(x,2\tau-x) + \int_{2\tau-x}^{2\tau} P(x,2\tau-t)\tilde{H}(x,t)\mathbf{1}_{[0,x]}(t)dt \\
&+ \int_{2\tau}^{x} P(x,t)\tilde{H}(x,t-2\tau)\, dt\mathbf{1}_{[2\tau,1]}(x) + \int_{2\tau}^{x} P(x,t-2\tau)\tilde{H}(x,t)\, dt\mathbf{1}_{[2\tau,1]}(x).
\end{aligned}$$

Lemma 4.20. *There is a constant $C_A > 0$ such that, for all h in $L^2(0,1)$:*

$$\forall n \in \mathbb{N}^*,\ \forall \tau \in [0,1], \quad |C^n h(\tau)| \leq C_A \frac{\big((1-\tau)\|H_1\|_{L^\infty}\big)^{n-1}}{(n-1)!}\|h\|_{L^2([0,1])}$$

Proof.

By induction :

• From the estimates of Proposition 4.9, H, $\tilde{H}$ and H_1 are bounded by a constant C_A. Using the triangle inequality and the Cauchy-Schwarz inequality, one immediately gets :

$$|Ch(\tau)| \leq C_A \int_\tau^1 |h(x)|dx \leq C_A(1-\tau)\|h\|_{L^2([0,1])} \leq C_A\|h\|_{L^2([0,1])}$$

• Assume it is true for some $n \in \mathbb{N}^*$. Then :

$$\begin{aligned}
|C^{n+1}h(\tau)| = \left| \int_\tau^1 H_1(x,t)C^n h(x)dx \right| &\leq \int_\tau^1 \|H_1\|_\infty C_A \frac{(1-x)^{n-1}\|H_1\|_\infty^{n-1}}{(n-1)!}\|h\|_{L^2(0,1)}dx \\
&= C_A \frac{\|H_1\|_\infty^n}{(n-1)!}\|h\|_{L^2(0,1)} \int_\tau^1 (1-x)^{n-1}dx \\
&= C_A \frac{\big((1-\tau)\|H_1\|_\infty\big)^n}{n!}\|h\|_{L^2([0,1])}
\end{aligned}$$

□

Thus $\|C^n\| \leq C_A \dfrac{\big((1-\tau)\|H_1\|_\infty\big)^{n-1}}{(n-1)!}$ for all $n \in \mathbb{N}^*$. It follows that the serie $\sum(-1)^n C^n$ is convergent. Consequently B is invertible, $B^{-1} = \displaystyle\sum_{n=0}^{+\infty}(-1)^n C^n$ and :

$$\|B^{-1}\| \leq C_A.$$

Hence :

$$\|q-\tilde{q}\|_{L^2(0,1)} = \|L\|_{L^2(0,1)} \leq \|B^{-1}\|\|BL\|_{L^2(0,1)} \leq C_A \frac{1}{\ln\left(\frac{1}{\varepsilon}\right)}$$

and the proof of Theorem 1.9 is complete.

Let us prove Corollary 1.13.

Proof. Let $s_1, s_2 \geq 0$ and $\theta \in (0,1)$. Using the Gagliardo-Nirenberg inequalities (see [3]), one can write

$$\|g\|_{H^s(0,1)} \leq \|g\|_{H^{s_1}(0,1)}^\theta \|g\|_{H^{s_2}(0,1)}^{1-\theta}$$

for every $g \in H^{s_1}(0,1) \cap H^{s_2}(0,1)$ and $s = \theta s_1 + (1-\theta)s_2$. As f and $\tilde{f}$ belong to $C(A)$ then $q - \tilde{q}$ belong to $H^2(0,1)$ and $\|q-\tilde{q}\|_{H^2(0,1)} \leq C_A$. Hence, for $s_1 = 0$ and $s_2 = 2$, we have:

$$\begin{aligned}
\|q-\tilde{q}\|_{H^s(0,1)} &\leq \|q-\tilde{q}\|^{\theta}_{L^2(0,1)} \|q-\tilde{q}\|^{1-\theta}_{H^2(0,1)} \\
&\leq C_A^{1-\theta} \|q-\tilde{q}\|^{\theta}_{L^2(0,1)} \\
&\leq C_A \frac{1}{\ln\left(\frac{1}{\varepsilon}\right)^{\theta}}
\end{aligned}$$

with $\theta = \dfrac{2-s}{2}$. Using the Sobolev embedding $H^1(0,1) \hookrightarrow C^0(0,1)$ with $\|.\|_\infty \leq 2\|.\|_{H^1(0,1)}$, one gets (for $s = 1$ and $\theta = 1/2$):

$$\|q-\tilde{q}\|_\infty \leq 2\|q-\tilde{q}\|_{H^1(0,1)} \leq C_A \sqrt{\frac{1}{\ln\left(\frac{1}{\varepsilon}\right)}}$$

□

4.4.3 Uniform estimate of the conformal factors

Now we give the proof of Corollary 1.11. Assume that $n \geq 3$, $\omega = 0$ and let us set $F = f^{n-2}$. We can write $q = \dfrac{F''}{F}$ and then

$$(\tilde{F}F' - \tilde{F}'F)'(t) = \tilde{F}F(q-\tilde{q})(t).$$

For all $t \in [0,1]$, we have :

$$\begin{aligned}
\tilde{F}(t)F'(t) - \tilde{F}'(t)F(t) &= (n-2)\tilde{f}^{n-2} f^{n-3}(t) f'(t) - (d-2)\tilde{f}^{n-3} f^{n-2}(t)\tilde{f}'(t) \\
&= (n-2) f^{n-3}(t)\tilde{f}^{n-3}(t)\Big(\tilde{f}(t)f'(t) - f(t)\tilde{f}'(t)\Big)
\end{aligned}$$

Assume that for all t in $[0,1]$, $\tilde{f}(t)f'(t) - f(t)\tilde{f}'(t) \neq 0$, for example $\tilde{f}(t)f'(t) > f(t)\tilde{f}(t)$. Then :

$$\frac{f'(t)}{f(t)} > \frac{\tilde{f}'(t)}{\tilde{f}(t)}.$$

Then, by integrating between 0 and 1, one gets :

$$\ln\big(f(1)\big) - \ln\big(f(0)\big) > \ln\big(\tilde{f}(1)\big) - \ln\big(\tilde{f}(0)\big).$$

and this is not true as $f(0) = f(1)$ and $\tilde{f}(0) = \tilde{f}(1)$. Consequently, there is $x_0 \in [0,1]$ such that $\big(\tilde{f}f' - f\tilde{f}'\big)(x_0) = 0$. Setting $G(x) = (\tilde{F}F' - \tilde{F}'F)(x)$, we have :

$$\forall x \in [0,1], \quad G(x) = \int_{x_0}^{x} \tilde{F}F(q-\tilde{q})(t)dt.$$

From the L^2 estimate previously established on $q - \tilde{q}$, one has :

$$\begin{aligned}
\forall x \in [0,1], \quad |G(x)| &\leq \sqrt{|x-x_0|} C_A \|q-\tilde{q}\|_2 \\
&\leq C_A \frac{1}{\ln\left(\frac{1}{\varepsilon}\right)}
\end{aligned}$$

Hence :

$$\left|\left(\frac{F}{\tilde{F}}\right)'(x)\right| = \left|\frac{G(x)}{\tilde{F}(x)^2}\right| \leq C_A \frac{1}{\ln\left(\frac{1}{\varepsilon}\right)},$$

and by integrating betwwen 0 and x :

$$\left|\frac{F(x)}{\tilde{F}(x)}-1\right| = \left|\int_0^x \left(\frac{F}{\tilde{F}}\right)'(t)dt\right| \le \int_0^1 \left|\frac{G(t)}{\tilde{F}(t)^2}\right|dt \le C_A \frac{1}{\ln\left(\frac{1}{\varepsilon}\right)}.$$

and this last inequality leads to the estimate :

$$\forall x \in [0,1], \quad |f^{n-2}(x) - \tilde{f}^{n-2}(x)| \le C_A \frac{1}{\ln\left(\frac{1}{\varepsilon}\right)}.$$

Setting $k = n-2$, thanks to the relation $a^k - b^k = (a-b)\sum_{j=0}^{k} a^j b^{k-j}$, we get at last :

$$\forall x \in [0,1], \quad |f(x) - \tilde{f}(x)| \le C_A \frac{1}{\ln\left(\frac{1}{\varepsilon}\right)}.$$

5 About the Calderón problem

Now, we prove Theorem 1.12. For $s \in \mathbb{R}$, $H^s(\partial M)$ can be defined as

$$H^s(\partial M) = \left\{\psi \in \mathcal{D}'(\partial M),\ \psi = \sum_{m\ge 0} \begin{pmatrix} \psi_m^1 \\ \psi_m^2 \end{pmatrix} \otimes Y_m, \quad \sum_{m \ge 0} (1+\mu_m)^s \Big(|\psi_m^1|^2 + |\psi_m^2|^2\Big) < \infty \right\}.$$

Recall that we have denoted $\mathcal{B}(H^{1/2}(\partial M))$ the set of bounded operators from $H^{1/2}(\partial M)$ to $H^{1/2}(\partial M)$ and equipped $\mathcal{B}(H^{1/2}(\partial M))$ with the norm

$$\|F\|_* = \sup_{\psi \in H^{1/2}(\partial M)\setminus\{0\}} \frac{\|F\psi\|_{H^{1/2}}}{\|\psi\|_{H^{1/2}}}.$$

Lemma 5.1. *We have the equivalence :*

$$\Lambda_g(\omega) - \Lambda_{\tilde{g}}(\omega) \in \mathcal{B}(H^{1/2}(\partial M)) \Leftrightarrow \begin{cases} f(0) = \tilde{f}(0) \\ f(1) = \tilde{f}(1). \end{cases}$$

Proof. Let us set

$$C_0 = \frac{1}{4\sqrt{f(0)}}\frac{h'(0)}{h(0)}, \quad C_1 = \frac{1}{4\sqrt{f(1)}}\frac{h'(1)}{h(1)}, \quad A_0 = \frac{1}{f(0)} - \frac{1}{\tilde{f}(0)} \quad \text{and} \quad A_1 = \frac{1}{f(1)} - \frac{1}{\tilde{f}(1)}.$$

For $m \ge 0$, one has, using the block diagonal representation of $\Lambda_g(\omega)$ and the asymptotics of $M(\mu_m)$ and $N(\mu_m)$ given in Theorem 2.2 and Corollary 2.3:

$$\begin{aligned}
\Lambda_g^m(\omega) - \Lambda_{\tilde{g}}^m(\omega) &= \begin{pmatrix} \frac{\tilde{M}(\mu_m)}{\sqrt{\tilde{f}(0)}} - \frac{M(\mu_m)}{\sqrt{f(0)}} + C_0 - \tilde{C}_0 & O\big(e^{-2\mu_m}\big) \\ O\big(e^{-2\mu_m}\big) & \frac{\tilde{N}(\mu_m)}{\sqrt{\tilde{f}(1)}} - \frac{N(\mu_m)}{\sqrt{f(1)}} + \tilde{C}_1 - C_1 \end{pmatrix} \\
&= \begin{pmatrix} A_0\sqrt{\mu_m} + (C_0 - \tilde{C}_0) & 0 \\ 0 & A_1\sqrt{\mu_m} + (\tilde{C}_1 - C_1) \end{pmatrix} + \begin{pmatrix} O\Big(\frac{1}{\sqrt{\mu_m}}\Big) & O\big(e^{-2\mu_m}\big) \\ O\big(e^{-2\mu_m}\big) & O\Big(\frac{1}{\sqrt{\mu_m}}\Big) \end{pmatrix}
\end{aligned}$$

Hence, for any $\left(\psi_m^1, \psi_m^2\right) \in \mathbb{R}^2$:

$$\left(\Lambda_g^m(\omega) - \Lambda_{\tilde{g}}^m(\omega)\right) \begin{pmatrix} \psi_m^1 \\ \\ \psi_m^2 \end{pmatrix} = \sqrt{\mu_m} \begin{pmatrix} A_0\psi_m^1 \\ \\ A_1\psi_m^2 \end{pmatrix} + \begin{pmatrix} (C_0 - \tilde{C}_0)\psi_m^1 \\ \\ (\tilde{C}_1 - C_1)\psi_m^2 \end{pmatrix} + O\Big(\frac{\psi_m^1 + \psi_m^2}{\sqrt{\mu_m}}\Big)$$

For $\psi = \sum_{m\geq 0} \begin{pmatrix} \psi_m^1 \\ \\ \psi_m^2 \end{pmatrix} \otimes Y_m \in H^{1/2}(\partial M)$, one has

$$\begin{aligned}
\|\left(\Lambda_g(\omega) - \Lambda_{\tilde{g}}(\omega)\right)\psi\|^2_{H^{1/2}(\partial M)} &= \sum_{m\geq 0} (1+\mu_m)^{1/2}\mu_m \Big(A_0^2|\psi_m^1|^2 + A_1^2|\psi_m^2|^2\Big) \\
&+ \sum_{m\geq 0} 2(1+\mu_m)^{1/2}\sqrt{\mu_m}\Big(|A_0(C_0 - \tilde{C}_0)||\psi_m^1|^2 + |A_1(\tilde{C}_1 - C_1)||\psi_m^2|^2\Big) \\
&+ \sum_{m\geq 0} (1+\mu_m)^{1/2} O\big(|\psi_m^1|^2 + |\psi_m^2|^2\big)
\end{aligned}$$

Then

$$\|\Lambda_g(\omega) - \Lambda_{\tilde{g}}(\omega)\|_* < \infty \Leftrightarrow \begin{cases} A_0 = 0 \\ A_1 = 0 \end{cases} \Leftrightarrow \begin{cases} \tilde{f}(0) = \tilde{f}(0) \\ \tilde{f}(1) = \tilde{f}(1). \end{cases}$$

$\square$

Under the assumptions of Theorem 1.12, the following estimate holds:

Proposition 5.2. *Let $\varepsilon > 0$. Assume that $\|\Lambda_g(\omega) - \Lambda_{\tilde{g}}(\omega)\|_* \leq \varepsilon$. There is $C_A > 0$ such that :*

$$\forall m \in \mathbb{N}, \qquad \Big| N(\kappa_m) - \tilde{N}(\kappa_m) \Big| \leq C_A \varepsilon.$$

Proof. For $m \in \mathbb{N}$, consider $\psi_m = \begin{pmatrix} 0 \\ \\ 1 \end{pmatrix} \otimes Y_m \in H^{1/2}(\partial M)$.

One has :

$$\begin{aligned}
\left(\Lambda_g(\omega) - \Lambda_{\tilde{g}}(\omega)\right)\psi_m &= \left(\Lambda_g^m(\omega) - \Lambda_{\tilde{g}}^m(\omega)\right) \begin{pmatrix} 0 \\ \\ 1 \end{pmatrix} \otimes Y_m \\
&= \begin{pmatrix} 0 & \frac{1}{\sqrt{f(0)}} \frac{h^{1/4}(1)}{h^{1/4}(0)} \big(\frac{1}{\tilde{\Delta}(\mu_m)} - \frac{1}{\Delta(\mu_m)}\big) \\ 0 & \Big(\frac{\tilde{N}(\mu_m)}{\sqrt{f(1)}} - \frac{N(\mu_m)}{\sqrt{f(1)}}\Big) + (\tilde{C}_1 - C_1) \end{pmatrix} \otimes Y_m
\end{aligned}$$

Then

$$\begin{aligned}
\|\left(\Lambda_g(\omega) - \Lambda_{\tilde{g}}(\omega)\right)\psi_m\|^2_{H^{1/2}(\partial M)} = (\mu_m + 1)^{1/2} \Bigg[\Big(\frac{\tilde{N}(\mu_m)}{\sqrt{f(1)}} - \frac{N(\mu_m)}{\sqrt{f(1)}} + (\tilde{C}_1 - C_1)\Big)^2 \\
+ \frac{1}{f(0)} \frac{h^{1/2}(1)}{h^{1/2}(0)} \Big(\frac{1}{\tilde{\Delta}(\mu_m)} - \frac{1}{\Delta(\mu_m)}\Big)^2 \Bigg].
\end{aligned}$$

so, for all $m \geq 0$:

$$
\begin{aligned}
(\mu_m+1)^{1/2}\left|\frac{1}{\sqrt{f(1)}}\big(\tilde{N}(\mu_m)-N(\mu_m)\big)+(\tilde{C}_1-C_1)\right|^2 &\leq \|\big(\Lambda_g(\omega)-\Lambda_{\tilde{g}}(\omega)\psi_m\|^2_{H^{1/2}(\partial M)} \\
&\leq \|\Lambda_g(\omega)-\Lambda_{\tilde{g}}(\omega)\|^2_*\|\psi_m\|^2_{H^{1/2}(\partial M)} \\
&= \|\Lambda_g(\omega)-\Lambda_{\tilde{g}}(\omega)\|^2_*(\mu_m+1)^{1/2} \\
&\leq \varepsilon^2(\mu_m+1)^{1/2}.
\end{aligned}
$$

Hence

$$
\left|\frac{1}{\sqrt{f(1)}}\big(\tilde{N}(\mu_m)-N(\mu_m)\big)+(\tilde{C}_1-C_1)\right| \leq \varepsilon. \tag{51}
$$

Using the asymptotic $N(\mu_m)=-\mu_m+o(1)$, we deduce from (51) that

$$
|\tilde{C}_1-C_1| \leq \varepsilon
$$

and then that there is $C_A>0$ such that, for all $m \in \mathbb{N}$:

$$
\left|N(\mu_m)-\tilde{N}(\mu_m)\right| \leq C_A\,\varepsilon.
$$

□

As in Lemma 4.5, one gets an integral relation between $N(z)-\tilde{N}(z)$ and $q-\tilde{q}$:

Lemma 5.3. *The following integral relation holds:*

$$
\big(N(z)-\tilde{N}(z)\big)\Delta(z)\tilde{\Delta}(z) = \int_0^1 \big(q(x)-\tilde{q}(x)\big)s_0(x,z)\tilde{s}_0(x,z)dx \tag{52}
$$

Proof. Let us define $\theta : x \mapsto s_0(x,z)\tilde{s_0}'(x,z)-s_0'(x,z)\tilde{s}_0(x,z)$. Then :

$$
\theta'(x) = \big(\tilde{q}(x)-q(x)\big)s_0(x,z)\tilde{s}_0(x,z)
$$

By integrating between 0 and 1, one gets:

$$
s_0'(1,z)\tilde{s}_0(1,z)-s_0(1,z)\tilde{s}_0'(1,z) = \int_0^1 \big(q(x)-\tilde{q}(x)\big)s_0(x,z)\tilde{s}_0(x,z)dx
$$

As $s_0'(1,z)=N(z)\Delta(z)$ and $s_0(1,z)=\Delta(z)$, one gets for all $z \in \mathbb{C}\backslash\mathcal{P}$:

$$
\big(N(z)-\tilde{N}(z)\big)\Delta(z)\tilde{\Delta}(z) = \int_0^1 \big(q(x)-\tilde{q}(x)\big)s_0(x,z)\tilde{s}_0(x,z)dx.
$$

□

Just as in Section 4, let us extend on $[-1,0]$ q and $\tilde{q}$ into even functions and denote $L(x)=q(x)-\tilde{q}(x)$. We recall that for all $m\in\mathbb{N}$, we have set $y_m=\sqrt{\kappa_m}$.

We will take advantage of this representation to write in another way the equalities

$$
\big(N(\kappa_m)-\tilde{N}(\kappa_m)\big)\Delta(\kappa_m)\tilde{\Delta}(\kappa_m) = \int_0^1 (q(x)-\tilde{q}(x)s_0(x,\kappa_m)\tilde{s}_0(x,\kappa_m)dx.
$$

Proposition 5.4. *There is an operator $D : L^2([0,1]) \to L^2([0,1])$ such that :*

1. *For all $m \in \mathbb{N}$,*

$$\big(N(\kappa_m) - \tilde{N}(\kappa_m)\big) s_0(1,\kappa_m)\tilde{s}_0(1,\kappa_m) = \frac{1}{y_m^2}\int_0^1 \cosh(2\tau y_m) DL(\tau)d\tau - \frac{1}{y_m^2}\int_0^1 L(\tau)d\tau.$$

2. *The function $\tau \mapsto DL(\tau)$ is C^1 on $[0,1]$ and DL and $(DL)'$ are uniformly bounded by a constant C_A.*

Proof. Using the same calculations as in Proposition 4.7 together with the representation formula for s_0

$$s_0(x,\kappa_m) = \frac{\sinh(y_m x)}{y_m} + \int_0^x H(x,t)\frac{\sinh(y_m t)}{y_m}dt$$

one can prove that the operator D is given by

$$\begin{aligned} DL(\tau) = L(\tau) + \int_\tau^1 \tilde{H}(x, 2\tau - x)L(x)dx &+ \int_\tau^1 H(x,2\tau - x)L(x)dx \\ &+ \int_\tau^1 L(x)\int_{2\tau - x}^{2\tau} \tilde{H}(x, 2\tau - t)H(x,t)\mathbf{1}_{[0,x]}(t)dtdx \\ &+ \int_{2\tau}^1 L(x)\int_{2\tau}^x \tilde{H}(x,t)H(x,t-2\tau)\,dtdx \\ &+ \int_{2\tau}^1 L(x)\int_{2\tau}^x \tilde{H}(x,t-2\tau)H(x,t)\,dtdx \end{aligned}$$

and so that DL and its derivative are bounded by some constant $C_A > 0$. □

For all $m \in \mathbb{N}$, one has

$$\begin{aligned} (N(\kappa_m) - \tilde{N}(\kappa_m))s_0(1,\kappa_m)\tilde{s}_0(1,\kappa_m) &= \frac{1}{y_m^2}\int_0^1 \cosh(2\tau y_m)DL(\tau)d\tau - \frac{1}{y_m^2}\int_0^1 L(\tau)d\tau \\ &= \frac{1}{2y_m^2}\int_0^1 e^{2\tau y_m}DL(\tau)d\tau + \frac{1}{2y_m^2}\int_0^1 e^{-2\tau y_m}DL(\tau)d\tau \\ &\quad - \frac{1}{y_m^2}\int_0^1 L(\tau)d\tau. \end{aligned}$$

Hence, by multiplying both sides by $2y_m^2 e^{-2y_m}$, one has :

$$\begin{aligned} 2y_m^2 e^{-2y_m}\big(N(\kappa_m) - \tilde{N}(\kappa_m)\big)s_0(1,\kappa_m)\tilde{s}_0(1,\kappa_m) &= \int_0^1 e^{2y_m(\tau-1)}DL(\tau)d\tau + \int_0^1 e^{-2y_m(\tau+1)}DL(\tau)d\tau \\ &\quad - 2e^{-2y_m}\int_0^1 L(\tau)d\tau \end{aligned}$$

The asymptotic

$$s_0(1,\kappa_m) \sim \frac{e^{y_m}}{y_m}, \qquad m \to +\infty,$$

ensures that $y_m^2 e^{-2y_m}s_0(1,\kappa_m)\tilde{s}_0(1,\kappa_m)$ is bounded uniformly in m. Moreover, by hypothesis:

$$\left|\int_0^1 L(\tau)d\tau\right| \leq \varepsilon$$

so

$$\left|\int_0^1 e^{2y_m(\tau-1)}DL(\tau)d\tau + \int_0^1 e^{-2y_m(\tau+1)}DL(\tau)d\tau\right| \leq C_A\varepsilon.$$

We write :

- $\displaystyle\int_0^1 e^{2y_m(\tau-1)}DL(\tau)d\tau = \int_0^1 e^{-2\tau y_m}DL(1-\tau)d\tau,$

- $\displaystyle\int_0^1 e^{-2y_m(t+1)}DL(\tau)d\tau = \int_1^2 e^{-2\tau y_m}DL(\tau-1)d\tau,$

Setting
$$RL(\tau) = DL(1-\tau)\mathbf{1}_{[0,1]}(\tau) + DL(\tau-1)\mathbf{1}_{[1,2]}(\tau)$$
one has for all $m \in \mathbb{N}$
$$\left| \int_0^{+\infty} e^{-2\tau y_m} RL(\tau)d\tau \right| \leq C_A\varepsilon$$

By the change of variable $\tau = -\ln(t)$, we obtain the moment problem :
$$\forall m \in \mathbb{N}, \qquad \left| \int_0^1 t^{2y_m} RL(-\ln(t))dt \right| \leq C_A\varepsilon$$

Using the same technique as in section 4.4.1, we prove the stability estimate
$$\|DL\|_{L^2([0,1])} \leq C_A \frac{1}{\ln\left(\frac{1}{\varepsilon}\right)}.$$

But D can be written as (see section 4.4.2)
$$D = I + C$$
with, for all $n \geq 1$, $\|C^n\| \leq \dfrac{\big(C_A(1-\tau)\big)^{n-1}}{(n-1)!}$. Consequently B is invertible and its inverse is bounded by some constant $C_A > 0$. Hence
$$\begin{aligned} \|q-\tilde{q}\|_2 &\leq \|D^{-1}\|\|DL\|_{L^2([0,1])} \\ &\leq C_A \frac{1}{\ln\left(\frac{1}{\varepsilon}\right)}. \end{aligned}$$

At last, if $\omega = 0$ and $n \geq 3$, we deduce as previously that
$$\|f-\tilde{f}\|_\infty \leq C_A \frac{1}{\ln\left(\frac{1}{\varepsilon}\right)}.$$

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