Basis Risk Management in an Index-Based Insurance Framework under Randomly Scaled Uncertainty

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Abstract

This paper is concerned with the quantification of basis risk in index-based insurance products using randomly scaled variables. To this extent, we first discuss the shape, the unimodality and the symmetry of randomly scaled variables depending on the distribution of the random scaling factor using Mellin transform. We explicitly obtain the distribution of a randomly scaled variable when the random scaling factor is either uniformly distributed or of Beta type. We then determine s-convex extremal distributions for randomly scaled variables and discuss the way of comparing it. Next, we define an Enterprise Risk Management framework that relies on randomly scaled variables to assess basis risk, introducing the class of generalized penalty functions. This ERM framework allows for setting up basis risk limits to eventually determine a Basis Risk Capital Requirement. The results are illustrated with particular cases that carefully challenge the methodology.

Key Words: Basis risk; ERM; Index-based insurance; Random scaling; s-convex extrema

1 Introduction and motivation

Basis risk and its impacts. Traditional indemnity transactions are familiar to risk management as they provide a perfect hedge for claims payable to policyholders. However, the payment process can be lengthy because it is based on the assessment of the claims adjuster. Due to the deductibles and excluded risks generally contained in indemnity transactions, some insurance and reinsurance companies fill the protection gap by concluding a parametric transaction. These parametric transactions, also called index-based insurance or parametric risk transfers, do not take into account the loss ultimately suffered by the protection buyer to trigger the payment, but physical parameters directly linked to the risk against which the protection buyer wishes to be covered. In other words, a parametric risk transfer has a pre-agreed parametric trigger, thus defining when the contract must be paid to the protection buyer. This parametric trigger is easily observable, making payment predictable and quick when conditions are met. However, a parametric risk transfer does not compensate for the actual loss suffered. This mismatch between the actual loss and the payout triggered by the index insurance gives rise

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to what is called basis risk. Following Ross and Williams [72], basis risk is defined as "the risk that a protection buyer's own losses exceed the payments under a risk transfer mechanism structured to hedge against these losses."

Most common forms of parametric risk transfers are both Non Life and Life parametric catastrophe bonds, weather derivatives and agricultural index-based insurance. Because of the lack of wind coverage following Hurricane Andrew, Insurance-Linked Securities (ILS) were developed to offset the decrease in (re)insurance capacity. These products were designed to facilitate the transfer of catastrophic insurance risk from insurers or reinsurers (referred to as "sponsors") to investors. Catastrophe or "cat" bonds are the most prominent form of ILS. The ILS market reaches \$13.8 billion of issuance in 2018 and \$11 billion in 2019. Cat bond and ILS capital outstanding was \$13.2 billion at the end of 2008, whereas it reached \$40.8 billion in 2019. Even if indemnity trigger is the most prevalent type of trigger in the outstanding catastrophe bonds and ILS market, parametric trigger represents 4.2% of the market, for an outstanding capital of \$1.7 billion. We refer the reader to the comprehensive handbook of Barrieu and Albertini [6] for a detailed presentation of the ILS market.

Even if Non Life Securitization represents the majority of ILS transactions, Life Securitization has to be mentioned too. Products like mortality bonds or longevity bonds has continued to expand in recent years, with larger transactions in a growing list of regions now being completed on an annual basis. We refer to Peard [68] for general features of Life ILS, and to e.g. Blake et al. [12] to address the problem of longevity risk using mortality-linked securities.

As opposed to a lack of insurance capacity, weather derivatives (WD) were directly tied to the deregulation of the U.S. energy industry. Introduced in 1997, the use of WD has quickly expanded beyond energy industry; see e.g. Bank and Wiesner [4] for a study in the Austrian winter tourism industry. WD can be defined as financial instrument to hedge against the risk of weather-related losses. Consequently, parametric trigger is at the basis of these products. Heating Degree Days (HDD) or Cooling Degree Days (CDD) are the most common indices used to trigger the payments, see e.g. Alaton et al. [1] or Benth and Benth [8] for issues related to modeling and pricing weather derivatives.

Agricultural insurance is particularly concerned with the development of index-based insurance products, mainly in rural regions of developing countries. According to Greatrex et al. [41] and Barnett and Mahul [5], the number of agricultural index insurance products is likely to be in the hundreds, spanning dozens of countries, and the number of farmers involved in these insurance products reaches at least tens of millions. Paradoxically, despite the obvious enthusiasm for index-based agricultural insurance, recent research points out that the demand for such coverage remains low; see e.g. Cole et al. [17] for a study in Bangladesh, Jensen et al. [52] in northern Kenya, Takahashi et al. [78] in Ethiopia and Elabed and Carter [33] in Mali. Even if barriers to demand depend on many factors such as culture, regulation or trust, Jensen and Barrett [51] and Jensen et al. [53] show that demand is reduced because of basis risk. In fact, due to the presence of basis risk, Elabed and Carter [33, 34, 35] and Elabed et al. [36] present index-based agricultural insurance as a sort of compound lottery which is perceived as a complex mechanism by farmers, preventing them from signing such a contract. Therefore, basis risk is at the heart of index-based agricultural insurance. We refer to Binswanger-Mkhize

[11] for a general overview of the challenges of this insurance.

According to Dalhaus and Finger [21], basis risk can be decomposed into three different sources, namely the design basis risk, the spatial basis risk and the temporal basis risk. This decomposition is consistent with the description of basis risk given by Ross and Williams [72]. For example, in natural disaster risk transfers, the buyer and seller of protection must agree that the index calibration takes into account the amplification of losses and secondary risks. Note that the basis risk can turn into a favorable risk for the protection buyer when the payment given is greater than the loss suffered. A large literature describes how to build appropriate indices, depending on the underlying risk covered and the data available; see Leblois and Quirion [58] for a review and Gornott and Wechsung [40] or Biffis and Chavez [10] for recent developments.

Since the cat-bond market has become mainstream, or because of low demand for index agricultural insurance due to basis risk, several ways to quantify basis risk have been proposed. As indicated by Ross and Williams [72], quantifying the basis risk usually helps to answer questions like this: in cases where there is a modeled recovery in the context of parametric risk transfer, what is the probability that this modeled recovery does not offer adequate protection? This can be measured by the negative false probability (NFP), used by e.g. Jensen et al. [53], Takahashi et al. [78] and Elabed et al. [36]. Another question is: what is the expected amount by which the loss modeled exceeds the recovery modeled under the parametric risk transfer (without giving weight to cases where recovery exceeds the loss)? Jensen et al. [52], Woodard and Garcia [84] and Vedenov and Barnett [82] answer by proposing the semi-variance to measure basis risk. Harrington and Niehaus [44] provide a way to measure two-sided basis risk using the correlation coefficient between the loss ratio of the protection buyer and the loss ratio of the index. As pointed out by Zeng [86], this measure is not actuarial-oriented. More recently, Morsink et al. [64], via a study carried out by the World Bank, analyze the reliability of index insurance by defining two indicators to assess the effectiveness of insurance, namely the probability of catastrophic basis risk and the catastrophic performance ratio. The probability of catastrophic basis risk gives the probability that a farmer will suffer more than 70% of agricultural production loss due to the non-triggering of the index. The catastrophic performance ratio reflects what, on average, a farmer recovers per $1 of commercial premium paid in the event that he experiences a catastrophic crop loss. The common point of all these measures is to focus on average amounts or on acceptable thresholds.

In this paper, we do not add any contribution on how to build an index, and we do not enter into data considerations. However, in the continuation of Lefèvre et al. [60], the added value lies in the development of a systematic methodology to quantify basis risk, trying to partially respond to the following remark from Jensen et al. [52]: "Detecting the magnitude and distribution of basis risk should be of utmost importance for organizations promoting index insurance products, lest they inadvertently peddle lottery tickets under an insurance label."

Quantifying basis risk. Consider a protection buyer who faces a possible loss represented by a positive continuous random variable L. Assume the loss can be modeled by an index represented by a positive continuous random variable Z.

Once the index is calibrated, the basis risk X comes from the difference between the loss L

and the payout Z triggered by the index. In other words,

$$X =_d L - Z. \tag{1.1}$$

Note that since the loss may or may not be greater than the index, the variable X may take positive values or not.

A priori, we would prefer that there is no basis risk, i.e. $L =_d Z$. In practice, this does not happen and a difference does exist in most situations. The main drawbacks of defining basis risk by $X =_d L - Z$ is that the distribution of X is in general unknown or at least not perfectly known. In fact, on the one hand the distribution of the loss is obtained by statistical studies on the historical losses, if they are available. For instance, when the underlying risk covered by the index-based structure is new, there is no recorded loss. On the other hand, the data used to build the index may not be perfectly related to the underlying risk. For instance, for very localized risks, the index requires very localized data and sometimes there are no available weather stations in the area. Consequently, the index is built using weighted weather data available in closest weather stations. To this extent, the difference between the loss and the payouts triggered by the index does not reflect the basis risk, but rather the simulated basis risk. Nevertheless, it might be reasonable to expect the difference $X =_d L - Z$ to be unimodal with a mode at 0. For example, X could have a normal distribution of mean 0 or, more generally, a symmetric unimodal distribution. We recall that a random variable X has a symmetric and unimodal distribution if and only if it can be represented as $X =_d SY$ where S and Y are two independent variables such that S is uniform on $(-1, 1)$ and Y is non-negative (see e.g. Dharmadhikari and Joag-Dev [32], Theorem 1.5). The property of symmetry can however be quite restrictive for real applications. For this reason, we choose in this paper to represent X rather as a continuous randomly scaled variable.

Definition 1.1. *A real-valued random variable X has a randomly scaled distribution if it has the product representation*

$$X =_d SY, \tag{1.2}$$

where Y is a positive random variable and S is a real-valued random scaling factor, bounded and independent of Y.

Randomly scaled variables are the natural extension of the characterization of unimodal distributions due to Khintchine [56] and Shepp [75].

It is interesting here to mention that the usual concept of unimodality has been generalized in different ways. Thus, X has a distribution called α-unimodal, $\alpha > 0$, when $X =_d S^{1/\alpha}Y$ for some variable Y and with S uniform on $(0, 1)$ independently of Y (Olshen and Savage [66]). On the other hand, X has a β-unimodal distribution, $\beta > 0$, when $X =_d (1 - S^{1/\beta})Y$ with S still uniform on $(0, 1)$ and independent of Y (following the work of Williamson [83] on multiple monotone functions). More generally, X has a Beta-unimodal distribution when $X =_d SY$ where S has a Beta distribution (see the book of Bertin et al. [9] for a detailed analysis).

Random scaling relations like (1.2) have received some attention in theoretical and applied probability. Distributional characterizations by such relations are established by Pakes and

Navarro [67]. Asymptotics of products of random variables are studied by e.g. Cline and Samorodnitsky [16], Hashorva et al. [48] and Yang and Wang [85]. Various applications aim to represent an economic environment (Tang and Tsitsiashvili [81], Asimit et al. [3]), a systemic background risk (Côté and Genest [18]) or a dependency structure for claim sizes (Hashorva [45], Hashorva and Ji [46]).

By (1.2), the basis risk X corresponds to an underlying positive random variable Y coupled with an independent random scaling factor S. Because the "true" basis risk is not available, the methodology introduced in this paper assumes the distribution of Y is known. Typically, Y is a given as positive function of the index Z. For example, we consider later the cases where $Y =_d Z$ and $Y =_d \sqrt{Z}$. On the contrary, the methodology proposes different distributions for the random scaling factor S so that it modifies the shape of the distribution of the product $X =_d SY$, for a given Y. Then, we provide a way to control this additional uncertainty by taking into account the information available on the values of the first moments of S.

Quantifying the basis risk is part of an Enterprise Risk Management (ERM) framework. In the approach developed here, the first step is to define the worst possible scenarios from the available information on uncertainty. This is done using the theory of s-convex extremal distributions studied by Denuit et al. [28, 29] and Hürlimann [49]. Note that bounding problems are a classsical subject in actuarial risk theory; see e.g. De Vylder [22, 23], De Vylder and Goovaerts [25], Kaas and Goovaerts [54, 55], Brockett and Cox [13], Denuit and Lefèvre [27], Lefèvre and Utev [59], among many others. Once the scenarios have been obtained, the second step consists in measuring the consequences of the basis risk using a family of flexible and representative penalty functions. These functions generalize more realistically the penalty functions examined in Lefèvre et al. [60] since they add a criteria based on the size of the uncertainty with respect to the size of the index. Finally, the third step is to determine limits for the basis risk. These limits are introduced as a capital requirement to cope with the consequences of basis risk.

The paper is organized as follows. In Section 2, we deduce some results related to the shape of the distribution of a randomly scaled variable. We first deal with the case where the distribution of X is gaussian and we explicitly identify the positive random variable Y hidden in X. We then provide the distribution of a randomly scaled variable where the random scaling factor is of Beta type. We close this Section by discussing the unimodality and the symmetry of a randomly scaled variable. Section 3 is devoted to the convex ordering of randomly scaled relations. In this Section, we use the (known) s-convex orders and s-convex extremal distributions and we adapt it to random scaling relations. In Section 4, we present an Enterprise Risk Management approach by introducing the class of generalized penalty functions to quantify basis risk. We continue the description of the ERM framework in Section 5 by providing a systematical way to set up basis risk limits. We eventually define a basis risk capital requirement. Section 6 illustrates the methodology for several forms of basis risk, including light and heavy tailed distributions for the index.

2 On the distribution, the unimodality and the symmetry of randomly scaled variables

In this Section, we justify the use of randomly scaled variables in the basis risk assessment as a way to drop the symmetry and unimodality assumptions inherent to the decomposition given by Dharmadhikari and Joag-Dev [32].

In subsection 2.1.1 we begin by discussing the ideal case where the distribution of the basis risk is unimodal and symmetric and we provide the distribution function of the random variable Y hidden in X.

In subsection 2.1.2 we consider the general case of randomly scaled variables and we explicitly give the distribution of a randomly scaled variable when the random scaling factor is of Beta type with integer parameters. From now, we are concerned with the resulting shape of the distribution of a randomly scaled variable. Consequently, we first deal with the ordering of two randomly scaled variables in the stochastic dominance sense, to eventually recall (known) results on the asymptotics of randomly scaled variables in subsection 2.1.3. Tail behavior of randomly scaled variables is directly related to our risk management purpose. Later in the paper we illustrate the role of light and heavy tails in the basis risk assessment.

Once we move from unimodal and symmetric random variables to randomly scaled ones, we wonder if a randomly scaled variable can still be unimodal, symmetric or both at the same time and we provide the conditions that have to be fulfilled. As announced in the introduction, we show that a randomly scaled variable is not necessarily unimodal nor symmetric. Consequently, randomly scaled variables stand for a quite malleable class of random variables that allow for the modification of the distribution of the basis risk to eventually generate different scenarios for its assessment.

2.1 On the distribution of $X =_d SY$

As it is stated in the introduction Khintchine [56] and then Shepp [75] give a characterization of unimodal distributions, namely a random variable X is said to be unimodal if and only if it can be written as $X =_d UV$, where V and U are independent and U is uniformly distributed on the unit interval. Later, it has been shown that X has a symmetric about 0 and unimodal distribution if and only if it can be represented as $X =_d U_1 Y$ where U_1 and Y are two independent variables such that Y is non-negative and U_1 is uniformly distributed over $(-1, 1)$ (see e.g. Dharmadhikari and Joag-Dev [32], Theorem 1.5). This case is of utmost importance in our work since it corresponds to the ideal case where the difference between the loss and the payout triggered by the index is for instance normally distributed. It is then worth describing the distribution of the random variable Y hidden in X when X is symmetric and unimodal.

2.1.1 On the decomposition of Dharmadhikari and Joag-Dev [32]

We begin by providing the distribution of $X =_d U_1 Y$ for a general positive random variable Y.

Proposition 2.1. *Let $X =_d U_1 Y$ have a symmetric about* 0 *and unimodal distribution. Let F_Y be the distribution function of Y and f_Y its density. Then*

$$F_X(x) = \begin{cases} \frac{1}{2}\left(x \int_x^\infty \frac{f_Y(y)}{y} dy + F_Y(x) + 1 \right), & x \geq 0, \\ \\ \frac{1}{2}\left(x \int_{-x}^\infty \frac{f_Y(y)}{y} dy + (1 - F_Y(-x)) \right), & x \leq 0. \end{cases} \tag{2.1}$$

Proof. The proof is based on the independence between U_1 and Y. Only the case $x \leq 0$ is shown. Let $x \leq 0$, then

$$\begin{aligned} \mathbb{P}(X \leq x) &= \frac{1}{2} \int_{-1}^0 \mathbb{P}\left(Y \geq \frac{x}{u} \right) du + \frac{1}{2} \int_0^1 \mathbb{P}\left(Y \leq \frac{x}{u} \right) du \\ &= \frac{1}{2} \int_{-1}^0 \int_{x/u}^\infty f_Y(y) dy du = \frac{1}{2} \int_{-x}^\infty \int_{-1}^{x/y} f_Y(y) du dy \\ &= \frac{1}{2}\left(x \int_{-x}^\infty \frac{f_Y(y)}{y} dy + (1 - F_Y(-x)) \right). \end{aligned}$$

□

Equation (2.1) can be reversed in the sense that the distribution of Y can be recovered if the one of X is known.

Proposition 2.2. *Let $X =_d U_1 Y$ have a symmetric about* 0 *and unimodal distribution. Let F_X be the distribution function of X and f_X its density. Then*

$$F_Y(x) = 2F_X(x) - 1 - 2x f_X(x), \quad x \geq 0, \tag{2.2}$$

Proof. From the case $x \geq 0$ in Proposition 2.1, we obtain

$$\int_x^\infty \frac{f_Y(y)}{y} dy = \frac{2F_X(x)}{x} - \frac{1}{x} - \frac{F_Y(x)}{x}. \tag{2.3}$$

Derivating (2.3) with respect to x yields the announced result. □

This property gives the form of the random variable Y hidden in X when X is symmetric and unimodal. When X is assumed to be normally distributed with mean 0 and variance σ^2, the following example provides the form of the density of the corresponding Y.

Example 2.1. *Let $X =_d U_1 Y$ be normally distributed with mean* 0 *and variance σ^2. We directly obtain the density of Y by differentiating* (2.2) *with respect to x, yielding*

$$f_Y(x) = \frac{2x^2}{\sigma^3 \sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2} \right), \quad x \geq 0.$$

This random variable Y has a mode at $\sigma\sqrt{2}$.

2.1.2 Extension to general randomly scaled variables

Whatever the form of S and Y, the moments of the general randomly scaled variable $X =_d SY$ are given by

$$\mathbb{E}[X^k] = \mathbb{E}[S^k]\mathbb{E}[Y^k], \quad k \in \mathbb{N}.$$

As Y is a positive random variable, then $\mathbb{E}[X] = 0$ if and only if $\mathbb{E}[S] = 0$. We will see later in this section that the random scaling factor is responsible for the unimodality and the symmetry of the overall distribution of X.

From here and subsequently, let U_a be a uniformly distributed random variable over $(-a, a)$ and S be a general random variable distributed over $(-a, a)$. In the sequel of the paper, we are particularly concerned with the case where S is of scaled Beta type. This case is referred to as S_a. The scaled Beta distribution, also known as the four-parameter Beta distribution or the translated Beta distribution, enlarges the support of the usual Beta distribution; see e.g. Hanson [43] and Carnahan [15]. In fact, if S_a follows a scaled Beta distribution with parameters $\alpha, \beta > 0$ over $[-a, a]$, denoted by $S_a \sim \mathcal{B}eta([-a, a], \alpha, \beta)$, then

$$S_a =_d 2a\tilde{S} - a = a(2\tilde{S} - 1), \tag{2.4}$$

where $\tilde{S} \sim \mathcal{B}eta([0, 1], \alpha, \beta)$. Thus, S_a takes values over $[-a, a]$ and the moments of S_a are directly obtained with (2.4). In addition the density of S_a is given by

$$f(x) = \frac{(x + a)^{\alpha-1}(a - x)^{\beta-1}}{B(\alpha, \beta)(2a)^{\alpha+\beta-1}}, \quad -a \leq x \leq a \ \ and \ \ \alpha, \beta > 0.$$

From now, we shift our attention to the case where Y is known (for instance given by (2.2)) but we allow for the random scaling factor to be modified. We provide some results on the shape of the resulting randomly scaled variable $X =_d SY$. First of all, the distribution of $X =_d U_a Y$ can be easily obtained whereas the case where $X =_d S_a Y$ with $S_a \sim \mathcal{B}eta([-a, a], \alpha, \beta)$ is not straightforward. The next proposition gives the distribution of X when $S_a \sim \mathcal{B}eta([-a, a], n, m)$, $n, m \geq 1$.

Proposition 2.3. *Let $X =_d S_a Y$ with $S_a \sim \mathcal{B}eta([-a, a], n, m)$, n and m are integers veryfing $n, m \geq 1$. Then*

$$F_X(x) = \begin{cases} \mathbb{P}(S_a \leq 0) + \dfrac{1}{2^{n+m-1}} \displaystyle\sum_{k=0}^{n-1} \binom{n+m-1}{n-k-1} \mathbb{P}\left(Y\tilde{S}_k \leq \frac{x}{a}\right), & x \geq 0, \\ \\ \dfrac{1}{2^{n+m-1}} \displaystyle\sum_{k=0}^{m-1} \binom{n+m-1}{m-k-1} \mathbb{P}\left(Y\hat{S}_k > \frac{-x}{a}\right), & x \leq 0, \end{cases} \tag{2.5}$$

where

$$\begin{cases} \tilde{S}_k \sim \mathcal{B}eta([0, 1], k+1, m), \\ \hat{S}_k \sim \mathcal{B}eta([0, 1], k+1, n). \end{cases}$$

The proof of Proposition 2.3 is given in Appendix. The next point raised is the ordering of random scaling variables in the stochastic dominance order. Let X_1 and X_2 be two random variables, not necessarily randomly scaled. Then X_1 is said to be smaller than X_2 in stochastic dominance, denoted as $X_1 \leq_{st} X_2$, if and only if $\mathbb{P}(X_2 \leq x) \leq \mathbb{P}(X_1 \leq x)$; see Denuit et al. [31] for an entire presentation of this stochastic order.

Proposition 2.4. *Let S be distributed over (s_{min}, s_{max}) with $s_{min} \leq 0 \leq s_{max}$ and let F_S be its distribution function. Let Y_1 and Y_2 be two positive random variables independent from S. If $Y_1 \leq_{st} Y_2$, then*

$$\mathbb{P}(SY_1 \leq x) \geq \mathbb{P}(SY_2 \leq x), \quad x \geq 0,$$
$$\mathbb{P}(SY_1 \leq x) \leq \mathbb{P}(SY_2 \leq x), \quad x \leq 0.$$

Proof. For $x \geq 0$,

$$\begin{aligned}\mathbb{P}(SY_1 \leq x) &= F_S(0) + \int_0^{s_{max}} \mathbb{P}(Y_1 \leq x/s) dF_S(s) \\ &\geq F_S(0) + \int_0^{s_{max}} \mathbb{P}(Y_2 \leq x/s) dF_S(s) = \mathbb{P}(SY_2 \leq x).\end{aligned}$$

For $x \leq 0$,

$$\mathbb{P}(SY_1 \leq x) = \int_{s_{min}}^0 \mathbb{P}(Y_1 \geq x/s) dF_S(s) \leq \int_{s_{min}}^0 \mathbb{P}(Y_2 \geq x/s) dF_S(s) = \mathbb{P}(SY_2 \leq x).$$

□

An equivalent result can be obtained when the random scaling factors are ordered in the stochastic dominance sense.

Proposition 2.5. *Let Y be a positive random variable with distribution function F_Y and let S_1 and S_2 be distributed over (s_{min}, s_{max}) with $s_{min} \leq 0 \leq s_{max}$. Assume both S_1 and S_2 are independent from Y. If $S_1 \leq_{st} S_2$, then*

$$\mathbb{P}(S_1 Y \leq x) \geq \mathbb{P}(S_2 Y \leq x), \quad x \in \mathbb{R}.$$

2.1.3 Tails of randomly scaled variables

A natural question is the nature of the tail of the random scaling variable X. For instance, we wonder if it is possible that $X =_d YS$ is light-tailed knowing that Y is heavy-tailed. Actually, this issue has already been studied in Embrechts and Goldie [37], Cline and Samorodnitsky [16], Tang [79], Tang [80], Hashorva and Pakes [47] and Hashorva et al. [48]. We recall the definition of three classes of interest.

Definition 2.1. *Let F be a cumulative distribution function and $\bar{F} = 1 - F$.*

(i) $F \in \mathscr{R}_\alpha$ *if* $\bar{F}$ *is regularly varying, i.e.*

$$\lim_{t\to\infty} \frac{\bar{F}(\lambda t)}{\bar{F}(t)} = \lambda^{-\alpha} \text{ for some } \alpha \geq 0, \text{ all } \lambda \geq 1.$$

(ii) $F \in \mathscr{I}$ *if* $\bar{F}$ *is intermediate regularly varying, i.e.*

$$\lim_{\lambda\downarrow 1} \liminf_{t\to\infty} \frac{\bar{F}(\lambda t)}{\bar{F}(t)} = 1.$$

(iii) $F \in \mathscr{S}$ *if* $\bar{F}$ *is subexponential, i.e.*

$$\bar{F}(t) > 0 \text{ for every } t \text{ and } \lim_{t\to\infty} \frac{\overline{F * F}(t)}{\bar{F}(t)} = 2,$$

where $*$ *denotes convolution.*

Besides, we recall that $\mathscr{R}_\alpha \subset \mathscr{I} \subset \mathscr{S}$. Denoting by F_Y the distribution function of Y, by F_S the one of S and by F_X the one of the product $X =_d YS$, Embrechts and Goldie [37] prove that if $F_Y \in \mathscr{R}_\alpha$ for some $\alpha > 0$ and either $\bar{F_S}(x) = o(\bar{F_Y}(x))$ or $F_S \in \mathscr{R}_\alpha$ then $F_X \in \mathscr{R}_\alpha$. Then, Cline and Samorodnitsky [16] prove that if $F_Y \in \mathscr{I}$ and $\bar{F_S}(ux) = o(\bar{F_X}(x))$ for some $u > 0$, then $F_X \in \mathscr{I}$. Besides, they find four conditions that must be verified to conclude that if $F_Y \in \mathscr{S}$ then $F_X \in \mathscr{S}$. In the same spirit, [79] removes the last condition from Cline and Samorodnitsky [16] with the only cost of adding a mild condition to the distribution F_Y. Furthermore, the case where S is unbounded is carefully watched in Tang [80]. Cline and Samorodnitsky [16] also deal with the particular case where F_S has a bounded support, which corresponds actually to our framework of randomly scaled relations. In this case, they obtained (corollary 2.5) that if $F_Y \in \mathscr{S}$ and S is a bounded random variable, then $F_X \in \mathscr{S}$. Hashorva et al. [48] discuss the asymptotic behavior of F_X when $S \in [0, 1]$, considering it as a random discount factor. The particular case of tail asymptotics under beta random scaling has been highlighted in Hashorva and Pakes [47] through the determination of which maximal domain of attraction contains F_X when the membership of F_Y is known. In a nutshell, $X =_d SY$ belongs to the same class as Y. The result of Cline and Samorodnitsky [16] concerning bounded random scaling factor is of particular interest because it allows for adding heavy tails in the uncertainty, or equivalently, it allows for basis risk to take great values.

2.2 On the unimodality of $X =_d SY$

Then we focus on the unimodality of a randomly scaled variable. In the genesis of the forthcoming result, the problem of the unimodality of the sum of independent unimodal random variables has been tackled by Ibragimov [50] who show that in general convolutions of unimodal distributions are not unimodal. To this extent, the set of strongly unimodal distributions is introduced and defined as follows: the distribution of a random variable A is strongly unimodal if $A+B$ has a unimodal distribution whenever B is independent of A and has a unimodal distribution. Furthermore, Ibragimov [50] prove that a continuous random variable is strongly unimodal if its probability density function f is logconcave (i.e. $\log f$ is concave).

Later Cuculescu and Theodorescu [20] introduce the notion of multiplicative strong unimodality which is defined as the preservation of unimodality in product of independent random variables. They first show that all unimodal distribution at 0 are multiplicative strong unimodal and that the resulting product is unimodal at 0. Indeed, take Y with positive support and unimodal about 0 so that $Y =_d UV$ where U is uniform on $(0,1)$, V is non-negative and U and V are independent. Then, the randomly scaled variable $X =_d SY$ of the form (1.2) can be written as $X =_d SUV =_d U(SV)$, and therefore X (real-valued) is unimodal about 0.

They then demonstrate that a unimodal random variable $\tilde{S}$ is multiplicative strong unimodal if and only if it is one-sided and absolutely continuous, with a density $f_{\tilde{S}}$ having the property that $t \mapsto f_{\tilde{S}}(e^t)$ is log-concave in $\mathbb{R}$ when $\tilde{S}$ is non-negative. For positive variables, a change of variable entails that $\tilde{S}$ is multiplicative strong unimodal if and only $\log \tilde{S}$ is strongly unimodal. Since location and scale do not affect log-concavity, if $\tilde{S}$ is multiplicative strong unimodal so does $S = u\tilde{S} + v$, $u \neq 0$ and $v \in \mathbb{R}$; see e.g. Simon [76] or Alimohammadi et al. [2] for recent contributions.

We hence deduce the next proposition by combining results from Ibragimov [50] and Cuculescu and Theodorescu [20].

Proposition 2.6. *Let $X =_d SY$ be a randomly scaled variable.*

i) If both S and Y are unimodal and either S or Y is multiplicative strong unimodal, then X is unimodal.

ii) In particular, if S or Y is unimodal at 0, then X is unimodal at 0.

Therefore, we obtain the following corollary when $X =_d U_a Y$.

Corollary 2.1. *If $X =_d U_a Y$ then X is necessarily unimodal at 0.*

However unimodality of a randomly scaled variable is not always guaranteed. To this extent, we illustrate Proposition 2.6 with an insightful example involving the scaled Beta distribution over $[-1,1]$.

Example 2.2. *Let Y be a positive unimodal random variable. Let $S_1 \sim \mathcal{B}eta([-1,1], \alpha, \beta)$.*

It is known form e.g. Alimohammadi et al. [2] that S_1 is multiplicative strong unimodal if and only if $\alpha > 0$ and $\beta \geq 1$. In this case, $X =_d S_1 Y$ is unimodal.

Now, assume $\hat{S}_1$ is Beta distributed with parameters $\alpha < 1$ and $\beta < 1$. Then $\hat{X} = \hat{S}_1 Y$ may not be unimodal anymore. For instance, Figure 2.1 represents the density of $\hat{X}$ (solid blue line) and X (dashed black line) when $Y \sim \Gamma(4,2)$ and $\hat{S}$ and S are Beta distributed over $[-1,1]$ with parameters $(0.2, 0.2)$ and $(2,2)$ respectively.

As a result, we obtain the following corollary when the random scaling factor follows a scaled Beta distribution.

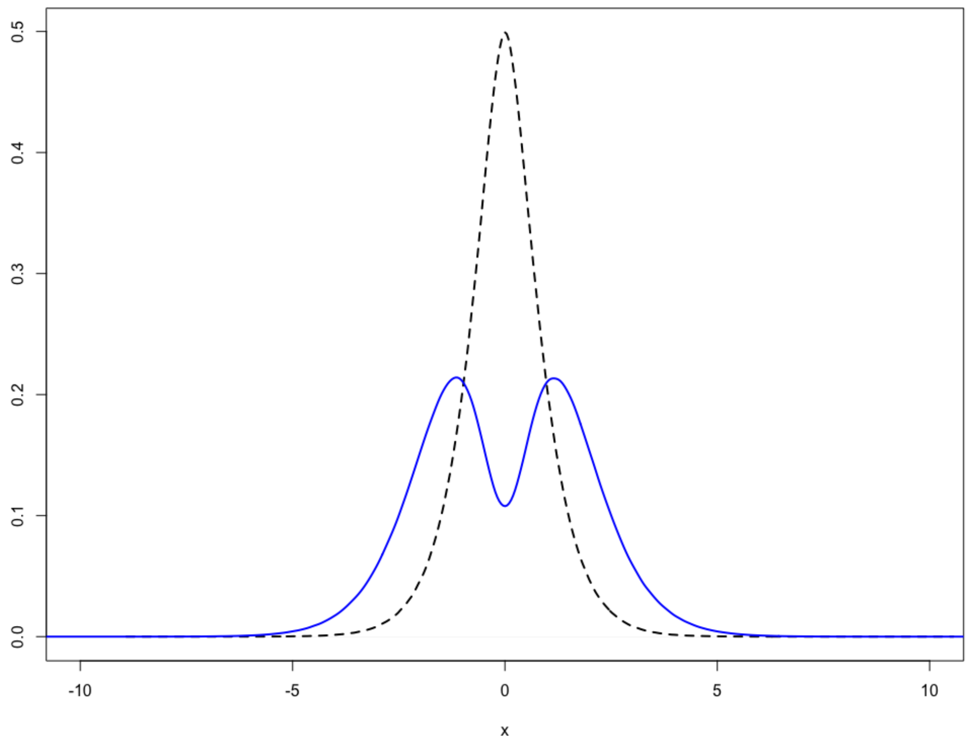

Figure 2.1: Densities of $\hat{X}$ (solid blue line) and X (dashed black line) when $Y \sim \Gamma(4,2)$ and $\hat{S}$ and S are Beta distributed over $[-1,1]$ with parameters $(0.2, 0.2)$ and $(2,2)$ respectively.

Corollary 2.2. *Let $\alpha > 0$ and $\beta \geq 1$. If $S_a \sim \mathcal{B}eta([-a,a],\alpha,\beta)$ then $X =_d S_a Y$ is unimodal (not necessarily at 0).*

The question of the quantification of the mode of the randomly scaled variable X is far from easy. The mode is defined as the maximum value taken by the density and it rapidly involves intractable calculations. Nevertheless, when the distribution of X is known then the mode can be relatively easily obtained numerically; see e.g. Basu and DasGupta [7], Sato [73] or Gavriliadis [39]. The question of the mode is related to the one of the symmetry of the distribution of X.

2.3 On the symmetry of $X =_d SY$

The last point we deal with is the symmetry of a randomly scaled variable. This topic has received some attention in the literature; see among many others the book by Galambos and Simonelli [38] and Simonelli [77] or Hamedani and Volkmer [42]. There exist many characterizations of the symmetry such as X, or equivalently its distribution function, is said to be symmetric about x_0 if and only if $X - x_0 =_d x_0 - X$ or X is said to be symmetric if and only if there exists a value x_0 such that $f_X(x_0 + x) = f_X(x_0 - x)$ for all real x where f_X is the density of the random variable X (not necessarily randomly scaled). These two characterizations, although they are intuitive, do not allow for straightforward calculation in our case since there is no simple expression for the density nor the distribution function of a randomly scaled variable.

However, the Mellin transform stands for a powerful tool when it comes to dealing with the product of (independent) random variables.

Definition 2.2. *(Galambos and Simonelli [38])*
Let Y be a nonnegative random variable with distribution function F_Y. The Mellin transform of Y is defined as

$$\mathcal{M}_Y(s) = \int_0^{+\infty} t^s dF_Y(t),$$

where s is a complex number and $\mathcal{M}_Y(s)$ is assumed finite.

To extend the definition of the Mellin transform to arbitrary random variable X (not necessarily randomly scaled), let X^+ and X^- denote the nonnegative and the negative part of X, respectively. Since both X^+ and X^- are nonnegative their Mellin transforms, denoted respectively by $\mathcal{M}_{X^+}(s)$ and $\mathcal{M}_{X^-}(s)$, are well defined. Then the Mellin transform of X is given by

$$\mathcal{M}_X(s) = \mathcal{M}_{X^+}(s) + \gamma \mathcal{M}_{X^-}(s),$$

where $\gamma^2 = 1$ is a formal indeterminate.

Among all the results related to Mellin transform, we point out that the Mellin transform of the product of independent random variables is the product of their Mellin transforms. In particular for a randomly scaled variable $X =_d SY$,

$$\mathcal{M}_X(s) = \mathcal{M}_S(s)\mathcal{M}_Y(s) = \left(\mathcal{M}_{S^+}(s) + \gamma \mathcal{M}_{S^-}(s)\right) \mathcal{M}_Y(s).$$

Further, from Simonelli [77] a random variable X is symmetric about 0 if and only if $\mathcal{M}_{X^+}(s) = \mathcal{M}_{X^-}(s)$ for all s in some neighborhood of 0. In particular when X is randomly scaled, Y is positive thus the symmetry of X only depends on the bounded random variable S. The following proposition characterizes the symmetry of $X =_d SY$.

Proposition 2.7. *Let X be randomly scaled with $X =_d SX$ where S is any random variable distributed over $[-a, a]$ independent from Y.*

If X is symmetric, then X is symmetric about 0*.*

In addition, X is symmetric about 0 *if and only if S is symmetric about* 0*.*

Proof. Let $X =_d SY$ with Y is a positive random variable and S is distributed over $[-a, a]$ and independent from Y. Note that $\mathcal{M}_{Y^-}(s) = 0$ for all s. Then, X is symmetric about 0 if and only if

$$\begin{aligned} 0 &= \mathcal{M}_{X^+}(s) - \mathcal{M}_{X^-}(s) \\ &= \mathcal{M}_{S^+}(s)\mathcal{M}_Y(s) - \mathcal{M}_{S^-}(s)\mathcal{M}_Y(s) \\ &= \mathcal{M}_{S^+}(s) - \mathcal{M}_{S^-}(s), \end{aligned}$$

yielding the announced result. □

We recall that the symmetry of X about 0 and $\mathbb{P}(X < 0) = 1/2$ are not equivalent; see Galambos and Simonelli [38] page 27 for a counter example. Combining all the results, we obtain the following corollaries depending on the form of the random scaling factor.

Corollary 2.3. *Let $X =_d S_a Y$ be a randomly scaled variable, where S_a is distributed over $(-a, a)$. The following assertions are equivalent:*

i) $S_a \sim \mathcal{B}eta([-a, a], \alpha, \alpha)$, $\alpha \geq 1$,

ii) S_a *is unimodal about* 0,

iii) S_a *is symmetric about* 0,

iv) X *is unimodal about* 0,

v) X *is symmetric about* 0.

Corollary 2.4. *Let S be a non-symmetric random variable distributed over $(-a, a)$ and let Y be unimodal at 0. Then $X =_d SY$ is unimodal at 0 but is not symmetric.*

For instance, a suitable distribution for Y is the exponential one.

To close the discussion on the shape of randomly scaled variables, we now come back to the positive random variable Y when X is symmetric and unimodal. In fact, the symmetry of X can be used to obtain the distribution of Y. We deduce the Mellin transform of Y using the fact that if X is symmetric, so does S.

Proposition 2.8. *Let $X =_d SY$ be a symmetric randomly scaled variable about 0. Then the Mellin transform of Y is given by*

$$\mathcal{M}_Y(s) = \frac{\mathcal{M}_{X^+}(s)}{\mathcal{M}_{S^+}(s)}.$$

Proof. Since both X and S are symmetric, then $\mathcal{M}_X(s) = \mathcal{M}_{X^+}(s)(1 + \gamma)$ and $\mathcal{M}_S(s) = \mathcal{M}_{S^+}(s)(1 + \gamma)$. Besides, the independence between S and Y yields

$$\mathcal{M}_Y(s) = \frac{\mathcal{M}_X(s)}{\mathcal{M}_S(s)} = \frac{\mathcal{M}_{X^+}(s)}{\mathcal{M}_{S^+}(s)}.$$

□

Once the Mellin transform of Y is known, the density of Y can be obtained by reversing the Mellin transform. The resulting density is then given by,

$$f_Y(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |x|^{-it-1} \mathcal{M}_Y(t) dt, \quad x \geq 0. \tag{2.6}$$

However, solving (2.6) is in general not an easy task; see Galambos and Simonelli [38] for some examples. In the next example we deal with the case where X is normally distributed with mean 0 and variance σ^2. We hence complement Example 2.2 by providing the Mellin transform of the positive random variable Y hidden in X.

Example 2.3. *Let X be a normally distributed random variable with mean 0 and variance σ^2. From Theorem 1.5 of Dharmadhikari and Joag-Dev [32], $X =_d U_1 Y$. By definition, X and U_1 are symmetric and their Mellin transform are given by*

$$\mathcal{M}_{X^+}(s) = \frac{\sigma^s 2^{\frac{s-1}{2}}}{\sqrt{2\pi}} \Gamma\left(\frac{s+1}{2}\right), \quad \textit{and} \quad \mathcal{M}_{U_1^+}(s) = \frac{1}{2(s+1)}, \quad \mathbb{R}e(s) > -1.$$

By Proposition 2.8, the Mellin transform of Y is given by

$$\mathcal{M}_Y(s) = \frac{\sigma^s 2^{\frac{s+1}{2}}(s+1)}{\sqrt{2\pi}} \Gamma\left(\frac{s+1}{2}\right), \quad \mathbb{R}e(s) > -1.$$

In addition, from Example 2.2 we have,

$$f_Y(x) = \frac{2x^2}{\sigma^3\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right), \quad x \geq 0. \tag{2.7}$$

In other words, we have shown that

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} |x|^{-it-1} \frac{\sigma^t 2^{\frac{t+1}{2}}(t+1)}{\sqrt{2\pi}} \Gamma\left(\frac{t+1}{2}\right) dt = \frac{2x^2}{\sigma^3\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right), \quad x \geq 0.$$

In our framework, both the distributions of Y and S are known. However, the random scaling factor plays the role of the link between the chosen Y and the corresponding X. The methodology proposed in this work focuses on the role of the random scaling factor to assess basis risk. The next section is devoted to the theory of convex orders, adapted for random scaling relations.

3 Convex ordering of randomly scaled relation

The theory of stochastic orders has a variety of uses in theoretical and applied probability, and especially in actuarial science. Much of this theory can be found in the comprehensive books by Shaked and Shanthikumar [74] and Müller and Stoyan [65]. An important class of stochastic orders is provided by the orders called s-convex whose properties have been studied in depth by Denuit et al. [28, 29]. These orders are the key tool used in this section to compare randomly scaled relations. For the sake of brevity, we will only recall a few elements on the s-convex orders when necessary. The purpose of this section is twofold. It first enlarges the result given by Lefèvre et al. [60] on transferring s-convex order for any random scaling relation, it then provides the s-convex extremal distributions in this general framework. In an ERM approach, these extremal distributions stand for the worst case scenarios.

3.1 Transferring the s-convex order

Before proceeding further, let $\mathscr{S}$ be a subinterval of the real line $\mathbb{R}$ and $\mathcal{U}^{\mathscr{S}}_{s-cx}$ be the class of all s-convex functions $\phi : \mathscr{S} \to \mathbb{R}$. A function $\phi : \mathscr{S} \to \mathbb{R}$ is said to be s-increasing convex if ϕ is k-convex, $k = 1, \dots, s$. We denote by $\mathscr{U}^{\mathscr{S}}_{s-icx}$ the class of s-increasing convex functions. All random variables discussed in this paper are defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and we denote by $Supp(X)$ the support of the random variable X. In this context, let S and Y be two random variables such that $Supp(S) = [a, b] \subset \mathbb{R}$ and $Supp(Y) = \mathscr{D} \subseteq \mathbb{R}_+$. In addition, let $\bar{y} = sup(\mathscr{D})$ and $\underline{y} = inf(\mathscr{D})$ be the supremum and the infimum of $\mathscr{D}$ respectively. For given $[a, b]$ and $\mathscr{D} \subseteq \mathbb{R}_+$, the subinterval $\mathscr{A}$ is given by

$$\mathscr{A} = \begin{cases} (a\underline{y}, b\bar{y}) & \text{if} \quad 0 \le a < b < +\infty, \\ (a\bar{y}, b\bar{y}) & \text{if} \quad a \le 0 < b < +\infty, \\ (a\bar{y}, b\underline{y}) & \text{if} \quad -\infty < a < b \le 0. \end{cases}$$

Consequently when $X =_d SY$, then $Supp(X) = \mathscr{A}$. From here and subsequently, we assume $\mathcal{U}^{\mathscr{A}}_{s-cx}$ is non-empty.

When two random scaling factors S_1 and S_2 are ordered in the s-convex sense, a natural question is to wonder if this stochastic order can be transferred to the corresponding randomly scaled variable X. Proposition 3.1 provides the answer.

Proposition 3.1. *Let $X_1 = S_1 Y$ and $X_2 = S_2 Y$ two randomly scaled variables, with $S_i \perp Y$, $i = 1, 2$. If $S_1 \le^{[a,b]}_{s-cx} S_2$ then $X_1 \le^{\mathscr{A}}_{s-cx} X_2$.*

Proof. From Denuit et al. [28], if $S_1 \le^{[a,b]}_{s-cx} S_2$, then $yS_1 \le^{[ya,yb]}_{s-cx} yS_2$ for $y > 0$ which is verified since $Y \in \mathscr{D} \subseteq \mathbb{R}_+$. For $y = 0$, $\mathbb{E}[\phi(yS_1)] = \mathbb{E}[\phi(yS_2)] = \mathbb{E}[\phi(0)]$. Consequently, the independence between Y and S_i, $i = 1, 2$ yields

$$\begin{aligned} \mathbb{E}[\phi(X_1)] &= \int \mathbb{E}[\phi(yS_1)] dF_Y(y), \\ &\le \int \mathbb{E}[\phi(yS_2)] dF_Y(y), \\ &\le \mathbb{E}[\phi(X_2)]. \end{aligned}$$

Then, $X_1 \le^{\mathscr{A}}_{s-cx} X_2$.

□

In the literature, the roles of S and Y are usually reversed. In fact, it is assumed that the Y's are ordered in the s-convex sense, and that the distribution of the random scaling factor is completely known (unimodality in e.g. Denuit et al. [28, 29], α-unimodality in Brockett et al. [14] and Beta-unimodality in Lefèvre et al. [60]). Proposition 3.1 still holds when S and Y are reversed providing the hypotheses on the variables playing the role of the S_i's hold.

Again, Proposition 3.1 remains valid when we consider two random variables Y_1 and Y_2 that are identically distributed instead of the same Y.

Proposition 3.2. *Let $X_1 = S_1Y_1$ and $X_2 = S_2Y_2$ be two randomly scaled variables, with Y_1, Y_2 identically distributed and independent of (S_1, S_2). If $S_1 \leq_{s-cx}^{[a,b]} S_2$, then $X_1 \leq_{s-cx}^{\mathscr{A}} X_2$.*

The dependence structures between S_1 with S_2 and Y_1 with Y_2 are not involved in the transfer of the s-convex order from the S's to the X's. The proposition can be pushed one step further considering a product of random scaling factors. An interesting point emerges in insurance and finance when it comes to iterating interest rates for several periods. In this spirit, Courtois and Denuit [19] derives 2-convex bounds on multiplicative processes in an option pricing purpose. Denoting by Y_i the random discount factor of the period $1 \leq i \leq n$, ordering quantities such as $\mathbb{E}[Y_1...Y_nS_1]$ and $\mathbb{E}[Y_1...Y_nS_2]$ could be of particular interest. Then the latter property is still valid if $Y_1...Y_n \perp S_i$, $i = 1, 2$. The hypothesis of independence between S_i, $i = 1, 2$ and Y is crucial when it comes to transferring the s-convex order from the S's to the X's. Generally speaking, Proposition 3.1 does not hold any more when the random scaling factors are not independent from the Y's.

Once the way of transferring the initial s-convex order from a variable to another is presented, it can be used to derive s-convex extremal distributions. In a nutshell, these extremal distributions are based on s-moment spaces and on the support of the initial random variable. Denuit et al. [30] describes the whole methodology to built suitable s-moment spaces. For a random variable X such that $Supp(X) = \mathscr{A}$ and $\mathbb{E}[X^k] = \nu_k$, $k = 1, \ldots, s-1$, we denote by $\mathcal{B}(\mathscr{A}, \nu_1, \ldots, \nu_{s-1})$ its s-moment space.

3.2 s-convex extrema on random scaling relations

As presented in Denuit et al. [29], dealing with s-convex stochastic order provides so-called extremal s-convex distributions defined on a s-moment space. These extremal distributions are of particular interest when it comes to bounding quantities of the form $\mathbb{E}[\phi(X)]$ for $\phi \in \mathcal{U}_{s-cx}^{\mathscr{A}}$, assuming $X \in \mathcal{B}(\mathscr{A}, \nu_1, \ldots, \nu_{s-1})$. In the same vein, s-convex extrema for a randomly scaled variable are derived in the following proposition when the initial s-convex order comes from the bounded random variable S. From here and subsequently, $\mathcal{B}([a, b], \rho_1, ..., \rho_{s-1})$ and $\mathcal{B}(\mathscr{D}, \mu_1, ..., \mu_{s-1})$ denote the s-moment spaces of the random variables S and Y respectively.

Proposition 3.3. *s-convex extrema for randomly scaled variables*
Let $X =_d SY$ be a randomly scaled variable. If $S \in \mathcal{B}([a, b], \rho_1, ..., \rho_{s-1})$ and $Y \in \mathcal{B}(\mathscr{D}, \mu_1, ..., \mu_{s-1})$, then $X \in \mathcal{B}(\mathscr{A}, \nu_1, ..., \nu_{s-1})$ where

$$\nu_k = \rho_k \, \mu_k, \; k = 1, s-1. \tag{3.1}$$

Furthermore, within $\mathcal{B}(\mathscr{A}, \nu_1, ..., \nu_{s-1})$ the s-convex extremal distributions of X are

$$X_{min}^{(s)} = S_{min}^{(s)} Y \;\; and \;\; X_{max}^{(s)} = S_{max}^{(s)} Y.$$

Proof. Equation (3.1) is a direct consequence of the independence between S and Y. The existence of the extremal distributions $S_{min}^{(s)}$ and $S_{max}^{(s)}$ are due to Denuit et al. [29]. The result is obtained using Proposition 3.1. $\square$

By definition of the extremal distributions, $X_{min}^{(s)} \preceq_{s-cx}^{\mathscr{A}} X \preceq_{s-cx}^{\mathscr{A}} X_{max}^{(s)}$. The s-convex extremal distributions of any randomly scaled variable are obtained in two steps. First the extremal distributions of the random scaling factor S are easily obtained using results given by De Vylder [24] or Denuit et al. [29, 30], providing its s-moment space is known. Obviously, the distribution of S impacts the resulting extremal distribution. Second, Proposition 3.3 ensures s-convex extremal distributions on X are obtained by multiplying the resulting random variable by Y. The presentation of the bounds is omitted for reasons of brevity.

Proposition 3.4 illustrates the fact that knowing information either on S or Y leads to sharper bounds. The result can be extended to any product as long as the factors are mutually independent.

Proposition 3.4. *Let $X =_d SY$ be a randomly scaled variable and assume $X \in \mathcal{B}(\mathscr{A}, \nu_1, ..., \nu_{s-1})$. Then,*

$$X \preceq_{s-cx}^{\mathscr{A}} \begin{cases} S_{max}^{(s)} Y \\ S Y_{max}^{(s)} \end{cases} \preceq_{s-cx}^{\mathscr{A}} S_{max}^{(s)} Y_{max}^{(s)} \preceq_{s-cx}^{\mathscr{A}} \tilde{X}_{max}^{(s)}, \tag{3.2}$$

where $\tilde{X}_{max}^{(s)} \in \mathcal{B}(\mathscr{A}, \nu_1, ..., \nu_{s-1})$ denotes the general s-convex maximum of X, i.e. without knowing X is randomly scaled.

Proof. By definition of $S_{max}^{(s)}$, then $X \preceq_{s-cx}^{\mathscr{A}} S_{max}^{(s)} Y$. Besides, Proposition 3.1 yields $\mathbb{E}[\phi(S_{max}^{(s)} Y)] \leq \mathbb{E}[\phi(S_{max}^{(s)} Y_{max}^{(s)})]$. The result is the same when we begin by Y. The last inequality comes from the fact that $S_{max}^{(s)} Y_{max}^{(s)} \in \mathcal{B}_s(\mathscr{A}, \nu_1, ..., \nu_{s-1})$. Combining it with the definition of $\tilde{X}_{max}^{(s)}$ yields the announced result. □

3.3 Comparison of s-convex extrema

The previous subsections give the worst case scenarios represented by the extremal distributions. In our ERM approach, we would like worst case scenarios to be sorted in function of the available information. In other words, since the 2-convex extremal distribution is built only with the first moment whereas the 4-convex distribution uses the three first moments, it seems relevant to require the resulting 4-convex worst case scenario to be less serious than the 2-convex one.

Assume $X \in \mathcal{B}(\mathscr{A}, \nu_1, \ldots, \nu_{s-1})$ is a random variable, not necessarily randomly scaled. By definition, $X_{max}^{(3)} \in \mathcal{B}(\mathscr{A}, \nu_1, \nu_2) \subset \mathcal{B}(\mathscr{A}, \nu_1)$ and for $\phi \in \mathscr{U}_{2-cx}^{\mathscr{A}}$, both $\mathbb{E}[\phi(X_{max}^{(3)})] \leq \mathbb{E}[\phi(X_{max}^{(2)})]$ and $\mathbb{E}[\phi(X)] \leq \mathbb{E}[\phi(X_{max}^{(2)})]$ hold. However nothing guarantees $\mathbb{E}[\phi(X)] \leq \mathbb{E}[\phi(X_{max}^{(3)})] \leq \mathbb{E}[\phi(X_{max}^{(2)})]$ is verified; see Lefèvre et al. [60] for an insightful example. This constraint, referred to as the ERM requirement is at the basis of our methodology. To this extent, we provide theoretical results that make the s-convex extremal distributions comparable.

Formally, since $\mathcal{B}(\mathscr{A}, \nu_1, \ldots, \nu_{s-1}) \subset \mathcal{B}(\mathscr{A}, \nu_1, \ldots, \nu_{s-2})$ given the same ν_k, $1 \le k \le s-2$, then

$$X_{max}^{(s)} \le_{(s-1)-cx} X_{max}^{(s-1)}.$$

If all the available information is used, assuming $s \ge 2$, the same reasoning yields

$$X \le_{s-cx} X_{max}^{(s)} \le_{(s-1)-cx} X_{max}^{(s-1)} \le_{(s-2)-cx} \cdots \le_{2-cx} X_{max}^{(2)},$$

which can be rewritten as

$$\mathbb{E}[\phi_s(X)] \le \mathbb{E}[\phi_s(X_{max}^{(s)})] \le \mathbb{E}[\phi_{s-1}(X_{max}^{(s-1)})] \le \ldots \le \mathbb{E}[\phi_2(X_{max}^{(2)})], \tag{3.3}$$

where $\phi_k \in \mathscr{U}_{k-cx}^{\mathscr{A}}$, $k = 2, \ldots, s$. It is clear that unless

$$\phi \in \bigcap_{k=2}^{s} \mathscr{U}_{k-cx}^{\mathscr{A}}, \tag{3.4}$$

inequalities in (3.3) involve *a priori* different functions. We are now ready to state the following result, which shows the s-convex extremal distributions can be compared in the s-convex order if ϕ is k-convex, $k = 2, \ldots, s$.

Proposition 3.5. *Let $X \in \mathcal{B}(\mathscr{A}, \nu_1, \ldots, \nu_{s-1})$, not necessarily randomly scaled. Then*

$$\mathbb{E}[\phi(X)] \le \mathbb{E}[\phi(X_{max}^{(s)})] \le \mathbb{E}[\phi(X_{max}^{(s-1)})] \le \ldots \le \mathbb{E}[\phi(X_{max}^{(2)})],$$

hold for $\phi \in \bigcap_{k=2}^{s} \mathscr{U}_{k-cx}^{\mathscr{A}}$.

Proof. Let $X \in \mathcal{B}(\mathscr{A}, \nu_1, \ldots, \nu_{s-1})$ and assume $\phi \in \bigcap_{k=2}^{s} \mathscr{U}_{k-cx}^{\mathscr{A}},$. By definition of $X_{max}^{(s)}$ and ϕ, $\mathbb{E}[\phi(X)] \le \mathbb{E}[\phi(X_{max}^{(s)})]$. In addition, $X_{max}^{(s)} \in \mathcal{B}(\mathscr{A}, \nu_1, \ldots, \nu_{s-1}) \subset \mathcal{B}(\mathscr{A}, \nu_1, \ldots, \nu_{s-2})$, then $\mathbb{E}[\phi(X_{max}^{(s)})] \le \mathbb{E}[\phi(X_{max}^{(s-1)})]$, leading to $\mathbb{E}[\phi(X)] \le \mathbb{E}[\phi(X_{max}^{(s)})] \le \mathbb{E}[\phi(X_{max}^{(s-1)})]$. We proceed in the same way for $s-1, \ldots, 2$. $\square$

Basically, comparing $X_{max}^{(s)}$ with $X_{max}^{(s-1)}$ in the s-convex order can be done only if the function is both s and $s-1$ convex. The higher degree convex functions is not a new concept, see e.g. Popoviciu [70, 69], Kuczma [57] and Denuit et al. [28] and references therein. In Proposition 3.5, if the function ϕ is increasing, then ϕ is said to be s-increasing convex. The class of s-increasing convex functions on $\mathscr{A}$ is denoted by $\mathscr{U}_{s-icx}^{\mathscr{A}}$. In the literature, s-increasing convex functions are referred to as s-times monotone functions, see e.g. Maksa and Páles [61], McNeil and Nešlehová [62] and Rajba [71].

When $\mathscr{A}$ admits a finite left end point Popoviciu [69] give a characterization of s-increasing convex functions. Assume the left endpoint of $\mathscr{A}$, say a, is finite, and let us denote by $\psi_{s,a} : \mathscr{A} \mapsto \mathbb{R}^+$ the function defined by $\psi_{s,a}(x) = (x-a)^s$, $s = 1, 2, \ldots$. Also, let $\psi_{s-1,t,+}$ denote the function defined by $\psi_{s-1,t,+}(x) = (x-t)_+^{s-1}$. Popoviciu [69] (Theorem 8) showed that the functions $\psi_{k,a}$, $k = 0, \ldots, s-1$ and $\psi_{s-1,t,+}$, $t \in \mathscr{A}$, span $\mathscr{U}^{\mathscr{A}}_{s-icx}$. More precisely, for $n \geq s$, let the function φ_n be of the form

$$\varphi_n(x) = \sum_{j=0}^{s-1} \gamma_j (x-a)^j + \sum_{j=0}^{n-s} \beta_j (x - t_j)_+^{s-1}, \tag{3.5}$$

where $\gamma_0, \ldots, \gamma_{s-1}$ are non-negative constants, $\beta_0, \ldots, \beta_{n-s}$ are non-negative constants, and $t_0 < t_1 < \ldots < t_{n-s} \in \mathscr{A}$. Then every $\phi \in \mathscr{U}^{\mathscr{A}}_{s-icx}$ is the uniform limit of a sequence $\{\varphi_n, n \geq s\}$, where the φ_n's are of the form (3.5).

Before we shift our attention to the next section, let us deal with the product of increasing convex functions. According to Popoviciu [70], it is in general impossible to characterize the order of convexity of the product of two functions. Nevertheless, it is direct to deal with the particular case of two positive s-increasing convex functions.

Proposition 3.6. *Let $f : \mathscr{A} \mapsto \mathbb{R}^+$ and $\phi : \mathscr{A} \mapsto \mathbb{R}^+$ be two positive s-increasing convex functions. Then $f \times \phi \in \mathscr{U}^{\mathscr{A}}_{s-icx}$.*

Proof. From Popoviciu [70], the divided difference of order s of $f \times \phi$ is given by

$$[x_0, \ldots, x_s; f \times \phi] = \sum_{k=0}^{s} [x_0, \ldots, x_{s-k}; f][x_{s-k}, \ldots, x_s; \phi],$$

where $x_0, \ldots, x_s \in \mathscr{A}$. Since $f \in \mathscr{U}^{\mathscr{A}}_{s-icx}$, $[x_0, \ldots, x_{s-k}; f] \leq 0$ for $k \leq s$. For the same reason, $[x_{s-k}, \ldots, x_s; \phi] \geq 0$ for $k \leq s$. Consequently $[x_0, \ldots, x_s; f \times \phi] \geq 0$. For $n > s$,

$$[x_0, \ldots, x_n; f \times \phi] = [x_0, \ldots, x_n; f]\phi(x_n) + \sum_{k=1}^{n} [x_0, \ldots, x_{n-k}; f][x_{n-k}, \ldots, x_n; \phi].$$

This time, the sign of $[x_0, \ldots, x_n; f]$ is unknown and we can not conclude $[x_0, \ldots, x_n; f \times \phi] \geq 0$. $\square$

To close the discussion on the comparison of s-convex extremal distributions, the ERM requirement is fulfilled if the function used to measure the consequences of basis risk belongs to the class introduced in (3.4). The next section is devoted to the definition of the fonctions used to measure the consequences of the basis risk, referred to as generalized penalty functions.

4 Quantifying two-sided basis risk

The aim of this section is to provide the second and third steps of our basis risk quantification method. The second step consists in quantifying basis risk thanks to generalized penalty functions. In the third step, we derive basis risk limits and propose a way to cope with consequences of basis risk. For a fixed basis risk budget, basis risk limits are defined as the affordable maximum window of uncertainty.

4.1 Motivation

Lefèvre et al. [60] introduce a particular class of functions, referred to as penalty functions. These functions worsen the consequences of a great difference between the real loss and the value taken by the parametric index. In other words, a penalty function transforms linear consequences into convex ones. For the sake of clarity, we formalize the required conditions to define a penalty function in the following definition.

Definition 4.1. *Initial Penalty Function (IPF)*
Let n be a positive integer. A function $g_n : \mathbb{R} \to \mathbb{R}_+$ is called an initial penalty function if both $g_n \in \bigcap_{k=2}^{n} \mathscr{U}_{k-cx}^{\mathbb{R}}$ and $g_n(0) = 0$ hold.

From Proposition 3.6 and Definition 4.1, we provide some examples of Initial Penalty Functions.

Example 4.1. *Let $x \in \mathbb{R}$. The function $g_2 : x \mapsto \lambda x^2$, $\lambda > 0$ is the simplest IPF.*

Because the symmetry of g_2 may be too restrictive, it can be enlarged to $g_n : x \mapsto \lambda x^2 (x-d)_+^n$, or to $g_n : x \mapsto \lambda x^2(1 + (x-d)_+^n)$. Such a form takes into account both negative and positive values of uncertainty, i.e. measures the impact of both upside and downside basis risks. In this representation, the parameter d stands for a critical level. When the uncertainty is greater than the critical level, the consequences of basis risk worsen dramatically (depending on the value of n).

Of course, it is possible to set several layers of impacts say $d_1, \ldots, d_k$, k being a positive integer since the function $g_n : x \mapsto \lambda x^2 \prod_{i=1}^{k}(1 + (x-d_i)_+^n)$ is an IPF.

It is even possible to increase the power n according to the layer. In fact, let $n_i \geq n$ be the power associated to the layer i, $i = 1, \ldots, k$. The function $g_n : x \mapsto \lambda x^2 \prod_{i=1}^{k}(1 + (x-d_i)_+^{n_i})$ is an IPF.

If one is interested in modifying the impact when $x < 0$, he has to take care of the resulting order of convexity of the function. For instance, $f : x \mapsto g_2(x)(1 + (c-x)_+^4)$, $c < 0$ is only 2 and 4-convex. Therefore, f is not an IPF and only the 2 and 4-convex extrema can be computed.

Whatever the form of the chosen IPF, it only deals with the difference between the loss and the index by fixing an additive constraint. In other words, penalty functions introduced in Definition 4.1 lay the stress on the size of the uncertainty, without taking into account the size of the loss (or equivalently, the size of the index). In order to illustrate the issue, consider any of the IPF given in Example 4.1. On the one hand, assume the value of the loss is 200 whereas the value of the index is 100 : then the difference is 100. The penalty function yields, say $g_n(100)$. On the other hand, assume the value of the loss is 2 000 000 and the one of the index is 1 999 900. Once again the difference is 100, and the penalty function leads to the same level of consequences, $g_n(100)$. In the first case the difference represents half of the loss, whereas it represents only 0.005% in the second case. Comparing the size of the uncertainty with the size of the loss is equivalent to comparing the size of the uncertainty with the size of the index. By adopting this point of view, we focus on the quality of the index. If the ratio "uncertainty on index" is high, the index does not reflect the loss properly. On the other hand, if the ratio is low, then the index is efficient. Therefore, the upcoming section generalizes the initial penalty function by increasing the impact of the uncertainty when the ratio "uncertainty on index" is greater than a predefined acceptable value.

4.2 Generalized penalty functions

To overpass the disadvantage of the initial penalty function, we propose a way to improve the measurement of the consequences of basis risk. This improvement involves the ratio "uncertainty on index" so that the impact of the uncertainty is increased when the ratio is greater than an acceptable value. Therefore, we introduce an impact function that takes three parameters in input.

Definition 4.2. *Impact Function*
Let n be a positive integer. A function

$$h_n : \begin{cases} \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+ & \to \quad [1, +\infty[\\ (x, z, l) & \mapsto \quad h_n(x/z, l) \end{cases},$$

is called an impact function if it verifies the following conditions,

- h_n *is continuous,*
- $h_n(x/z, l) = 1$ *for* $l \geq x/z$,
- $h_n(x/z, l) > 1$ *for* $l < x/z$,
- $h_n \in \mathscr{U}_{n-icx}^{\mathbb{R}_+}$.

The parameter x stands for the size of the uncertainty whereas z represents the size of the index. The positive parameter l represents the maximal acceptable ratio. When the ratio "uncertainty on index" remains under l, the consequences of the difference between the loss and the index are not modified ($h_n(x/z, l) = 1$). On the other hand, the impact is increased when the ratio is greater than the acceptable value (($h_n(x/z, l) > 1$). Let $l > 0$, and $n \in \mathbb{N}^\star$. A suitable h_n is for example

$$h_n\left(\frac{x}{z}, l\right) = 1 + \left(\frac{x}{z} - l\right)_+^n, \; x \in \mathbb{R}, \; z \in \mathbb{R}_+. \tag{4.1}$$

From now on, the greater the ratio the greater the consequences. Once the initial penalty function and the impact function are formalized, we can introduce the generalized penalty function ξ_n as

Definition 4.3. *Generalized Penalty Function*
A function

$$\xi_n : \begin{cases} \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+ & \to \quad \mathbb{R}_+ \\ (x, z, l) & \mapsto \quad \xi_n(x, z, l) \end{cases},$$

is called a generalized penalty function if it can be written as,

$$\xi_n(x,z,l) = g_n(x)h_n\left(\frac{x}{z}, l\right),\ x \in \mathbb{R},\ z \in \mathbb{R}_+,\ l > 0,$$

where g_n is an initial penalty function, and h_n is an impact function, n being a positive integer.

Therefore, the impact function h_n behaves like a multiplicative factor which increases the initial consequences of a difference between the loss and the index. This multiplicative factor intervenes only if the size of the uncertainty is greater than a level based on the size of the index. Consequently, because of the form of both g_n and h_n, $\xi_n(x,z,l) \geq g_n(x)$, $\forall x \in \mathbb{R}$, $\forall z \in \mathbb{R}_+$ and $\forall l > 0$. The generalized penalty functions are used to measure the consequences of the uncertainty in the worst case scenarios. Consequently, an important point to look at is the order of convexity of theses functions to make sure the ERM requirement is fulfilled.

Proposition 4.1. *Let ξ_n be a generalized penalty function built with an initial penalty function g_n and an impact function h_n. Then for a given $(z,l) > 0$,*

$$\xi_n \in \bigcap_{k=2}^{n} \mathscr{U}_{k-cx}^{\mathbb{R}}.$$

Proof. The proof is a direct consequence of Proposition 3.6 □

Obviously, by definition of l, when $l \to \infty$, $\xi_n(x,z,l) = g_n(x)$ for all $z \in \mathbb{R}_+$. This means when the acceptable ratio is infinite, the generalized penalty function is merely the initial penalty function. For instance,

$$\xi_n(x,z,l) = \lambda\left(1 + (x-d)_+^n\right)\left(1 + \left(\frac{x}{z} - l\right)_+^n\right), \tag{4.2}$$

for $x \in \mathbb{R}$, $z \in \mathbb{R}_+$, $n \in \mathbb{N}^\star$, $d > 0$ and $l > 0$ is a generalized penalty function. Note that ξ_n defined in (4.2) penalizes only positive differences.

4.3 Computation of basis risk measurement in worst case scenarios

We must now quantify the consequences of the basis risk X in each worst case scenario. For this, we calculate the (positive) measure associated with X, denoted $C_n \equiv C_n(X)$, which corresponds to the average value of the generalized penalty for X, i.e. $C_n = \mathbb{E}[\xi_n(X,Z,l)]$. Note that for Z fixed, $C_n(X)$ is convex in X.

The generalized penalty function ξ_n has been built so that $\xi_n \in \bigcap_{k=2}^{n} \mathscr{U}_{k-cx}^{\mathbb{R}}$. Consequently Proposition 3.5 holds, and the ERM requirement is fulfilled. The following proposition gives the general form of the maximal basis risk measurement.

Proposition 4.2. *Let n be an integer and let ξ_n be a generalized penalty function. Assume the uncertainty is a randomly scaled variable with $S \perp Y$, $S \perp Z$ and $S \in \mathcal{B}([-a,a],\rho_1,\ldots,\rho_{n-1})$. The s-convex basis risk measurement is hence defined as a function of the scenario s, namely*

$$C_n^{(s)}(a) = C_n\left(X_{a,max}^{(s)}\right) = C_n\left(S_{a,max}^{(s)}Y\right), \quad s=2,\ldots,n, \quad a>0. \tag{4.3}$$

In addition,

$$C_n(a) \leq C_n^{(s)}(a) \leq C_n^{(s-1)}(a) \leq \ldots \leq C_n^{(2)}(a). \tag{4.4}$$

Proof. Equation (4.4) is obtained combining the fact that $\xi \in \bigcap_{k=2}^{n} \mathscr{U}_{k-cx}^{\mathbb{R}}$ and Proposition 3.5. $\square$

As presented in Denuit et al. [29], s-convex extremal distributions are atomic ones and in particular, the number of points of the s-convex maximal distribution is $k+1$ where $k \in \mathbb{N}^\star$ is such that $s=2k$ if s is an even number or $s=2k+1$ if s is odd. Therefore, denoting s_i, $i=1\ldots,k$ the points of the s-convex maximal distributions and p_i their associated mass, then

$$C_n^{(s)}(a) = \sum_{i=1}^{k} p_i C_n\left(s_i Y\right), \tag{4.5}$$

and the difficulty lies in the computation of $C_n\left(s_i Y\right)$. We point out that when the distributions of both Y and Z are fixed, $C_n^{(s)}(0)=0$ and $C_n^{(s)}$ is strictly increasing in a. This allows for defining basis risk limits in the following way.

5 Setting up basis risk limits

The way to measure basis risk presented in previous sections lays the stress on the ratio between the value of the uncertainty and the value of the index. For each scenario, one is able to derive average consequences of basis risk. The last step in our methodology is to set up basis risk limits in order to be able to manage and to control basis risk. In this Section, we first define the basis risk limits, we then adopt two points of view to derive sound basis risk management.

5.1 Fixed basis risk budget

The first point of view is to fix beforehand a basis risk budget. In fact, the entity facing basis risk is able to define a basis risk budget based on an ERM criterion, for instance risk appetite. This basis risk budget can be defined as the monetary amount that allows for coping with consequences of basis risk when these consequences are measured with a generalized penalty function. To this extent, basis risk limits naturally appears as the maximal affordable basis risk.

In other words, let $S_a \in \mathcal{B}([-a,a],\rho_1,\rho_2,...,\rho_{s-1})$, $a>0$, and let $\pi>0$ be the fixed basis risk budget. Then the basis risk limit in scenario s, denoted by $a^{(s)}$, is defined as the value of the parameter a such that

$$C_n^{(s)}(a) = \pi. \tag{$\mathcal{P}1$}$$

From this point of view, solving ($\mathcal{P}1$) yields one value of $a^{(s)}$ for each scenario s. By doing so, we completely manage the weight of uncertainty in the size of the ultimate loss.

In fact, basis risk limits can be expressed as the acceptable probability that the ratio between the value of the uncertainty and the value of the index exceeds the predefined threshold l involved in the impact function of the generalized penalty function. By definition,

$$\mathbb{P}\left(\frac{S_aY}{Z} > l\right) = \gamma, \quad \gamma \in [0,1]. \tag{5.1}$$

We draw the attention on the fact that we do not deal with the s-convex maximum distribution of S_a but with its underlying distribution. In addition, let $T =_d Y/Z$ then S_aT is a randomly scaled variable. Since solving ($\mathcal{P}1$) yields one value of $a^{(s)}$ for each scenario s, we equivalently obtain one value of γ for each scenario, denoted by $\gamma^{(s)}$, given by

$$\gamma^{(s)} = \mathbb{P}\left(S_a^{(s)}T > l\right).$$

If the resulting $\gamma^{(s)}$ is not satisfactory from the point of view of the basis risk bearer the index can be rejected. Note from Proposition 2.4 that for a fixed S_a, if $T_1 \leq_{st} T_2$ then $\gamma_1^{(s)} \leq \gamma_2^{(s)}$. Consequently, T plays a key role in the determination of the basis risk limits. In the following subsection we deal with the reversed problem.

5.2 Basis risk capital requirement

Whereas in the previous subsection basis risk budget was fixed and we aimed at deducing the basis risk limits, here we are interested in the reversed problem, namely we want to obtain the basis risk budget knowing the basis risk limits. In other words, we begin by fixing the parameter γ introduced in (5.1), then we have to find the corresponding parameter a in S_a to eventually obtain one basis risk budget for each scenario.

The next proposition ensures that $h_l : a \mapsto \mathbb{P}(S_aT > l)$, can be reversed. In addition we provide an interesting result on the limit of this function.

Proposition 5.1. *Let $l>0$, S_a be a random scaling factor distributed over $[-a,a]$ and T be a positive random variable, independent from S_a, with density f_T. Assume the $[0,1]$-valued random variable $U = \dfrac{S_a+a}{2a}$ admits a strictly decreasing survival distribution function. Let*

$$h_l(a) = \mathbb{P}(S_aT > l) = \int_{l/a}^{\infty} \mathbb{P}\left(U > \frac{l}{2at} + \frac{1}{2}\right) f_T(t)dt, \forall a \in \mathbb{R}_+, \quad a>0.$$

Then h_l is continuous and strictly increasing. In addition,

$$\lim_{a \to +\infty} h_l(a) = \mathbb{P}\left(U > \frac{1}{2}\right), \tag{5.2}$$

providing the limit exists.

Proof. For all $a \in \mathbb{R}_+$, $\lim_{x \to a} h_l(x) = h_l(a)$, ensuring the continuity of h_l.
Let $\tilde{a} = a + \varepsilon$, $\varepsilon > 0$, and $t \in [l/\tilde{a}, +\infty[$. With the strict decrease of the survival distribution of U,

$$\mathbb{P}\left(U > \frac{l}{2\tilde{a}t} + \frac{1}{2}\right) > \mathbb{P}\left(U > \frac{l}{2at} + \frac{1}{2}\right). \tag{5.3}$$

Integrating (5.3) from l/a to $+\infty$ gives

$$\int_{l/a}^{\infty} \mathbb{P}\left(U > \frac{l}{2\tilde{a}t} + \frac{1}{2}\right) f_T(t)dt > \int_{l/a}^{\infty} \mathbb{P}\left(U > \frac{l}{2at} + \frac{1}{2}\right) f_T(t)dt = h_l(a). \tag{5.4}$$

Moreover,

$$h_l(\tilde{a}) = \int_{l/\tilde{a}}^{\infty} \mathbb{P}\left(U > \frac{l}{2\tilde{a}t} + \frac{1}{2}\right) f_T(t)dt > \int_{l/a}^{\infty} \mathbb{P}\left(U > \frac{l}{2\tilde{a}t} + \frac{1}{2}\right) f_T(t)dt. \tag{5.5}$$

Combining (5.4) and (5.5) yields

$$\forall a \geq 0,\ \forall \tilde{a} > 0,\ a < \tilde{a} \Rightarrow h_l(a) < h_l(\tilde{a}),$$

proving thus the function h_l is strictly increasing.
Let $t \in [l/a, +\infty[$. Since the survival distribution function of U is strictly decreasing,

$$\mathbb{P}(U > 1) < \mathbb{P}\left(U > \frac{l}{2at} + \frac{1}{2}\right) \leq \mathbb{P}\left(U > \frac{1}{2}\right). \tag{5.6}$$

Integrating (5.6) yields

$$0 < \int_{l/a}^{\infty} \mathbb{P}\left(U > \frac{l}{2at} + \frac{1}{2}\right) f_T(t)dt \leq \mathbb{P}\left(U > \frac{1}{2}\right) \int_{l/a}^{\infty} f_T(t)dt. \tag{5.7}$$

Taking the limits, providing it exits, of (5.7) leads to

$$0 < \lim_{a \to +\infty} h_l(a) \leq \mathbb{P}\left(U > \frac{1}{2}\right).$$

The function h_l is strictly increasing and admits a horizontal asymptote, proving thus Equation (5.2). $\square$

Now, we are allowed to reverse the function h_l, yielding

$$\hat{a} = h_l^{-1}(\gamma). \tag{$\mathcal{P}2$}$$

The parameter $\hat{a}$ denotes the value of a such that the probability that the ratio $S_{\hat{a}}Y/Z$ is greater than l equals γ. Solving ($\mathcal{P}2$) yields the basis risk budget induced by the parameter γ. For each scenario s, we obtain a basis risk budget $\pi_\gamma^{(s)}$ defined as

$$\pi_\gamma^{(s)} = C_n^{(s)}(\hat{a}) = C_n^{(s)}\left(h_l^{-1}(\gamma)\right).$$

From Proposition 2.5, if $S_a \leq_{st} \tilde{S}_a$, then $\pi_\gamma^{(s)} \leq \tilde{\pi}_\gamma^{(s)}$ highlighting thus the role of the random scaling factor in the determination of the basis risk budget.

Contrary to ($\mathcal{P}1$), solving ($\mathcal{P}2$) yields one value of the parameter a for all the scenarios, namely $\hat{a}$, but one value of the basis risk budget based on the initial known value of γ for each scenario. To this extent, we define the value $\pi_\gamma^{(s)}$ as a Basis Risk Capital Requirement of level γ. This capital requirement ensures to lead to affordable consequences of basis risk, and to uncertainties that do not weight more than l times the size of the index in the total loss with a given and chosen probability γ. In other words, this ensures to select efficient indices. From an ERM point of view (5.2) stands for a criterion to select or reject an index. For a given index, if the probability that the uncertainty exceeds l times the value of the index is closed to one half, then the uncertainty is maximal, the basis risk limits are huge and the resulting basis risk capital requirement is infinite. In other words, this particular index is not affordable.

6 Practical implementation of the methodology

In this section, we apply and illustrate our methodology. First we identify the s-convex extremal distributions of the basis risk then we chose a generalized penalty function and we end by solving both ($\mathcal{P}1$) and ($\mathcal{P}2$). As we focus so far on the importance of taking into account both the size of the uncertainty and the size of the index, we lay the stress on three types of uncertainty. For a given distribution of the index Z, we deal with i) $Y = Z$, ii) $Y = \sqrt{Z}$ and iii) $Y = \tilde{Z}$ where $\tilde{Z}$ is an independent copy of Z.

In case i) the loss L can be written as $L = (1+S)Z = \tilde{S}Z$, i.e. L is randomly scaled. In this case, the index Z is multiplied by a random scaling factor $\tilde{S}$ in order to take into account the unperfect link between the index and the loss. Cases ii) and iii) allow the basis risk to be dependent from the index.

In order to compare light with heavy tails, results are given for $Z \sim \Gamma(\kappa, r)$, $\kappa > 0$ and $r > 0$ and for $Z \sim Pareto(\omega, m)$, $m > 0$ and $\omega > 0$. We recall that the density of a Pareto distribution is given by

$$f(x) = \omega\frac{\omega^m}{x^{m+1}},\ for\ x \geq \omega,$$

where $m > 0$ is the shape parameter.

In this illustration, we focus on the case where the random scaling factor is either uniformly distributed over $(-a, a)$ or of Beta type over $(-a, a)$. As before, these two cases are denoted by U_a and S_a respectively. In short, we denote by $SB(\alpha, \beta)$ the case where $S_a \sim \mathcal{B}eta([-a, a], \alpha, \beta)$.

In order to keep the computation tractable, the generalized penalty function is given by

$$\xi_n(x, z, l) = (x - c)_+^n \left(1 + \left(\frac{x}{z} - l\right)_+^n\right), \quad x \in \mathbb{R}, z, l > 0.$$

We assume $n \geq 4$ so that ξ_n is at least 4 increasing convex and we set $l = 5\%$. Now we must focus on the existence of $C_n(s_i Y) = \mathbb{E}[\xi_n(s_i Y, Z, l)]$ introduced in (4.5) and given by

$$C_n(s_i Y) = \lambda \mathbb{E}\left[(s_i Y - c)_+^n \left(1 + \left(s_i \frac{Y}{Z} - l\right)_+^n\right)\right].$$

Obviously, if $s_i \leq 0$, $C_n(s_i Y) = 0$. For the sake of clarity, the general forms of $C_n(s_i Y)$ are given in Appendix for the three cases i), ii) and iii) and when Z is either Gamma or Pareto distributed. The following Proposition gives the form of the function h_l introduced in Proposition 5.1 when the random scaling factor is uniformly distributed over $(-a, a)$.

Proposition 6.1. *Let $S_a \sim \mathcal{U}[-a, a]$, and $l > 0$.*
The function h_l is

i) for $Y = Z$,

$$h_l(a) = \frac{a - l}{2a}.$$

ii) For $Y = \sqrt{Z}$, when $Z \sim \Gamma(\kappa, r)$, $\kappa > 0$ $r > 0$,

$$h_l(a) = \frac{1}{2}\mathbb{P}\left(Z < \left(\frac{a}{l}\right)^2\right) - \frac{l}{2a}\frac{\Gamma(\kappa + 1/2)}{r^{1/2}\Gamma(\kappa)}\mathbb{P}\left(Z_{\kappa+1/2} < \left(\frac{a}{l}\right)^2\right),$$

where $Z_{\kappa+1/2} \sim \Gamma(\kappa + 1/2, r)$.

When $Z \sim Pareto(\omega, m)$, and $\omega > 1/2$,

$$h_l(a) = \frac{1}{2}\left(1 - \omega^m \left(\omega \vee \left(\frac{a}{l}\right)^2\right)^{-m}\right) - \frac{lm\omega^m}{2a(m - 1/2)}\left(\omega^{1/2-m} - \left(\omega \vee \left(\frac{a}{l}\right)^2\right)^{1/2-m}\right).$$

iii) For $Y = \tilde{Z}$, $\tilde{Z}$ being an independent copy of Z, when $Z \sim \Gamma(\kappa, r)$, $\kappa > 1$ $r > 0$,

$$h_l(a) = \frac{1}{2}\mathbb{P}\left(\frac{\tilde{Z}}{Z} > \frac{l}{a}\right) - \frac{l}{2a}\frac{\Gamma(\kappa - 1)\Gamma(\kappa + 1)}{\Gamma(\kappa)^2}\mathbb{P}\left(\frac{\tilde{Z}_{\kappa-1}}{Z_{\kappa+1}} > \frac{l}{a}\right),$$

where $\tilde{Z}_{\kappa-1} \sim \Gamma(\kappa-1, r)$ *and* $Z_{\kappa+1} \sim \Gamma(\kappa+1, r)$.

When $Z \sim Pareto(\omega, m)$*, the function* h_l *is*

$$h_l(a) = \begin{cases} \dfrac{1}{4(m+1)} \left(\dfrac{a}{l}\right)^m & \text{if } \dfrac{l}{a} \geq 1, \\ \dfrac{1}{2} - \dfrac{lm^2}{2a(m^2-1)} + \dfrac{1}{4(m-1)} \left(\dfrac{l}{a}\right)^m & \text{if } \dfrac{l}{a} < 1. \end{cases}$$

The function h_l is the survival function of the randomly scaled variable $S \times Y/Z$. To this extent, the general form of h_l when the random scaling factor is of Beta type with integer parameters is given by Proposition 2.3.

6.1 Light tailed randomly scaled uncertainty

We are ready to solve ($\mathcal{P}1$) and ($\mathcal{P}2$) in cases i) $Y = Z$, ii) $Y = \sqrt{Z}$ and iii) $Y = \tilde{Z}$, $\tilde{Z}$ being an independent copy of Z when Z is Gamma distributed with parameters, say $\kappa = 5$ and $r = 2$. We set $c = 0.1\mathbb{E}[Z] = 0.25$ and $\pi = 0.25\mathbb{E}[Z] = 0.625$.

First of all, Figure 6.2 illustrates the shape of $C_n^{(s)}(a)$ in case i) $Y = Z$ for $s = 2$ (solid line), $s = 3$ (dashed line) and $s = 4$ (dotted line) with random scaling factors $SB(1,4)$ (top left), $SB(4,1)$ (top right), U_a (bottom left) and $SB(0.5, 0.5)$ (bottom right). As expected, the more moment, the sharper the bounds, i.e. the solid curve is always above the dashed one which is itself above the dotted one. Further, we can see the bounds for $SB(4,1)$ take the greater values. This is due to the fact that when $S_a \sim \mathcal{B}eta([-a,a], 4, 1)$, $\mathbb{P}(S_a > 0) = 0.9375$. The resulting $X =_d S_a Y$ is right shifted (strictly positive mode) and not symmetric, increasing thus the size of the uncertainty. On the contrary, the case where $S_a \sim \mathcal{B}eta([-a,a], 1, 4)$ yields $\mathbb{P}(S_a > 0) = 0.0625$. The resulting $X =_d S_a Y$ is left shifted (strictly negative mode) and not symmetric. Therefore the bounds remain low. Further, the case where the random scaling factor is uniformly distributed over $(-a, a)$ looks like the one where $S_a \sim \mathcal{B}eta([-a,a], 0.5, 0.5)$. In fact, since $\mathbb{E}[U_a] = \mathbb{E}[S_a] = 0$, the 2 convex bounds are the same.

From now, we focus on the cases where the random scaling factor is either uniformly distributed over $(-a, a)$ or is $SB(4,1)$. Table 1 summarizes the basis risk limits for each scenario $s = 2, 3, 4$ for the three different cases i), ii) and iii) for U_a and $SB(4,1)$.

	U_a			$SB(4,1)$		
s-cx order	$Y = Z$	$Y = \sqrt{Z}$	$Y = \tilde{Z}$	$Y = Z$	$Y = \sqrt{Z}$	$Y = \tilde{Z}$
2	0.3894	0.7743	0.2802	0.3529	0.7045	0.2641
3	0.4508	0.8904	0.3055	0.4077	0.8094	0.2882
4	0.4914	0.9663	0.3212	0.4218	0.8367	0.2947

Table 1: Basis risk limits $a^{(s)}$ for $s = 2, 3, 4$ in cases i), ii) and iii) for $Z \sim \Gamma(5, 2)$.

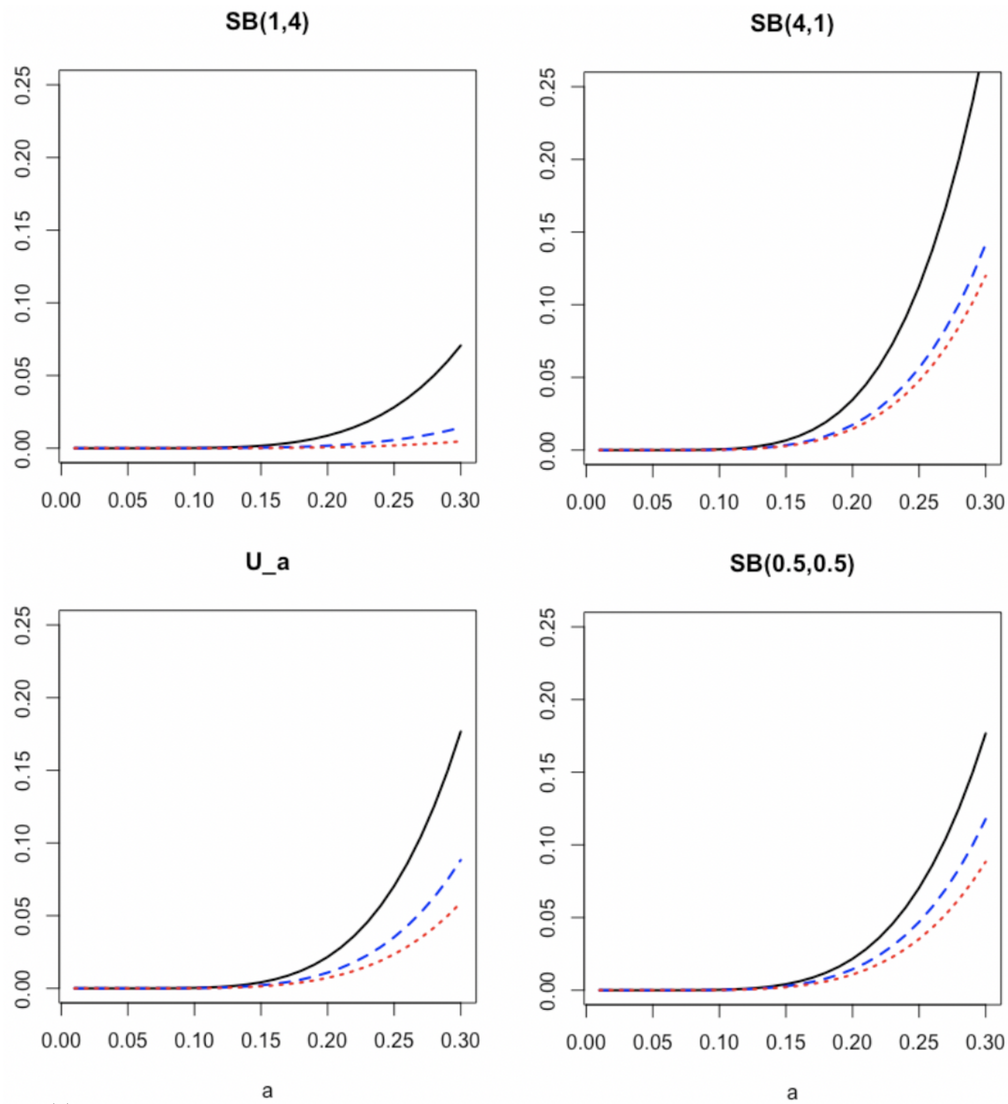

Figure 6.2: $C_4^{(s)}(a)$ for $s = 2$ (solid line), $s = 3$ (dashed line) and $s = 4$ (dotted line) with random scaling factors $SB(1, 4)$ (top left), $SB(4, 1)$ (top right), U_a (bottom left) and $SB(0.5, 0.5)$ (bottom right) in case i) $Y = Z$.

One notes for a given type of uncertainty, $a^{(s)}$ increases with the order of convexity. This is explained by the available information in scenario, or equivalently by the number of moments used to compute the s-convex extremal distributions. The higher the convex order the more information, and consequently the greater the affordable a for a given basis risk budget. Furthermore, for a fixed convex order, i.e. for a given scenario, case iii) leads to the lowest values of a, whereas the greatest are reached in case ii). This is partially explained using the fact $\mathbb{P}(\sqrt{Z} > x) \leq \mathbb{P}(Z > x)$ for $x \geq 1$. Case iii) highlights the importance of having taken into account the ratio $T = Y/Z$. Since $\tilde{Z}$ is an independent copy of Z, then $\mathbb{E}[g_n(Z)] = \mathbb{E}[g_n(\tilde{Z})]$ where g_n is the IPF contained in the generalized penalty function, i.e. $g_n(x) = (x - c)_+^n$. Consequently case i) and case iii) would have lead to the same consequences. In other words, an uncertainty behaving as the index leads to the same consequences than an independent uncertainty. When one adopts an ERM point of view, these two cases do not have the same signification.

Now, we move to the resolution of Problem ($\mathcal{P}2$). We fix $\gamma = 0.05$ and we first find the values of $\hat{a}$ in each cases i), ii) and iii). Table 2 sums up the resulting basis risk limits (the value of $\hat{a}$) in cases i), ii) and iii) for $S_a \sim \mathcal{U}[-\hat{a}, \hat{a}]$.

	$Y = Z$	$Y = \sqrt{Z}$	$Y = \tilde{Z}$
$\hat{a}$	0.0555	0.06324	0.0338

Table 2: Values of $\hat{a}$ corresponding to $\gamma = 0.05$ for cases i), ii) and iii), when $Z \sim \Gamma(5, 2)$.

It remains to read on Figure 6.2 the corresponding values of $\pi_\gamma^{(s)}$. Small γ's yield small $\hat{a}$'s leading hence to small $\pi_\gamma^{(s)}$. For instance, the corresponding $\pi_\gamma^{(s)}$ for case i) are 1.6343×10^{-6}, 0.8171×10^{-6} and 0.5447×10^{-6} for $s = 2, 3, 4$, respectively. From a basis risk point of view, for a fixed type of uncertainty, $\pi_\gamma^{(s)}$ decreases when the convex order increases. Since the corresponding value of a is constant, and because the scenarios are sorted in function of the convex order, then $\pi_\gamma^{(2)} > \pi_\gamma^{(3)} > \pi_\gamma^{(4)}$.

6.2 Heavy tailed randomly scaled uncertainty

Here, we move to a heavy tailed distribution for Z, assuming $Z \sim Pareto(m, \omega)$. Studies conducted for the Gamma distribution can easily be replicated for the Pareto case. However here, the existence of C_n in case i), ii) and iii) is guaranteed for $m > n$, $m > n/2$ and $m > 2n$, respectively. These conditions are quite restrictive since they require the eight first moments of Z to exist in case iii). If this condition is not fulfilled, then the only acceptable value for a is 0. In other words, only where there is no basis risk is affordable (and costs 0). To this extent, assume the shape parameter of the Pareto distribution is lower than 2, then the methodology can be applied but only for $n = 1$. In this case, the generalized penalty function is only 2 convex. Consequently, only the 2 convex bound can be computed. To reach the 3 convexity of the generalized penalty function and hence to compute the 3 convex bound, the methodology requires the shape parameter to be greater than 6. In Table 3, we close the discussion by providing the basis risk limits in case i), ii) and iii) for $m = 1.8$ and $\omega = 1$ (only the first moment of Z exists) for $c = 0.1\mathbb{E}[Z] = 0.225$ and $\pi = 0.25\mathbb{E}[Z] = 0.5625$.

	U_a			$SB(4,1)$		
s-cx order	$Y = Z$	$Y = \sqrt{Z}$	$Y = \tilde{Z}$	$Y = Z$	$Y = \sqrt{Z}$	$Y = \tilde{Z}$
2	0.4557	0.7200	0	0.3419	0.5445	0

Table 3: Basis risk limits $a^{(s)}$ for $s = 2$ in cases i), ii) and iii) for $Z \sim Pareto(1.8, 1)$.

In summary. We have adopted an ERM point of view to quantify the consequences of basis risk inherent to an index-based insurance transaction. The quantification methodology consists in a three steps procedure. Wort-case scenarios are derived from the available information on the uncertainty. Measurement of consequences of basis risk is done thanks to generalized penalty functions, to ultimately derive basis risk capital requirement. This value stands for the price of the average consequences of a mismatching between the value taken by the index and the real incurred loss. In other words, this is the required amount to mop up losses due to the use of index-based insurance instead of an indemnity-based insurance. From this point of view, the "real" price of index-based insurance products is greater than the net premiums ask by the supplier of the coverage. Consequently, this may be an explanation of the low demand in several countries.

Appendix

- **Proof Proposition 2.3** Before we proceed further, we recall from Pakes and Navarro [67] the following result. Let $S \sim \mathcal{B}eta([0,1],\alpha,\beta)$ and let Y be a random variable independent from S. Then $X =_d SY$ is said to be Beta unimodal, and

$$\mathbb{P}(SY \leq x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} x^{\alpha} (I_\beta \phi)(x), \tag{6.1}$$

where the function ϕ depends on F_Y by

$$\phi(t) = F_Y(t) t^{-\alpha-\beta},$$

and the function $I_\beta \phi$ is defined as

$$(I_\beta \phi)(x) = \frac{1}{\Gamma(\beta)} \int_x^\infty (t-x)^{\beta-1} \phi(t) dt,$$

i.e., it corresponds to the Weyl fractional integral of ϕ; see Miller and Ross [63] or Debnath and Bhatta [26].

Now, let n and m be two integers greater than 1 such that $S_a \sim \mathcal{B}([-a,a],n,m)$. First, let $x \geq 0$. Then,

$$\mathbb{P}(X \leq x) = \mathbb{P}(S_a \leq 0) + \int_0^a \mathbb{P}(Y \leq x/u) f_{S_a}(u) du. \tag{6.2}$$

The proof is now based on the calculation of the integral in the right hand side of (6.2). Let

$I_a = \int_0^a \mathbb{P}(Y \leq x/u) f_{S_a}(u)du.$

$$
\begin{aligned}
I_a &= \frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)(2a)^{m+n-1}} \sum_{k=0}^{n-1} \binom{n-1}{k} a^{n-k-1} \int_0^a F_Y(x/u) u^k (a-u)^{m-1} du \\
&= \frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)(2a)^{m+n-1}} \sum_{k=0}^{n-1} \binom{n-1}{k} a^{n+m-k-2} x^{k+1} \int_{x/a}^{+\infty} F_Y(t) t^{-m-(k+1)} \left(t - \frac{x}{a}\right)^{m-1} dt, \\
&= \frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)(2)^{m+n-1}} \sum_{k=0}^{n-1} \binom{n-1}{k} \left(\frac{x}{a}\right)^{k+1} \int_{x/a}^{+\infty} F_Y(t) t^{-m-(k+1)} \left(t - \frac{x}{a}\right)^{m-1} dt, \\
&= \frac{1}{2^{m+n-1}} \sum_{k=0}^{n-1} \frac{(n+m-1)!}{(n-k-1)!(m+k)!} \frac{\Gamma(m+k+1)}{\Gamma(k+1)} \left(\frac{x}{a}\right)^{k+1} (I_m \phi_k) \left(\frac{x}{a}\right), \\
&= \frac{1}{2^{m+n-1}} \sum_{k=0}^{n-1} \binom{n+m-1}{n-k-1} \mathbb{P}\left(Y \tilde{S}_k \leq \frac{x}{a}\right),
\end{aligned}
$$

where $\tilde{S}_k \sim \mathcal{B}eta([0,1], k+1, m)$ according to (6.1).

When $x \leq 0$,

$$
\mathbb{P}(X \leq x) = \int_{-a}^{0} \bar{F}_Y(x/u) f_{S_a}(u)du = J_a.
$$

This time, we obtain

$$
\begin{aligned}
J_a &= \frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)(2a)^{m+n-1}} \int_{-a}^{0} \bar{F}_Y(x/u)(a-u)^{m-1}(a+u)^{n-1} du \\
&= \frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)(2a)^{m+n-1}} \sum_{k=0}^{m-1} \binom{m-1}{k} \int_0^a \bar{F}_Y(-x/u) u^k a^{m-k-1} (a-u)^{n-1} du \\
&= \frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)(2a)^{m+n-1}} \sum_{k=0}^{m-1} \binom{m-1}{k} \left(\frac{-x}{a}\right)^{k+1} \int_{-x/a}^{+\infty} \bar{F}_Y(u) u^{-n-(k+1)} \left(u - \frac{-x}{a}\right)^{n-1} du \\
&= \frac{1}{2^{m+n-1}} \sum_{k=0}^{m-1} \frac{(n+m-1)!}{(n-k-1)!(n+k)!} \frac{\Gamma(n+k+1)}{\Gamma(k+1)} \left(\frac{-x}{a}\right)^{k+1} (I_n \phi_k) \left(\frac{-x}{a}\right), \\
&= \frac{1}{2^{m+n-1}} \sum_{k=0}^{m-1} \binom{n+m-1}{m-k-1} \mathbb{P}\left(Y \hat{S}_k > \frac{-x}{a}\right),
\end{aligned}
$$

where $\hat{S}_k \sim \mathcal{B}eta([0,1], k+1, n)$ according to (6.1).

- **Existence of C_n**

As a reminder,

$$
\xi_n(x,z,l) = (x-c)_+^n \left(1 + \left(\frac{x}{z} - l\right)_+^n\right), \quad x \in \mathbb{R}, \quad z, l > 0,
$$

and $C_n^{(s)}(a) = \sum_{i=1}^{k} p_i C_n\left(s_i Y\right)$. We must then calculate

$$\mathbb{E}[(s_i Y - c)_+^n] \quad and \quad \mathbb{E}\left[(s_i Y - c)_+^n \left(s_i \frac{Y}{Z} - l\right)_+^n\right].$$

To lighten the reading, we drop the index i.

i) $Y = Z$

$$\mathbb{E}[(sY - c)_+^n] = \sum_{k=0}^{n} \binom{n}{k} (-c)^{n-k} s^k \int_{c/s}^{+\infty} y^k dF_Y(y).$$

When $Z \sim \Gamma(\kappa, r)$,

$$\int_{c/s}^{+\infty} y^k dF_Y(y) = \frac{\Gamma(\kappa + k)}{r^k \Gamma(\kappa)} \mathbb{P}\left(Y_k > \frac{c}{s}\right),$$

where $Y_k \sim \Gamma(\kappa + k, r)$.

When $Z \sim Pareto(\omega, m)$,

$$\int_{c/s}^{+\infty} y^k dF_Y(y) = \frac{m}{m-k} \omega^k \mathbb{P}(Y_k > y_m), \quad m > k, \quad k = 0, \ldots, n.$$

where $Y_k \sim Pareto(\omega, m - k)$ and $z_m = max\left(\frac{c}{s}, \omega\right)$.

ii) $Y = \sqrt{Z}$

$$\mathbb{E}[(s\sqrt{Z} - c)_+^n] = \sum_{k=0}^{n} \binom{n}{k} (-c)^{n-k} s^k \int_{(c/s)^2}^{+\infty} z^{k/2} dF_Z(z),$$

$$\mathbb{E}\left[(s\sqrt{Z} - c)_+^n \left(\frac{s}{\sqrt{Z}} - l\right)_+^n\right] = \sum_{k=0}^{n} \sum_{j=0}^{n} \binom{k}{n} \binom{n}{j} s^{k+j} (-c)^{n-k} (-l)^{n-j} \int_u^v z^{\frac{k-j}{2}} dF_Z(z),$$

where $u = \frac{c^2}{s^2}$ and $v = \frac{s^2}{l^2}$.

When $Z \sim \Gamma(\kappa, r)$,

$$\int_{(c/s)^2}^{+\infty} z^{k/2} dF_Z(z) = \frac{\Gamma(\kappa + k/2)}{r^{k/2} \Gamma(\kappa)} \mathbb{P}\left(Z_{k/2} > \left(\frac{c}{s}\right)^2\right),$$

where $Z_k \sim \Gamma(\kappa + k/2, r)$ and

$$\int_u^v z^{\frac{k-j}{2}} dF_Z(z) = \frac{\Gamma\left(\kappa + \frac{k-j}{2}\right)}{\Gamma(\alpha) r^{\frac{k-j}{2}}} \left(\mathbb{P}\left(Z_{\frac{k-j}{2}} \leq v\right) - \mathbb{P}\left(Z_{\frac{k-j}{2}} \leq u\right)\right) \mathbb{1}_{v>u},$$

where $Z_{\frac{k-j}{2}} \sim \Gamma\left(\kappa + \frac{k-j}{2}, r\right)$ with $\kappa > n/2$.

When $Z \sim Pareto(m, \omega)$,

$$\int_{(c/s)^2}^{+\infty} z^{k/2} dF_Z(z) = \frac{m}{m - k/2} \omega^{k/2} \mathbb{P}(Z_{k/2} > z_m), \quad m > k/2, \quad k = 0, \ldots, n.$$

where $Z_{k/2} \sim Pareto(\omega, m - k/2)$ and $z_m = max\left(\left(\frac{c}{s}\right)^2, \omega\right)$ and

$$\int_u^v z^{\frac{k-j}{2}} dF_Z(z) = \frac{m\omega^m}{m - \frac{k-j}{2}} \left(z_m^{\frac{k-j}{2} - m} - v^{\frac{k-j}{2} - m}\right) \mathbb{1}_{v>\omega},$$

where $z_m = max(u, \omega)$, $u = (c/s)^2$, $v = (s/l)^2$.

iii) $Y = \tilde{Z}$

$$\mathbb{E}\left[(s\tilde{Z} - c)_+^n \left(s\frac{\tilde{Z}}{Z} - l\right)_+^n\right] = \sum_{k=0}^n \sum_{j=0}^n \binom{k}{n} \binom{n}{j} s^{k+j} (-c)^{n-k} (-l)^{n-j} \mathcal{I}(k, j),$$

where

$$\mathcal{I}(k, j) = \int_{(c/s)}^{+\infty} \int_0^{sy/l} y^{k+j} z^{-j} dF_Z(z) dF_Z(y).$$

When $Z \sim \Gamma(\kappa, r)$,

$$\mathcal{I}(k, j) = \frac{\Gamma(\kappa - j)}{\Gamma(\kappa)} r^j \int_{c/s}^{+\infty} y^{k+j} \mathbb{P}\left(Y_j \leq \frac{sy}{l}\right) dF_Z(y),$$

where $Y_j \sim \Gamma(\kappa - j, r)$. Since $\mathbb{P}\left(Y_j \leq \frac{sy}{l}\right) \leq 1$, for all $y > 0$ and all $k \in \mathbb{N}$, then

$$\mathcal{I}(k, j) \leq \frac{\Gamma(\kappa - j)}{\Gamma(\kappa)^2} \frac{\Gamma(\kappa + k + j)}{r^k} \mathbb{P}\left(Y_{k+j} > \frac{c}{s}\right),$$

where $Y_{k+j} \sim \Gamma(\kappa + k + j, r)$. Then $\mathcal{I}(k, j) < +\infty$ for $\kappa > n$.

When $Z \sim Pareto(\omega, m)$,

$$\mathcal{I}(k,j) = \frac{m^2\omega^{m-j}}{(m+j)(m-k-j)} z_m^{k+j-m} - \mathcal{J}(k,j), \quad m > 2n,$$

where $z_m = max\left(\frac{c}{s}, \omega\right)$ and

$$\mathcal{J}(k,j) = \frac{m^2\omega^{2m}}{m+j} \times \begin{cases} \left(\frac{l}{s}\right)^{m+j} \frac{z_m^{k-2m}}{2m-k}, & z_m > l\omega/s, \\ \frac{\omega^{-(m+j)}}{m-k-j}\left(z_m^{k+j-m} - \left(\frac{l\omega}{s}\right)^{k+j-m}\right) + \left(\frac{l}{s}\right)^{k+j-m} \frac{\omega^{k-2m}}{2m-k}, & z_m < l\omega/s. \end{cases}$$

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