Joint Spectral Radius and Ternary Hermite Subdivision

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Abstract In this paper we construct a family of ternary interpolatory Hermite subdivision schemes of order 1 with small support and $\mathcal{HC}^2$-smoothness. Indeed, leaving the binary domain, it is possible to derive interpolatory Hermite subdivision schemes with higher regularity than their binary counterparts. The family of schemes we construct is a two-parameter family whose $\mathcal{HC}^2$-smoothness is guaranteed whenever the parameters are chosen from a certain polygonal region. The construction of this new family is inspired by the geometric insight into the ternary interpolatory scalar three-point subdivision scheme by Hassan and Dodgson. The smoothness of our new family of Hermite schemes is proven by means of joint spectral radius techniques.

Keywords Hermite subdivision · Hermite interpolation · Joint spectral radius · Taylor operator

Mathematics Subject Classification (2000) MSC 41A60 · MSC 65D15 · 13P05

1 Introduction

This paper focuses on Hermite subdivision schemes which are iterative algorithms for approximation or interpolation of given vector-valued discrete data consisting of function values and associated consecutive derivatives. Our main goal is to

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show, by applying the joint spectral radius techniques, that even in the Hermite case ternary subdivision schemes achieve higher smoothness than their binary counterparts. Furthermore we show that this can be done without a significant increase of the support of the mask. It is well known that such a phenomenon occurs in the scalar case, see e.g. [24]. However, the Hermite ternary case is more challenging.

1.1 Motivation and novel contributions

As well known, it is a difficult task to keep the support size of a subdivision scheme small while increasing its smoothness. These two notions are mutually conflicting because high smoothness generally requires masks of large support. This leads to an undesired more global influence of each initial data value on the limit function. Increasing the a-rity of the subdivision scheme is one of the possible ways to overcome this problem.

In this paper, we propose a new two-parameter family of *ternary three-point Hermite* schemes for vector-valued data consisting of function values and associated first derivatives. These parameters are used to control the convergence and the regularity of the corresponding Hermite scheme. We show that, if the parameters are chosen from a certain polygonal domain of the parameter plane, the associated Hermite scheme is $\mathcal{HC}^2$-convergent instead of $\mathcal{HC}^1$-convergent as usually expected for a scheme of order 1. In section 2.1, to derive our family of Hermite schemes we provide a new geometric interpretation of the family of ternary interpolatory scalar three-point subdivision schemes in [23]. This family is a one-parameter family which is C^1-smooth for a certain parameter range. In Section 2.2, we extend this scalar scheme to the Hermite case. The analysis of the convergence and regularity of the Hermite family is given in Section 3 where we present two different approaches: The classical approach for regularity analysis of *vector* subdivision schemes combined with the $\mathcal{HC}^0$ convergence analysis of $H_{\mathcal{A}}$ and the classical approach for regularity analysis of *Hermite* subdivision schemes. These approaches involve the so-called joint spectral radius methods, difference operator technique and the corresponding extended scheme, respectively. The combination of these different methods, allows us to identify the convergence and regularity regions of the parameter plane and to identify the optimal parameters. Thus, we identify a whole class of Hermite interpolatory schemes whose regularity is higher than that of their binary counterparts of the same order in [28].

1.2 Background on vector and Hermite subdivision schemes

Let $d \in \mathbb{N}_0$ be an integer number. *Vector subdivision schemes* of dimension $d+1$ are iterative algorithms based on subdivision operators that generate denser and denser sequences $\boldsymbol{g}_n$, $n \in \mathbb{N}$, of vector-valued data from some initial vector-valued sequence $\boldsymbol{g}_0 = \{\boldsymbol{g}_0(\alpha) \in \mathbb{R}^{d+1},\ \alpha \in \mathbb{Z}\}$ in $\ell^{d+1}(\mathbb{Z})$, the set of $d+1$-dimensional vector sequences indexed by $\mathbb{Z}$. Loosely speaking, the ratio between the number of elements in $\boldsymbol{g}_{n+1}$ to the number of elements in $\boldsymbol{g}_n$, assumed to be independent of n, is called the *a-rity* of the corresponding scheme.

In this paper we study *ternary* (3-arity) vector subdivision schemes. The associated level-dependent linear *subdivision operators* $S_{\mathcal{A}_n} : \ \ell^{d+1}(\mathbb{Z}) \to \ell^{d+1}(\mathbb{Z})$, for any $n \in \mathbb{N}_0$, map the sequences $\boldsymbol{g}_n = \{\boldsymbol{g}_n(\alpha), \ \alpha \in \mathbb{Z}\}$ into $\boldsymbol{g}_{n+1} = \{\boldsymbol{g}_{n+1}(\alpha), \ \alpha \in \mathbb{Z}\}$ and are defined by

$$\boldsymbol{g}_{n+1}(\alpha) := (S_{\mathcal{A}_n}\boldsymbol{g}_n)(\alpha)\,\boldsymbol{g}_n(\beta), \qquad \alpha \in \mathbb{Z}, \qquad n \in \mathbb{N}_0. \tag{1}$$

$$(S_{\mathcal{A}_n}\boldsymbol{g}_n)(\alpha) := \sum_{\beta \in \mathbb{Z}} \boldsymbol{A}_n(\alpha - 3\beta)\ \boldsymbol{g}_n(\beta), \qquad \alpha \in \mathbb{Z}, \qquad n \in \mathbb{N}_0. \tag{2}$$

At each level n of the subdivision recursion, the matrix coefficients in (2) are taken from the matrix-valued sequence $\mathcal{A}_n := \{\boldsymbol{A}_n(\alpha) \in \mathbb{R}^{(d+1)\times(d+1)}, \ \alpha \in \mathbb{Z}\}$, called the level n *subdivision mask*. In our case, the sequence $\{\mathcal{A}_n, \ n \in \mathbb{N}_0\}$ contains masks $\mathcal{A}_n$ of the same finite support, i.e. $\operatorname{supp} \mathcal{A}_n := \{\alpha \in \mathbb{Z} : \ \boldsymbol{A}_n(\alpha) \neq \boldsymbol{0}\} \subseteq [-N, N]$ for some $N \in \mathbb{N}$. The $(d+1) \times (d+1)$ matrix-valued *Laurent polynomials*

$$\boldsymbol{A}_n^*(z) := \sum_{\alpha \in \mathbb{Z}} A_n z^{\alpha}, \quad z \in \mathbb{C} \setminus \{0\}, \quad n \in \mathbb{N}_0,$$

are the associated *mask symbols*.

The vector *subdivision scheme* is the repeated application of the subdivision operators in (2) to the starting vector-valued sequence $\boldsymbol{g}_0 \in \ell^{d+1}(\mathbb{Z})$, i.e.

$$\begin{cases} \text{Input} \quad \{\mathcal{A}_n, \ n \in \mathbb{N}_0\} \text{ and } \boldsymbol{g}_0 \\ \text{For} \quad n = 0, 1, \ldots \\ \qquad\quad \boldsymbol{g}_{n+1} := S_{\mathcal{A}_n}\boldsymbol{g}_n \end{cases} \tag{3}$$

This paper deals with two kinds of vector schemes of dimension $d+1$. The first one is a *stationary vector subdivision scheme* $S_{\mathcal{A}}$ with the level-independent masks $\mathcal{A}_n = \mathcal{A} = \{\boldsymbol{A}(\alpha) \in \mathbb{R}^{(d+1)\times(d+1)}, \ \alpha \in \mathbb{Z}\}$, $n \in \mathbb{N}_0$, and with the level-independent subdivision operators $S_{\mathcal{A}_n} = S_{\mathcal{A}}$, $n \in \mathbb{N}_0$. For simplicity, we call this stationary subdivision scheme $S_{\mathcal{A}}$ as it is fully determined by the corresponding subdivision operator $S_{\mathcal{A}}$. Such vector subdivision schemes have been studied by many authors e.g. [1], [2] , [3] , [5], [7] ,[11], [20], [26] and reference therein.

The second type of vector subdivision schemes that we consider are *Hermite subdivision schemes* $H_{\mathcal{A}}$ of dimension $d+1$ and *order* d. For a given mask $\mathcal{A} = \{\boldsymbol{A}(\alpha) \in \mathbb{R}^{(d+1)\times(d+1)}, \ \alpha \in \mathbb{Z}\}$ with finite support, the associated level-dependent (non-stationary) linear Hermite subdivision operators $H_{\mathcal{A}_n} : \ell^{d+1}(\mathbb{Z}) \to \ell^{d+1}(\mathbb{Z})$ are given by

$$\boldsymbol{f}_{n+1}(\alpha) := (H_{\mathcal{A}_n}\boldsymbol{f}_n)(\alpha) = \sum_{\beta \in \mathbb{Z}} \boldsymbol{D}^{-n-1}\boldsymbol{A}(\alpha - 3\beta)\boldsymbol{D}^n \boldsymbol{f}_n(\beta), \quad \alpha \in \mathbb{Z}, \quad n \in \mathbb{N}_0, \tag{4}$$

with the diagonal matrix $\boldsymbol{D} = \operatorname{diag}(1, 1/3, \ldots, 1/3^d) \in \mathbb{R}^{(d+1)\times(d+1)}$.
Note that the masks

$$\boldsymbol{D}^{-n-1}\mathcal{A}\boldsymbol{D}^n = \{\boldsymbol{D}^{-n-1}\boldsymbol{A}(\alpha)\boldsymbol{D}^n, \ \alpha \in \mathbb{Z}\}, \quad n \in \mathbb{N}_0,$$

in (4) have a very special type of level-dependence. For simplicity, we call this Hermite subdivision scheme $H_{\mathcal{A}}$ to emphasize this special type of the dependency on the mask $\mathcal{A}$. Moreover, if $H_{\mathcal{A}}$ is convergent, in the sense of Definition 2, the vector-valued elements $\boldsymbol{f}_n(\alpha)$ are to be interpreted, for large n, as approximations to the function values and the successive derivatives of the corresponding limit function $\Phi_{\boldsymbol{f}_0} = \lim_{n\to\infty} H_{\mathcal{A}_n} \dots H_{\mathcal{A}_0}\boldsymbol{f}_0$ evaluated at $3^{-n}\alpha$, i.e.

$$bff_n(\alpha) = \begin{pmatrix} f_n(\alpha) \\ f_n'(\alpha) \\ \vdots \\ f_n^{(d)}(\alpha) \end{pmatrix} \approx \Phi_{\boldsymbol{f}_0}(3^{-n}\alpha) = \begin{pmatrix} \phi(3^{-n}\alpha) \\ \phi'(3^{-n}\alpha) \\ \vdots \\ \phi^{(d)}(3^{-n}\alpha) \end{pmatrix}, \quad \alpha \in \mathbb{Z}.$$

Hermite schemes, introduced [28] in have been studied by several authors [9], [10], [14], [16], [17], [21], [25].

We observe that, since (4) can be rewritten as

$$\boldsymbol{D}^{n+1}\boldsymbol{f}_{n+1}(\alpha) = \sum_{\beta\in\mathbb{Z}} \boldsymbol{A}(\alpha - 3\beta)\boldsymbol{D}^n\boldsymbol{f}_n(\beta), \quad \alpha \in \mathbb{Z}, \quad n \in \mathbb{N}_0, \tag{5}$$

each Hermite subdivision operator $H_{\mathcal{A}_n}$ of order d and the stationary vector operator $S_{\mathcal{A}}$ of dimension $d+1$ are related by

$$\boldsymbol{D}^{n+1}\left(H_{\mathcal{A}_n}\boldsymbol{f}_n\right)(\alpha) = \left(S_{\mathcal{A}}\boldsymbol{D}^n\boldsymbol{f}_n\right)(\alpha), \quad \alpha \in \mathbb{Z}, \quad n \in \mathbb{N}_0. \tag{6}$$

Thus, the (stationary) vector subdivision scheme $S_{\mathcal{A}}$ is called the *stationary counterpart* of $H_{\mathcal{A}}$.

We continue by defining the convergence and the regularity of $S_{\mathcal{A}}$ and $H_{\mathcal{A}}$.

Definition 1 A ternary vector subdivision scheme $S_{\mathcal{A}}$is called

i) *convergent*, if for every initial vector sequence $\boldsymbol{g}_0 \in \ell^{d+1}(\mathbb{Z})$ and the corresponding sequence of refinements in (2), $\boldsymbol{g}_n = S^n_{\mathcal{A}}\,\boldsymbol{g}_0$, $n \in \mathbb{N}_0$, there exists a continuous vector function $\boldsymbol{\Phi}_{\boldsymbol{g}_0} : \mathbb{R} \to \mathbb{R}^{d+1}$, such that for every compact subset $K \subset \mathbb{R}$

$$\lim_{n\to\infty} \max_{\alpha\in\mathbb{Z}\cap 3^nK} \|\boldsymbol{\Phi}_{\boldsymbol{g}_0}(3^{-n}\alpha) - \boldsymbol{g}_n(\alpha)\|_\infty = 0;$$

ii) $\mathcal{C}^\ell$*–convergent*, $\ell \in \mathbb{N}$, if $\boldsymbol{\Phi}_{\boldsymbol{g}_0} \in \mathcal{C}^\ell(\mathbb{R})$ for every initial vector sequence $\boldsymbol{g}_0$ in $\ell^{d+1}_\infty(\mathbb{Z})$;

iii) *contractive*, if $\boldsymbol{\Phi}_{\boldsymbol{g}_0} = \boldsymbol{0}$ for every initial sequence $\boldsymbol{g}_0$ in $\ell^{d+1}(\mathbb{Z})$.

We also make use of the following notion of convergence that better captures the intrinsic structure of Hermite subdivision schemes.

Definition 2 A ternary Hermite subdivision scheme $H_{\mathcal{A}}$ of order d, is $\mathcal{HC}^\ell$*–convergent* with $\ell \ge d$, if for any initial vector sequence $\boldsymbol{f}_0 \in \ell^{d+1}(\mathbb{Z})$ and the corresponding sequence of refinements $\boldsymbol{f}_n = H_{\mathcal{A}_n} \cdots H_{\mathcal{A}_0}\,\boldsymbol{f}_0$, $n \in \mathbb{N}_0$, in (4), there exists a vector-valued function $\boldsymbol{\Phi}_{\boldsymbol{f}_0} : \mathbb{R} \to \mathbb{R}^{d+1}$, $\boldsymbol{\Phi}_{\boldsymbol{f}_0} = [\phi^{(i)}]_{i=0,\dots,d} \in C^{\ell-d}(\mathbb{R})$ with $\phi = \phi^{(0)} \in C^\ell(\mathbb{R})$ and $\phi^{(i)} = \dfrac{d^i\phi^{(0)}}{dx^i}$, $i = 1,\dots,d$ such that for every compact subset $K \subset \mathbb{R}$

$$\lim_{n\to\infty} \max_{\alpha\in\mathbb{Z}\cap 3^nK} \|\boldsymbol{\Phi}_{\boldsymbol{f}_0}\left(3^{-n}\alpha\right) - \boldsymbol{f}_n(\alpha)\|_\infty = 0\,.$$

Note that the notions of C^ℓ-convergence for a vector scheme $S_{\mathcal{A}}$ and HC^ℓ-convergence for a Hermite scheme $H_{\mathcal{A}}$ are intrinsically different. The first notion in Definition 1 refers to the *minimal* smoothness of the entries in $\boldsymbol{\Phi}_{\boldsymbol{g}_0}$ while the second one in Definition 2 to the *maximal* one of the entries in $\boldsymbol{\Phi}_{\boldsymbol{f}_0}$. In other words, due to (6), if we look at a $\mathcal{HC}^\ell$-convergent Hermite scheme simply as a vector scheme it is only $\mathcal{C}^{\ell-d}$-convergent. Moreover, again due to (6), convergence and regularity $S_{\mathcal{A}}$ does not imply convergence and regularity $\mathcal{H}_{\mathcal{A}}$.
In this paper we make use of the following concept of ternary interpolatory subdivision scheme.

Definition 3 A ternary Hermite subdivision scheme $H_{\mathcal{A}}$ is called *interpolatory*, if

$$\boldsymbol{A}(0) = \boldsymbol{D} \quad \text{and} \quad \boldsymbol{A}(3\alpha) = \boldsymbol{0}, \quad \alpha \in \mathbb{Z}, \quad \alpha \neq 0.$$

Note that a Hermite interpolatory scheme in (4) generates vector sequences satisfying

$$\boldsymbol{f}_{n+1}(3\alpha) = \boldsymbol{f}_n(\alpha), \quad \text{for} \quad \alpha \in \mathbb{Z}, \quad n \in \mathbb{N}_0.$$

2 Construction of ternary Hermite subdivision schemes

In this section, we construct a two-parameter family of interpolatory ternary Hermite subdivision schemes of order $d = 1$, i.e. of dimension 2. By construction, each scheme from this family reproduces quadratic polynomials. The suitable range of the parameters is given in Section 3. To derive this new family of Hermite scheme, we first present, see Subsection 2.1, an alternative way for defining the ternary interpolatory scalar 3-point subdivision scheme in [23]. This new interpretation of the scheme is generalized to the Hermite case in Subsection 2.2.

2.1 Scalar case

2.1.1 Lagrange interpolants

We start by determining two special cubic interpolants P_ℓ and P_r to the data (α, p_α) with $p_\alpha \in \mathbb{R}$, $\alpha \in \{-1, 0, 1\}$. The corresponding interpolation problems are underdetermined and we choose

$$\begin{aligned} P_\ell(t) &:= p_0 + \ell_1 t + \ell_2 t^2 + \lambda(p_{-1} - 2p_0 + p_1)t^3, \quad t \in \mathbb{R}, \\ P_r(t) &:= p_0 + r_1 t + r_2 t^2 - \lambda(p_{-1} - 2p_0 + p_1)t^3, \quad t \in \mathbb{R}, \end{aligned} \tag{7}$$

to ensure $P_\ell(0) = P_r(0) = p_0$ and to introduce a free parameter $\lambda \in \mathbb{R}$. Note that λ controls the second difference $p_{-1} - 2p_0 + p_1$ and will be useful for controlling the regularity of the scalar ternary scheme in Subsection 2.1.2. The coefficients

$$\begin{aligned} \ell_1 &= -\lambda(p_{-1} - 2p_0 + p_1) + \tfrac{1}{2}(p_1 - p_{-1}), \quad \ell_2 = \tfrac{1}{2}(p_{-1} - 2p_0 + p_1), \\ r_1 &= \lambda(p_{-1} - 2p_0 + p_1) + \tfrac{1}{2}(p_1 - p_{-1}), \quad r_2 = \tfrac{1}{2}(p_{-1} - 2p_0 + p_1), \end{aligned} \tag{8}$$

are determined from two linear systems of equations derived from the interpolation conditions $P_\ell(\alpha) = P_r(\alpha) = p_\alpha$ for $\alpha \in \{-1, 1\}$.

Remark 1 If $\pi \in \mathbb{P}_1$ and $p_\alpha = \pi(\alpha)$, $\alpha \in \{-1,0,1\}$, then, by construction,

$$P_\ell = P_r = \pi. \tag{9}$$

Moreover, for $\lambda = 0$, Equation (9) holds for $\pi \in \mathbb{P}_2$, see Figure 1.

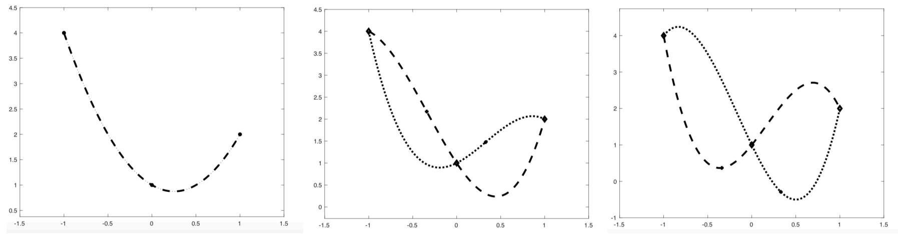

Fig. 1 Interpolants P_ℓ (dashed) and P_r (dotted) for $\lambda = 0, 0.5, -1$, respectively, with the corresponding evaluation points.

2.1.2 Ternary scalar subdivision scheme

To define the corresponding ternary scalar subdivision operator $S_a : \ell(\mathbb{Z}) \to \ell(\mathbb{Z})$,

$$f_{n+1} := S_a f_n = \sum_{\beta\in\mathbb{Z}} a(\cdot - 3\beta) f_n(\beta), \quad f_n = \{f_n(\alpha),\, \alpha \in \mathbb{Z}\}, \quad n \in \mathbb{N}_0,$$

we choose the initial sequence $f_0 = \{f_0(\alpha),\, \alpha \in \mathbb{Z}\}$ with

$$f_0(\alpha) = p_\alpha, \quad \alpha \in \{-1,0,1\} \quad \text{and} \quad f_0(\alpha) = 0, \quad \alpha \neq -1,0,1,$$

and set

$$f_1(-1) = P_\ell(-1/3), \quad f_1(0) = P_r(0) = P_\ell(0) = p_0, \quad f_1(1) = P_r(1/3).$$

Therefore, the following linear system of equations

$$\sum_{\beta\in\{-1,0,1\}} a(-1-3\beta)p_\beta = P_\ell(-1/3) = \frac{1}{27}\big((6+8\lambda)p_{-1} + (24-16\lambda)p_0 + (-3+8\lambda)p_1\big),$$

$$\sum_{\beta\in\{-1,0,1\}} a(-3\beta)p_\beta = p_0,$$

$$\sum_{\beta\in\{-1,0,1\}} a(1-3\beta)p_\beta = P_r(1/3) = \frac{1}{27}\big((-3+8\lambda)p_{-1} + (24-16\lambda)p_0 + (6+8\lambda)p_1\big)$$

uniquely identifies the mask

$$a = \{\ldots, 0, u, 0, v, 1-u-v, 1, 1-u-v, v, 0, u, 0, \ldots\}, \tag{10}$$

where $u := \dfrac{-3+8\lambda}{27}$, $v := \dfrac{6+8\lambda}{27}$ and where 1 is at the position $\alpha = 0$. For every $\lambda \in \mathbb{R}$, the mask a is symmetric and is supported on $[-4,4]$.
Note that $u = v - 1/3$ and that (10) defines the one-parameter family of subdivision schemes in [23] whose convergence and C^1 smoothness is proved for $2/9 < v < 3/9$.

2.2 A family of Hermite subdivision schemes of order 1

2.2.1 Hermite interpolants

We generalize the idea described in Subsection 2.1 to the Hermite case. We start by solving the Hermite interpolation problem for the given data $(\alpha, p_\alpha, p'_\alpha)$, $\alpha \in \{-1, 0, 1\}$, and for two sextic Hermite interpolants of the form

$$\begin{aligned} P_{H,\ell}(t) &:= p_0 + p'_0 t + \sum_{i=1}^{4} \ell_i t^{i+1} + (\lambda\delta' + \mu\delta)t^6, \\ P_{H,r}(t) &:= p_0 + p'_0 t + \sum_{i=1}^{4} r_i t^{i+1} - (\lambda\delta' + \mu\delta)t^6, \end{aligned} \tag{11}$$

with

$$\delta' = p'_{-1} - 2p'_0 + p'_1 \quad \text{and} \quad \delta = \frac{p_1 - p_{-1}}{2} - p'_0 \,.$$

Note that the polynomials in (11) automatically satisfy the Hermite-type interpolation conditions

$$P_{H,\ell}(0) = P_{H,r}(0) = p_0 \quad \text{and} \quad P'_{H,\ell}(0) = P'_{H,r}(0) = p'_0.$$

Note also that the parameters $\lambda, \mu \in \mathbb{R}$ in (11) are introduced to control the differences $p'_{-1} - 2p'_0 + p'_1$ and $\frac{p_1 - p_{-1}}{2} - p'_0$ and influence the regularity of the corresponding Hermite subdivision scheme in Subsection 2.2.2. In fact, for every function $\varphi \in C^3([-1, 1])$, both differences $\varphi'(-1) - 2\varphi'(0) + \varphi'(1)$ and $6\left(\frac{\varphi(1) - \varphi(-1)}{2} - \varphi'(0)\right)$ are approximations of $\varphi^{(3)}(0)$.

The remaining coefficients in (11)

$$\begin{aligned} &\ell_1 = A - B + \lambda\delta' + \mu\delta, \quad r_1 = A - B - \lambda\delta' - \mu\delta, \\ &\ell_2 = r_2 = 5\delta/2 - \delta'/4, \\ &\ell_3 = B - A/2 - 2\lambda\delta' - 2\mu\delta, \quad r_3 = B - A/2 + 2\lambda\delta' + 2\mu\delta, \\ &\ell_4 = r_4 = \delta'/4 - 3\delta/2, \\ &A = p_{-1} - 2p_0 + p_1, \quad B = (p'_1 - p'_{-1})/4, \end{aligned} \tag{12}$$

are determined by solving two linear systems of equations derived from the remaining Hermite interpolating conditions $P_{H,\ell}(\alpha) = P_{H,r}(\alpha) = p_\alpha$ and ${P_{H,\ell}}'(\alpha) = {P_{H,r}}'(\alpha) = p'_\alpha$ for $\alpha = -1, 1$.

Remark 2 If $\pi \in \mathbb{P}_2$ and $p_\alpha = \pi(\alpha)$, $p'_\alpha = \pi'(\alpha)$, $\alpha \in \{-1, 0, 1\}$, then, by construction, the polynomial reproduction property of the Hermite interpolants is guaranteed and

$$P_{H,\ell} = P_{H,r} = \pi. \tag{13}$$

Moreover, for $\lambda = \mu = 0$, Equation (13) holds for every $\pi \in \mathbb{P}_5$, see Figure 2.

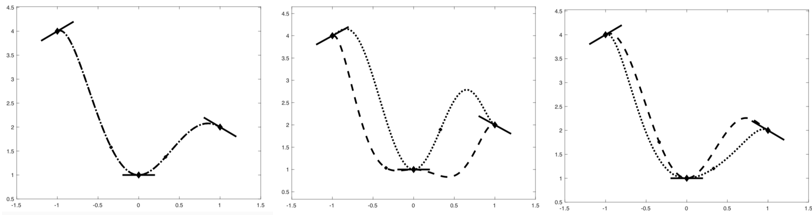

Fig. 2 Hermite interpolants $P_{H,\ell}$ (dashed) and $P_{H,r}$ (dotted) for $(\mu,\lambda) = (0,0)$, $(2,6),(-6,-2)$, respectively, with corresponding evaluation points and derivatives.

2.2.2 Ternary Hermite subdivision scheme

Similarly to Subsection 2.1.2, for $\boldsymbol{D} = \begin{pmatrix} 1 & 0 \\ 0 & 1/3 \end{pmatrix}$, to define the corresponding ternary Hermite subdivision scheme in (4), we choose the initial vector-valued sequence $\boldsymbol{f}_0 = \{\left(f_0(\alpha), f_0'(\alpha)\right)^T, \alpha \in \mathbb{Z}\}$ with

$$f_0(\alpha) = p_\alpha, \quad f_0'(\alpha) = p_\alpha', \quad \alpha \in \{-1,0,1\} \text{ and } f_0(\alpha) = f_0'(\alpha) = 0, \quad \alpha \neq -1,0,1,$$

and choose the sequence $\boldsymbol{f}_1 = \{\left(f_1(\alpha), f_1'(\alpha)\right)^T, \alpha \in \mathbb{Z}\}$ with

$$\begin{aligned} f_1(-1) &= P_{H,\ell}(-1/3), \quad f_1(0) = p_0, \quad f_1(1) = P_{H,r}(1/3), \\ f_1'(-1) &= P_{H,\ell}'(-1/3), \quad f_1'(0) = p_0', \quad f_1'(1) = P_{H,r}'(1/3). \end{aligned}$$

Due to interpolation, we immediately deduce that

$$\boldsymbol{A}(3\alpha) = \delta_{0\alpha}\boldsymbol{D}, \quad \alpha \in \mathbb{Z}.$$

To compute the other components of the mask $\mathcal{A} = \{\boldsymbol{A}(\alpha), \alpha \in \mathbb{Z}\}$, we express the polynomials in (11) as

$$P_{H,s}(t) = \sum_{\alpha \in \{-1,0,1\}} p_\alpha H_{\alpha,s,0}(t) + \tag{14}$$

where the polynomials $H_{\alpha,s,k}$ for $\alpha \in \{-1,0,1\}$ and $k \in \{0,1\}$ are obtained from (11) imposing the *cardinal* Hermite interpolation conditions

$$\begin{aligned} &H_{\alpha,s,0}(\beta) = \delta_{\alpha\beta}, \; H_{\alpha,s,0}'(\beta) = 0, \quad \beta = -1,0,1, \quad s \in \{\ell, r\}, \\ &H_{\alpha,s,1}(\beta) \;\; = 0, \;\; H_{\alpha,s,1}'(\beta) = \delta_{\alpha\beta}, \quad \beta = -1,0,1, \quad s \in \{\ell, r\}. \end{aligned}$$

Then, for $s \in \{\ell, r\}$, the matrix form of the conditions on $\boldsymbol{f}_1$ at $\alpha_\ell = -1$ and $\alpha_r = 1$ read as follows

$$\begin{aligned} \begin{pmatrix} f_1(\alpha_s) \\ f_1'(\alpha_s) \end{pmatrix} &= \begin{pmatrix} H_{-1,s,0}(\alpha_s/3) & H_{-1,s,1}(\alpha_s/3) \\ H_{-1,s,0}'(\alpha_s/3) & H_{-1,s,1}'(\alpha_s/3) \end{pmatrix} \begin{pmatrix} f_0(-1) \\ f_0'(-1) \end{pmatrix} \\ &+ \begin{pmatrix} H_{0,s,0}(\alpha_s/3) & H_{0,s,1}(\alpha_s/3) \\ H_{0,s,0}'(\alpha_s/3) & H_{0,s,1}'(\alpha_s/3) \end{pmatrix} \begin{pmatrix} f_0(0) \\ f_0'(0) \end{pmatrix} \\ &+ \begin{pmatrix} H_{1,s,0}(\alpha_s/3) & H_{1,s,1}(\alpha_s/3) \\ H_{1,s,0}'(\alpha_s/3) & H_{1,s,1}'(\alpha_s/3) \end{pmatrix} \begin{pmatrix} f_0(1) \\ f_0'(1) \end{pmatrix}. \end{aligned}$$

The linear system corresponding to

$$\boldsymbol{D}\boldsymbol{f}_1(\alpha) = \sum_{\beta\in\{-1,0,1\}} \boldsymbol{A}(\alpha-3\beta)f_0(\beta), \quad \alpha=-1,1,$$

for the chosen $\boldsymbol{f}_0$ and $\boldsymbol{f}_1$, uniquely identifies the remaining entries of the matrix mask $\mathcal{A}$ supported on $[-4,4]$ by taking into account that the grid spacing reduces at each subdivision step by 1/3 (so that the factor 1/3 appears when computing derivatives of (14))

$$\begin{aligned}
&\boldsymbol{A}(-4) = \boldsymbol{D}\cdot\tfrac{1}{729}\begin{pmatrix} 32\mu+45 & 64\lambda-12\\ -144\mu-162 & -288\lambda+45\end{pmatrix}, \quad \boldsymbol{A}(-3)=\begin{pmatrix}0&0\\0&0\end{pmatrix},\\
&\boldsymbol{A}(-2) = \boldsymbol{D}\cdot\tfrac{1}{729}\begin{pmatrix} -32\mu+108 & -64\lambda-24\\ -144\mu+702 & -288\lambda-144\end{pmatrix},\\
&\boldsymbol{A}(-1) = \boldsymbol{D}\cdot\tfrac{1}{729}\begin{pmatrix} 576 & -128\lambda-64\mu-192\\ 864 & 576\lambda+288\mu+288\end{pmatrix},\\
&\boldsymbol{A}(0)=\boldsymbol{D}, \quad \boldsymbol{A}(1) = \boldsymbol{D}\cdot\tfrac{1}{729}\begin{pmatrix} 576 & 128\lambda+64\mu+192\\ -864 & 576\lambda+288\mu+288\end{pmatrix},\\
&\boldsymbol{A}(2) = \boldsymbol{D}\cdot\tfrac{1}{729}\begin{pmatrix} -32\mu+108 & 64\lambda+24\\ 144\mu-702 & -288\lambda-144\end{pmatrix},\\
&\boldsymbol{A}(3)=\begin{pmatrix}0&0\\0&0\end{pmatrix}, \quad \boldsymbol{A}(4) = \boldsymbol{D}\cdot\tfrac{1}{729}\begin{pmatrix} 32\mu+45 & -64\lambda+12\\ 144\mu+162 & -288\lambda+45\end{pmatrix}.
\end{aligned} \tag{15}$$

Note that, by the support estimates in [8] which are also valid for Hermite schemes, the support of the basic limit function of the scheme defined in (15) is $[-2,2]$.

3 Regularity analysis of the Hermite family

We recall that our motivation for the construction in Section 2 is a derivation of a class of ternary Hermite schemes of order $d=1$ which for specific parameter values of λ and μ are at least $\mathcal{HC}^2$-smooth.

To this purpose, several regularity analysis approaches are available based on joint spectral radius, difference (Taylor) operators or their combination. We observe that while the joint spectral radius approach is fully developed for general dilation factors and therefore applicable in the ternary case (see [4], [6], [7] for more details), the approaches based on Taylor operators are stated for the binary case only. In any case, the generalization of the Taylor theory to the case of general a-rity is believed to be straightforward.

The analysis of our Hermite family of subdivision schemes $H_{\mathcal{A}}$ via joint spectral radius techniques could be done in three different ways (detailed below): firstly, by the analysis of $\mathcal{C}^2$-regularity of the stationary counterpart $S_{\mathcal{A}}$ combined with convergence analysis of the Taylor subdivision scheme $S_{\mathcal{B}}$ associated with $S_{\mathcal{A}}$; secondly, by the analysis of $\mathcal{C}^1$-regularity of $S_{\mathcal{B}}$; lastly, by the analysis of contractivity of the complete Taylor subdivision scheme $S_{\tilde{\mathcal{B}}_+}$ associated with the extended scheme $S_{\mathcal{A}_+}$ derived from $S_{\mathcal{A}}$.

Using the first approach, we observe that the domain of $\mathcal{C}^2$-convergence of $S_{\mathcal{A}}$ intersected with the convergence domain of $S_{\mathcal{B}}$ provides the parameter range for $\mathcal{HC}^2$-convergence of $H_{\mathcal{A}}$. This parameter range coincides with the parameter range obtained by $\mathcal{C}^1$-regularity analysis of $S_{\mathcal{B}}$ corresponding to the second approach. Therefore, we decided not to present the related details. The third approach, based on the contractivity analysis of $S_{\tilde{\mathcal{B}}_+}$ is used for checking the correctness of our computations.

In particular, in Subsection 3.1 we define and construct the Taylor operator $S_{\mathcal{B}}$ associated with $S_{\mathcal{A}}$ while in Subsection 3.2, we shortly recall the basic facts about the joint spectral radius techniques and apply them to $S_{\mathcal{A}}$ and to $S_{\mathcal{B}}$ to identify a polygonal parameter region of $\mathcal{HC}^2$-convergence, see Figure 3. In Subsection 3.3, we follow the approach based on the construction of the complete Taylor subdivision scheme $S_{\tilde{\mathcal{B}}_+}$ associated with the extended scheme $S_{\mathcal{A}_+}$ derived from $S_{\mathcal{A}}$ and identify what we call optimal parameter values which ensure better visual quality of the limit curves.

3.1 Taylor subdivision operator $S_{\mathcal{B}}$

In this section, we construct the Taylor operator $S_{\mathcal{B}}$ which is used for the regularity analysis of $H_{\mathcal{A}}$ in subsection 3.2.2. The Taylor operator was proposed in [29], and its symbol satisfies the following identity

$$\begin{pmatrix} z-1 & -1 \\ 0 & 1 \end{pmatrix} \boldsymbol{A}^*(z) = \frac{1}{3}\boldsymbol{B}^*(z) \begin{pmatrix} z^3-1 & -1 \\ 0 & 1 \end{pmatrix}, \quad z \in \mathbb{C} \setminus \{0\},$$

where $\boldsymbol{A}^*(z)$ is the symbol associated with the subdivision operator $S_{\mathcal{A}}$ and $\boldsymbol{B}^*(z)$ the symbol associated with the Taylor operator $S_{\mathcal{B}}$. In our specific situation, computation of the symbol $\boldsymbol{B}^*(z)$ gives the mask $\mathcal{B}$ (to be divided by the integer 243)

$$\begin{aligned}
&\boldsymbol{B}(-5) = \begin{pmatrix} -16\mu-9 & 32\lambda-16\mu-12 \\ 48\mu+54 & 48\mu-96\lambda+69 \end{pmatrix}, &&\boldsymbol{B}(-4) = \begin{pmatrix} -32\mu-45 & 64\lambda-32\mu-57 \\ 0 & 0 \end{pmatrix},\\
&\boldsymbol{B}(-3) = \begin{pmatrix} 342-80\mu & 160\lambda-80\mu+414 \\ 48\mu-234 & 48\mu-96\lambda-282 \end{pmatrix}, &&\boldsymbol{B}(-2) = \begin{pmatrix} 16\mu+747 & 819-16\mu-128\lambda \\ 48\mu-234 & 192\lambda+144\mu-138 \end{pmatrix},\\
&\boldsymbol{B}(-1) = \begin{pmatrix} 108-32\mu & -128\lambda-96\mu-327 \\ 0 & 243 \end{pmatrix}, &&\boldsymbol{B}(0) = \begin{pmatrix} -80\mu-99 & -320\lambda-240\mu-387 \\ 48\mu+54 & 192\lambda+144\mu+150 \end{pmatrix},\\
&\boldsymbol{B}(1) = \begin{pmatrix} 32\mu+45 & 160\lambda+96\mu+261 \\ 0 & -96\lambda-48 \end{pmatrix}, &&\boldsymbol{B}(2) = \begin{pmatrix} 0 & 64\lambda+24 \\ 0 & 0 \end{pmatrix},\\
&\boldsymbol{B}(3) = \begin{pmatrix} 0 & 160\lambda-27 \\ 0 & 15-96\lambda \end{pmatrix}, &&\boldsymbol{B}(4) = \begin{pmatrix} 0 & 12-64\lambda \\ 0 & 0 \end{pmatrix}.
\end{aligned} \tag{16}$$

3.2 $\mathcal{HC}^2$ regularity via joint spectral radius

The joint spectral radius approach for regularity analysis of scalar binary refinable functions was introduced in [13] and has been generalized to various different situations ever since. We recall that the joint spectral radius of a finite matrix set $\mathcal{T} = \{T_j \in \mathbb{R}^{n\times n} : j = 1, \dots, J\}$, $J \in \mathbb{N}$, was introduced in [33] and is defined by

$$\rho(\mathcal{T}) = \lim_{n\to\infty} \max_{T_k\in\mathcal{T}} \Big\| \prod_{k=1}^{n} T_k \Big\|^{1/n}. \tag{17}$$

The limit in (17) exists and is independent of the matrix norm. It is well known that the joint spectral radius measures the joint contractivity of the matrices in $\mathcal{T}$.

The study of $\mathcal{HC}^2$-regularity of $H_{\mathcal{A}}$ consists of several steps. Firstly, the algorithm in [6, Lemma 3.8] allows to determines the transition matrices for $S_{\mathcal{A}}$ and $S_{\mathcal{B}}$ derived from the parameter dependent subdivision masks in (15) and (16). The invariant common subspaces of these matrices, crucial for our regularity analysis, are determined from the polynomial reproduction properties of the Hermite schemes. Then, the recent advances in numerical linear algebra allow for exact computations of the joint spectral radius of finite sets of transition matrices restricted to common difference subspaces [18], [19], [32]. Moreover, to treat the parameter dependency of the transition matrices, we use the techniques presented in [4, Theorem 3.2, Remark 3.5 (iii)].

3.2.1 Transition matrices for $S_{\mathcal{A}}$ and $S_{\mathcal{B}}$

We start by constructing the transition matrices used for the regularity analysis of the vector subdivision scheme $S_{\mathcal{A}}$ associated with the matrix mask $\boldsymbol{A}$ in (15). These transition matrices $T_\varepsilon^A \in \mathbb{R}^{8\times 8}$ are 4×4 block matrices consisting of 2×2 matrix blocks $\boldsymbol{A}^T(\varepsilon + 3\alpha - \beta)$

$$T_\varepsilon^A = \Big(\boldsymbol{A}^T(\varepsilon + 3\alpha - \beta)\Big)_{\alpha,\beta\in\Omega^A} \qquad \varepsilon \in \{0,1,2\}, \quad \Omega^A = \{-2,-1,0,1\}.$$

The polynomial reproduction properties of $H_{\mathcal{A}}$ guarantee that the polynomial sequences

$$u_0 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} : \alpha \in \mathbb{Z} \right\},\ u_1 = \left\{ \begin{pmatrix} \alpha \\ 1 \end{pmatrix} : \alpha \in \mathbb{Z} \right\} \text{ and } u_2 = \left\{ \begin{pmatrix} \alpha^2 \\ 2\alpha \end{pmatrix} : \alpha \in \mathbb{Z} \right\}, \tag{18}$$

are polynomial eigensequences of the subdivision operator $S_{\mathcal{A}}$, i.e.

$$S_{\mathcal{A}} u_m = \left(\frac{1}{3}\right)^m u_m, \quad m = 0, 1, 2.$$

The structure of u_0 indicates [12] that the *basic limit function* of the scheme $S_{\mathcal{A}}$ has the form

$$\Phi_{\boldsymbol{G}_0} = \lim_{n\to\infty} S_{\mathcal{A}}^n \boldsymbol{G}_0 = \begin{pmatrix} \phi_1 & \phi_2 \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{G}_0 = \delta I_2,$$

with ϕ_1 and ϕ_2 being first components of the corresponding Hermite limits of $H_{\mathcal{A}}$. Therefore, to show that the Hermite scheme $H_{\mathcal{A}}$ is $\mathcal{HC}^2$-smooth, we need to show that ϕ_1, ϕ_2 belong to $C^2(\mathbb{R})$.

By [1, Theorem 3.1], the subspaces of $\mathbb{R}^8$ spanned by

$$\left\{\begin{pmatrix}1\\0\end{pmatrix}\right\}_{\alpha\in\Omega^A} = \begin{pmatrix}1\\0\\1\\0\\1\\0\\1\\0\end{pmatrix}, \quad \left\{\begin{pmatrix}\alpha\\1\end{pmatrix}\right\}_{\alpha\in\Omega^A} = \begin{pmatrix}-2\\1\\-1\\1\\0\\1\\1\\1\end{pmatrix}, \quad \left\{\begin{pmatrix}\alpha^2\\2\alpha\end{pmatrix}\right\}_{\alpha\in\Omega^A} = \begin{pmatrix}4\\-4\\1\\-2\\0\\0\\1\\2\end{pmatrix},$$

are right-invariant under all T_ε^A, $\varepsilon\in\{0,1,2\}$, and the difference subspaces V_0^A, V_1^A and V_2^A

$$V_0^A = \begin{pmatrix} 1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0\\ -1&1&0&0&0&0&0\\ 0&0&0&0&1&0&0\\ 0&-1&1&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&-1&0&0&0&0\\ 0&0&0&0&0&0&1 \end{pmatrix}, \quad V_1^A = \begin{pmatrix} 1&0&0&0&0&1\\ 0&0&1&0&0&-1\\ -2&1&0&0&0&-1\\ 0&0&-1&1&0&0\\ 1&-2&0&0&0&0\\ 0&0&0&-1&1&0\\ 0&1&0&0&0&0\\ 0&0&0&0&-1&0 \end{pmatrix},$$

and

$$V_2^A = \begin{pmatrix} 1&0&0&-2&0\\ 0&1&0&1&-1\\ -3&0&0&2&-2\\ 0&-2&1&1&3\\ 3&0&0&0&2\\ 0&1&-2&0&0\\ -1&0&0&0&0\\ 0&0&1&0&0 \end{pmatrix},$$

satisfying

$$\sum_{\alpha\in\mathbb{Z}} v(\alpha)u_j(-\alpha) = 0, \quad v\in V_k^A, \quad 0\le j\le k, \quad k\in\{0,1,2\},$$

are left-invariant under T_ε^A, $\varepsilon\in\{0,1,2\}$. Since we are interested in the C^2 regularity of $S_{\mathcal{A}}$, in Proposition 1 below, we analyze the joint spectral properties of the matrix set $\{T_\varepsilon^A|_{V_2^A} : \varepsilon \in \{0,1,2\}\}$ respectively given by

$$\frac{1}{729}\begin{pmatrix} 64\mu+243 & 96\mu+108 & 180-96\mu & -112\mu-144 & -80\mu-72 \\ 64\lambda-128\mu-480 & -96\lambda-192\mu-57 & 192\mu-360 & 224\mu-224\lambda+462 & 96\lambda+160\mu-6 \\ 128\lambda-60 & 192\lambda-30 & -192\lambda-33 & 39-224\lambda & 21-160\lambda \\ 16\mu+99 & 72\mu+81 & 90-48\mu & 0 & -64\mu-63 \\ -112\mu-369 & 9-120\mu & 144\mu-270 & 224\mu+423 & 96\mu-54 \end{pmatrix},$$
$$\frac{1}{729}\begin{pmatrix} -96\mu-135 & -48\mu-54 & 96\mu+108 & 0 & -64\mu-90 \\ 192\mu-192\lambda+576 & 192\lambda+96\mu+222 & -96\lambda-192\mu-57 & 32\lambda & 128\mu-416\lambda+240 \\ 36-192\lambda & 15-96\lambda & 192\lambda-30 & 0 & 24-128\lambda \\ 0 & -72\mu-81 & 72\mu+81 & -8\mu-18 & 72\mu+81 \\ 192\mu+540 & 24\mu+171 & 9-120\mu & -8\mu-9 & 200\mu+252 \end{pmatrix},$$
$$\frac{1}{729}\begin{pmatrix} 32\mu+45 & 0 & -48\mu-54 & 0 & 0 \\ 128\lambda-64\mu-132 & -192\lambda-33 & 192\lambda+96\mu+222 & 32\lambda-9 & 224\lambda+33 \\ 64\lambda-12 & 0 & 15-96\lambda & 0 & 0 \\ -16\mu-99 & 48\mu-90 & -72\mu-81 & -8\mu-9 & 72-56\mu \\ -80\mu-171 & 48\mu-90 & 24\mu+171 & -8\mu-18 & 81-56\mu \end{pmatrix}. \tag{19}$$

We continue by defining the transition matrices of $S_{\mathcal{B}}$. These transition matrices $T_\varepsilon^B \in \mathbb{R}^{10\times 10}$ are 5×5 block matrices consisting of 2×2 matrix blocks $\boldsymbol{B}^T(\varepsilon+3\alpha-\beta)$

$$T_\varepsilon^B = \Big(\boldsymbol{B}^T(\varepsilon+3\alpha-\beta)\Big)_{\alpha,\beta\in\Omega^B} \qquad \varepsilon\in\{0,1,2\}, \quad \Omega^B=\{-3,-2,-1,0,1\}.$$

The polynomial reproduction property of $H_{\mathcal{A}}$ that we use further on guarantees that the polynomial sequence

$$u_0 = \left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} : \alpha\in\mathbb{Z} \right\} \tag{20}$$

is a polynomial eigensequence of the subdivision operator $S_{\mathcal{B}}$, i.e. $S_{\mathcal{B}} u_0 = u_0$. The size of the corresponding difference subspace can be reduced by removing further common invariant subspaces of the transition matrices and we get V_0^B identified by the column vectors

$$V_0^B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 & -1 & 1 \\ 1 & 0 & -2 & 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & -2 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

V_0^B is left-invariant under all transition matrices T_ε^B, $\varepsilon\in\{0,1,2\}$. Since we are interested in the C^0 regularity of $S_{\mathcal{B}}$, in Proposition 2 below, we analyze the joint spectral properties of the matrix set $\{T_\varepsilon^B|_{V_0^B} : \varepsilon\in\{0,1,2\}\}$ respectively given by

(to be divided by the integer number 243)

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
423-32\mu & 96\mu-180 & -27 & -96\mu-108 & 48\mu+54 & 16\mu+9 & 96\mu+135 \\
64\lambda+32\mu+138 & -192\lambda-33 & -144\mu-195 & 192\lambda+48\mu+24 & -96\lambda-96\mu-93 & -32\lambda-64\mu-87 & 176\mu-192\lambda+270 \\
12-64\lambda & 0 & 288\lambda-51 & 15-96\lambda & 192\lambda-30 & 128\lambda-24 & 63-352\lambda \\
64\mu+72 & 0 & -288\mu-792 & 96\mu+252 & -192\mu-72 & -128\mu-459 & 352\mu+882 \\
450 & 96\mu-180 & -144\mu-630 & 90-48\mu & 90-48\mu & -48\mu-360 & 272\mu+783
\end{pmatrix},
$$

$$
\begin{pmatrix}
-16\mu-9 & 48\mu+54 & 0 & 0 & 0 & 0 & 0 \\
32\lambda-16\mu-12 & 48\mu-96\lambda+69 & 0 & 0 & 0 & 0 & 0 \\
112\mu & 48\mu+342 & 423-32\mu & 96\mu-180 & -96\mu-108 & 16\mu+324 & -32\mu-360 \\
-320\lambda-32\mu-90 & 192\lambda+96\mu+222 & 64\lambda+32\mu+138 & -192\lambda-33 & 192\lambda+48\mu+24 & 32\mu-32\lambda+117 & 64\lambda-32\mu-126 \\
0 & 0 & 12-64\lambda & 0 & 15-96\lambda & 12-64\lambda & 64\lambda-12 \\
32\mu+180 & -96\mu-252 & 64\mu+72 & 0 & 96\mu+252 & 64\mu+81 & -64\mu-72 \\
144\mu+180 & 90-48\mu & 450 & 96\mu-180 & 90-48\mu & 48\mu+360 & -64\mu-387
\end{pmatrix},
$$

$$
\begin{pmatrix}
-27 & -96\mu-108 & -16\mu-9 & 48\mu+54 & 0 & -48\mu-54 & 48\mu+54 \\
-33 & 192\lambda-96\mu-138 & 32\lambda-16\mu-12 & 48\mu-96\lambda+69 & 0 & 96\lambda-48\mu-69 & 48\mu-96\lambda+69 \\
-144\mu-1098 & 18-240\mu & 112\mu & 48\mu+342 & 96\mu-180 & -96\mu-261 & 80\mu+243 \\
288\lambda-519 & -96\lambda-192\mu-57 & -320\lambda-32\mu-90 & 192\lambda+96\mu+222 & -192\lambda-33 & -96\lambda-96\mu-210 & 128\lambda+96\mu+210 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
468 & 192\mu+72 & 32\mu+180 & -96\mu-252 & 0 & 96\mu+261 & -96\mu-270 \\
-144\mu-630 & 90-48\mu & 144\mu+180 & 90-48\mu & 96\mu-180 & 0 & -16\mu-27
\end{pmatrix}. \tag{21}
$$

3.2.2 Parameter domain for $\mathcal{HC}^2$-regularity of $H_{\mathcal{A}}$

The main result of this section, Theorem 1, shows that the Hermite scheme $H_{\mathcal{A}}$ in (15) is $\mathcal{HC}^2$-smooth for $(\mu, \lambda) \in K_2$, where $K_2 \subset \mathbb{R}^2$ is a closed convex hull

$$K_2 = \mathrm{co}\{(\mu_{2,m}, \lambda_{2,m}) \in \mathbb{R}^2 \,:\, m = 1, \dots, 6\}$$

of the points

$$
\begin{aligned}
(\mu_{2,1}, \lambda_{2,1}) &= (\ -271/100, \quad 519/971\ ), & (\mu_{2,2}, \lambda_{2,2}) &= (\ -159/73, \quad 78/175\ ), \\
(\mu_{2,3}, \lambda_{2,3}) &= (\ -2417/2062, \quad 509/2217\ ), & (\mu_{2,4}, \lambda_{2,4}) &= (\ -185/91, \quad 92/293\ ), \\
(\mu_{2,5}, \lambda_{2,5}) &= (\ -271/100, \quad 311/661\ ), & (\mu_{2,6}, \lambda_{2,6}) &= (\ -277/100, \quad 113/222\ ),
\end{aligned}
$$

shown on Figure 3. The proof of Theorem 1 is based on Proposition 1 and Proposition 2 whose combination guarantee $\mathcal{HC}^2$-regularity of $H_{\mathcal{A}}$.

Proposition 1 *The scheme $S_{\mathcal{A}}$ in* (15) *is $\mathcal{C}^2$-smooth for $(\lambda, \mu) \in K_2$.*

Proof To stress the parameter dependence, we set $T^A_{\varepsilon,\mu,\lambda}|_{V_2^A} = T^A_{\varepsilon}|_{V_2^A}$ for $\varepsilon \in \{0, 1, 2\}$. Define $U = \mathrm{span}\{u_1, u_2, u_3\}$ with u_j, $j = 1, 2, 3$, in (18). Note that the columns of V_2^A in section 3.2.1 can be identified with sequences in $\ell^{2\times 1}(\Omega)$ with $\Omega^A = \{-2, -1, 0, 1\}$. Then, straightforward computations show that

$$\sum_{\alpha \in \mathbb{Z}} v(\alpha) u(-\alpha) = 0, \quad v \in V_2^A, \quad u \in U.$$

Therefore, due to the minimality of Ω^A and by [26, Theorem 4.1], it suffices to show that

$$\rho(\mathcal{T}^A) < \frac{1}{9} \quad \text{for} \quad \mathcal{T}^A = \{T^A_{\varepsilon,\mu,\lambda}|_{V_2^A} \,:\, \varepsilon \in \{0,1,2\},\, (\mu, \lambda) \in K_2\}.$$

To do that we compute the Delauney triangulation $\varDelta$ of K_2

$$\varDelta = \left\{ \varDelta_j = \mathrm{co}\{k_j^1, k_j^2, k_j^3\} \,:\, k_j^m \in K_2,\, j = 1, \dots, J \right\}, \quad J \in \mathbb{N},$$

with vertices $k_j^m \in K_2$. Next, for each triangle $\Delta_j \in \Delta$, we define the following set of nine matrices

$$\mathcal{T}_j^A = \{T^A_{\varepsilon,\mu,\lambda}|_{V_2^A} \, : \, \varepsilon \in \{0,1,2\},\, (\mu,\lambda) = k_j^m,\, m = 1,2,3\}.$$

By [4, Theorem 3.2, Remark 3.5 (iii)], if

$$\rho(\mathcal{T}_j^A) < \frac{1}{9}, \tag{22}$$

then

$$\rho(\mathcal{T}^A_{\Delta_j}) < \frac{1}{9} \quad \text{for} \quad \mathcal{T}_{\Delta_j} = \{T^A_{\varepsilon,\mu,\lambda}|_{V_2^A} \, : \, \varepsilon \in \{0,1,2\},\, (\mu,\lambda) \in \Delta_j\}.$$

To prove (22), we use the results in [18], [19], [32] and the modified Gripenberg and modified invariant polytope algorithms in [32]. For each triangle Δ_j in the triangulation Δ of K_2, or for its dyadically refined version, the algorithms terminate successfully determining the spectrum maximizing matrix product from $\mathcal{T}_j$. This proves (22) for all $j = 1, \ldots, J$ and, thus, the claim follows.

The regularity analysis if $S_{\mathcal{A}}$ is not sufficient to conclude $\mathcal{HC}^2$-regularity of $\mathcal{H}_{\mathcal{A}}$. Therefore, we proceed with the analysis of the Taylor operator $S_{\mathcal{B}}$ constructed in (16). For this scheme we can prove the following convergence result, again based on a JSR approach.

Proposition 2 *The scheme $S_{\mathcal{B}}$ in* (16) *is convergent for $(\lambda,\mu) \in K_0$.*

Proof The proof mimics the proof of Proposition 1 where we replace the transition matrices by the one in (21).

Combining Propositions 1 and 2 and we are finally ready to prove our main result.

Theorem 1 *The scheme $H_{\mathcal{A}}$ in* (15) *is $\mathcal{HC}^2$-smooth for $(\lambda,\mu) \in K_2$.*

Proof Since, by Proposition 2, $H_{\mathcal{A}}$ is convergent its vector-valued limit function $\boldsymbol{\Phi} : \mathbb{R} \to \mathbb{R}^2$, has structure $\boldsymbol{\Phi} = \begin{pmatrix} \phi \\ \phi' \end{pmatrix}$ with $\phi' \in C^0$. On the other hand, since, by Proposition 1, $S_{\mathcal{A}}$ is C^2-convergent it follows that $\phi \in C^2$. Hence $\phi' \in C^1$, which implies the $\mathcal{HC}^2$-convergence of $H_{\mathcal{A}}$.

Remark 3 Note that the use of the triangulation Δ of K_2 in the proof of Proposition 1 is unavoidable. In fact, if we compute the joint spectral radius of a matrix set defined by the vertices of the whole domain K_2 the sufficient conditions in [4, Theorem 3.2, Remark 3.5 (iii)] are not satisfied.

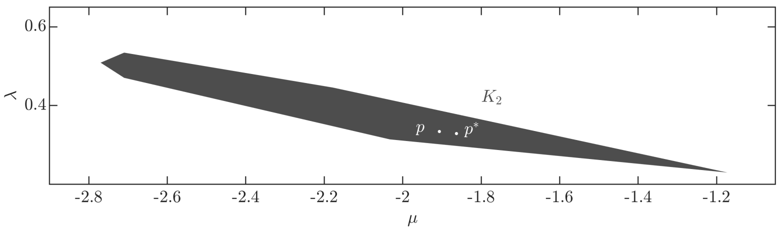

Fig. 3 Domain K_2.

3.2.3 $\mathcal{C}^0$ and $\mathcal{C}^1$-regularity of $S_{\mathcal{A}}$

For completeness, we also describe the parameter domain K_0 and K_1 for $\mathcal{C}^0$ and $\mathcal{C}^1$-convergence of $S_{\mathcal{A}}$, respectively. They are depicted in Figure 4. The domains K_0, K_1 are obtained using the argument similar to the one in Proposition 1. The set K_0 is a closed convex hull

$$K_0 = \text{co}\{(\mu_{0,m}, \lambda_{0,m}) \in \mathbb{R}^2 \,:\, m = 1, \ldots, 7\}$$

of the points

$$\begin{array}{llll}
(\mu_{0,1}, \lambda_{0,1}) = (\,-513/50, & 231/50\,), & (\mu_{0,2}, \lambda_{0,2}) = (\,357/125, & 216/125\,),\\
(\mu_{0,3}, \lambda_{0,3}) = (\,230/50, & 310/250\,), & (\mu_{0,4}, \lambda_{0,4}) = (\,520/129, & -748/217\,),\\
(\mu_{0,5}, \lambda_{0,5}) = (\,-230/50, & -301/200\,), & (\mu_{0,6}, \lambda_{0,6}) = (\,-950/80, & 403/100\,),\\
(\mu_{0,7}, \lambda_{0,7}) = (\,-627/53, & 211/50\,). & &
\end{array}$$

The domain K_1 is a closed convex hull

$$K_1 = \text{co}\{(\mu_{1,m}, \lambda_{1,m}) \in \mathbb{R}^2 \,:\, m = 1, \ldots, 7\}$$

of the points

$$\begin{array}{llll}
(\mu_{1,1}, \lambda_{1,1}) = (\,-51/10, & 31/22\,), & (\mu_{1,2}, \lambda_{1,2}) = (\,-159/73, & 9/7\,),\\
(\mu_{1,3}, \lambda_{1,3}) = (\,4/5, & 3/8\,), & (\mu_{1,4}, \lambda_{1,4}) = (\,13/14, & 6/25\,),\\
(\mu_{1,5}, \lambda_{1,5}) = (\,-13/8, & -7/20\,), & (\mu_{1,6}, \lambda_{1,6}) = (\,-19/8, & -2/13\,),\\
(\mu_{1,7}, \lambda_{1,7}) = (\,-59/11, & 9/7\,). & &
\end{array}$$

3.3 Complete Taylor operator approach for the extended scheme

In this subsection, we derive the *complete* Taylor factorization [29] of the extended Hermite scheme $H_{\mathcal{A}+}$ in (23) of order 2 and study its contractivity for different parameter values $(\mu, \lambda) \in K_2$. We determine what we call optimal parameters from K_2 such that the corresponding Hermite scheme $H_{\mathcal{A}}$ yields visually smoother curves in fewer iterations, see Figure 5.

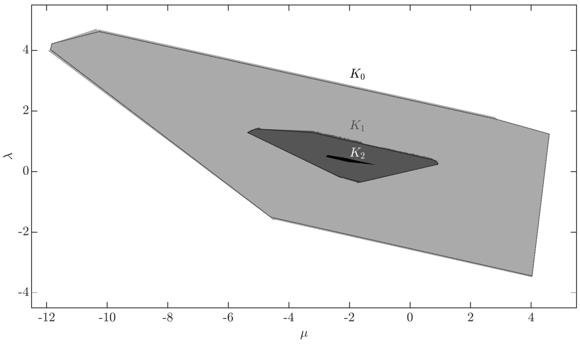

Fig. 4 Parameter domains $K_2 \subset K_1 \subset K_0$.

To define the scheme $H_{\mathcal{A}+}$ we use the results from [29] and [30] and start by completing the sequences $\boldsymbol{f}_n \in \ell^2(\mathbb{Z})$ generated by $H_{\mathcal{A}}$ by one additional component determined from the following approximations of the second derivative. In particular, for $\varphi \in C^3(\mathbb{R})$ and for small $h \in \mathbb{R}$, we use the following approximations

$$\begin{aligned} \varphi''(x) &\approx \frac{1}{2h}\left(\varphi'(x+h) - \varphi'(x-h)\right), \\ \varphi''(x+h/3) &\approx \frac{1}{6h}\left(5\varphi''(x+h) - 4\varphi''(x) - \varphi''(x-h)\right), \\ \varphi''(x-h/3) &\approx \frac{1}{6h}\left(-5\varphi'(x-h) + 4\varphi'(x) + \varphi'(x+h)\right), \quad x \in \mathbb{R}. \end{aligned}$$

Note that for an arbitrary initial sequence $\{\boldsymbol{f}_0(\alpha) = \left(f_0(\alpha)\ f_0'(\alpha)\right)^T,\ \alpha \in \mathbb{Z}\} \in \ell^2(\mathbb{Z})$, at the n-th step of the subdivision recursion, the above approximations suggest the construction of a new sequence in $\ell^3(\mathbb{Z})$ with additional component $f''_{n+1}(\alpha)$ for $\alpha \in \mathbb{Z}$ defined by

$$\begin{aligned} 3^{-2(n+1)} f''_{n+1}(3\alpha) &= \tfrac{3^{-2}}{2}\big(3^{-n} f'_n(\alpha+1) - 3^{-n} f'_n(\alpha-1)\big), \\ 3^{-2(n+1)} f''_{n+1}(3\alpha+1) &= \tfrac{3^{-3}}{2}\big(5\cdot 3^{-n} f'_n(\alpha+1) - 4\cdot 3^{-n} f'_n(\alpha) - 3^{-n} f'_n(\alpha-1)\big), \\ 3^{-2(n+1)} f''_{n+1}(3\alpha-1) &= \tfrac{3^{-3}}{2}\big(-5\cdot 3^{-n} f'_n(\alpha-1) + 4\cdot 3^{-n} f'_n(\alpha) + 3^{-n} f'_n(\alpha+1)\big), \end{aligned}$$

These identities define additional subdivision rules for the scheme of type (4) with the corresponding subdivision mask $\mathcal{A}_+ \in \ell^{3\times 3}(\mathbb{Z})$ given by its symbol

$$\boldsymbol{A}_+^*(z) = \sum_{\alpha\in\mathbb{Z}} \boldsymbol{A}_+(\alpha) z^\alpha = \begin{pmatrix} a_{11}(z) & a_{12}(z) & 0 \\ a_{21}(z) & a_{22}(z) & 0 \\ 0 & q_2(z) & 0 \end{pmatrix}, \quad z \in \mathbb{C}\setminus\{0\}, \tag{23}$$

with

$$\begin{pmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{pmatrix} = \sum_{\alpha\in\mathbb{Z}} \boldsymbol{A}(\alpha) z^\alpha$$

defined by using (15) and with

$$q_2(z) = -\frac{z^{-4} - z^4 + 3(z^{-3} - z^3) + 5(z^{-2} - z^2) + 4(z^{-1} - z)}{54}, \quad z \in \mathbb{C}\setminus\{0\}.$$

The *complete* Taylor factorization of $\boldsymbol{A}_+^*(z)$ defines the operator $S_{\tilde{\boldsymbol{B}}_+} : \ell^3(\mathbb{Z}) \to \ell^3(\mathbb{Z})$ whose symbol $\tilde{\boldsymbol{B}}_+^*(z)$ satisfies

$$\begin{pmatrix} z^{-1}-1 & -1 & -1/2 \\ 0 & z^{-1}-1 & -1 \\ 0 & 0 & z^{-1}-1 \end{pmatrix} \boldsymbol{A}_+^*(z) = \frac{1}{9}\tilde{\boldsymbol{B}}_+^*(z) \begin{pmatrix} z^{-3}-1 & -1 & -1/2 \\ 0 & z^{-3}-1 & -1 \\ 0 & 0 & z^{-3}-1 \end{pmatrix}. \tag{24}$$

Computation of the entries of $\tilde{\boldsymbol{B}}_+^*(z)$ provides

$$\begin{aligned}
\tilde{b}_{00}(z) &= \frac{80\mu z^6+99z^6-32\mu z^5-45z^5+16\mu z^4-126z^4+112\mu z^3+279z^3-32\mu z^2+108z^2+16\mu z+9z+32\mu+45}{81z^2} \\
\tilde{b}_{01}(z) &= -\frac{320\mu z^6+640\lambda z^6+315z^6-128\mu z^5-256\lambda z^5-51z^5+64\mu z^4+128\lambda z^4-273z^4}{324z^2} \\
&\quad -\frac{128\mu z^3-384l z^3+483z^3+256\lambda z^2+177z^2-128\lambda z+39z-256\lambda+48}{324z^2} \\
\tilde{b}_{02}(z) &= \frac{z\left(160\mu z^3+640\lambda z^3+117z^3-64\mu z^2-256\lambda z^2+39z^2+32\mu z+128\lambda z-21z+64\mu+256\lambda+42\right)}{324} \\
\tilde{b}_{10}(z) &= \frac{2(z-1)(z+1)\left(8\mu z^4+9z^4-8\mu z^3-9z^3+16\mu z^2-30z^2-8\mu z-9z+8\mu+9\right)}{27z^2} \\
\tilde{b}_{11}(z) &= -\frac{(z-1)\left(32\mu z^5+64\lambda z^5+35z^5+36z^4+32\mu z^3+64\lambda z^3-43z^3-64\lambda z^2-68z^2-9z-64\lambda+10\right)}{54z^2} \\
\tilde{b}_{12}(z) &= \frac{z\left(16\mu z^3+64\lambda z^3+17z^3-16\mu z^2-64\lambda z^2+19z^2+16\mu z+64\lambda z-z-16\mu-64\lambda-8\right)}{54} \\
\tilde{b}_{20}(z) &= 0 \qquad \tilde{b}_{21}(z) = -\frac{(z-1)(z+1)\left(z^2+z+1\right)^2}{6z^2}, \quad \tilde{b}_{22}(z) = \frac{z(z+1)\left(z^2+z+1\right)}{6}.
\end{aligned} \tag{25}$$

By [29], the scheme $H_{\mathcal{A}}$ is $\mathcal{HC}^2$-smooth, if the stationary vector subdivision scheme $S_{\tilde{\mathcal{B}}_+}$ is contractive, i.e there exists $R \in \mathbb{N}$ such that $\|S^R_{\tilde{\boldsymbol{B}}_+}\|_\infty < 1$. By [15], the latter is equivalent to show that

$$\max \left\{ \left\| \sum_{\beta\in\mathbb{Z}} |\tilde{\boldsymbol{B}}_+^{[R]}(\varepsilon - 3\beta)| \right\|_\infty, \quad \varepsilon \in \{0,1,2\} \right\} < 1$$

with the R-iterated mask $\tilde{\boldsymbol{B}}_+^{[R]}$ computed by the iteration

$$\tilde{\boldsymbol{B}}_+^{[1]} = \tilde{\boldsymbol{B}}_+, \quad \tilde{\boldsymbol{B}}_+^{[r]} = \sum_{\beta\in\mathbb{Z}} \tilde{\boldsymbol{B}}_+(\cdot - 3\beta)\tilde{\boldsymbol{B}}_+^{[r-1]}(\beta), \quad r = 1, \dots, R.$$

Matlab computations for

$$\begin{aligned} \mu &= -1.90660580626993, \quad \lambda = 0.333939685329603, \\ \mu^* &= -1.86303965004445, \quad \lambda^* = 0.328737778603242, \end{aligned} \tag{26}$$

yield the results presented in the following table where R is the smallest integer such that $\|S^R_{S_{\tilde{\mathcal{B}}_+}}\|_\infty < 1$; The real value $\alpha_\phi \geq -\log_3(\rho(\mathcal{T}))$ is the Hölder exponent of ϕ, the first component of the vector limit function, while s.m.p. is the spectrum maximizing product $\prod$ of length n that attains the joint spectral radius, i.e. the joint spectral radius $\rho(\mathcal{T}) = \rho(\prod)^{1/n}$.

	R	$\lVert S^R_{\tilde{\mathcal{B}}}\rVert_\infty$	α_ϕ	s.m.p.
$(\mu,\ \lambda)$	10	0.682990725	2.40722...	$T_0^6 T_1 T_0 T_1^3 T_2^6 T_1 T_2 T_1^3\vert_{V_2}$
$(\mu^*,\ \lambda^*)$	9	0.722679251	2.40289...	$T_0^4 T_1 T_0 T_1^3 T_2^4 T_1 T_2 T_1^3\vert_{V_2}$

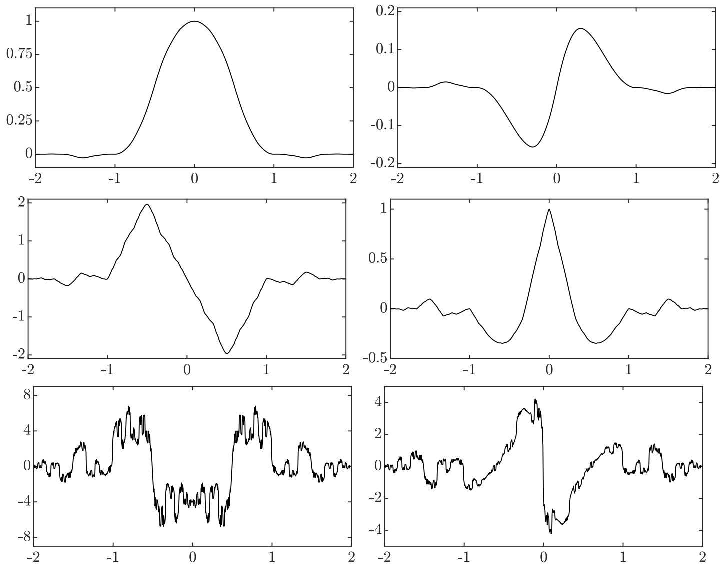

Fig. 5 Basic limit functions ϕ_1 and ϕ_2 for (μ, λ) as in (26)

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