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On the mod-2 cohomology of some 2-Postnikov towers

Nguyễn Thế Cường *

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Abstract

The present note presents some results about the mod-2 cohomology, modulo nilpotent elements elements of the fiber E of a decomposable map $\psi : K(\mathbb{Z}, 2) \rightarrow K(\mathbb{Z}/2, p)$. This is more an announcement and a brief description of the tools that are used: Lannes' T functor and the Eilenberg-Moore spectral sequence.

1 Introduction

Let E be the homotopy fiber of a map $\psi : K(\mathbb{F}_2, q) \rightarrow K(\mathbb{F}_2, p)$ between two Eilenberg-Mac Lane spaces, the homotopy class $[\psi]$ belongs to $H^q(K(\mathbb{F}_2, q), \mathbb{F}_2)$. If ψ is a stable map the work of Larry Smith, David Kraines [22] [23] [10] and others describes the mod-2 cohomology of E . The results of L. Smith concern the fiber of an H -map between H -spaces. Those of Kraines concern more specific examples. One of their main tool is the Eilenberg-Moore spectral sequence.

To make notation shorter we will write K_n for the Eilenberg-Mac Lane space $K(\mathbb{F}_2, n)$ in all the sequel. In all this note the mod-2 singular cohomology of a space X will be denoted by $H^*(X)$.

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In this note we study the case where (the homotopy class of) $\psi : X \rightarrow K_p$ is a decomposable element in the cohomology group $H^p(X)$: $\psi = \sum_i a_i b_i$, with $|a_i| > 0$, $|b_i| > 0$. One investigates what is possible to say on the cohomology of the homotopy fiber modulo the ideal of nilpotent elements.

After general results one focuses on the case $X = K_q$.

The main statements are:

Theorem 1.1. *There is a monomorphism:*

$$\frac{(H^*(X)/(\psi^*(\iota_p)))}{Nil} \hookrightarrow \frac{H^*(E)}{Nil}$$

where Nil denotes the ideal of nilpotents elements.

This a reformulation of of [13, Théorème 0.5]

Theorem 1.2. *There is an epimorphism, modulo nilpotent elements:*

$$H^*(E) \rightarrow H^*(K_{p-1})$$

This means that for any element $x \in H^*(K_{p-1})$ some power x^{2^h} , h large enough, is in the image.

Part of the results are to some extent, already contained in [19] and [20], but are made here more precise. The next questions are to decide how these two theorems "match" together and to describe the right hand side quotient more explicitly in some cases. As we have said we are going to discuss this modulo nilpotent elements.

The techniques described in the note makes the following conjecture very plausible. The notations are explained after the statement.

Conjecture 1.3. *(Weak form) As graded algebras one has:*

$$\sqrt{H^*(E)} \cong \sqrt{H^*(X)/(\psi^*(\iota_p))} \otimes H^*(K_{p-1}) \quad .$$

Conjecture 1.4. *(Strong form) As unstable algebras one has:*

$$\sqrt{H^*(E)} \cong \sqrt{H^*(X)/(\psi^*(\iota_p))} \otimes H^*(K_{p-1}) \quad .$$

The algebra $\sqrt{H^*(Y)}$, for any space Y , denotes the quadratic closure of the quotient of the (graded) algebra $H^*(Y)$ by the ideal of nilpotent elements Nil . The unstable algebra $\sqrt{H^*(Y)}$ is the smallest (graded) algebraic extension, in the sense of J. F. Adams and C. Wilkerson [1], of $H^*(Y)/Nil$ which contains all square roots.

For example:

- $\sqrt{H^*(K_p)} = H^*(K_p)$;

- consider $H^*(\mathbb{C}P^\infty) \cong \mathbb{F}_2[u^2]$ then $\sqrt{H^*(\mathbb{C}P^\infty)} = H^*(\mathbb{R}P^\infty) = \mathbb{F}_2[u]$;
- $\sqrt{K} \cong \mathbb{F}_2$ for all finite connected unstable algebra K .

From now on we discuss examples.

The mod-2 cohomology of an elementary abelian 2-group V_n of rank n denoted by H^*V_n , is a graded polynomial algebra on n generators of degree 1, $\mathbb{F}_2[x_1, \dots, x_n]$, these generators may also be denoted also by u, v, w, \dots .

Let X be the Eilenberg-Mac Lane space K_2 . Let Q_i , $i \geq 0$ be the Milnor operation of degree $2^{i+1} - 1$ [15]. One has [21]:

$$H^*(K_2) \cong \mathbb{F}_2[\iota_2, Q_0(\iota_2), Q_1(\iota_2), \dots, Q_i(\iota_2), \dots] \quad .$$

We now describe the right hand side algebra $H^*(K_2)/(\psi^*(\iota_p))$.

Proposition 1.5. *If $[\psi] = \iota_2^t \in H^*(K_2)$, $t > 1$, then*

$$\sqrt{H^*(K_2) \otimes_{H^*K_p} \mathbb{F}_2} \cong \mathbb{F}_2$$

Proposition 1.6. *If $[\psi] \in H^*(K_2)$ is a nontrivial monomial in the polynomial generators of $H^*(K_2)$, involving at least one $Q_i(\iota_2)$, then*

$$\sqrt{H^*(K_2) \otimes_{H^*K_p} \mathbb{F}_2} \cong H^*(B\mathbb{Z}/2) \cong \mathbb{F}_2[u]$$

Proposition 1.7. *If $[\psi] = \alpha d_2$, with $d_2 = \iota_2 Q_0(\iota_2) + Q_1(\iota_2)$, α any non-trivial monomial, then*

$$\sqrt{H^*(K_2) \otimes_{H^*K_p} \mathbb{F}_2} \cong D(2)$$

where $D(2) = H^*(V_2)^{\text{GL}_2(\mathbb{F}_2)}$ is the mod-2 Dickson algebra.

Proposition 1.8. *If $[\psi] = \alpha h_2$ with $h_2 = \iota_2^2 Q_1(\iota_2) + Q_0(\iota_2)^3 + \iota_2^3 Q_0 \iota_2$, α any nontrivial monomial, then*

$$\sqrt{H^*(K_2) \otimes_{H^*K_p} \mathbb{F}_2} \cong H_2$$

where $H_2 \subset H^*(B(\mathbb{Z}/2 \times \mathbb{Z}/2)) \cong \mathbb{F}_2[u, v]$, $|u| = |v| = 1$ is the unstable subalgebra generated by

$$\{uv, uv(u+v), uv(u^3+v^3), \dots, uv(u^{2^n-1}+v^{2^n-1}), \dots\} \quad .$$

This is also the subalgebra of $\mathbb{F}_2[u, v]$ generated by

$$w_2, w_1 w_2, w_1^2 w_2, \dots, w_1^n w_2, \dots \text{ where}$$

w_1, w_2 are the universal Stiefel-Whitney classes.

Proposition 1.9. *If $[\psi] = \alpha d_2 h_2$, α any monomial, then*

$$\sqrt{H^*(K_2) \otimes_{H^*K_p} \mathbb{F}_2} \cong \sqrt{M_2}$$

where $M_2 \subset H^*(B(\mathbb{Z}/2 \times \mathbb{Z}/2))^{\times 2} \cong \mathbb{F}_2[u, v]^{\times 2}$, is the unstable subalgebra which is the fiber product of $D(2)$ and H_2 over $\mathbb{F}_2[u]$ over the nontrivial morphisms $D(2) \rightarrow \mathbb{F}_2[u]$ and $H_2 \rightarrow \mathbb{F}_2[u]$.

In fact all of the preceding results (1.5 to 1.8) hold without the assumption "nontrivial monomial". This hypothesis is used in the sequel.

Exercice 1.10. *Find classes $[\psi] \in H^p(K_2)$ such that $\sqrt{H^*(K_2)/(\psi^*(\iota_p))}$ is the unstable subalgebra of $\mathbb{F}_2[u, v, w]$ generated by $uv + w^2$.*

Proposition 1.11. *In all of these examples the weak form of the conjecture olds. The strong form also holds except in Proposition 1.9, in which case it is true if $p - 1$ is odd.*

We will use the T -technology of J. Lannes [11], [20] and also the Eilenberg-Moore spectral sequence [18], [23]. We make the necessary recollections in Section 1. In Section 2 we discuss the right hand side quotients in 3.3. In Section 3 we prove Theorems 1.1 and 1.2. In Section 4 we prove 1.11. The methods developed is very much likely to extend to solve Conjecture 1.3.

This work will be continued in order to generalize the results, in two directions. The first direction is to extend the results on the reduced part of the cohomology to some more general examples. The second one (and this is the reason for Theorem 4.5) is to extend the results to the 1-nilpotent part of the cohomology. It would be also worth, harder but possible, to try to get some informations about the space of maps $\text{map}(BV \wedge \mathbb{R}P^2, E)$, this would give insights on the 2-nilpotent part of the cohomology of E .

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2 Recollections about unstable modules and algebras.

We recall here some notations and facts about unstable modules and unstable algebras over the Steenrod algebra.

One denotes as usual \mathcal{U} the abelian category of unstable modules over the Steenrod algebra \mathcal{A}_2 , and by \mathcal{K} the category of unstable algebras. An unstable module M is a (graded) module over the (graded) Steenrod algebra which satisfies $\text{Sq}^i(m) = 0$ for any $m \in M$ such that $|m| < i$. An unstable algebra is a (graded) algebra which is also an object in \mathcal{U} , the two structures being compatible : the Cartan formula and the restriction axiom hold.

As usual, Σ denotes the suspension functor in the category \mathcal{U} . An unstable module M is a suspension if and only if the map $\text{Sq}_0 : M \rightarrow M$, $m \mapsto \text{Sq}^{|m|}(m)$ is trivial. Let $\mathcal{N}il_i$ be the Serre classe in \mathcal{U} generated by i -th suspensions : it is closed by sub-objects, quotients, extensions and colimits [19], [20]; $\mathcal{N}il_1$ is more commonly denoted by $\mathcal{N}il$. In an unstable algebra K the ideal consisting of all nilpotent elements is stable under the action of the Steenrod algebra, and is the largest submodule of K in $\mathcal{N}il$.

Details can be found, for example, in [24], [11], [19], [14], [5].

We will say that a morphism $f : M \rightarrow N$ of unstable modules is an F -monomorphism (resp. epimorphism) if the kernel (resp. cokernel) are in $\mathcal{N}il$. The definition restricts (and is more classical) to unstable algebras: $f : K \rightarrow L$, is an F -monomorphism if any element in the kernel is nilpotent, and an F -epimorphism if for any $x \in L$ there exists n such that x^{2^n} is in the image.

Being an F -monomorphism and an F -epimorphism at the same time is called an F -isomorphism.

As the category $\mathcal{N}il$ is a localizing subcategory [7], in the quotient category $\mathcal{U}/\mathcal{N}il$ an F -isomorphism becomes an isomorphism. The canonical functor $\mathcal{U} \rightarrow \mathcal{U}/\mathcal{N}il$ has a right adjoint and there is a localisation functor $\ell : \mathcal{U} \rightarrow \mathcal{U}$. The unstable module $\ell(M)$ is $\mathcal{N}il$ -closed [7]. If K is an unstable algebra, then so is $\ell(K)$. Moreover the unit of the adjunction $\mathcal{K} : K \rightarrow \ell(K)$ is a morphism of unstable algebras [2].

The ideal of nilpotent elements in $\ell(K)$ is trivial and $\ell(K)$ is quadratically closed. This last property means that one cannot find $J \supset \ell(K)$, an element $x \in J \setminus \ell(K)$, with $x^2 \in \ell(K) \setminus \{0\}$ [14] For this reason we are going to denote $\ell(K)$ by \sqrt{K} .

Now, recall the functor $f : \mathcal{U} \rightarrow \mathcal{F}$, where \mathcal{F} is the category of functors from the category of finite dimensional \mathbb{F}_2 - vector spaces to the category of \mathbb{F}_2 - vector spaces. The functor f sends an unstable module M to the functor $V \mapsto \text{Hom}_{\mathcal{U}}(M, \mathbb{H}^*(V))^{\#}$. It is shown in [9] that f induces an equivalence (also denoted by f) $f : \mathcal{U}/\mathcal{N}il \rightarrow \mathcal{F}_{\omega}$ of $\mathcal{U}/\mathcal{N}il$ with the subcategory of analytic functors \mathcal{F}_{ω} .

The dual considered above is a continuous one. Indeed, $\text{Hom}_{\mathcal{U}}(M, \mathbb{H}^*(V))$ has a natural profinite structure ([9]). However in the cases we consider all vector spaces are finite dimensional . Thus we will ignore profinite structures.

Recall that Lannes' functor $T_V : \mathcal{U} \rightarrow \mathcal{U}$, left adjoint to $M \mapsto \mathbb{H}^*(V) \otimes M$,

is exact and commute with tensor product [11]. Following [9] $f(M)(V)$ is defined to be $T_V(M)^0$.

As in ([9]), let $g : \mathcal{K} \rightarrow \mathcal{G}$, where \mathcal{G} is the category of contravariant functors from the category of finite dimensional \mathbb{F}_2 -vector spaces to the category of sets, be defined by $K \mapsto \{V \mapsto \text{Hom}_{\mathcal{K}}(K, \mathbb{H}^*(V))\}$. It induces an equivalence (also denoted by g) $g : \mathcal{K}/\mathcal{N}il \rightarrow \mathcal{G}_\omega$ to the category of set valued analytic functors.

Recall that Lannes' "linearization theorem" writes as:

$$f(K)(V) \cong \mathbb{F}_2^{\text{Hom}_{\mathcal{K}}(K, \mathbb{H}^*(V))}$$

where the right hand side denotes the Boolean algebra of set maps from $\text{Hom}_{\mathcal{K}}(K, \mathbb{H}^*(V))$ to \mathbb{F}_2 .

Again one should need profinite structures on sets, but again here sets are finite.

Let $U : \mathcal{U} \rightarrow \mathcal{K}$ be the Steenrod-Epstein functor: the universal enveloping algebra [20]. Recall ([21]) the isomorphism of unstable algebras $\mathbb{H}^*(K_n) \cong U(F(n))$ where the free unstable modules $F(n)$ is characterized by

$$\text{Hom}_{\mathcal{U}}(F(n), M) \cong M^n \quad .$$

We will also use later the fact the functor T_V commutes with U : $T_V(U(M)) \cong U(T_V(M))$ and the following isomorphisms (which can be deduced from part 1 of [20] and of [5]):

$$T_V(F(p)) \cong \bigoplus_{0 \leq i \leq p} \Gamma^i(V) \otimes F(p-i) \quad (2.1)$$

where $\Gamma^i \in \mathcal{F}$ is i -th divided power functor. And as $\mathbb{H}^*(K_p) \cong U(F(p))$ one gets:

$$T_V(\mathbb{H}^*(K_p)) \cong \bigotimes_{0 \leq i \leq p} \mathbb{H}^*(K(\Gamma^i(V^\#), p-i)) \quad . \quad (2.2)$$

3 Examples: subfunctors of S^2 and quotients of $\mathbb{H}^*(K_2)$

Next consider the examples alluded to in the introduction:

Example 3.1. *The set valued functor $g(\mathbb{H}^*(K_2))$:*

$$V \mapsto \text{Hom}_{\mathcal{K}}(\mathbb{H}^*(K_2), \mathbb{H}^*V) \cong S^2(V^\#)$$

is filtered by the rank r of the associated bilinear form (which is an even integer $r = 2k$) by subfunctors:

$$S^{2,0} \subset S^{2,0} \subset S^{2,2} \subset S^{2,4} \subset \dots S^{2,2k} \subset \dots$$

A subfunctor G is said to be generated in dimension less than d if any element in $G(V)$ is in the image of f^* , for some linear map $f : V \rightarrow V_d$. There is a diagram of subfunctors:

$$\begin{array}{ccc}
 & \langle \sum_{1 \leq i \leq r-1} u_i v_i + u_r^2 + u_r v_r + v_r^2 \rangle & \\
 \swarrow \cup & & \searrow \cup \\
 S^{2,2(r-1)} & & \langle \sum_{1 \leq i \leq r} u_i v_i + w^2 \rangle \rightarrow S^{2,2r} \\
 \searrow \cup & & \swarrow \cup \\
 & S^{2,2(r-1)} &
 \end{array}$$

In this diagram quadratic forms generating the subfunctors are written as usual. In rank $2r$ and dimension $2r$ there are two non isomorphic forms, one of Arf invariant 1, the other of invariant 0. In dimension $2r + 1$ and rank $2r$ there is the form $\langle \sum_{1 \leq i \leq r} u_i v_i + w^2 \rangle$.

Example 3.2. The subfunctors generated in dimension less than 2 are:

- the subfunctor $S^{2,0}$ itself is filtered by the constant subfunctor generated by $S^*(0)$, and the subfunctor generated by the form $\langle u^2 \rangle \in S^2(\mathbb{F}_2^\#)$.
- the subfunctor $S^{2,2}$ contains as subfunctor the one generated by the form $\langle u^2 + uv + v^2 \rangle \in S^2((\mathbb{F}_2^{\oplus 2})^\#)$,
- the one generated by the form $\langle uv \rangle \in S^2((\mathbb{F}_2^{\oplus 2})^\#)$;
- the subfunctor generated by both forms $\langle uv, u^2 + uv + v^2 \rangle \in S^2((\mathbb{F}_2^{\oplus 2})^\#)$;
- $S^{2,2}$ is generated by the form $\langle u^2 + uv + v^2 + w^2 \rangle \in S^2((\mathbb{F}_2^{\oplus 3})^\#)$.

We come back now to $X = K_2$. The quotients (in $\mathcal{K}/\mathcal{N}il$) of $H^*(K_2)$ are classified by subfunctors of $V \mapsto S^2(V^\#)$. The correspondance:

$$\mathcal{K}/\mathcal{N}il \rightarrow \mathcal{G}_\omega, \quad K \mapsto \{V \mapsto \text{Hom}_{\mathcal{K}}(H, H^*(V))\}$$

is an equivalence of categories. Therefore there is a one-to-one correspondance between subfunctors of $S^2(-)^\#$ and the quotients in $\mathcal{K}/\mathcal{N}il$ of $H^*(K_2)$. Recall:

$$H^*(K_2) \cong \mathbb{F}_2[\iota_2, Q_0(\iota_2), Q_1(\iota_2), \dots, Q_i(\iota_2), \dots]$$

We compute explicitly the quotients $\sqrt{H^*(K_2)/(\psi^*(\iota_p))}$ mentioned in Example 3.2. We do not claim that they are in any sense the best examples and this deserves further investigations.

As $H^*(V) \cong S^*(V^\#)$, a (homotopy class of) map: $\psi : K_p \rightarrow K_2$ determines a natural transformation:

$$\psi_* : S^2(-)^\# \rightarrow S^p(-)^\#$$

It determines a subfunctor in S^2 as follows: it consists of the set of elements of $S^2(V^\#)$ which are sent to 0 by this class:

$$\{s \in S^2(V^\#) \mid \psi_*(s) = 0\}$$

The map ψ_* is just a cohomology operation: a polynomial in the Steenrod powers. The computation is done using the usual rules with the Steenrod algebra.

We now illustrate some examples.

- Let $[\psi]$ be ι_2^t , for some integer $t > 1$. In this case

$$\psi_* : S^2(-)^\# \rightarrow S^p(-)^\#$$

is injective. The kernel of this morphism is the constant functor on one point. The corresponding quotient is \mathbb{F}_2 .

- Let $[\psi]$ be $\alpha Q_i(\iota)$ for some monomial. As In the case of the subfunctor generated by $\langle u^2 \rangle$ any monomial involving one $Q_i(\iota_2)$ works because it is zero on u^2 and non zero on uv and $u^2 + uv + v^2$:

$$Q_i(u^2) = 0$$

$$Q_i(uv) = uv(u^{2^{i+1}-1} + v^{2^{i+1}-1})$$

$$Q_i(u^2 + uv + v^2) = uv(u^{2^{i+1}-1} + v^{2^{i+1}-1})$$

the kernel of ψ_* is generated by $\langle u^2 \rangle$. The corresponding quotient is therefore $\mathbb{F}_2[u]$, whence Proposition 1.6.

- Similarly if $[\psi] = \alpha d_2$, and $d_2 = (\iota_2 Q_0(\iota_2) + Q_1(\iota_2))$, α any monomial, a direct computation shows that the class ψ_* is generated by $\langle u^2 + uv + v^2 \rangle$. The corresponding quotient is the Dickson algebra generated by the Dickson invariants $u^2 + uv + v^2, uv(u + v)$. Proposition 1.7 follows.

The other cases follow. Proposition 3.3 will be proved in section 4. The subfunctors of Example 3.2 classify (modulo $\mathcal{N}il$) the quotients of transcendence degree at most 2 and 3 of $H^*(K_2)$, $D(2)$ and H_2 are the only quotients of $H^*(K_2)$ which are integral domain and of transcendence degree 2. This follows from the classification of quadratic forms over \mathbb{F}_2 (described above) and [9].

Proposition 3.3. *Let $\psi : K_2 \rightarrow K_p$ be one of the maps in Propositions 1.5 to 1.8 E the homotopy fiber of ψ , then there is an isomorphism of unstable algebras:*

$$\sqrt{H^*(E)} \cong \sqrt{H^*(K_2) \otimes_{H^*K_p} \mathbb{F}_2}$$

For Proposition 1.9 this holds if $p - 1$ is odd.

This will be proved in the last section.

4 Sets of homotopy classes of maps, applications

Let us start with a classical remark about homotopy classes of maps ([8], Additionnal topics : Base points and Homotopy 4A).

Proposition 4.1. *Let X and Y be two connected pointed CW-complexes then the natural map from the set of pointed homotopy classes from X to Y to homotopy classes :*

$$[X, Y]_* \rightarrow [X, Y]$$

is always surjective. It is a bijection if either X is 1-connected or X is H -space.

Recall that for an elementary abelian 2-group V , BV is the classifying space of V . The fibration $E \rightarrow X \xrightarrow{\psi} K_p$ yields a short exact sequence of pointed sets:

$$\begin{aligned} [\Sigma BV, E]_* &\rightarrow [\Sigma BV, X]_* \rightarrow [\Sigma BV, K_p]_* \rightarrow \\ &\rightarrow [BV, E]_* \rightarrow [BV, X]_* \rightarrow [BV, K_p]_* \end{aligned} \quad (4.1)$$

If X is 1-connected it is equivalent to :

$$\begin{aligned} [\Sigma BV, E] &\rightarrow [\Sigma BV, X] \rightarrow [\Sigma BV, K_p] \rightarrow \\ &\rightarrow [BV, E] \rightarrow [BV, X] \rightarrow [BV, K_p] \end{aligned} \quad (4.2)$$

Where:

- The first 3 terms are groups and the sequence is exact in the usual sense.
- The last three terms are pointed sets, the composite morphisms are trivial.

- Exactness at $[BV, X]_*$ means that the (homotopy class) of a map $f : BV \rightarrow X$ lifts to E if and only if the composite $BV \rightarrow X \rightarrow K_p$ is trivial.
- Exactness at $[BV, E]_*$ means that the (homotopy class) of a map $f : BV \rightarrow E$ lifts to $[\Sigma BV, K_p]$ if and only if the composite $BV \rightarrow E \rightarrow X$ is trivial.
- Moreover $[\Sigma BV, K_p]_*$ acts on $[BV, E]_*$, and two homotopy classes $f, g : BV \rightarrow E_d$ which have the same image in $[BV, X]_*$ are in the same orbit under the action of $[\Sigma BV, K_p]$.
- An element of $[\Sigma BV, K_p]$ which acts trivially on $[BV, E]_*$ comes from $[\Sigma BV, X]$.

We are going to use the following two theorems:

Theorem 4.2 ([11]). *Let X be a connected CW-complex such that $H^*(X)$ is of finite dimension in any degree and that $\pi_1(X)$ is a finite 2-group. Then the natural map*

$$[BV, X] \rightarrow \text{Hom}_{\mathcal{K}}(H^*(BV), H^*(X)), \quad f \mapsto f^*$$

is a bijection

The second theorem below is outside of the main stream of the note. It is there to show that this type of strategy also allows to get informations on the ideal $Nil \subset H^*(E)$. This will be done in future articles.

Theorem 4.3. [12] *The natural map*

$$[\Sigma BV, X] \rightarrow \text{Hom}_{\mathcal{K}}(H^*(X), H^*(\Sigma BV)), \quad f \mapsto f^*$$

is a surjection

For an augmented unstable algebra K , let \tilde{K} be the ideal of augmentation. Denote by

$$Q(K) := \frac{\tilde{K}}{\tilde{K} \cdot \tilde{K}}$$

the unstable module of indecomposable elements. For any unstable algebra K , $Q(K)$ is a suspension. Remark that this is true without connectivity hypothesis on K because K^0 is a Boolean algebra. One has:

$$\text{Hom}_{\mathcal{K}}(H^*(X), H^*(\Sigma BV)) \cong \text{Hom}_{\mathcal{U}}(\Sigma^{-1}Q(H^*(X)), \tilde{H}^*(BV))$$

Thanks to 4.2 and 4.3, sequences 4.1 and 4.2 yield the following exact sequences:

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{K}}(\mathrm{H}^*(E), \mathrm{H}^*(\Sigma BV)) & & \mathrm{Hom}_{\mathcal{K}}(\mathrm{H}^*(E), \mathrm{H}^*(BV)) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathcal{K}}(\mathrm{H}^*(X), \mathrm{H}^*(\Sigma BV)) & & \mathrm{Hom}_{\mathcal{K}}(\mathrm{H}^*(X), \mathrm{H}^*(BV)) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathcal{K}}(\mathrm{H}^*(K_p), \mathrm{H}^*(\Sigma BV)) & \xrightarrow{\quad} & \mathrm{Hom}_{\mathcal{K}}(\mathrm{H}^*(K_p), \mathrm{H}^*(BV))
\end{array}$$

Note that we have the following isomorphisms:

$$[\Sigma BV, Y]_* \cong [BV, \Omega X]_* \cong \mathrm{Hom}_{\mathcal{K}}(\mathrm{H}^*(\Omega X), \mathrm{H}^*(BV))$$

Now we use the hypothesis that the class ψ is decomposable. The following is classical.

Lemma 4.4. *If $\psi \in \mathrm{H}^p(X)$ is decomposable the class $\Omega\psi \in \mathrm{H}^{p-1}(\Omega X)$ is trivial. Consequently ΩE is homotopically equivalent to $\Omega X \times K_{p-2}$*

Proof. One has a commutative diagram :

$$\begin{array}{ccc}
\Sigma\Omega X & \xrightarrow{\Sigma\Omega\psi} & \Sigma\Omega K_p \cong \Sigma K_{p-1} \\
\downarrow ev & & \downarrow ev \\
X & \xrightarrow{\psi} & K_p
\end{array}$$

Denote by ι_p the generator of $\mathrm{H}^p(K_p)$. The evaluation map ev sends (t, ω) , $t \in [0, 1]$, $\omega \in \Omega X$ (resp. ΩK_p) to $\omega(t)$. As all nontrivial cup-products are trivial in a suspension $\psi \circ ev \in \mathrm{H}^p(\Sigma\Omega X)$ is trivial. By commutativity of the diagram $ev \circ \Sigma\Omega\psi$ is trivial. As $eV^\#(\iota_p) = \Sigma\iota_{p-1}$ this implies that $\Sigma(\Omega\psi)^*(\iota_{p-1}) = 0$, and $(\Omega\psi)^*(\iota_{p-1}) = 0$

□

As $\Omega\psi \sim *$ one gets an inclusion

$$\mathrm{Hom}_{\mathcal{K}}(\mathrm{H}^*(K_{p-1}), \mathrm{H}^*(BV)) \hookrightarrow \mathrm{Hom}_{\mathcal{K}}(\mathrm{H}^*(E), \mathrm{H}^*(BV))$$

In other words, we have an injection $g(K_{p-1}) \rightarrow g(\mathrm{H}^*(E))$. Because of the equivalence of categories $\mathcal{K}/\mathcal{N}il \simeq \mathcal{G}_\omega$, we get an F -epimorphism $\mathrm{H}^*(E) \rightarrow \mathrm{H}^*(K_{p-1})$, whence Theorem 1.2. Thus, $\mathrm{H}^*(E) \rightarrow \mathrm{H}^*(K_{p-1})$ is an F -epimorphism: a certain power $\iota_{p-1}^{2^h}$ of the fundamental class is in the image of the morphism, however, to find a representative may be hard.

The following result is a consequence of Theorem 4.3, it is mentioned for future investigations:

Proposition 4.5. *Let E be the homotopy fiber of a non homotopically trivial decomposable map $\psi : X \rightarrow K_p$. There is a monomorphism of set valued functors:*

$$f(\Sigma^{-1}Q(\mathbb{H}^*E))/\mathcal{N}il \hookrightarrow S^{p-2} \oplus g(\mathbb{H}^*(\Omega X)) \quad .$$

It follows from 4.3 that one has a surjection:

$$[\Sigma BV, E] \twoheadrightarrow \text{Hom}_{\mathcal{K}}(\mathbb{H}^*(E), \mathbb{H}^*(\Sigma BV)) \cong \text{Hom}_{\mathcal{U}}(\Sigma^{-1}Q(\mathbb{H}^*(E)), \tilde{\mathbb{H}}^*V) \quad ,$$

as

$$[\Sigma BV, E] \cong [BV, \Omega E] \cong \text{Hom}_{\mathcal{K}}(\mathbb{H}^*(\Omega E), \mathbb{H}^*V)$$

and as we have isomorphisms

$$\mathbb{H}^*(\Omega E) \cong \mathbb{H}^*(\Omega X) \otimes \mathbb{H}^*(K_{p-2}) \quad ,$$

the inclusion follows.

5 The Eilenberg-Moore spectral sequence

We are now going to prove Proposition 3.3. The methods are very much likely to adapt to prove Conjecture 1.3. We will use the Eilenberg-Moore spectral sequence. Recall various facts about the spectral sequence, references are [18], [23] and [4]. Let $F \rightarrow E \rightarrow B$ be a fiber sequence of pointed spaces, assume that $\pi_1(B)$ is a finite 2-group. The spectral sequence has the following properties :

- i. There is a decreasing filtration on $\mathbb{H}^*(F)$:

$$\dots \supset F_{-s} \supset \dots \supset F_{-1} \supset F_0$$

by \mathcal{A}_2 -modules;

- ii. The $E_{s,*}^2$ page is isomorphic to $\text{Tor}_{\mathbb{H}^*(B)}^{-s}(H^*(E), \mathbb{F}_2)$, $s \geq 0$;
- iii. the differentials are of bidegree $(r, 1 - r)$ and are \mathcal{A}_2 -linear;
- iv. the spectral sequence is a multiplicative one, the product being given by the shuffle product;
- v. it converges towards $\mathbb{H}^*(F)$: the k -th term of the graded object associated to the filtration of $\mathbb{H}^k(F)$ is $\Sigma^s E_{s,-s+k}^\infty$;
- vi. $\text{Tor}_{\mathbb{H}^*(B)}^{-s}(H^*(E), \mathbb{F}_2)$ is $(s - 1)$ -connected and in $\mathcal{N}il_s$ ([20] Theorem 6.4.1).

About the convergence, it is obvious when B is 1-connected, there is a vanishing line of slope 2 : everything below the line $2u + v = 0$ is trivial and

the filtration of $H^*(F)$ in a given degree is finite. For the general case see [4].

About (vi), $\text{Tor}_{H^*(B)}^{-s}(H^*(E), \mathbb{F}_2)$ is not in general an s -suspension, even if it is true for $s = 1$ and $p = 2$ (this is not true if $p > 2$). Being in $\mathcal{N}il_{-s}$ follows from the fact that if M is a module over an unstable module K , such that the action satisfies the Cartan formula, then $\text{Tor}_K^s(M, \mathbb{F}_2)$ is $(-s - 1)$ -connected (see [20] Lemma 6.4.3).

It follows from (ii) and (vi) that $d_r(E_{s,*}^2) \in \mathcal{N}il_{-s-r+1}$.

Applications of these results are given in [19], see for example Theorem 8.7.1 and Proposition 8.7.7.

It is worth and it will be useful to give some more informations on the construction of the spectral sequence. It follows from [18] (see Section 3, the construction of the space X^1) that the -1 step of the filtration on $H^*(F)$ is, up to suspension, the image of $H^*(X^n)$, is defined by the diagram of cofibrations below:

$$\begin{array}{ccccccc} F & \longrightarrow & \bullet & \longrightarrow & \Sigma F & \longrightarrow & \Sigma F \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ E/\emptyset & \xrightarrow{\text{diag}} & E/\emptyset \wedge B & \longrightarrow & X^1 & \longrightarrow & \Sigma E/\emptyset \end{array}$$

In our case $E = K_2$ and $B = K_p$.

This works as well when replacing all spaces Z in this diagram by the corresponding mapping spaces $\text{map}(BV, Z)$ ([3]).

In order to prove 3.3 one computes the associated graded object of $f(\sqrt{H^*(F)})$ with respect to the Eilenberg-Moore filtration. Part (a) of the proposition follows directly. For part (b) one needs to control extensions.

Proposition 5.1. *The Eilenberg-Moore filtration on $H^*(F)$ induces a filtration on the functor $f(H^*F) \cong f(\sqrt{H^*F})$. The graded functor associated to this filtration is a (graded) subquotient of the graded functor:*

$$\left\{ T_V \left(\text{Tor}_{H^*(B)}^{-s}(H^*(E), \mathbb{F}_2) \right)^s \right\}_{s \geq 0} .$$

(b) *In our cases the spectral sequence degenerates at E^2 . Thus the associated graded functor is equal to the preceding one and isomorphic to:*

$$\left\{ (\mathbb{F}_2^{\Gamma^2(V\#)} \otimes_{\mathbb{F}_2^{\Gamma^p(V\#)}} \mathbb{F}_2) \otimes \Lambda^s(\Gamma^{p-1}) \right\}_{s \geq 0} .$$

This isomorphism respect the multiplicative structure. In terms of unstable modules this corresponds to:

$$\sqrt{H^*(X)/(\psi^*(\iota_p))} \otimes \Lambda^*(F(p-1))$$

We come back to the case of a decomposable map $\psi : K_2 \rightarrow K_p$. One needs to compute: $T_V(\text{Tor}_{\mathbb{H}^*(K_p)}^{-s}(\mathbb{H}^*(K_2), \mathbb{F}_2))^s$. Recall that: $\text{Tor}_{\mathbb{H}^*(K_p)}^{-s}(\mathbb{H}^*(K_2), \mathbb{F}_2)$ is the $-s$ -th cohomology of the reduced bar complex

$$\dots \rightarrow \mathbb{H}^*(K_2) \otimes \tilde{\mathbb{H}}^*(K_p)^{\otimes s} \rightarrow \mathbb{H}^*(K_2) \otimes \tilde{\mathbb{H}}^*(K_p)^{\otimes s-1} \rightarrow \dots \rightarrow \mathbb{H}^*(K_2) \quad (5.1)$$

As the functor T_V is exact and commutes with tensor products, applying it to the sequence 5.1 yields a complex whose $-s$ -th cohomology is

$$\text{Tor}_{T_V(\mathbb{H}^*(K_p))}^{-s}(T_V(\mathbb{H}^*(K_2)), \mathbb{F}_2)$$

(see[20]).

This complex simplifies because $\mathbb{H}^*(K_p)$ is an unstable algebra (see the proof of 6.4.3 in [20]). Recall formulae (2.1) and (2.2):

$$T_V(F(p)) \cong \bigoplus_{0 \leq i \leq p} \Gamma^i(V) \otimes F(p-i)$$

and as $\mathbb{H}^*(K_p) \cong U(F(p))$ one gets:

$$T_V(\mathbb{H}^*(K_p)) \cong \bigotimes_{0 \leq i \leq p} \mathbb{H}^*(K(\Gamma^i(V^\#), p-i))$$

The Boolean of maps from a set X to \mathbb{F}_2 is denoted by \mathbb{F}_2^X , $\overline{\mathbb{F}_2^X}$ denotes the ideal of maps taking the value 0 at a chosen base point; in our case as the sets are vector spaces one chooses 0 as base point.

The evaluation map at the base point $\mathbb{F}_2^X \rightarrow \mathbb{F}_2$ makes \mathbb{F}_2 a projective module over \mathbb{F}_2^X . As $T_V(\mathbb{H}^*(K_p))^0 \cong \mathbb{F}_2^{\Gamma^p(V^\#)}$, it makes

$$L_p = \bigotimes_{1 \leq i \leq p} \mathbb{H}^*(K(p-i, \Gamma^i(V^\#)))$$

a projective module over

$$T_V(\mathbb{H}^*(K_p)) \cong \bigotimes_{0 \leq i \leq p} \mathbb{H}^*(K(\Gamma^i(V^\#), p-i)) \quad .$$

Thus:

Lemma 5.2. *The reduced bar resolution of \mathbb{F}_2 over L_p is also a projective resolution of \mathbb{F}_2 over $T_V(\mathbb{H}^*(K_p))$.*

Define the unstable algebra L_2 by $L_2 = T_V(\mathbb{H}^*(K_2)) \otimes_{\mathbb{F}_2^{\Gamma^p(V^\#)}} \mathbb{F}_2$. It follows from the preceding lemma that $T_V(\text{Tor}_{\mathbb{H}^*(K_p)}^{-s}(\mathbb{H}^*(K_2), \mathbb{F}_2))$ is also the $-s$ -th homology of the reduced bar complex:

$$\dots \rightarrow L_2 \otimes \bar{L}_p^{\otimes s} \rightarrow L_2 \otimes \bar{L}_p^{\otimes s-1} \rightarrow \dots \rightarrow L_2 \rightarrow 0 \quad (5.2)$$

In degree 0 this is :

$$\dots \rightarrow 0 \rightarrow L_2^0 \cong \mathbb{F}_2^{\Gamma^2(V^\#)} \otimes_{\mathbb{F}_2^{\Gamma^p(V^\#)}} \mathbb{F}_2 \rightarrow 0$$

One has

$$L_2 \cong L_2^0 \otimes H^*(K_2) \otimes H^*(K(V^\#, 1)) \quad ,$$

thus

$$\begin{aligned} L_2^1 &\cong V^\# \otimes L_2^0, \\ L_p^1 &\cong \Gamma^{p-1}(V^\#), \\ L_p^2 &\cong S^2(\Gamma^{p-1}(V^\#)), \end{aligned}$$

in degree s , place $-s$, Complex (5.2) becomes:

$$\begin{array}{c} 0 \\ \downarrow \\ L_2^0 \otimes \Gamma^{p-1}(V^\#)^{\otimes s} \\ \downarrow \\ V^\# \otimes L_2^0 \otimes \Gamma^{p-1}(V^\#)^{\otimes s-1} \oplus L_2^0 \otimes \left(\bigoplus_i \Gamma^{p-1}(V^\#)^{\otimes i} \otimes S^2(\Gamma^{p-1}(V^\#)) \otimes \Gamma^{p-1}(V^\#)^{\otimes s-3-i} \right) \end{array}$$

We will show that the morphism is trivial for $s = 1$, the map is the one of the module structure. We will show below that the composite:

$$L_{L_2} \otimes \Gamma^{p-1}(V^\#) \hookrightarrow L_2^0 \otimes L_p^1 \otimes V^\# \rightarrow L_2^0 \otimes V^\#$$

takes values into $L_{L_2}^0 \otimes V^\#$. We will show also it factors through nontrivial the tensor products. As there are no nontrivial morphism from a nontrivial tensor product to Γ^1 (see in [6], "Algèbre de Steenrod, modules instables et foncteurs polynomiaux") the morphism will be shown to be trivial because of the module structure.

It follows, using the multiplicative structure on the bar resolution, that as an algebra, we have a natural isomorphism in V :

$$\left\{ T_V(\mathrm{Tor}_{H^*(K_p)}^{-s}(H^*(K_2), \mathbb{F}_2))^s \right\}_{s \geq 0} \cong \left\{ L_2^0 \otimes \Lambda^s(\Gamma^{p-1}) \right\}_{s \geq 0} \quad .$$

This identifies with the E^∞ -term on the diagonal $p+q = 0$ and therefore is the associated graded of $f(\sqrt{H^*(E)})$, 3.3 (a) follows. Let us make this a bit more precise: applying the functor T_V to the Eilenberg-Moore spectral sequence yields the Eilenberg-Moore spectral sequence for the mapping spaces as observed in [3].

In order to complete the proof one needs to describe how to compute the induced morphism $T_V(\psi)$ and gets some informations about the module

structure of $T_V(\mathbb{H}^*(K_2))$ over $T_V(\mathbb{H}^*(K_p))$. We will make this a bit longer than necessary for future use.

We will show two facts:

- the first one is that $T_V(\psi)$ writes as $T_V(\psi)^0 \otimes T_V(\psi)^{>0}$,
- the second one is that $T_V(\psi)^1 : \Gamma^{p-1}(V^\#) \otimes F(1)^1 \rightarrow V^\# \otimes F(1)^1$ under the hypothesis ψ is decomposable, factors through (a direct sum of) nontrivial tensor products $A \otimes B \otimes F(1)^1$. As there are no nontrivial morphisms ([6]) from a nontrivial tensor product $A \otimes B$ to the identity functor we are done.

In the discussion below, we check the first affirmation in all cases: stable classes, sums of classes, products of classes. This is done by inspection. For the second fact one just needs to look at the argument about product of classes.

In degree 0 the morphism is induced by composition of maps: ψ induces a set map, natural in V , $S^2(V^\#) \rightarrow S^p(V^\#)$. To a (set) map $a : S^p(V^\#) \rightarrow \mathbb{F}_2$ one associates $a \circ \psi$.

We discuss now the behavior in non zero degrees.

If ψ is a stable class, it is induced by a morphism $F(p) \rightarrow F(2)$ which we will denote also by ψ , (*i.e.* equivalently $\psi = \theta(\iota_2)$ where θ is a Steenrod operation). The induced morphism $\mathbb{H}^*(K_p) \rightarrow \mathbb{H}^*(K_2)$ is obtained by applying the Steenrod-Epstein functor U to ψ .

The morphism $T_V(\psi) : T_V(\mathbb{H}^*(K_p)) \rightarrow T_V(\mathbb{H}^*(K_2))$ is obtained by applying U to:

$$T_V(F(p)) \cong \bigoplus_{0 \leq i \leq p} \Gamma^i(V) \otimes F(p-i) \rightarrow T_V(F(2)) \cong \bigoplus_{0 \leq j \leq 2} \Gamma^j(V) \otimes F(2-j)$$

As there are no nontrivial morphisms to (resp. from) $F(0)$ from (resp. to) $F(n)$, $n > 0$, this morphism splits up as a direct sum $T_V(\psi)^0 \oplus T_V(\psi)^{>0}$.

Next, as there are no nontrivial morphism from $F(k)$ to $F(\ell)$ if $k < \ell$ the morphism $T_V(\psi)^{>0}$ has a triangular matrix in the following sense: the morphisms from $\Gamma^i(V) \otimes F(p-i)$ to $\Gamma^j(V) \otimes F(2-j)$ can be nontrivial only $p-i \geq 2-j$.

The following steps are to describe the situation on a sums and products of classes.

Let $\psi \in \mathbb{H}^p(K_2)$, suppose that $\psi = \alpha + \beta$, $\alpha \in \mathbb{H}^p(K_2)$, $\beta \in \mathbb{H}^p(K_2)$. The morphism

$$T_V(\psi) : T_V(\mathbb{H}^*(K_p)) \rightarrow T_V(\mathbb{H}^*(K_2))$$

is the composite:

$$T_V(\mathbb{H}^*(K_p)) \rightarrow T_V(\mathbb{H}^*(K_p)) \otimes T_V(\mathbb{H}^*(K_p)) \xrightarrow{T_V(\alpha \otimes T_V(\beta))} \dots$$

$$\dots \xrightarrow{T_V(\alpha) \otimes T_V(\beta)} T_V(\mathbf{H}^*(K_2) \otimes T_V(\mathbf{H}^*(K_2))) \xrightarrow{\mu} T_V(\mathbf{H}^*(K_2))$$

where $T_V(\mathbf{H}^*(K_p)) \rightarrow T_V(\mathbf{H}^*(K_p)) \otimes T_V(\mathbf{H}^*(K_p))$ is obtained by applying $T_V \circ U$ to $F(p) \xrightarrow{diag} F(p) \oplus F(p)$.

The key step is the one of a product of classes. Let $\psi \in \mathbf{H}^p(K_2)$, suppose that $\psi = \alpha\beta$, $\alpha \in \mathbf{H}^u(K_2)$, $\beta \in \mathbf{H}^v(K_2)$, $u, v > 0$. The morphism

$$T_V(\psi) : T_V(\mathbf{H}^*(K_p)) \rightarrow T_V(\mathbf{H}^*(K_2))$$

is the composite:

$$T_V(\mathbf{H}^*(K_p)) \rightarrow T_V(\mathbf{H}^*(K_u)) \otimes T_V(\mathbf{H}^*(K_v)) \xrightarrow{T_V(\alpha) \otimes T_V(\beta)} T_V(\mathbf{H}^*(K_2)) \otimes T_V(\mathbf{H}^*(K_2)) \xrightarrow{\mu} T_V(\mathbf{H}^*(K_2))$$

where $T_V(\mathbf{H}^*(K_p)) \rightarrow T_V(\mathbf{H}^*(K_u)) \otimes T_V(\mathbf{H}^*(K_v))$ is obtained, using the universal property of U , from $F(p) \rightarrow F(u) \otimes F(v) \hookrightarrow U(F(u)) \otimes U(F(v))$.

Finally checking each step one gets:

Proposition 5.3. *For any class ψ the morphism $T_V(\psi)$ is the tensor product of $T_V(\psi)^0$ and of $T_V(\psi)^{>0}$*

Corollary 5.4. *The morphism $T_V(\psi)^1$ identifies is the tensor product of $T_V(\psi)^0$ and $\Gamma^{p-1}(V^\#) \otimes F(1) \rightarrow V^\# \otimes F(1)$.*

Consider now the induced morphism on $\Gamma^{p-1}(V^\#) \otimes F(1)^1 \rightarrow V^\# \otimes F(1)^1$. By construction it factors as:

$$\begin{array}{c} \Gamma^{p-1}(V^\#) \otimes F(1)^1 \\ \downarrow \\ \Gamma^u(V^\#) \otimes F(1)^1 \otimes \Gamma^{v-1}(V^\#) \otimes F(1)^0 \oplus \Gamma^u(V^\#) \otimes F(1)^0 \otimes \Gamma^{v-1}(V^\#) \otimes F(1)^1 \\ \downarrow \\ V^\# \otimes F(1)^1 \end{array}$$

with $u + v = p$. As $u, v > 1$ in our cases the second technical result follows.

This proves part (a) of the conjecture.

In order to prove the strong form in 3.3 it is enough to show that the -1 step of the filtration of $f(\mathbf{H}^*(E))$, F_{-1} , is isomorphic to

$$\tilde{L}_2^0 \otimes \Gamma^{p-1}(V^\#) \oplus \Gamma^{p-1}(V^\#) \oplus L_2^0$$

Indeed, if so the direct summand $\Gamma^{p-1}(V^\#)$ generates multiplicatively an exterior algebra $\Lambda^*(\Gamma^{p-1}(V^\#))$, and thus the graded associated to $\sqrt{\mathbf{H}^*(E)}$

contains a subfactor $\Lambda^*(F(p-1))$ generated on the -1 -column of the spectral sequence. The result follows.

Saying it differently it means that a certain power $\iota_{p-1}^{2^k}$ is not only in the image of $H^*(E) \rightarrow H^*(K_{p-1})$, but that there is a lifting x such that the map

$$\frac{\mathcal{A}_2 \cdot x}{Nil} \rightarrow \mathcal{A}_2 \cdot \iota_{p-1}^{2^k} \subset F(p-1)$$

is an isomorphism.

Example 5.5. *Such a lifting may be very hard to find. In the case of the class $\psi = \iota_2 Q_1(\iota_2)$, the class $Sq^4 \iota_5 \otimes 1 + Sq^2 \iota_5 \otimes \iota_2 + \iota_5 \otimes \iota_2^2$ is a candidate.*

One knows there is a short exact sequence:

$$0 \rightarrow L_2^0 \rightarrow F_{-1} \rightarrow L_2^0 \otimes \Gamma^{p-1}(V^\#) \cong \Gamma^{p-1}(V^\#) \oplus \overline{L_2^0} \otimes \Gamma^{p-1}(V^\#) \rightarrow 0$$

and one shows that it splits.

Because of the multiplicative structure it is enough to show that,

$$\text{Ext}_{\mathcal{F}}^1(\Gamma^{p-1}, L_2^0) = 0 \tag{5.3}$$

As is shown below this is true in all the cases we consider, however it is hopeless to expect such a condition to be true in full generality. We will show that the corresponding extension group is non trivial if $p-1$ is odd in the case of some fiber E of a map from K_3 to K_{p-1} .

There is an isomorphism ([5] théorème 12.13 page 95):

$$\text{Ext}_{\mathcal{F}}^1(\Gamma^{p-1}, L_2^0) \cong (\ell^1 \circ m(L_2^0))^{p-1}$$

m is the right adjoint of f , $\ell = m \circ f$, $\ell = m \circ f$, ℓ is left exact and ℓ^1 is the first right derived functor.

The work of G. Powell [16] shows all the of L_2^0 in our list have a finite filtration whose quotients are in the list below. Therefore it is enough to check (5.3) for the functors:

- the constant functor \mathbb{F}_2 ,
- $I : V \mapsto \mathbb{F}_2^{V^\#}$;
- $I_2 : V \mapsto (\mathbb{F}_2^{V^\#})^{\otimes 2}$;
- their direct summands;
- $B_2 : V \mapsto ((\mathbb{F}_2^{V^\#})^{\otimes 2})^{\mathbb{Z}/2}$, note that $B_2 \cong S^2(I)$;
- the kernel \bar{B}_2 of the epimorphism $B_2 \rightarrow I$;
- $D_2 : V \mapsto ((\mathbb{F}_2^{V^\#})^{\otimes 2})^{\text{GL}_2(\mathbb{F}_2)}$;

- the kernel \bar{D}_2 of the epimorphism $D_2 \rightarrow I$.

The first three ones are injective.

For the two following cases, one has $m(B_2) \cong \mathbb{F}_2[w_1, w_2] \cong \mathbb{F}_2[u, v]^{\mathbb{Z}/2}$. The unstable module $\mathbb{F}_2[w_1, w_2]$ has an injective resolution which starts like:

$$\mathbb{F}_2[w_1, w_2] \hookrightarrow \mathbb{F}_2[u, v] \xrightarrow{1+\tau^*} \mathbb{F}_2[u, v] \xrightarrow{1+\tau^*} \mathbb{F}_2[u, v] \oplus \mathbb{F}_2[u] \rightarrow \dots$$

τ is the transposition. In order to prove this one uses the exact sequence of functors:

$$0 \rightarrow \Gamma^2 \rightarrow (\Gamma^1)^{\otimes 2} \rightarrow (\Gamma^1)^{\otimes 2} \rightarrow S^2$$

applied to $\mathbb{F}_2[u] \cong H^*(V_1)$, with $\mathbb{F}_2[u, v] \cong H^*(V_1)^{\otimes 2}$ and $\mathbb{F}_2[w_1, w_2] \cong \Gamma^2(H^*(V_1))$. The morphism $(\Gamma^1)^{\otimes 2} \rightarrow (\Gamma^1)^{\otimes 2}$ being the norm map.

From the definition of ℓ^1 it follows that $\ell^1(m(B_2)) = 0$ and the result follows for B_2 . As B_2 is the direct sum of \bar{B}_2 and I result holds for \bar{B}_2 .

This gives us the proof of Propositions 1.6 and 1.8.

For the two last cases one needs to study D_2 and \bar{D}_2 . It is done as follows, the subfunctor

$$A_2 : V \mapsto ((\mathbb{F}_2^{V^\#})^{\otimes 2})^{\mathbb{Z}/3}$$

is a direct summand in $\subset (\mathbb{F}_2^{V^\#})^{\otimes 2} \cong \mathbb{F}_2^{\text{Hom}(V, \mathbb{F}_2^{\oplus 2})}$. The action of $\mathbb{Z}/3$ is *via* the inclusion into $\text{GL}_2(\mathbb{F}_2)$. It is known ([17]) to be inserted in a short exact sequence:

$$E \hookrightarrow A_2 \twoheadrightarrow I$$

where

$$\bar{D}_2 \hookrightarrow E \twoheadrightarrow \bar{D}_2$$

and one has also

$$D_2 \hookrightarrow A_2 \twoheadrightarrow \bar{D}_2 \quad .$$

in terms of unstable modules one gets corresponding sequences applying the functor m, \cdot . As m they may not be exact on the right, by inspection it is not in the first case and it is in the the two other cases. In the first case :

$$m(E) \hookrightarrow \mathbb{F}_2[u, v]^{\mathbb{Z}/3} \twoheadrightarrow \mathbb{F}_2[u]$$

the image on the right being $\mathbb{F}_2[u^2]$, and

$$\bar{D}(2) \hookrightarrow m(E) \twoheadrightarrow \bar{D}(2) \quad ,$$

and

$$D(2) \hookrightarrow \mathbb{F}_2[u, v]^{\mathbb{Z}/3} \twoheadrightarrow \bar{D}(2) \quad ,$$

$\bar{D}(2)$ being the kernel of the unique nontrivial morphism $D(2) \rightarrow \mathbb{F}_2[u]$. It follows that:

$$\ell^1(m(E)) \cong \mathbb{F}_2[u]/\mathbb{F}_2[u^2] \quad ,$$

and that:

$$\ell^1(m(\bar{D}_2)) \cong \mathbb{F}_2[u]/\mathbb{F}_2[u^2]$$

In our examples we are first concerned with $D(2)$. In this case we can exploit the short exact sequence: which, as $\bar{D}(2)$ is *Nil*-closed, yields immediately $\ell^1 \circ m(D(2)) = 0$. This is 1.7. There remains case 1.9, keeping the notations one has a short exact sequence of unstable algebras:

$$M_2 \hookrightarrow D(2) \oplus H_2 \twoheadrightarrow \mathbb{F}_2[u^2]$$

As $\ell^1(D(2)) = \ell^1(H_2) = 0$ it implies that:

$$\text{Ext}_{\mathcal{F}}^1(\Gamma^{p-1}, \bar{D}_2) \cong \ell^1 \circ m(\bar{D}(2))^{p-1} \cong \mathbb{F}_2[u]/\mathbb{F}_2[u^2]$$

and the strong form of the conjecture is true if $p - 1$ is odd.

Remark 5.6. *As \bar{D}_2 is not a quotient of $H^*(K_2)$, the bottom class in degree 3, it does not appear (alone) in our list. However it is a quotient of $H^*(K_3)$, thus in such a case the proof of the strong form of the conjecture would not follow because*

$$\text{Ext}_{\mathcal{F}}^1(\Gamma^{p-1}, \bar{D}_2) \cong \ell^1 \circ m(\bar{D}(2))^{p-1} \cong \mathbb{F}_2[u]/\mathbb{F}_2[u^2]$$

is trivial only if $p - 1$ is odd.

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