Bounding basis risk using s-convex orders on Beta-unimodal distributions

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Abstract

This paper is concerned with properties of Beta-unimodal distributions and their use to assess the basis risk inherent to index-based insurance or reinsurance contracts. To this extent, we first characterize s-convex stochastic orders for Beta-unimodal distributions in terms of the Weyl fractional integral. We then determine s-convex extrema for such distributions, focusing in particular on the cases $s = 2, 3, 4$. Next, we define an Enterprise Risk Management framework that relies on Beta-unimodality to assess these hedge imperfections, introducing several penalty functions and worst case scenarios. Some of the results obtained are illustrated numerically via a representative catastrophe model.

Keywords: Risk management; Parametric index; Basis risk; Beta-unimodality; s-convex stochastic orders; s-convex extrema.

1 Introduction

In catastrophe modeling applied to insurance-linked securities, so-called Beta-unimodal distributions naturally appear in the assessment of basis risks since destruction rates are often represented by Beta random variables. This paper provides bounds of convex-type for Beta-unimodal distributions and aims to build a framework for measuring basis risk which involves penalty functions of different forms. Before recalling Beta-unimodal distributions, let us explain how they appear in catastrophe modeling and how they can be useful for basis risk measurement in index-based securitization mechanisms.

Thanks to reinsurance or risk transfers, insurers can cede parts of their risk portfolios to other parties through some form of agreement to reduce the likelihood of paying a large obligation resulting from an insurance claim. Insurers need to hedge and reinsurance is a well-known risk management tool. We refer the reader to Cole and McCullough [16], Garven and Lamm-Tennant [36] and Gron [39], for example, and to the books by Carter [11] for a general presentation and by Albrecher et al. [1] for the actuarial aspects.

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Alternative risk transfers (ART) correspond to the part of the insurance market that allows companies to purchase coverage and transfer risks without having to resort to traditional commercial insurance. The ART market includes risk retention groups, insurance pools, captive insurers and a number of insurance products issued on the capital market such as contingent capital, derivatives and insurance-linked securities (ILS). In this paper we are particularly concerned with a specific class of ILS, namely the parametric transactions. We refer to Cummins and Weiss [22] and Cummins and Barrieu [21], for example, and to the books by Banks [4] and Culp [20] for a clear presentation of the ART and by Barrieu and Albertini [5] for a complete focus on ILS.

The securitization of insurance risk is today deeply rooted as an alternative transfer of risks; see e.g. Golden et al. [37], Cummins and Weiss [22] and Cummins and Barrieu [21]. In fact, the capital market provides additional capacity for insurance risk, many products offer multi-year coverages, and trading of insurance risk on the secondary market creates price transparency. Moreover, capital market transactions reduce the credit risk of the cedent vis-à-vis reinsurers, since the proceeds of the transaction remain in the appropriate vehicle and are invested as collateral. Within the ILS class, the catastrophe bond has been in the spotlight since Hurricane Andrew in 1992.

In fact, the insurance industry was pushed to the limits of its solvency in 1992. Given the infrequent nature of large-scale catastrophic (cat) events, actuarial and statistical methods do not provide a complete picture. As a result, the disaster modeling industry has grown rapidly, making cat models key input for measuring, pricing and managing cat risks. A cat model is generally broken down into four components, namely hazard, inventory, vulnerability and loss.

The hazard component consists of a set of stochastic events, i.e. a large deterministic number N of catastrophic event scenarios which together provide a representation of the events that can cause losses, and an associated modeled occurrence rate for each.

The inventory component represents the exposure. It consists of the portfolio of sites subject to catastrophic risk. The items in the inventory are classified in terms of aspects that affect the amount of damage suffered by a structure for a given level of hazard. In the paper, we assume that the exposure in each scenario is not perfectly known, in agreement with several works by Ewing [33], Pinelli et al. [59], Hamid et al. [41] and Grossi [40]. This may be due to the fact that the trajectory of the event is uncertain (spatial uncertainty) or because the buyer of the protection does not know its exposure when the event occurs: for example, information about the property is collected for a typical sampled study area and the results are extrapolated to the regional level. Moreover, the sites subject to cat risks generated in each scenario may differ. So, let Y_i be a positive random variable representing the exposure of the scenario $i = 1, \ldots, N$. This variable is obviously bounded by the value of the entire portfolio, i.e. the loss cannot be greater than the total portfolio $m > 0$, say. The hazard component generating localized events, the portfolio is not (necessarily) fully exposed in each scenario. Consequently, we will assume that for each scenario i, there is an interval (a_i, b_i) $(0 \leq a_i < b_i \leq m)$ for which $a_i \leq Y_i \leq b_i$ holds almost surely.

The vulnerability component assesses the degree to which structures, their contents and other insured property are likely to be damaged by the hazard. In other words, this component provides the intensity exerted by the cat event on the portfolio under exposure. A common way to represent the intensity of the event is to use the damage ratio, also called destruction ratio or proportion of loss, and which is represented by a random variable with Beta distribution. We refer e.g. to Sampson et al. [62], Walker [66], Cossette et al. [18], Dutang et al. [31] and to the books by Gorge [38] and Charpentier [12]. Thus, the destruction ratio for each scenario i is a variable S_i whose distribution is Beta (α_i, β_i). In what follows, the effective loss in scenario i will then be a bounded positive random variable X_i defined as $X_i = S_i Y_i$.

The loss component translates the expected physical damage into monetary loss taking into account any insurance structures. Within the loss component, the insurance structures can be of different forms depending on the type of trigger. Three main types of trigger can be distinguished, namely indemnity, industry loss and parametric. In indemnity transactions, modeling is based on the insurance loss itself. Industry loss based structures are essentially a pool-indemnity solution. A simple form is the industry loss warranties whereby the trigger is the total industry loss in a particular region. A parametric transaction uses the measured physical properties of a cat event as the trigger. It is typically based on an index of the event hazard: a payment to the protection buyer depends on the values taken by an index built from the physical parameters of a natural disaster (such as wind speed, precipitation level, earthquake magnitude). As with any non-indemnity structure, a parametric index contains the probability that the payment under the structure does not match the loss experience. In the present paper, this parametric risk is called basis risk.

Basis risk is a well-established concept in insurance and finance. This corresponds to the difference in payment between own losses incurred and a structured risk transfer mechanism to protect against these losses (Ross and Williams [61]). In finance, basis risk is often borne by banks managing deposits payable on demand and short-term interest rate risk, or by financial institutions using cross-hedging techniques. In life insurance, the payoffs of most mortality bonds are based on national populations mortality evolution, leaving the cedent (insurer or reinsurer) with basis risk also. Insurance and financial firms are not the only ones facing basis risk. The recent development of parametric insurance, particularly in emerging countries and in agriculture insurance, represents a great hope for the rise of a more inclusive insurance, provided that the basis risk to which the policyholder is exposed remains under control.

As announced above, in our cat modeling framework, the loss X_i for scenario i is represented as the product $X_i = S_i Y_i$ of a random variable Y_i with values in a bounded positive inteval (a_i, b_i) by a random destruction ratio S_i with a Beta $(\alpha_i,\ \beta_i)$ distribution independent of Y_i. This independence assumption between the destruction rate and the exposure seems to be reasonable for any given scenario. Note however that when considering all scenarios, there can exist some dependence since the parameters of the Beta distribution are not always identical. For example, certain scenarios j corresponding to an extreme storm would probably imply a more significant exposure as well as a higher destruction rate. This would lead to using a severe distribution for Y_j and Beta parameters (α_j, β_j) which make S_j also important.

In probability terminology, X_i is then said to have a Beta-unimodal distribution. The relation $X_i = S_i Y_i$ corresponds to a Beta random scaling. Such relations with a scaling of arbitrary distribution have been proposed to describe the effects of various factors, including an economic environment (Galambos and Simonelli [35], Tang and Tsitsiashvili [65], Asimit et al. [2]), a systemic background risk (Côté and Genest [19]) or a dependence structure of claim sizes (Hashorva [42], Hashorva and Ji [44]). Special attention has been paid to asymptotics of these relations; see e.g. Yang and Wang [69], Cline and Samorodnitsky [15], Hashorva et al. [46]. The case of Beta-unimodality considered here has been thoroughly investigated in the book by Bertin et al. [7]; see also Hashorva and Pakes [45], Hashorva [43], Pakes and Navarro [58]. This concept covers several forms of unimodality introduced in the literature: usual unimodal distributions (Khintchine [53]), β-monotone distributions (Williamson [67]; see also McNeil and Nešlehová [55], Lefèvre and Loisel [54] in applied probability) and α-unimodal distributions (Olshen and Savage [57]; see Brockett et al. [9] in insurance).

The motivation of the present work is to propose a simple way to quantify the basis risk inherent to parametric risk transfers by adopting an Enterprise Risk Management (ERM) point of view. The ERM approach typically involves the measurement of hedge imperfections and the identification of worst case scenarios. In our method, we measure the consequences of these hedge imperfections by introducing penalty functions. In their paper, Brockett et al. [9] derived lower and upper bounds for expectations of the form $\mathbb{E}[\phi(X)]$ when X has an α-unimodal distribution and ϕ is a function of convex-type. In fact, this topic is important to our risk management problem. We want to pursue its study when X has now a Beta-unimodal distribution, using explicitly a class of convex type stochastic orders called s-convex. The theory of s-convex orders is developed in Denuit et al. [29, 30], and associated bounding problems were examined by e.g. De Vylder [23, 24], De Vylder and Goovaerts [25], Brockett and Cox [8], Hürlimann [47], Kaas and Goovaerts [49, 50].

Our main contribution is twofold. First, we characterize s-convex orderings and derive s-convex bounds for Beta-unimodal distributions. Second, we use these results and penalty functions to estimate the consequences of basis risk inherent to parametric transactions in an ERM framework. In particular, we will show that the knowledge of higher moments of the random exposure is really useful to reduce the uncertainty around basis risk. More precisely, the paper is organized as follows. In Section 2, we recall the concept of Beta-unimodality and give some key properties in terms of the Weyl fractional integral. In Section 3, we introduce the (known) class of s-convex stochastic orders and we use it to compare Beta-unimodal distributions. In Section 4, we continue the analysis by deriving s-convex extrema for Beta-unimodal distributions, with particular interest on the cases $s = 2, 3, 4$. In Section 5, we propose a framework that uses Beta-unimodality to assess the basis risk. We consider different penalty functions, symmetric or not, to measure the impact of the basis risk and to deduce the worst case scenarios. In Section 6, we numerically illustrate some of the results obtained via a simplified but representative catastrophe model.

2 Beta-unimodal distributions

We present in this section key definitions and properties on Beta-unimodal distributions. A detailed analysis with applications can be found in the book by Bertin et al. [7] and e.g. the papers by Pakes and Navarro [58], Hashorva [43], Hashorva and Pakes [45].

Definition 2.1. *An $\mathbb{R}_+$-valued random variable X has a continuous Beta-unimodal distribution if it has the product representation*

$$X =_d SY, \tag{2.1}$$

where Y is a positive continuous random variable, and S is a random contracting factor independent of Y and of Beta distribution.

The Beta density function of S is

$$f_S(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\mathcal{B}(\alpha,\beta)}, \quad 0 \leq x \leq 1, \tag{2.2}$$

where α and β are positive shape parameters ($\mathcal{B}(\alpha,\beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$). For clarity, it is convenient to set $S \sim S(\alpha,\beta)$. If X is Beta-unimodal, then

$$\mathbb{E}[X^i] = \mathbb{E}[S^i]\mathbb{E}[Y^i] = \left(\prod_{j=0}^{i-1} \frac{\alpha+j}{\alpha+\beta+j}\right) \mathbb{E}[Y^i], \quad i \in \mathbb{N}^\star. \tag{2.3}$$

The relation (2.3) will play a key role in the sequel, in particular when it comes to building convex type bounds.

2.1 Particular unimodalities

The cases where $\alpha = 1$ and/or $\beta = 1$ are the subject of special attention. Let U denote a $[0,1]$-uniform random variable.

(1) If $\alpha = \beta = 1$, $f_S(x) = 1$, i.e., $S(1,1) =_d U$. Then, (2.1) becomes

$$X =_d UY, \tag{2.4}$$

and by a famous theorem of Khintchine [53], X has a unimodal distribution in the usual sense (with a mode at 0).

(2) If $\alpha = 1$, $f_S(x) = \beta(1-x)^{\beta-1}$, i.e., $S(1,\beta) =_d (1-U^{1/\beta})$. Then, (2.1) becomes

$$X =_d (1-U^{1/\beta})Y, \tag{2.5}$$

and X is often said to have a β-monotone distribution. Multiple monotonicity of functions was analyzed in details by Williamson [67] when β is any real ≥ 1. It has various fields of application, especially when β is a positive integer n. In statistics, the estimation problem of n-monotone densities was examined, e.g., by Balabdaoui and Wellner [3]. As shown by McNeil and Nešlehová [55], the generator of a n-dimensional Archimedean copula generator is a n-monotone fonction. Applications to ruin problems were discussed by Constantinescu et al. [17]. Similarly, Chi et al. [13] pointed out that the same monotonicity holds for the generator of a n-dimensional Schur-constant vector. Lefèvre and Loisel [54] studied properties and stochastic orderings for n-monotone densities, with illustrations in insurance.

(3) If $\beta = 1$, $f_S(x) = \alpha x^{\alpha-1}$, i.e., $S(\alpha, 1) =_d U^{1/\alpha}$. Then, (2.1) becomes

$$X =_d U^{1/\alpha} Y, \tag{2.6}$$

and X is generally said to have an α-unimodal distribution. This concept of generalized monotonicity was introduced and investigated by Olshen and Savage [57]. Recently, Brockett et al. [9] deal with bounding problems on the expectation of such distributions.

2.2 Link between X and Y

Beta-unimodality allows us to obtain information on X or Y depending on the initial information. As shown e.g. in Pakes and Navarro [58], the two survival functions $\bar{F}_X$ of X and $\bar{F}_Y$ of Y are closely related. The result is worth proving again.

Proposition 2.1. *If X is Beta-unimodal with $X =_d SY$, then $\bar{F}_X$ is given by*

$$\bar{F}_X(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} x^\alpha (I_\beta \phi_Y)(x), \tag{2.7}$$

where the function ϕ_Y depends on $\bar{F}_Y$ by

$$\phi_Y(t) = \bar{F}_Y(t) t^{-\alpha-\beta},$$

and the function $I_\beta \phi_Y$ is defined as

$$(I_\beta \phi_Y)(x) = \frac{1}{\Gamma(\beta)} \int_x^\infty (t-x)^{\beta-1} \phi_Y(t) dt, \tag{2.8}$$

i.e., it corresponds to the Weyl fractional integral of ϕ_Y.

Proof. Since X is Beta-unimodal, we get

$$\begin{aligned} \mathbb{P}(X > x) &= \mathbb{P}(YS > x) = \int_0^1 \mathbb{P}(sY > x) f_S(s) ds \\ &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} x^\alpha \frac{1}{\Gamma(\beta)} \int_x^\infty (t-x)^{\beta-1} \bar{F}_Y(t) t^{-\alpha-\beta} dt, \end{aligned} \tag{2.9}$$

using the definition (2.2) of f_S, which is (2.7) in the notation of the statement. □

For the Weyl integral calculus, the reader is referred e.g. to the books by Miller and Ross [56], Debnath and Bhatta [26]. A converse to (2.7) holds too.

Proposition 2.2. *If X is Beta-unimodal with $X =_d SY$, then $\bar{F}_Y$ is given by*

$$\bar{F}_Y(x) = (-1)^n x^{\alpha+\beta} \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)} (I_\delta D^n \psi_X)(x), \tag{2.10}$$

where the function ψ_X depends on $\bar{F}_X$ by

$$\psi_X(t) = \bar{F}_X(t) t^{-\alpha},$$

and D^n is the derivative of order n which is the smallest integer $\geq \beta$, and $\delta = n - \beta \in [0, 1)$.

Proof. From (2.7), we have

$$(I_\beta \phi_Y)(x) = \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)} \psi_X(x), \tag{2.11}$$

where ψ_X is defined above. As annouced, let $\delta \in [0,1)$ be such that $\beta + \delta = n \in \mathbb{N}$. An important property of the fractional integral operator is that

$$I_\delta I_\beta = I_{\delta+\beta} = I_n.$$

Thus, applying I_δ to both sides of (2.11) yields

$$(I_n \phi_Y)(x) = \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)} (I_\delta \psi_X)(x). \tag{2.12}$$

Another well-known property of the integral operator is that

$$D^k I_n = (-1)^k I_{n-k},\ 1 \leq k \leq n, \quad \text{and} \quad D^n(I_\delta \psi_X) = I_\delta D^n \psi_X.$$

Taking the derivative D^n in (2.12) then gives

$$(-1)^n \phi_Y(x) = \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)} (I_\delta D^n \psi_X)(x).$$

It remains to insert the definition of ϕ_Y, and we deduce the formula (2.10). □

Such a link between X and Y can be interesting in practice. Indeed, it provides us with the distribution of X when X is known to be a Beta version of a given risk, and vice versa.

3 Orderings of convex-type

A main objective of our work is to determine bounds on the expectation $\mathbb{E}[\phi(X)]$ for a class of functions of convex-type and when X has a Beta-unimodal distribution. This subject is treated by Brockett et al. [9] in the particular case where X is α-unimodal. The authors there apply a method based on the Markov-Krein theorem and Chebychev systems (see Brockett and Cox [8] and, for a general theory, Karlin and Studden [51]). Their method, strong enough, requires the function ϕ to be differentiable, which can be restrictive for various applications (see later). When this assumption does not hold, they follow an approach based on the results of Kemperman [52] for the geometry of the moment problem.

The method we propose below is developed in the context of the s-convex extrema discussed in Denuit et al. [29, 30]. An advantage is that the function ϕ does not have to be differentiable to construct bounds on $\mathbb{E}[\phi(X)]$. Basic reminders and their applications are presented in Section 3.1 and 3.2, respectively.

3.1 s-convex orders

Let us start by giving the definition of an s-convex function (s being a positive integer). We use for that a strong concept that does not rely on differentiability (see e.g. Popoviciu [60], Farwig and Zwick [34], Karlin and Studden [51]).

Consider a compact interval $[a, b]$ with $-\infty < a < b < \infty$. Let $[x_0, \dots, x_s; \phi]$ be the forward divided difference of the function ϕ on the nodes $x_0, \dots, x_s \in [a, b]$. Specifically, $[x_k; \phi] = \phi(x_k)$, $k = 0, \dots, s$, and by recursion,

$$[x_0, \dots, x_k; \phi] = \frac{[x_1, \dots, x_k; \phi] - [x_0, \dots, x_{k-1}; \phi]}{x_k - x_0}, \quad k = 1, \dots, s.$$

This operator is a standard mathematical tool in numerical analysis. It can be expressed as a linear combination of $\phi(x_0), \dots, \phi(x_k)$ by

$$[x_0, \dots, x_k; \phi] = \sum_{j=0}^{k} \frac{\phi(x_j)}{(x_j - x_0) \dots (x_j - x_{j-1})(x_j - x_{j+1}) \dots (x_j - x_k)}.$$

Definition 3.1. *A real function ϕ defined on $[a, b]$ is s-convex if*

$$[x_0, ..., x_s; \phi] \geq 0,$$

holds for any choice of distinct points $x_0, ..., x_s$ in $[a, b]$.

Nevertheless, certain properties of differentiability are verified by s-convex functions ϕ on $[a, b]$. So, $\phi^{(s-2)}$, $s \geq 2$, exists and is continuous, $\phi_+^{(s-1)}$ exists, is right-continuous and is increasing. If $\phi^{(s)}$ exists, then ϕ is s-convex if and only if $\phi^{(s)} \geq 0$.

We can now give the definition of the s-convex orders (see e.g. Denuit et al. [29, 30], Shaked and Shanthikumar [63]). Let $\mathcal{U}_{s-cx}^{[a,b]}$ be the class of all s-convex real functions ϕ on $[a, b]$.

Definition 3.2. *Let Y_1 and Y_2 be two random variables valued in $[a, b]$. Then, Y_1 is smaller than Y_2 in the s-convex order ($Y_1 \leq_{s-cx}^{[a,b]} Y_2$) if*

$$\mathbb{E}[\phi(Y_1)] \leq \mathbb{E}[\phi(Y_2)] \quad \textit{for all functions } \phi \in \mathcal{U}_{s-cx}^{[a,b]}. \tag{3.1}$$

Note that the 1-convex is the classic order in distribution ($\leq_d$) and the 2-convex order is the usual convex order ($\leq_{cx}$). Since a polynomial of degree $s - 1$ (or less) is an s-convex function, the s-convex order (3.1) implies that

$$\mathbb{E}[Y_1^i] = \mathbb{E}[Y_2^i], \quad 1 \leq i \leq s - 1. \tag{3.2}$$

Thus, only the variables that share the same first $s-1$ moments can be compared in the s-convex sense. That property must be completed to provide a characterization of the s-convex order. This can be done using the stop-loss transforms of order $s - 1$ (for Y: $\mathbb{E}[(Y - t)_+^{s-1}]$, $t \in [a, b]$) or the iterated survival functions of order $s - 1$ (for Y: $\bar{F}_Y^{[s-1]}(x)$ with $\bar{F}_Y^{[0]}(x) = \bar{F}_Y(x)$ and for $k \geq 1$, $\bar{F}_Y^{[k]}(x) = \int_x^\infty \bar{F}_Y^{[k-1]}(t)dt$, $x \in \mathbb{R}$).

Characterization 3.1. *$Y_1 \leq_{s-cx}^{[a,b]} Y_2$ when (3.2) holds and*

$$\mathbb{E}[(Y_1 - t)_+^{s-1}] \leq \mathbb{E}[(Y_2 - t)_+^{s-1}], \quad t \in [a,b], \tag{3.3}$$

or, equivalently,

$$\bar{F}_{Y_1}^{[s-1]}(x) \leq \bar{F}_{Y_2}^{[s-1]}(x), \quad x \in \mathbb{R}. \tag{3.4}$$

3.2 Under Beta-unimodality

Let us go back to our framework of Beta-unimodal random variables $X =_d SY$ with Y valued in $[a,b] \in \mathbb{R}_+$. A natural question is to know under which conditions a stochastic order on Y can be transferred on X. The following result will often be used later.

Proposition 3.1. *Let $X_1 =_d SY_1$ and $X_2 =_d SY_2$ be Beta-unimodal with the same S independent of (Y_1, Y_2). If $Y_1 \leq_{s-cx}^{[a,b]} Y_2$, then $X_1 \leq_{s-cx}^{[0,b]} X_2$.*

Proof. From (3.2), (3.3), it is clear that if $Y_1 \leq_{s-cx}^{[a,b]} Y_2$, then $xY_1 \leq_{s-cx}^{x[a,b]} xY_2$ for any $x \in [0,1]$, where

$$x[a,b] = \left\{ u \in \mathbb{R} \middle| \frac{u}{x} \in [a,b] \right\} = [xa, xb].$$

For $\phi \in \mathcal{U}_{s-cx}^{[0,b]}$, then

$$\begin{aligned} \mathbb{E}[\phi(X_1)] &= \int_0^1 \mathbb{E}[\phi(xY_1)|S = x] dF_S(x) \\ &\leq \int_0^1 \mathbb{E}[\phi(xY_2)|S = x] dF_S(x) \;=\; \mathbb{E}[\phi(X_2)], \end{aligned} \tag{3.5}$$

as desired. □

A similar result is valid when the scaling factors S_1, S_2 are identically distributed.

Proposition 3.2. *Let $X_1 =_d S_1Y_1$ and $X_2 =_d S_2Y_2$ be Beta-unimodal with S_1, S_2 identically distributed and independent of (Y_1, Y_2). If $Y_1 \leq_{s-cx}^{[a,b]} Y_2$, then $X_1 \leq_{s-cx}^{[0,b]} X_2$.*

Proof. This can be shown using a simple coupling argument. Instead, we proceed directly and exploit the assumptions made on S_1, S_2 to write

$$\begin{aligned} \mathbb{E}[\phi(X_1)] &= \int_0^1 \mathbb{E}[\phi(xY_1)|S_1 = x] dF_{S_1}(x) \;=\; \int_0^1 \mathbb{E}[\phi(xY_1)] dF_{S_1}(x) \\ &= \int_0^1 \mathbb{E}[\phi(xY_1)] dF_{S_2}(x) \\ &\leq \int_0^1 \mathbb{E}[\phi(xY_2)] dF_{S_2}(x) \;=\; \mathbb{E}[\phi(X_2)], \end{aligned}$$

the inequality following from $Y_1 \leq_{s-cx}^{[a,b]} Y_2$. This yields the announced result. □

Note that the results of Propositions 3.1 and 3.2 do not use that the scaling factors are of Beta distribution. In fact, they remain true for any scaling distribution. Suppose now that Y_1, Y_2 are not ordered and, for instance, valued in $\mathbb{R}_+$. Then, X_1, X_2 can be compared in a convex sense through the characterization 3.1.

Proposition 3.3. *Let $X_1 =_d S_1Y_1$ and $X_2 =_d S_2Y_2$ be Beta-unimodal with S_1, S_2 independent of (Y_1, Y_2). Then, $X_1 \le_{s-cx}^{[0,\infty)} X_2$ when (3.2) holds with X_1 and X_2 and, for $x \ge 0$,*

$$\frac{1}{\mathcal{B}(\alpha_1,\beta_1)}\int_x^\infty \bar{F}_{Y_1}(t)t^{s-2}[(I_{s-1}\chi_{S_1})(x/t)]dt \le \frac{1}{\mathcal{B}(\alpha_2,\beta_2)}\int_x^\infty \bar{F}_{Y_2}(t)t^{s-2}[(I_{s-1}\chi_{S_2})(x/t)]dt, \quad (3.6)$$

where $I_{s-1}(\chi_{S_i})$ is the Weyl fractional integral (2.8) of the function

$$\chi_{S_i}(t) = t^{\alpha_i}(1-t)^{\beta_i-1}, \quad i = 1, 2.$$

Proof. We will require that the condition (3.4) be satisfied. For this, we must explicitly express the iterated survival function $\bar{F}_X^{[s]}(x)$, $x \ge 0$. The proof is based on two reasoning by induction.

First, we show that for $x \ge 0$,

$$\bar{F}_X^{[s]}(x) = \frac{\alpha}{\alpha+\beta}\int_x^\infty \bar{F}_Y(t)\, t^{s-1}\bar{F}_{\hat{S}}^{[s-1]}(x/t)dt, \quad (3.7)$$

where $\hat{S} \sim S(\alpha+1,\beta)$ is a Beta variable of parameters $(\alpha+1,\beta)$.

For $s = 1$, we have from (2.9)

$$\begin{aligned}
\bar{F}_X^{[1]}(x) &= \int_{t=x}^\infty P(X>t)dt \;=\; \frac{1}{\mathcal{B}(\alpha,\beta)}\int_{t=x}^\infty\int_{u=t}^\infty t^\alpha(u-t)^{\beta-1}\bar{F}_Y(u)u^{-(\alpha+\beta)}dudt \\
&= \frac{1}{\mathcal{B}(\alpha,\beta)}\int_{u=x}^\infty \bar{F}_Y(u)u^{-(\alpha+\beta)}\int_{t=x}^u t^\alpha(u-t)^{\beta-1}dtdu \\
&= \frac{1}{\mathcal{B}(\alpha,\beta)}\int_{u=x}^\infty \bar{F}_Y(u)u^{-(\alpha+1)}\int_{t=x}^u t^\alpha(1-t/u)^{\beta-1}dtdu \\
&= \frac{1}{\mathcal{B}(\alpha,\beta)}\int_{u=x}^\infty \bar{F}_Y(u)\int_{y=x/u}^1 y^\alpha(1-y)^{\beta-1}dydu \\
&= \frac{\alpha}{\alpha+\beta}\int_{u=x}^\infty \bar{F}_Y(u)\bar{F}_{\hat{S}}(x/u)du,
\end{aligned}$$

using the definition above of $\hat{S}$. This corresponds to the relation (3.7) for $s = 1$. Assuming that (3.7) holds for the k-th iterated, we obtain

$$\begin{aligned}
\bar{F}_X^{[k+1]}(x) &= \int_{y=x}^\infty \bar{F}_X^{[k]}(y)dy \;=\; \frac{\alpha}{\alpha+\beta}\int_{y=x}^\infty\int_{t=y}^\infty \bar{F}_Y(t)t^{k-1}\bar{F}_{\hat{S}}^{[k-1]}(y/t)dtdy \\
&= \frac{\alpha}{\alpha+\beta}\int_{t=x}^\infty \bar{F}_Y(t)t^{k-1}\int_{y=x}^t \bar{F}_{\hat{S}}^{[k-1]}(y/t)dydt
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\alpha}{\alpha+\beta} \int_{t=x}^{\infty} \bar{F}_Y(t) t^k \int_{u=x/t}^{1} \bar{F}_{\hat{S}}^{[k-1]}(u) du dt \\
&= \frac{\alpha}{\alpha+\beta} \int_{t=x}^{\infty} \bar{F}_Y(t)\, t^{k-1} \bar{F}_{\hat{S}}^{[k]}(x/t) dt,
\end{aligned}
$$

which gives (3.7) for the $k+1$-th iterated as desired.

Second, we show that for $x \in [0,1]$,

$$
\bar{F}_{\hat{S}}^{[s]}(x) = \frac{1}{\mathcal{B}(\alpha+1,\beta)} \frac{x^{\alpha+1+s}}{s!} \int_x^1 t^{-(\alpha+\beta+1+s)} (t-x)^{\beta-1} (1-t)^s dt. \tag{3.8}
$$

For $s = 1$, we have from (2.2)

$$
\begin{aligned}
\bar{F}_S^{[1]}(x) &= \int_x^1 \mathbb{P}(S > y) dy \;=\; \frac{1}{\mathcal{B}(\alpha+1,\beta)} \int_{y=x}^{1} \int_{u=y}^{1} u^{\alpha}(1-u)^{\beta-1} du dy \\
&= \frac{1}{\mathcal{B}(\alpha+1,\beta)} \int_{u=x}^{1} u^{\alpha}(1-u)^{\beta-1}(u-x) du \\
&= \frac{1}{\mathcal{B}(\alpha+1,\beta)} \int_{u=x}^{1} u^{\alpha+1}(1-u)^{\beta-1}(1-x/u) du \\
&= \frac{1}{\mathcal{B}(\alpha+1,\beta)} x^{\alpha+2} \int_x^1 t^{-(\alpha+\beta+2)}(t-x)^{\beta-1}(1-t) dt,
\end{aligned}
$$

which is in accordance with (3.8). Assuming (3.8) for the k-th iterated, we get

$$
\begin{aligned}
\bar{F}_S^{[k+1]}(x) &= \frac{1}{\mathcal{B}(\alpha+1,\beta)} \frac{1}{k!} \int_{y=x}^{1} y^{\alpha+1+k} \int_{t=y}^{1} t^{-(\alpha+\beta+1+k)}(t-y)^{\beta-1}(1-t)^k dt dy \\
&= \frac{1}{\mathcal{B}(\alpha+1,\beta)} \frac{1}{k!} \int_{t=x}^{1} t^{-(\alpha+\beta+1+k)}(1-t)^s \int_{y=x}^{t} y^{\alpha+1+s}(t-y)^{\beta-1} dy dt \\
&= \frac{1}{\mathcal{B}(\alpha+1,\beta)} \frac{1}{k!} \int_{t=x}^{1} t^{-(\alpha+2+k)}(1-t)^k \int_{y=x}^{t} y^{\alpha+1+k}(1-y/t)^{\beta-1} dy dt \\
&= \frac{1}{\mathcal{B}(\alpha+1,\beta)} \frac{1}{k!} \int_{t=x}^{1} (1-t)^k \int_{u=x/t}^{1} u^{\alpha+1+k}(1-u)^{\beta-1} du dt \\
&= \frac{1}{\mathcal{B}(\alpha+1,\beta)} \frac{1}{k!} \int_{u=x}^{1} u^{\alpha+1+k}(1-u)^{\beta-1} \int_{t=x/u}^{1} (1-t)^s dt du \\
&= \frac{1}{\mathcal{B}(\alpha+1,\beta)} \frac{1}{(k+1)!} \int_x^1 u^{\alpha+1+k}(1-u)^{\beta-1}(1-x/u)^{k+1} du,
\end{aligned}
$$

which becomes (3.8) for $s = k+1$ after the change of variable $x/u = t$.

It now suffices to insert (3.8) in (3.7) and we find that

$$
\bar{F}_X^{[s]}(x) = \frac{1}{\mathcal{B}(\alpha,\beta)} \frac{1}{(s-1)!} \int_{t=x}^{\infty} \bar{F}_Y(t)\, t^{s-1} (x/t)^{\alpha+s}
$$

$$\int_{y=x/t}^{1} y^{-(\alpha+\beta+s)}(y-x/t)^{\beta-1}(1-y)^{s-1}dydt$$

$$
\begin{aligned}
&= \frac{1}{\mathcal{B}(\alpha,\beta)}\frac{1}{(s-1)!}\int_{t=x}^{\infty}\bar{F}_Y(t)\,t^{s-1}\int_{u=x/t}^{1}u^{\alpha+s-1}(1-x/tu)^{s-1}(1-u)^{\beta-1}dudt \\
&= \frac{1}{\mathcal{B}(\alpha,\beta)}\frac{1}{(s-1)!}\int_{t=x}^{\infty}\bar{F}_Y(t)\,t^{s-1}\int_{u=x/t}^{1}u^{\alpha}(1-u)^{\beta-1}(u-x/t)^{s-1}dudt \\
&= \frac{1}{\mathcal{B}(\alpha,\beta)}\frac{1}{(s-1)!}\int_{x}^{\infty}\bar{F}_Y(t)\,t^{s-1}[(I_s\chi_S)(x/t)]dt, \qquad (3.9)
\end{aligned}
$$

using the above notation for s. The condition (3.6) then follows from (3.4), (3.9) with $\bar{F}_X^{[s-1]}$. $\square$

4 Bounds of convex-type

Having introduced the class of s-convex stochastic orders, we can now focus on the construction of lower or upper s-convex bounds for a given set of risks. Remember that by virtue of (3.2), an s-convex ordering is only possible between random variables which possess the same first $s-1$ moments. Therefore, we consider the space $\mathcal{B}_s([a,b],\mu_1,\mu_2,...,\mu_{s-1})$ of all the variables valued in $[a,b]$ and with the $s-1$ fixed moments $\mu_i = \mathbb{E}[Y^i]$, $1 \le i \le s-1$.

Of course, for this space to be non-empty, the μ_i must satisfy some constraints between them. This question is discussed e.g. in Denuit [28]. We only mention here that the constraints for the first three are

$$
\begin{aligned}
a < \mu_1 < b,& \\
\mu_1^2 < \mu_2 < \mu_1(a+b)-ab,& \qquad (4.1)\\
\frac{(\mu_2-\mu_1 a)^2 + a\mu_2(\mu_1-a)}{\mu_1-a} < \mu_3 < \frac{b\mu_2(b-\mu_1)-(b\mu_1-\mu_2)^2}{b-\mu_1}.&
\end{aligned}
$$

4.1 s-convex extrema

The problem of bounding quantities of the form $\mathbb{E}[\phi(Y)]$ when $Y \in \mathcal{B}_s([a,b],\mu_1,\mu_2,\dots,\mu_{s-1})$ has been widely studied in the literature. Hereafter, we follow Denuit et al. [30] and define the random variables $Y_{min}^{(s)}$ and $Y_{max}^{(s)}$ such that for all random variables $Y \in \mathcal{B}_s([a,b],\mu_1,\mu_2,\dots,\mu_{s-1})$,

$$Y_{min}^{(s)} \le_{s-cx}^{[a,b]} Y \le_{s-cx}^{[a,b]} Y_{max}^{(s)},$$

meaning that for all real functions $\phi \in \mathcal{U}_{s-cx}^{[a,b]}$,

$$\mathbb{E}[\phi(Y_{min}^{(s)})] \le \mathbb{E}[\phi(Y)] \le \mathbb{E}[\phi(Y_{max}^{(s)})].$$

As pointed out earlier, ϕ does not need to be differentiable as is supposed in the recent paper of Brockett and Cox [8]. Other references on this subject are e.g. De Vylder [23], De Vylder

[24], De Vylder and Goovaerts [25], Hürlimann [47], Kaas and Goovaerts [49, 50]. We recall the result proved by Denuit et al. [30] which distinguishes the even ($2s$) and odd ($2s+1$) degrees of convex ordering. For $j \in \mathbb{N}$, $x, y \in \mathbb{R}$, define the reals $\mu_0 = 0$ and

$$\mu_{j,x} = \mu_j - x\mu_{j-1} \text{ and } \mu_{j,xy} = \mu_{j,x} - y\mu_{j-1,x}.$$

Proposition 4.1. *([30]) In $\mathcal{B}_{2s}([a,b], \mu_1, \mu_2, \ldots, \mu_{2s-1})$, $Y_{min}^{(2s)}$ has s support points $\theta_1 < \theta_2 < \ldots < \theta_s$ inside (a,b) which are the roots of the equation*

$$\begin{vmatrix} 1 & x & x^2 & \cdots & x^s \\ \mu_0 & \mu_1 & \mu_2 & \cdots & \mu_s \\ \mu_1 & \mu_2 & \mu_3 & \cdots & \mu_{s+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_{s-1} & \mu_s & \mu_{s+1} & \cdots & \mu_{2s-1} \end{vmatrix} = 0,$$

with positive masses $q_1, q_2, \ldots, q_s$ which are the solution of the Vandermonde system

$$q_1\theta_1^k + q_2\theta_2^k + \cdots + q_s\theta_s^k = \mu_k, \quad k = 0, 1, \ldots, s-1,$$

while $Y_{max}^{(2s)}$ has $s+1$ support points $a < \theta_1 < \theta_2 < \ldots < \theta_{s-1} < b$, the θ_i being the roots of the equation

$$\begin{vmatrix} 1 & x & x^2 & \cdots & x^{s-1} \\ \mu_{2,ab} & \mu_{3,ab} & \mu_{4,ab} & \cdots & \mu_{s+1,ab} \\ \mu_{3,ab} & \mu_{4,ab} & \mu_{5,ab} & \cdots & \mu_{s+2,ab} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_{s,ab} & \mu_{s+1,ab} & \mu_{s+2,ab} & \cdots & \mu_{2s-1,ab} \end{vmatrix} = 0,$$

with positive masses $p_a, q_1, q_2 \ldots, q_{s-1}, p_b$ which are the solution of the Vandermonde system

$$p_a a^k + q_1\theta_1^k + q_2\theta_2^k + \cdots + q_{s-1}\theta_{s-1}^k + p_b b^k = \mu_k, \quad k = 0, 1, \ldots, s.$$

In $\mathcal{B}_{2s+1}([a,b], \mu_1, \mu_2, \ldots, \mu_{2s})$, $Y_{min}^{(2s+1)}$ has $s+1$ support points $a < \theta_1 < \theta_2 < \ldots < \theta_s$, the θ_i being the roots of the equation

$$\begin{vmatrix} 1 & x & x^2 & \cdots & x^s \\ \mu_{1,a} & \mu_{2,a} & \mu_{3,a} & \cdots & \mu_{s+1,a} \\ \mu_{2,a} & \mu_{3,a} & \mu_{4,a} & \cdots & \mu_{s+2,a} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_{s,a} & \mu_{s+1,a} & \mu_{s+2,a} & \cdots & \mu_{2s,a} \end{vmatrix} = 0,$$

with positive masses $p_a, q_1, q_2, \ldots, q_s$ which are the solution of the Vandermonde system

$$p_a a^k + q_1\theta_1^k + q_2\theta_2^k + \cdots + q_{s-1}\theta_{s-1}^k + q_s\theta_s^k = \mu_k, \quad k = 0, 1, \ldots, s,$$

while $Y_{max}^{(2s+1)}$ *has* $s+1$ *support points* $\theta_1 < \theta_2 < ... < \theta_s < b$*, the* θ_i *being the roots of the equation*

$$\begin{vmatrix} 1 & x & x^2 & \cdots & x^s \\ \mu_{1,b} & \mu_{2,b} & \mu_{3,b} & \cdots & \mu_{s+1,b} \\ \mu_{2,b} & \mu_{3,b} & \mu_{4,b} & \cdots & \mu_{s+2,b} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_{s,b} & \mu_{s+1,b} & \mu_{s+2,b} & \cdots & \mu_{2s,b} \end{vmatrix} = 0,$$

with positive masses $q_1, q_2, \ldots, q_s, p_b$ *which are the solution of the Vandermonde system*

$$q_1\theta_1^k + q_2\theta_2^k + \cdots + q_{s-1}\theta_{s-1}^k + q_s\theta_s^k + p_b b^k = \mu_k, \quad k = 0, 1, \ldots, s.$$

4.2 Comparison of extrema

Once the bounds are obtained for a given s, a natural question is the order of the bounds themselves. As an additional moment reduces the space of moments and improves knowledge on the random variable, one might expect that the s-convex bound is sharper than the $(s-1)$-convex in the $(s-1)$ -convex sense.

Obviously, $\mathcal{B}_s([a,b], \mu_1, \mu_2, ..., \mu_{s-1}) \subset \mathcal{B}_{s-1}([a,b], \mu_1, \mu_2, ..., \mu_{s-2})$ given the same μ_i, $1 \le i \le s-2$. As $Y_{min}^{(s)}, Y_{max}^{(s)} \in \mathcal{B}_s([a,b], \mu_1, \mu_2, ..., \mu_{s-1})$, we thus have

$$Y_{min}^{(s-1)} \preceq_{(s-1)-cx}^{[a,b]} Y_{min}^{(s)}, \quad \text{and} \quad Y_{max}^{(s)} \preceq_{(s-1)-cx}^{[a,b]} Y_{max}^{(s-1)}.$$

However, we will show with a counterexample that the implication $Y \preceq_{s-cx}^{[a,b]} Y_{max}^{(s)} \Rightarrow Y \preceq_{(s-1)-cx}^{[a,b]} Y_{max}^{(s)}$ does not always hold.

Example. Let $Y =_d U$, a $[0,1]$-uniform random variable. Thus, $\mu_1 = 1/2$ and $\mu_2 = 1/3$ (which satisfy the condition of (4.1)). Consider the function $\phi : x \in [0,1] \to \phi(x) = \exp(-\lambda x)$ where $\lambda > 0$. Then, $\phi^{(2)}(x) > 0$ and $\phi^{(3)}(x) < 0$, i.e. ϕ is 2-convex, but not 3-convex.

We look at the convex upper bound, for instance. From the extreme distributions obtained by Proposition 4.1, we directly obtain for $s = 2$ and $s = 3$

$$Y_{max}^{(2)} = \begin{cases} 0 & \text{with probability} \quad \dfrac{1}{2}, \\ 1 & \text{with probability} \quad \dfrac{1}{2}, \end{cases} \quad \textit{and} \quad Y_{max}^{(3)} = \begin{cases} \dfrac{1}{3} & \text{with probability} \quad \dfrac{3}{4}, \\ 1 & \text{with probability} \quad \dfrac{1}{4}. \end{cases}$$

Figure 4.1 gives the differences $\mathbb{E}[\phi(Y_{max}^{(2)})] - \mathbb{E}[\phi(Y)]$ (solid red curve) and $\mathbb{E}[\phi(Y_{max}^{(3)})] - \mathbb{E}[\phi(Y)]$ (dashed black curve) as a function of $\lambda > 0$. We see that $\mathbb{E}[\phi(Y_{max}^{(2)})] \ge \mathbb{E}[\phi(Y)]$ while $\mathbb{E}[\phi(Y)] \ge \mathbb{E}[\phi(Y_{max}^{(3)})]$. The first inequality was expected since the function ϕ is 2-convex. The second inequality means that the 3-convex maximum is a lower, and not upper, 2-convex bound.

When does knowing more moments than the degree of convexity allow for sharper bounds? Suppose that λ may be negative. Figure 4.2 gives the same differences as Figure 4.1 but this

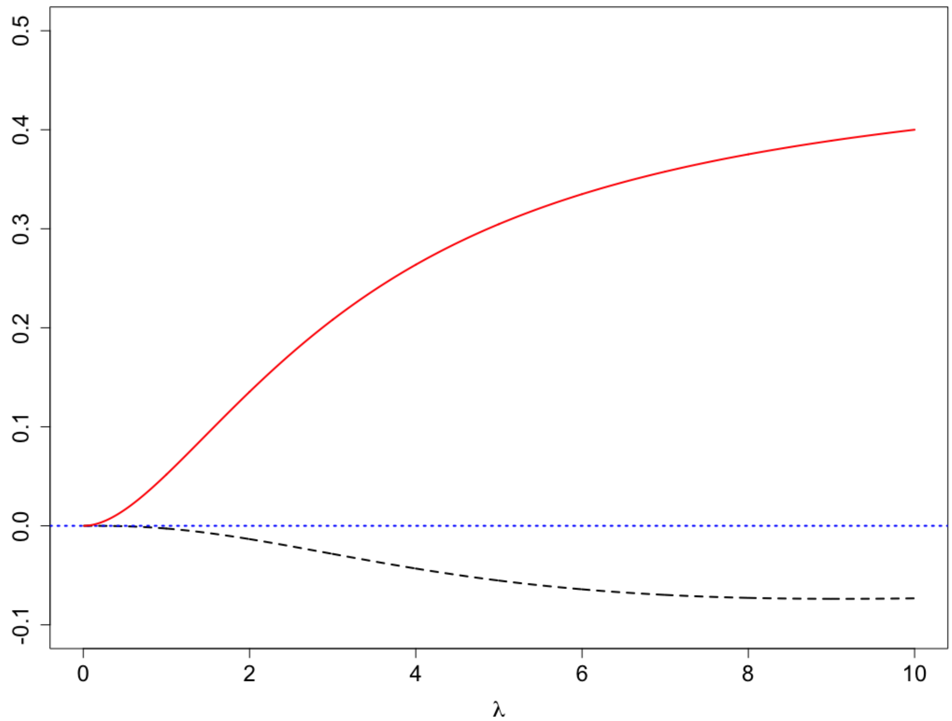

Figure 4.1: Differences $\mathbb{E}[\phi(Y_{max}^{(2)})]-\mathbb{E}[\phi(Y)]$ (solid red curve) and $\mathbb{E}[\phi(Y_{max}^{(3)})]-\mathbb{E}[\phi(Y)]$ (dashed black curve) when $\phi(x)=\exp(-\lambda x)$, $x\in[0,1]$, and with $\lambda>0$.

time with $\lambda \in \mathbb{R}$. When $\lambda < 0$, we observe that both curves of differences are positive. Besides, as the dashed black curve is always under the solid red one, we deduce that $\mathbb{E}[\phi(Y_{max}^{(2)})] \geq \mathbb{E}[\phi(Y_{max}^{(3)})] \geq \mathbb{E}[\phi(Y)]$. Thus, knowing the moment μ_2 implies a decrease of the upper bound. This is not surprising since the function ϕ is both 2 and 3-convex when $\lambda < 0$.

In general, the convex bounds are ordered as follows: for $k \leq s$,

$$Y_{min}^{(k)} \leq_{k-cx}^{[a,b]} \cdots \leq_{(s-2)-cx}^{[a,b]} Y_{min}^{(s-1)} \leq_{(s-1)-cx}^{[a,b]} Y_{min}^{(s)} \leq_{s-cx}^{[a,b]} Y,$$
$$Y \leq_{s-cx}^{[a,b]} Y_{max}^{(s)} \leq_{(s-1)-cx}^{[a,b]} Y_{max}^{(s-1)} \leq_{(s-2)-cx}^{[a,b]} \cdots \leq_{k-cx}^{[a,b]} Y_{max}^{(k)}.$$

Since each ordering is for a different convexity degree, the extremal distributions cannot be compared with the same function ϕ. However, when the function ϕ is both $(s-1)$ and s-convex, then $Y \leq_{(s-1)-cx}^{[a,b]} Y_{max}^{(s)} \leq_{(s-1)-cx}^{[a,b]} Y_{max}^{(s-1)}$, i.e. knowing the $(s-1)$-th moment reduces the $(s-1)$-convex upper bound. This result can be formalized by using the notion of s-increasing convex order (see e.g. Shaked and Shanthikumar [63]). A standard example is with the moment generating function considered by e.g. Denuit [27] and Brockett and Cox [8], Brockett et al. [9]. This point is not detailed for the sake of brevity.

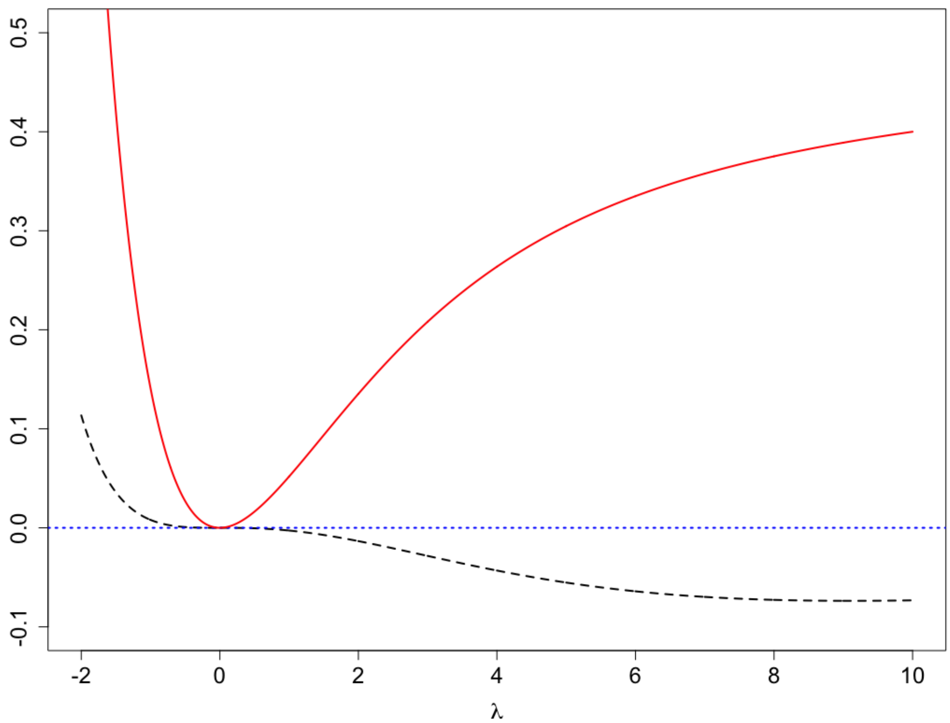

Figure 4.2: Differences $\mathbb{E}[\phi(Y_{max}^{(2)})] - \mathbb{E}[\phi(Y)]$ (solid red curve) and $\mathbb{E}[\phi(Y_{max}^{(3)})] - \mathbb{E}[\phi(Y)]$ (dashed black curve) when $\phi(x) = \exp(-\lambda x)$, $x \in [0, 1]$, and with $\lambda \in \mathbb{R}$.

4.3 Under Beta-unimodality

A simple way to construct convex extrema for Beta-unimodal variables is to apply the scaling factor S to the convex extrema of the starting variable.

Proposition 4.2. *If X is Beta-unimodal with $X =_d SY$, and $Y \in \mathcal{B}_s([a, b], \mu_1, \mu_2, \ldots, \mu_{s-1})$, then $X \in \mathcal{B}_s([0, b], \nu_1, \nu_2, \ldots, \nu_{s-1})$ where*

$$\nu_i = \left(\prod_{j=0}^{i-1} \frac{\alpha + j}{\alpha + \beta + j} \right) \mu_i, \quad 1 \leq i \leq s - 1, \tag{4.2}$$

and the s-convex extrema for X are

$$X_{min}^{(s)} =_d SY_{min}^{(s)}, \quad \text{and} \quad X_{max}^{(s)} =_d SY_{max}^{(s)}, \tag{4.3}$$

where $Y_{min}^{(s)}$ and $Y_{max}^{(s)}$ are provided by the extreme distributions given in Proposition 4.1.

Proof. The relation (4.2) between the moments $\mu_i = \mathbb{E}[Y^i]$ and $\nu_i = \mathbb{E}[X^i]$ comes from (2.3). The distributional identity (4.3) is a direct consequence of Proposition 3.1 and the one-to-one correspondence between X and Y for S given (see Section 2.2). $\square$

From Propositions 4.1 and 4.2, we obtain the s-convex extrema for Beta-unimodal random variables. This is done below for $s = 2, 3, 4$. Denote by $\mathcal{B}_s(\alpha, \beta; [0, b]; \nu_1, \ldots, \nu_{s-1})$ the space of all the random variables which are Beta-unimodal with scaling factor $S(\alpha, \beta)$, are valued in the interval $[0, b]$ and have the first $s-1$ fixed moments $\nu_i = \mathbb{E}[X^i]$.

Corollary 4.1. *Let* $Y \in \mathcal{B}_s([a, b], \mu_1, \ldots, \mu_{s-1})$.

In $\mathcal{B}_2(\alpha, \beta; [0, b]; \nu_1)$, $X_{min}^{(2)} \leq_{2-cx}^{[0,b]} X \leq_{2-cx}^{[0,b]} X_{max}^{(2)}$, *where*

$$X_{min}^{(2)} = S\mu_1,$$

$$X_{max}^{(2)} = \begin{cases} Sa & \text{with probability} \quad p_{max}^{(2)} = \dfrac{b-\mu_1}{b-a}, \\ Sb & \text{with probability} \quad q_{max}^{(2)} = \dfrac{\mu_1 - a}{b-a}. \end{cases}$$

In $\mathcal{B}_3(\alpha, \beta; [0, b]; \nu_1, \nu_2)$, $X_{min}^{(3)} \leq_{3-cx}^{[0,b]} X \leq_{3-cx}^{[0,b]} X_{max}^{(3)}$, *where*

$$X_{min}^{(3)} = \begin{cases} Sa & \text{with probability} \quad p_{min}^{(3)} = \dfrac{\mu_2 - \mu_1^2}{\mu_2 - \mu_1^2 + (\mu_1 - a)^2}, \\ S\dfrac{\mu_2 - a\mu_1}{\mu_1 - a} & \text{with probability} \quad q_{min}^{(3)} = \dfrac{(\mu_1 - a)^2}{\mu_2 - \mu_1^2 + (\mu_1 - a)^2}, \end{cases}$$

$$X_{max}^{(3)} = \begin{cases} S\dfrac{b\mu_1 - \mu_2}{b - \mu_1} & \text{with probability} \quad p_{max}^{(3)} = \dfrac{(b-\mu_1)^2}{\mu_2 - \mu_1^2 + (b-\mu_1)^2}, \\ Sb & \text{with probability} \quad q_{max}^{(3)} = \dfrac{\mu_2 - \mu_1^2}{\mu_2 - \mu_1^2 + (b-\mu_1)^2}. \end{cases}$$

In $\mathcal{B}_4(\alpha, \beta; [0, b]; \nu_1, \nu_2, \nu_3)$, $X_{min}^{(4)} \leq_{4-cx}^{[0,b]} X \leq_{4-cx}^{[0,b]} X_{max}^{(4)}$, *where*

$$X_{min}^{(4)} = \begin{cases} Sr_- & \text{with probability} \quad p_{min}^{(4)} = \dfrac{r_+ - \mu_1}{r_+ - r_-}, \\ Sr_+ & \text{with probability} \quad q_{min}^{(4)} = \dfrac{\mu_1 - r_-}{r_+ - r_-}, \end{cases}$$

where

$$\begin{cases} r_- = \dfrac{\mu_3 - \mu_1\mu_2 - \sqrt{(\mu_3 - \mu_1\mu_2)^2 - 4(\mu_2 - \mu_1)^2(\mu_1\mu_3 - \mu_2^2)}}{2(\mu_2 - \mu_1^2)}, \\ r_+ = \dfrac{\mu_3 - \mu_1\mu_2 + \sqrt{(\mu_3 - \mu_1\mu_2)^2 - 4(\mu_2 - \mu_1)^2(\mu_1\mu_3 - \mu_2^2)}}{2(\mu_2 - \mu_1^2)}, \end{cases}$$

and

$$X_{max}^{(4)} = \begin{cases} Sa & \text{with probability} \quad 1 - p_{max}^{(4)} - q_{max}^{(4)}, \\ Sx_{max}^{(4)} & \text{with probability} \quad p_{max}^{(4)}, \\ Sb & \text{with probability} \quad q_{max}^{(4)}, \end{cases}$$

where

$$\begin{cases} x_{max}^{(4)} = \dfrac{\mu_3 - (a+b)\mu_2 + ab\mu_1}{\mu_2 - (a+b)\mu_1 + ab}, \\ p_{max}^{(4)} = \dfrac{b\mu_1 - \mu_2 + a(\mu_1 - b)}{(x_{max}^{(4)} - a)(b - x_{max}^{(4)})}, \\ q_{max}^{(4)} = \dfrac{\mu_2 - x_{max}^{(4)}\mu_1 + a(x_{max}^{(4)} - \mu_1)}{(b-a)(b - x_{max}^{(4)})}. \end{cases}$$

Obviously, the extrema have been improved when we know that X is Beta-unimodal. If the extrema in $\mathcal{B}_s([0,b];\nu_1,\ldots,\nu_{s-1})$ without knowing the Beta-unimodality of X are denoted by $\tilde{X}_{min}^{(s)}, \tilde{X}_{max}^{(s)}$, then

$$\tilde{X}_{min}^{(s)} \leq_{s-cx}^{[0,b]} X_{min}^{(s)}, \quad \text{and} \quad X_{max}^{(s)} \leq_{s-cx}^{[0,b]} \tilde{X}_{max}^{(s)}.$$

5 Basis risk assessment

As explained in the Introduction, the objective of this article is to provide a means of quantifying the basis risk inherent in an index-based transaction. This type of transaction involves using a catastrophe modeling software to generate stochastic events. Let N be the number of stochastic events and p_i be the occurence rate of the scenario $i = 1,\ldots,N$. In each of them, the event trajectory is uncertain, which leads to an uncertain exposure (see e.g. Ewing [33], Pinelli et al. [59], Hamid et al. [41] or Grossi [40]). Therefore, the exposure is represented by a positive bounded random variable Y_i depending on the event i. This random variable is bounded, of course, by the value m (> 0) of the whole portfolio. Moreover, the hazard component generates localized events so that the portfolio is not necessarily fully exposed. Thus, we assume that each Y_i takes values almost surely in some interval (a_i, b_i) where $0 \leq a_i < b_i \leq m$.

Following a common practice (see e.g. Sampson et al. [62], Walker [66], Cossette et al. [18], Dutang et al. [31], Gorge [38], Charpentier [12]), the destruction ratio in scenario i is modeled by a random variable S_i with Beta distribution of parameters (α_i, β_i) and which is independent of the exposure Y_i. This assumption of independence is reasonable since the destruction ratio depends on the event rather than the exposure. Of course, the two variables S_i and Y_i depend on i, so that the parameters (α_i, β_i) can be adapted to have a relationship between the destruction ratio and the exposure level. The effective loss X_i on scenario i is then given by $X_i = S_i Y_i$.

In the paper, we focus on the so-called parametric or basis risk. For each scenario, the physical parameters corresponding to the reference index can be recovered. Thus, if the index in scenario i takes the value c_i, the basis risk in this scenario is $X_i - c_i$ with $X_i = S_i Y_i$. As the exposure Y_i takes values in (a_i, b_i) by hypothesis (and $S_i \in (0,1)$) by definition), we almost surely have

$$a_i - c_i \leq X_i - c_i \leq b_i - c_i.$$

The basis risk can therefore be two-sided in practice. When the reimbursement is greater than the actual loss, this means that additional protection has been purchased and an additional risk premium has been paid, which is usually suboptimal. When the refund is less than the actual loss, the problem is clearly more serious. To some extent, we thus expect the loss component to provide a means of measuring the impact of basis risk. The literature provides some key indicators for measuring basis risk such as the expected conditional shortfall or the probability of certain levels of shortfall, also known as false negative probability, expected non-negative shortfall given loss, probability of shortfall in excess of a threshold given non-zero loss. We refer e.g. to Brookes [10], Ross and Williams [61], Jensen et al. [48], Takahashi et al. [64] and Elabed et al. [32].

In the spirit of these measures, we introduce below different penalty functions which allow us to quantify the basis risk $X_i - c_i$ through measuring its effects. For this, we also assume that certain first moments of the loss $X_i = S_i Y_i$ are known or can be estimated. By bounding the expected value of the penalty functions, we are then able to determine the worst scenarios that actually correspond to the extrema s-convex extrema. More precisely, the global impact of the basis risks $X_i - c_i$ for the n scenarios is measured by the quantity

$$BR_\phi = \sum_{i=1}^{N} p_i \mathbb{E}[\phi(X_i - c_i)], \tag{5.1}$$

where p_i is the likelihood of scenario i and ϕ represents some specific penalty function. We use the s-convex extrema for each X_i, but for convenience we will limit ourselves to the cases where $s = 2, 3, 4$. These extrema allow us to derive lower and upper bounds for the measurement BR. From now on, we consider only one scenario for clarity; thus, the index i is superfluous and deleted.

5.1 Symmetric penalty function

We start by assuming a symmetric impact of the basis risk $X - c$. To this end, we use the absolute value as a penalty function g_1, i.e.

$$g_1(X - c) = |X - c|. \tag{5.2}$$

The function g_1 is 2-convex (only) on $\mathbb{R}$. Thus, if $\nu_1 = \mathbb{E}[X]$ is known, we can reason in the space $\mathcal{B}_2(\alpha, \beta; [0, m]; \nu_1)$ for which the 2-convex extrema are provided by Corollary 4.1. Set $J_{\alpha,\beta}(x) \equiv \mathbb{P}[S(\alpha, \beta) > x]$, the Beta survival function. After a simple calculation of integrals, we obtain the following bounds for $\mathbb{E}[g_1(X - c)]$.

Proposition 5.1. *In $\mathcal{B}_2(\alpha, \beta; [0, b]; \nu_1)$, the lower bound for $\mathbb{E}[|X - c|]$ is*

$$\mathbb{E}[|X^{(2)}_{min} - c|] = c[1 - 2J_{\alpha,\beta}(c/\mu_1)] - \nu_1[1 - 2J_{\alpha+1,\beta}(c/\mu_1)], \tag{5.3}$$

and the upper bound is

$$\begin{aligned} \mathbb{E}[|X^{(2)}_{max} - c|] = \; & p^{(2)}_{max}[c\left(1 - 2J_{\alpha,\beta}(c/a)\right) - \left(a\alpha/(\alpha + \beta)\right)\left(1 - 2J_{\alpha+1,\beta}(c/a)\right)] \\ & + q^{(2)}_{max}[c\left(1 - 2J_{\alpha,\beta}(c/b)\right) - \left(b\alpha/(\alpha + \beta)\right)\left(1 - 2J_{\alpha+1,\beta}(c/b)\right)]. \end{aligned} \tag{5.4}$$

Note that without the assumption of Beta-unimodality, i.e. in the space $\mathcal{B}_2([0,b];\nu_1)$, the bounds for $\mathbb{E}[|X-c|]$ are

$$\mathbb{E}[|\tilde{X}^{(2)}_{min}-c|] = |\nu_1 - c|, \quad \text{and} \quad \mathbb{E}[|\tilde{X}^{(2)}_{max}-c|] = c + \nu_1(b-2c)/b. \tag{5.5}$$

Example. Let $Y =_d U$, a $[0,b]$-uniform random variable, and choose the parameters $\alpha = 4, \beta = 2$, $b = 10$ and $c = 2$. Table 1 gives the values of $\mathbb{E}[|X-c|]$ and its bounds (5.3), (5.4), (5.5). As indicated before, the assumption of Beta-unimodality allows us to get sharper bounds. Incidentally, we note that $\mathbb{E}[|\tilde{X}^{(3)}_{min}-c|] = 2.3528$ and $\mathbb{E}[|\tilde{X}^{(3)}_{max}-c|] = 0.3999$, which again illustrates the discussion made in Section 4.2.

Bounds	Beta-unimodality	Values for $\mathbb{E}[g_1]$
$\mathbb{E}[|\tilde{X}^{(2)}_{min}-c|]$	No	1.3333
$\mathbb{E}[|X^{(2)}_{min}-c|]$	Yes	1.4084
$\mathbb{E}[|X-c|]$	-	2.0005
$\mathbb{E}[|X^{(2)}_{max}-c|]$	Yes	3.3361
$\mathbb{E}[|\tilde{X}^{(2)}_{max}-c|]$	No	4.0000

Table 1: The values of $\mathbb{E}[|X-c|]$ and its lower and upper bounds under Beta-unimodality or not, when Y is uniform on $[0,b]$, $b=10, c=2$ and $\alpha=4, \beta=2$.

Remark. Brockett et al. [9] derived 2-cx bounds on $\mathbb{E}[|X-c|]$ when X is α-unimodal, with Y takes values over $[0,b]$. This situation is a particular case of Beta-unimodality that corresponds to the case where $S \sim S(\alpha,1)$ (see Section 2.1). We are going to show that our bounds correct a (minor) error in their result.

Under α-unimodality (i.e., when $\beta = 1$), defining a function f given by

$$f(y) = \begin{cases} c - \dfrac{\alpha}{\alpha+1} y, & y \in [0,c], \\ -c + \dfrac{\alpha}{\alpha+1} y + 2\dfrac{c^{\alpha+1}}{(\alpha+1)y^\alpha}, & y \in [c,b]. \end{cases} \tag{5.6}$$

our bounds (5.3), (5.4) take the simple form

$$\mathbb{E}[|X^{(2)}_{min}-c|] = f(\nu_1(\alpha+1)/\alpha), \tag{5.7}$$
$$\mathbb{E}[|X^{(2)}_{max}-c|] = (1-\nu_1(\alpha+1)/\alpha b) f(0) + (\nu_1(\alpha+1)/\alpha b) f(b). \tag{5.8}$$

For their part, Brockett et al. [9] defined a function g by

$$g(y) = \frac{\alpha}{y^\alpha} \int_0^y t^{\alpha-1}|t-c|dt, \tag{5.9}$$

when, as here, the mode is 0 and $0 \le y \le b$, and proved that the 2-convex bounds are given by (5.7), (5.8) with g substituted for f. In addition, they claimed that the function (5.9) is equivalent to

$$g(y) = \begin{cases} c - \dfrac{\alpha}{\alpha+1} y, & y \in [0,c], \\ \dfrac{\alpha}{\alpha+1}(y-c) + \dfrac{c^{\alpha+1}}{(\alpha+1)y^{\alpha}}, & y \in [c,b], \end{cases} \tag{5.10}$$

and the expression (5.10) was used in their Theorem 4, pages 776-777. In fact, an integral calculation shows that, as axpected, g given by (5.9) is well identical to f given by (5.6). However, (5.9) differs from (5.10) when $y \in [c,b]$.

5.2 Piecewise linear penalty function

From (5.2), the function g_1 similarly accounts for overestimation and underestimation of the risk. This can be too simplistic because minimizing a risk is often very dangerous. To remedy it, a natural generalization of g_1 is to consider different slopes for the positive and negative differences of the basis risk $X - c$. Thus, introducing the functions

$$f_1(x) = \eta(c-x)_+, \quad \text{and} \quad f_2(x) = \gamma(x-c)_+, \quad x \in [0,b],$$

where $\eta, \gamma > 0$, we define a penalty function g_2 by

$$g_2(X-c) = f_1(X) + f_2(X). \tag{5.11}$$

As the functions f_1, f_2 are 2-convex on $\mathbb{R}$, we may argue exactly as for Proposition 5.1 to get bounds for $\mathbb{E}[f_1(X)], \mathbb{E}[f_2(X)]$. From (5.11), bounds for $\mathbb{E}[g_2(X-c)]$ directly follow.

Proposition 5.2. *In $\mathcal{B}_2(\alpha, \beta; [0,b]; \nu_1)$, the lower bounds for $\mathbb{E}[f_1(X)]$ and $\mathbb{E}[f_2(X)]$ are*

$$\mathbb{E}[f_1(X^{(2)}_{min})] = \eta[c\left(1 - J_{\alpha,\beta}(c/\mu_1)\right) - \nu_1\left(1 - J_{\alpha+1,\beta}(c/\mu_1)\right)],$$
$$\mathbb{E}[f_2(X^{(2)}_{min})] = \gamma[\nu_1 J_{\alpha+1,\beta}(c/\mu_1) - cJ_{\alpha,\beta}(c/\mu_1)],$$

and the upper bounds are

$$\begin{aligned} \mathbb{E}\{f_1(X^{(2)}_{max})] = \eta \Big\{ & p^{(2)}_{max}\left[c\left(1 - J_{\alpha,\beta}(c/a)\right) - (a\alpha/(\alpha+\beta))\left(1 - J_{\alpha+1,\beta}(c/a)\right)\right] \\ & + q^{(2)}_{max}\left[c\left(1 - J_{\alpha,\beta}(c/b)\right) - (b\alpha/(\alpha+\beta))\left(1 - J_{\alpha+1,\beta}(c/b)\right)\right]\Big\}, \\ \mathbb{E}[f_2(X^{(2)}_{max})] = \gamma \Big\{ & p^{(2)}_{max}\left[(a\alpha/(\alpha+\beta))J_{\alpha+1,\beta}(c/a) - cJ_{\alpha,\beta}(c/a)\right] \\ & + q^{(2)}_{max}\left[(b\alpha/(\alpha+\beta))J_{\alpha+1,\beta}(c/b) - cJ_{\alpha,\beta}(c/b)\right]\Big\}. \end{aligned}$$

5.3 Power-type penalty function

Nevertheless, the function $g_2(x-c)$ is piecewise linear, which can still be too constraining. For example, after the level c, when the uncertainty increases by ϵ, the consequences will remain the same, regardless of the initial level. This is not particularly troublesome for small values of ϵ, but this can be unrealistic for larger values of ϵ. It is therefore useful to have a penalty function that impacts the basis risk in a more general non-linear way.

As a general rule, a penalty function should increase the impact of a basis risk $X-c$ that occurs at a higher level. To model such an effect, we introduce a power-type function

$$\phi_n(x) = \gamma[(x-c)-d]_+^n, \quad x \in [0,b],$$

which depends on two new parameters $d > 0$ and $n \in \mathbb{N}_0$. The parameter d can be viewed as a critical level because the difference $x-c$ only has a negative impact if it is greater than d. The parameter n is used to manage the level of consequences produced by the observed difference. Note that $x-c > d+1$ implies $\phi_{n+1}(x) > \phi_n(x)$ for all n, i.e. the higher the n, the greater the impact.

By combining ϕ_n with the previous g_2, we define the penalty function

$$h_n(X-c) = f_1(X) + f_2(X) + \phi_n(X), \tag{5.12}$$

which shows three types of possible consequences. First, a negative value for the difference $X-c$ means that the payout given by the index c overestimates the loss X (Zone 1 in Figure 5.3). In this case, if an insurer wishes to purchase parametric reinsurance coverage based on the same index, the premium paid will be too high and the insurer will be over-hedged. In fact, the more negative the difference, the more unnecessary reassurance is important. Second, a positive value for the difference $X-c$ means that the value taken by the index c underestimates the loss X, and the penalty ϕ_n displays two different levels of consequences. If $X-c$ remains under the critical level d (Zone 2 in Figure 5.3), the amount of parametric reinsurance coverage will be too small and the insurer will be under-hedged. Conversely, if the $X-c$ exceeds the critical level d (Zone 3 in Figure 5.3), the consequences get worse. One possible reason could be that the insurer decides not to take reinsurance at all because the index is never greater than its acceptable loss, which forces it to draw on its reserve or even to drive it to the bankruptcy.

We want to get bounds on the expected penalty $\mathbb{E}[h_n(X-c)]$. For that, we will deduce bounds on $\mathbb{E}[\phi_n(X)]$. It can be checked that the function $\phi_n(x)$ is n-increasing convex on $\mathbb{R}$, i.e. it is k-convex for all $k = 2, \dots, n$. Therefore, s-convex bounds can only be considered for $s = 2, \dots, n$. We assume below that $n \geq 4$ and $s = 2, 3, 4$.

As a preliminary, we determine the expectation $l_n(x,y) \equiv \mathbb{E}[(Sx-y)_+^n]$, $x, y > 0$, that will serve to express the bounds.

Lemma 5.1.

$$l_n(x,y) = \mathbf{1}_{\{x>y\}} \sum_{k=0}^{n} \binom{n}{k} (-1)^k y^k x^{n-k} J_{\alpha+n-k,\beta}\left(\frac{y}{x}\right) \frac{\Gamma(\alpha+n-k)\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+n-k)\Gamma(\alpha)}. \tag{5.13}$$

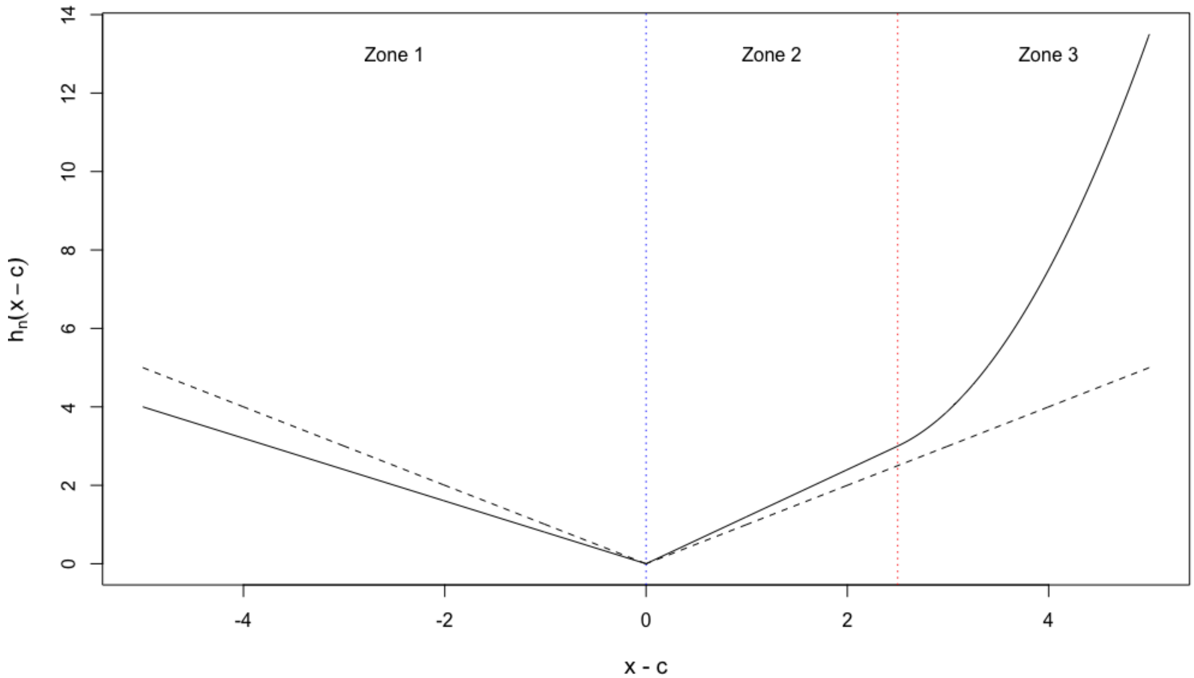

Figure 5.3: The penalty function $h_n(x-c)$ when $\eta = 0.8, \gamma = 1.2, c = 5, d = 2.5, b = 10$ and $n = 2$. Dashed line represents $|x-c|$, i.e. $g_1(x-c)$.

Proof. Since $S \sim S(\alpha, \beta)$, $l_n(x, y) = 0$ if $x \leq y$. For $x > y$,

$$\begin{aligned} l_n(x,y) &= \int_{y/x}^{1} (sx-y)^n f_S(s)ds, \\ &= \sum_{k=0}^{n} \binom{n}{k} (-1)^k y^k x^{n-k} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_{y/x}^{1} s^{\alpha+n-k-1}(1-s)^{\beta-1} ds, \end{aligned}$$

hence (5.13) using again the notation $J_{\alpha,\beta}(x) = \mathbb{P}[S(\alpha,\beta) > x]$. $\square$

From Corollary 4.1 and using Lemma 5.1, we then find the following bounds for $\mathbb{E}[\phi_n(X)]$, $n \geq 4$, in the spaces $\mathcal{B}_s(\alpha, \beta; [0, b]; \nu_1, \ldots, \nu_{s-1})$ for $s = 2, 3, 4$.

Proposition 5.3. *Choose $n \geq 4$. In $\mathcal{B}_2(\alpha, \beta; [0, b]; \nu_1)$, the lower bound for $\mathbb{E}[\phi_n(X)]$ is*

$$\mathbb{E}[\phi_n(X^{(2)}_{min})] = \gamma\, l_n(\mu_1, c+d),$$

and the upper bound is

$$\mathbb{E}[\phi_n(X^{(2)}_{max})] = \gamma \left[p^{(2)}_{max} l_n(a, c+d) + q^{(2)}_{max} l_n(b, c+d) \right].$$

In $\mathcal{B}_3(\alpha, \beta; [0,b]; \nu_1, \nu_2)$*, the lower bound for* $\mathbb{E}[\phi_n(X)]$ *is*

$$\mathbb{E}[\phi_n(X^{(3)}_{min})] = \gamma \left[p^{(3)}_{min} l_n(a, c+d) + q^{(3)}_{min} l_n(\frac{\mu_2 - a\mu_1}{\mu_1 - a}, c+d) \right],$$

and the upper bound is

$$\mathbb{E}[\phi_n(X^{(3)}_{max})] = \gamma \left[p^{(3)}_{max}\, l_n(\frac{b\mu_1 - \mu_2}{b - \mu_1}, c+d) + q^{(3)}_{max}\, l_n(b, c+d) \right].$$

In $\mathcal{B}_4(\alpha, \beta; [0,b]; \nu_1, \nu_2, \nu_3)$*, the lower bound for* $\mathbb{E}[\phi_n(X)]$ *is*

$$\mathbb{E}[\phi_n(X^{(4)}_{min})] = \gamma \left[\frac{r_+ - \mu_1}{r_+ - r_-}\, l_n(r_-, c+d) + \frac{\mu_1 - r_-}{r_+ - r_-}\, l_n(r_+, c+d) \right],$$

and the upper bound is

$$\mathbb{E}[\phi_n(X^{(4)}_{max})] = \gamma \left[(1 - p^{(4)}_{max} - q^{(4)}_{max})\, l_n(a, c+d) + p^{(4)}_{max}\, l_n(x^{(4)}_{max}, c+d) + q^{(4)}_{max}\, l_n(b, c+d) \right].$$

For the desired expectation $\mathbb{E}[h_n(X - c)]$, we now deduce from (5.12) and Propositions 5.2, 5.3 that for $n \geq 4$ and $s = 2, 3, 4$,

$$\begin{aligned} \mathbb{E}[f_1(X^{(2)}_{min})] + \mathbb{E}[f_2(X^{(2)}_{min})] + \mathbb{E}[\phi_n(X^{(s)}_{min})] &\leq \mathbb{E}[h_n(X - c)] \\ &\leq \mathbb{E}[f_1(X^{(2)}_{max})] + \mathbb{E}[f_2(X^{(2)}_{max})] + \mathbb{E}[\phi_n(X^{(s)}_{max})]. \end{aligned}$$

By the n-increasing convexity of ϕ_n, we note also that for $s = 2, 3$,

$$\mathbb{E}[\phi_n(X^{(s)}_{min})] \leq \mathbb{E}[\phi_n(X^{(s+1)}_{min})] \leq \mathbb{E}[\phi_n(X)], \text{ and } \mathbb{E}[\phi_n(X)] \leq \mathbb{E}[\phi_n(X^{(s+1)}_{max})] \leq \mathbb{E}[\phi_n(X^{(s)}_{max})].$$

5.4 Exponential-type penalty function

The function $\phi_n(x)$ being increasing convex of finite order n, the number of usable moments of X is necessarily limited. This motivates us to consider a function $\phi_\infty(x)$ which is increasing convex of infinite order, i.e. typically of exponential form. Thus, we define $\phi_\infty(x)$ by

$$\phi_\infty(x) = \gamma \left\{ \exp\left(\kappa[(x - c) - d]_+\right) - 1 \right\}, \quad x \in [0, b],$$

in which the term -1 is introduced to guarantee $\phi_\infty(x) = 0$ for $x \leq c + d$ as before. The new parameter $\kappa > 0$ handles the impact of the difference $x - c$. The resulting penalty function is then given by

$$h_\infty(X - c) = f_1(X) + f_2(X) + \phi_\infty(X). \tag{5.14}$$

The exponential form of ϕ_∞ has the advantage of allowing the use of all available moments of X. For example, if the first three moments are known, it is natural to consider the function ϕ_4. Suppose now that the fourth moment can be estimated later. Two choices are then possible: either ignore this new information since ϕ_4 is not 5-convex, or replace ϕ_4 with ϕ_5 in order to exploit this information, which leads to modify the model. With the function ϕ_∞ function, having an extra moment does not change the model, it only allows for sharper bounds.

To obtain bounds on $\mathbb{E}[h_\infty(X-c)]$, we must first derive bounds on $\mathbb{E}[\phi_\infty(X)]$. As a preliminary, we determine the expectation $\mathbb{E}\{\exp[\kappa(Sx-y)_+]\}$, $x, y > 0$, that is involved in these calculations. For that, we will use the concept of partial moment generating function of a variable X (see e.g. Winkler et al. Winkler et al. [68]). It is the integral, if it exists, defined from y to z by

$$\mathcal{M}_y^z(t) = \int_y^z e^{tx} dF_X(x), \quad t \in \mathbb{R}. \tag{5.15}$$

Lemma 5.2.

$$\mathbb{E}\{\exp[\kappa(Sx-y)_+]\} = 1 - J_{\alpha,\beta}(y/x) + \exp(-\kappa y)\, \mathcal{M}^1_{y/x\wedge 1}(\kappa x)\,. \tag{5.16}$$

Proof. Since $S \sim S(\alpha, \beta)$, a direct integration yields

$$\mathbb{E}\{\exp[\kappa(Sx-y)_+]\} \;\; = \int_0^{y/x\wedge 1} f_S(s)ds + \exp(-\kappa y) \int_{y/x\wedge 1}^1 \exp(\kappa x s) f_S(s) ds,$$

which becomes (5.16) using the notation $J_{\alpha,\beta}$ and (5.15). □

From Corollary 4.1 and applying Lemma 5.2, we deduce the following bounds for $\mathbb{E}[\phi_\infty(X)]$ in the spaces $\mathcal{B}_s(\alpha, \beta; [0, b]; \nu_1, \ldots, \nu_{s-1})$ for $s = 2, 3, 4$.

Proposition 5.4. *In $\mathcal{B}_2(\alpha, \beta; [0, b]; \nu_1)$, the lower bound for $\mathbb{E}[\phi_\infty(X)]$ is*

$$\mathbb{E}[\phi_\infty(X^{(2)}_{min})] = \gamma \left[-J_{\alpha,\beta}(\frac{c+d}{\mu_1}) + \exp(-\kappa(c+d)) \mathcal{M}^1_{\frac{c+d}{\mu_1}\wedge 1}(\kappa\mu_1) \right],$$

and the upper bound is

$$\mathbb{E}[\phi_\infty(X^{(2)}_{max})] = \gamma q^{(2)}_{max} \left[-J_{\alpha,\beta}(\frac{c+d}{b}) + \exp(-\kappa(c+d)) \mathcal{M}^1_{\frac{c+d}{b}}(\kappa b) \right].$$

In $\mathcal{B}_3(\alpha, \beta; [0, b]; \nu_1, \nu_2)$, the lower bound for $\mathbb{E}[\phi_\infty(X)]$ is

$$\mathbb{E}[\phi_\infty(X^{(3)}_{min})] = \gamma q^{(3)}_{max} \left[-J_{\alpha,\beta}(\frac{c+d}{x^{(3)}_{min}}) + \exp(-\kappa(c+d)) \mathcal{M}^1_{\frac{c+d}{x^{(3)}_{min}}\wedge 1}(\kappa x^{(3)}_{min}) \right],$$

with $x^{(3)}_{min} = (\mu_2 - a\mu_1)/(\mu_1 - a)$, and the upper bound is

$$\mathbb{E}[\phi_\infty(X^{(3)}_{max})] = \gamma p^{(3)}_{max} \left[-J_{\alpha,\beta}(\frac{c+d}{x^{(3)}_{max}}) + \exp(-\kappa(c+d)) \mathcal{M}^1_{\frac{c+d}{x^{(3)}_{max}}\wedge 1}(\kappa x^{(3)}_{max}) \right]$$

$$+\gamma q_{max}^{(3)}\left[-J_{\alpha,\beta}(\frac{c+d}{b})+\exp(-\kappa(c+d))\mathcal{M}^1_{\frac{c+d}{b}}(\kappa b)\right],$$

with $x_{max}^{(3)}=(b\mu_1-\mu_2)/(b-\mu_1)$.

In $\mathcal{B}_4(\alpha,\beta;[0,b];\nu_1,\nu_2,\nu_3)$, *the lower bound for* $\mathbb{E}[\phi_\infty(X)]$ *is*

$$\begin{aligned}\mathbb{E}[\phi_\infty(X_{min}^{(4)})]=\gamma p_{min}^{(4)}&\left[-J_{\alpha,\beta}(\frac{c+d}{r_-})+\exp(-\kappa(c+d))\mathcal{M}^1_{\frac{c+d}{r_-}\wedge 1}(\kappa r_-)\right]\\&+\gamma q_{min}^{(4)}\left[-J_{\alpha,\beta}(\frac{c+d}{r_+})+\exp(-\kappa(c+d))\mathcal{M}^1_{\frac{c+d}{r_+}\wedge 1}(\kappa r_+)\right],\end{aligned}$$

and the upper bound is

$$\begin{aligned}\mathbb{E}[\phi_\infty(X_{max}^{(4)})]=\gamma p_{max}^{(4)}&\left[-J_{\alpha,\beta}(\frac{c+d}{x_{max}^{(4)}})+\exp(-\kappa(c+d))\mathcal{M}^1_{\frac{c+d}{x_{max}^{(4)}}\wedge 1}(\kappa x_{max}^{(4)})\right]\\&+\gamma q_{max}^{(4)}\left[-J_{\alpha,\beta}(\frac{c+d}{b})+\exp(-\kappa(c+d))\mathcal{M}^1_{\frac{c+d}{b}}(\kappa b)\right].\end{aligned}$$

For the expectation $\mathbb{E}[h_\infty(X-c)]$, we now get from (5.14) and Propositions 5.2, 5.4 that for $s=2,3,4$,

$$\begin{aligned}\mathbb{E}[f_1(X_{min}^{(2)})]+\mathbb{E}[f_2(X_{min}^{(2)})]+\mathbb{E}[\phi_\infty(X_{min}^{(s)})]&\leq\mathbb{E}[h_\infty(X-c)]\\&\leq\mathbb{E}[f_1(X_{max}^{(2)})]+\mathbb{E}[f_2(X_{max}^{(2)})]+\mathbb{E}[\phi_\infty(X_{max}^{(s)})].\end{aligned}$$

The remark made before about the comparison of the bounds when $s=2,3$ remains valid for $\mathbb{E}(\phi_\infty)$. For an appropriate penalty function, the greater the number of known moments, the sharper the bounds on the expected penalty functions.

6 Numerical illustrations

Most licenses for cat modeling software are expensive, which has prevented us from using actual data. However, thanks to the close links between the authors and certain reinsurance companies, we were able to generate $N=100$ scenarios of simulated data very similar to typical real-world data sets (which often include several thousand scenarios).

Each scenario i is characterized by its occurrence rate p_i, the parameters (α_i,β_i) of the Beta distribution of the destruction ratio S_i, and the information available on the exposure Y_i, i.e. its range (a_i,b_i) and several first moments. To generate this information on the moments of Y_i, we used the moments of distributions that are traditional in cat modeling. So, the event set involves 67 scenarios where the exposure moments are obtained from a scaled Beta distribution and 33 scenarios where the exposure moments are derived from a truncated Pareto distribution.

These two distributions are discussed in an actuarial context in e.g. Clark [14] and Beirlant et al. [6]; a short presentation is given in the Appendix.

We considered five types of scenarios, depending on the size of uncertainty, the intensity of the destruction ratio and the link between the loss and the destruction ratio.

- Scenarios 1 to 5 represent cases where the exposure is limited and of Beta type, but the intensity of the event is either small or large. Thus, a_i, b_i are relatively small whereas $\alpha_i, \beta_i < 1$.

- In scenarios 6 to 66, the destruction ratios are sorted according to their riskiness. In fact, the higher α_i and the lower β_i, the greater the probability that the destruction ratio is close to 1. In addition, the corresponding Y_i is a scaled Beta variable and is generated so that its mean is decreasing. In other words, a large event is linked to a huge exposure.

- Scenarios 67 to 96 look like the previous ones, except that the exposure Y_i is a truncated Pareto variable. Here too, the probability that the destruction ratio takes values close to 1 decreases from the scenario 67 to 96.

- Until now, all scenarios $i = 1, \ldots, 96$ had the same probability of occurrence $p_i = 1.0\%$. This time, we take $p_{97} = 3.0\%$. That scenario represents the case where the event potentially involves a huge exposure in the sense that Y_{97} is Beta$([40, 80], 0.5, 0.5)$ with a lot of uncertainty (which could come, for example, from the fact that a very big city can be hit or not). To that extent, the exposure is somehow close to 40 or 80. In addition, the destruction ratio is often close to 1.

- Finally, the scenarios $98, 99, 100$ involve significant exposures as well as destruction ratios shifted to the right. Thus, these three scenarios can be considered the most dangerous. It seems reasonable to assume that these scenarios do not all have the same frequency, hence the choices $p_{98} = 0.5\%$, $p_{99} = 0.4\%$, $p_{100} = 0.1\%$.

Those 100 scenarios encompass lighter and heavier tailed risks (even if there is a finite total exposure), and feature different levels of risk, uncertainty, and correlations between destruction rates and exposures, like in the real world. The full description of scenarios is given in Table 6 of the Appendix.

A major objective of this illustration is to highlight the role of the penalty function, the index and the impact of information on exposure on the sharpness of the bounds. The penalty functions h_n given by (5.12) and h_∞ given by (5.14) are defined with the parameters $\eta = 0.8$, $\gamma = 1.2$, $n = 4$ and $\kappa = 0.2$. Tables 2 to 5 provide the s-convex bounds for X_i when $s = 2, 3, 4$ and the corresponding impact of the basis risk $BR_\phi^{(s)}$ given by (5.1) when ϕ is h_4 or h_∞.

In Table 2, the index value is $c_i = \mathbb{E}[S_i Y_i]$ and the critical level is $d_i = 0.2\mathbb{E}[S_i Y_i]$. As expected, the 4-convex bounds are much sharper than the 3 and 2-convex bounds. Equivalently, the worst case scenario in the 4-convex sense is less severe than in the 3 or 2-convex sense. This conclusion is common to the following three tables. To reduce the basis risk, the insurer must logically refine the information on exposures, for example by knowing higher moments.

In Tables 3 to 5, we choose $c_i = 0.95\mathbb{E}[S_iY_i]$, $c_i = 1.05\mathbb{E}[S_iY_i]$, $c_i = 1.15\mathbb{E}[S_iY_i]$ respectively, with the same d_i as before. We observe that the bounds are very sensitive to the value of c_i. This is natural since the index defines the basis risk. We draw attention to the fact that the choice of penalty function, its form and its parameters should be driven by the risk appetite of the protection buyer and not for a purpose of optimization. In other words, the threshold level d_i should be the same whatever the retained index-based structure, while the minimization of the basis risk measurement should be done by modifying the index value c_i, and not the penalty function nor its parameters.

s-convex bounds	$BR^{(s)}_{h_4}$	$BR^{(s)}_{h_\infty}$
2-cx lower bound	317.599	4.436
3-cx lower bound	534.484	4.581
4-cx lower bound	1 053.163	4.754
X	1 356.515	7.785
4-cx upper bound	1 863.618	26.737
3-cx upper bound	3 389.789	103.215
2-cx upper bound	18 647.031	1 577.784

Table 2: Values of $BR^{(s)}_{h_4}$ and $BR^{(s)}_{h_\infty}$ for $s = 2, 3, 4$, when $c = \mathbb{E}[SY]$ and $d = 0.2\mathbb{E}[SY]$.

s-convex order	$BR^{(s)}_{h_4}$	$BR^{(s)}_{h_\infty}$
2-cx lower bound	487.918	4.842
3-cx lower bound	870.181	5.045
4-cx lower bound	1 603.361	5.276
X	1 956.715	8.589
4-cx upper bound	2 540.901	30.907
3-cx upper bound	4 299.803	118.657
2-cx upper bound	21 149.301	1 779.437

Table 3: Values of $BR^{(s)}_{h_4}$ and $BR^{(s)}_{h_\infty}$ for $s = 2, 3, 4$, when $c = 0.95\mathbb{E}[SY]$ and $d = 1.05\mathbb{E}[SY]$.

s-convex order	$BR^{(s)}_{h_4}$	$BR^{(s)}_{h_\infty}$
2-cx lower bound	215.974	4.194
3-cx lower bound	342.337	4.299
4-cx lower bound	706.460	4.431
X	959.211	7.163
4-cx upper bound	1 399.546	23.395
3-cx upper bound	2 741.775	90.257
2-cx upper bound	16 697.651	1 401.555

Table 4: Values of $BR^{(s)}_{h_4}$ and $BR^{(s)}_{h_\infty}$ for $s = 2, 3, 4$, when $c = 1.05\mathbb{E}[SY]$ and $d = 1.05\mathbb{E}[SY]$.

s-convex order	$BR^{(s)}_{h_4}$	$BR^{(s)}_{h_\infty}$
2-cx lower bound	110.008	4.173
3-cx lower bound	164.702	4.229
4-cx lower bound	350.635	4.311
X	526.320	6.703
4-cx upper bound	860.462	18.580
3-cx upper bound	1 940.151	70.117
2-cx upper bound	13 885.964	1 111.376

Table 5: Values of $BR^{(s)}_{h_4}$ and $BR^{(s)}_{h_\infty}$ for $s = 2, 3, 4$, when $c = 1.15\mathbb{E}[SY]$ and $d = 1.05\mathbb{E}[SY]$.

In conclusion. We started by identifying s-convex extrema for bounded random variables obtained by a random scaling of Beta type.. Thanks to this, we were able to evaluate the basis risk in a parametric transaction using penalty functions of different foms. We also showed the impact of information, in terms of moments, on the assessment of the basis risk. So, within the proposed framework, we have provided a (partial) answer to the key question: how much does the basis risk cost? In a future paper, we will examine other situations where the basis risk is unbounded and where the independence between the destruction ratio and the exposure level is questionable.

Appendix. Let us recall the definition of the two distributions used to model the exposure Y_i.

A scaled Beta distribution enlarges the support of the usual Beta distribution. It again depends on two positive parameters, u_i, v_i say, but is now defined on an interval $[a_i, b_i]$. Specifically, if W_i is a Beta(u_i, v_i) variable, then Y_i is a Beta$([a_i, b_i], u_i, v_i)$ variable means that

$$Y_i = (b_i - a_i)W_i + a_i.$$

The moments of a scaled Beta distribution follow directly.

The variable Y_i has a truncated Pareto distribution of real parameter u_i when its distribution function is

$$F_{Y_i}(x) = \begin{cases} \dfrac{1-(a_i/x)^{u_i}}{1-(a_i/b_i)^{u_i}}, & a_i \leq x \leq b_i, \quad \text{if } \; u_i \neq 0, \\ \dfrac{\ln(x/a_i)}{\ln(b_i/a_i)}, & a_i \leq x \leq b_i, \quad \text{if } \; u_i = 0. \end{cases}$$

Note that for $u_i = -1$, this is reduced to the uniform distribution over $[a_i, b_i]$. All moments of a truncated Pareto exist and are given by

$$\mathbb{E}[Y_i^k] = \frac{u_i a_i^k}{u_i - k} \frac{1-(a_i/b_i)^{u_i-k}}{1-(a_i/b_i)^{u_i}}, \quad k \in \mathbb{N}, \quad \text{if } \; u_i \neq 0, k.$$

When $u_i = 0$ or k, the moments become $\ln(b_i/a_i)$.

For reproductibility purposes, we give in detail below the parameters used for the 100 scenarios of the illustration described in Section 6. The notations S-B and T-P refer to the scaled Beta and truncated Pareto distributions.

i	p_i	α_i	β_i	Y_i	a_i	b_i	u_i	v_i
1	0.01	0.5	0.5	S-B	10	20	5	1
2	0.01	0.6	0.5	S-B	12	30	5	1
3	0.01	0.4	0.6	S-B	5	35	4	2
4	0.01	0.6	0.6	S-B	8	15	3	2
5	0.01	0.5	0.5	S-B	8	20	3	2
6	0.01	7	2	S-B	60	80	5	2
7	0.01	6.9	2.1	S-B	58.8	79	5	2
8	0.01	6.8	2.2	S-B	56.8	78	5	2
9	0.01	6.7	2.3	S-B	54.8	77	5	2
10	0.01	6.6	2.4	S-B	52.8	76	5	2
11	0.01	6.5	2.5	S-B	50.8	75	5	2
12	0.01	6.4	2.6	S-B	48.8	74	5	2
13	0.01	6.3	2.7	S-B	46.8	73	5	2
14	0.01	6.2	2.8	S-B	44.8	72	5	2
15	0.01	6.1	2.9	S-B	42.8	71	5	2
16	0.01	6.0	3.0	S-B	40.8	70	5	2
17	0.01	5.9	3.1	S-B	38.8	69	5	2
18	0.01	5.8	3.2	S-B	36.8	68	5	2
19	0.01	5.7	3.3	S-B	34.8	67	5	2
20	0.01	5.6	3.4	S-B	32.8	66	5	2
21	0.01	5.5	3.5	S-B	30.8	65	5	2
22	0.01	5.4	3.6	S-B	28.8	64	5	2
23	0.01	5.3	3.7	S-B	26.8	63	5	2
24	0.01	5.2	3.8	S-B	24.8	62	5	2
25	0.01	5.1	3.9	S-B	22.8	61	5	2

i	p_i	α_i	β_i	Y_i	a_i	b_i	u_i	v_i
26	0.01	5.0	4.0	S-B	20.8	60	5	2
27	0.01	4.9	4.1	S-B	20	59	5	2
28	0.01	4.8	4.2	S-B	20	58	5	2
29	0.01	4.7	4.3	S-B	20	57	5	2
30	0.01	4.6	4.4	S-B	20	56	5	2
31	0.01	4.5	4.5	S-B	20	55	5	2
32	0.01	4.4	4.6	S-B	20	54	5	2
33	0.01	4.3	4.7	S-B	20	53	5	2
34	0.01	4.2	4.8	S-B	20	52	5	2
35	0.01	4.1	4.9	S-B	20	51	5	2
36	0.01	4.0	5.0	S-B	20	50	5	2
37	0.01	3.9	5.1	S-B	20	49	5	2
38	0.01	3.8	5.2	S-B	20	48	5	2
39	0.01	3.7	5.3	S-B	20	47	5	2
40	0.01	3.6	5.4	S-B	20	46	5	2
41	0.01	3.5	5.5	S-B	20	45	5	2
42	0.01	3.4	5.6	S-B	20	45	4.8	2.3
43	0.01	3.3	5.7	S-B	20	45	4.6	2.6
44	0.01	3.2	5.8	S-B	20	45	4.4	2.9
45	0.01	3.1	5.9	S-B	20	45	4.2	3.2
46	0.01	3.0	6.0	S-B	20	45	4.0	3.5
47	0.01	2.9	6.1	S-B	20	45	3.8	3.8
48	0.01	2.8	6.2	S-B	20	45	3.6	4.1
49	0.01	2.7	6.3	S-B	20	45	3.4	4.4
50	0.01	2.6	6.4	S-B	20	45	3.2	4.7

i	p_i	α_i	β_i	Y_i	a_i	b_i	u_i	v_i
51	0.01	2.5	6.5	S-B	20	45	3.0	5.0
52	0.01	2.4	6.6	S-B	20	45	2.8	5.3
53	0.01	2.3	6.7	S-B	20	45	2.6	5.6
54	0.01	2.2	6.8	S-B	20	45	2.4	5.9
55	0.01	2.1	6.9	S-B	20	45	2.2	6.2
56	0.01	2.0	7.0	S-B	20	45	2.0	6.5
57	0.01	1.9	7.1	S-B	20	45	1.8	6.8
58	0.01	1.8	7.2	S-B	20	45	1.6	7.1
59	0.01	1.7	7.3	S-B	20	45	1.4	7.4
60	0.01	1.6	7.4	S-B	20	45	1.2	7.7
61	0.01	1.5	7.5	S-B	20	45	1	8.0
62	0.01	1.4	7.6	S-B	20	45	1	8.3
63	0.01	1.3	7.7	S-B	20	45	1	8.6
64	0.01	1.2	7.8	S-B	20	45	1	8.9
65	0.01	1.1	7.9	S-B	20	45	1	9.2
66	0.01	1.0	8.0	S-B	20	45	1	9.5
67	0.01	4	1.1	T-P	10	60	4	-
68	0.01	3.9	1.2	T-P	10	50	4	-
69	0.01	3.8	1.3	T-P	10	40	4	-
70	0.01	3.7	1.4	T-P	15	35	4	-
71	0.01	3.6	1.5	T-P	10	35	4	-
72	0.01	3.5	1.6	T-P	10	35	4.5	-
73	0.01	3.4	1.7	T-P	10	35	4.5	-
74	0.01	3.3	1.8	T-P	10	35	4.5	-
75	0.01	3.2	1.9	T-P	10	35	4.5	-
76	0.01	3.1	2.0	T-P	10	35	4.5	-
77	0.01	3.0	2.1	T-P	10	35	4.5	-
78	0.01	2.9	2.2	T-P	10	35	4.5	-
79	0.01	2.8	2.3	T-P	10	35	4.5	-
80	0.01	2.7	2.4	T-P	10	35	4.5	-
81	0.01	2.6	2.5	T-P	10	35	4.5	-
82	0.01	2.5	2.6	T-P	10	35	4.5	-
83	0.01	2.4	2.7	T-P	2	30	4.5	-
84	0.01	2.3	2.8	T-P	2	30	5	-
85	0.01	2.2	2.9	T-P	2	30	5	-
86	0.01	2.1	3.0	T-P	2	30	5	-
87	0.01	2.0	3.1	T-P	2	30	5	-
88	0.01	1.9	3.2	T-P	2	30	5	-
89	0.01	1.8	3.3	T-P	2	30	5	-
90	0.01	1.7	3.4	T-P	2	30	5	-
91	0.01	1.6	3.5	T-P	2	30	5	-
92	0.01	1.5	3.6	T-P	2	30	5	-
93	0.01	1.4	3.7	T-P	2	30	5	-
94	0.01	1.3	3.8	T-P	2	30	5	-
95	0.01	1.2	3.9	T-P	2	30	5	-
96	0.01	1.1	4.0	T-P	2	30	5	-
97	0.03	4	1	S-B	40	80	0.5	0.5
98	0.005	5	1	T-P	10	100	5	-
99	0.004	6	1	T-P	20	100	4	-
100	0.001	8	1	T-P	35	100	3.1	-

Table 6: The parameters used in the 100 scenarios of the illustration given in Section 6.

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