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# A Formal Account of Structuring Motor Actions With Sensory Prediction for a Naive Agent

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## 2 ABSTRACT

3 For naive robots to become truly autonomous, they need a means of developing their perceptive  
4 capabilities instead of relying on hand crafted models. The sensorimotor contingency theory  
5 asserts that such a way resides in learning invariants of the sensorimotor flow. We propose a  
6 formal framework inspired by this theory for the description of sensorimotor experiences of a naive  
7 agent, extending previous related works. We then use said formalism to conduct a theoretical  
8 study where we isolate sufficient conditions for the determination of a sensory prediction function.  
9 Furthermore, we also show that algebraic structure found in this prediction can be taken as a  
10 proxy for structure on the motor displacements, allowing for the discovery of the combinatorial  
11 structure of said displacements. Both these claims are further illustrated in simulations where a toy  
12 naive agent determines the sensory predictions of its spatial displacements from its uninterpreted  
13 sensory flow, which it then uses to infer the combinatorics of said displacements.

14 **Keywords:** Sensory prediction, sensorimotor contingencies, interactive perception, bootstrapping, developmental robotics.

15 **textcount statistics:** words in text = app. 12000, words outside text (captions, etc.) = 767, number of  
16 floats/tables/figures: 9.

## 1 INTRODUCTION

17 *Autonomous* robots need possess the cognitive capabilities to face realistic and uncertain environments.  
18 Classical approaches deal with this problem by giving them *a priori* models of their interaction with their  
19 environment. These rely on carefully crafted models of the agent's body (Mutambara and Litt, 1998), its  
20 sensors, the environments it will encounter and the nature of the tasks it is setting to perform (Marconi  
21 et al., 2011). But said models are notoriously difficult to obtain (Lee et al., 2017), by definition incom-  
22 plete (Nguyen et al., 2017) and often fail to generalize to interactions varying in unknown spatial and  
23 temporal scales. As it has been previously studied, models of an agent sensorimotor apparatus (Censi and  
24 Murray, 2012) or of a mobile robot interaction with its environment (Jonschkowski and Brock, 2015) can  
25 alternatively be learned. In particular, these capabilities crucially depend on the robot correctly learning its  
26 *perception* in that it represents the interface layer between the raw readings of its sensors and its higher  
27 level cognitive capabilities, e.g. decision making or task solving layers.

28 While there certainly is an established practice of mostly treating perception as processing the sensory  
29 signal, multiple cues argue that perception can only emerge from the joint *sensorimotor* experience (Noë,  
30 2004). The field of interactive perception, reviewed in (Bohg et al., 2017), indeed displays several  
31 approaches which roughly adhere to this principle. One particular theoretical framework is that of *Sensori-*  
32 *motor Contingencies Theory* (O’Regan and Noë, 2001) (SMCT for short) which asserts that perception is  
33 the mastery of invariant structures in the sensorimotor flow an agent discovers during its interaction with  
34 the environment. Their theoretical origin as an abstract, generic cognitive construct lends them desirable  
35 properties for robotic applications: namely, they could support bootstrapping the learning of perceptual  
36 capabilities in a way that does *not* depend on the implementation of the artificial agent considered as well as  
37 on the environments in which said learning is done. In this regard, it differs significantly from the modern  
38 developmental approaches supported by Deep Learning (Ruiz-del-Solar et al., 2018) which rely on specific  
39 structural details of neural networks, i.e. the numeric forms of inputs, outputs and activation of the neurons.

40 Early works on SMCT have been shown to lead to discovery of the color spectrum (Philipona and  
41 O’Regan, 2006) and of the dimensionality of ambient space (Philipona et al., 2003, 2004; Laflaquière  
42 et al., 2010; Laflaquière et al., 2012) in “naive” agents, which can be extended to that of an internal, path  
43 independent, notion of space (Terekhov and O’Regan, 2016). We already proposed different contributions  
44 in this field, successively dealing with peripersonal space characterization (Laflaquière et al., 2015), self-  
45 contact and body representation (Marcel et al., 2017) and the emergence of a topological representation  
46 of sensors poses (Marcel et al., 2019). These works, as well as (Laflaquière et al., 2018; Laflaquière  
47 and Ortiz, 2019), devote a significant effort to providing formalisms suited to make explicit –and, where  
48 applicable, formally prove– not only the processing required to capture the contingencies, i.e. invariants in  
49 the sensorimotor flow of the agent, but also the mechanisms by which said contingencies should appear.  
50 This is an attempt to pinpoint the exact conditions of validity of the proposed processes in order to deliver  
51 on the promises of genericity of SMCT.

52 Many recent contributions drawing from SMCT revolve around sensorimotor prediction in some way:  
53 the ability to discover a sensorimotor prediction is empirically shown to arise from both the temporal  
54 structure of the sensorimotor experience (Maye and Engel, 2012) and the spatial coherence of a natural  
55 visual environment for a sensor based on a retina (Laflaquière, 2017). Moreover, said ability to predict  
56 sensory outcomes has been shown to provide in robots a basis for an egocentric representation of ambient  
57 space (Laflaquière and Ortiz, 2019), object perception (Maye and Engel, 2011; Le Hir et al., 2018), action  
58 selection (Maye and Engel, 2012), motor control (Schröder-Schetelig et al., 2010) and motor sequence  
59 compression (Ortiz and Laflaquière, 2018). Along the “Bayesian brain” approach, predictive processing is  
60 even argued to form the mechanistic implementation of sensorimotor contingencies (Seth, 2014). This is  
61 very much in line with classical findings in cognitive psychology, both those regarding the physiological  
62 implementations of sensorimotor prediction via *efference copies* (von Helmholtz et al., 1925; Sperry,  
63 1950; von Holst and Mittelstaedt, 1950) and how it supports, albeit incompletely, a number of perceptual  
64 processes (Bridgeman, 1995; Imamizu, 2010; Pynn and DeSouza, 2012; Bhanpuri et al., 2013), as well as  
65 those supporting ideomotor theory (Stock and Stock, 2004) according to which actions are equated to their  
66 perceptual consequences from a cognitive standpoint.

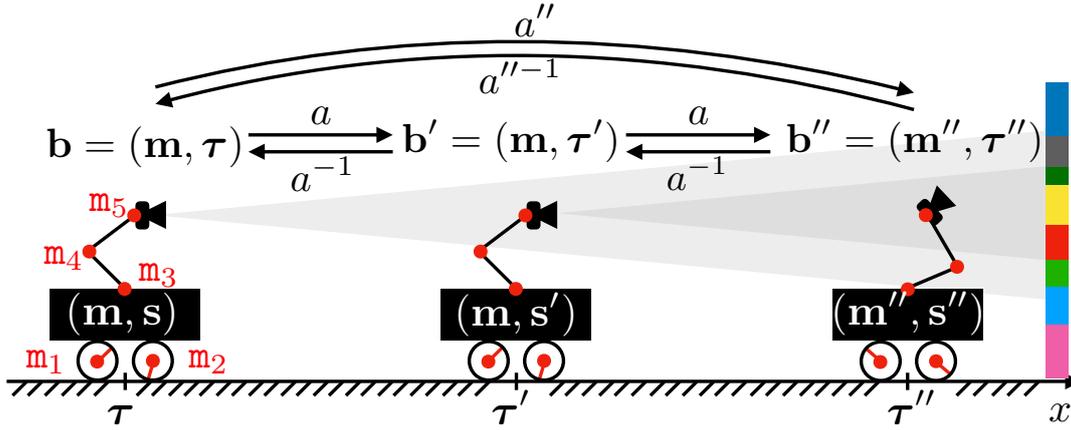
67 This article follows much of the same approach begun in (Philipona et al., 2003) and subsequently  
68 developed in e.g. (Marcel et al., 2017; Laflaquière et al., 2018; Laflaquière and Ortiz, 2019; Marcel et al.,  
69 2019). In particular, it sets out to mathematically describe from an *exterior*, “objective”, point of view  
70 some properties of the interaction between the robot and its environment which should appear in its  
71 sensorimotor flow. To this end we propose a revised formalism building upon the previous instances in

72 these contribution. One notable contribution indeed resides in our proposal remedying their requirement  
73 of the agent having a fixed base by transposing the location of sensorimotor contingencies in sets of  
74 “displacements” instead of that of motor or sensory configurations. In accordance with the previous remark  
75 about genericity, a particular attention is given to the construction of said formalism with assumptions and  
76 proofs explicitly detailed. Moreover, the bootstrapping aspect is emphasized throughout the work, much  
77 in the spirit of (Marcel et al., 2017; Marcel et al., 2019), highlighting the distinction between the points  
78 of view of the agent and of the observer in the description of the problems and an explicit discussion of  
79 the degree of *a priori* knowledge given to the agent, in terms of both data and computations available to  
80 it. There lie two contributions of this article: while the formalism is used to formally describe why and  
81 how spatial coherence lead to the discovery of sensory prediction very much like alluded to in (Laflaquière,  
82 2017) and how this sensory prediction encodes spatial structure akin to that of (Terekhov and O’Regan,  
83 2016; Laflaquière and Ortiz, 2019), it does so with a greater emphasis put on the precise relations between  
84 the algebraic structures at play and with much weakened assumptions about *a priori* capabilities of the  
85 agent, much closer to those put forward in (O’Regan and Noë, 2001). We argue that this formalism unifies  
86 and extends those found in previous works; that the formal structures its expressive power makes explicit  
87 (e.g. group morphisms between action and prediction) give a conceptual explanation of results previously  
88 achieved by more complex means in experimental contexts (Ortiz and Laflaquière, 2018; Laflaquière and  
89 Ortiz, 2019); and on a somewhat “philosophical” level that it allows for a clearer picture of the applicability  
90 and function of SMCT in the process of bootstrapping perception *via* its systematic distinction of points of  
91 view.

92 The paper is organized as follows: to begin with, we introduce in Section 2 all the notations and concepts  
93 used for describing the sensorimotor experience. On this basis, Section 3 defines the two distinct points of  
94 view and enunciates generic properties of the sensorimotor experience that motivate the proposed study of  
95 internal sensorimotor prediction. In particular, the equivalence between the combinatorial structures of  
96 actions and sensory prediction is proved. Then, some simulations are proposed in Section 4 to assess the  
97 mathematical formalism through a careful evaluation of each step of the proposed framework. We establish  
98 that the spatial shifts mediating the sensory experience of a naive agent allow it to determine the sensory  
99 outcome of particular actions, in particular those corresponding to displacements of the agent. Further,  
100 we show that the ability to predict said outcomes can be used as a proxy to the hidden combinatorial  
101 structure of its motor actions. We argue that the theoretical focus adopted in this work provides some new  
102 valuable insight into the mechanisms supporting these results, as well as several similar findings presented  
103 in aforementioned related works.

## 2 DEFINING A FORMALISM FOR SENSORIMOTOR INTERACTION

104 This first section aims at expanding several previous results in *Interactive Perception* as obtained for  
105 example in (Bohg et al., 2017). These have made use of several classical objects such as the *pose* (or  
106 *working*) *space* and the *forward* (either geometrical or sensory) *maps*, at times rearranging their definitions  
107 or making them more precise to allow for formal proofs to be derived. Such work is followed upon in this  
108 contribution, with a somewhat significant overhaul of the formal definitions. This section is thus devoted to  
109 the definitions of the terms we will use to describe a sensorimotor problem, showing during the exposition  
110 how they appear in a simple classical example and how they differ from previous theoretical formulations.  
111 We then leverage these definitions to propose and prove new perceptive bootstrapping algorithms in the  
112 following section.



**Figure 1.** Illustration of the motor actions effects. The agent actuator states  $m_i$  are regrouped into its motor configuration  $\mathbf{m}$ , while  $\mathbf{s}$  denotes its sensor output. Both define the internal agent configuration i.e. the sensorimotor flow the agent has access to. The agent position and orientation in space is captured by  $\tau$ , which together with  $\mathbf{m}$  makes the absolute configuration  $\mathbf{b}$ , partially unknown to the agent. Note that one motor configuration  $\mathbf{m}$  can be associated to distinct sensory outputs  $\mathbf{s}$  and  $\mathbf{s}'$  provided a displacement in space from  $\tau$  to  $\tau'$ , thus preventing the existence of a mapping  $\mathbf{m} \mapsto \mathbf{s}$ . The agent may also perform an action  $a$  to modify its absolute configuration. One such action  $a$  is partially represented, mapping  $\mathbf{b}$  to  $\mathbf{b}' = (\mathbf{m}' = \mathbf{m}, \tau')$ , and  $\mathbf{b}'$  to  $\mathbf{b}'' = (\mathbf{m}'', \tau'')$ . Note that it induces a rigid displacement in the first case, while a change of posture occurs in the second. Also depicted are the inverse action  $a^{-1}$  of  $a$  and the combination  $a''$  obtained by repeating  $a$  twice.

## 113 2.1 Motor actions

114 As a first step, this subsection is devoted to the introduction of all the notions and definitions of the  
 115 *motor* side of the proposed sensorimotor framework. After highlighting the limitations of the previous  
 116 approaches, as seen in e.g. (Marcel et al., 2017; Laflaquière et al., 2018), we show how to reparameterize  
 117 the sensorimotor interaction by introducing motor actions. Their definition and properties are then carefully  
 118 discussed.

### 119 2.1.1 A look back to previous formalisms

120 Let us consider in all the following an agent endowed with motor and sensing capabilities. In the refer-  
 121 enced previous contributions, the sensorimotor interaction is defined by the *internal* motor configuration  $\mathbf{m}$   
 122 and the sensory configuration  $\mathbf{s}$  of this agent, which lie respectively into some sets  $\mathcal{M}$  and  $\mathcal{S}$ . Both of them  
 123 define the *internal* agent configuration  $(\mathbf{m}, \mathbf{s})$ , i.e. the sensorimotor flow the agent has access to. There is a  
 124 clear dependency between the sensory and motor configurations that can be captured by the sensorimotor  
 125 maps  $\psi : \mathcal{M} \times \mathcal{E} \rightarrow \mathcal{S}$ , such that  $\psi(\mathbf{m}, \epsilon) = \mathbf{s}$ , where  $\epsilon \in \mathcal{E}$  represents the state of the environment. As  
 126 said in the introduction, other contributions already exploited this kind of parameterization (Philipona  
 127 et al., 2003; Laflaquière et al., 2015; Marcel et al., 2017). In all these contributions, only fixed base agents  
 128 are considered, since a single internal motor configuration  $\mathbf{m} \in \mathcal{M}$  is only mapped to a single sensory  
 129 configuration  $\mathbf{s} \in \mathcal{S}$  for a fixed environment configuration  $\epsilon$ .

130 To illustrate this point, Figure 1 represents a 2D-agent able to translate itself only along one dimension  $x$ .  
 131 This agent is able to move inside an environment made of colored walls thanks to 5 rotating joints whose  
 132 states  $m_i$ ,  $i = 1, \dots, 5$ , are captured in its motor configuration  $\mathbf{m} = (m_i)_i$  (where the second  $i$  subscript  
 133 in  $(\cdot)_i$  denotes the collection being taken with  $i$  for ranging variable). To begin, let us consider the case  
 134 where  $m_1$  and  $m_2$  are fixed, so that the agent is only able to move its arm supporting a camera-like sensor  
 135 generating a sensation  $\mathbf{s}$ , i.e.  $\mathbf{m}$  is restricted to  $(m_3, m_4, m_5)$  only. In such a scenario, one has a fixed base

136 agent for which each motor configuration  $\mathbf{m}$  can be mapped to one corresponding sensor pose, which is  
137 itself mapped to a sensation  $\mathbf{s}$ . This simple statement allows to build structures in  $\mathcal{M}$  by exploiting only the  
138 sensorimotor flow  $(\mathbf{m}, \mathbf{s})$ , structures that can be leveraged to build an internal representation of the agent  
139 body; they can be further refined into a representation of its peripersonal space (Marcel et al., 2017; Marcel  
140 et al., 2019). In these works,  $\mathbf{m}$  carries all spatial data, possibly with some redundancy, about the coupling  
141 between the agent and its environment: the combination of states  $\mathbf{m}$  and  $\mathbf{e}$  is sufficient to determine the  
142 resulting sensory output  $\mathbf{s}$  (as described by the formal sensorimotor map  $\psi$ ).

143 However, what would happen if the same agent was able to perform translations in its environment? Let  
144 us now focus on the case where *all* motor states  $m_i$  are actually used, as depicted in Figure 1. Indeed, one  
145 can imagine a case where the agent moves in its environment along the  $x$  axis from (external) position  
146  $\tau$  (with internal configuration  $\mathbf{m}$ ) to  $\tau'$  (same  $\mathbf{m}$ ). In this case, the sensor samples two different parts of  
147 the color wall so that its generated sensations  $\mathbf{s}$  and  $\mathbf{s}'$  from these two different positions are different.  
148 Then two identical internal configurations  $\mathbf{m}$  give two different sensations: there is no more mapping  
149 between  $\mathbf{m}$  and  $\mathbf{s}$ , and all the mathematical developments performed in previous works can no longer apply.  
150 Therefore, it seems necessary to generalize these formalisms to cope with agents able to move freely in  
151 their environment. In this paper, one proposes a *variational* formulation of motor actions to deal with this  
152 issue. Importantly, the term *variational* refers in all the following to the focus given on specific sequences  
153 of states (e.g. motor, sensory or external states) rather than any specific one of said states. It is introduced  
154 in the next subsections.

### 155 2.1.2 Dealing with mobile agents: reparameterizing the sensorimotor interaction

156 From previous arguments, the internal motor configuration  $\mathbf{m}$  can not be mapped unambiguously to  
157 sensations without additional considerations. If one still insists on having a functional relation between  
158 motor data and sensations, one then needs to enrich the initial motor set. In this paper, one proposes to  
159 introduce some superset  $\mathcal{B}$  of  $\mathcal{M}$  as initial parameter space. This new set  $\mathcal{B}$  can be thought of as the set  
160 of all *absolute* configurations  $\mathbf{b}$  made of pairs  $(\mathbf{m}, \tau)$  where  $\mathbf{m}$  is the internal motor configuration and  $\tau$   
161 represents an absolute measure of the pose of the agent in its ambient space (which would most commonly  
162 be position and orientation in 3D space). While posture or proprioception give  $\mathbf{m}$  a definite meaning,  
163 one should instead only think of  $\tau$  as a choice of reference frame in space. Indeed for spaces much like  
164 ours it ought to be somewhat arbitrary since any “displacement” from pose  $\tau$  to  $\tau'$  could be instead  
165 realized as an opposite motion of the whole ambient space like for compensatory movements as initially  
166 introduced in (Poincaré, 1895) and dealt with in (Philippona et al., 2003). This equivalency argument has  
167 been mentioned in previous contributions as a possible way to deal with mobile agents as proposed in this  
168 contribution. While this could be formalized in a consistent way, either from a quasi static or variational  
169 perspective, we argue the proposed point of view offers usability advantages, like for multi-agent situations  
170 (where space should emerge as a shared common playground), an easier formulation of compensability, or  
171 a clear separation between the internal and external points of view. It is important to understand that the  
172 agent itself has no knowledge of the current absolute configuration  $\mathbf{b}$  of its interaction with its environment,  
173 retaining the same hypotheses about a priori structure. However we may then consider the sensorimotor  
174 map as a function  $\psi : \mathcal{B} \times \mathcal{E} \rightarrow \mathcal{S}$  instead of  $\psi : \mathcal{M} \times \mathcal{E} \rightarrow \mathcal{S}$  to account for possible displacements in the  
175 environment. Defining such a new set  $\mathcal{B}$  allows then to introduce the notion of *external* agent configuration  
176 as the tuple  $(\mathbf{b}, \mathbf{s})$ . As such, two different points of view must be stressed out: (i) the external point of  
177 view (i.e. coming from the designer of the system) will allow to characterize some properties of the agent  
178 interaction with its environment (through modelization, hypotheses, etc.), and (ii) the internal point of view

179 which represents which data and concepts are available to the agent for its operations. This specific point is  
 180 discussed in §3.1.

181 Coming back to Figure 1, the agent moves to three successive absolute configurations  $\mathbf{b}$ ,  $\mathbf{b}'$  and  $\mathbf{b}''$ . All of  
 182 them are now different, which was not the case of the internal motor configurations: introducing  $\mathbf{b} \in \mathcal{B}$   
 183 apparently solves the issue raised at the end of §2.1.1. Let us now explain how the agent actually reaches  
 184 some given absolute configuration  $\mathbf{b}$ .

### 185 2.1.3 Going variational: introducing motor actions

186 As explained previously, the agent has no direct access to the configuration data  $\mathbf{b}$ : it cannot know where  
 187 it *is* in  $\mathcal{B}$ . Instead we suppose it starts with some (very limited) knowledge of how it *moves* in this set,  
 188 i.e. it is capable of performing some moves in  $\mathcal{B}$  and of comparing any two moves for equality. To this  
 189 end, we propose to introduce some new set  $\mathcal{A}$  behaving in the following manner: an element  $a \in \mathcal{A}$  can be  
 190 applied to any absolute configuration  $\mathbf{b} \in \mathcal{B}$  to give a new configuration  $\mathbf{b}' = a\mathbf{b} = a(\mathbf{b})$ . Therefore,  $a$  can  
 191 be seen as a function  $\mathcal{B} \rightarrow \mathcal{B}$ . We will usually denote  $\mathbf{b} \xrightarrow{a} \mathbf{b}'$  this situation, and call  $a$  a *motor action*.  
 192 Such a definition for “actions” differs from many intuitions since it is restricted to quasistatic differences in  
 193 posture and position; it does not account for a notion of dynamical effort exerted by actuators. In particular,  
 194 no dynamical effects are considered at this level and no specification is made of the precise motor path  
 195 taken from  $\mathbf{b}$  to  $a\mathbf{b}$ . Instead, only these pairs of related  $(\mathbf{b}, a\mathbf{b})$  endpoints are relevant to characterizing any  
 196 action  $a$ . This constitutes a present limitation pervading much of similar works to which a future –and  
 197 assuredly significant– contribution shall be devoted.

198 Now as we intend to represent the way in which the agent can move in its environment, one can take for  
 199 granted the existence of a special action  $e \in \mathcal{A}$  that verifies  $\forall \mathbf{b} \in \mathcal{B}, \mathbf{b} \xrightarrow{e} \mathbf{b}$ : the agent may decide to stay  
 200 still. Note that for certain systems, e.g. drones or bipedal walkers, this is distinctly different from doing  
 201 nothing since constant posture and position must still be maintained. Moreover, considering it is able to do  
 202 any moves  $a$  and  $a'$ , it may then chain them in one single move  $a'' = a'a \in \mathcal{A}$  which satisfies

$$\forall \mathbf{b} \in \mathcal{B}, \quad \begin{array}{ccc} \mathbf{b} & \xrightarrow{a''} & \mathbf{b}'' \\ & \searrow a & \nearrow a' \\ & \mathbf{b}' & \end{array} \quad (1)$$

203 so that  $\mathcal{A}$  naturally carries the structure of a *monoid*. Remark that this composition operation is necessarily  
 204 associative since motor actions are assumed to behave as functions  $\mathcal{B} \rightarrow \mathcal{B}$ . In the following, we will  
 205 further restrict ourselves to the case where individual actions are *reversible*, that is for any action  $a$  there  
 206 exists an action  $a^{-1}$  such that

$$\forall \mathbf{b} \in \mathcal{B}, \quad \mathbf{b} \xrightleftharpoons[a^{-1}]{a} \mathbf{b}', \quad (2)$$

207 making  $\mathcal{A}$  into a *group*. Seeing as how actions can be thought of as mappings  $\mathcal{B} \rightarrow \mathcal{B}$ , a necessary (and  
 208 sufficient) condition is for all mappings in  $\mathcal{A}$  to be bijective. It is clear that this assumption of invertibility  
 209 may not apply in some experimental contexts, e.g. an agent may jump down a height which it cannot jump  
 210 up. This constitutes a current limitation of the proposed framework, although several factors may limit its  
 211 severity. In this example, even if the agent can not jump up the height directly, it may still find a sequence  
 212 of actions allowing it to climb back up to its original position. Corresponding to the definition of Eq. (2),  
 213 this would result in an inverse action in the formal sense, as illustrated later in Section 4.3.3

214 Figure 1 illustrates these notions, with the agent moving from external configuration  $\mathbf{b}$  to  $\mathbf{b}'$  through  
 215 an action  $a$ . This action, applied at  $\mathbf{b} = (\mathbf{m}, \tau)$ , happens to produce a translation of the agent so that its  
 216 internal motor configuration finishes at the same  $\mathbf{m}$ . Note that the agent would be able to return back to  
 217 its initial absolute configuration by applying the inverse action  $a^{-1}$  of  $a$ . Moreover, since  $a$  is a function  
 218 defined on the whole  $\mathcal{B}$  set, the same action can be applied at  $\mathbf{b}' = (\mathbf{m}, \tau')$  to reach a third configuration  
 219  $\mathbf{b}'' = (\mathbf{m}'', \tau'')$ . This time, the same action  $a$  has conducted to a global displacement of the agent in the  
 220 environment, combined with a change in its internal motor configuration. Indeed, while it represents cases  
 221 which are mostly avoided for practical reasons, it is not required for  $a$  to only depend on  $\mathbf{m}$  in the general  
 222 case: the outcome of the same action  $a$  may depend on the position  $\tau$  of the agent in the environment.  
 223 Finally, the agent would have been able to move from  $\mathbf{b}$  to  $\mathbf{b}''$  by applying the action  $a'' = a^2$ , as per  
 224 Equation (1).

225

226 With these structure assumptions, for a given subset of *motor primitives*  $\mathcal{A}' \subset \mathcal{A}$  available to the agent,  
 227 we can search for the set of composed moves the agent can actually reach by iteration of its known ones.  
 228 We shall say an action  $a \in \mathcal{A}$  *decomposes over*  $\mathcal{A}' = \{a_i\}_{i \in I}$  if it can be written in the form

$$a = a_{i_n} \dots a_{i_1} = \prod_{1 \leq k \leq n} a_{i_k}, i_k \in I \quad (3)$$

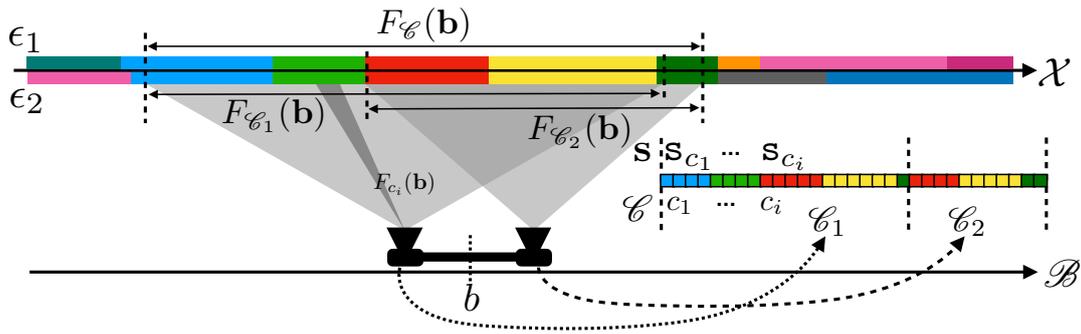
229 This represents a formal property functionally similar to that of compositionality of motor trajectories,  
 230 with  $\mathcal{A}'$  filling for actions the role of primitives (Flash and Hochner, 2006). Indeed, the interest of these  
 231 decompositions appears because the effect of composed moves on motor configurations boils down to the  
 232 effects of its components as per the following diagram:

$$\forall \mathbf{b} \in \mathcal{B}, \quad \begin{array}{ccc} \mathbf{b} & \xrightarrow{a} & \mathbf{b}' \\ & \searrow^{a_{i_1}} & \nearrow^{a_{i_n}} \\ & \mathbf{b}_1 & \xrightarrow{a_{i_2}} \dots \xrightarrow{a_{i_{n-1}}} \mathbf{b}_{n-1} & \end{array} \quad (4)$$

233 In the example in Figure 1, it may well be that the agent can move to any configuration  $\mathbf{b}$  i.e. that its  
 234 action set is  $\mathcal{A} = \mathbb{R}^5$  (for the 5 possible angular increments of its 5 joints). But it may also be restricted  
 235 to a limited set of moves, for example if it only can send discrete commands to its joints. For instance, if  
 236 each actuator is a stepper motor, then its action set turns into  $\mathcal{A} = \mathbb{Z}^5$ . In this case,  $a$  would be written  
 237 as the tuple  $(\Delta q_i)_i, i = 1, \dots, 5$ , where  $\Delta q_i \in \mathbb{Z}$  is the  $i^{\text{th}}$  motor increment expressed in step increments.  
 238 Consequently, any action  $a$  would decompose over  $\mathcal{A}' = \{a_i\}_i$  where action  $a_i$  corresponds to adding  
 239 one step to the  $i^{\text{th}}$  actuator. In this specific case, while  $\mathcal{A}$  is infinite, it is sufficient for the agent to know  
 240 the 5 motor primitives  $a_i$  to generate any action  $a \in \mathcal{A}$ . This is very similar to the notion of reducing a  
 241 (finite dimensional) vector space, which is usually infinite, to the very finite subset of a generating set or if  
 242 possible a base. However it can be proven that any finite subset of  $\mathbb{R}$  will *not* generate it as a group, and that  
 243 it will often only generate a discrete  $k\mathbb{Z}$  subgroup. This occurs in the proposed simulations in §4, where all  
 244 combinations of a finite subset of starting actions lead to the discovery of a discrete generated action group.

## 245 2.2 Grounding sensations in space

246 The previous subsection was devoted to the introduction of actions on the motor side of the proposed  
 247 sensorimotor framework. This subsection accordingly deals with the *sensory* side of it, and more particularly



**Figure 2.** Illustration of the receptive fields for a sensor made of two rigidly linked cameras at configuration **b**. Each pixel  $c_i$  of either camera produces a sensory value  $s_{c_i}$  in the overall sensory array  $\mathbf{s}$  explained only by a small subset of space  $F_{c_i}(\mathbf{b})$ . The same applies for both cameras, thus explaining how a sensation for the agent can be explained by the perception of a subset of space.

248 with its relation to a persistent “space” which was entirely absent from previous considerations. After a  
 249 more precise definition of the meaning of “environment configuration”, the link between local perception  
 250 and spatial considerations is formalized. This will constitute the root of the theoretical developments  
 251 proposed in the next section.

### 252 2.2.1 Decoupling space and environment : the *where* and the *what*

253 In previous works, a traditional way for parameterizing the environment was to introduce the environment  
 254 configuration  $\epsilon$ . The meaning of such a variable was often left unspecified, almost without any formal  
 255 semantics linking it to the sensorimotor experience of the agent (Laflaquière et al., 2015). In this paper  
 256 it is proposed to stress the difference between the ambient geometrical *space*—in which sensorimotor  
 257 experience occurs—and the *environment* itself—that is the state of “things” lying in this space. The former  
 258 takes the form of some set  $\mathcal{X}$  endowed with a spatial structure as encoded by a group  $\mathcal{G}(\mathcal{X})$  of *admissible*  
 259 *transformations*. These spatial transformations are mappings  $\mathcal{X} \rightarrow \mathcal{X}$  preserving some “geometry” of  $\mathcal{X}$ .  
 260 The most common illustration is the usual affine geometry of  $\mathbb{R}^3$  given by the group  $\text{SE}_3(\mathbb{R}) = \text{SO}_3(\mathbb{R}) \rtimes \mathbb{R}^3$   
 261 of its rigid transformations, made of 3D rotations, translations and their compositions. On this basis, one  
 262 chooses to particularize a “state of the environment” as a valuation that maps each point of  $\mathcal{X}$  to its  
 263 corresponding physical properties such as temperature, color, luminance, etc. These states are therefore  
 264 best represented as functions  $\epsilon : \mathcal{X} \rightarrow \mathcal{P}$  where  $\mathcal{P}$  is a set describing the different physical properties the  
 265 agent can observe. Consequently,  $\epsilon(\mathbf{x})$  represents the observable physical properties at point  $\mathbf{x} \in \mathcal{X}$ . We  
 266 will henceforth denote  $\mathcal{E}$  the set of *environment states*, i.e. a set of such functions  $\epsilon$ .

267 Figure 2 illustrates these considerations. In this simple case, the geometrical space  $\mathcal{X}$  is monodimensional,  
 268 represented as an axis where each point  $\mathbf{x}$  is assigned a color through a function  $\epsilon_1$  or  $\epsilon_2$ . Interestingly,  
 269 one can now distinguish points in  $\mathcal{X}$  on which the physics described by different environment states  $\epsilon_1$   
 270 and  $\epsilon_2$  locally coincide, as far as the agent is able to observe this coincidence. Particularizing the former  
 271 unspecified  $\epsilon$  state to a function  $\epsilon$  of the spatial variable will allow to express new properties of the  
 272 sensorimotor experience, as in the next subsection.

### 273 2.2.2 Local perception and receptive fields

274 Now that we have formally defined what is “out there” from an external point of view, let us now focus on  
 275 the sensory capabilities of the agent. On this specific point, most previous contributions were considering  
 276 the full sensory output as atomic data: although it is implemented as a possibly high dimensional vector,

277 elements and subarrays were generally kept from scrutiny. On the contrary, we now take interest at the  
278 subarray level and accordingly adapt the formalism. Therefore,

279 in all the following the sensorimotor map is written as  $\psi_c : \mathcal{B} \times \mathcal{E} \rightarrow \mathcal{S}_c$  where the  $c$  subscript outlines  
280 that the sensory map is explicitly written for a sensory element  $c$  (or *sensel*, i.e. one pixel for a camera, the  
281 cochlea cell coding for one sound frequency, etc.). Thus, the sensorimotor map  $\psi_{\mathcal{C}}$  for the entire sensory  
282 apparatus is made of the aggregate of all sensels along  $\psi_{\mathcal{C}} : \mathcal{B} \times \mathcal{E} \rightarrow \mathcal{S} = \prod_{c \in \mathcal{C}} \mathcal{S}_c$  with  $\mathcal{C}$  the set of  
283 all sensels<sup>1</sup>. An illustration of these points is proposed in Figure 2 for a similar agent endowed this time  
284 with two cameras so as to better show the descriptive capabilities of the formalism. In this case, the sensels  
285  $c_i$ —each depicted as elements in a color array— represent the pixels of either camera. Separate sensors in  
286 the apparatus thus appear as sub-arrays in  $\mathcal{C}$ : the first (resp. second) camera is figured by  $\mathcal{C}_1$  (resp.  $\mathcal{C}_2$ ).  
287 Note that this decomposition of  $\mathcal{C}$  as  $\mathcal{C}_1 \cup \mathcal{C}_2$  directly comes from our external understanding of the agent  
288 structure (i.e. with one camera corresponding to one set of sensels, i.e. one sensor). One could have selected  
289 others sub-arrays to form a distinct set of (virtual) sensors not necessarily corresponding to their (physical)  
290 implementation on the agent.

291 With space and sensors made formally precise we can now proceed with the (spatial) *receptive field* of  
292 sensor  $\mathcal{C}' \subset \mathcal{C}$ , that is a region of space which environment state suffices to determine the output of  $\mathcal{C}'$ .  
293 This region as a subset of  $\mathcal{X}$  should naturally depend on the current configuration  $\mathbf{b}$  since moving causes  
294 one's sensors to sample new parts of space, so that it takes the form of a map  $\mathbf{b} \in \mathcal{B} \mapsto F_{\mathcal{C}'}(\mathbf{b}) \subset \mathcal{X}$ .  
295 Then, its characteristic property is

$$\begin{aligned} \forall \epsilon_1, \epsilon_2 \in \mathcal{E}, \forall \mathbf{b} \in \mathcal{B}, \\ \epsilon_1|_{F_{\mathcal{C}'}(\mathbf{b})} = \epsilon_2|_{F_{\mathcal{C}'}(\mathbf{b})} \Rightarrow \psi_{\mathcal{C}'}(\mathbf{b}, \epsilon_1) = \psi_{\mathcal{C}'}(\mathbf{b}, \epsilon_2). \end{aligned} \quad (5)$$

296 Figure 2 represents some of the receptive fields for the two cameras agent. The first one,  $F_{c_i}(\mathbf{b})$ , is the  
297 receptive field of a single sensel/pixel  $c_i \in \mathcal{C}$ . The receptive fields  $F_{\mathcal{C}_1}(\mathbf{b})$  and  $F_{\mathcal{C}_2}(\mathbf{b})$  of each camera can  
298 be obtained as the union of the receptive field  $F_{c_j}(\mathbf{b})$  of their respective pixels. In the same vein, the overall  
299 receptive field of the agent  $F_{\mathcal{C}}(\mathbf{b})$  is also given by  $F_{\mathcal{C}_1}(\mathbf{b}) \cup F_{\mathcal{C}_2}(\mathbf{b})$ . From the same figure, it is clear that  
300 even if  $\epsilon_1 \neq \epsilon_2$  (since there are areas of different colors on the  $\mathcal{X}$  axis), the sensation captured by the agent  
301 is the same since the aforementioned differences are restricted to areas of space unseen to the agent.

302 It is important to notice that this is the formal step where the notion of receptive field formalizes an  
303 implicit relation between the sensations of the agent and spatial structure. This constitutes one fundamental  
304 property sufficient to leverage spatial knowledge from the agent interaction with its environment. The  
305 application of these theoretical elements is proposed in the next section.

### 3 A ZERO-TH LAYER OF SENSORIMOTOR CONTINGENCIES: SPATIAL REGULARITIES THROUGH VARIATIONS

306 In this section, we proceed by describing how the formal elements from Section 2 can be arranged  
307 to enunciate some interesting properties of the sensorimotor interaction. First, to keep in line with  
308 considerations of minimalist bootstrapping, the assumptions we use relative to the model of knowledge of  
309 the agent are discussed, and compared to that of previous contributions. Then, the definitions provided in  
310 the previous sections are used to isolate conditions where the spatial structure of the receptive fields can be

<sup>1</sup> In all the following, the map  $\psi_{\mathcal{C}}$  will be shortened to  $\psi$  when there is no ambiguity, consistently with the initial definition of the sensorimotor map of the agent recalled in §2.1.1.

311 leveraged, in particular *via* a certain class of “conservative” actions which are themselves defined. We prove  
 312 that under these conditions a naive agent may achieve the determination of sensory prediction functions  
 313 for said conservative actions, and that the algebraic structure of these prediction functions matches that  
 314 of their actions. The corresponding results are of two distinct but equally important sorts: some, taking  
 315 the viewpoint of an external observer, assert that certain particular objects of interest (such as a *sensory*  
 316 *prediction function*) *exist*; others guarantee these objects to be *computable* in the boundaries set by our  
 317 model of knowledge. This endeavor is made in an effort to keep *a priori* knowledge to a minimum, and  
 318 these proofs are generally of a constructive nature.

### 319 3.1 Model of knowledge of the agent

320 In the authors’ previous works (Marcel et al., 2017; Marcel et al., 2019), sensorimotor interaction  
 321 occurred as a sequence of (generally discrete) steps where at each point, the agent could access both  
 322 its proprioception  $\mathbf{m} \in \mathcal{M}$  (seen as an array of current joint configuration states) and its corresponding  
 323 exteroceptive array  $\mathbf{s} = \psi(\mathbf{m}, \epsilon)$ . These sensory arrays were then compared for equality (and for equality  
 324 *only*) *as total vectors*, that is the agent may not access the vectors component by component. It is crucial  
 325 to note that, much like in the referenced articles —and following the argument that “there is no *a priori*  
 326 reason why similar neural processes should generate similar percepts” as found in (O’Regan and Noë,  
 327 2001)— we will assume here that the sensory signals are *uninterpreted* in the very strong sense that they  
 328 retain no other structure than equality. This presents an *a priori* significant hurdle since this includes e.g.  
 329 order comparisons, substractions, metric structures and precludes us from using objects such as gradients  
 330 or clusterings, which are required in almost all comparable works ((Censi and Murray, 2012; Montone  
 331 et al., 2015; Laflaquière, 2017) among others). This knowledge was then used for example to compute  
 332 set-theoretic motor kernels (Marcel et al., 2017) which were shown to be a structural invariant of the  
 333 sensorimotor interaction (Marcel et al., 2019). By contrast, in this paper slight modifications are applied.  
 334 Indeed, from the external point of view we now have  $\mathcal{B}$  as a functional analogue to the previous  $\mathcal{M}$ ,  
 335 that is the set of “parameter” data that entirely determines the state of the interaction between agent  
 336 and environment. However, as the definition of  $\mathcal{B}$  refers to some explicitly *external* data (i.e. the  $\tau$  in  
 337  $\mathbf{b} = (\mathbf{m}, \tau)$ ), we cannot assume its knowledge from the point of view of the agent. We could however elect,  
 338 on the same basis previous contributions used, to assume internal knowledge of the  $\mathbf{m}$  part of  $\mathbf{b} = (\mathbf{m}, \tau)$ .  
 339 Instead, we even assume no direct access to “proprioceptive” data and treat it as unknown to the agent. Our  
 340 hypothesis is that the agent should learn to isolate what part of its proprioception lies in its unified sensory  
 341 array  $\mathbf{s}$  from the statistics of its sensorimotor experience.

342 As for remedies, it is instead where a variational approach, as defined in Section 2.1.1, is preferred: while  
 343 configuration data represented by  $\mathbf{b} \in \mathcal{B}$  still exists as an *external* object, the agent may only choose a  
 344 *motor action*  $a \in \mathcal{A}$  which, applied at  $\mathbf{b}$ , yields the following configuration  $\mathbf{b}' = a\mathbf{b} = a(\mathbf{b})$ . The agent is  
 345 therefore given the capacity to compare any two elements of  $\mathcal{A}$  for equality, so that it may tell whether  
 346 at any two steps of its sensorimotor experience it performed two identical or distinct actions. Moreover,  
 347 much deeper change in knowledge occurs at the level of sensory readings: in the following we not only  
 348 ask that the agent be able to compare its entire sensory output  $\mathbf{s} = (s_c)_{c \in \mathcal{C}}$  for equality as a vector, but  
 349 that it also can check for equality two values of any given sensel. That is, for every sensel  $c \in \mathcal{C}$ , for  
 350 every values  $s_c, s'_c$  this sensel may output, the agent may test whether  $s_c == s'_c$ . In this contribution, it  
 351 will be further assumed that the values output by distinct sensels are themselves *a priori* comparable for  
 352 equality. While it is a common property in many classical applications, this limitation has been partially  
 353 tackled in (Laflaquière, 2017) via sensory prediction. However, this solution relies on clustering methods

354 implicitly exploiting structure assumptions we do not yet consider available. Therefore, this remains a  
 355 current limitation of our approach which will be addressed in a future ongoing work.

### 356 3.2 Sensorimotor binding: a marketplace for spatial information

357 The formalism introduced in Section 2 makes space appear as a variable in the sensorimotor equations  
 358 via the receptive field, which we will use in this section to prove that under some reasonable assumptions  
 359 we can talk about the spatial information content of a sensory signal. This in turn is used to form the basis  
 360 of a *sensory prediction* the agent can use to try and infer the sensory consequences of its motor actions,  
 361 mirroring the psychological construct of forward sensory model which is at the heart of ideomotor theories.  
 362 This is the core idea we will further develop in the simulations of Section 4 to see how a naive agent can  
 363 derive such a prediction function from its sensorimotor flow.

364 Recall that for any given sensel  $c \in \mathcal{C}$  and environment state  $\epsilon \in \mathcal{E}$ , we introduced  $F_c$  the receptive field  
 365 of sensel  $c$  as the function which given agent configuration  $\mathbf{b} \in \mathcal{B}$  yields  $\mathcal{X}' = F_c(\mathbf{b}) \subset \mathcal{X}$  the minimal  
 366 region of space which entirely determines the output of  $\psi_c$ . Therefore we can write

$$\forall \mathbf{b} \in \mathcal{B}, \forall \epsilon \in \mathcal{E}, \psi_c(\mathbf{b}, \epsilon) = f_c(\epsilon|_{F_c(\mathbf{b})}), \quad (6)$$

367 where  $f_c$  is a “sensitivity” function (or filter) which converts the physical properties of environment sampled  
 368 into a sensory output, both selecting to which property the sensor reacts and how. Equation (6) describes  
 369 the sensorimotor dynamics by dissociating the spatial dependency (which is given by  $F_c$ ) and the sensitivity  
 370 one (as seen with  $f_c$ ), so that the observer can now speak of sensels that *look at the same region of space*.  
 371 Let us then consider a particular condition, in which for a given action  $a$ , some sensel  $c_i$  samples *after*  $a$  the  
 372 same point some other sensel  $c_j$  was sampling before the agent began to move. This is formally described  
 373 by the relation

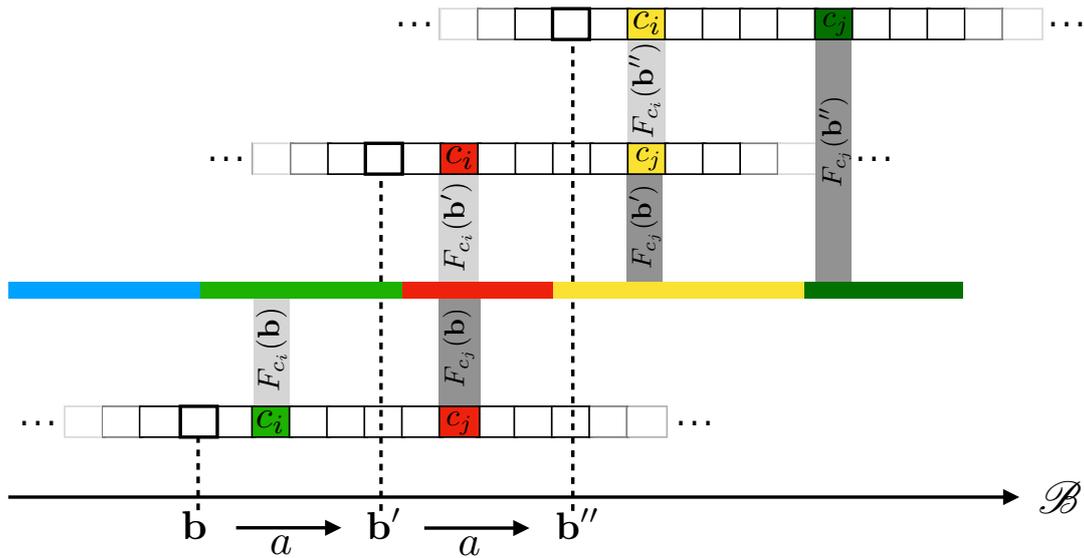
$$\forall \mathbf{b} \in \mathcal{B}, F_{c_i}(a\mathbf{b}) = F_{c_j}(\mathbf{b}). \quad (7)$$

374 This situation where the spatial difference between two sensels can be bridged by a displacement of the  
 375 agent over time to make their respective sensory experiences coincide has already proven to yield interesting  
 376 structures as in (Montone et al., 2015; Laflaquière, 2017). However, these works largely dealt with the  
 377 geometry of sensels and sensors, while we aim to elaborate on how this relates to the structure of actions.  
 378 As for us, to have this relation apparent to the agent we also require that the output of these particular  
 379 sensels be comparable, as already discussed in §3.1. In the strictest sense, this can be by requiring that their  
 380 sensitivity functions  $f_{c_i}$  and  $f_{c_j}$  are equal. It follows that

$$\begin{aligned} \forall \mathbf{b} \in \mathcal{B}, \quad \forall \epsilon \in \mathcal{E}, \\ \psi_{c_i}(a\mathbf{b}, \epsilon) &= f_{c_i}(\epsilon|_{F_{c_i}(a\mathbf{b})}) \\ &= f_{c_j}(\epsilon|_{F_{c_j}(\mathbf{b})}) = \psi_{c_j}(\mathbf{b}, \epsilon). \end{aligned} \quad (8)$$

381 While there are reasons to hope that a working relation could be found even for dissimilar  $f_{c_i}$  and  $f_{c_j}$ , one  
 382 should remember that at the moment the outputs of sensels  $c_i$  and  $c_j$  lie in some sets totally devoid of  
 383 structure. Therefore, even though a conversion function  $C_{i,j}$  such that  $\psi_{c_i}(a\mathbf{b}, \epsilon) = C_{i,j}(\psi_{c_j}(\mathbf{b}, \epsilon))$  might  
 384 exist, we would lack the means to represent it in any way but the collection of the related sensory outputs,  
 385 e.g. as opposed to the already resource heavy clustering done in (Laflaquière, 2017).

386 Equations (7) and (8) are both illustrated in Figure 3, where a 1D (infinite) pixel array is placed in front  
 387 of a 1D colored line along which the sensor can translate itself thanks to actions  $a$ . Equation (7) is captured



**Figure 3.** Illustration of how the underlying 1D space induces transitions between cross-sensory outputs. In this case, under action  $a$ , sensel  $c_i$  takes the place of sensel  $c_j$ : the output of  $c_i$  after  $a$  (red) is the same as the output of  $c_j$  before  $a$  (red). The same applies when performing  $a$  a second time: the yellow color is transferred from  $c_j$  to  $c_i$ .

388 by the fact that both receptive fields  $F_{c_j}(\mathbf{b})$  and  $F_{c_i}(\mathbf{b}')$ , drawn as two rectangular shapes, project on the  
 389 same area on the environment. Then, Equation (8) explains how, provided the environment state  $\epsilon$  at these  
 390 locations stays constant through any one execution of  $a$ , it causes both sensels to actually generate the same  
 391 sensory (red) output. It is clear that the spatial relation being forwarded to sensory transitions depends on  
 392 the sensels actually outputting the same (red) value. This might be argued to be a restrictive assumption.  
 393 Nevertheless, being able to deal with different sensitivity functions is a sizable development to which an  
 394 ongoing contribution shall be devoted. To conclude, a key point here is that a property entirely defined from  
 395 the external point of view through receptive fields is accessible from the internal one by the constraints it  
 396 imposes on the sensels outputs values during exploration. Equation (8) therefore shows how space, insofar  
 397 as it is common to all sensels and actions, makes this phenomenon of shifts of receptive fields into an  
 398 observable contingency of the agent’s sensorimotor experience.

### 399 3.3 A motor and sensory account of spatial conservation

#### 400 3.3.1 Conservation through permutation: conservative actions

401 The result obtained in the previous subsection exhibits an important property making internally available  
 402 spatial matching between receptive fields at different timesteps of motor exploration. But given that the  
 403 actual motor exploration follows the algebraic structure of actions  $\mathcal{A}$ , it still remains to be shown how these  
 404 two structures are consistent. This can be made apparent by introducing *conservative actions* as those  $a$  of  
 405  $\mathcal{A}$  for which *all* sensels of the agent exchange the places they sample: there is conservation of the (spatial)  
 406 information available. In terms of the formalism,  $a \in \mathcal{A}$  is conservative if it verifies

$$\forall c \in \mathcal{C}, \exists c' \in \mathcal{C} \text{ such that } \forall \mathbf{b} \in \mathcal{B}, F_c(a\mathbf{b}) = F_{c'}(\mathbf{b}), \quad (9)$$

407 generalizing somewhat Equation (7). This characterization makes apparent that many actions can’t be  
 408 conservative: for example, “turning back” may only be conservative for the rare agent that “sees” precisely  
 409 as much forwards as it does backwards. In fact, the spatiality of the condition on receptive fields makes it

410 so that all readily found conservative actions correspond to displacements of the body of the agent. In the  
 411 following, “ $\forall \mathbf{b} \in \mathcal{B}, F_c(a\mathbf{b}) = F_{c'}(\mathbf{b})$ ” will be shortened to the more legible  $c \xrightarrow{a} c'$ , and  $c$  (resp.  $c'$ ) is said  
 412 to be the *predecessor* (resp. *successor*) of  $c'$  (resp.  $c$ ) by  $a$ . It is proven in Appendix 1 that for conservative  
 413 actions  $a$ , the relation  $\xrightarrow{a}$  can be made into a successor function

$$\begin{aligned} \sigma_a: \mathcal{C} &\rightarrow \mathcal{C} \\ c &\mapsto c' \end{aligned} \quad (10)$$

414 where  $c' = \sigma_a(c)$  is a sensel verifying  $c \xrightarrow{a} c'$ . Therefore, conservative actions can equivalently be thought  
 415 of as *permutation of sensels*. Importantly, conservative actions provide a natural framework for exploiting  
 416 Equations (7) and (8) during motor exploration. Indeed, it is proven in Appendix 2 that conservative actions  
 417 form a *subgroup*  $\mathcal{A}_{\mathcal{C}} \subset \mathcal{A}$  for its composition operation. That is to say, chaining conservative actions yield  
 418 other actions which are necessarily conservative, and the inverses of conservative actions are themselves  
 419 conservative.

420 At this stage, it has been shown how the spatial property of permutation of the receptive fields relates to  
 421 the intrinsic motor structure of the agent. However, this does not suffice to make this group structure of  
 422 conservative actions accessible to the agent given the *a priori* knowledge we discussed in Subsection 3.1,  
 423 since the dependency of the sensorimotor process on the spatial variable is implicit. We must therefore  
 424 go through one final step to relate the available informational content (i.e. sensory reading) to the motor  
 425 structure.

### 426 3.3.2 From permutation to prediction: making it into sensory territory!

427 Let us consider the agent at any point  $(\mathbf{b}, \mathbf{s})$  of its sensorimotor experience. Its sensory output is  
 428  $\mathbf{s} = \psi_{\mathcal{C}}(\mathbf{b}, \epsilon) = (s_c)_{c \in \mathcal{C}}$ , and for any action  $a$  it may perform this sensory output should shift to  
 429  $\mathbf{s}' = \psi(\mathbf{b}' = a\mathbf{b}, \epsilon)$  provided the environment state stays constant throughout the action. If we now restrict  
 430 ourselves to the case of conservative actions, we get

$$\begin{aligned} \mathbf{s}' &= (s'_c)_{c \in \mathcal{C}} \\ &= (\psi_c(a\mathbf{b}, \epsilon))_{c \in \mathcal{C}} = (f_c(\epsilon|_{F_c(a\mathbf{b})}))_{c \in \mathcal{C}} \\ &= (f_c(\epsilon|_{F_{\sigma_a(c)}(\mathbf{b})}))_{c \in \mathcal{C}} = (\psi_{\sigma_a(c)}(\mathbf{b}, \epsilon))_{c \in \mathcal{C}} \\ &= (s_{\sigma_a(c)})_{c \in \mathcal{C}} \end{aligned} \quad (11)$$

431 so that performing motor action  $a$  only results in a permutation of the components of the sensory output.  
 432 This permutation is exactly  $\sigma_a$ , and therefore is a constant of the agent which does not depend on the actual  
 433 current configuration  $(\mathbf{b}, \epsilon)$ . Equation (11) shows that any *conservative* action  $a \in \mathcal{A}_{\mathcal{C}}$  corresponds to a  
 434 *sensory* function

$$\begin{aligned} \Pi_a: \mathcal{S} &\rightarrow \mathcal{S} \\ (s_c)_{c \in \mathcal{C}} &\mapsto (s_{\sigma_a(c)})_{c \in \mathcal{C}} \end{aligned} \quad (12)$$

435 which verifies the property

$$\forall \mathbf{b} \in \mathcal{B}, \forall \epsilon \in \mathcal{E}, \psi_{\mathcal{C}}(a\mathbf{b}, \epsilon) = \Pi_a(\psi_{\mathcal{C}}(\mathbf{b}, \epsilon)). \quad (13)$$

436 Per this property,  $\Pi_a$  is a function which given *any* starting sensory reading of the agent can determine  
 437 the sensation it would experience after performing action  $a$  (provided the environment state stays constant

438 during  $a$ ). It must be reiterated that a crucial part is that this function operates on *sensory* data, which is  
 439 precisely the only data available to the agent.

### 440 3.4 Prediction as an internal proxy of the action group

441 From there, let us now consider

$$\begin{aligned} \Pi: \mathcal{A}_{\mathcal{E}} &\rightarrow \text{Bij}(\mathcal{S}) \\ a &\mapsto \Pi_a \end{aligned} \quad (14)$$

442 with  $\text{Bij}(\mathcal{S})$  the set of all bijections from  $\mathcal{S}$  onto itself, i.e.  $\Pi$  maps abstract motor actions to their sensory  
 443 prediction functions. As proven in Appendix 3, it establishes a group isomorphism between conservative  
 444 actions  $a \in \mathcal{A}_{\mathcal{E}}$  and their associated sensory prediction maps  $\Pi_a \in \Pi(\mathcal{A}_{\mathcal{E}})$ , so that

$$\mathcal{A}_{\mathcal{E}} \cong \Pi(\mathcal{A}_{\mathcal{E}}). \quad (15)$$

445 While Equation (15) written as is might easily pass as benign, it is actually a very powerful result and the  
 446 centerpiece of our argument. In a similar fashion to Equation (8) before it, this specifies how the algebraic  
 447 structure of (conservative) actions—which largely governs the sensorimotor experience—appears as  
 448 a contingency of the sensorimotor flow which can be picked up on by an agent as naive as outlined in  
 449 Section 3.1. Using the terminology introduced there, it shows how some *external* structures describing the  
 450 interaction between agent and environment can be captured from the *internal* point of view. In turn, it is the  
 451 enunciation—and the proof—of this result that motivate developing the formalism as in Section 2, going  
 452 as far back as absolute configurations  $\mathbf{b} \in \mathcal{B}$  and ambient space  $\mathcal{X}$ . Equation (15) will be leveraged as part  
 453 of the simulations in the following.

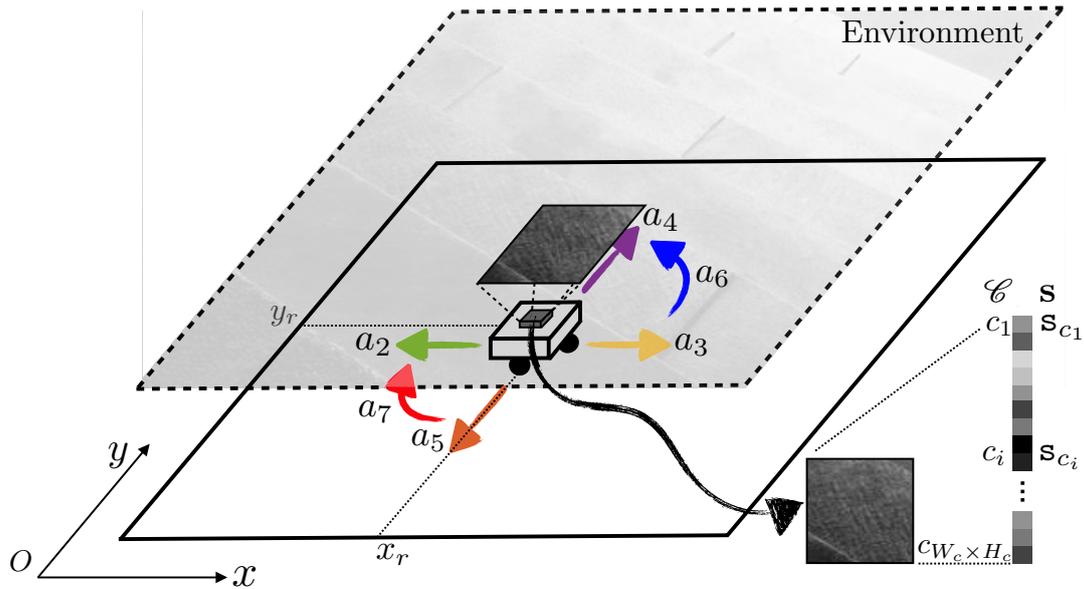
## 4 SIMULATING A 2D VERSION OF OUR TOY MODEL

454 Up until this point, the discussion has been kept to a purely theoretical level. The following section is now  
 455 devoted to a simulated experiment illustrating the new proposed formalism. To this end it starts with a  
 456 description of the experimental setup, highlighting how it manifests in the proposed formalism of Section  
 457 2. Then, we describe what steps the agent goes through and how they relate to the theoretical results we put  
 458 forth in the previous section. Finally, we review the observable results of these experiments to inspect how  
 459 our earlier theoretical claims appear in practical cases.

### 460 4.1 Description of the experimental setup

461 In the following, we will consider the 2D generalization of the illustrating case used in the previous sec-  
 462 tions. That is, the studied agent body is now made of a planar, rectangular camera sat atop omnidirectional  
 463 wheels, see Figure 4. These allow for translations along both  $x$  and  $y$  coordinates, as well as rotations in  
 464 the plane. The pixels of the camera are sensitive to the luminance of the ambient stimulus, which for our  
 465 experimental purposes is a fixed grayscale image placed above the moving camera. Describing the problem  
 466 in the terms of the developed formalism gives:

- 467 • the ambient space  $\mathcal{X}$  is the plane  $\mathbb{R}^2$ ;
- 468 • the set of physical properties of space  $\mathcal{P}$  is  $[0; 255]$  the set of luminance values. Therefore, a state of  
 469 the environment  $\epsilon \in \mathcal{E}$  is a function which takes points  $(x, y)$  of the ambient plane and map them to  
 470 luminances as given by the data of the acquired image;



**Figure 4.** Experimental setup used in simulation to assess the proposed formalism. A holonomic agent is placed in a 2D environment which ceiling is made of a fixed grayscale image. The agent can move in this environment by applying seven different actions  $a_k$ . A  $10 \times 10$  camera pointed towards the ceiling is placed on the top of the agent and generates a sensory array  $\mathbf{s} = (s_{c_i})_i$ .

- 471 • the configuration space  $\mathcal{B}$  is  $\mathbb{R}^2 \times S_1 \cong \mathbb{R}^2 \times ]-\pi; \pi]$  to account for both position  $(x, y)$  and orientation
- 472  $\theta$  of the robot on the plane;
- 473 • the sensory output of the agent is an array  $\mathbf{s} \in [0; 255]^{W_c \times H_c}$ , with  $W_c$  (resp.  $H_c$ ) the number of
- 474 sensels/pixels in one row (resp. one column) of the camera. In the simulation, the image dimension is
- 475 set to  $W_c = H_c = 10$ . Each of the components  $s_{c_i}$  of  $\mathbf{s}$  are the sensory output of pixel  $c_i$ , given by the
- 476 luminance of the spatial location in the environment it is currently looking at. Importantly, the order of
- 477 each pixel in  $\mathbf{s}$  is chosen arbitrarily.

478 Let us define a set  $A$  of *seven* basic actions  $a_k, k = 1, \dots, 7$ :

- 479 1. one identity action  $a_1$ , mapping any current absolute configuration to itself;
- 480 2. four translations  $a_2, a_3, a_4, a_5$ , one for each direction of the basis axes on the plane, all of amplitude
- 481 the size of 1 pixel. These are defined relative to the *current orientation of the agent*, which can end up
- 482 distinct from external systems of axes when the agent rotates;
- 483 3. two  $90^\circ$  rotations  $a_6, a_7$ , to account for both clockwise and counter-clockwise turns.

484 These actions are depicted in Figure 4 with colored arrows. Note that the color convention used in this

485 figure is the same used in the forthcoming figures for coherence.

486 Relative to the prior discussion about properties of motor actions, these are not strictly *conservative* as

487 per the definition (9): indeed, consider  $a_5$  the elementary “forward” translation. While inner pixels of the

488 camera will certainly exchange receptive fields, those in the front row will necessarily observe new areas

489 of space after the agent has moved forward. Therefore none of these front row pixels has any successor for

490  $a_5$ , which precludes it from being strictly conservative. The same phenomenon of border unpredictability

491 occurs for all translations, each with their respective side failing to verify the conservation property. We

492 nevertheless proceed with the formalism on the basis that actions are at worst, informally speaking, “quasi”

493 conservative. This is based on the quick analysis that, for a  $N$ -by- $N$  square camera, this defect only occurs  
 494 in  $N$  pixels which remains an order of magnitude fewer than the  $N^2$  total.

495 Representing the sensory configuration as numerical arrays makes the permutation of sensels into  $N_c$ -by-  
 496  $N_c$  sparse matrices, where  $N_c = W_c \times H_c$  is the number of sensels. Indeed, starting with any permutation  
 497  $\phi : \llbracket 1, N_c \rrbracket \rightarrow \llbracket 1, N_c \rrbracket$  we can define a matrix  $M_\phi \in M_{N_c, N_c}(\mathbb{R})$  by

$$M_{\phi,i,j} = \begin{cases} 1 & \text{iff } j = \phi(i), \\ 0 & \text{else.} \end{cases} \quad (16)$$

498 It can then be checked that for any array  $\mathbf{s} = (s_i)_{i \in \llbracket 1, N_c \rrbracket}$ , the array  $\mathbf{s}_\phi = (s_{\phi(i)})_{i \in \llbracket 1, N_c \rrbracket}$  obtained by  
 499 permutating the components of  $\mathbf{s}$  by  $\phi$  verifies  $\mathbf{s}_\phi = M_\phi \mathbf{s}$ . It is clear that working with such a representation  
 500 incurs a large memory overhead (with only  $N_c$  of all  $N_c^2$  coefficients being non null). Furthermore, finding  
 501 a permutation is known to be a problem of exponential complexity. However we do not aim to propose a  
 502 scalable implementation in the following, but rather to illustrate as a proof of concepts the developments in  
 503 Section 3.

## 504 4.2 Description of the experiments

505 The proposed simulation can be decomposed as a sequence of 2 related, successive, experiments. First,  
 506 these are briefly described in a global manner so as to go through the flow of the experiment. Then, each  
 507 experiment is described in greater detail with respect to its implementation. It is in this second part that  
 508 relevant proofs ensuring both completion and correctness of the endeavor are provided. In this setup, the  
 509 robot is given a set  $\mathcal{A}_{\text{init}}$  of  $n_{\mathcal{A}}$  unknown actions drawn in the set of *combinations* of actions of  $A$ . Although  
 510  $A$  was designed for convenience from an external point of view,  $\mathcal{A}_{\text{init}}$  may not accurately reflect it. Indeed,  
 511 for random draws there is a high likelihood of missing actions when  $n_{\mathcal{A}}$  is small, of duplicate actions  
 512 when it is large. However, as discussed previously these notions do not yet make sense to the agent, which  
 513 can only “run” actions drawn. Importantly, at first the considerations will be restricted to the case where  
 514  $\mathcal{A}_{\text{init}} = A$ . This is a possibly strong assumption about the initial fitness of readily available commands  
 515 to the “objective” capabilities of the agent. The influence of this choice and the effect of less optimally  
 516 designed starting command shall be discussed in the final part of this section.

517 The first part of the experiment is one of *motor babbling*. During it, the agent effectively runs its available  
 518 actions  $a_k \in \mathcal{A}_{\text{init}}$  multiple times and tries to figure out whether they are *conservative* by computing  
 519 their associated sensel permutation map. This is realized as a sequential process: at timestep  $t_n$ , the agent  
 520 chooses and runs an action  $a_k = a[t_n] \in \mathcal{A}_{\text{init}}$ , and the absolute configuration  $\mathbf{b}[t_n] = (x[t_n], y[t_n], \theta[t_n])$   
 521 is accordingly changed to  $\mathbf{b}[t_{n+1}] = a_k \mathbf{b}[t_n]$ . Corresponding sensory array  $\mathbf{s}[t_{n+1}] = (s_i[t_{n+1}])_i$  is then  
 522 used to proceed in the computation of the (candidate) permutation matrix  $M_{a_k}$  of  $a_k$ , with the details of  
 523 the update rule discussed in the following subsection. It must again be stressed that we do not consider  
 524 the actual time  $t_{n+1} - t_n$  required to perform the action  $a$  as a relevant information of the proposed  
 525 sensorimotor framework. It may vary for distinct actions without it affecting whatsoever the sequence  
 526 of experienced absolute configurations  $\mathbf{b}$ . During this exploration, the state of the environment  $\epsilon$  is also  
 527 allowed to vary with time so long as it is not updated during the generation of  $a_k$ , i.e.  $\epsilon$  can change *between*  
 528 actions. This is achieved during the simulation by entirely changing the grayscale image presented to the  
 529 agent between each action. In the following, the choice of action is randomly made at each timestep. This  
 530 may at times slow the learning of the permutation matrices and could certainly be improved, for instance

531 by introducing a necessarily intrinsic criterion like curiosity as in (Oudeyer et al., 2005). However, the  
532 specific case studied here is simple enough that a random strategy suffices to obtain good results.

533 Once this first step is complete, the agent computes all products of (quasi) permutation matrices to make  
534 the resulting set of matrices. As per Equation (4), this set is precisely the one of all matrices that decompose  
535 over the  $M_{a_k}$ ,  $a_k \in \mathcal{A}_{\text{init}}$ . Following our argument about the groups of prediction functions and motor  
536 actions being isomorphic, this set can be taken as the *global* understanding of its motor capabilities the  
537 agent has acquired. Here “global” denotes that new structure, absent from the first empirical phase which  
538 was limited to  $\mathcal{A}_{\text{init}}$ , emerged from the computation of products. Finally, the effect of changing the set of  
539 actions available at start on the structure graph discovered in the second experiment is studied in a third  
540 part, see 4.3.3.

#### 541 4.2.1 Learning the prediction through sensorimotor interaction

542 The first experiment performed by the agent is computing, where possible, the permutation matrix  
543 associated to each of its available motor actions. This is done according to the following procedure: at the  
544 beginning of the sensorimotor experience, to each starting action  $a_k \in \mathcal{A}_{\text{init}}$  associate a  $N_c \times N_c$  matrix  
545  $M_{a_k}$  where  $N_c$  is the number of sensels. This matrix is initialized so that all of its coefficients are 1. Then,  
546 at the end of timestep  $t_n$  where it performed action  $a_k$  (that is  $a[t_n] = a_k$ ), the agent uses its sensory output  
547 arrays both previous ( $\mathbf{s}[t_n]$ ) and current ( $\mathbf{s}[t_{n+1}]$ ) as per the update rule:

$$(M_{a_k}[t_{n+1}])_{i,j} = \begin{cases} 1 & \text{iff } s_j[t_{n+1}] = s_i[t_n] \text{ and } (M_{a_k}[t_n])_{i,j} = 1 \\ 0 & \text{else.} \end{cases} \quad (17)$$

548 Let us first observe that in this rule the only possible change in coefficients is going from 1 to 0: whenever  
549 a coefficient  $(M_{a_k}[t_n])_{i,j}$  is already 0, the condition of the first case automatically fails so that its value  
550 stays at 0. Therefore, the rough dynamics of the update is that while all coefficients start at 1, some are  
551 eventually switched to 0 upon exploration until matrices converge to a final (possibly null) form.

552 One can note that this is a very drastic choice compared to the more usual soft incremental rules. This  
553 offers increased simplicity such as in Appendix 4 where an argument is provided that for any conservative  
554 motor action this algorithm makes the empirical matrix  $M_{a_k}$  converge to the associated permutation matrix  
555  $M_{\sigma_{a_k}}$ . Moreover, we argue that obtaining said convergence with such an unforgiving rule is strong evidence  
556 towards the systematic, rather than statistical, nature of the supporting mechanism. The argument also  
557 proves that for non conservative actions, under the same richness hypothesis the associated empirical matrix  
558 will converge to the null matrix. This fact allows the robot to naively distinguish between conservative and  
559 non conservative actions, should he be given the capability to perform both on startup.

#### 560 4.2.2 Inferring motor structure from learned interaction

561 In the second phase of the experiment, the agent uses the prediction functions it discovered for elementary  
562 conservative moves to infer how combinations of these moves relate to each other. Indeed, it was proved in  
563 the previous part that for any conservative actions  $a$  and  $a'$  with associated permutation matrices  $M_{\sigma_a}$  and  
564  $M_{\sigma_{a'}}$ , it is true that

$$M_{\sigma_{a'}} M_{\sigma_a} = M_{\sigma_{a'a}}. \quad (18)$$

565 In the case of actions which are not strictly conservative such as those in the simulation, equality in the  
566 previous equation is not guaranteed. This happens because in the  $M_{\sigma_{a'}} M_{\sigma_a}$  expression, all the loss of  
567 information of  $a$  and  $a'$  on their respective boundaries is accumulated, whereas  $a'a$  might recoup some of

568 it e.g. when  $a' = a^{-1}$ . However, multiple expressions should at least yield non contradictory prediction,  
 569 that is whenever one specifies a pair  $c_i \rightarrow c_j$  another other cannot assert  $c_i \rightarrow c_k$  with  $j \neq k$ . As long  
 570 as these combinations are kept short enough to limit the accumulation, this non contradiction criterion  
 571 can be used by the agent to internally infer the sensory prediction of *any* combination of the moves it  
 empirically learned. This is used in a Dijkstra-like process to build a graph of prediction matrices, which

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**Algorithm 1** Dijkstra-like algorithm for live construction of action group graph.

---

**Input**

- A The set of all matrices learned in exp.1
- D A bound on length of matrix combinations used
- O A reference matrix around which to explore

**Output**

- G A local view of the combinatorial graph of matrix products around O, using edges in A

Add O to collection U

O.depth  $\leftarrow$  0

Add node O to G

**while** U is not empty **do** ▷ True iff the neighborhood of some node K is still Unexplored

  K  $\leftarrow$  node in U

**for all**  $M_a$  in A **do** ▷ Test all learned predictions starting from node K

    P  $\leftarrow$   $M_a$ K

    P.depth  $\leftarrow$  K.depth + 1

**if** P.depth  $\leq$  D **then**

      B  $\leftarrow$  False

**for all** node C in G **do** ▷ Test previously discovered nodes for equality

**if** predictions for P and C match **then**

          B  $\leftarrow$  True

          Set edge  $M_a$ : K  $\rightarrow$  P in G

**end if**

**end for** ▷ END for all node C in G

**if** B is False **then** ▷ Branch taken iff P :=  $M_a$ K was not previously discovered

        Add P to U

        Add node P to G

        Set edge  $M_a$ : K  $\rightarrow$  P in G

**end if**

**end if** ▷ END if P.depth  $\leq$  D

**end for** ▷ END for all a in A

  Remove K from U

**end while** ▷ END while U is not empty

---

572 runs as follows, see Algorithm 1: starting from a prediction matrix  $M_0$  corresponding to any origin action  
 573  $a_0$ , each of the known matrices  $M_{a_k}$ ,  $a_k \in \mathcal{A}_{\text{init}}$  are applied to yield both a set of new neighboring “end  
 574 points”  $N_{M_0} := \{M_{a_k}M_0, a_k \in \mathcal{A}_{\text{init}}\}$  and for each pair  $(M_0, M_{a_k}M_0)$  a directed edge  $M_{a_k}$ . This is then  
 575 recursively applied to all newly discovered end points, while those that were previously visited (as the  
 576 prediction matrices *can* be compared for equality) are discarded. However, the resulting graph would in  
 577 most cases be infinite, therefore a stopping rule must be chosen. In our case, we chose to explore up to a  
 578 given depth parameter in graph edge distance.

### 580 4.3 Results

581 This subsection is devoted to the evaluation in simulation of the previous points, divided in three successive  
 582 experiments. The first one illustrates how the agent can build the permutation matrices associated to each

583 of its conservative actions; a discussion about the convergence and the statistics of this experiment is then  
 584 proposed. The second one exploits the permutation matrices just obtained to structure its own actions  
 585 through a graph of their combinations; a discussion about its fidelity as a representation of the action  
 586 group  $\mathcal{A}$  is proposed. In the third and final one, the effect of varying starting action sets on the structure  
 587 discovered is studied, concluding the subsection.

#### 588 4.3.1 Experiment 1: discovering the permutations

##### 589 4.3.1.1 Building the permutation matrices

590 To begin with, the simulated robot in Figure 4 is placed at a random 2D position inside the image to be  
 591 explored. The available action set is defined as  $\mathcal{A}_{\text{init}} = A$  so that  $n_{\mathcal{A}} = 7$ . Then, at each time step  $t_n$ , a  
 592 random action  $a_k = a[t_n] \in \mathcal{A}_{\text{init}}$  is run, and the associated permutation matrix  $M_{a_k}$  is updated according  
 593 to (17). After this update, the agent is able to evaluate if these matrices have finished converging and  
 594 therefore can decide when to stop the exploration. An entropy-like internal criterion is proposed to quantify  
 595 this convergence, along

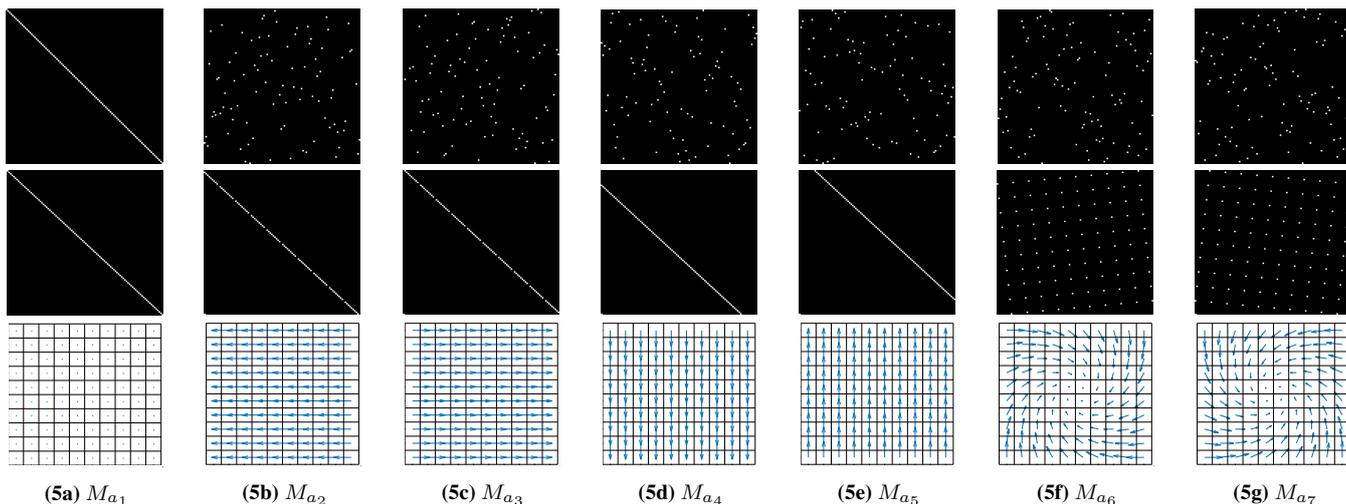
$$C(M) = 1 - \frac{1}{N_c \log_2(N_c)} \sum_{i=1}^{N_c} H_i,$$

$$\text{where } H_i = - \sum_{j=1}^{N_c} \frac{M_{i,j}}{\mu_i} \log_2\left(\frac{M_{i,j}}{\mu_i}\right), \quad (19)$$

$$\text{and } \mu_i = \frac{1}{\max(1, \sum_{j=1}^{N_c} M_{i,j})}.$$

596 In this criterion,  $H_i$  is the entropy of the post-action output of sensel  $c_i$  as a random variable of the pre-  
 597 action outputs of all sensels  $c_j$ . Therefore it measures which degree of surprise remains in the determination  
 598 of which (if any) sensel is successor to  $c_i$ . Finally, this makes  $C$  into an average measure of certainty in  
 599 the discovery of successor sensel pairs, going in nondecreasing trajectories from 0 at initialization to 1 at  
 600 permutation matrices. Consequently when it obtains the updated matrices  $M_{a_k}[t_{n+1}]$ ,  $a_k \in \mathcal{A}_{\text{init}}$  the agent  
 601 computes all  $C_k[t_{n+1}] = C(M_{a_k}[t_{n+1}])$  to assess the state of its discovery, stopping its exploration when  
 602 all the  $C_k$  have reached 1.

603 After convergence, the resulting matrices for all seven actions shown in Figure 4 are depicted in Figure 5.  
 604 In this figure, a 0 (resp. 1) is represented in black (resp. white). Since the agent has no knowledge of  
 605 its sensor geometry, the position of its sensels (i.e. pixels) inside the sensory array  $s$  (i.e. the flattened  
 606 image) is randomly chosen. In this case, the resulting permutation matrices for each action is depicted in  
 607 Figure 5 (top), demonstrating the fact that those matrices are not easy to understand from an external point  
 608 of view. If one now selects a more natural ordering of the pixels inside  $s$ , like a line by line arrangement, one  
 609 then gets the permutation matrices in Figure 5 (middle). With such an arrangement, an external observer is  
 610 now able to get a clearer intuition about the effects of each action on the pixels permutations. Nevertheless,  
 611 these two different sets of matrices, as two contingent images of the same underlying structure, are purely  
 612 equivalent from an internal point of view. This can be illustrated by mapping the permutation on the  
 613 overall sensor to better catch how the agent has been able to discover the underlying spatial transfer  
 614 between sensels. This is done by plotting the sensel pairs along which values are transferred as proposed in  
 615 Figure 5 (bottom). In this figure, the  $10 \times 10$  pixel grid of the simulated camera is represented together with  
 616 arrows connecting each sensel to its successor. While such a representation requires external knowledge in  
 617 the sensor geometry, the arrows are entirely determined by the internal permutation matrices from either of

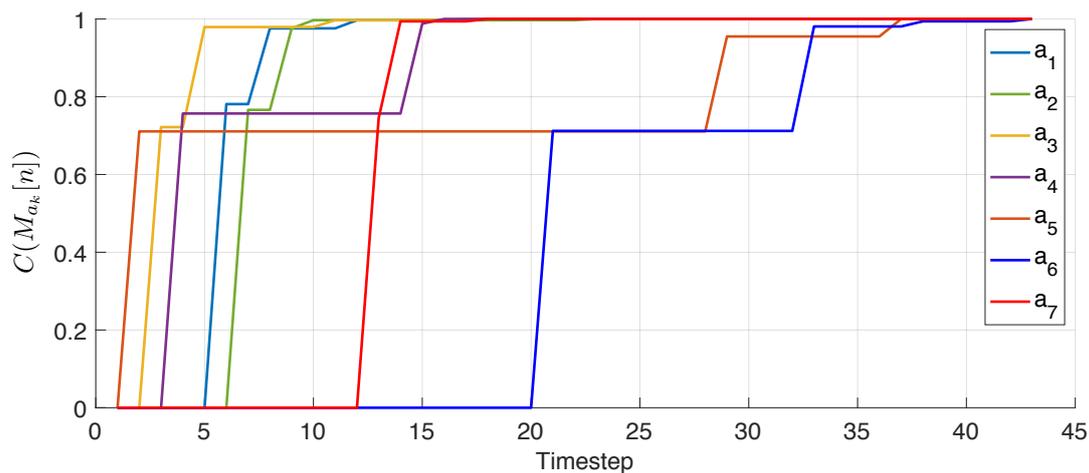


**Figure 5.** Representation of the seven binary  $10^2 \times 10^2$  permutation matrices  $M_{a_k}$  corresponding to the actions  $a_k$  possibly generated by the agent, where a 0 (resp. a 1) is represented as black (resp. white). **(first row)** Matrices obtained for a random organization of the sensels outputs  $s_i$  inside the sensory array  $\mathbf{s}$ . **(second row)** Matrices obtained for a well chosen sensels arrangement, where each pixel values are stored line by line in  $\mathbf{s}$ . **(third row)** Interpretation of the permutation matrices (either from the first or second row) directly on the physical  $10 \times 10$  pixel array: if a 1 is present at line  $i$  and column  $j$  of a matrix  $M_{a_k}$ , then an arrow joining pixel  $i$  to  $j$  is plotted. Note that the arrow length has been resized for actions 6 and 7 (i.e. rotations) to enhance readability.

618 the two sets presented. It is thus a convenient external way to display that each matrix has actually captured  
 619 the pixel shift induced by each action. For instance, with such a visualization, it is now very clear that  
 620  $(a_2, a_3)$ ,  $(a_4, a_5)$  or even  $(a_6, a_7)$  are all found to be pairs of inverse actions; this specific capability will  
 621 actually be exploited in §4.3.2 to structure the agent set of actions.

#### 622 4.3.1.2 A discussion about the dynamics of convergence

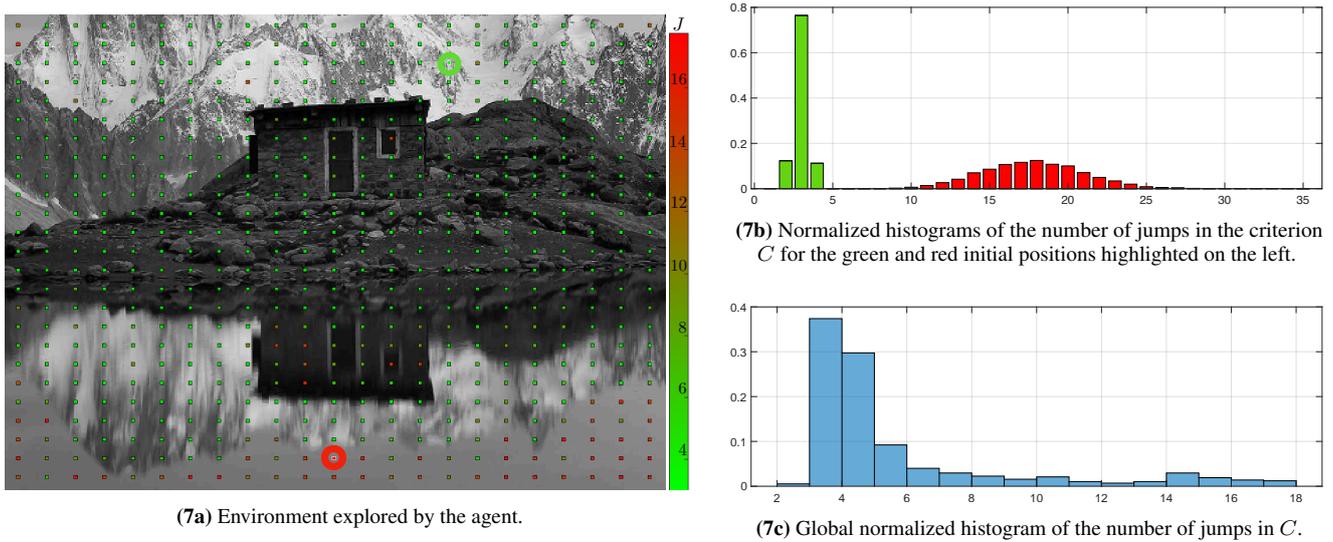
623 It is clear from Figure 5 that at some point the agent captured the permutation to the best of its capabilities.  
 624 One therefore proposes to study the dynamics of the convergence of the approach w.r.t. the experimental  
 625 time step  $t_n$ . For the remainder of experiment 1, we now keep the image constant during all the simulation,  
 626 so as to better assess the influence of the experienced environment on the results. First, the internal criterion  
 627  $C_k = (M_{a_k})$  defined in Equation (19) is evaluated at each  $t_n$  and each  $a_k$ , resulting in the plot in Figure 6.  
 628 One can then confirm that the  $C_k$  increase from 0 (all elements in the matrices are initialized at 1) to 1 (all  
 629 successor pairs have been discovered). It also appears that for each particular action  $a_k$ , the associated  
 630 criterion increases in sparse jumps because its matrix  $M_{a_k}$  is only actually updated at the random time steps  
 631 when  $a_k$  is drawn. Figure 6 also illustrates the fact that the amplitude of these jumps decreases over the  
 632 experiment. For the starting conditions of this experiment, a detailed analysis shows that about 7 realizations  
 633 of each action are necessary to fully discover the target permutation matrices. But it also appears that most  
 634 of the initial 1s in the matrices are wiped out very early, with a criterion value  $C_k[t_n] \approx 0.7$  after only  
 635 one execution of the corresponding action  $a_k$ . However, one still questions whether the differences in the  
 636 dynamic of all actions is a random occurrence of this particular exploration, or there is an intrinsic variance  
 637 in difficulty in learning between actions.



**Figure 6.** Representation of the criterion  $C_k = C(M_{a_k})$  for the seven actions  $a_k$ . Each jump in this figure corresponds to a reevaluation of the criterion happening at a timestep when the corresponding action has been drawn in the set in  $\mathcal{A}_{\text{init}}$ . As expected, the criterion starts from 0 to reach 1, indicating that all possible permutations have been found.

### 638 4.3.1.3 A statistical analysis about richness of the environment

639 The answer to the previous question can be obtained by performing an empirical survey (i) by averaging  
 640 over random explorations for given starting conditions, and (ii) by varying these starting conditions and  
 641 comparing the resulting performances. With such a study, (i) will allow to quantify the influence of the  
 642 randomness in exploration, while (ii) lets us assess how the properties of the environment influence the  
 643 discovery of permutations. For this experiment, the environment is made of the image shown in Figure 7a,  
 644 where the starting points of each exploration is depicted as a grid of points on it. At each of these points,  
 645 1000 random explorations are conducted, each of them consisting in a random run of actions  $a_k$  as in  
 646 §4.3.1.1, resulting in 1000 sets of seven  $C_k$  curves as in Figure 6. For each random exploration  $l$  and each  
 647 action  $a_k$ , the number of jumps  $J_{l,a_k}$  in the  $C_k$  curve obtained is taken as a measure of difficulty in learning  
 648 the permutation. The average  $J = \frac{1}{L} \sum_l \sum_k J_{l,a_k}$  of  $J_{l,a_k}$  over all actions  $a_k$  and explorations  $l$  at a given  
 649 starting position is depicted as the color of the grid in Figure 7a, with  $L = 1000$  (runs)  $\times$  7 (actions). Green  
 650 points correspond to a low number of jumps  $J$ , while red ones are representing higher values. One can  
 651 observe that the points are overwhelmingly green, and that the red ones are restricted to precise areas in the  
 652 picture. These correspond to areas with locally low contrast, such as the sky (in the top left corner) or its  
 653 reflection (in the bottom). The extremal conditions corresponding to the two green and red highlighted  
 654 points are further compared. For each of them, the distribution of the  $J_{l,a_k}$  is plotted as an histogram in  
 655 Figure 7b. Clearly, green points correspond to areas in the environment where the permutation matrices  
 656 can be discovered in at most 5 executions of actions. On the contrary, at red points the agent must wait  
 657 for about 17 on average, and up to 35, executions before it has obtained the same results. This illustrates  
 658 how the richness of the environment might influence the agent ability to capture the structure of its sensory  
 659 prediction. On a more global scale, Figure 7c shows the distribution of the  $J_{l,a_k}$  for all random explorations,  
 660 indiscriminately of the starting position. This corroborates the observation that most positions in the image  
 661 are green, i.e. lead to easy convergence. It appears that for a randomly selected starting position, there  
 662 is more than 66% of chance of permutation matrices being discovered in less than 4 executions of their  
 663 corresponding actions.



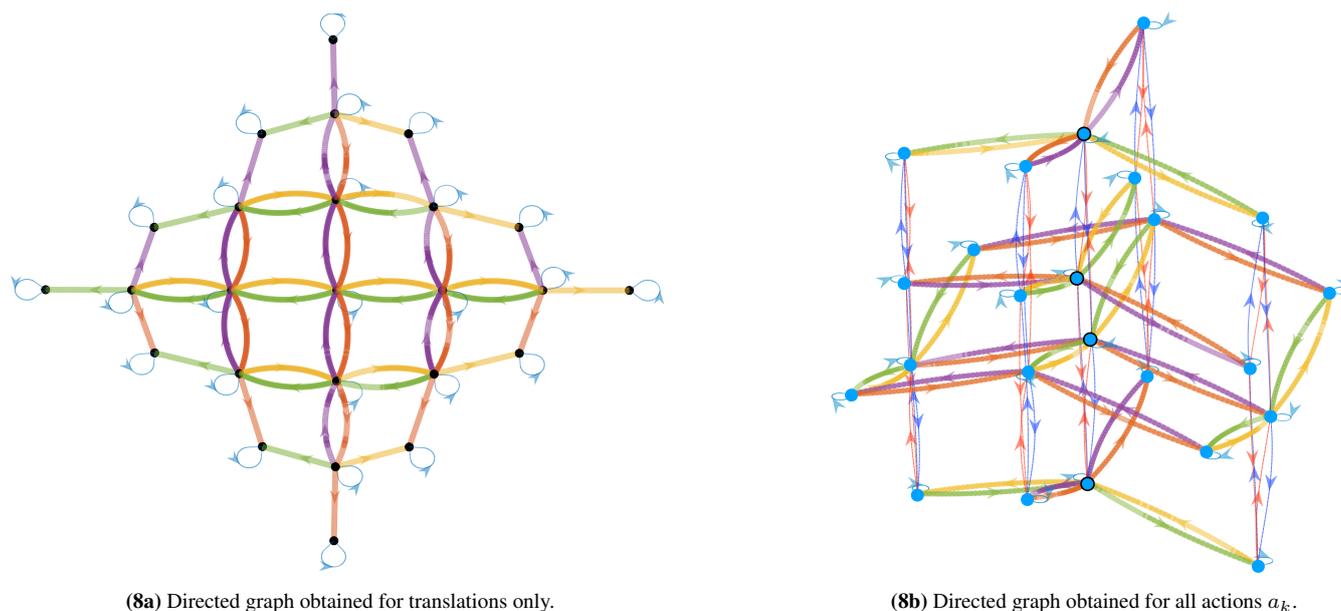
**Figure 7.** Statistical analysis of the permutation matrices building process. **(left)** Environment explored by the agent. Each point in this environment corresponds to a starting position around which the agent draws actions to build the permutation matrices  $M_{a_k}$ . Counting the mean number of jumps in the criterion curves  $C_k$  for each realization of the exploration around a given starting point and each action leads to the value  $J$  representing the difficulty to build the corresponding matrices. A high (resp. low)  $J$  value in red (resp. in green) corresponds to areas in the environment harder (resp. easier) to exploit for sensory prediction. **(right)** Normalized histograms of the number of jumps in the criterion curves  $C_k$  averaged across actions. **(top right)** Focus on the histograms obtained around two different starting conditions corresponding to low (resp. high)  $J$  value highlighted by a green (resp. red) circle in the environment. **(bottom right)** Overall normalized histogram for all actions and all starting positions in the environment, showing that most of the permutation matrices are correctly obtained after a low number of action run.

### 664 4.3.2 Experiment 2: structuring actions by combination

665 From the previous experiment, we now have as many permutation matrices  $M_{a_k}$  as we have actions in  
 666  $\mathcal{A}_{\text{init}}$ . As outlined in §4.2.2, one can then use them to build a *graph of prediction matrices* by following  
 667 Algorithm 1. Recall that in this graph, a node is a permutation matrix obtained as a combination of the  
 668  $M_{a_k}$  matrices, while there is a  $M_{a_k}$  edge from matrix  $M$  to  $M'$  iff  $M' = M_{a_k}M$ . Therefore, all edges in  
 669 the graph correspond to the permutation matrices built during experiment 1. According to Equation (15),  
 670 this graph is isomorphic to the graph of corresponding actions, meaning all properties discovered of any  
 671 combination of matrices holds true for the corresponding combination of actions. As an example, if one  
 672 discovers that  $M_{a_1} = M_{a_2}M_{a_3}$ , then one also has  $a_1 = a_2a_3$ .

673 As a first step, let us consider only the actions corresponding to translations in the environment, i.e.  $a_6$   
 674 and  $a_7$  are discarded from  $\mathcal{A}_{\text{init}}$ . This *a priori* selection is only made to simplify the visualization of the  
 675 graph at first. After applying Algorithm 1 to the matrices shown in Figure 5, one gets the directed graph in  
 676 Figure 8a, where all the color conventions are consistent with experiment 1. This particular graph has been  
 677 built for a maximum depth set to 3 and with  $M_{a_1}$  taken as the origin of the graph. Note that the depth of  
 678 this graph has been maintained voluntarily low so as to help in the reading of the graph. Note also that the  
 679 arbitrary choice of origin makes all of its neighbors themselves correspond to one of the  $M_{a_k}$  discovered in  
 680 experiment 1 since they all occurred as  $M_{a_k}M_{a_1} = M_{a_k}$  products, whereas all other nodes are indeed new  
 681 matrices.

682 This graph mirrors many algebraic properties of the  $M_{a_k}$  as captured by the internal experience. Indeed  
 683 one can first observe that the light blue arrow leads from any given node  $M$  to itself, which corresponds to



**Figure 8.** Directed graph of permutation matrices  $M_{a_k}$ —and thus also of corresponding actions  $a_k$  as per Equation (15)—obtained by combination of these matrices. Color conventions for edges match the color of each action in Figure 4. The depth up to which new nodes are explored has been limited for visualization purposes by setting low depth parameters in Algorithm 1

684  $M_{a_1}$  being the identity matrix  $I_{N_c}$ . Furthermore one can note that the graph obtained is, up to its borders,  
 685 completely homogeneous; that is the neighborhoods of each interior nodes share the same geometry. This  
 686 even extends to the color of edges matching, so that some of them form pairs. One can for example verify  
 687 that whenever a yellow edge goes from node  $M$  to  $M'$ , there is a green edge from  $M'$  to  $M$  and no other  
 688 one. This identifies the corresponding actions to be inverses w.r.t. successive execution since from any  
 689 starting node, taking first the green (resp. yellow) edge then the yellow (resp. green) one forms a loop. The  
 690 same can be said of the red and purple colors, which are found to correspond to another pair of inverse  
 691 actions. At last, the four central squares correspond to the commutativity of the  $a_k$  used: indeed one can  
 692 see on the graph that taking the red edge first, then the green one always leads to the same node as green  
 693 first, red second.

694 While those observations were discussed as properties of the permutation matrices, the actual result is  
 695 their representing properties of the abstract motor actions  $a_k$ . And indeed one can check that the blue arrow  
 696 corresponds to the identity action  $a_1$ , that the inverse pairs (yellow, green) and (orange, purple) respectively  
 697 correspond to (rightward, leftward) and (forward, backward) translations, and that the commutativity  
 698 discussed is that of “forward then left” being the same as “left then forward”. While these facts seem  
 699 obvious from an external point of view, they were not part of the initial knowledge of the agent discussed  
 700 in §3.1. This only appears as a consequence of the agent capability to predict the sensory consequences of  
 701 its own actions built during experiment 1. On a functional level, this is very similar to the property of motor  
 702 sequence compression exhibited by RNNs performing sensorimotor prediction in (Ortiz and Laflaquière,  
 703 2018); in fact we argue that it is the same phenomenon that is picked up on by the neural networks and that  
 704 it is intrinsically related to sensorimotor prediction as developed in Section 3.

705 This also applies to the graph shown in Figure 8b obtained when considering all seven actions, i.e. the  
 706 two rotations corresponding to actions  $a_6$  and  $a_7$  are now included in the analysis. This plot, obtained  
 707 through a classical force-directed algorithm, shows the same 2D graph of translations obtained before, but

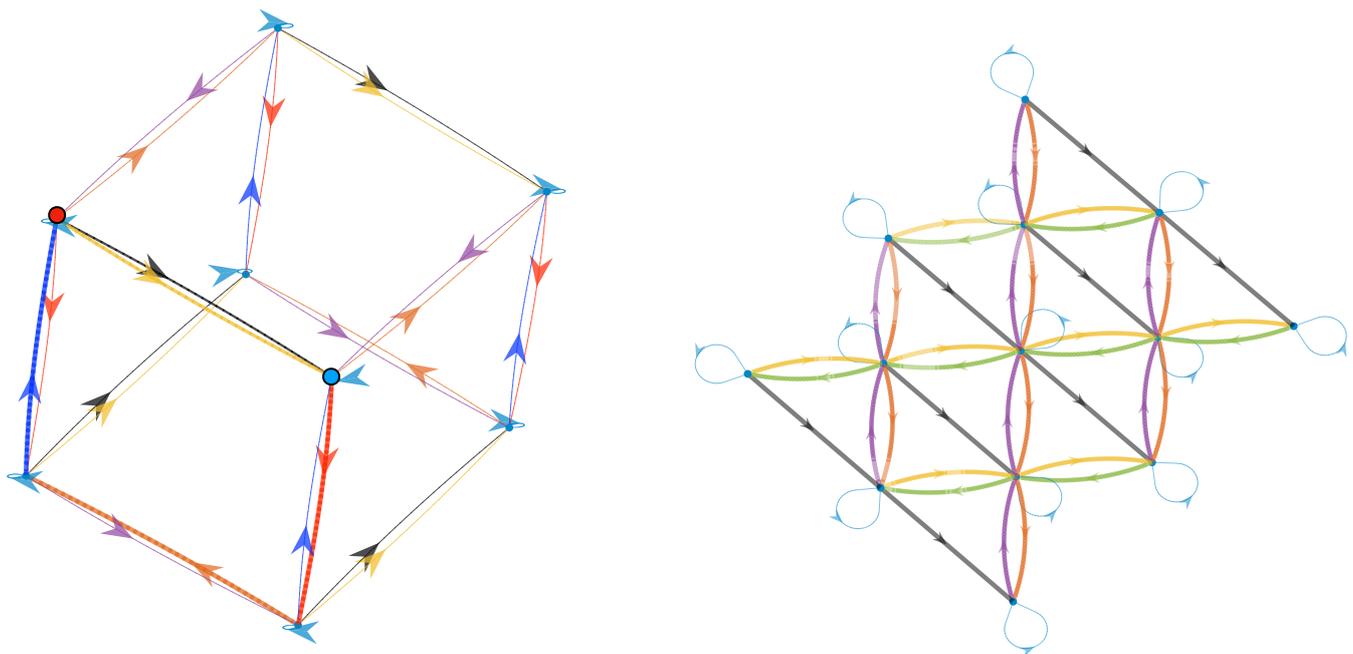
708 enriched with a third dimension supporting the change of orientation induced by rotations. Again, the depth  
 709 of the graph is maintained low to keep things legible. The global structure of the graph can be described as  
 710 a disjunction of 2D subgraphs corresponding to translations at fixed orientation. Each subgraph is therefore  
 711 equivalent to each other up to a rotation as can be seen by the edges colors shifting between the planes.  
 712 As an example, one can see that the same node in the graph can be reached by following either (green,  
 713 dark blue) and (dark blue, purple), or (left, turn left) and (turn left, forward) in terms of actions seen from  
 714 an external point of view. Figure 8b also shows that rotations are limited to the third vertical dimension  
 715 in which they form cycles at constant position in the planar subgraph. This property is highlighted in the  
 716 graph by the 4 circled nodes which figure the same agent position for the four possible orientations. The  
 717 cycle simply mirrors the external observation that taking four  $\pi/2$  rotations successively takes one back to  
 718 the initial orientation. Importantly, this could constitute an internal signature of rotations as opposed to  
 719 translations.

720 In the end of this second experiment, the agent has thus been able to discover a structure of its actual group  
 721 of motor actions. The agent now has access to algebraic relations between its own actions which relate to  
 722 its motor capabilities. This knowledge also allows it to generalize the sensory prediction it discovered in  
 723 experiment 1 to all the combinations considered in the graph. Nevertheless, one has to keep in mind that all  
 724 the actions considered in these experiments are not exactly conservative in the sense of Equation (9). Indeed,  
 725 they fail at conserving spatial information on the border of the simulated camera. However, results show  
 726 that conservativity holding true for all internal pixels allows for discovering the aforementioned properties.  
 727 Because all calculations are made on actual current outputs of the pixels, one obvious consequence is that  
 728 the agent has no way to predict what happens outside of its field of view, and so far it keeps no memory of  
 729 it. Therefore, the results only hold for a very local movement w.r.t. the dimension of the agent. However,  
 730 the discovered structure stays true whatever the initial position of the exploration.

### 731 4.3.3 Experiment 3: exploiting the graph to improve the representation of the action set

732 In the previous results, we have run the experiments with an experimental starting action set  $\mathcal{A}_{\text{init}}$   
 733 conveniently set to  $A$ , the set of externally defined movements. While this has been useful at first to yield  
 734 easily recognized structure in Experiment 2, it is a crucial point that the results do not depend on this strong  
 735 assumption. Therefore the same two part experiment is conducted with the difference that the starting  
 736 action set  $\mathcal{A}_{\text{init}}$  the agent can run is not arbitrarily set to  $A$  anymore. Instead, it is now drawn in the set  
 737 of *combinations of* actions  $a_k \in A$ . Three important cases are now possible : first, it may be that some  
 738 of the  $a_k$  are “missing” in  $\mathcal{A}_{\text{init}}$ ; on the contrary, duplicates may have been drawn so that the agent can  
 739 run  $a, a' \in \mathcal{A}_{\text{init}}$  which are effectively the same action (i.e.  $\forall \mathbf{b} \in \mathcal{B}, a\mathbf{b} = a'\mathbf{b}$ ). Finally, it may even have  
 740 drawn “complex” actions  $a \notin A$ , that is actions that can only be obtained by combining some of the  $a_k$ .

741 The three situations are illustrated in Figure 9, where the graphs obtained at the end of Experiment 2  
 742 are drawn for various starting action sets. In both cases, both the complexity of the starting action set and  
 743 the depth of the depicted graph have been limited to keep the discussed structure as readable as possible.  
 744 Figure 9a depicts the first two cases: the agent was given a duplicate action from  $A$  as well as missing one.  
 745 This can be assessed in the resulting prediction graph by the yellow and black arrows which relate the  
 746 same ordered pairs of nodes, e.g. from the highlighted red to blue nodes, and the lack of an inverse arrow  
 747 that would match them. Note that while the absence of this inverse (green) arrow represents the lack of a  
 748 “direct” inverse action *in*  $\mathcal{A}_{\text{init}}$ , the emerging structure from the graph allows for the determination of an  
 749 inverse *path* as highlighted by the bold (red, orange, blue) arrows. From an external point of view, this  
 750 basically means that if the agent has no action to translate itself to its left, it can instead rotate clockwise,  
 751 then move backward, and finally rotate counter-clockwise to reach the correct orientation. Interestingly, this



(9a) Graph obtained with a duplicate starting action (depicted in black) and a missing action ( $a_2$ , green in other figures). A path made of 3 edges (red, orange, blue) equivalent to the missing starting action is highlighted, providing an inverse to the (also highlighted) yellow edge.

(9b) Graph obtained by adding a combination (depicted in black) to the set of starting actions. Here, the action set was limited to translations to keep a clear visual.

**Figure 9.** Illustration of the effect of drawing starting actions at random on the discovered structure.

752 phenomenon where a missing inverse can be otherwise obtained by combination of other actions can *only*  
 753 occur when the agent is able to rotate. The third situation is depicted in Figure 9b, where the experiment  
 754 was conducted with the robot given the additional action  $a_8$  “forward then rightward” along  $a_8 = a_5a_3$  in  
 755 addition to the translations of  $A$ . The choice not to give the agent its defined rotations  $a_6$  and  $a_7$  serves  
 756 only to get an easily legible picture of the resulting graph, much like in Figure 8a, and does not impact the  
 757 following. This additional move  $a_8$ , which we as an external observer know to be a combination, is studied  
 758 like all other basic actions by the agent during the motor babbling phase. It means that the agent has no cue  
 759 about  $a_8$  being an actual combination of two other actions. In the end of the experiment, the obtained graph  
 760 of action exhibits this additional action  $a_8$  as black edges, as shown in Figure 9b. From this graph, one can  
 761 easily see that, from any point, it is indeed equivalent to follow either the black arrow or first the orange  
 762 and then the yellow ones. The agent has thus been able to discover the action combination property.

763 These graphs therefore show two important results. The first conceptual comment is that the validity of  
 764 the proposed experiment is not conditional to a perfect match between ideal, “objective” moves of the  
 765 agent and actions it is effectively able to perform at start. This is a desirable property for genericity and our  
 766 goal of bootstrapping, for it allows to avoid justifying said match. The second, more practical, comment is  
 767 that the graph resulting from the experiment can be used by the agent to select an action set “better” than  
 768  $\mathcal{A}_{\text{init}}$ . Indeed, the redundancy between edges (or paths of edges) in Figures 9a and 9b represent the agent  
 769 discovering it can discard the actions corresponding to black edges *without losing capabilities*, i.e. while  
 770 keeping all nodes reachable. It can then be used to prune the available action set to a *minimal* set generating  
 771 the same group, in the sense of Equation (4). Determining a criterion for selecting which actions are kept  
 772 and which are discarded could functionally correspond to a basis for invariant principles in motor actions  
 773 of the agent (Flash and Hogan, 1998). The agent may also *expand* its action set with new actions that verify

774 useful properties: for example, in the case depicted in Figure 9a it can package the bolded path of edges  
775 into a single action so that it gets a missing inverse.

## 5 CONCLUSION

776 This paper was devoted to the introduction of a variational extension of a previous framework into the  
777 sensorimotor framework, extending the scope of such approaches to naive agents able to move freely in  
778 their environment. We demonstrated how, despite their extremely limited starting capabilities, these agents  
779 could exploit said framework not only to perform sensory prediction, but also to structure their own actions.  
780 The proposed formalism has been assessed in simulation as a proof of concept, with a naive agent able to (i)  
781 build for each of its action some permutation matrices associated to its own sensory array, and (ii) exploit  
782 them to structure its own set of actions. These experiments were conducted here in a somewhat simplistic  
783 experimental setting to keep the simulated situation as close as possible to the theoretical exposition.  
784 However, their transparent exploitation of the formal mechanisms we explicitly isolated yields valuable  
785 insight as to similar results, which related works otherwise achieved in more realistic conditions.

786 Implementing a formal version of sensory prediction comes with many interesting perspectives, as it was  
787 shown to yield crucial properties both in the original cognitive psychology literature and in the previous  
788 robotic contexts. We hypothesize that it can be used to better understand the emergence and properties  
789 of capabilities often related to that of sensory prediction, both from robotics and from cognitive sciences,  
790 such as those mentioned in the Introduction. These include e.g. motor control, motor planning, isolating  
791 proprioception, suppression of self-induced changes or object perception. These capabilities therefore  
792 constitute potential applications to which further study could be devoted from there.

793 Nevertheless, the applicability of the proposed paradigm to real agents or robots is still an opened  
794 question. First, it is clear that most of the actions an agent will be dealing with are not strictly conservative,  
795 but rather *quasi-conservative* like in the simulations conducted in this paper. While not extensively studied  
796 in this paper, some ongoing mathematical developments show that their properties still allow to reach the  
797 same concepts of sensory prediction and action structuration. Then, the fact that the sensory prediction  
798 relies on exact sensory values shifts inside the un-noisy sensory array is not very realistic. Introducing  
799 stochastic matrices instead of permutation ones constitutes a promising way to deal with such an issue,  
800 also pulling all these developments inside a probability territory (Rao and Ballard, 2005; Seth, 2014) in  
801 which a lot of development still needs to be done. Moreover, the way this framework can be extended  
802 to agents exhibiting dynamical effects, e.g. when performing kinematic or dynamical control, must still  
803 be investigated. This requires some clarification about the structure and role of time in the sensorimotor  
804 experience, a point which is still largely eluded in the SMCT context. Finally, actions can also be noisy, and  
805 the question of their repeatability over time needs to be addressed so as to face realistic conditions. This  
806 poses significant challenges in the SMCT context of minimal *a priori* knowledge outlined in the present  
807 contribution. However, ongoing exploratory work tends to show that topological structure grounding  
808 some continuity of the sensorimotor experience can be found as a contingency in said naive context. All  
809 these paths constitute future promising works in the field and will undoubtedly extend the scope of these  
810 approaches to naive adaptive and robust agents able to build by themselves their own understanding of  
811 their interaction with their environment.

## APPENDIX

812 In the following proofs, we will assume

$$(\forall \mathbf{b} \in \mathcal{B}, \forall \epsilon \in \mathcal{E}, \psi_{\mathcal{E}}(a'\mathbf{b}, \epsilon) = \psi_{\mathcal{E}}(a\mathbf{b}, \epsilon)) \Rightarrow a = a' \quad (20)$$

813 and

$$\forall c, c' \in \mathcal{C}, F_c = F_{c'} \Rightarrow c = c'. \quad (21)$$

814 These assumptions are of minimal importance for two reasons

- 815 • Eq. (21) only mandates that any two distinct sensels have different receptive fields *if only one time*,  
816 while Eq. (20) asks for actions to have no difference (as denoted by =) except that which can be  
817 assessed by the sensory capabilities  $\psi_{\mathcal{E}}$ . These conditions only fail to hold in very particular cases and  
818 can be found to be true in the presented examples.
- 819 • In any case where they indeed *fail* to hold, the exact same results can be found with suitable equivalence  
820 relations for actions (for Eq. (20)) and sensels (for Eq. (21)) at the cost of more loaded notations.

821 Therefore these conditions only serve as a way to streamline the presentation of the results with an at most  
822 negligible impact on generality.

### 823 1 Equivalency between conservative and permutative

824 Here is provided a proof that conservative actions can be described as permutations of sensels, as  
825 discussed in Section 3.3.1.

826 PROPOSITION 1. *Let  $a$  be a conservative action  $\in \mathcal{A}$ , there exists a unique map*

$$\begin{aligned} \sigma_a: \mathcal{C} &\rightarrow \mathcal{C} \\ c &\mapsto c' \end{aligned} \quad (22)$$

827 such that

$$\sigma_a(c) = c' \Leftrightarrow c \xrightarrow{a} c' \quad (23)$$

PROOF. Let  $a \in \mathcal{A}$  conservative and  $c \in \mathcal{C}$ . By conservativity  $\exists c' \in \mathcal{C}$  such that  $c \xrightarrow{a} c'$ . Let  $c'' \in \mathcal{C}$  such that  $c \xrightarrow{a} c''$ , then

$$\forall \mathbf{b} \in \mathcal{B}, F_c(\mathbf{b}) = F_{c'}(a\mathbf{b}) \text{ and } F_c(\mathbf{b}) = F_{c''}(a\mathbf{b})$$

so that

$$\forall \mathbf{b} \in \mathcal{B}, F_{c'}(a\mathbf{b}) = F_{c''}(a\mathbf{b}).$$

828 But  $a: \mathcal{B} \rightarrow \mathcal{B}$  must be surjective because it is bijective, so that *all*  $\mathbf{b} \in \mathcal{B}$  can be written  $a\mathbf{b}'$  for some  
829  $\mathbf{b}' \in \mathcal{B}$ . Therefore  $F_{c'} = F_{c''}$ , from which  $c' = c''$ : successor sensels are necessarily unique. We therefore  
830 declare  $\sigma_a$  to be the map that takes each sensel  $c \in \mathcal{C}$  to its *unique* successor sensel.

831 PROPOSITION 2. *For any conservative action  $a \in \mathcal{A}$ , its successor map  $\sigma_a$  is bijective.*

PROOF. Let  $a$  be a conservative action, and let  $c, c' \in \mathcal{C}$  be sensels such that  $\sigma_a(c) = \sigma_a(c')$ . From this it follows that

$$\forall \mathbf{b} \in \mathcal{B}, F_c(\mathbf{b}) = F_{c''}(a\mathbf{b}) \text{ and } F_{c'}(\mathbf{b}) = F_{c''}(a\mathbf{b})$$

for some common successor  $c'' \in \mathcal{C}$ . But it entails in particular

$$\forall \mathbf{b} \in \mathcal{B}, F_c(\mathbf{b}) = F_{c'}(\mathbf{b})$$

832 that is  $F_c = F_{c'}$ , which further yields  $c = c'$ :  $\sigma_a$  is injective.

833 From injectivity of  $\sigma_a$ , it follows that  $|\sigma_a(\mathcal{C})| = |\mathcal{C}|$ . But because  $\mathcal{C}$  is finite it in turns follows from this  
834 equality that  $\sigma_a(\mathcal{C}) = \mathcal{C}$ , i.e.  $\sigma_a$  is also surjective.

## 835 2 Conversing of conserving

836 We provide here the proof, as used in Section 3.3.1 and following, that conservative actions are themselves  
837 a subgroup of  $\mathcal{A}$  for its succession operation.

838 PROPOSITION 3. Let  $\mathcal{A}_{\mathcal{C}} \subset \mathcal{A}$  be the subset of all conservative actions. Then  $\mathcal{A}_{\mathcal{C}}$  is in fact a subgroup  
839 of  $\mathcal{A}$ .

840 PROOF.  $\mathcal{A}_{\mathcal{C}} \subset \mathcal{A}$  by its very definition, therefore we only need prove it is actually a group.

- 841
- $\forall c \in \mathcal{C}, c \xrightarrow{e} c$  with  $e$  the identity action:  $e$  is conservative.
  - Let  $a$  and  $a'$  be conservative actions, and  $c \in \mathcal{C}$ : since  $a \in \mathcal{A}_{\mathcal{C}}, \exists c' \in \mathcal{C}$  such that  $c \xrightarrow{a} c'$ . But since  $a' \in \mathcal{A}_{\mathcal{C}}$  too, there also exists  $c'' \in \mathcal{C}$  verifying  $c' \xrightarrow{a'} c''$ , so that finally

$$\forall c \in \mathcal{C}, \forall a, a' \in \mathcal{A}_{\mathcal{C}}, \exists c'' \text{ such that } c \xrightarrow{a'a} c''$$

842 that is  $a'a$  is conservative itself.

- Let  $a \in \mathcal{A}_{\mathcal{C}}$  and let  $\sigma_a$  be its successor map  $\mathcal{C} \rightarrow \mathcal{C}$ .  $\forall c \in \mathcal{C}$  since  $\sigma_a$  is surjective (see proof in 1) we have  $c = \sigma_a(c')$  for some  $c' \in \mathcal{C}$ , or equivalently

$$\forall c \in \mathcal{C}, \exists c' \text{ such that } c' \xrightarrow{a} c.$$

843 Finally, since  $c' \xrightarrow{a} c \Leftrightarrow c \xrightarrow{a^{-1}} c'$  it follows that  $a^{-1}$  is conservative too.

## 844 3 Conservation or prediction, it is all the same

845 Here is provided a proof that mapping conservative actions  $a$  to their respective *sensory prediction*  
846 *functions*  $\Pi_a$  provides a group isomorphism, as per Equation (15). To this end, let us recall the essential  
847 property of these functions:

$$\forall \mathbf{b} \in \mathcal{B}, \forall \epsilon \in \mathcal{E}, \psi_{\mathcal{C}}(a\mathbf{b}, \epsilon) = \Pi_a(\psi_{\mathcal{C}}(\mathbf{b}, \epsilon)). \quad (24)$$

848 From this we get:

849 PROPOSITION 4. The map

$$\begin{aligned} \Pi: \mathcal{A}_{\mathcal{C}} &\rightarrow \text{Bij}(\mathcal{S}) \\ a &\mapsto \Pi_a \end{aligned} \quad (25)$$

850 is a group morphism. Moreover it is injective, so that it induces a group isomorphism  $\mathcal{A}_{\mathcal{C}} \cong \Pi(\mathcal{A}_{\mathcal{C}})$ .

PROOF. Let  $a, a'$  be two conservative actions, we have

$$\begin{aligned}\forall \mathbf{b} \in \mathcal{B}, \forall \epsilon \in \mathcal{E}, \\ \Pi_{a'^{-1}a}(\psi_{\mathcal{E}}(\mathbf{b}, \epsilon)) &= \psi_{\mathcal{E}}(a'^{-1}a\mathbf{b}, \epsilon) \\ &= \Pi_{a'^{-1}}(\psi(a\mathbf{b}, \epsilon)) \\ &= \Pi_{a'^{-1}}(\Pi_a(\psi(\mathbf{b}, \epsilon))) \\ &= (\Pi_{a'^{-1}} \circ \Pi_a)(\psi(\mathbf{b}, \epsilon))\end{aligned}$$

851 so that  $\Pi(a'^{-1}a) = \Pi(a'^{-1}) \circ \Pi(a)$ :  $\Pi$  is a group morphism.

Now let  $a, a' \in \mathcal{A}_{\mathcal{E}}$  such that  $\Pi_a = \Pi_{a'}$ . It follows from Equation (24) that

$$\forall \mathbf{b} \in \mathcal{B}, \forall \epsilon \in \mathcal{E}, \psi_{\mathcal{E}}(a'\mathbf{b}, \epsilon) = \psi_{\mathcal{E}}(a\mathbf{b}, \epsilon)$$

852 so that under hypothesis (20)  $\Pi$  is indeed injective.

#### 853 4 Convergence of experiment 1

854 This part is devoted to the proof of the relevance of Equation (17) in §4.2.1, that is the convergence of  
855 matrices  $M_a$  towards the associated permutation matrices  $M_{\sigma_a}$  for all conservative actions  $a$ .

856 LEMMA 1. For any coefficient  $m_{a_{i,j}}$  of  $M_{\sigma_a}$ , the associated sequence  $(m_{a_{i,j}}[t_n])_n$  of values taken in  
857  $(M_a[t_n])_n$  during exploration is nonincreasing with values in  $\{0, 1\}$ .

*Proof.* Let us consider an arbitrary timestep  $t_n$ ,  $n \in \mathbb{N}$  in the exploration. If  $a$  is not drawn at this timestep, then

$$m_{a_{i,j}}[t_{n+1}] = m_{a_{i,j}}[t_n] \leq m_{a_{i,j}}[t_n].$$

858 If it is instead chosen, assuming  $m_{a_{i,j}}[t_n] \in \{0, 1\}$  then as per the update rule of  $M_a$ , either

- 859 •  $m_{a_{i,j}}[t_n] = 0$  and then  $m_{a_{i,j}}[t_{n+1}] = 0$  too,  
860 • or  $m_{a_{i,j}}[t_n] = 1$  and  $m_{a_{i,j}}[t_{n+1}] \in \{0, 1\}$

861 so that the lemma follows by induction on  $n$ .

862 LEMMA 2. For any coefficient  $m_{a_{i,j}} = 1$  in  $M_{\sigma_a}$ , the associated sequence  $(m_{a_{i,j}}[t_n])_n$  is constant with  
863 value  $m_{a_{i,j}}[t_n] = 1$ .

*Proof.* At any timestep  $t_n$  of the exploration, if  $a$  is not chosen then  $m_{a_{i,j}}[t_{n+1}] = m_{a_{i,j}}[t_n]$ .  
If it is instead drawn, then by Equation (11) we know that

$$s_j[t_{n+1}] = s_i[t_n]$$

864 because  $m_{a_{i,j}} = 1$  implies that  $j = \sigma_a(i)$  as per the definition of  $M_{\sigma_a}$ . Then by the update rule of  $M_a[t_n]$ ,  
865  $m_{a_{i,j}}[t_{n+1}] = 1$ .

866 The lemma then follows by induction on  $n$ .

867 We now proceed with the last part of our argument, that is showing that coefficients of the empirical  
868 matrices  $M_{a_k}$  which do not correspond to successor sensel pairs will actually be nulled during exploration.  
869 This specific part is provided in the specific case of the simulated experiment presented, allowing us to

870 formulate the relevant equations in the vector geometry of  $\mathcal{X} = \mathbb{R}^2$ . The same idea could also be adapted  
 871 for generalized spaces  $\mathcal{X}$ ,  $\mathcal{B}$  and actions  $\mathcal{A}_{\text{init}}$ , more in line with the previous theoretical descriptions.  
 872 However, such a development is out of the scope of this contribution.

Let us define

$$\forall \theta \in \mathbb{R}, R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

the matrix corresponding to the rotation in  $\mathbb{R}^2$  of angle  $\theta$ . As per the definitions provided for our particular example, we may assume the properties:

$$\forall c \in \mathcal{C}, \forall (x, y, \vec{\theta}) \in \mathcal{B}, F_c(x, y, \vec{\theta}) = \begin{pmatrix} x \\ y \end{pmatrix} + R_\theta F_c(0, 0, \vec{0})$$

and

$$\forall a \in \mathcal{A}_{\text{init}}, \exists \vec{u}_a = \begin{pmatrix} x_a \\ y_a \end{pmatrix} \in \mathbb{R}^2, \exists \theta_a \in \mathbb{R}$$

$$\text{such that } \forall \mathbf{b} = (x, y, \vec{\theta}) \in \mathcal{B}, \mathbf{ab} = (x', y', \vec{\theta}')$$

$$\text{where } \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + R_\theta \begin{pmatrix} x_a \\ y_a \end{pmatrix} \text{ and } \theta' = \theta + \theta_a.$$

873 Therefore we have

LEMMA 3. Let  $a \in \mathcal{A}_{\text{init}}$ ,  $c, c' \in \mathcal{C}$ . There exists a unique vector  $\vec{d}_{a,c,c'} \in \mathbb{R}^2$  such that

$$\forall \mathbf{b} = (x, y, \vec{\theta}) \in \mathcal{B}, \overrightarrow{F_c(\mathbf{ab})F_{c'}(\mathbf{b})} = R_\theta \vec{d}_{a,c,c'}.$$

874 PROOF. Let  $\mathbf{b} = (x, y, \vec{\theta}) \in \mathcal{B}$ . We therefore have

$$875 \quad 1. F_{c'}(\mathbf{b}) = \begin{pmatrix} x \\ y \end{pmatrix} + R_\theta F_{c'}(0, 0, \vec{0}),$$

$$876 \quad 2. F_c(\mathbf{ab}) = F_c\left(\begin{pmatrix} x \\ y \end{pmatrix} + R_\theta \begin{pmatrix} x_a \\ y_a \end{pmatrix}, \vec{\theta} + \vec{\theta}_a\right) \\ = \begin{pmatrix} x \\ y \end{pmatrix} + R_\theta \begin{pmatrix} x_a \\ y_a \end{pmatrix} + R_{\theta+\theta_a} F_c(0, 0, \vec{0})$$

so that

$$\overrightarrow{F_c(\mathbf{ab})F_{c'}(\mathbf{b})} = R_\theta \left( F_{c'}(0) - \begin{pmatrix} x_a \\ y_a \end{pmatrix} - R_{\theta_a} F_c(0) \right).$$

877 which proves taking  $\vec{d}_{a,c,c'} = \left( F_{c'}(0) - \begin{pmatrix} x_a \\ y_a \end{pmatrix} - R_{\theta_a} F_c(0) \right)$  satisfies the property.

878 It should be noted that  $\vec{d}_{a,c,c'}$  captures some geometry of conservation: indeed, from the definition of  $\xrightarrow{a}$   
 879 it can easily be shown that

$$\forall a \in \mathcal{A}_{\text{init}}, \forall c, c' \in \mathcal{C}, \left( c \xrightarrow{a} c' \Leftrightarrow \vec{d}_{a,c,c'} = 0 \right). \quad (26)$$

880 This greatly serves the conclusion of our argument with

881 PROPOSITION 5. Let  $a \in \mathcal{A}_{\text{init}}$ ,  $c_i, c_j \in \mathcal{C}$ . Then for given environment configuration  $\epsilon \in \mathcal{E}$  the two  
882 statements

883 1. There exists an absolute configuration  $\mathbf{b} \in \mathcal{B}$  such that if  $\mathbf{b}[t_n] = \mathbf{b}$  and  $a[t_n] = a$  for some  $n \in \mathbb{N}$ ,  
884 then  $m_{a_i,j}[t_k] = 0 \forall k \geq n + 1$

885 2.  $\epsilon$  is not doubly periodic with periods  $\vec{d}_{a,c_i,c_j}$  and  $R_{\frac{\pi}{2}}\vec{d}_{a,c_i,c_j}$

886 are equivalent.

887 PROOF.

888 • Assume that  $\epsilon$  is both  $\vec{d}_{a,c_i,c_j}$ - and  $R_{\frac{\pi}{2}}\vec{d}_{a,c_i,c_j}$ -periodic. Let  $n \in \mathbb{N}$  such that  $a[t_n] = a$ , let  $\mathbf{b} = \mathbf{b}[t_n]$ .

889 We have  $s_i[t_n] = \epsilon(F_c(\mathbf{b}[t_n]))$  and  $s_j[t_{n+1}] = \epsilon(F_{c'}(\mathbf{b}[t_{n+1}])) = \epsilon(F_c(\mathbf{b}[t_n]) + R_{\theta}\vec{d}_{a,c_i,c_j})$ . But since  
890  $\theta \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$ ,  $R_{\theta} = \pm I_2$  or  $R_{\theta} = \pm R_{\frac{\pi}{2}}$ . Therefore by periodicity of  $\epsilon$  we have  $s_j[t_{n+1}] = s_i[t_n]$ ,  
891 from which by induction on  $n$  we get  $\forall n \in \mathbb{N}$ ,  $m_{a_i,j}[t_n] = 1: 1) \Rightarrow 2)$ .

892 • Without loss of generality, let us assume that  $\epsilon$  is not  $\vec{d}_{a,c_i,c_j}$ -periodic (if it is instead only not  $R_{\frac{\pi}{2}}\vec{d}_{a,c_i,c_j}$ -  
893 periodic, the same argument follows up to a rotation).

894 Let  $X_0 \in \mathbb{R}^2$  such that  $\epsilon(X_0) \neq \epsilon(X_0 + \vec{d}_{a,c_i,c_j})$ ,  $\mathbf{b}_0 = (x, y, \vec{0}) \in \mathcal{B}$  such that  $F_c(\mathbf{b}_0) = X_0$ . By  
895 definition  $F_{c'}(a\mathbf{b}_0) = X_0 + \vec{d}_{a,c_i,c_j}$  so that if  $\mathbf{b}[t_n] = \mathbf{b}_0$  for some  $t_n \in \mathbb{N}$ ,  $s_j[t_{n+1}] \neq s_i[t_n]$ . From the  
896 update rule of  $M_a$  we then get  $m_{a_i,j}[t_{n+1}] = 0$ , which by Lemma 1 concludes the proof.

897 Finally, simultaneously applying this proof to *all* actions and pair of sensels of the agent has us deduce:

COROLLARY. If  $\epsilon : \mathbb{R}^2 \rightarrow \mathcal{P}$  is aperiodic, then there exists a sequence of drawings of actions  $(a[t_n])_{n \in \mathbb{N}}$   
such that

$$\forall a_k \in \mathcal{A}_{\text{init}}, \lim_n M_{a_k}[t_n] = M_{\sigma_{a_k}}.$$

898 While the converse strictly speaking is not true, we can see from the preliminary lemma that problems in  
899 the algorithm arise from very particular periodicity properties which relate to the geometry of (receptive  
900 fields of) sensels. It therefore should be noted already how most experiments in live specimens made  
901 use of specifically engineered symmetric and *periodic* environments to try and impair the development  
902 of perception (Held and Hein, 1963). Future works could expand on the effects of such “pathological”  
903 environment configurations on the proposed algorithm.

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