Asymptotic Independence ex machina - Extreme Value Theory for the Diagonal Stochastic Recurrence Equation

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Abstract

We consider multivariate stationary processes $(\boldsymbol{X}_t)$ satisfying a stochastic recurrence equation of the form

$$\boldsymbol{X}_t = \boldsymbol{m} M_t \boldsymbol{X}_{t-1} + \boldsymbol{Q}_t,$$

where (M_t) and $(\boldsymbol{Q}_t)$ are iid random variables and random vectors, respectively, and $\boldsymbol{m} = \mathrm{diag}(m_1, \dots, m_d)$ is a deterministic diagonal matrix. We obtain a full characterization of the multivariate regular variation properties of $(\boldsymbol{X}_t)$, proving that coordinates $X_{t,i}$ and $X_{t,j}$ are asymptotically independent if and only if $m_i \neq m_j$; even though all coordinates rely on the same random input (M_t). We describe extremal properties of $(\boldsymbol{X}_t)$ in the framework of vector scaling regular variation. Our results are applied to multivariate autoregressive conditional heteroskedasticity (ARCH) processes.

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1 Introduction

We consider multivariate stationary processes $(\boldsymbol{X}_t)$, satisfying a diagonal Stochastic Recurrence Equation (SRE) of the form

$$\boldsymbol{X}_t = \mathrm{Diag}(m_1, \dots, m_d) M_t \boldsymbol{X}_{t-1} + \boldsymbol{Q}_t, \qquad t \in \mathbb{Z}, \tag{1.1}$$

where (M_t) is an iid sequence of real-valued random variables and $(\boldsymbol{Q}_t)$ is an iid sequence of $\mathbb{R}^d$ random vectors with marginals $Q_{t,i}$, $1 \le i \le d$, independent of (M_t). Stationary solutions of SRE have attracted a lot of research in the past few years, see Buraczewski et al. (2016b) and references therein. However, in the present setting of diagonal matrices, only marginal tail behavior has been investigated so far using the result of the seminal paper of Goldie (1991).

Due to the diagonal multiplicative term in (1.1), the marginals of $\boldsymbol{X}_t = (X_{t,1}, \dots, X_{t,d})^\top$ are satisfying the marginal SREs $X_{t,i} = m_i M_t X_{t-1,i} + Q_{t,i}$, $t \in \mathbb{Z}$ for $1 \le i \le d$. We work under the following set of assumptions that implies the ones of Goldie (1991) on the marginal SREs. Denoting by $(M, \boldsymbol{Q})$ a generic copy of $(M_t, \boldsymbol{Q}_t)$, we assume for all $1 \le i \le d$,

$$\mathbb{E}\big[\log\big(m_i |M|\big)\big] < 0. \tag{A1}$$

This guarantees that the Markov chain $(\boldsymbol{X}_t)$ has a unique stationary distribution. It is given by the law of the random variable

$$\boldsymbol{X} = \begin{pmatrix} X_1 \\ \vdots \\ X_d \end{pmatrix} := \sum_{k=1}^{\infty} \begin{pmatrix} m_1^{k-1} & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \dots & m_d^{k-1} \end{pmatrix} M_1 \cdots M_{k-1} \boldsymbol{Q}_k. \tag{1.2}$$

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We further assume that there exist positive constants $\alpha_1, \ldots, \alpha_d$ such that for $1 \le i \le d$

$$\mathbb{E}\big[(m_i|M|)^{\alpha_i}\big] \;=\; 1. \tag{A2}$$

Given these $\alpha_1, \ldots, \alpha_d$, we assume for $1 \le i \le d$

$$\mathbb{E}\big[|M|^{\alpha_i+\epsilon}\big] < \infty, \quad \mathbb{E}\big[\|\boldsymbol{Q}\|^{\alpha_i+\epsilon}\big] < \infty \;\text{ for some } \epsilon > 0. \tag{A3}$$

Of course, it suffices to check this condition for the maximal α_i. Note that (A1) follows from (A2)-(A3) as shown by (Goldie, 1991, Lemma 2.2). We also need the technical assumption that

$$\text{the law of } \log|M| \text{ is non-arithmetic, i.e. not supported on } \lambda\mathbb{Z},\ \lambda > 0. \tag{A4}$$

Finally, to avoid degeneracy, we require for $1 \le i \le d$ that

$$\mathbb{P}(m_i M x + Q_i = x) < 1 \qquad \text{for all } x \in \mathbb{R}\,, \tag{A5}$$

where Q_i, $1 \le i \le d$ denote the marginals of $\boldsymbol{Q}$. For all pairs $1 \le i, j \le d$ such that $\alpha_i > \alpha_j$, we will require that

$$\lim_{u\to\infty} \log(u)\, \mathbb{P}\Big(\frac{|Q_j|}{|Q_i|} > u^{\varepsilon}\Big) = 0 \qquad \text{for all } \epsilon > 0. \tag{A6}$$

With no loss of generality, we assume throughout the paper that $\mathbb{P}(M < 0) > 0$, the case of positive multiplicative factors M_t following from simpler arguments.

For the specific case (M_t) are iid $\mathcal{N}(0,1)$ and $(\boldsymbol{Q}_t)$ are iid $\mathcal{N}(0, \boldsymbol{C})$ and independent of (M_t), the diagonal SRE coincides with the BEKK-ARCH(1) model as in Pedersen and Wintenberger (2018)

$$\boldsymbol{X}_t = \boldsymbol{H}_t^{1/2} \boldsymbol{Z}_t, \qquad t \in \mathbb{Z}, \tag{1.3}$$

$$\boldsymbol{H}_t = \boldsymbol{C} + \mathrm{Diag}(m_1, \ldots, m_d) \boldsymbol{X}_{t-1} \boldsymbol{X}_{t-1}^\top \mathrm{Diag}(m_1, \ldots, m_d)\,, \tag{1.4}$$

where $(\boldsymbol{Z}_t)$ is an iid sequence of Gaussian random vectors $\mathcal{N}_d(0, I)$. The model depends on few parameters, the ones in the symmetric positive semi-definite matrix $\boldsymbol{C}$ and the diagonal coefficients $m_i > 0$ for $1 \le i \le d$. The Diagonal BEKK-ARCH(1) model is very interesting as it generates different tail index marginals. This freedom is not offered by other BEKK-ARCH model specification which marginals have the same tail index, see Pedersen and Wintenberger (2018). This feature is important for modeling: Heavy tailed data, such as in finance, may exhibit different tail indices indicating different responses during financial crisis. Under the top-Lyapunov condition

$$m_i^2 < 2e^{\gamma}, \qquad 1 \le i \le d, \tag{1.5}$$

where $\gamma \approx 0.5772$ is the Euler constant, it exists a stationary solution $(\boldsymbol{X}_t)$ of the system (1.3)-(1.4); see e.g. Nelson (1990). Moreover the Diagonal BEKK-ARCH(1) model satisfies the assumptions (A2) – (A5) (and thus (A1)) as soon as (1.5) holds; see Pedersen and Wintenberger (2018) for details. Condition (A6) is checked in Corollary 5.2 below.

Back to the diagonal SRE (1.1), under (A1) – (A6) the marginal stationary distributions are regularly varying with possibly different tail indices; following Goldie (1991) we have that

$$\mathbb{P}(X_{0,i} > x) \sim \mathbb{P}(X_{0,i} \le -x) \sim \frac{c_i}{2} x^{-\alpha_i}, \qquad x \to \infty, \tag{1.6}$$

where $c_i > 0$ and $\alpha_i > 0$ is the unique solution of the equation $\mathbb{E}[|m_i M|^{\alpha_i}] = 1$. Here and below, $f(x) \sim g(x)$ means that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$. As α_i is a decreasing function of m_i, the tail indices are distinct when the diagonal terms are distinct. Moreover, serial extremal dependence of the marginal sequences $(X_{t,i})_{t\in\mathbb{Z}}$ for any $1 \le i \le d$ is well known since the pioneer work of De Haan et al. (1989).

The main goal of this paper is to understand the **joint extremal behaviour**, *i.e.*, multivariate regular variation of $(\boldsymbol{X}_t)$ and the interplay between marginals that have distinct tail indices. As an example, consider the case of a couple $(X_{0,i}, X_{0,j})$ of marginals such that $m_i \neq m_j$ and then $\alpha_i \neq \alpha_j$. Our first main result in **Section 3** states that $X_{0,i}$ and $X_{0,j}$ are asymptotically *independent* in the sense that

$$\lim_{x\to\infty} \mathbb{P}(X_{0,i} > x^{1/\alpha_i} \mid X_{0,j} > x^{1/\alpha_j}) = \lim_{x\to\infty} \mathbb{P}(X_{0,j} > x^{1/\alpha_j} \mid X_{0,i} > x^{1/\alpha_i}) = 0. \tag{1.7}$$

This result remains true also when $Q_i = Q_j$. Thus, even though $X_{0,i}$ and $X_{0,j}$ are perfectly dependent in the sense that all their randomness comes from the same random variables, extremes never occur simultaneously in these marginals. This result also allows us to derive that the random vector $(X_{0,i}, X_{0,j})$ is non-standard regularly varying in the sense of Resnick (2007). This result extends easily to $d \geq 2$ and we will show the following.

Theorem 1.1. *Suppose* (A1) – (A6)*, and that $m_i \neq m_j$ for $i \neq j$. Then all components of $\boldsymbol{X}_0$ are asymptotically independent,* i.e.*,* (1.7) *holds.*

This theorem is a particular case of the more general Theorem 5.1, proved in Section 5.

Section 4 concerns the case where the diagonal terms m_i are identically equal to m and hence the tail indices of the marginals $X_{0,i}$ are the same. Applying Theorem 1.6 of Buraczewski et al. (2009) on the SRE equation (1.3)-(1.4) with multiplicative similarity matrix $mM_0\boldsymbol{I}_d$ Pedersen and Wintenberger (2018) derived multivariate regular variation of the process (X_t). We refined this result by characterizing the angular properties of the tail measure.

To study the general diagonal SRE where some diagonal elements are identical and others are distinct, we use and extend the framework of *Vector Scaling Regular Variation (VSRV)*, introduced in Pedersen and Wintenberger (2018). It is defined in full generality in **Section 2**. It describes the joint extremal behaviour via a spectral tail process $(\widetilde{\boldsymbol{\Theta}}_t)_{t\geq 0}$, satisfying

$$\widetilde{\boldsymbol{\Theta}}_t = \mathrm{Diag}(m_1, \ldots, m_d) M_t \widetilde{\boldsymbol{\Theta}}_{t-1}, \qquad t \geq 1$$

from some initial value $\widetilde{\boldsymbol{\Theta}}_0$. In **Section 5**, we derive the characterization of $\widetilde{\boldsymbol{\Theta}}_0$, proving asymptotic independence between blocks with different tail indices, and asymptotic dependence within blocks.

In **Section 6**, we extend our results by studying second order properties, *i.e.*, we show - under more restrictive assumptions - that there exist two rates $\Delta > \delta > 0$, depending on the coefficients m_i and m_j, so that

$$\lim_{x\to\infty} x^{1+\delta} \mathbb{P}(X_{0,i} > x^{1/\alpha_i}, X_{0,j} > x^{1/\alpha_j}) = 0 \tag{1.8}$$

$$\liminf_{x\to\infty} x^{1+\Delta} \mathbb{P}(X_{0,i} > x^{1/\alpha_i}, X_{0,j} > x^{1/\alpha_j}) > 0. \tag{1.9}$$

Notation

There and in the rest of the paper we will denote by $\overset{\text{law}}{=}$ the equality in distribution (between random variables on both sides), $\|\cdot\|$ will denote the infinity norm on $\mathbb{R}^d$ and $\|\cdot\|_2$ the euclidean norm. For vectors, we use bold notation $\boldsymbol{x} = (x_1, \ldots, x_d)$. Operations between vectors or scalar and vector are interpreted coordinate wise, *e.g.*, $x^{-1/\boldsymbol{\alpha}} = (x^{-1/\alpha_1}, \ldots, x^{-1/\alpha_d})$ for positive x and $\boldsymbol{ab} = (a_i b_i)_{1\leq i\leq d}$. A notation that will be used frequently is vector scaling of a sequence of $\mathbb{R}^d$-valued random variables, e.g.

$$\begin{aligned} x^{-1/\boldsymbol{\alpha}}(\boldsymbol{X}_0, \ldots, \boldsymbol{X}_t) &= \big(x^{-1/\boldsymbol{\alpha}}\boldsymbol{X}_0, \ldots, x^{-1/\boldsymbol{\alpha}}\boldsymbol{X}_t\big) \\ &= \big((x^{-1/\alpha_i}X_{0,i})_{1\leq i\leq d}, \ldots, (x^{-1/\alpha_i}X_{t,i})_{1\leq i\leq d}\big). \end{aligned} \tag{1.10}$$

For some potentially distinct $\alpha_1, \dots, \alpha_d$ we define the following notion of a radial distance:

$$\|\boldsymbol{x}\|_{\boldsymbol{\alpha}} = \max_{1\le i\le d} |x_i|^{\alpha_i} = \|\boldsymbol{x}^{\boldsymbol{\alpha}}\|, \qquad \boldsymbol{x} = (x_i)_{1\le i\le d} \in \mathbb{R}^d. \tag{1.11}$$

Here $\boldsymbol{x}^{\boldsymbol{\alpha}}$ denotes the vector $(\mathrm{sign}(x_i)|x_i|^{\alpha_i})_{1\le i\le d}$ in $\mathbb{R}^d$. We want to stress that $\|\boldsymbol{x}\|_{\boldsymbol{\alpha}}$ is neither homogeneous nor does it satisfy the triangle inequality for general values of $\alpha_1, \dots, \alpha_d$. Thus, it is not a (pseudo-)norm but it will provide a meaningful scaling function. Note that $\boldsymbol{x} \mapsto \|\boldsymbol{x}\|_{\boldsymbol{\alpha}}$ is a continuous function and is $1/\boldsymbol{\alpha}$-homogeneous in the following sense:

$$\|\lambda^{1/\boldsymbol{\alpha}} \boldsymbol{X}_0\|_{\boldsymbol{\alpha}} \;=\; \max_{1\le i\le d} \left|\lambda^{1/\alpha_i} X_{0,i}\right|^{\alpha_i} \;=\; \lambda \|\boldsymbol{X}_0\|_{\boldsymbol{\alpha}} \tag{1.12}$$

The components of the vector

$$\|\boldsymbol{X}_0\|_{\boldsymbol{\alpha}}^{-1/\boldsymbol{\alpha}} \boldsymbol{X}_t \;=\; \big(\|\boldsymbol{X}_0\|_{\boldsymbol{\alpha}}^{-1/\alpha_i} X_{t,i}\big)_{1\le i\le d} \tag{1.13}$$

have $\|\cdot\|_{\boldsymbol{\alpha}}$ and max-norm equal to one when $t=0$ thus belongs to $\mathcal{S}_\infty^{d-1} = \{\boldsymbol{x} \in \mathbb{R}^d;\, \|\boldsymbol{x}\|_{\boldsymbol{\alpha}} = 1\}$ the max-norm-unit sphere.

2 Vector Scaling Regular Variation (VSRV) Markov chains

2.1 Vector Scaling Regular Variation

2.1.1 Regular variation and the tail process

Let $(\boldsymbol{X}_t) \in \mathbb{R}^d$ be a stationary time series. Its regular variation properties are defined in different ways. The most usual way is to define the tail process as in Basrak and Segers (2009).

Definition 2.1. The stationary time series $(\boldsymbol{X}_t)$ is regularly varying if and only if $\|\boldsymbol{X}_0\|$ is regularly varying and there exist weak limits

$$\lim_{x\to\infty} \mathbb{P}\Big(\|\boldsymbol{X}_0\|^{-1}(\boldsymbol{X}_0, \dots, \boldsymbol{X}_t) \in \cdot \,\Big|\, \|\boldsymbol{X}_0\| > x\Big) = \mathbb{P}\big((\boldsymbol{\Theta}_0, \dots, \boldsymbol{\Theta}_t) \in \cdot\big), \quad t \ge 0\,.$$

By stationarity and using Kolmogorov consistency theorem one can extend the trajectories $(\boldsymbol{\Theta}_0, \dots, \boldsymbol{\Theta}_t)$ into a process $(\boldsymbol{\Theta}_t)$ called the *spectral tail process*. To be stationary regularly varying time series does not depend on the choice of the norm. We work with the max-norm in the following for convenience.

2.1.2 Non-standard Regular Variation

If there exists $1 \le i \le d$ such that

$$\mathbb{P}(|X_{0,i}| > x) = o\big(\mathbb{P}(\|\boldsymbol{X}_0\| > x)\big)\,, \qquad x \to \infty\,,$$

then the marginals of $\boldsymbol{X}_0$ are *not tail equivalent*. In this case, the notion introduced above is not suitable, since then the corresponding coordinate of the spectral tail process is degenerated, *i.e.*, $\Theta_{0,i} = 0$ a.s. Hence, information about extremes in this coordinate is lost.

To circumvent this issue, the notion of *non-standard regular variation* was introduced (see Resnick (2007) and reference therein). It is based on a standardization of the coordinates which holds as follows. Assume that marginals are positive and (one-dimensional) regularly varying with possibly different tail indices α_i and cdf F_i, $1 \le i \le d$. Then non-standard regular variation holds if and only if

$$\lim_{x\to\infty} x \cdot \mathbb{P}\big(x^{-1}\widetilde{\boldsymbol{X}_0} \in \cdot\big)$$

exists in the vague sense, where the standardized vector $\widetilde{\boldsymbol{X}_0}$ is defined as

$$\widetilde{\boldsymbol{X}_0} = (1/(1 - F_i(X_{0,i})))_{1\leq i\leq d}\,. \tag{2.1}$$

Following (de Haan and Resnick, 1977, Theorem 4), we note that $\widetilde{\boldsymbol{X}_0}$ is regularly varying in the classical sense, i.e. $\|\widetilde{\boldsymbol{X}_0}\|$ is regularly varying with tail index 1 and there exists an angular measure which is the weak limit of

$$\lim_{x\to\infty} \mathbb{P}\Big(\|\widetilde{\boldsymbol{X}_0}\|^{-1}\widetilde{\boldsymbol{X}_0} \in \cdot \,\Big|\, \|\widetilde{\boldsymbol{X}_0}\| > x\Big)\,.$$

Note that the standardization is made so that all coordinates $\widetilde{\boldsymbol{X}_0}$ of are tail equivalent

$$\mathbb{P}\big(\widetilde{X_{0,i}} > x\big) \sim x^{-1}\,, \qquad x\to\infty\,, \qquad 1\leq i\leq d\,.$$

2.1.3 Vector Scaling Regular Variation

When dealing with time series such as diagonal SRE, temporal dependencies between extremes are of particular interest. As it turns out, neither of the notions discussed above is fully adequate for the investigation of these. Indeed, the SRE representation (1.1) of the diagonal BEKK-ARCH(1) model appeals for an analysis of the serial extremal dependence directly on $(\boldsymbol{X}_t)$ rather than on a standardized version. For SRE Markov chains such as (1.1), it has been shown by Janssen and Segers (2014) that the spectral tail process satisfies the simple recursion

$$\boldsymbol{\Theta}_t = \mathrm{Diag}(a_1,\ldots,a_d)\boldsymbol{\Theta}_{t-1}, \qquad t\geq 1.$$

This multiplicative property has nice consequences and allows to translate the properties of multiplicative random walks to the extremes of multivariate time series. However, the degeneracy of the coordinates with lower tails discussed in Section 2.1.2 propagates through time; If $\Theta_{0,i} = 0$ a.s. then $\Theta_{t,i} = 0$ a.s. as well for any $t\geq 1$. On the other hand, the standardized version does not satisfy an SRE and its serial extremal dependence is less explicit; see Perfekt (1997) for details.

In order to treat the temporal dependence of the stationary solution $(\boldsymbol{X}_t)$, we will use the notion of Vector Scaling Regular Variation (VSRV) introduced in Pedersen and Wintenberger (2018). We slightly extend the original notion of Pedersen and Wintenberger (2018), suppressing the requirement that the marginal tails are equivalent to power functions. This wider definition of VSRV writes in a simpler form as follows:

Definition 2.2 (VSRV)**.** A stationary time series $(\boldsymbol{X}_t)$ is VSRV of order $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_d)$ if $\|\boldsymbol{X}_0\|_{\boldsymbol{\alpha}}$ is regularly varying and there exists weak limits

$$\lim_{x\to\infty} \mathbb{P}\Big(\|\boldsymbol{X}_0\|_{\boldsymbol{\alpha}}^{-1/\boldsymbol{\alpha}}(\boldsymbol{X}_0,\ldots,\boldsymbol{X}_t) \in \cdot \,\Big|\, \|\boldsymbol{X}_0\|_{\boldsymbol{\alpha}} > x\Big) = \mathbb{P}\big((\widetilde{\boldsymbol{\Theta}}_0,\ldots,\widetilde{\boldsymbol{\Theta}}_t) \in \cdot\big)\,, \tag{2.2}$$

for any $t\geq 0$.

A few remarks are in order. A nonnegative VSRV time series $(\boldsymbol{X}_t)$ with indices $\alpha_1,\ldots,\alpha_d$ is VSRV if and only if $(\boldsymbol{X}_t^{\boldsymbol{\alpha}})$ is regularly varying. Moreover, $(\boldsymbol{X}_t)$ is also VSRV with indices $\beta\alpha_1,\ldots,\beta\alpha_d$ for any $\beta > 0$. Note that a times series is VSRV with indices $\alpha_1 = \ldots = \alpha_d$ if and only if it is standard regularly varying. For general indices, the marginals $X_{0,i}$ have distributions F_i with different tail indices.

The next proposition show that indeed any VSRV random vector $\boldsymbol{X}_0 \in \mathbb{R}^d$ (i.e. $\|X_0\|_{\boldsymbol{\alpha}}$ is regularly varying and (2.2) holds for $t = 0$) is also non-standard regularly varying.

Proposition 2.3. *Let $\boldsymbol{X}_0$ be a positive VSRV random vector of order $\alpha = (\alpha_1,\ldots,\alpha_d)$. Then $\boldsymbol{X}_0$ has regularly varying marginals satisfying*

$$\mathbb{P}(X_{0,i} > x) \sim c_i\,\mathbb{P}(\|\boldsymbol{X}_0\|_{\boldsymbol{\alpha}} > x^{\alpha_i})\,, \qquad x\to\infty\,, \tag{2.3}$$

Moreover it is non-standard regularly varying if and only if $\mathbb{P}(\widetilde{\Theta}_{0,i}=0)<1$ *for all* $1\le i\le d$ *and the angular measure is given by*

$$\frac{\mathbb{E}\big[\|\boldsymbol{c}^{-1}\widetilde{\boldsymbol{\Theta}}_0^{\beta\boldsymbol{\alpha}}\|\,\mathbf{1}\big(\|\boldsymbol{c}^{-1}\widetilde{\boldsymbol{\Theta}}_0^{\beta\boldsymbol{\alpha}}\|^{-1}\boldsymbol{c}^{-1}\widetilde{\boldsymbol{\Theta}}_0^{\beta\boldsymbol{\alpha}}\in\cdot\big)\big]}{\mathbb{E}\big[\|\boldsymbol{c}^{-1}\widetilde{\boldsymbol{\Theta}}_0^{\beta\boldsymbol{\alpha}}\|\big]}\,,$$

where $\boldsymbol{c}=(c_1,\dots,c_d)$,

$$c_i=\mathbb{E}\big[\widetilde{\boldsymbol{\Theta}}_{0,i}^{\beta\alpha_i}\big],\qquad 1\le i\le d\,,$$

with $\beta>0$ *the index of regular variation of* $\|\boldsymbol{X}_0\|_\alpha$.

We remark that the angular measure of $\boldsymbol{X}_0$ is completely determined by the spectral tail process $(\widetilde{\boldsymbol{\Theta}}_t)$. However its expression is intricate because of the standardization whereas we will derive explicit expressions of $(\widetilde{\boldsymbol{\Theta}}_t)$ for many Markov chains in Section 2.2. We emphasize that this simplicity is the main motivation for introducing the notion of VSRV rather than using the more general notion of non-standard regular variation.

Proof. Fix $1\le i\le d$. Denoting $y=x^{\alpha_i}$, using (1.12) and the definition of the weak convergence as $\{x\in\mathbb{R}^d;\ x_i>1\}$ is a continuity set by homogeneity of the limiting measure, we obtain

$$\begin{aligned}\frac{\mathbb{P}(X_{0,i}>y^{1/\alpha_i})}{\mathbb{P}(\|\boldsymbol{X}_0\|_{\boldsymbol{\alpha}}>y)}&=\frac{\mathbb{P}\big(y^{-1/\alpha_i}X_{0,i}>1,\ \|\boldsymbol{X}_0\|_{\boldsymbol{\alpha}}>y\big)}{\mathbb{P}(\|\boldsymbol{X}_0\|_{\boldsymbol{\alpha}}>y)}\\&=\mathbb{P}\Big(y^{-1/\alpha_i}X_{0,i}>1\,\Big|\,\|\boldsymbol{X}_0\|_{\boldsymbol{\alpha}}>y\Big)\\&=\mathbb{P}\Big((\|\boldsymbol{X}_0\|_{\boldsymbol{\alpha}}/y)^{1/\alpha_i}\|\boldsymbol{X}_0\|_{\boldsymbol{\alpha}}^{-1/\alpha_i}X_{0,i}>1\,\Big|\,\|\boldsymbol{X}_0\|_{\boldsymbol{\alpha}}>y\Big)\\&\to\ \mathbb{P}(Y^{1/\alpha_i}\widetilde{\boldsymbol{\Theta}}_{0,i}>1)\,,\qquad y\to\infty\,,\end{aligned}$$

where $\mathbb{P}(\|\boldsymbol{X}_0\|_{\boldsymbol{\alpha}}/y\in\cdot\,|\,\|\boldsymbol{X}_0\|_{\boldsymbol{\alpha}}>y)\to\mathbb{P}(Y\in\cdot)$ with Y Pareto β distributed. We compute

$$\mathbb{P}(Y^{1/\alpha_i}\widetilde{\boldsymbol{\Theta}}_{0,i}>1)=\int_1^\infty\mathbb{P}(\widetilde{\boldsymbol{\Theta}}_{0,i}^{\beta\alpha_i}>y)d(-y^{-1})=\mathbb{E}\big[\widetilde{\boldsymbol{\Theta}}_{0,i}^{\beta\alpha_i}\big]$$

as $\widetilde{\boldsymbol{\Theta}}_{0,i}^{\beta\alpha_i}\le\|\widetilde{\boldsymbol{\Theta}}\|_{\boldsymbol{\alpha}}=1$ a.s. by definition and the first assertion follows.

Denoting $F_{\boldsymbol{\alpha}}(x)$ the cdf of $\|\boldsymbol{X}_0\|_\alpha$ and $\bar F_{\boldsymbol{\alpha}}=1-F_{\boldsymbol{\alpha}}$, we get

$$\bar F_i(y)\sim\bar F_{\boldsymbol{\alpha}}\big(c_i^{-1/\beta}y^{\alpha_i}\big),\qquad y\to\infty,\tag{2.4}$$

using the regular variation properties of order $\beta>0$ of $\bar F_{\boldsymbol{\alpha}}$. The standardized vector

$$\widetilde{\boldsymbol{X}}_0=\left(\frac{1}{\bar F_{\boldsymbol{\alpha}}\big(c_i^{-1/\beta}X_{0,i}^{\alpha_i}\big)}\right)_{1\le i\le d}$$

has marginal tails equivalent to the standard Pareto marginally distributed vector $\big(1/(1-F_i(X_{0,i}))\big)_{1\le i\le d}$ by (2.4). Thus, $\boldsymbol{X}+0$ is regularly varying if the vague convergence holds

$$\mathbb{P}\big(x^{-1}\widetilde{\boldsymbol{X}}_0\in\cdot\,\big|\,\|\widetilde{\boldsymbol{X}}_0\|>x\big)\to\nu_*\,,\qquad x\to\infty,$$

with ν_* the standardized tail measure. It is implied by

$$\mathbb{P}\left(y^{-1}\boldsymbol{c}^{-1/\beta}\boldsymbol{X}_0^{\boldsymbol{\alpha}}\in\cdot\,\Big|\,\|\boldsymbol{c}^{-1/\beta}\boldsymbol{X}_0^{\boldsymbol{\alpha}}\|>y\right)\to\nu\,,\qquad x\to\infty\,,$$

with ν the tail measure of $\boldsymbol{X}_0^{\boldsymbol{\alpha}}$ which is standard regularly varying of order $\beta>0$. From (Resnick, 2007, (6.46) p. 205) we notice that ν_* exists and is such that $\nu([\mathbf{0},\boldsymbol{x}^{1/\beta}]^c)=\nu_*([\mathbf{0},\boldsymbol{x}]^c)$. Then the relation between $\widetilde{\boldsymbol{\Theta}}_0^{\boldsymbol{\alpha}}$, the spectral component of the tail measure μ, and the angular measure of the standardized tail measure ν_* follows from standard calculations. □

2.2 VSRV Markov chains

We adapt the work of Janssen and Segers (2014) to our framework. We consider a Markov chain $(\boldsymbol{X}_t)_{t\geq 0}$ with values in $\mathbb{R}^d$ satisfying the recursive equation

$$\boldsymbol{X}_t = \boldsymbol{\Phi}(\boldsymbol{X}_{t-1}, Z_t), \qquad t \geq 0\,, \tag{2.5}$$

where $\boldsymbol{\Phi} : \mathbb{R}^d \times \mathcal{E} \mapsto \mathbb{R}^d$ is measurable and (Z_t) is an iid sequence taking values in a Polish space $\mathcal{E}$. We work under the following assumption, which is the vector scaling adaptation of (Janssen and Segers, 2014, Condition 2.2). As above, we fix in advance the positive indices $\alpha_1, \dots, \alpha_d$.

VS Condition for Markov chains: *There exists a measurable function* $\boldsymbol{\phi} : \mathcal{S}_\infty^{d-1} \times \mathcal{E} \mapsto \mathbb{R}^d$ *such that, for all* $e \in \mathcal{E}$*,*

$$\lim_{x\to\infty} x^{-1/\boldsymbol{\alpha}} \boldsymbol{\Phi}(x^{1/\boldsymbol{\alpha}} \boldsymbol{s}(x), e) \to \boldsymbol{\phi}(\boldsymbol{s}, e)\,,$$

whenever $\boldsymbol{s}(x) \to \boldsymbol{s}$ *in* $\mathcal{S}_\infty^{d-1}$*. Moreover, if* $\mathbb{P}(\boldsymbol{\phi}(\boldsymbol{s}, Z_0) = 0) > 0$ *for some* $\boldsymbol{s} \in \mathcal{S}_\infty^{d-1}$ *then* $Z_0 \in \mathcal{W}$ *a.s. for a subset* $\mathcal{W} \subset \mathcal{E}$ *such that, for all* $e \in \mathcal{W}$*,*

$$\sup_{\|\boldsymbol{y}\|_{\boldsymbol{\alpha}} \leq x} \|\boldsymbol{\Phi}(\boldsymbol{y}, e)\|_{\boldsymbol{\alpha}} = O(x) \qquad x \to \infty\,.$$

We extend $\boldsymbol{\phi}$ over $\mathbb{R}^d \times \mathcal{E}$ thanks to the relation

$$\boldsymbol{\phi}(\boldsymbol{v}, e) = \begin{cases} \|\boldsymbol{v}\|_{\boldsymbol{\alpha}}^{1/\boldsymbol{\alpha}} \boldsymbol{\phi}\Big(\|\boldsymbol{v}\|_{\boldsymbol{\alpha}}^{-1/\boldsymbol{\alpha}} v, e\Big) & \text{if} \quad \boldsymbol{v} \neq 0, \\ 0 & \text{if} \quad \boldsymbol{v} = 0\,. \end{cases}$$

We have the following result which extends Theorem 2.1 of Janssen and Segers (2014)

Theorem 2.4. *If the Markov chain* $(\boldsymbol{X}_t)$ *satisfies the recursion* (2.5) *with* $\boldsymbol{\Phi}$ *satisfying the VS condition and if the vector* $\boldsymbol{X}_0$ *is VSRV with positive indices* $\alpha_1, \dots, \alpha_d$ *then* $(\boldsymbol{X}_t)_{t\geq 0}$ *is a VSRV process and its spectral tail process satisfies the relation*

$$\widetilde{\boldsymbol{\Theta}}_t = \boldsymbol{\phi}(\widetilde{\boldsymbol{\Theta}}_{t-1}, Z_t)\,, \qquad t \geq 0\,. \tag{2.6}$$

started from $\widetilde{\boldsymbol{\Theta}}_0$*, the spectral component of* $\boldsymbol{X}_0$*.*

Proof. The result follows by an application of Theorem 2.1 in Janssen and Segers (2014) to the Markov chain $(\boldsymbol{Y}_t)_{t\geq 0} = (\boldsymbol{X}_t^{\boldsymbol{\alpha}})_{t\geq 0}$. We have $\boldsymbol{Y}_0$ regularly varying since $\boldsymbol{X}_0^{\boldsymbol{\alpha}}$ is VSRV. Moreover

$$\boldsymbol{Y}_t = \widetilde{\boldsymbol{\Phi}}(\boldsymbol{Y}_{t-1}, Z_t)\,, \qquad t \geq 0\,,$$

with $\widetilde{\boldsymbol{\Phi}}(x, z) = (\boldsymbol{\Phi}(x^{1/\boldsymbol{\alpha}}, z))^{\boldsymbol{\alpha}}$. As the VS condition for Markov chain is the vector scaling version of the condition 2.2. of Janssen and Segers (2014) on $\widetilde{\boldsymbol{\Phi}}$ associated to the limit $\widetilde{\boldsymbol{\phi}}((x, z)) = \boldsymbol{\phi}((x^{1/\boldsymbol{\alpha}}, z))^{\boldsymbol{\alpha}}$, i.e.

$$\lim_{x\to\infty} x^{-1} \widetilde{\boldsymbol{\Phi}}(x\boldsymbol{s}(x), e) \to \widetilde{\boldsymbol{\phi}}(\boldsymbol{s}, e)$$

whenever $\boldsymbol{s}(x) \to \boldsymbol{s}$ in $\mathcal{S}_\infty^{d-1}$. We obtain that the spectral tail process of $(\boldsymbol{Y}_t)_{t\geq 0}$ satisfies the recursion

$$\boldsymbol{\Theta}_t^{\boldsymbol{Y}} = \widetilde{\boldsymbol{\phi}}(\boldsymbol{\Theta}_{t-1}^{\boldsymbol{Y}}, Z_t)\,, \qquad t \geq 1\,.$$

The desired result follows as $\widetilde{\boldsymbol{\phi}}((x, z)) = \widetilde{\boldsymbol{\phi}}((x^{1/\boldsymbol{\alpha}}, z))^{\boldsymbol{\alpha}}$ and $\widetilde{\boldsymbol{\Theta}}_t^{\boldsymbol{\alpha}} = \boldsymbol{\Theta}_t^{\boldsymbol{Y}}$, $t \geq 0$. □

We are specially interested in Stochastic Recurrence Equations (SRE) corresponding to the Markov chains

$$\boldsymbol{X}_t = \boldsymbol{\Phi}(\boldsymbol{X}_{t-1}, (\boldsymbol{M}, \boldsymbol{Q})_t) = \boldsymbol{M}_t \boldsymbol{X}_{t-1} + \boldsymbol{Q}_t\,, \qquad t \geq 0\,.$$

In this setting $(\boldsymbol{M}_t)$ are iid random $d \times d$ matrices and $(\boldsymbol{Q}_t)$ iid random vectors in $\mathbb{R}^d$. We have

Proposition 2.5. *The SRE Markov chain $(\boldsymbol{X}_t)_{t\geq 0}$ satisfies Condition VS for positive indices $\alpha_1,\dots,\alpha_d$ if and only if $M_{ij}=0$ a.s. for any (i,j) so that $\alpha_i>\alpha_j$. Then*

$$\boldsymbol{\phi}\big(\boldsymbol{s},(\boldsymbol{M},\boldsymbol{Q})\big)=\Big(\sum_{j=1}^d M_{ij}\mathbf{1}_{\alpha_i=\alpha_j}s_j\Big)_{1\leq i\leq d}.$$

Proof. As $x\to\infty$ and $\boldsymbol{s}(x)\to\boldsymbol{s}$, we have

$$\lim_{x\to\infty} x^{-1/\boldsymbol{\alpha}}\boldsymbol{\Phi}\big((x^{1/\boldsymbol{\alpha}})\boldsymbol{s}(x),(\boldsymbol{M},\boldsymbol{Q})\big) \;=\; \lim_{x\to\infty} x^{-1/\boldsymbol{\alpha}}\Big(\boldsymbol{M}(x^{1/\boldsymbol{\alpha}}\boldsymbol{s}(x))+\boldsymbol{Q}\Big)$$
$$=\lim_{x\to\infty}\Big(\sum_{j=1}^d M_{ij}s(x)_j x^{1/\alpha_j-1/\alpha_i}\Big)_{1\leq i\leq d}.$$

Each coordinate converges to $\sum_{j=1}^d M_{ij}1_{\alpha_i=\alpha_j}s_j$ for any $\boldsymbol{s}\in\mathcal{S}_\infty^{d-1}$ if and only if $M_{ij}=0$ a.s. for any (i,j) so that $\alpha_i>\alpha_j$. □

Remark 2.6. In case of distinct α_i's, it means that the dynamic tail process depends only on the diagonal elements of $\boldsymbol{M}$. In general, specifying $\boldsymbol{M}_t$ to be diagonal, we ensure that if $\boldsymbol{X}_0$ is VSRV then the SRE process is VSRV with

$$\widetilde{\boldsymbol{\Theta}}_t=\boldsymbol{M}_t\widetilde{\boldsymbol{\Theta}}_{t-1},\qquad t\geq 1\,,$$

whatever are the positive indices $\alpha_1,\dots,\alpha_d$.

3 The diagonal SRE with distinct coefficients

In this section we will show that the marginals of the diagonal SRE with distinct coefficients are asymptotically independent. A standard argument reduces the discussion to the bivariate case.

More precisely, we consider the bivariate random recursive process $\boldsymbol{X}_t=\boldsymbol{M}_t\boldsymbol{X}_{t-1}+\boldsymbol{Q}_t$, defined by $\boldsymbol{X}_0=0$ and

$$\begin{pmatrix} X_{t,1}\\ X_{t,2}\end{pmatrix} \;=\; \begin{pmatrix} m_1 & 0\\ 0 & m_2\end{pmatrix} M_t \begin{pmatrix} X_{t-1,1}\\ X_{t-1,2}\end{pmatrix}+Q_t, \tag{3.1}$$

where $(M_t)_{t\in\mathbb{N}}$ are iid real-valued random variables, $(Q_t)_{t\in\mathbb{N}}$ are iid random vectors independent of (M_t) while

$$0<m_1<m_2$$

are positive constants. We assume assumptions (A1) – (A6) to hold for $i=1,2$, which gives that

$$\alpha_1>\alpha_2.$$

We are going to study partial sums converging to the random variables X_1, X_2 given by (1.2), namely

$$X_{n,i}:=\sum_{k=1}^n m_i^{k-1}M_1\cdots M_{k-1}Q_{k,i},\qquad i=1,2.$$

Note the distinction between the Markov chain $(X_{t,i})$ (the *forward process*) and the almost surely convergent series $(X_{n,i})$ defined above (the *backward process*); see Letac (1986). Within this section, we will always consider the backward process $(X_{n,i})$.

Under our assumptions, by the Kesten-Goldie-Theorem of Goldie (1991); Kesten (1973) applied to multiplicative factors with $\mathbb{P}(m_iM<0)>0$, $i=1,2$, we have

$$\lim_{u\to\infty}u^{\alpha_1}\mathbb{P}(\pm X_1>u)=C_1,\qquad \lim_{u\to\infty}u^{\alpha_2}\mathbb{P}(\pm X_2>u)=C_2 \tag{3.2}$$

for positive constants C_1, C_2. Note that C_1 and C_2 are the same for the left and right tails.

3.1 Preliminaries

The asymptotic independence is proved studying the generation of extreme values in the marginal SREs that are exponential random walks with negative drifts. Extremal values of an exponential random walk occur after a change of measure, turning the negative drift into a positive one, applied during a certain period of time. The proof is based on the fact that the periods of time necessary for the exponential random walks to exhibit extremes are different on each marginals. Thus the extremes among marginals are asynchronous. Using the explicit formulation of the backward process we show that this asynchrony is responsible of the asymptotic independence among marginals.

Let us define the cumulant generating function $\Lambda(\alpha) = \log(\mathbb{E}|M_1|^\alpha)$ for $\alpha > 0$. Under Assumption (A5), the distribution of $\log|M_1|$ is non degenerate and Λ is a strictly convex function on its domain of definition. On this domain one can define $m(\alpha)$ as the unique solution $\alpha > 0$ of the equation $\mathbb{E}[(m|M_1|)^\alpha] = 1$, i.e. satisfying the relation

$$\alpha \log(m(\alpha)) + \Lambda(\alpha) = 0. \tag{3.3}$$

The following quantity

$$\mu(\alpha) = \mathbb{E}[\log(m(\alpha)|M_1|)(m(\alpha)|M_1|)^\alpha],$$

will play an important role in the proof of the asymptotic independence. It corresponds to the positive drift under the change of measure. We have the relation

Lemma 3.1. *The relation*

$$\alpha_2\Big(1 + \frac{\log(m(\alpha_2)) - \log(m(\alpha_1))}{\mu(\alpha_1)}\Big) < \alpha_1. \tag{3.4}$$

is always satisfied for $\alpha_2 < \alpha_1$ in the domain of definition of Λ.

Proof. We rewrite the condition (3.4) as follows:

$$\begin{aligned}
& \alpha_2\Big(1 + \frac{\log(m(\alpha_2)) - \log(m(\alpha_1))}{\mu(\alpha_1)}\Big) < \alpha_1 \\
\Leftrightarrow \quad & \alpha_2 \frac{\log(m(\alpha_2)) - \log(m(\alpha_1))}{\alpha_1 - \alpha_2} < \mu(\alpha_1) \\
\Leftrightarrow \quad & \frac{\alpha_1 \log(m(\alpha_1)) - \alpha_2 \log(m(\alpha_2))}{\alpha_2 - \alpha_1} + \log(m(\alpha_1)) < \mu(\alpha_1) \\
\Leftrightarrow \quad & \frac{\Lambda(\alpha_2) - \Lambda(\alpha_1)}{\alpha_2 - \alpha_1} < \mu(\alpha_1) - \log(m(\alpha_1)),
\end{aligned} \tag{3.5}$$

identifying $\alpha_1 \log(m(\alpha_1)) = -\Lambda(\alpha)$ using the identity (3.3). As Λ is infinitely differentiable on its domain of definition we can replace the difference quotient by $\Lambda'(\xi)$ for some $\xi \in (\alpha_2, \alpha_1)$ due to the intermediate value theorem. We further identify $\mu_1 - \log(m_1)$ as $\Lambda'(\alpha_1)$ and (3.5) becomes

$$\Lambda'(\xi) < \Lambda'(\alpha_1) \qquad \text{with } \alpha_1 > \xi.$$

Hence, the assertion follows as Λ is strictly convex. □

3.2 Proof of the asymptotic independence

The asymptotic independence of (X_1, X_2) is proved assuming $m_2 > m_1$, which implies $\alpha_1 > \alpha_2$. We denote $\mu_i = \mu(\alpha_i)$ for $i = 1, 2$.

Theorem 3.2. *Assume* (A1)–(A6) *for $i = 1, 2$. Then we have*

$$\lim_{u \to \infty} \mathbb{P}\Big(|X_2| > u^{1/\alpha_2} \,\Big|\, |X_1| > u^{1/\alpha_1}\Big) \;=\; 0,$$

i.e., $|X_1|$ and $|X_2|$ are asymptotically independent.

Remark 3.3. In particular, $\pm X_1$ and $\pm X_2$ are asymptotically independent. Indeed, from (3.2) we have

$$\mathbb{P}\Big(\pm X_i > u^{1/\alpha_i} \Big) \sim \frac{1}{2}\mathbb{P}\Big(|X_i| > u^{1/\alpha_i} \Big), \qquad i = 1, 2\,,$$

so that immediately we obtain as well

$$\lim_{u\to\infty} \mathbb{P}\Big(\pm X_2 > u^{1/\alpha_2} \,\Big|\, \pm X_1 > u^{1/\alpha_1} \Big) \;=\; 0\,.$$

Proof. Thanks to the Kesten-Goldie theorem (see Eq. (3.2)) it is enough to prove

$$\lim_{u\to\infty} u\,\mathbb{P}\Big(|X_2| > u^{1/\alpha_2}\,,\, |X_1| > u^{1/\alpha_1} \Big) \;=\; 0. \tag{3.6}$$

Step 1. We reduce to the study of a dominating sequence with nonnegative coefficients:

$$|X_{n,i}| \;\le\; X^*_{n,i} \;:=\; \sum_{k=1}^{n} m_i^{k-1} |M_1 \cdots M_{k-1}||Q_{k,i}|, \qquad i = 1, 2.$$

We notice that $X^*_i := \lim_{n\to\infty} X^*_{n,i}$ satisfies the fixed point equation, in distribution,

$$X^*_i \overset{\text{law}}{=} m_i |M| X^*_i + |Q_i|\,, \qquad i = 1, 2\,.$$

In particular, thanks to (A1)–(A4), the Kesten-Goldie theorem, now used in the case of positive coefficients, applies and yields

$$\lim_{u\to\infty} u\,\mathbb{P}\Big(X^*_2 > u^{1/\alpha_2} \Big) = C^*_2 > 0 \qquad \lim_{u\to\infty} u\mathbb{P}\Big(X^*_1 > u^{1/\alpha_1} \Big) = C^*_1 > 0. \tag{3.7}$$

Note that the tail indices α_1, α_2 remain unchanged thanks to their definition in (A2). Since $|X_i| \le X^*_i$, $i = 1, 2$, the result will follow from the relation

$$\lim_{u\to\infty} u\,\mathbb{P}\Big(X^*_2 > u^{1/\alpha_2}\,,\, X^*_1 > u^{1/\alpha_1} \Big) \;=\; 0.$$

Step 2. We gain additional control by introducing the first exit time for $(X^*_{n,1})$,

$$T_u := \inf\Big\{ n \in \mathbb{N} \,:\, X^*_{n,1} > u^{1/\alpha_1} \Big\}.$$

As $X^*_1 = \sup_{n\ge 0} X^*_{n,1}$ we have $\{X^*_i > u^{1/\alpha_1}\} = \{T_u < \infty\}$. By (3.7) we have

$$\lim_{u\to\infty} u \cdot \mathbb{P}(T_u < \infty) > 0. \tag{3.8}$$

Thus, the desired result will follow from the relation

$$\lim_{u\to\infty} \mathbb{P}\Big(X^*_2 > u^{1/\alpha_2} \,\Big|\, T_u < \infty \Big) \;=\; 0. \tag{3.9}$$

Introducing the following notation for partial sums,

$$X^*_{j:m,i} \;:=\; \sum_{k=j+1}^{m} m_i^{k-(j+1)} |M_{j+1}| \cdots |M_{k-1}||Q_{k,i}|, \qquad i = 1, 2, \tag{3.10}$$

we have on the set $\{T_u < \infty\}$,

$$X^*_2 = X^*_{T_u,2} + {m_2}^{T_u} |M_1 \cdots M_{T_u}| X^*_{T_u:\infty,2}. \tag{3.11}$$

The simple inclusion

$$\{X^*_2 > s\} \subset \Big\{ X^*_{T_u,2} > s/2 \Big\} \cup \Big\{ {m_2}^{T_u} |M_1 \cdots M_{T_u}| X^*_{T_u:\infty,2} > s/2 \Big\} \;=:\; I \cup II$$

allows us to consider the contributions in (3.11) separately. The following lemma, to be proved subsequently, is the crucial ingredient for evaluating the contributions of I and II.

Lemma 3.4. *For any $\epsilon > 0$, define the set $C_u(\epsilon)$ as the intersection*

$$\Big\{T_u \le L_u\Big\} \cap \Big\{X^*_{T_u,1} \le u^{\frac{1+\epsilon}{\alpha_1}}\Big\} \cap \Big\{\max_{1\le k\le L_u} \frac{|Q_{k,2}|}{|Q_{k,1}|} \le u^{\varepsilon/\alpha_1}\Big\} \cap \Big\{m_2|M_{T_u}| \le u^{\epsilon}\Big\}$$

where $L_u := \log(u)/(\mu_1\alpha_1) + Cf(u)$, $f(u) := \sqrt{log(u)\cdot\log(\log(u))}$ and C is a (suitably large) constant that can be chosen independent of ϵ.
Then it holds that

$$\lim_{u\to\infty} \mathbb{P}\Big(\Big\{X^*_2 > u^{1/\alpha_2}\Big\} \cap C_u(\epsilon) \,\Big|\, T_u < \infty\Big) = \lim_{u\to\infty} \mathbb{P}\Big(X^*_2 > u^{1/\alpha_2} \,\Big|\, T_u < \infty\Big)$$

if one of the limit exists.

Step 3. Considering I, we have, using $m_2 > m_1$ and the controls provided by $C_u(\varepsilon)$, that

$$\begin{aligned} X^*_{T_u,2} &\le \Big(\frac{m_2}{m_1}\Big)^{T_u} \max_{1\le k\le T_u} \frac{|Q_{k,2}|}{|Q_{k,1}|} X^*_{T_u,1} \\ &\le \Big(\frac{m_2}{m_1}\Big)^{L_u} u^{(1+2\epsilon)/\alpha_1} \\ &= \exp\Big\{\big(\log(m_2) - \log(m_1)\big)L_u\Big\} u^{(1+2\epsilon)/\alpha_1} \\ &\le u^{\frac{1}{\alpha_1}\left(1+\frac{\log(m_2)-\log(m_1)}{\mu_1}+3\epsilon\right)} . \end{aligned} \tag{3.12}$$

Here we have used that

$$\sqrt{\log(u)\cdot\log(\log(u))} \le \varepsilon \log u$$

for any fixed $\varepsilon > 0$, as soon as u is large enough. Abbreviate

$$\eta := \frac{1}{\alpha_1}\Big(1 + \frac{\log(m_2) - \log(m_1)}{\mu_1} + 4\epsilon\Big).$$

From an application of Lemma 3.1, (3.4) ensures that $\eta < 1/\alpha_2$ (choose ϵ sufficiently small) so that by (3.12),

$$\{X^*_{T_u,2} > \frac{u^{1/\alpha_2}}{2}\} \cap C_u(\epsilon) \subset \{u^\eta \ge X^*_{T_u,2} > \frac{u^{1/\alpha_2}}{2}\} = \emptyset$$

for u sufficiently large. It follows that the first term I in (3.11) does not contribute.

Step 4. Turning to II, we note that the multiplicative factor is almost the last summand in $X^*_{T_u,2}$, so we use the previous result to estimate on $C_u(\epsilon)$

$$m_2^{T_u}|M_1\cdots M_{T_u}| \;\le\; \Big(\Big(\frac{m_2}{m_1}\Big)^{T_u-1} \max_{1\le k\le T_u-1} \frac{|Q_{k,2}|}{|Q_{k,1}|} X^*_{T_u,1}\Big) m_2|M_{T_u}| \;\le\; u^\eta\,,$$

for u sufficiently large. Hence

$$\begin{aligned} &\mathbb{P}\Big(\Big\{m_2^{T_u}|M_1\cdots M_{T_u}|X^*_{T_u:\infty,2} > \frac{1}{2}u^{1/\alpha_2}\Big\} \cap C_u(\epsilon) \,\Big|\, T_u < \infty\Big) \\ &\le \mathbb{P}\Big(X^*_{T_u:\infty,2} > \frac{1}{2}u^{1/\alpha_2-\eta}\Big| {T_u}^* < \infty\Big) = \mathbb{P}\Big(X^*_2 > \frac{1}{2}u^{1/\alpha_2-\eta}\Big). \end{aligned}$$

since $X^*_{T_u:\infty,2}$ is independent of $\{T_u < \infty\}$. But as long as $1/\alpha_2 > \eta$, which is ensured by (3.4), the probability II tends to zero.

We conclude that

$$\lim_{u\to\infty} \mathbb{P}\Big(X^*_2 > u^{1/\alpha_2} \,\Big|\, T_u < \infty\Big) = 0$$

as soon as (3.4) holds and the desired result follows. □

3.3 Proof of Lemma 3.4

Fix $\epsilon > 0$ and write $C_u = C_u(\epsilon)$.

Step 1. It is enough to show that $\lim_{u\to\infty} \mathbb{P}(C_u^c \,|\, T_u < \infty) = 0$. Indeed, we can sandwich the conditional probabilities as follows

$$\begin{aligned}
&\mathbb{P}\Big(X_2^* > u^{1/\alpha_2} \,\Big|\, T_u < \infty\Big) \;\geq\; \mathbb{P}\Big(\Big\{X_2^* > u^{1/\alpha_2}\Big\} \cap C_u \,\Big|\, T_u < \infty\Big) \\
=\;& \mathbb{P}\Big(X_2^* > u^{1/\alpha_2} \,\Big|\, T_u < \infty\Big) - \mathbb{P}\Big(\Big\{X_2^* > u^{1/\alpha_2}\Big\} \cap C_u^c \,\Big|\, T_u < \infty\Big) \\
\geq\;& \mathbb{P}\Big(X_2^* > u^{1/\alpha_2} \,\Big|\, T_u < \infty\Big) - \mathbb{P}\Big(C_u^c \,\Big|\, T_u < \infty\Big).
\end{aligned}$$

Then the desired result follows by letting $u \to \infty$. We will consider each of the four contributions to C_u^c separately:

$$\begin{aligned}
C_u^c \;=\; & \Big\{T_u > L_u\Big\} \cup \Big\{X_{T_u,1}^* > u^{(1+\epsilon)/\alpha_1}\Big\} \\
& \cup \Big\{\max_{1\leq k\leq L_u} \frac{|Q_{k,2}|}{|Q_{k,1}|} > u^{\varepsilon/\alpha_1}\Big\} \cup \Big\{m_2|M_{T_u}| > u^{\epsilon}\Big\} \\
=& A \cup B \cup D \cup E.
\end{aligned}$$

By (3.8), the required assertion $\lim_{u\to\infty} P(B|T_u < \infty) = 0$ will as well follow from

$$\lim_{u\to\infty} u \cdot P(B \cap \{T_u < \infty\}) \leq \lim_{u\to\infty} u \cdot P(B) = 0.$$

Step 2. The negligibility of A is a direct consequence of (Buraczewski et al., 2016a, Lemma 4.3)) which provides that for a sufficiently large constant C,

$$\lim_{u\to\infty} \mathbb{P}\bigg(\Big|T_u - \frac{\log u}{\mu_1\alpha_1}\Big| \geq Cf(u), \,\bigg|\, T_u < \infty\bigg) = 0 \tag{3.13}$$

where $f(u) = \sqrt{\log(u)\cdot\log(\log(u))}$.

Turning to B, we have by (3.7) that $\lim_{u\to\infty} u\mathbb{P}(X_1^* > u^{(1+\epsilon)/\alpha_1}) = 0$ implying that $\lim_{u\to\infty} u\mathbb{P}(X_{T_u,1}^* > u^{(1+\epsilon)/\alpha_1}) = 0$, since $X_1^* = \sup_n X_{n,1}^*$.

D will be considered below; the neglibility of E is ensured by the independece of M_{T_u} and T_u,

$$\lim_{u\to\infty} P\big(m_2|M_{T_u}| > u^{\epsilon} \,\big|\, T_u < \infty\big) \;=\; \lim_{u\to\infty} P\big(m_2|M| > u^{\epsilon}\big) \;=\; 0 \tag{3.14}$$

Step 3. Now we turn to D. A union bound yields

$$\mathbb{P}\Big(\max_{1\leq k\leq L_u} \frac{|Q_{k,2}|}{|Q_{k,1}|} > u^{\varepsilon/\alpha_1},\, T_u < \infty\Big) \leq \sum_{k=1}^{L_u} \mathbb{P}\Big(u^{\varepsilon/\alpha_1}|Q_{k,1}| < |Q_{k,2}|,\, T_u < \infty\Big).$$

We decompose for any $k \geq 0$

$$\begin{aligned}
&\mathbb{P}(u^{\varepsilon/\alpha_1}|Q_{k,1}| < |Q_{k,2}|,\, T_u < \infty) \;\leq\; \mathbb{P}(u^{\varepsilon/\alpha_1}|Q_{k,1}| < |Q_{k,2}|, X_1^* > u^{1/\alpha_1}) \\
\leq\;& \mathbb{P}\Big(u^{\varepsilon/\alpha_1}|Q_{k,1}| < |Q_{k,2}|, \\
& \sum_{j\neq k} m_1^{j-1}|M_1\cdots M_{j-1}||Q_{j,1}| + m_1^{k-1}|M_1\cdots M_{k-1}||Q_{k,1}| > u^{1/\alpha_1}\Big).
\end{aligned}$$

We bound this probability by the sum of two terms

$$\begin{aligned}
&\mathbb{P}\Big(u^{\varepsilon/\alpha_1}|Q_{k,1}| < |Q_{k,2}|,\, \sum_{j\neq k} m_1^{j-1}|M_1\cdots M_{j-1}||Q_{j,1}| > \frac{1}{2}u^{1/\alpha_1}\Big) \\
&+\mathbb{P}\Big(u^{\varepsilon/\alpha_1}|Q_{k,1}| < |Q_{k,2}|m_1^{k-1}|M_1\cdots M_{k-1}||Q_{k,1}| > \frac{1}{2}u^{1/\alpha_1}\Big),
\end{aligned} \tag{3.15}$$

and have to show that both contributions, when summed over $k = 0, \ldots, L_u$, are of order $o(u^{-1})$.

By independence, the first term in (3.15) is equal to

$$\mathbb{P}(u^{\varepsilon/\alpha_1}|Q_1| < |Q_2|) \cdot \mathbb{P}\Big(\sum_{j\neq k} m_1^{j-1}|M_1\cdots M_{j-1}||Q_{j,1}| > \frac{1}{2}u^{1/\alpha_1}\Big)$$
$$\leq \mathbb{P}(u^{\varepsilon/\alpha_1}|Q_1| < |Q_2|) \cdot \mathbb{P}\Big(X_1^* > \frac{1}{2}u^{1/\alpha_1}/2\Big) = o\big((\log(u)u)^{-1}\big)$$

thanks to the regular variation properties of X_1^* and the assumption on $|Q_2|/|Q_1|$, see (3.7) and (A6), respectively. Since $L_u = O(\log(u))$, we may sum over $k = 0, \ldots, L_u$ and obtain a contribution of order $o(u^{-1})$, as required.

We estimate the second term in (3.15) thanks to Markov's inequality of order $\alpha_1/(1+\varepsilon) < \kappa < \alpha_1$:

$$\mathbb{P}\Big(u^{\varepsilon/\alpha_1}|Q_{k,1}| < |Q_{k,2}|,\, m_1^{k-1}|M_1\cdots M_{k-1}||Q_{k,1}| > \frac{1}{2}u^{1/\alpha_1}\Big)$$
$$\leq \mathbb{P}\Big(m_1^{k-1}|M_1\cdots M_{k-1}||Q_{k,2}| > \frac{1}{2}u^{(1+\varepsilon)/\alpha_1}\Big)$$
$$\leq \frac{2^\kappa \big(m_1^\kappa \mathbb{E}[|M|^\kappa]\big)^k \mathbb{E}\|\boldsymbol{Q}\|^\kappa}{u^{\kappa((1+\varepsilon)/\alpha_1)}}.$$

As $\alpha_1/(1+\varepsilon\alpha_1) < \kappa < \alpha_1$ we have that $m_1^\kappa\mathbb{E}[|M|^\kappa] < 1$ and conclude

$$\sum_{k=0}^\infty \mathbb{P}\Big(u^{\varepsilon/\alpha_1}|Q_k^1| < |Q_k^2|, m_1^{k-1}|M_1\cdots M_{k-1}||Q_k^1| > u^{1/\alpha_1}/2\Big)$$
$$\leq \sum_{k=0}^\infty \frac{2^k(m_1^\kappa\mathbb{E}[|M_1|^\kappa])^k\,\mathbb{E}\|\boldsymbol{Q}\|^\kappa}{u^{\kappa((1+\varepsilon)/\alpha_1)}} = o(u^{-1})\,.$$

4 The diagonal SRE with coefficients that are equal

In this section we focus on the case where $m_i = m > 0$ for any $1 \leq i \leq d$ so that

$$\boldsymbol{X}_t = mM_t\boldsymbol{X}_{t-1} + \boldsymbol{Q}_t, \qquad t \in \mathbb{Z}.$$

We can interprete the multiplicative factor mM_t as multiplication with the random similarity matrix $mM_t\mathrm{Diag}(1,\ldots,1)$, thus we are in the framework of Buraczewski et al. (2009). From there, we obtain the following result:

Theorem 4.1. *Assume* (A1)–(A5) *for all* $1 \leq i \leq d$. *Let* $\boldsymbol{X}_0$ *have the stationary distribution. Then* $\boldsymbol{X}_0$ *is VSRV and* $(\boldsymbol{X}_t)_{t\geq 0}$ *is a VSRV process of order* $\boldsymbol{\alpha} = (\alpha,\ldots,\alpha)$, *and its spectral tail process satisfies the relation*

$$\widetilde{\boldsymbol{\Theta}}_t = mM_t\widetilde{\boldsymbol{\Theta}}_{t-1}, \quad t \geq 1.$$

Proof. By (Buraczewski et al., 2009, Theorem 1.6), there is a non-null Radon measure μ on $[-\infty,\infty]^d \setminus \{0\}$ such that

$$x^\alpha\mathbb{P}(x^{-1}\boldsymbol{X}_0 \in \cdot) \stackrel{v}{\to} \mu, \qquad x \to \infty.$$

[See (Buraczewski et al., 2016b, Theorem 4.4.21) for a reformulation of the quoted result which is more consistent with our notation.] Hence, $\boldsymbol{X}_0$ is (standard) regularly varying and also VSRV of order $\boldsymbol{\alpha} = (\alpha,\ldots,\alpha)$ since $\boldsymbol{\Theta}_0$ and $\widetilde{\boldsymbol{\Theta}}_0$ coincide then.

The remaining assertions follow from a direct application of Proposition 2.5. □

In order to determine whether the components of $\boldsymbol{X}_0$ are asymptotically independent or dependent, we are interested in information about the support of $\mathbb{P}(\widetilde{\boldsymbol{\Theta}}_0 \in \cdot)$. We write $\mathrm{supp}(\boldsymbol{Q})$ for the support of the law of $\boldsymbol{Q}$ and $\mathrm{span}(E)$ for the linear space spanned by set $E \subset \mathbb{R}^d$. Let S_∞^{d-1} denote the unit sphere in $\mathbb{R}^d$ with respect to $\|\cdot\|_{\boldsymbol{\alpha}}$ which coincides with the unit sphere for the max-norm whatever is $\boldsymbol{\alpha}$.

Corollary 4.2. *Under the assumptions of Theorem 4.1,*

$$\mathrm{supp}(\widetilde{\boldsymbol{\Theta}}_0) \ \subset \ \mathrm{span}\big(\mathrm{supp}(\boldsymbol{Q})\big) \cap S_\infty^{d-1}. \tag{4.1}$$

If there is a group $\mathbb{G}$ *of matrices, such that* $\boldsymbol{g}\boldsymbol{Q} \stackrel{\text{law}}{=} \boldsymbol{Q}$ *for all* $\boldsymbol{g} \in \mathbb{G}$*, i.e., the law of* $\boldsymbol{Q}$ *is invariant under the action of* $\mathbb{G}$*, then* $\mathrm{supp}(\widetilde{\boldsymbol{\Theta}}_0)$ *is invariant under the action of* $\mathbb{G}$*.*
In particular, if the law of $\boldsymbol{Q}$ *is rotationally invariant, then* $\mathrm{supp}(\widetilde{\boldsymbol{\Theta}}_0) = S_\infty^{d-1}$*.*

Proof. The first assertion follows immediately from the series representation of $\boldsymbol{X}_0$:

$$\boldsymbol{X}_0 = \sum_{k=0}^{\infty} m^{k-1} M_1 \cdots M_{k-1} \boldsymbol{Q}_k,$$

where the right hand side is a sum of vectors in $\mathrm{span}\big(\mathrm{supp}(\boldsymbol{Q})\big)$.

If $\boldsymbol{g}\boldsymbol{Q} \stackrel{\text{law}}{=} \boldsymbol{Q}$, then

$$\boldsymbol{g}\boldsymbol{X} \stackrel{\text{law}}{=} \boldsymbol{g}\big(mM\boldsymbol{X} + \boldsymbol{Q}\big) \ = \ mM\boldsymbol{g}\boldsymbol{X} + \boldsymbol{g}\boldsymbol{Q} \stackrel{\text{law}}{=} mM(g\boldsymbol{X}) + Q,$$

i.e., the law of $\boldsymbol{X}$ satisfies the same equation as the law of $\boldsymbol{g}\boldsymbol{X}$. But the solution to $\boldsymbol{X} \stackrel{\text{law}}{=} mM\boldsymbol{X} + \boldsymbol{Q}$ is unique in law, hence $\boldsymbol{g}\boldsymbol{X} \stackrel{\text{law}}{=} \boldsymbol{X}$. Thus, the law of $\boldsymbol{X}$ is invariant under the action of $\mathbb{G}$, which implies the same invariance for its tail spectral measure $\mathbb{P}(\widetilde{\boldsymbol{\Theta}}_0 \in \cdot)$. □

Finally, we provide sufficient conditions in order to have equality in (4.1).

Lemma 4.3. *Assume* (A1)–(A5)*. Then the following implications hold:*

(a) If $\mathrm{supp}(M)$ *is dense in* $\mathbb{R}$*, then* $\mathrm{supp}(\widetilde{\boldsymbol{\Theta}}_0) \ = \ \mathrm{span}\big(\mathrm{supp}(\boldsymbol{Q})\big) \cap S_\infty^{d-1}$*.*

(b) If $\mathrm{supp}(\boldsymbol{Q})$ *is dense in* $\mathbb{R}^d$*, then* $\mathrm{supp}(\widetilde{\boldsymbol{\Theta}}_0) \ = \ S_\infty^{d-1}$*.*

Proof of Lemma 4.3. The proof consists of deriving a represention of $\mathrm{supp}(\boldsymbol{\Theta}_0)$, from which both implications can be read off. It is based on (Buraczewski et al., 2009, Remark 1.9), which gives that the support of the spectral measure σ_∞ with respect to the Euclidean norm is given by the directions (subsets of the unit sphere S^{d-1}) in which the support of $\boldsymbol{X}_0$ is unbounded. More precisely, consider the measures

$$\sigma_t(A) \ := \ \mathbb{P}\Big(\|\boldsymbol{X}_0\|_2 > t, \frac{\boldsymbol{X}_0}{\|\boldsymbol{X}_0\|_2} \in A\Big)$$

Then $\mathrm{supp}(\sigma_\infty) = \bigcap_{t>0} \mathrm{supp}(\sigma_t)$. The surprising part of this result is that all directions, in which the support of $\boldsymbol{X}_0$ is unbounded, do matter. One does not need a lower bound on the decay of mass at infinity. But if we know that the support of the spectral measure w.r.t. the Euclidean norm is the intersection of a particular subspace with the unit sphere, we immediately deduce the same for the spectral measure w.r.t the max-norm, *i.e.,* for $\mathbb{P}(\boldsymbol{\Theta}_0 \in \cdot)$.

Thus, to proceed, we have to study the support of $\boldsymbol{X}_0$. For simplicity, we work with $m = 1$, which is equivalent to replacing M by mM. This allows us to write, for the remainder of the proof, $(m, \boldsymbol{q})$ for a realization of the random variables $(M, \boldsymbol{Q})$. We identify a pair $(m, \boldsymbol{q})$ with the affine mapping $h(\boldsymbol{x}) = m\boldsymbol{x} + \boldsymbol{q}$, we say that $h \in \operatorname{supp}\big((M, \boldsymbol{Q})\big)$ if $(m, \boldsymbol{q}) \in \operatorname{supp}\big((M, \boldsymbol{Q})\big)$. We consider the semigroup generated by mappings in $\operatorname{supp}\big((M, \boldsymbol{Q})\big)$,

$$\mathcal{G} \ := \ \Big\{ h_1 \cdots h_n \ : \ h_i \in \operatorname{supp}\big((M, \boldsymbol{Q})\big), \ 1 \le i \le n, \ n \ge 1 \Big\}.$$

Then, by (Buraczewski et al., 2009, Lemma 2.7)

$$\operatorname{supp}\big(\boldsymbol{X}_0\big) \ = \ \text{closure of } \Big\{ \tfrac{1}{1-m}\boldsymbol{q} \ : \ (m, \boldsymbol{q}) \in \mathcal{G}, \ |m| < 1 \Big\}.$$

[Again, see (Buraczewski et al., 2016b, Proposition 4.3.1) for a reformulation of the quoted result which is more consistent with our notation.]

Since M and $\boldsymbol{Q}$ are independent, $\operatorname{supp}\big((M, \boldsymbol{Q})\big) = \operatorname{supp}(M) \times \operatorname{supp}(\boldsymbol{Q})$ and a general element in $\mathcal{G}$ is of the form

$$h(\boldsymbol{x}) = m_1 \cdots m_n \boldsymbol{x} + \Big(\boldsymbol{q}_1 + \sum_{k=2}^{n} m_1 \cdots m_{k-1} \boldsymbol{q}_k \Big)$$

with $m_i \in \operatorname{supp}(M)$, $\boldsymbol{q}_i \in \operatorname{supp}(\boldsymbol{Q})$. Thus, a generic point in $\operatorname{supp}(\boldsymbol{X}_0)$ is of the form

$$\frac{1}{1 - m_1 \cdots m_n} \Big(\boldsymbol{q}_1 + \sum_{k=2}^{n} m_1 \cdots m_{k-1} \boldsymbol{q}_k \Big), \tag{4.2}$$

with

$$m_i \in \operatorname{supp}(M), \ \ \boldsymbol{q}_i \in \operatorname{supp}(\boldsymbol{Q}), \ \ |m_1 \cdots m_n| < 1.$$

The prefactor in (4.2) is scalar, while the bracket term represents a linear combination of $q_k \in \operatorname{supp}(\boldsymbol{Q})$. Now we can prove the two implications.

Concerning (a), if $\operatorname{supp}(M)$ is dense in $\mathbb{R}$, then the bracket term in (4.2) can be chosen such that its direction approximates any direction of $y \in \operatorname{span}\big(\operatorname{supp}(\boldsymbol{Q})\big)$. Then, given $t > 0$, m_n can be chosen arbitrarily small, such that $|m_1 \dots m_n| < 1$ and moreover, the norm of (4.2) exceeds t. It follows that $\operatorname{supp}(\sigma_t) = \operatorname{span}\big(\operatorname{supp}(\boldsymbol{Q})\big) \cap S^{d-1}$ for all t, which yields the assertion since $\operatorname{supp}(\sigma_\infty) = \bigcap_{t>0} \operatorname{supp}(\sigma_t)$.

Concering (b), if $\operatorname{supp}(\boldsymbol{Q})$ is dense in $\mathbb{R}^d$, then the bracket term can be chosen such that it approximates an arbitrary element of $\mathbb{R}^d$ and its modulus is larger than t, while (A1) entails that there are $m_i \in \operatorname{supp}(M)$ such that $|m_1 \dots m_n| < 1$. □

5 The diagonal SRE - the general case

In this section we study the vector scaling regular variation properties of the diagonal SRE in full generality. We suppose that coordinates are chosen in such a way that m_i are increasing (hence, α_i decreasing) with i. We partition $\{1, \dots, d\} = I_1 \cup I_2 \cup \dots \cup I_r$ such that $m_i = m_j$ if and only if $i, j \in I_\ell$ for some $1 \le \ell \le r$. We further denote by

$$\mathbb{R}^{|I_\ell|} \ = \ \{x \in \mathbb{R}^d; \ x_i = 0 \text{ for } i \notin I_\ell\}$$

the (embedded) subspace corresponding with coordinates indexed by I_ℓ and by

$$S_\infty^{|I_\ell|-1} \ = \ \{x \in \mathbb{R}^d; \max_{i \in I_\ell} |x_i| = 1 \text{ and } x_i = 0 \text{ for } i \notin I_\ell\}$$

its max-norm-unit sphere. Note that if $I_\ell = \{i\}$ is a singleton, then $S_\infty^{|I_\ell|-1} = \{\boldsymbol{e}_i, -\boldsymbol{e}_i\}$.

Theorem 5.1. *Let $(\boldsymbol{X}_t)$ a stationary process satisfying the diagonal SRE* (1.1) *and assume that* (A1)–(A6) *hold. Then $(\boldsymbol{X}_t)$ is a VSRV process satisfying*

$$\operatorname{Supp}(\widetilde{\boldsymbol{\Theta}}_0) \ \subset \ \cup_{1 \le \ell \le r} S_\infty^{|I_\ell|-1} \tag{5.1}$$

and

$$\widetilde{\boldsymbol{\Theta}}_t = M_t \operatorname{Diag}(m_1, \dots, m_d) \widetilde{\boldsymbol{\Theta}}_{t-1}, \qquad t \ge 1. \tag{5.2}$$

Proof. We start by proving that $(\boldsymbol{X}_t)$ is a VSRV process. According to Proposition 2.5 and Remark 2.6, it suffices to prove that $\boldsymbol{X}_0$ is VSRV, then (5.2) and the VSRV of $(\boldsymbol{X}_t)$ follow.

We use the following short-hand notation: For $\boldsymbol{x} \in \mathbb{R}^d$, let $\boldsymbol{x}_\ell = (x_i)_{i\in I_\ell}$, $\|\boldsymbol{x}\|_\ell := \max_{i\in I_\ell} |x_i|$ and $\alpha(\ell)$ is the common tail index of all coordinates in I_ℓ.

Let $\epsilon > 0$, $\ell \neq k$. By Eq. (3.6) of Theorem 3.2, it holds that

$$\begin{aligned}
&\lim_{x\to\infty} x \cdot \mathbb{P}\Big(\|\boldsymbol{X}_0\|_\ell > \epsilon \,\|\boldsymbol{X}_0\|_{\boldsymbol{\alpha}}^{1/\alpha(\ell)}\,,\ \|\boldsymbol{X}_0\|_k > \epsilon\,\|\boldsymbol{X}_0\|_{\boldsymbol{\alpha}}^{1/\alpha(k)}\,,\ \|\boldsymbol{X}_0\|_{\boldsymbol{\alpha}} > x\Big)\\
\le\ &\lim_{x\to\infty} x \cdot \mathbb{P}\Big(\|\boldsymbol{X}_0\|_\ell > \epsilon x^{1/\alpha(\ell)},\ \|\boldsymbol{X}_0\|_k > \epsilon x^{1/\alpha(k)}\Big)\\
\le\ &\sum_{i\in I_\ell,\, j\in I_k} \lim_{x\to\infty} x \cdot \mathbb{P}\Big(|X_{0,i}| > \epsilon x^{1/\alpha_i},\ |X_{0,j}| > \epsilon x^{1/\alpha_j}\Big) \ = \ 0 \qquad (5.3)
\end{aligned}$$

We note from the results of Section 4 that there are positive constants c_ℓ and probability measures ξ_ℓ on the $|I_\ell|$-dimensional unit sphere (w.r.t. the max-norm), such that for all $1 \le \ell \le r$

$$\lim_{x\to\infty} x \cdot \mathbb{P}\Big(\|\boldsymbol{X}_0\|_\ell > x^{1/\alpha(\ell)},\ \|\boldsymbol{X}_0\|_\ell^{-1}\boldsymbol{X}_{0,\ell} \in \cdot\Big) \ = \ c_\ell\, \xi_\ell(\cdot). \qquad (5.4)$$

Applying the inclusion-exclusion principle, we have

$$\begin{aligned}
&\lim_{x\to\infty} x \cdot \mathbb{P}\big(\,\|\boldsymbol{X}_0\|_{\boldsymbol{\alpha}} > x\big) \ = \ \lim_{x\to\infty} x \cdot \mathbb{P}\Big(\bigvee_{1\le\ell\le r} \|\boldsymbol{X}_0\|_\ell > x^{1/\alpha(\ell)}\Big)\\
&= \sum_{1\le\ell\le r} \lim_{x\to\infty} x \cdot \mathbb{P}\Big(\|\boldsymbol{X}_0\|_\ell > x^{1/\alpha(\ell)}\Big) \qquad (5.5)\\
&\quad - \sum_{1\le\ell<k\le r} \lim_{x\to\infty} x \cdot \mathbb{P}\Big(\|\boldsymbol{X}_0\|_\ell > x^{1/\alpha(\ell)},\ \|\boldsymbol{X}_0\|_k > x^{1/\alpha(k)}\Big) + \dots\\
&= c_1 + \dots + c_r \ =: \ c, \qquad (5.6)
\end{aligned}$$

since all intersection terms vanish asymptotically due to Eq. 5.3 (with $\epsilon = 1$).

Thus we have shown that $\|\boldsymbol{X}_0\|_{\boldsymbol{\alpha}}$ is regularly varying. We claim that

$$\lim_{x\to\infty} \mathbb{P}\Big(\,\|\boldsymbol{X}_0\|_{\boldsymbol{\alpha}}^{-1/\boldsymbol{\alpha}}\, \boldsymbol{X}_0 \in \cdot \,\Big|\, \|\boldsymbol{X}_0\|_{\boldsymbol{\alpha}} > x\Big) \ = \ \frac{1}{c} \sum_{1\le\ell\le r} c_\ell\, \widetilde{\xi}_\ell(\cdot), \qquad (5.7)$$

where

$$\widetilde{\xi}_\ell \ = \ \delta_{\mathbf{0}_1} \otimes \dots \otimes \delta_{\mathbf{0}_{\ell-1}} \otimes \xi_\ell \otimes \delta_{\mathbf{0}_{\ell+1}} \otimes \dots \otimes \delta_{\mathbf{0}_r}$$

is the extension of ξ_ℓ to a measure on the unit sphere S_∞^{d-1} in $\mathbb{R}^d$ by putting unit mass in the origin of the additional coordinates. Hence, its support is contained in $S_\infty^{|I_\ell|-1}$. In particular, (5.1) follows once this claim is proved.

By the Portmanteau lemma, it suffices to study closed sets. Note that for any closed set $B \subset S_\infty^{d-1}$, it holds that

$$B_{\ell,\epsilon} \ := \ \{\boldsymbol{x}_\ell \,:\, \boldsymbol{x} \in B,\, |x_j| < \epsilon \text{ for } j \notin I_\ell\} \ \to \ \{\boldsymbol{x}_\ell \,:\, \boldsymbol{x} \in B \cap S^{|I_\ell|-1}\} \ =: \ B_\ell$$

as $\epsilon \to 0$. Using (5.3) and the inclusion-exclusion principle, we obtain

$$
\begin{aligned}
&\limsup_{x\to\infty} x \cdot \mathbb{P}\Big(\|\boldsymbol{X}_0\|_{\boldsymbol{\alpha}}^{-1/\boldsymbol{\alpha}}\, \boldsymbol{X}_0 \in B,\, \|\boldsymbol{X}_0\|_{\boldsymbol{\alpha}} > x\Big)\\
&= \limsup_{x\to\infty} x \cdot \mathbb{P}\Big(\|\boldsymbol{X}_0\|_{\boldsymbol{\alpha}}^{-1/\boldsymbol{\alpha}}\, \boldsymbol{X}_0 \in B,\, \bigvee_{1\le k\le r} \|\boldsymbol{X}_0\|_k > x^{1/\alpha(k)}\Big)\\
&= \sum_{1\le \ell\le r} \limsup_{x\to\infty} x \cdot \mathbb{P}\Big(\|\boldsymbol{X}_0\|_{\boldsymbol{\alpha}}^{-1/\boldsymbol{\alpha}}\, \boldsymbol{X}_0 \in B, \|\boldsymbol{X}_0\|_\ell > x^{1/\alpha(\ell)},\, \bigwedge_{k\neq \ell} \|\boldsymbol{X}_0\|_k \le \epsilon\, \|\boldsymbol{X}_0\|_{\boldsymbol{\alpha}}^{1/\alpha(k)}\Big)\\
&\le \sum_{1\le \ell\le r} \lim_{x\to\infty} x \cdot \mathbb{P}\Big(\|\boldsymbol{X}_0\|_\ell^{-1} \boldsymbol{X}_{0,\ell} \in B_{\ell,\epsilon},\, \|\boldsymbol{X}_0\|_\ell > x^{1/\alpha(\ell)},\, \bigwedge_{k\neq \ell} \|\boldsymbol{X}_0\|_k \le \epsilon x^{1/\alpha(k)}\Big)\\
&= \sum_{1\le \ell\le r} \lim_{x\to\infty} x \cdot \mathbb{P}\Big(\|\boldsymbol{X}_0\|_\ell^{-1} \boldsymbol{X}_{0,\ell} \in B_{\ell,\epsilon},\, \|\boldsymbol{X}_0\|_\ell > x^{1/\alpha(\ell)}\Big) \;=\; \sum_{1\le \ell\le r} c_l\, \xi_\ell(B_\ell,\epsilon)
\end{aligned}
$$

This holds for all $\epsilon > 0$. Since the sequence $B_{\ell,\epsilon}$ is decreasing, we conclude by the continuity of ξ_ℓ that

$$
\begin{aligned}
\limsup_{x\to\infty} x \cdot \mathbb{P}\Big(\|\boldsymbol{X}_0\|_{\boldsymbol{\alpha}}^{-1/\boldsymbol{\alpha}}\, \boldsymbol{X}_0 \in B,\, \|\boldsymbol{X}_0\|_{\boldsymbol{\alpha}} > x\Big) \;&\le\; \sum_{1\le \ell\le r} c_\ell\, \xi_\ell(B_\ell)\\
&=\; \sum_{1\le \ell\le r} c_\ell \widetilde{\xi}_\ell(B).
\end{aligned}
$$

Combined with (5.6), this proves the weak convergence by an application of the Portmanteau lemma.

□

Corollary 5.2. *If the stationarity assumption* (1.5) *is satisfied, then the stationary solution* $(\boldsymbol{X}_t)$ *of the diagonal BEKK-ARCH(1) model is a VSRV process satisfying*

$$\mathrm{Supp}(\widetilde{\boldsymbol{\Theta}}_0) \;=\; \cup_{1\le \ell\le r} S_\infty^{|I_\ell|-1}$$

and

$$\widetilde{\boldsymbol{\Theta}}_t = M_t\, \mathrm{Diag}(m_1,\dots,m_d)\widetilde{\boldsymbol{\Theta}}_{t-1}\,, \qquad t \ge 1\,. \tag{5.8}$$

Proof. We have to check the assumptions of the previous theorem. This is readily done for (A1)-(A5), see Pedersen and Wintenberger (2018) for details. Considering (A6), let $\sigma_i^2 = \mathrm{Var}(Q_i)$ and ρ_{ij} be the correlation coefficient of Q_i and Q_j; $\mathbb{E}Q_i = \mathbb{E}Q_j = 0$. Then the ratio Q_i/Q_j has a Cauchy distribution with location parameter $a = \rho_{ij}\frac{\sigma_i}{\sigma_j}$ and scale parameter $b = \frac{\sigma_i}{\sigma_j}\sqrt{1-\rho_{ij}^2}$; see e.g. (Curtiss, 1941, Eq. (3.3)). The Cauchy distributions are 1-stable, hence

$$\mathbb{P}\Big(\frac{|Q_i|}{|Q_j|} > u\Big) = O(u)$$

and (A6) follows if I, J are singletons. To compare $Q_I^* = \max_{i\in I}|Q_i|$ with $Q_J^* = \max_{j\in J}|Q_j|$ we use the simple bound (fix any $j \in J$)

$$\Big\{\frac{Q_I^*}{Q_J^*} > u\Big\} \;\subset\; \bigcup_{i\in I}\Big\{\frac{|Q_i|}{|Q_k|} > u\Big\}$$

to conclude that the probability of this event still decays as $O(u)$. Thus (A6) also holds in this case.

It remains to show that $\mathrm{supp}\big(\widetilde{\boldsymbol{\Theta}}_0\big)$ is *equal* to $\cup_{1\le \ell\le r} S^{|I_\ell|-1}$. Therefore, we can focus on a particular block I and show that the spectral measure of the restriction $(X_{0,i})_{i\in I}$ has full support $S^{|I|-1}$.

If I is a singleton, then this means nothing but that left and right tails are regularly varying with the same index; which already follows from the Goldie-Kesten theorem, see (3.2). If $|I| > 1$ then we are in the setting of Section 4. The result follows from the first assertion of Lemma 4.3, since M and $(Q_i)_{i\in I}$ are independent Gaussians, and $\mathrm{span}\big(\mathrm{supp}((Q_i)_{i\in I})\big) = \mathbb{R}^{|I|}$ since C, the variance of $\boldsymbol{Q}$, has full rank. □

The multivariate regular variation properties of the BEKK-ARCH(1) process is quite simple as the support is preserved by the multiplicative form of the tail process: The tail process is a mixture of multiplicative random walks with distinct supports. Each support corresponds to the span of the diagonal coefficients of the multiplicative matrix that are equal. From a risk analysis point of view, it means that the extremal risks are dependent and of similar intensity only in the directions of equal diagonal coefficients. Our multivariate analysis appeals for an extreme financial risk analysis based on the estimation of the diagonal coefficients of the BEKK-ARCH(1) process accompanied with a test of their equality.

The asymptotic independence between directions with distinct diagonal coefficients may be seen as artificially due to the diagonal restriction imposed on the multiplicative matrices. However we suspect it is the case in any situation of VSRV Markov chains as in Proposition 2.5. More precisely, we conjecture in the upper triangular matrices case:

Remark 5.3. Damek et al. (2019) study bivariate stochastic recurrence equations with upper triangular matrices, including the following model:

$$\begin{pmatrix} X_{t,1} \\ X_{t,2} \end{pmatrix} = \begin{pmatrix} m_1 & m_{12} \\ 0 & m_2 \end{pmatrix} M_t \begin{pmatrix} X_{t-1,1} \\ X_{t-1,2} \end{pmatrix} + Q_t,$$

here (M_t) and (Q_t) are iid, taking values in $[0,\infty)$ and $[0,\infty)^2$, respectively. Defining α_i as before by the condition $\mathbb{E}\big(m_i M_1\big)^{\alpha_i} = 1$ and assuming (A1)–(A5), they study the marginal tail behavior under the assumption that $m_{12} \neq 0$.

Let X_0 have the stationary distribution. Since $(X_{t,2})$ satisfies a one-dimensional SRE, it holds $P(X_{0,2} > x) \sim c_2 x^{-\alpha_2}$ by the Kesten-Goldie theorem. Since all random variables are nonnegative, it is clear that $X_{t,1} \geq (m_1 M_t X_{t-1,1} + Q_{t,1}) \vee (m_{12} M_t X_{t-1,2} + Q_{t,1})$; in particular, the tails of $X_{0,1}$ have to be at least as heavy as $t^{-\alpha_1}$ which would be the case if we had $m_{12} = 0$ but also as heavy as $t^{-\alpha_2}$ as $m_{12} \neq 0$. In fact, it is proved in Damek et al. (2019) that

$$P(X_{0,1} > x) \sim \begin{cases} c_1 x^{-\alpha_1} & \text{if } \alpha_1 < \alpha_2 \quad (\text{CASE } 1) \\ \tilde{c}_1 x^{-\alpha_2} & \text{if } \alpha_1 > \alpha_2 \quad (\text{CASE } 2) \end{cases}$$

with positive constants c_1, $\tilde{c_1}$. In Case 2, $X_{0,1}$ and $X_{0,2}$ are obviously dependent (also asymptotically), while we *conjecture* that our methods will carry over to prove asymptotic independence in Case 1. We expect similar results to hold in the higher-dimensional setup studied in Matsui and Swiatkowski (2018).

6 Second order results

In this section, we work in the setup of Section 3, *i.e.*, in the two-dimensional stationary distribution with generic element (X_1, X_2) with distinct coefficients $m_1 < m_2$. As the marginals are asymptotically independent, we need a second measure quantifying the extremal dependence. The coefficient of tail dependence, $1/2 \leq \eta \leq 1$, was introduced by Ledford and Tawn (1996) assuming there exists a slowly varying function L so that

$$\mathbb{P}(X_1 > u^{1/\alpha_1},\ X_2 > u^{1/\alpha_2}) \sim u^{1/\eta} L(u)\,, \qquad u \to \infty\,.$$

We were not able to prove such a result but rather two second-order results of the following form. We prove that there are $0 < \delta < \Delta$ such that

$$\lim_{u\to\infty} u^{1+\delta}\,\mathbb{P}(X_1 > u^{1/\alpha_1},\ X_2 > u^{1/\alpha_2}) \ = 0\ , \tag{6.1}$$

$$\liminf_{u\to\infty} u^{1+\Delta}\,\mathbb{P}(X_1 > u^{1/\alpha_1},\ X_2 > u^{1/\alpha_2}) \ > 0\ . \tag{6.2}$$

One gets the range $1/(1+\delta) < \eta \leq 1/(1+\Delta)$ if the coefficient of tail dependence existed. We decided not to treat the most general case here, but rather consider these two results as illustration of the possible second-order behavior. The reason is that both proofs use as a crucial ingredient deep results on the exceedance times of the a.s. convergent series $X_{n,1}$ and $X_{n,2}$. Such estimates are not available in full generality, see Buraczewski et al. (2018, 2016a) for a discussion and counterexamples. This is why we refrained from striving for optimal assumptions here.

6.1 Asymptotic independence

Our first result considers "second-order-independence", *i.e.*, (6.1). We start with a simple, but useful observation.

Lemma 6.1. *Consider a sequence of events A_u, B_u, D_u such that there is $\delta \geq 0$ with*

$$\lim_{u\to\infty} u\cdot\mathbb{P}(D_u) \in (0,\infty), \qquad \lim_{u\to\infty} u^{1+\delta}\cdot\mathbb{P}(B_u^c) = 0. \tag{6.3}$$

Then $u^\delta\cdot\mathbb{P}\big(A_u \,\big|\, D_u\big)$ converges if and only if $u^\delta\cdot\mathbb{P}\big(A_u\cap B_u \,\big|\, D_u\big)$ converges (as $u\to\infty$) and if either of the limits exists, it holds

$$\lim_{u\to\infty} u^\delta\cdot\mathbb{P}\big(A_u \,\big|\, D_u\big) \ = \ \lim_{u\to\infty} u^\delta\cdot\mathbb{P}\big(A_u\cap B_u \,\big|\, D_u\big).$$

Proof. Using the elementary definition of conditional probabilities (the denominators are positive by Assumption (6.3) as soon as u is large enough),

$$\begin{aligned}
&\Big|u^\delta\mathbb{P}\big(A_u \,\big|\, D_u\big) - u^\delta\mathbb{P}\big(A_u\cap B_u \,\big|\, D_u\big)\Big| \\
&= \Big|\frac{u^{1+\delta}\mathbb{P}\big(A_u\cap D_u\big)}{u\mathbb{P}(D_u)} - \frac{u^{1+\delta}\mathbb{P}\big(A_u\cap B_u\cap D_u\big)}{u\mathbb{P}(D_u)}\Big| \ \leq \ \frac{u^{1+\delta}\mathbb{P}\big(B_u^c\big)}{u\mathbb{P}(D_u)},
\end{aligned}$$

and the last expression tends to 0 by Assumption (6.3). □

The proof of the subsequent result proceeds by exploiting further the estimates used in the proof of Theorem 3.2. As a main ingredient, we need upper large deviation bounds for the exceedence time T_u, which are only available under additional regularity assumptions on $\boldsymbol{Q}$ and M.

Theorem 6.2. *In addition to* (A1), (A2), (A4), (A5), *assume that $\boldsymbol{Q} = (1,1)^t$ and that the law of M has compact support and is absolutely continuous with a bounded density. Then there exists $\delta > 0$ such that*

$$\lim_{u\to\infty} u^\delta\cdot\mathbb{P}\Big(X_2 > u^{1/\alpha_2}\,\Big|\, X_1 > u^{1/\alpha_1}\Big) \ = \ 0.$$

Proof. We will proceed along the same lines as in the proof of Theorem 3.2. Therefore, we will abbreviate some arguments and focus on the new ingredients. Without loss of generality, we may assume that M is nonnegative by studying dominating sequences (see Step 1 in the proof of Theorem 3.2). Let

$$T_u \ = \ \inf\{n\in\mathbb{N}\,:\, X_{n,1} > u^{1/\alpha_1}\}$$

Step 1. We introduce sets B_u satisfying

$$\begin{aligned}&\lim_{u\to\infty} u^{\delta}\cdot \mathbb{P}\Big(X_2 > u^{1/\alpha_2}\,\Big|\, X_1 > u^{1/\alpha_1}\Big)\\ &= \lim_{u\to\infty} u^{\delta}\cdot \mathbb{P}\Big(\Big\{X_2 > u^{1/\alpha_2}\Big\}\cap B_u \,\Big|\, T_u<\infty\Big),\end{aligned} \tag{6.4}$$

(given that one out of the two limits exists), chosen in such a way that they provide further control over T_u and $X_{T_u,1}$.

In order to define B_u, consider the function $\Lambda_1(s) := \log \mathbb{E}\big[(m_1 M)^s\big]$, with Fenchel-Legendre transform $\Lambda_1^*(x) := \sup_{s\in\mathbb{R}}\big(sx-\Lambda_1(s)\big)$. For any $0<\mu<\mu_1$ there is α such that $\mu=\Lambda_1'(\alpha)$. For such corresponding α and μ, it holds by a standard calculation in large deviation theory that

$$I(\mu) \;:=\; \frac{\Lambda_1^*(\mu)}{\mu} \;=\; \alpha - \frac{\Lambda_1(\alpha)}{\Lambda_1'(\alpha)} \;>\; \alpha_1.$$

Choose $0<\mu_*<\mu_1$ and $\epsilon>0$ such that the following restrictions are satisfied:

$$\alpha_2\Big(1+\frac{\log(m_2)-\log(m_1)}{\mu_*}+\epsilon\Big) < \alpha_1, \tag{6.5}$$

$$\lim_{u\to\infty} u^{I(\mu_*)/\alpha_1}\mathbb{P}\big(X_1 > u^{\frac{1}{\alpha_1}(1+\epsilon)}\big) = 0. \tag{6.6}$$

This is possible by Lemma 3.1 and the fact that μ_* and $I(\mu_*)$ deviate continuously from μ and α_1, respectively. The additional conditions of Theorem 6.2 ensure that the assumptions of (Buraczewski et al., 2016a, Theorem 2.4) are satisfied, which yields

$$\lim_{u\to\infty} u^{I(\mu_*)/\alpha_1}\mathbb{P}\Big(T_u \ge \frac{\log u}{\alpha_1\mu_*}\Big) = 0. \tag{6.7}$$

Note that Λ_1 is a convex function with $\Lambda_1'(0)<0$, hence $\mu_*=\Lambda_1'(\alpha_*)>0$ implies that there is $\beta<\min\{1,\alpha_*\}$ with $\Lambda_1(\beta)<\Lambda_1(\alpha_*)$. Thus, Condition (2.26) of (Buraczewski et al., 2016a, Theorem 2.4) is satisfied.

Set

$$\delta \;:=\; \frac{I(\mu_*)}{\alpha_1}-1>0, \qquad B_u := \Big\{T_u < \frac{\log u}{\alpha_1\mu_*}\Big\}\cap\Big\{X_{T_u,1}\le u^{\frac{1}{\alpha_1}(1+\epsilon)}\Big\}.$$

By Eq.s (6.6), (6.7) and the fact that $X_{T_u,1}\le X_1$, we have

$$\lim_{u\to\infty} u^{1+\delta}\mathbb{P}(B_u^c) = 0.$$

Thus (6.4) follows by an application of Lemma 6.1.

Step 2. Decomposing as in (3.11), we estimate

$$\begin{aligned}&\limsup_{u\to\infty} u^{\delta}\cdot\mathbb{P}\Big(\Big\{X_2>u^{1/\alpha_2}\Big\}\cap B_u\,\Big|\,T_u<\infty\Big)\\
&\le \limsup_{u\to\infty} u^{\delta}\cdot\mathbb{P}\Big(\Big\{X_{T_u,2}>\frac{1}{2}u^{1/\alpha_2}\Big\}\cap B_u\,\Big|\,T_u<\infty\Big) \qquad (6.8)\\
&\quad+\limsup_{u\to\infty} u^{\delta}\cdot\mathbb{P}\Big(\Big\{m_2^{T_u}M_1\cdots M_{T_u}X_{T_u:\infty,2}>\frac{1}{2}u^{1/\alpha_2}\Big\}\cap B_u\,\Big|\,T_u<\infty\Big). \qquad (6.9)\end{aligned}$$

On the set B_u,

$$X_{T_u,2} \;\le\; \Big(\frac{m_2}{m_1}\Big)^{T_u} X_{T_u,1} \;\le\; \Big(\frac{m_2}{m_1}\Big)^{\frac{\log u}{\alpha_1\mu_*}} u^{\frac{1}{\alpha_1}(1+\epsilon)} \;=\; u^{\eta} \tag{6.10}$$

with

$$\eta \;=\; \frac{1}{\alpha_1}\Big(1 + \frac{\log(m_2) - \log(m_1)}{\mu_*} + \epsilon\Big) \;<\; \frac{1}{\alpha_2},$$

see Eq. (6.5). Hence the term in (6.8) vanishes.

Turning to (6.9), we have on B_u

$$m_2^{T_u} M_1 \cdots M_{T_u} X_{T_u:\infty,2} \le X_{T_u,2} \cdot (m_2 M_{T_u}) \cdot X_{T_u:\infty,2} \le u^{\eta} \cdot X_{T_u:\infty,2}$$

(recall that M and thus M_{T_u} have bounded support). Using the independence of $X_{T_u:\infty,2}$ and T_u, we find that the term in (6.9) is bounded by

$$\limsup_{u\to\infty} u^{\delta} \cdot \mathbb{P}\Big(X_{T_u:\infty,2} > u^{1/\alpha_2 - \eta} \,\Big|\, T_u < \infty\Big) \;=\; \limsup_{u\to\infty} u^{\delta}\mathbb{P}\Big(X_2 > u^{1/\alpha_2-\eta}\Big).$$

Since $1/\alpha_2 > \eta$, we can choose $0 < \delta^* \le \delta$ such that

$$\delta^* < 1 - \alpha_2\eta \qquad \text{or, equivalently,} \qquad \frac{\alpha_2}{\delta^*}\Big(\frac{1}{\alpha_2} - \eta\Big) > 1.$$

But then

$$\lim_{u\to\infty} u^{\delta^*}\mathbb{P}\Big(X_2 > u^{1/\alpha_2-\eta}\Big) = 0.$$

We conclude [note that the previous estimates also hold with δ replaced by δ^*, since $\delta^* \le \delta$] that

$$\lim_{u\to\infty} u^{\delta^*} \cdot \mathbb{P}\Big(X_2 > u^{1/\alpha_2} \,\Big|\, X_1 > u^{\frac{1}{\alpha_1}}\Big) \;=\; 0.$$

□

Remark 6.3. Considering the estimates (6.10) and (3.12), it would be possible to weaken the assumptions on $\boldsymbol{Q}$, in particular, allowing for random $\boldsymbol{Q}$. However, we would have to require that Q_2/Q_1 has very light tails in order to deduce that

$$\lim_{u\to\infty} u^{\delta}\mathbb{P}\Big(\max_{1\le k\le T_u} \frac{|Q_{2,k}|}{|Q_{1,k}|} > u^{\epsilon/\alpha_1}\Big) \;=\; 0.$$

The regularity assumptions on M are a requirement of the quoted result (Buraczewski et al., 2016a, Theorem 2.4) and cannot be weakened without reproving that (very technical) result.

6.2 Asympotic Dependence

Finally, we consider the possibility of "second-order-dependence", *i.e.*, we study (6.2). Since we will use bounds from below, we cannot work with dominating sequences here, so we have to assume that M is positive. The requirement that $\boldsymbol{Q}$ is constant could be weakened by assuming some lower bounds on the ratio of Q_1/Q_2.

Theorem 6.4. *Assume* (A1), (A2), (A4), (A5), *that* $\boldsymbol{Q} = (1,1)$ *and* $\mathbb{P}(M > 0) = 1$ *and satisfies*

$$\mathbb{E}M^s < \infty \quad \textit{for all } s > 0. \tag{6.11}$$

Then there is $\Delta > 0$ *such that*

$$\liminf_{u\to\infty} u^{1+\Delta} \cdot \mathbb{P}\Big(X_2 > u^{1/\alpha_2},\; X_1 > u^{1/\alpha_1}\Big) > 0. \tag{6.12}$$

Proof. In contrast to the previous proofs, we now study the exceedence time of $X_{n,2}$,

$$N_u \; := \; \inf\{n \,:\, X_{n,2} > u^{1/\alpha_2}\}$$

in order to bound $X_{N_u,1}$ from *below* by comparing it to $X_{N_u,2}$ on the set $\{N_u < \infty\}$.

Step 1. Once again, we want to control N_u and introduce the events

$$B_u \; := \; \Big\{ N_u \le \frac{\log u}{\mu^* \alpha_2} \Big\},$$

where μ^* is a parameter to be chosen below in Step 2, where we are going to show the existence of $\Delta > 0$ with

$$\liminf_{u\to\infty} u^{1+\Delta} \cdot \mathbb{P}(B_u) > 0. \tag{6.13}$$

Using $X_1 \ge X_{N_u,1}$, it holds that

$$\begin{aligned} &\liminf_{u\to\infty} u^{1+\Delta} \cdot \mathbb{P}\Big(X_2 > u^{1/\alpha_2}, \; X_1 > u^{1/\alpha_1} \Big) \\ \ge \; &\liminf_{u\to\infty} u^{1+\Delta} \cdot \mathbb{P}\Big(B_u \cap \{ \; X_{N_u,1} > u^{1/\alpha_1} \} \Big) \end{aligned}$$

The result will follow from (6.13) if we can show that B_u implies $X_{N_u,1} > u^{1/\alpha_1}$. Namely, on B_u we have the following estimate

$$\begin{aligned} X_{N_u,1} \; &\ge \; \Big(\frac{m_1}{m_2} \Big)^{N_u} X_{N_u,2} \\ &> \; \exp\Big(\big(\log(m_1) - \log(m_2) \big) \frac{\log u}{\mu^* \alpha_2} \Big) \cdot u^{1/\alpha_2} \; = \; u^{\aleph} \end{aligned}$$

with

$$\aleph \; = \; \frac{1}{\alpha_2} \Big(1 + \frac{\log(m_1) - \log(m_2)}{\mu^*} \Big)$$

The proof concludes by

Step 2. We can choose $\mu^* > 0$ and $\Delta > 0$ satisfying (6.13) and such that $\aleph \ge \frac{1}{\alpha_1}$.

The condition $\aleph \ge \frac{1}{\alpha_1}$ is equivalent to

$$\mu^* \; \ge \; \frac{\alpha_2 \log(m_2) - \alpha_1 \log(m_1)}{\alpha_1 - \alpha_2} + \log(m_2). \tag{6.14}$$

We choose μ^* such that we have equality in (6.14). It follows from the calculations in the proof of Lemma 3.1 that (for some $\xi \in (\alpha_2, \alpha_1)$)

$$\begin{aligned} \mu^* \; &= \; \frac{\alpha_2 \log(m_2) - \alpha_1 \log(m_1)}{\alpha_1 - \alpha_2} + \log(m_2) \; = \; \Lambda'(\xi) + \log(m_2) \\ &> \Lambda'(\alpha_2) + \log(m_2) \; = \; \mu_2. \end{aligned}$$

Defining $\Lambda_2(s) = \log \mathbb{E}\big[(m_2 M)^s \big]$, this function is finite for all $s > 0$ due to (6.11) and moreover, it is strictly convex; $\Lambda_2'(\alpha_2) = \mu_2$. Hence there is $\alpha^* > \alpha_2$ with $\Lambda_2'(\alpha^*) = \mu^*$. In this case, (Buraczewski et al., 2016a, Theorem 2.1, (2.14)) yields that

$$\liminf_{u\to\infty} u^{\frac{J(\alpha^*)}{\alpha_2}} \cdot \mathbb{P}\Big(N_u \le \frac{\log u}{\mu^* \alpha_2} \Big) \; > \; 0 \tag{6.15}$$

where

$$J(\alpha^*) \; = \; \alpha^* - \frac{\Lambda_2(\alpha^*)}{\Lambda_2'(\alpha^*)}$$

and $J(\alpha^*) > \alpha_2$ as soon as $\Lambda_2'(\alpha^*) > 0$, which is satisfied here. Thus, (6.13) holds with $\Delta := J(\alpha^*)/\alpha_2 - 1$, and the assertion follows. □

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