Electromagnetism from 5D gravity: Beyond the Maxwell equations

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Abstract

Ever with the work of Kaluza, it has been known that 4D Einstein- and Maxwell-type equations can emerge from the field equations of 5D gravity in Ricci-flat space-times, having a space-like Killing vector. We revisit these equations and compare them with the Maxwell equations and the Ohm's law. Although 5D gravity and traditional electromagnetic theory are mathematically related, a paradigm shift in traditional electromagnetic concepts is required for unification. First, both the Maxwell equations and the Ohm's law are found among the field equations for 5D gravity, which gives Ohm's law the status of a field equation. Second, the concept of 4-current density, bringing together electrical charge and current density, is not fundamental for the electromagnetic theory resulting from 5D gravity. Third, the 5D gravity field equations suggest a generalization for the model of the perfect conductor, to describe a broader range of electromagnetic phenomena and bypass the concept of 4-current density. To see these ideas at work in a simple context, we propose an amendment to the Maxwell equations, based on the field equations of 5D gravity, and discuss a few applications. Namely, we discuss the propagation of electromagnetic waves through an imperfect conductor and topics in electrostatics and magnetostatics, touching ground with the Hall effect and the phenomenon of persistent current.

1 Introduction

The history of the electromagnetic theory is marked by philosophical choices, standardizations and leaps in understanding [1]. These may be revisited with progress in physics, to shed new light on the exceptional body of work in electromagnetism. The core equations for explaining the phenomenology are the Maxwell equations, written in space and time. The electromagnetic field is defined using three spatial coordinates, x^j ($j, k, l, \dots = 1, 2, 3$), in 3D Euclidian geometry, while the time t stands only as a parameter. However, the very structure of the Maxwell equations defines a 4D Lorentzian geometry, where time appears as a coordinate, x^0 . In CGS units, the modern formulation of the Maxwell equations is (e.g., Refs. [2, Ch. 1] and [3, Ch. 6])

$$\nabla \mathbf{D} = 4\pi\rho, \quad (1.1)$$

$$\nabla \mathbf{B} = 0, \quad (1.2)$$

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \quad (1.3)$$

$$\nabla \times \mathbf{H} = \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} + \frac{4\pi}{c} \mathbf{j}, \quad (1.4)$$

supplemented by constitutive/material equations

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}, \quad (1.5)$$

$$\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}, \quad (1.6)$$

(n.b., in vacuum, $\mathbf{P} = \mathbf{M} = 0$) and Ohm's law

$$\mathbf{j} = \varsigma \mathbf{E}. \quad (1.7)$$

For mathematical convenience, the potentials A_0 and $\mathbf{A}$ are introduced for the electromagnetic field

$$\mathbf{E} = -\nabla A_0 - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}, \quad (1.8)$$

$$\mathbf{B} = \nabla \times \mathbf{A}. \quad (1.9)$$

A recent critique of the Maxwell equations can be found in Ref. [4]. We make two additional notes. First, the Maxwell equations require the concepts of both field and source. Briefly, source generates field; the observer can act *directly* on the source, but not on the field. It is conjectured that electrical charge in motion is the source of the electromagnetic field. Second, as evidenced by Eqs. (1.5)-(1.7), the Maxwell equations make explicit assumptions about the structure of matter. In fact, many concepts such as conductor, dielectric, electron, were developed in parallel with the equations for the electromagnetic field, itself. Electromagnetic phenomena are attributed to particles, crystalline/amorphous structures and domain dynamics of matter. The relationship between electromagnetic phenomenology and the Maxwell equations remains complex; e.g., Ref. [5]. Here, our discussion is limited to fields, sources and particle electromagnetism in vacuum.

Features of the Maxwell equations can be recovered from a 5D geometry without compact dimensions. In 1921, Kaluza [6] proposed to unify the description of electromagnetic field in vacuum, with that of the 4D gravitational field, using a 5D space-time geometry where the metric is independent of the fifth dimension, x^5 . Later, Wesson [7] elaborated the work of Kaluza, inspired by progress in 4D gravitation theory, particularly resisting Klein's idea [8] that the fifth dimension be compact. Specifically, Wesson [7] proposed an interpretation for the covariance breaking in the gravitational field variables, from five to four dimensions. We adopt the Kaluza-Wesson interpretation of *induced matter*, whereby sourceless 5D gravity can be regarded, in four dimensions, as electromagnetic field in vacuum, onto a space-time hosting 4D gravity. Thus, an empty (i.e., Ricci-flat) 5D space-time appears to contain *active matter* (i.e., sources of fields), when it is interpreted in four dimensions. Several authors published field equations resulting for 5D Ricci-flat space-times, which are compatible with the induced-matter interpretation [6, 9–12]. However, covariance breaking of the 5D gravitational field, to obtain field equations of type Einstein and Maxwell, is not unique. There are many ways to parameterize the 5D metric and still obtain 4D field equations with desired properties. Here, we impose two requirements, which make the 5D field parameterization unique. First, the parameterization submits to the geometric picture of the foliation of the 5D metric along x^5 and, second, it provides an anomaly-free theory for single-particle propagation in 4D space-time, according to the principles of general relativity [13–16].

Geometrically speaking, there is nothing peculiar about the fifth dimension. Rather than using an x^5 -independent 5D metric, we can use an x^0 -independent 5D metric, foliate along x^0 , and obtain Einstein- and Maxwell-type equations for 4D fields from the condition of 5D Ricci flatness;¹ It is claimed that time-independent particle propagation can be interpreted within a statistical physics formalism [13–15]. However, the Maxwell-type equations resulting in this case are associated with a 4D Euclidian metric rather than a 4D Lorentzian metric and it remains unclear how they link to the current phenomenology of electromagnetism. An alternative is to maintain the parameterization induced by the foliation along x^5 , when discussing x^0 -independent physics, as it guarantees an interpretation for the field variables [13–15]. These topics remain for further work.

The paper is structured as follows. In Sec. 2, we briefly review the theory for 5D null-propagation of *passive* particles; i.e., material particles which are not sources of fields.² In Sec. 3, we explain how, in the case where the 5D metric is x^5 -independent, 5D physics can undergo covariance breaking, to be interpreted as 4D mechanics. Indeed, the resulting 4D theory of passive particle propagation is anomaly free, in full agreement to the traditional equations of 4D, classical and quantum, relativistic mechanics. The reader familiar with these developments may proceed directly to Sec. 4, where we discuss the field equations resulting in Ricci-flat 5D gravity, from the perspective of the Maxwell equations and Ohm's law. As with previous work [6, 9–11], we make the case that 5D gravity is a successful physical theory, in touch with experiment. However, the traditional electromagnetic theory can gain, as well, from the parallel with field equations for 5D gravity. In particular, it can gain sharper concepts and, perhaps, a richer basis, as phenomenology finds reason to be included with the fundamentals. These aspects and others are discussed in Secs. 4.1 and 5. In Sec. 6, we conclude our work.

¹This type of foliation is performed in the Hamiltonian formulation of 4D general relativity, for which the Arnowitt-Deser-Misner foliation [17, 18] was originally intended.

²We do not consider the propagation of *active* massive (charged) particles propagating within the gravitational (electromagnetic) field that themselves produce.

2 5D physics for 4D formalisms

We postulate that reality is 5D, yet an observer perceives/acknowledges only 4 dimensions out of 5. For a complete physical picture or formalism, the observer relies on 4D geometry and an array of non-geometrical concepts. The physical pictures employed by a 4D observer result from different foliations of a 5D geometry [15]. It is postulated that, if the 5D geometry is independent of the fifth coordinate then the 5D physics appears as 4D mechanics, while if the 5D geometry is independent of time then the 5D physics appears as 4D statistical physics. Throughout the manuscript, we use bold face to indicate that some indices of the corresponding mathematical objects are suppressed, particularly for 3D vectors. We start with a 5D Lorentzian space-time, $\mathcal{M}^{(5)}$, with metric h_{MN} ($M, N, \dots = 0, 1, 2, 3, 5$), having the signature $(-, +, +, +, +)$, that is sufficiently smooth and subject to the principle of general covariance. We further consider that $\mathcal{M}^{(5)}$ is Ricci flat; i.e., $R_{MN}^{(5)} = 0$ and the 5D gravity fields have no sources. This is just a minimalist choice, not required by the induced matter principle.

We assume that the observed particle is passive, not a source of field interacting with the space-time metric. Particle propagation can be described using suitable equations defined on $(\mathcal{M}^{(5)}, \mathbf{h})$. We are particularly interested in null propagation, therefore 5D optics, defined in two flavors. First, we can define *classical* or geometrical optics, based on 5D null geodesics. Second, we can define quantum optics. A spinless formulation is provided by the 5D wave equation, where the wavefunction $\Psi(\cdot, \cdot)$ may be regarded as the number of null paths between any two given points of $\mathcal{M}^{(5)}$. A spin-1/2 quantum optics requires a wavefunction Ψ , which is a pair of bispinors or a *biquaternion* [13], a vector with two quaternionic components, at every point of $\mathcal{M}^{(5)}$. It can be argued that Ψ represents two tangent null vectors, one incoming and one outgoing, at every point of $\mathcal{M}^{(5)}$ [13]. The wavefunction equation is of Dirac-type, based on a fünfbein field³ and a special Clifford algebra; see Eq. (14) in Ref. [13]. The focus is, thus, solving null-propagation problems for passive particles, given the 5D gravity field.⁴ The propagation problems are invariant to conformal transformations of h_{MN} , which cannot be fully recovered by observing only particles in null propagation. We add tilde to the metric symbol, $\tilde{h}_{MN}$, to denote a conformally transformed metric h_{MN} .

Among space-times suitable for applications are 5D black holes, where the null structure has been a topic of major interest [19–22]. It is interesting to note that, while the event horizon of a 4D black hole is always of spherical topology, the event horizon of a 5D black hole can have spherical, toroidal, or other topologies [22]. Here, we propose 4D interpretations of 5D space-times and passive-particle null-propagation in 5D space-times [13–15].

3 4D classical and quantum mechanics from 5D physics

Interpretation of the 5D geometry and dimensional reduction are greatly facilitated by symmetry. The isometries of $(\mathcal{M}^{(5)}, \mathbf{h})$ form a Lie group, denoted $\text{Iso}(\mathcal{M}^{(5)}, \mathbf{h})$. The generators of this group are called *Killing vectors*, ξ , satisfying the Killing equation $\partial_M \xi_N + \partial_N \xi_M = 0$. We select a nontrivial subgroup of $\text{Iso}(\mathcal{M}^{(5)}, \mathbf{h})$, which we call the *formalism isometry group*, $\text{FIso}(\mathcal{M}^{(5)}, \mathbf{h})$, and its elements, *formalism isometries*. Here, we consider that $(\mathcal{M}^{(5)}, \mathbf{h})$ has only one formalism isometry: the translation along a space-like dimension, x^5 .⁵ We foliate $(\mathcal{M}^{(5)}, \mathbf{h})$ along the formalism Killing field, into 4D space-times, using the Arnowitt-Deser-Misner procedure [17, 18], and indicate the lapse and shift fields of the foliation in the 5D metric. In turn, this yields a geometrical parameterization of h_{MN} , which we further write in adapted coordinates, so it does not depend on x^5 . We thus obtain

$$h^{MN} = \begin{pmatrix} g^{\mu\nu} & N^\mu \\ N^\nu & N^\rho N_\rho + N^2 \end{pmatrix} \Leftrightarrow h_{MN} = \begin{pmatrix} g_{\mu\nu} + N_\mu N_\nu / N^2 & -N_\mu / N^2 \\ -N_\nu / N^2 & 1 / N^2 \end{pmatrix},$$

³The fünfbein field specifies a choice for a local frame at each point of the 5D manifold.

⁴The 5D observer has no notion of particle charge or mass and does not distinguish between electric and gravitational fields.

⁵Mathematics allows for much more general theories. For example, we may consider a space-time with a time-like Killing vector or several Killing vectors as formalism isometries, possibly requiring foliation over several dimensions of the 5D manifold. The difficulty resides in the physical interpretation and link to experiment.

where $(\mathcal{M}^{(4)}, \mathbf{g})$ is the foliated 4D manifold and, N and N_μ ($\mu, \nu, \dots = 0, 1, 2, 3$), respectively, are the lapse and shift of the 4D foliation. We introduce *physical notation*

$$N_\mu \equiv -\frac{q}{c^2} A_\mu, \quad N \equiv \Phi^{-1},$$

where A_μ represents the electromagnetic potential in CGS units; q is specific charge and c is the speed of light in vacuum. As it turns out, the 4D metric, which yields the well-known laws of 4D particle propagation, is not $g_{\mu\nu}$, but $\tilde{g}_{\mu\nu} \equiv g_{\mu\nu}/\Phi^2$ [15]. The 5D metric observed through particle propagation in 4D mechanics is $\tilde{h}_{MN} = h_{MN}/\Phi^2$; this gives the second requirement for the parameterization of the 5D metric. We write the metric h_{MN} using the fields that are observed in particle propagation; see Sec. 4.

The 4D observer may perform any *cylindrical* transformation of coordinates, consisting of diffeomorphisms of $(\mathcal{M}^{(4)}, \tilde{\mathbf{g}})$ leaving x^5 unchanged; i.e.,

$$x^\mu \rightarrow \mathcal{F}^\mu(x^\nu), \quad x^5 \rightarrow x^5. \quad (3.10)$$

Other coordinate transformations, not changing 4D physical observables, are gauge transformations of the shift of the foliation, N_μ , or the electromagnetic potential, A_μ

$$\left\{ \begin{array}{l} x^\mu \rightarrow x^\mu, \\ x^5 \rightarrow x^5 - \frac{q}{c^2} \mathcal{A}(x^\mu), \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} \tilde{g}_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu}, \\ A_\mu \rightarrow A_\mu + \partial_\mu \mathcal{A}(x^\nu), \\ \Phi \rightarrow \Phi, \end{array} \right. \quad (3.11)$$

where $\mathcal{A}(x^\nu)$ is the gauge function for A_μ . It is unclear how the 4D observer would use other coordinate transformations and what is their physical interpretation within 4D mechanics.

The symmetry of $(\mathcal{M}^{(5)}, \mathbf{h})$ to translation along x^5 is key for describing 5D passive particles of null mass (i.e., 5D photons), in four dimensions, as 4D passive particles carrying mass and electrical charge. The 4D observer considers himself/herself uncharged, which leads to important choices in the process of dimensional reduction. For classical propagation, 5D null geodesics in $(\mathcal{M}^{(5)}, \mathbf{h})$ are given 4D interpretation [15]. If the uncharged 4D observer adopts the path element of $(\mathcal{M}^{(4)}, \tilde{\mathbf{g}})$ for the infinitesimal proper time, then 5D null geodesics are represented as 4D time-like Randers geodesics [23, 24] in $(\mathcal{M}^{(4)}, \tilde{\mathbf{g}})$, where the eccentricity of the path element is determined by A_μ . Furthermore, since the x^5 -translational symmetry yields a conserved quantity for 5D geodesic motion, we say that the 4D particle propagates with constant passive mass [15]. For quantum propagation, we perform Fourier transforms with respect with x^5 of the 5D wave equation and the 5D Dirac-type equation, which, in addition, requires a foliated fünfbein field [13].⁶ It is further postulated that the coordinate x^5 is conjugated with the inverse Compton wavelength $\lambda^{-1} = mc/\hbar$ [13, 14]. In this fashion, 5D photons are interpreted as 4D *on-shell* massive quantum particles.

A priori, there is no reason for the 5D space-time be symmetric, in which case, we do not offer a physical picture for the 5D geometry. However, if the 5D space-time is *nearly* symmetric, then a interpretation can be constructed based on the alleged symmetry and a top-down hierarchy of physical concepts. The first order concepts are those of the physical picture assuming that the symmetry is exact. They provide a satisfactory description for particle propagation in manifold fields, for a small enough proper-time interval and space-time neighborhood. Next to the first order physical concepts, second order concepts can be added for improving the description of particle propagation, so it holds for longer proper-time intervals and larger space-time neighborhoods. In general, the physical concepts of second order rely on those of first order. Therefore, the hierarchical physical interpretation is not covariant. Reference [16] demonstrates how particle propagation in a 5D space-time with approximate x^5 -translation symmetry can be given a physical picture. In the first order, we use 4D mechanics [15]. For the second order, we invoke additional phenomena such as Stokes friction and non-hermitian quantum decoherence [16].

4 Field equations in 4D mechanics

Several authors [9–12] published the field equations for x^5 -independent 5D gravity, $R_{MN}^{(5)} = 0$, in the parametrization given by $g_{\mu\nu}$, A_μ and Φ . We performed an additional conformal transformation and rewrote

⁶The choice for the foliated fünfbein rests with the uncharged 4D observer [14].

the equations for 5D gravity in the fields $\tilde{g}_{\mu\nu}$, A_μ and Φ , which have more physical content because $\tilde{g}_{\mu\nu}$ is the 4D metric observed for 4D passive particle propagation [15]

$$\tilde{R}_{\mu\nu}^{(4)} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{R}^{(4)} = \frac{q^2}{2c^4}\tilde{T}_{\mu\nu}^{EM} + \tilde{T}_{\mu\nu}^\Phi, \quad (4.12)$$

$$\tilde{\nabla}^\mu \tilde{F}_{\mu\nu} = -3\tilde{F}_{\mu\nu}\tilde{\nabla}^\mu \ln \Phi, \quad (4.13)$$

$$\frac{1}{\Phi}\tilde{\nabla}^\mu \tilde{\nabla}_\mu \Phi = \frac{q^2}{4c^4}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} + \frac{1}{6}\tilde{T}_{\mu\nu}^\Phi\tilde{g}^{\mu\nu}, \quad (4.14)$$

where $\tilde{R}_{\mu\nu}^{(4)}$ and $\tilde{\nabla}_\mu$ are, respectively, the 4D Ricci tensor and the Levi-Civita connection of the metric $\tilde{g}_{\mu\nu}$. Furthermore, $\tilde{F}_{\mu\nu} \equiv F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and

$$\tilde{T}_{\mu\nu}^{EM} = \tilde{g}_{\nu\rho}\tilde{F}_{\mu\sigma}\tilde{F}^{\sigma\rho} + \frac{1}{4}\tilde{g}_{\mu\nu}\tilde{F}_{\rho\sigma}\tilde{F}^{\rho\sigma}, \quad (4.15)$$

$$\tilde{T}_{\mu\nu}^\Phi = \frac{1}{\Phi}\left[\tilde{\nabla}_\mu\tilde{\nabla}_\nu\Phi - \frac{2}{\Phi}(\tilde{\nabla}_\mu\Phi)(\tilde{\nabla}_\nu\Phi)\right] - \frac{1}{\Phi}\tilde{g}_{\mu\nu}\tilde{g}^{\rho\sigma}\left[\tilde{\nabla}_\rho\tilde{\nabla}_\sigma\Phi - \frac{2}{\Phi}(\tilde{\nabla}_\rho\Phi)(\tilde{\nabla}_\sigma\Phi)\right], \quad (4.16)$$

are the stress-energy tensors for the electromagnetic and Φ fields, respectively. These equations are interpreted according to the induced matter principle. An empty 5D space-time, equipped with a space-like Killing field, is regarded as a 4D space-time containing electromagnetic and Φ fields, as sources of 4D gravity. Equation (4.12) represents the 4D Einstein equations where the dichotomy between fields and sources occurs only with the 4D induced-matter interpretation of the 5D geometry. The RHS of Eq. (4.12) is separated into electromagnetic and Φ sources. It is important to note that the 4D gravity sources are different in nature: $\tilde{T}_{\mu\nu}^{EM}$ can be inferred from passive particle propagation, while $\tilde{T}_{\mu\nu}^\Phi$ cannot. Therefore, from this perspective, Φ appears like a *dark field* [25, 26].

4.1 The Maxwell equations and Ohm's law from the equations for 5D gravity

Equation (4.13) is of type Maxwell and helps recover Eqs. (1.1) and (1.4), considered in vacuum.⁷ For this, we break Eq. (4.13) to introduce the 4-current density, j_ν

$$\tilde{\nabla}^\mu \tilde{F}_{\mu\nu} \equiv \frac{4\pi}{c}j_\nu, \quad (4.17)$$

$$\frac{4\pi}{c}j_\nu \equiv -3\tilde{F}_{\mu\nu}\tilde{\nabla}^\mu \ln \Phi. \quad (4.18)$$

Eq. (4.17) corresponds to the Maxwell equations (1.1) and (1.4); the RHS of Eq. (4.17) expresses the sources for the electromagnetic field in the LHS. Equation (4.18) further explicates j_μ in terms of Φ and the electromagnetic field, and is interpreted in the same way: the RHS designates the sources for the LHS. The alternative interpretation, where the LHS designates the source for the electromagnetic and Φ fields in the RHS, yields a contender to the Maxwell equations, and a line of argument far from the traditional electromagnetic theory. We will not pursue it here. Applying $\tilde{\nabla}^\nu$ on both sides of Eq. (4.18) and using Eq. (4.13) yields the 4D continuity equation for j_ν as an identity⁸

$$\frac{4\pi}{c}\tilde{\nabla}^\nu j_\nu = -3(\tilde{\nabla}^\nu \tilde{F}_{\mu\nu})(\tilde{\nabla}^\mu \ln \Phi) - 3\tilde{F}_{\mu\nu}\tilde{\nabla}^\nu \tilde{\nabla}^\mu \ln \Phi = 6\tilde{F}_{\mu\nu}\tilde{\nabla}^\nu \tilde{\nabla}^\mu \ln \Phi = 0. \quad (4.19)$$

It is significant that j_ν depends on Φ , which does not appear in the LHS of Eq. (4.17). Still, as a divergence-free vector field, j_ν retains more freedom than Φ . From the perspective of a 4D field theory, j_ν requires no

⁷The concept of vacuum is vague in terms of Lorentzian geometry. According to the induced matter principle, $F_{\mu\nu}$ is an active matter field defined on 4D space-time. However, the reference, empty 4D space-time is Ricci flat, with field equations $R_{MN}^{(4)} = 0$, where $F_{\mu\nu}$ and Φ do not exist. Therefore, the reference 4D space-time is not sufficient for describing particle propagation in electromagnetic field, in vacuum. We are lead to postulate that vacuum represents 5D emptiness rather than 4D emptiness; it is unclear how this relates to experiment. Another possibility is to associate vacuum to the limit of weak fields evolving in a 4D space-time. The term *electrovacuum* is often used for solutions of both Einstein and Maxwell equations.

⁸This is also obtained by applying $\tilde{\nabla}^\nu$ on both sides of Eq. (4.17).

further analysis. It is very important to note that, in the theory of 5D gravity, j_ν results from a dispensable ansatz (i.e., Eqs. (4.17) and (4.18)), of no particular benefit for solving the 5D field equations. This is in stark contrast with the fundamental role that j_ν plays in the theories of electromagnetism and condensed matter, where a fundamental SI unit is dedicated to measurements of electrical current. These theories further describe j_ν , postulating conduction mechanisms which rely on the structure of matter. For example, in solid state physics (e.g., Refs. [27, Chs. 1, 2] and [28, Ch. 6]), the current density is described using classical or quantum formalisms for passive particles/quasiparticles in motion, trapped within matter, subject to the Lorentz force. In addition, the particles/quasiparticles are subject to collisions or phenomenological friction forces, due to additional electromagnetic interactions associated to the atomic structure of matter.⁹ In plasma physics (e.g., Refs. [29, Ch. 1] and [30, Chs. 5, 6]), conduction mechanisms have similar building principles, although, sometimes, they simply postulate that the Lorentz force is zero for particles within plasma. A 3D velocity field for charged carriers is often defined using 4D active matter $\mathbf{v}/c \equiv \mathbf{j}/j^0$.¹⁰

There are two main uses for the microscopic pictures of j_ν . First, they explain the Laplace force [31], the force acting on a conductor placed in electromagnetic field, in terms of the Lorentz force;¹¹ see e.g., [3, Ch. 5.2] for an argument based on Ampère’s work [32]. However, the Laplace force does not require an explanation via the Lorentz force, because the Laplace force can be obtained from the divergence of the electromagnetic stress-energy tensor, $\tilde{\nabla}^\mu \tilde{T}_{\mu\nu}^{EM}$ [3, Ch. 11.11]. Second, they provide microscopic explanations for the Ohm’s law and other laws in condensed matter physics. However, as we explain below, a conduction mechanism is not required to explain Ohm’s law, which here appears as a field equation. To see this, we provide a physical interpretation for Eq. (4.18), breaking 4D covariance in the limit of weak fields.

5 A minimal amendment to the Maxwell equations

The rigorous approach to Eqs. (4.12)-(4.14) is through exact solutions, analyzed for their relevance to experiment. Many exact solutions for empty 5D space-times without compact dimensions have already been found [20, 33–39], but they primarily describe time-independent geometries, which are not invariant to translation along x^5 , and do not satisfy Eqs. (4.12)-(4.14). Perhaps, the only exception is a class of time-dependent solutions, obtained through the Wick rotation of the 5D Myers-Perry black-hole metric [12, 40, 41].¹²

An alternate approach is using the 5D field equations for an amendment to the Maxwell equations, which are already close to experiment. Evidently, the physics of the amended Maxwell equations is superseded by that of corresponding exact 5D solutions. We propose a strong approximation of the 5D gravity equations (4.12)-(4.14) in the limit of weak fields. We consider that the 4D metric, $\tilde{g}_{\mu\nu}$, is approximately the Minkowski metric, $\eta_{\mu\nu}$; n.b., $(q/c^2)^2 \tilde{T}^{EM}/2 + \tilde{T}^\Phi \approx 0$. We further break covariance to distinguish 3D space and time, and write the 4D connection as $\tilde{\nabla}^\nu \equiv (\tilde{\partial}^0, \nabla)$. In (pseudo)cartesian coordinates, the 3D space metric is $diag(1, 1, 1)$, so we no longer distinguish covariant and contravariant spatial components of tensors. Furthermore, we use the Levi-Civita symbol, ϵ_{jkl} , as a cartesian tensor, and introduce the 3D curl operator and vector product. Setting $x^0 \equiv -ct$, and using Eqs. (1.8) and (1.9), we introduce the 3D vector field variables

$$\mathbf{E} = \tilde{F}_{0j}, \quad B_j = \epsilon_{jkl} \tilde{F}_{kl}/2. \quad (5.20)$$

Next to Eqs. (1.2) and (1.3), Eq. (4.13) yields

$$\nabla \mathbf{E} = \frac{4\pi}{c} \boldsymbol{\varrho} \mathbf{E}, \quad (5.21)$$

$$\nabla \mathbf{B} = 0, \quad (5.22)$$

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \quad (5.23)$$

$$\nabla \times \mathbf{B} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c} (\varsigma \mathbf{E} + \boldsymbol{\varrho} \times \mathbf{B}), \quad (5.24)$$

⁹These atomic electromagnetic fields are not included in the Maxwell equations with the sources j_ν .

¹⁰Our theory does not require 4D active matter satisfy the 4D on-shell constraint, designed for 4D passive particles.

¹¹The Lorentz force acts on passive particles.

¹²Identifying z with x^5 in Eq. (1) of Ref. [12] shows that the 5D metric has nontrivial fields $\tilde{g}_{\mu\nu}$, A_μ and Φ . The induced matter interpretation of 5D metric remains for further work.

where the fields

$$\boldsymbol{\varrho} \equiv -\frac{c}{4\pi} \nabla \ln \Phi^3, \quad \varsigma \equiv \frac{1}{4\pi} \frac{\partial \ln \Phi^3}{\partial t}, \quad (5.25)$$

are regarded to be *material properties*, satisfying

$$-\frac{1}{c} \frac{\partial \boldsymbol{\varrho}}{\partial t} = \nabla \varsigma. \quad (5.26)$$

Alternatively, one can directly choose a model for the field Φ .¹³ Electrical charge and current densities are not needed for the amended Maxwell equations (5.21)-(5.24), but they result from Eq. (4.18) as

$$\rho = \boldsymbol{\varrho} \cdot \mathbf{E}/c, \quad (5.27)$$

$$\mathbf{j} = \varsigma \mathbf{E} + \boldsymbol{\varrho} \times \mathbf{B}. \quad (5.28)$$

Equations (5.27) and (5.28) can be regarded as a generalization of the Ohm's law. Conceptually, a perfect conductor has very high conductivity, does not hold charge density in volume, nor allows for magnetic field inside the material [3, Ch. 8.1]. This model is recovered in the above equations by assuming that Φ is approximately independent of space (i.e., $\nabla \ln \Phi^3 \approx 0$ or $\boldsymbol{\varrho} \approx 0$) in some region, where we say the perfect conductor is found. According to Eq. (5.27), the charge density is approximately zero, and Eq. (5.28) yields the Ohm's law (1.7). Equations (5.27) and (5.28) thus describe a more realistic, *imperfect* conductor, which holds electric charge in volume and allows penetration by the magnetic field. Hence, the conduction current $\mathbf{j}$ has an ohmic component and a non-ohmic component, which depends on the local magnetic field. These phenomena are characterized by the 3D vector field $\boldsymbol{\varrho}$, which is regarded here as a material property, next to the conductivity ς .

Taking an imperfect conductor for being perfect can have important consequences for the interpretation of experimental data. The *ordinary* Hall effect [27, 28, 42] clearly demonstrates that an external magnetic field penetrates conductor materials, in contradiction to the model for the perfect conductor.¹⁴ Still, if the model of a perfect conductor is used to interpret experimental data, then all conduction current is considered to be ohmic and, apparently, the conductivity depends linearly on the local magnetic field; c.f., Eqs. (5.28). Furthermore, if the external magnetic field is large compared to the magnetic field generated by the electric current $\mathbf{j}$, then it appears that the conductivity depends linearly on the external magnetic field. These considerations seem to explain experimental observations by Kapitza [43–45], regarding a large number of chemical elements. It may however be possible to reinterpret Kapitza's experimental data using the model of an imperfect conductor (c.f., Eq. (5.27) and (5.28)), with the benefit of a simpler picture for the material properties. The phenomena by which an applied magnetic field changes the conductivity¹⁵ of a material are known under the generic name of *magnetoresistance* [27, 28]. This is a large and complex chapter in condensed matter physics. The imperfect conductor model may help disentangle this complexity, while making no assumptions about the structure of matter, only using 5D field theory.

Another example of a phenomenon where the concept of imperfect conductor may be important is that of persistent current [46–48], which was discovered in gold and other metals. Consider an imperfect conductor in zero electric field and nonzero external magnetic field; this may be, for example, the magnetic field of a probe. According to Eq. (5.28), the external magnetic field causes a current density $\boldsymbol{\varrho} \times \mathbf{B}$ inside the conductor, where $\mathbf{B}$ is the local magnetic field inside the conductor. In turn, this current density causes an additional component to the external magnetic, as described by the Maxwell equations. In the model of the perfect conductor, this looks like an instantaneous field response, otherwise, one is lead to assume a strange ohmic current inside the perfect conductor, even in the absence of electric field. We hope that the model of an imperfect conductor presented here can help provide a simple and natural explanation for the phenomenon of persistent current, within the framework of classical field theory.

¹³The weak field limit of Eq. (4.14) may provide an equation for Φ . Here, we adopt a more general view, where the choice of Φ is only limited by its success in applying Eqs. (5.21)-(5.24) to experiment.

¹⁴Hall discovered the effect in gold, which does not behave like a perfect conductor. This raises questions about the applicability of the model of the perfect conductor.

¹⁵The conductivity remains defined by the Ohm's law, within the perfect conductor model.

Lastly, we briefly comment on the Hall effect [27, 28, 42]. Using the perfect conductor model, the effect can be described that applying magnetic field transverse to ohmic current flow triggers an additional current component, transverse to both the initial ohmic current and the applied magnetic field. The traditional explanation of the Hall effect is based on conduction mechanisms, violating both the perfect and imperfect conductor models. It is assumed that the external magnetic field penetrates the conductor material and changes the motion of the conducting particles inside. Consequently, the conducting particles, considered as being passive, deviate from their initial trajectories, creating an additional current component inside the conductor.¹⁶ However, within an imperfect conductor, even without applying a magnetic field, the conducting particles form a current, and already propagate within the magnetic field which they, themselves, create (Ampère's law). Therefore, the conducting particles within an imperfect conductor constitute active matter and cannot be described as passive particles. An alternate explanation of the Hall effect, free of conduction mechanisms, based only on the classical field theory and the concept of the imperfect conductor, would not have these difficulties. In fact, the phenomenon of persistent current and the Hall effect are very much related, both regarding penetration of magnetic field inside an imperfect conductor. The difference is that, for the Hall effect, the conductor is subject to ohmic current, next to the external magnetic field. In the remaining of this section, we discuss several applications of the amended Maxwell equations (5.21)-(5.24) to traditional problems formulated for the Maxwell equations.

5.1 On the propagation of electromagnetic waves within a conducting material

The most natural application of Eqs. (5.21)-(5.24) may be to describing the propagation of electromagnetic waves through conducting materials. Traditionally, the problem is formulated by assuming a perfect conductor and merging the Maxwell equations with the Ohm's law. This results in Eqs. (5.21)-(5.24), where ς is a constant and $\boldsymbol{\varrho}$ is zero; see e.g., Refs. [3, Ch. 7.7], [49, Ch. 9.4] and [50, Ch. 13]. We slightly generalize this application to the case where both ς and $\boldsymbol{\varrho}$ are constants.¹⁷ We proceed by deriving an equation for the electric field, $\boldsymbol{E}$.¹⁸ First, we take a curl on both sides of Eq. (5.23) and use Eq. (5.24) to substitute $\nabla \times \boldsymbol{B}$

$$\nabla(\nabla \boldsymbol{E}) - \nabla^2 \boldsymbol{E} = -\frac{1}{c} \frac{\partial}{\partial t} \left(\frac{1}{c} \frac{\partial \boldsymbol{E}}{\partial t} + \frac{4\pi}{c} \varsigma \boldsymbol{E} + \frac{4\pi}{c} \boldsymbol{\varrho} \times \boldsymbol{B} \right). \quad (5.29)$$

Second, in Eq. (5.29), we substitute $\nabla \boldsymbol{E}$ using Eq. (5.21) and $\partial \boldsymbol{B} / \partial t$ using Eq. (5.23)

$$\frac{4\pi}{c} \nabla(\boldsymbol{E} \boldsymbol{\varrho}) = \nabla^2 \boldsymbol{E} - \frac{1}{c^2} \frac{\partial^2 \boldsymbol{E}}{\partial t^2} - \frac{4\pi \varsigma}{c^2} \frac{\partial \boldsymbol{E}}{\partial t} + \frac{4\pi}{c} \boldsymbol{\varrho} \times (\nabla \times \boldsymbol{E}). \quad (5.30)$$

We now search for plane wave solutions of type

$$\boldsymbol{E} = \boldsymbol{E}_0 \exp(i\boldsymbol{k}\boldsymbol{x} - i\omega t), \quad (5.31)$$

where i is the complex unit, $\boldsymbol{E}_0$ and ω are real constants, but we allow $\boldsymbol{k}$ to be a complex constant. We thus obtain the following dispersion equation in the propagation vector $\boldsymbol{k}$

$$\frac{\omega^2}{c^2} - \boldsymbol{k}^2 + \frac{4\pi i}{c} \left(\frac{\varsigma \omega}{c} - \boldsymbol{\varrho} \boldsymbol{k} \right) = 0. \quad (5.32)$$

In the case where $\boldsymbol{\varrho} = 0$, we simply recover the formula for a perfect conductor [3, Ch. 7.7]. The propagation vector $\boldsymbol{k}$ is complex, indicating that the electromagnetic wave can propagate within a perfect conductor, but it is attenuated over the length scale $1/\text{Im}|\boldsymbol{k}| \propto \varsigma^{-1/2}$, called skin or penetration depth. Equation (5.32) also shows that an imperfect conductor behaves like a perfect conductor, provided $\boldsymbol{\varrho} \boldsymbol{k} = 0$. This represents an anisotropy in the electromagnetic wave propagation. Notably, the anisotropy does not occur in the model of a perfect conductor and can help measure $\boldsymbol{\varrho}$, assessing how imperfect the conductor material is. Wave propagation through an imperfect conductor is intrinsically anisotropic, where the anisotropy relates to $\boldsymbol{\varrho}$, or

¹⁶It is intriguing that magnetoresistance never enters the discussion about the Hall effect. According to the experiments of Kapitza [43–45], magnetoresistance always occurs next to the Hall effect.

¹⁷It remains unclear how applicable to experiment is this choice for ς and $\boldsymbol{\varrho}$.

¹⁸The same results are obtained by analyzing the equations for the magnetic field.

the dark field Φ . This is different from other sources of anisotropy, generally associated to the tensor nature of the relative electric permittivity or magnetic permeability, which here are taken for unit scalars. It is remarkable that, according to Eq. (5.32), electromagnetic waves can propagate loss-free within an imperfect conductor given that

$$|\mathbf{k}| = \omega/c, \quad \cos \theta = \varsigma/|\boldsymbol{\varrho}|, \quad (5.33)$$

where θ is the angle between $\boldsymbol{\varrho}$ and $\mathbf{k}$, which has no imaginary part. The propagating wave solution requires $\varsigma \leq |\boldsymbol{\varrho}|$, which cannot occur for a perfect conductor. Finally, we note that recent studies of electromagnetic wave propagations [51–56] take precautions to model realistic features of matter, yet do not go beyond the concept of perfect conductor.

5.2 On electrostatics and magnetostatics

We submit that Eqs. (5.21)-(5.24) can describe the electromagnetic field in a large number of applications, not just wave propagation. To discuss electrostatic and magnetostatic solutions, we consider a 5D metric which is independent of both x^5 and x^0 . Therefore, Φ is time-independent, the conductivity ς is zero, and the ohmic component of the electric current density is zero, as well. Consequently, the framework cannot address the traditional magnetostatic theory [3, 32], unless the ohmic component of the electric current density is zero. This is too restrictive to recover the law of Biot and Savart, or that of Ampère. The field equations become

$$\nabla \mathbf{E} = \frac{4\pi}{c} \boldsymbol{\varrho} \mathbf{E}, \quad (5.34)$$

$$\nabla \mathbf{B} = 0, \quad (5.35)$$

$$\nabla \times \mathbf{E} = 0, \quad (5.36)$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \boldsymbol{\varrho} \times \mathbf{B}, \quad (5.37)$$

where, based on the definitions (5.25), $\boldsymbol{\varrho} \equiv -(c/4\pi)\nabla \ln \Phi^3$ has zero curl; i.e., $\nabla \times \boldsymbol{\varrho} = 0$. The electrostatic and magnetostatic sources depend on the very fields they determine

$$\rho = \boldsymbol{\varrho} \mathbf{E}/c, \quad (5.38)$$

$$\mathbf{j} = \boldsymbol{\varrho} \times \mathbf{B}. \quad (5.39)$$

However, these concepts are not strictly needed for the field equations. The equations for the electric and magnetic fields are determined by the same vector field,¹⁹ $\boldsymbol{\varrho}$, yet they are decoupled. Either $\mathbf{E}$ or $\mathbf{B}$ can be zero, irrespective of the other. There are, of course, solutions where both $\mathbf{E}$ and $\mathbf{B}$ are nonzero, determined by $\boldsymbol{\varrho}$. The following results through straightforward calculations.

Proposition. Under the change of variables

$$\mathbf{E} \rightarrow \mathbf{E}^* \Phi^{-3}, \quad \mathbf{B} \rightarrow \mathbf{B}^* \Phi^{-3}, \quad (5.40)$$

Eqs. (5.34)-(5.37) remain invariant, where

$$(\mathbf{E}, \mathbf{B}, \boldsymbol{\varrho}) \rightarrow (\mathbf{B}^*, \mathbf{E}^*, -\boldsymbol{\varrho}). \quad (5.41)$$

In consequence, it is sufficient to solve the equations for either electrostatics (i.e., Eqs. (5.34) and (5.36)) or magnetostatics (i.e., Eqs. (5.35) and (5.37)) for a complete solution. For example, the electrostatic equations can be combined into a single one, for the electrostatic potential,

$$\nabla^2 A_0 = \frac{4\pi}{c} \boldsymbol{\varrho} \nabla A_0, \quad (5.42)$$

¹⁹Taking $\boldsymbol{\varrho} = 0$ accommodates all field configurations where $\mathbf{E}$ and $\mathbf{B}$ are independent constant vectors.

that can be used to formulate homogeneous problems of type Dirichlet and Neumann; see Appendix A for alternate formulations of the Eq. (5.42). The solutions for A_0 generate all solutions of Eqs. (5.34)-(5.37).

The typical problem in electrostatics and magnetostatics is finding field configurations due to localized sources; it is assumed that the resulting field is also localized, thus vanishes at infinity. We discuss two elementary solutions for Eqs. (5.34)-(5.37). If all the vector fields are spherically symmetric, we obtain $\mathbf{B} = 0$ and $\mathbf{E} = (E_r, E_\theta = 0, E_\phi = 0)$, where

$$E_r(r) = \frac{\mathcal{C}\Phi^{-3}(r)}{r^2}, \quad (5.43)$$

and $\mathcal{C}$ is an integration constant. The radial component of the spherically symmetric, Coulomb electric field, resulting from the Maxwell equations, is qualitatively the same [3, Ch. 1]

$$E_r^M(r) = \frac{Q(r)}{r^2}, \quad (5.44)$$

where $Q(r) = 4\pi \int_0^r \rho(r')r'^2 dr'$ is the electric charge contained by the sphere of radius r , centered at the origin and $\rho(\cdot)$ denotes a spherically symmetric, electrical charge density. Comparing Eqs. (5.43) and (5.44) yields $\mathcal{C}\Phi^{-3}(r) = Q(r)$, addressing the integration constant $\mathcal{C}$ and providing an interpretation for Φ , in this restricted geometry. The amended Maxwell equations recover the concept of point-like source electric charge, which is remarkably successful with the model of the hydrogenic atom.

In magnetostatics, we say $\mathbf{E} = 0$. If $\mathbf{q} = (q_\rho, q_\phi = 0, q_z = 0)$, in cylindrical coordinates, then the solution of Eq. (5.37) is $\mathbf{B} = (B_\rho = 0, B_\phi, B_z = 0)$, where

$$B_\phi(\rho) = \frac{\mathcal{C}\Phi^{-3}(\rho)}{\rho}, \quad (5.45)$$

is reminiscent of Ampère's law. The solution predicts nonzero magnetic field in the absence of ohmic current. In fact, taking a curl on both hand sides of Eq. (5.37) yields a generalized Helmholtz equation

$$\nabla^2 \mathbf{B} = (\nabla \mathbf{q}) \mathbf{B}. \quad (5.46)$$

We assume that $\nabla \mathbf{q}$ is zero outside a compact domain and say that the magnetic sources, $\mathbf{q} \times \mathbf{B}$, are localized. Thus, Eq. (5.46) becomes the Laplace equation outside the compact domain, so we can choose magnetostatic solutions vanishing at infinity; localized magnetic sources yield localized magnetostatic fields. Traditionally, we acknowledge only ohmic current and spin magnetic moments as the sources of magnetic field. However, our 5D theory suggests that sources of magnetic field can also be represented as $\mathbf{q} \times \mathbf{B}$. Given suitable models for $\mathbf{q}$, Eq. (5.37) may serve as a basic tool to model field configurations due to magnetic materials [5]. It is very important to note that, the field equations of 4D mechanics (4.12)-(4.14), and, implicitly, Eq. (5.37), cannot describe the thermodynamics of the electromagnetic field response, addressing topics such as the Curie temperature, the Curie-Weiss law, etc. According to the principles of our 5D theory, this requires the translation along x^0 be formalism isometry, and a *statistic* interpretation of the electromagnetic field.

5.2.1 A new class of field-static problems

In contrast to the traditional electrostatic and magnetostatic equations, (5.34)-(5.37) allow for another class of problems, besides that of localized field configurations. We consider problems of field configurations, where static, constant field is applied from spatial infinity; n.b., the applied field has zero curl and divergence. These problems cannot be formulated using the Maxwell equations (1.1)-(1.4), where sources are considered to be independent of field, and applying field from spatial infinity has no consequence for the localized field component. In electrostatics, we write $\mathbf{E} = \mathbf{E}^{(0)} + \mathbf{E}^{(1)}$, where $\mathbf{E}^{(0)}$ is the field at infinity and $\mathbf{E}^{(1)}$ is localized field. Using Eqs. (5.34) and (5.36), and $\mathbf{E}^{(1)} \equiv -\nabla A_0^{(1)}$, we get

$$\nabla^2 A_0^{(1)} = \frac{4\pi}{c} \mathbf{q} \nabla A_0^{(1)} - \frac{4\pi}{c} \mathbf{q} \mathbf{E}^{(0)}, \quad (5.47)$$

a non-homogeneous equation in $A_0^{(1)}$, to be solved using Green's function, starting from the homogeneous equation (5.42). In magnetostatics, we write $\mathbf{B} = \mathbf{B}^{(0)} + \mathbf{B}^{(1)}$, where $\mathbf{B}^{(0)}$ is the field at infinity and $\mathbf{B}^{(1)}$ is a

localized. Introducing $\mathbf{B}^{(1)} \equiv \nabla \times \mathbf{A}^{(1)}$, under the gauge condition $\nabla \mathbf{A}^{(1)} = 0$, we obtain a non-homogeneous equation in $\mathbf{A}^{(1)}$,

$$-\nabla^2 \mathbf{A}^{(1)} = \frac{4\pi}{c} \boldsymbol{\varrho} \times (\nabla \times \mathbf{A}^{(1)}) + \frac{4\pi}{c} \boldsymbol{\varrho} \times \mathbf{B}^{(0)}, \quad (5.48)$$

which, again, can be approached through the method of Green's function. The RHS of Eq. (5.48) has two terms: $\boldsymbol{\varrho} \times (\nabla \times \mathbf{A}^{(1)})$ represents a current density due to the localized magnetic field $\mathbf{B}^{(1)}$, and $\boldsymbol{\varrho} \times \mathbf{B}^{(0)}$ represents the contribution to magnetic sources by the magnetic field applied from spatial infinity. Hopes are that Eq. (5.48) can serve to model penetration of magnetic field inside materials and permanent currents. For $\mathbf{B}^{(0)} = 0$, we choose the solution $\mathbf{B}^{(1)} = 0$, while for $\mathbf{B}^{(0)} \neq 0$, we solve Eq. (5.48) to obtain $\mathbf{B}^{(1)}$.

5.2.2 Another 5D space-time geometry for electrostatics and magnetostatics

Field configurations can appear static to the 4D observer, even though the 5D geometry does not have a Killing field along x^0 . In particular, the gravitational (i.e., $\tilde{g}_{\mu\nu}$) and electromagnetic fields can be observed as being static through observing passive particle propagation. However, time dependence can occur in the lapse field Φ , which is a dark field. The amended Maxwell equations admit time-independent solutions, if ς and $\boldsymbol{\varrho}$ are time-independent. Equation (5.26) further implies that ς must have zero gradient, thus the conductivity ς is a constant field, but $\boldsymbol{\varrho}$ remains a zero-curl 3D field.²⁰ The field equations become

$$\nabla \mathbf{E} = \frac{4\pi}{c} \boldsymbol{\varrho} \mathbf{E}, \quad (5.49)$$

$$\nabla \mathbf{B} = 0, \quad (5.50)$$

$$\nabla \times \mathbf{E} = 0, \quad (5.51)$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \varsigma \mathbf{E} + \frac{4\pi}{c} \boldsymbol{\varrho} \times \mathbf{B}, \quad (5.52)$$

accommodating better the traditional magnetostatics [3, Ch. 5], which results assuming a perfect conductor; i.e., $\boldsymbol{\varrho} = 0$. However, because translation along x^0 is not a 5D isometry, our 5D framework implies that the statistics of the electromagnetic field is ill-defined. In fact, experiment shows that, as long as conductivity is nonzero, the electromagnetic field continuously loses energy through ohmic heating. It may thus be intuitive that, in this case, the electromagnetic field cannot be in thermodynamic equilibrium, even though it is time-independent.

We comment on the original Hall effect, using Eqs. (5.49)-(5.52) to sketch a theory for the time-independent phenomenon, even though ς and $\boldsymbol{\varrho}$ provide just a crude model for the material properties. In particular, constant conductivity over the entire 3D space necessarily implies that the conducting material fills up the entire 3D space. We assume that the magnetic field has two components, $\mathbf{B} = \mathbf{B}^{(0)} + \mathbf{B}^{(1)}$, where $\mathbf{B}^{(0)}$ is a constant magnetic field, applied from infinity, and $\mathbf{B}^{(1)}$ is the localized magnetic field. Equation (5.52) yields

$$-\nabla^2 \mathbf{A}^{(1)} = \frac{4\pi}{c} \varsigma \mathbf{E} + \frac{4\pi}{c} \boldsymbol{\varrho} \times (\nabla \times \mathbf{A}^{(1)}) + \frac{4\pi}{c} \boldsymbol{\varrho} \times \mathbf{B}^{(0)}, \quad (5.53)$$

with the same notation as in Eq. (5.48). The electric field $\mathbf{E}$ is an input in Eq. (5.53), and results from Eq. (5.47), where we also distinguish two components for the electric field $\mathbf{E} = \mathbf{E}^{(0)} + \mathbf{E}^{(1)}$. The typical conceptualization of the field configuration for the Hall effect [27, 28] considers $\mathbf{B}^{(0)}$ pointing along the z -axis and $\mathbf{E}^{(0)}$ pointing along the x -axis. Equation (5.53) predicts the existence of a non-ohmic current component due to the magnetic field at infinity, $\boldsymbol{\varrho} \times \mathbf{B}^{(0)}$, next to a contribution from $\boldsymbol{\varrho} \times (\nabla \times \mathbf{A}^{(1)})$, which depends implicitly on $\mathbf{B}^{(0)}$ and $\mathbf{E}$. These additional current components could be translated into electric field components using Ohm's law, within the perfect conductor model. Still, this is very different from the position of the traditional theory of the Hall effect [27, 28], that the electric field $\mathbf{E}$ changes with $\mathbf{B}^{(0)}$. The Hall coefficient $R_H = E_y/(j_x B_z)$ quantifies this effect and is regarded as a measure of electron density within the conductor material. These developments, based on conduction mechanisms and the perfect conductor model,

²⁰The field $\ln \Phi^3$ depends linearly with time.

are incompatible to our theory. It is very important to note that the experimental measurements, themselves, provide readings of electric current and potential, based on Ohm's law and the perfect conductor model, as well. Therefore, as anticipated in Sec. 5, giving up the model of the perfect conductor, poses problems not only for theory, but also for measurement principles and data interpretation. Our field theoretic approach would require measurement principles based on the principles of general relativity; i.e., field probes using passive particle propagation, rather than measurements of electrical current and potential. Corroborating Eq. (5.53) with experimental results, remains a topic for further work.

6 Discussion and conclusions

The electromagnetic theory and experiment developed in parallel during the nineteenth century, both sides contributing to sharpening much needed physical concepts such as field, perfect conductor, and electron. Meanwhile, condensed matter physics also made considerable progress, particularly through combining elements of thermodynamics, electromagnetism and atomic theory. We submit that, owing to the choice of fundamental concepts in electromagnetism, some actual field phenomena²¹ might have been misclassified for describing properties of matter. In consequence, traditional electromagnetism seems very accessible and successful, wherever applicable by itself, whereas condensed matter physics struggles with describing complex properties for matter. A revision of the fundamental concepts in electromagnetism and condensed matter physics may be in order, involving both theory and experiment. Kaluza's proposal, that electromagnetism and gravity should be unified within the 5D general relativity framework, has inspired very much work on the theoretical fundamentals of electromagnetism and gravity. On the experimental side, a very broad set of measurement techniques became available, and may be critically evaluated from the fundamental point of view, taking into account philosophical choices in electromagnetism and general relativity.

In this work, we analyzed the equations of 5D gravity from the perspective of the Maxwell equations and Ohm's law. The covariance of the 5D metric field can be broken to distinguish 4D gravitational field, electromagnetic field and a dark scalar field. Among the field equations resulting from the 5D condition of Ricci flatness, we recognized Maxwell-type equations and a generalization of Ohm's law, modeling an imperfect conductor, which holds electric charge in volume and can be penetrated by an exterior magnetic field. Furthermore, our analysis makes it clear that the density of electrical charge and current should not be considered as fundamental concepts in electromagnetism. Still, the notion of electrical charge retains fundamental status. Indeed, passive specific charge occurs as a factor converting the electromagnetic field into 5D gravitational field; see Eq. (3.10). Active charge may occur as a parameter of the 5D metric. For example, black hole metrics have parameters such as mass, angular momenta, etc., often called *conserved charges*. Active electric charge may be found among these parameters.

Based on the 5D gravity equations, we proposed an amendment to the Maxwell equations, in the limit of weak fields, and commented on a few, direct applications. Notably, we discussed electromagnetic wave propagation through an imperfect conductor, electrostatics, magnetostatics of magnetic materials, permanent currents and the Hall effect. The goal of this manuscript is to convey that, even though the electromagnetic theory resulting from 5D gravity differs in many regards from the traditional theory, it is able to recover much of the formalism, albeit on a slightly different conceptual basis. We did not provide a mathematical definition for the limit of weak fields, rather, we capitalized on the resemblance between the Maxwell equations and 5D gravity equations. There are three main reasons why our discussion of physical applications is limited. First, the 5D space-time may not be Ricci-flat, and a stress-energy tensor representing 5D active matter may be postulated. Second, the x^5 -translational symmetry of the 5D space-time may be only approximate, and the field formalism could be adjusted using perturbation methods to account for the symmetry breaking. Third, the final word is left to exact solutions of the field equations. Therefore, the physics of electromagnetism resulting from 5D gravity could be significantly more complex.

In conclusion, the geometry of an empty 5D space-time with a space-like Killing field can serve to reconsider the fundamentals of electromagnetism and condensed matter theory. If the 5D space-time has a time-like Killing field, then it may be possible to construct a field thermodynamics. This remains a topic for further work.

²¹Hall regarded his discovery as a field effect, going against the Maxwell's conceptualization of electromagnetism [42].

A Alternate forms for the electrostatic potential equation

Equation (5.42), describing the electrostatic field versus $\boldsymbol{\varrho}$, is not found among the popular partial differential equations of mathematical physics. However, the equation can be reformulated to resemble more popular equations. For example, in the field variable $\alpha \equiv A_0 \Phi^{3/2}$, Eq. (5.42) becomes a generalized Helmholtz equation

$$\nabla^2 \alpha - \left[\frac{3}{2} \frac{\nabla^2 \Phi}{\Phi} + \frac{3}{4} \frac{(\nabla \Phi)^2}{\Phi^2} \right] \alpha = 0. \quad (\text{A.54})$$

Evidently, if A_0 vanishes at infinity, then so does α . The generalized Helmholtz equation has been previously studied, in both two and three dimensions [37, 57, 58]. However, general solutions for the 3D case are yet to be formulated.

Another possibility is to transform Eq. (5.42) into an integral equation, using the Green's function of the Laplace operator. We obtain

$$A_0(\mathbf{x}) = - \int d^3 x' \frac{\boldsymbol{\varrho}(\mathbf{x}') \nabla A_0(\mathbf{x}')}{c |\mathbf{x} - \mathbf{x}'|}, \quad (\text{A.55})$$

which, under the assumption that ∇A_0 is a localized field, can be integrated by parts to give

$$A_0(\mathbf{x}) = \int d^3 x' A_0(\mathbf{x}') K(\mathbf{x}, \mathbf{x}'), \quad (\text{A.56})$$

where

$$K(\mathbf{x}, \mathbf{x}') = \frac{\nabla \boldsymbol{\varrho}(\mathbf{x}')}{c |\mathbf{x} - \mathbf{x}'|} + \frac{\boldsymbol{\varrho}(\mathbf{x}') (\mathbf{x} - \mathbf{x}')}{c |\mathbf{x} - \mathbf{x}'|^3}. \quad (\text{A.57})$$

Therefore, A_0 appears as a fixed point (i.e., eigenvector of unit eigenvalue) of the integral operator with kernel $K(\cdot, \cdot)$. Fixed points of integral operators have been previously studied [59–62].

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