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Dedicated to the memory of my Ph.D. supervisor Alan Baker with admiration and gratitude.

Abstract. We develop a modular version of a super analogue of Schur's duality by means of supergroups, rather than Lie superalgebras, in preparation for a geometric analogue.

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1. Introduction

Schur [Sch27], reproving the results of his thesis [Sch1901], considered the permutation action of the group algebra $\mathbb{C}S_r$ over $\mathbb{C}$ of the symmetric group S_r on $r \geq 1$ letters, and the diagonal action of $\mathrm{GL}(n, \mathbb{C}) = \mathrm{GL}(V)$, $V = \mathbb{C}^n$, on $V^{\otimes r}$. The two actions commute, and Schur proved that these two actions have a double centralizing property in $\mathrm{End}(V^{\otimes r})$: the centralizer of one is the image of the other. Representations of $\mathrm{GL}(V)$ are thus determined from those of S_r, known from the work of Young.

Schur's work was continued by Weyl [Wey53], whose "strip theorem" showed for example that when $n \geq r$ there is a canonical bijection between the set of irreducible representations of S_r, and the set of irreducible polynomial representations of $\mathrm{GL}(n, \mathbb{C})$ in $V^{\otimes r}$.

A derivative of the Schur-Weyl duality, which started then as the study of the commuting actions of the symmetric group S_r and $\mathrm{GL}(n, \mathbb{C})$ on $V^{\otimes r}$ where $V = \mathbb{C}^n$, can be given in terms of the commuting actions of S_r and the *Lie algebra* $\mathrm{gl}(n, \mathbb{C})$ of $\mathrm{GL}(n, \mathbb{C})$. A *quantum deformation* of this duality was developed by Drinfeld [Dri85] and Jimbo [Jim86], to the context of the finite Iwahori-Hecke algebra $H_r(q^2)$ and the quantum algebra $U_q(\mathrm{gl}(n))$, on using universal R-matrices, that solve the Yang-Baxter equation. Chari and Pressley [ChPr96] extended this duality in the Hecke-quantum case to the *affine* case, relating the commuting actions of the affine Iwahori-Hecke algebra $H_r^a(q^2)$ and of the affine quantum Lie algebra $U_{q,a}(\mathrm{sl}(n))$.

In another direction, the study of commuting actions of the symmetric group S_r and the Lie algebra $\mathrm{gl}(n, \mathbb{C})$ on $(\mathbb{C}^n)^{\otimes r}$, was extended by Sergeev [Se85] and Berele and Segev [BeRe87] to the context of the diagonal action of the *superalgebra* $\mathrm{gl}(m|n, \mathbb{C})$ and of S_r, with a signed action. A quantum deformation of this work, as in Drinfeld and Jimbo, was given by Moon [Mo03] and Mitsuhashi [Mi06], who related the signed action of the Iwahori-Hecke algebra $H_r(q^2)$ with that of the quantum Lie superalgebra $U_q^\sigma(\mathrm{sl}(m|n))$. This chain of works is completed in [Fli20], dealing with the general *affine quantum super* case, relating the commuting actions of the affine Iwahori-Hecke algebra $H_r^a(q^2)$ and of the affine quantum Lie superalgebra $U_{q,a}^\sigma(\mathrm{sl}(m|n))$ using the presentation of the former by Bernstein (see [Fli11]) and of the later by Yamane [Yam99] in terms of generators and relations, acting on the rth tensor power of the superspace $V = \mathbb{C}^{m|n}$. Thus a functor is constructed and it is shown to be an equivalence of categories of $H_r^a(q^2)$ and $U_{q,a}^\sigma(\mathrm{sl}(m|n))$-modules when $r < m + n$.

The work of Schur was extended, or perhaps purified, in yet another – modular – direction. Motivated by R. Brauer, C. Chevalley, Serre [Ser68] and Carter and Lusztig [CaLu74], Green [Gr07] developed a modular – over $\mathbb{Z}$ – analogue of the original Schur duality, using polynomial representations of $\mathrm{GL}(n, \mathbb{C})$ homogeneous of degree r, on using the coalgebra structure of the algebra of finitary functions on this group. The aim of the present work is to explore a super analogue of this, namely develop – functorially in a superalgebra A – a modular theory of commuting actions of the group algebra AS_r and of the supergroup $\Gamma_A = \mathrm{GL}(m|n, A)$, or rather a signed permutation action of AS_r, and the supercoalgebra A^{Γ_A} of A-valued functions on Γ_A. We emphasize that we work with the supergroup $\mathrm{GL}(m|n)$, in contrast to most of the works after Schur and Weyl, that considered the Lie algebra derivative $\mathrm{gl}(m|n)$ of the group $\mathrm{GL}(m|n)$. It seems to us such a modular theory is needed for a geometric theory.

Thus we develop a modular version of a super analogue of Schur's duality by means of supergroups, rather than Lie superalgebras, in preparation for a geometric analogue.

2. Super world

In superalgebras, all objects are $\mathbb{Z}/2$-graded, and when the order of two odd objects is reversed in a product, a sign appears. This need not be a bad omen. We start by reviewing the basic definitions following the conceptual approach of [DeMo99], which in turn follows lectures of J. Bernstein as well as [Lei80] and [Man97]. Let F be an infinite field with $2 \neq 0$.

2.A. Superspaces

A *super vector space* is a $\mathbb{Z}/2$-graded F-vector space $V = V_0 \oplus V_1$. An element v of V_0, resp. V_1, is called *homogeneous even*, resp. *odd*, and we write $p(v) = 0$, resp. $= 1$; p is called the *parity function*, defined only on homogeneous vectors. A *morphism* $V \to W$ between two super vector spaces is a $\mathbb{Z}/2$-degree preserving linear map from V to W. Thus V_0 is mapped to W_0, and V_1 to W_1. We then obtain an abelian category of super F-vector spaces. The *dimension* of a finite dimensional such V is denoted by $m|n = (m, n)$, where $m = \dim_F V_0$, $n = \dim_F V_1$. The *parity reversing functor* Π is defined by $(\Pi V)_0 := V_1$, $(\Pi V)_1 := V_0$.

The *tensor product* of super vector spaces V and W is the tensor product of the underlying vector spaces, with the $\mathbb{Z}/2$-grading $(V \otimes W)_k = \oplus_{i+j=k} V_i \otimes W_j$; here $\otimes$ is $\otimes_F$. The tensor product functor is additive and exact in each variable, and has a unit object: if $\mathbf{1}$ is the vector space F in even degree, $\mathbf{1} \otimes V$ and $V \otimes \mathbf{1}$ are canonically isomorphic to V, by $1 \otimes v$, $v \otimes 1 \mapsto v$. It is associative: $(u \otimes v) \otimes w \mapsto u \otimes (v \otimes w)$ is a canonical isomorphism from $(U \otimes V) \otimes W$ to $U \otimes (V \otimes W)$. The sign appears in the definition of the commutativity isomorphism

$$c_{V,W} : V \otimes W \to W \otimes V, \qquad v \otimes w \mapsto (-1)^{p(v)p(w)} w \otimes v.$$

Here and below we assume homogeneity when writing formulae.

Let $(V_i;\, i \in I)$ be a finite family of $r = |I|$ super vector spaces. An *ordering* of I is a bijection σ from the ordered set $[1, r] = \{1, 2, \ldots, r\}$ to I. A tensor product of the V_i is obtained on choosing an ordering σ of I and parenthesis on $T_\sigma = V_{\sigma(1)} \otimes V_{\sigma(2)} \otimes \cdots \otimes V_{\sigma(r)}$. For two tensor products T_σ, T_τ of the V_i, and a way of composing associativity and commutativity isomorphisms to get an isomorphism from T_σ to T_τ, the same isomorphism is obtained. For v_i homogeneous in V_i, it is given by

$$v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(r)} \mapsto (-1)^N v_{\tau(1)} \otimes \cdots \otimes v_{\tau(r)},$$

$$N = \#\{(i, j) \in I \times I; \quad v_i, v_j \text{ odd}, \quad \sigma^{-1}(i) < \sigma^{-1}(j), \quad \tau^{-1}(i) > \tau^{-1}(j)\}.$$

2.B. Superalgebras

A *super algebra* over F is a super vector space A, together with a morphism $A \otimes A \to A$, $a \otimes b \mapsto ab$, called *product*. By definition of a morphism of superspaces, $p(ab) = p(a) + p(b)$, for homogeneous a, b in A. The superalgebra A is *associative* if $(ab)c = a(bc)$, a, b, $c \in A$. A *unit* is an even element $1 \in A_0$, thus a morphism $\mathbf{1} \to A$, with $1x = x = x1$. By a superalgebra (= super algebra) we shall mean an associative one, with a unit. For such a superalgebra A, a left (resp. right) A-*module* is a super vector space M, with a morphism, also called *product*: $A \otimes M \to M$ (resp. $M \otimes A \to M$), satisfying the usual identities expressing that M is a module over A considered as a usual algebra. The sign rule enters only in the definition of commutativity. The superalgebra A is called *commutative* if the product of homogeneous elements satisfies $ab = (-1)^{p(a)p(b)} ba$.

If A is commutative, a left A-module is also a right A-module, but the passage involves the sign rule:

$$m \cdot a := (-1)^{p(m)p(a)} a \cdot m.$$

The *tensor product* of A-modules $M \otimes_A N$ (M is a right module, N is a left module) is again an A-module. The tensor product functor is associative, commutative and has a unit: the A-module A. The commutativity isomorphism is given by $m \otimes n \mapsto (-1)^{p(m)p(n)} n \otimes m$.

The *opposite* algebra A° of A is A with the product $a \cdot_\circ b = (-1)^{p(a)p(b)} b \cdot a$. An element z of A is *central* if its homogeneous components satisfy $za = (-1)^{p(a)p(z)} az$ for all $a \in A$. The *tensor product* of superalgebras A, B is $A \otimes B$, with the product

$$(a \otimes b)(c \otimes d) := (-1)^{p(b)p(c)} ac \otimes bd.$$

2.C. Action of S_r on $V^{\otimes r}$

The action of the symmetric group S_r on the tensor product $V^{\otimes r} = V \otimes \cdots \otimes V$ of a superspace V of dimension $m|n$ can be explicitly described as in [Se85] and [BeRe87], as follows. Let A be a free associative commutative superalgebra, with a free family of generators $\{x_i;\, i \in I\}$. Define a function $c : (\mathbb{Z}/2)^r \times S_r \to \{\pm 1\}$ by

$$c(p(x), \sigma)x_1 \dots x_r = x_{\sigma(1)} \dots x_{\sigma(r)}, \qquad \text{where} \quad p(x) = (p(x_1), \dots, p(x_r))$$

is the parity vector of the elements x_i, and $\sigma \in S_r$. We check that

$$c(p(x), \sigma\tau)x_1 \dots x_r = x_{\sigma\tau(1)} \dots x_{\sigma\tau(r)} = y_{\tau(1)} \dots y_{\tau(r)}$$

$$= c(p(y), \tau)y_1 \dots y_r = c(\sigma^{-1}p(x), \tau)x_{\sigma(1)} \dots x_{\sigma(r)}$$

$$= c(\sigma^{-1}p(x), \tau)c(p(x), \sigma)x_1 \dots x_r, \qquad \text{where} \quad y_j = x_{\sigma(j)}, \quad p(y) = \sigma^{-1}p(x),$$

so

$$c(p(x), \sigma\tau) = c(p(x), \sigma) \cdot {}^{\sigma}c(p(x), \tau)$$

is a 1-cocycle, and in particular $c(p(x), \sigma^{-1}) = c(\sigma p(x), \sigma)$.

Put $p(v) = (p(v_1), \dots, p(v_r))$ if $v = v_1 \otimes \cdots \otimes v_r$. Define a (left) action π of S_r on $V^{\otimes r}$ (or right action $*$) by

$$\pi(\sigma)(v_1 \otimes \cdots \otimes v_r) = v_1 \otimes \cdots \otimes v_r * \sigma^{-1} = c(p(v), \sigma^{-1})v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(r)},$$

$\sigma \in S_r$. We verify this is an action, since the definition of [Se85, p. 420, l. 2] is different:

$$\pi((\sigma\tau)^{-1})v_1 \otimes \cdots \otimes v_r = c(p(v), \sigma\tau)v_{\sigma\tau(1)} \otimes \cdots \otimes v_{\sigma\tau(r)};$$

$$\pi(\tau^{-1})(\pi(\sigma^{-1})(v_1 \otimes \cdots \otimes v_r)) = \pi(\tau^{-1})(c(p(v), \sigma)v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(r)})$$

$$= c(p(v), \sigma)\pi(\tau^{-1})(u_1 \otimes \cdots \otimes u_r) = c(p(v), \sigma)c(p(u), \tau)u_{\tau(1)} \otimes \cdots \otimes_{\tau(r)}$$

$$= c(p(v), \sigma)c(\sigma^{-1}p(v), \tau)v_{\sigma\tau(1)} \otimes \cdots \otimes v_{\sigma\tau(r)}.$$

2.D. Free super module

A *free module* over a superalgebra is a module which is free as an ungraded module, with a homogeneous basis.

The *standard free module* $A^{m|n}$, where A is a commutative superalgebra, is the module freely generated by even elements $e_1, \dots, e_m$ and odd elements $e_{m+1}, \dots, e_{m+n}$. A morphism $T : A^{m|n} \to A^{p|q}$ can be represented by a matrix of size $(p+q) \times (m+n)$, with blocks of even and odd entries $\left(\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right)$, A' of size $p \times m$, B' of size $p \times n$, C' of size $q \times m$, D' of size $q \times n$, the entries of A' and D' are even, those of B' and C' are odd.

An element x of $A^{m|n}$ can be presented by a column vector $(x_i;\, 1 \le i \le m+n)$, if $x = \sum_{1 \le i \le m+n} e_i x_i$. The entries of T are defined by $T(e_j) = \sum_i e_i T_{ij}$. With these conventions $T(x)$ is given by the matrix product Tx, and composition of morphisms is given by matrix product:

$$S(T(e_j)) = S\left(\sum_i e_i T_{ij}\right) = \sum_i (Se_i)T_{ij} = \sum_i \sum_k e_k S_{ki} T_{ij} = \sum_i e_i \left(\sum_k S_{ik} T_{kj}\right),$$

so the (i,j)-entry of ST is $\sum_k S_{ik} T_{kj}$, as usual.

2.E. Super determinant

If M is a free module of finite type over a commutative super algebra A, write $\mathrm{GL}(M)$ for the group of automorphisms of the A-module M. Put $\mathrm{GL}(m|n, A) = \mathrm{GL}(A^{m|n})$. The *superdeterminant*, often called Berezinian, is a homomorphism $\mathrm{Ber} : \mathrm{GL}(M) \to \mathrm{GL}(1|0, A) = A_0^\times$, and with the choice of the standard basis, $\mathrm{sdet} : \mathrm{GL}(m|n, A) \to A_0^\times$, given as follows. Let T be an automorphism of $A^{m|n}$, represented by a matrix $\left(\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right)$. The entries of A', D' are even, those of B', C' are odd. The quotient B of A by the ideal $\langle A_1 \rangle$ generated by the odd elements of the superalgebra $A = A_0 \oplus A_1$ equals the quotient of A_0 by a nilpotent ideal J. After an extension of scalars to B (that is, applying $\otimes B$), the matrix of T takes the form $\left(\begin{smallmatrix} A' \bmod J & 0 \\ 0 & D' \bmod J \end{smallmatrix}\right)$. Hence A', D' are invertible modulo the nilpotent ideal J. So A', D' are invertible themselves, and one defines

$$\mathrm{sdet}(T) = \det(A' - B'D'^{-1}C') \det(D')^{-1},$$

a formula suggested by

$$\left(\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right) = \left(\begin{smallmatrix} I & B'D'^{-1} \\ 0 & I \end{smallmatrix}\right) \left(\begin{smallmatrix} A'-B'D'^{-1}C' & 0 \\ 0 & D' \end{smallmatrix}\right) \left(\begin{smallmatrix} I & 0 \\ D'^{-1}C' & I \end{smallmatrix}\right)$$

and $\mathrm{sdet}\left(\begin{smallmatrix} A' & 0 \\ 0 & D' \end{smallmatrix}\right) = \det A' \cdot \det(D')^{-1}$, which is compatible with the definition of the *supertrace* $\mathrm{str}\left(\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right) = \mathrm{tr}\, A' - \mathrm{tr}\, D'$, see [DeMo99]. The matrices A', D', $B'D'^{-1}C'$ have entries in the commutative ring A_0, so that their determinants are defined. That sdet is multiplicative is verified in detail in [Lei80].

2.F. Super rings

To work over $\mathbb{Z}$ in the next section we introduce a *commutative super ring* to be a $\mathbb{Z}/2$-graded ring $R = R_0 \oplus R_1$, associative and with a unit, satisfying $xy = (-1)^{p(x)p(y)} yx$, thus $x^2 = 0$ for any odd homogeneous x. This can be used to make super algebraic geometry over $\mathbb{Z}$. For example, if M is a free $\mathbb{Z}/2$-graded $\mathbb{Z}$-module, the commutative algebra freely generated by M is flat over $\mathbb{Z}$: it is $\mathrm{Sym}^*(M_0) \oplus \wedge^* M_1$.

We shall discuss below the free rank $m|n$-module $E = E^{m|n}$ as a functor mapping a superalgebra A to the free rank $m|n$ A-module $E_A = E_A^{m|n} = A^{m|n}$, and then the monoid $\mathrm{End}(E) : A \to \mathrm{End}_A(A^{m|n})$ and the group $\mathrm{GL}(m|n)$. Both are defined as functors, and are representable. The monoid $\mathrm{End}(E^{m|n})$ is represented by the affine scheme $\mathbb{A}^{m^2+n^2|2mn}$, with coordinates $a_{i,j}$, which are even if $1 \le i, j \le m$ or $m < i, j \le m+n$, odd otherwise. For $\mathrm{GL}(m|n)$ we need to invert $\det(A')$, $A' = (a_{i,j};\, 1 \le i, j \le m)$ and $\det(D')$, $D' = (a_{i,j};\, m < i, j \le m+n)$.

For Γ an affine group scheme over a superalgebra A, the group law $\Gamma \times \Gamma \to \Gamma$ becomes the coalgebra structure $\mathcal{O}(\Gamma) \to \mathcal{O}(\Gamma) \otimes \mathcal{O}(\Gamma)$.

An action of Γ on a free A-module M is, for each superalgebra B over A, a morphism $\Gamma(B) \to \mathrm{End}(M_B)$, functorial in B. The universal case is $B = \mathcal{O}(\Gamma)$, the affine algebra of Γ, for which we have the identity point $\mathrm{id} \in \Gamma(B) = \mathrm{Hom}(B, B)$, which gives an endomorphism of $\mathcal{O}(\Gamma) \otimes M$, namely $M \to \mathcal{O}(\Gamma) \otimes M$, as the endomorphism is $\mathcal{O}(\Gamma)$-linear. This is the comodule structure, describing the action of a general element g of the monoid or group. It transforms a basis vector e_p of M to $\sum_q e_q c_{q,p}(g)$, where the coefficients $c_{q,p}$ are in $\mathcal{O}(\Gamma)$.

3. Representation theory

We next check that basic definitions of representation theory extend to the super context, thus generalizing the modular exposition of [Gr07], where the usual, non-super case is studied. In contrast with the usual case, where it suffices to study the group of points of the algebraic group over a field, in the super case it is necessary to study the group of points of the super group at superalgebras, since the homogeneous odd elements are nilpotent. Thus the super case is similar to that of the study of non reduced schemes, which have nilpotents in their structure sheaf, not detected by their points in a field only.

3.A. Representations

Let us then begin as usual with a *semigroup* (a set with an associative multiplication) Γ with an identity 1_Γ, and a field F. A *representation* τ of Γ on an F-vector space V is a map $\tau : \Gamma \to \mathrm{End}_F V$ satisfying $\tau(gg') = \tau(g)\tau(g')$, $\tau(1_\Gamma) = 1_V$ (g, $g' \in \Gamma$; 1_V is the identity morphism $V \to V$). Our group of interest is $\Gamma = \mathrm{GL}(m|n)$, viewed as a scheme, or a functor, in the same way that an algebraic group is viewed. However, it takes values at superalgebras A, which have nilpotents ($e^2 = 0$ for a homogeneous odd element e). To study algebraic groups, such as $\mathrm{GL}(n)$, it suffices to study their points at an algebraically closed field, and Galois action. This does not suffice for the study of groups of automorphisms of superspaces: we need to consider A-valued points for general commutative superalgebras A over F, as in the study of non-reduced schemes one considers values in general commutative algebras, that have nilpotents. Thus we need to consider a functorially compatible family of maps $\tau_A : \Gamma_A \to \mathrm{End}_A(M_A)$, where M_A is an A-module for a superalgebra A over F, and $\Gamma_A = \mathrm{GL}(m|n, A)$, satisfying $\tau_A(gg') = \tau_A(g)\tau_A(g')$, $\tau_A(1_{\Gamma_A}) = 1_{M_A}$ (g, $g' \in \Gamma_A$). Extend τ_A linearly to a map of F-superalgebras $\tau_A : A\Gamma_A \to \mathrm{End}_A(M_A)$. Here $A\Gamma_A$ is the semigroup algebra of Γ_A over A, its elements are the formal linear combinations

$$\kappa = \sum_{g\in\Gamma_A} \kappa_g g, \qquad \kappa_g \in A,$$

whose *support* $\mathrm{supp}\,\kappa = \{g \in \Gamma_A;\ \kappa_g \neq 0\}$ is finite. Then $A\Gamma_A$ acts on M_A by $\kappa v = \tau_A(\kappa)v$, $\kappa \in A\Gamma_A$, $v \in M_A$. We get a left $A\Gamma_A$-module, denoted again by (M_A, τ_A), or simply M_A. An $A\Gamma_A$-*map* between such $A\Gamma_A$-modules (M_A, τ_A), (M'_A, τ'_A) is by definition an A-linear map $f : M_A \to M'_A$ satisfying $\tau'_A(g)f = f\tau_A(g)$ for all g in Γ_A. An $A\Gamma_A$-*isomorphism*, or an *equivalence*, between two representations τ_A, τ'_A, is a bijective $A\Gamma_A$-map.

Analogous definitions apply to right $A\Gamma_A$-modules. A right $A\Gamma_A$-module is a pair (M_A, τ_A), where $\tau_A : \Gamma_A \to \mathrm{End}_A(M_A)$ is an anti representation of Γ_A on the A-module M_A, thus $\tau_A(gg') = \tau_A(g')\tau_A(g)$ for all g, $g' \in \Gamma_A$, and $\tau_A(1_{\Gamma_A}) = 1_{M_A}$.

3.B. Comultiplication

The set $A^\Gamma = A^{\Gamma_A}$ of all maps $\Gamma_A \to A$, where A is a commutative superalgebra over F, is a commutative F-super algebra, with algebra operations defined pointwise, that is, $(ff')(g) := f(g)f'(g)$ for $g \in \Gamma_A$. The identity element 1 of A^Γ takes each $g \in \Gamma_A$ to the identity element 1_A of A.

If $s \in \Gamma_A$ and $f \in A^{\Gamma_A}$, then the left and right translates of f by s are defined to be the maps $L_s f$, $R_s f : \Gamma_A \to A$, given by

$$L_s f : g \mapsto f(sg), \qquad R_s f : g \mapsto f(gs), \qquad g \in \Gamma_A.$$

Each of the operators L_s, R_s maps A^Γ to itself, and is an F-super algebra morphism $A^\Gamma \to A^\Gamma$. In particular L_s and R_s both lie in the A-module $\mathrm{End}_A(A^\Gamma)$. Note that $R : s \mapsto R_s$ gives a representation of Γ_A on A^Γ, while $L : s \mapsto L_s$ gives an anti representation. Thus A^Γ can be made into a left $A\Gamma_A$-module using R, and a right $A\Gamma_A$-module using L. Denote both by $\circ$, so that if $s \in \Gamma_A$ and $f \in A^\Gamma$, we write $s \circ f = R_s f$, $f \circ s = L_s f$. These actions commute: $(s \circ f) \circ t = s \circ (f \circ t)$ for all s, $t \in \Gamma_A$, $f \in A^\Gamma$.

There is a linear map $A^\Gamma \otimes_A A^\Gamma \to A^{\Gamma\times\Gamma}$, which takes $f \otimes f'$ (f, $f' \in A^\Gamma$) to the function $\Gamma_A \times \Gamma_A \to A$ mapping $(s,t) \mapsto f(s)f'(t)$ (s, $t \in \Gamma_A$). This linear map is injective, we use it to identify $A^\Gamma \otimes_A A^\Gamma$ with a submodule of $A^{\Gamma\times\Gamma}$.

The semigroup structure on Γ_A gives rise to the *comultiplication* and *counit* maps

$$\Delta : A^\Gamma \to A^{\Gamma\times\Gamma}, \qquad \varepsilon : A^\Gamma \to A,$$

as follows. For $f \in A^\Gamma$, put $\Delta f(s,t) = f(st)$ and $\varepsilon(f) = f(1_{\Gamma_A})$. Both Δ and ε are F-super algebras maps.

3.C. Finitary maps

We say that $f \in A^\Gamma$ in *finitary*, or is a *representative function*, if it satisfies the following three equivalent conditions (cf. [Ho71, §2]):
F1. The left $A\Gamma_A$-submodule $A\Gamma_A \circ f$ generated by f is finite dimensional.
F2. The right $A\Gamma_A$-submodule $f \circ A\Gamma_A$ generated by f is finite dimensional.
F3. $\Delta f \in A^\Gamma \otimes A^\Gamma$, namely there exist finitely many pairs f_h, $f'_f \in A^\Gamma$ with

$$\Delta f = \sum_h f_f \otimes f'_h.$$

This equation is equivalent to the system of equations

$$f(st) = \sum_h f_h(s) f'_h(t) \qquad (s,\, t \in \Gamma_A),$$

as well as to each of the systems

$$t \circ f = \sum_h f_h f'_h(t) \quad (\forall t \in \Gamma_A); \qquad f \circ s = \sum_h f_h(s) f'_h \quad (\forall s \in \Gamma_A).$$

The set $F_A = F(A^\Gamma)$ of all finitary functions $f : \Gamma_A \to A$ is a subsuperalgebra of A^Γ, and is also closed under Δ in the sense that $\Delta F_A \subset F_A \otimes F_A$, namely if f is finitary, the functions f_h, f'_h can be chosen themselves to be finitary. Thus the A-module F_A, together with the maps $\Delta : F_A \to F_A \otimes F_A$, $\varepsilon : F_A \to A$, is an A-cosuperalgebra. The two structures, of super algebra and of cosuperalgebra, are linked by the fact that Δ and ε are both F-superalgebra maps.

3.D. Coefficient functions

Finitary functions on Γ_A appear as coefficient functions of finite dimensional representations of Γ_A. Suppose τ_A is a representation of Γ_A on a finite rank free A-module M_A. If $\{v_b;\, b \in B\}$ is a free set of generators of M_A over A, we have equations

$$\tau_A(g) v_b = g v_b = \sum_{a\in B} v_a r_{a,b}(g) \qquad g \in \Gamma_A, \quad b \in B. \tag{3.1}$$

Here $r_{a,b}(g) \in A$. We name $r_{a,b} : \Gamma_A \to A$ $(a,\, b \in B)$ the *coefficient functions* of τ_A, or of the $A\Gamma_A$-module $M_A = (M_A, \tau_A)$. The A-span of these functions is a submodule of A^Γ that we call the *coefficient module* of τ_A, or of the $A\Gamma_A$-module M_A. Denote this module by $\mathrm{cf}(M_A) = \sum_{a,b} A \cdot r_{a,b}$. It is independent of the choice of the basis $\{v_b\}$.

The matrix $R = (r_{a,b})$ gives a matrix representation of Γ_A, thus $R(gg') = R(g)R(g')$ and $R(1_{\Gamma_A}) = (\delta_{a,b})$ for all g, $g' \in \Gamma_A$, and $\delta_{a,b}$ is 1 if $a = b$, 0 otherwise. These relations can be expressed in terms of the coefficients $r_{a,b}$, as

$$\Delta r_{a,b} = \sum_{c\in B} r_{a,c} \otimes r_{c,b}, \qquad \varepsilon(r_{a,b}) = \delta_{a,b}, \qquad \text{for all } a, b \in B. \tag{3.2}$$

From the first equations, for Δ, it follows that all the coefficient functions $r_{a,b}$ are finitary. Hence $\mathrm{cf}(M_A)$ is a submodule of $F_A = F_A(A^\Gamma)$. These equations also show that $C_A = \mathrm{cf}(M_A)$ is a sub cosuperalgebra of F_A, namely that $\Delta C_A \subset C_A \otimes C_A$.

Note that every finitary function $f : \Gamma_A \to A$ lies in the coefficient space of some finite dimensional $A\Gamma_A$-module M_A: take $M_A = A\Gamma_A \circ f$. For this reason finitary functions are sometimes called *representative functions.*

If S is an F-superalgebra, possibly of infinite dimension as an F-superspace, denote by $\mathrm{mod}(S)$ the category of all finite dimensional left S-modules. Put $\mathrm{mod}'(S)$ for the category of all finite dimensional right S-modules.

3.E. Polynomial representations

An *algebraic representation theory* of Γ_A over A is defined as follows. Choose a subcosuperalgebra D of $F_A(A^\Gamma)$, thus D is an A-submodule of $F_A(A^\Gamma)$ satisfying $\Delta D \subset D \otimes D$. A *D-representation theory* of Γ_A is defined to be the study of the full subcategory $\mathrm{mod}_D(A\Gamma_A)$ of $\mathrm{mod}(A\Gamma_A)$ whose objects are all finite dimensional left $A\Gamma_A$-modules M_A such that $\mathrm{cf}(M_A) \subset D$. By definition, the morphisms $f : M_A \to M'_A$ between two objects M_A, M'_A of this category are just the $A\Gamma_A$-maps. We would also say that an $A\Gamma_A$-module M_A is *D-rational* if $\mathrm{cf}(M_A) \subset D$. Then $\mathrm{mod}_D(A\Gamma_A)$ is the category of finite dimensional D-rational left $A\Gamma_A$-modules. Submodules, quotient modules, and finite direct sums of D-rational modules are themselves D-rational. Similarly define the category $\mathrm{mod}'_D(A\Gamma_A)$ of finite dimensional right $A\Gamma_A$-modules which are D-rational.

The assumption $\Delta D \subset D \otimes D$ implies that if $f \in D$ then the functions f_h, f'_h can themselves be chosen to belong to D. It follows that D is a left and right $A\Gamma_A$-submodule of A^Γ. Any finite rank left (or right) $A\Gamma_A$-submodule M_A of D belongs to the category $\mathrm{mod}_D(A\Gamma_A)$ (or $\mathrm{mod}'_D(A\Gamma_A)$).

For example, take $\Gamma_A = \mathrm{GL}(m|n, A)$, where A is a superalgebra over an algebraically closed field F. Let $D = A[\Gamma_A]$ be the ring of A-valued regular functions on Γ_A. Then $\mathrm{mod}_D(A\Gamma_A)$ is the category of rational finite dimensional $A\Gamma_A$-modules.

The example of interest to us is of $\Gamma_A = \mathrm{GL}(m|n, A)$, A being a superalgebra over an infinite field F of characteristic $\neq 2$. Take D to be $C_A(m|n)$, the superalgebra of all polynomial functions $f : \Gamma_A \to A$. The objects (M_A, τ_A) in the category $\mathrm{mod}_D(A\Gamma_A)$, denoted later by $\mathfrak{M}_A(m|n)$, are called *polynomial* $A\Gamma_A$-modules. The associated representations, including the matrix representations $R = (r_{a,b})$ obtained by using the F-bases $\{v_b\}$ of M_A, are called *polynomial representations* of Γ_A.

Another category, later denoted by $\mathfrak{M}_A(m|n, r)$, is obtained by taking $D = C_A(m|n, r)$, the superspace of polynomial functions on Γ_A with values in A, homogeneous of degree r in the $(m+n)^2$ coefficients of a general element g in $\Gamma_A = \mathrm{GL}(m|n, A)$.

The super ring $C_A(m|n)$ can also be regarded as the affine super ring of the algebraic super semigroup $M(m|n, A)$ of all $(m+n) \times (m+n)$ matrices $\left(\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right)$, singular or not, with entries in A, even entries in A', D', odd entries in B', C', so we can regard polynomial representations of $\mathrm{GL}(m|n, A)$ as rational representations of $M(m|n, A)$, and conversely.

3.F. Comodules

Suppose now again that Γ is a semigroup with identity 1_Γ, and D is a sub super coalgebra of the A module $F_A(A^\Gamma)$ of all finitary functions $\Gamma \to A$, where A is a superalgebra over F. Then D itself is a super coalgebra, relative to the maps $\Delta : D \to D \otimes D$ and $\varepsilon : D \to A$. We may consider the category $\mathrm{com}(D)$ of all right D-comodules. An object of $\mathrm{com}(D)$ is a finite rank A-module, together with a structure map $\gamma : M_A \to M_A \otimes_A D$ which is left A-linear, and satisfies the identities

$$(\gamma \circ I_D)\gamma = (I_{M_A} \otimes \Delta)\gamma, \qquad (I_{M_A} \otimes \varepsilon)\gamma = I_{M_A}.$$

The category $\mathrm{mod}_D(A\Gamma_A)$ is equivalent to $\mathrm{com}(D)$ as follows. If $M_A \in \mathrm{mod}_D(A\Gamma_A)$ is free, take a basis $\{v_b\}$ of M_A over A, and consider the equations

$$\tau(g)v_b = gv_b = \sum_{a\in B} v_a r_{a,b}(g), \qquad g \in \Gamma_A, \qquad b \in B.$$

Then define $\gamma : M_A \to M_A \otimes D$ to be the A-linear map given by the equations

$$\gamma(v_b) = \sum_{a\in B} v_a \otimes r_{a,b}, \qquad b \in B. \tag{3.3}$$

Now γ is independent of the basis $\{v_b\}$. Using $\Delta r_{a,b} = \sum_{c\in B} r_{a,c} \otimes r_{c,b}$, $\varepsilon(r_{a,b}) = \delta_{a,b}$ $(a, b \in B)$ one checks that γ satisfies the comodule identities given a few lines above.

Conversely, given a D-comodule (M_A, γ), use (3.3) to define the $r_{a,b}$ in D. The comodule identities show that (3.2) holds, so we may use (3.1) to define the left $A\Gamma_A$-module $M_A = (M_A, \tau)$. Then $\mathrm{cf}(M_A) \subset D$. So every D-rational left $A\Gamma_A$-module can be regarded as a right D-comodule, and conversely. The definition of a morphism $f : M_A \to M'_A$ in $\mathrm{com}(D)$ is such that these morphisms are the same as $A\Gamma_A$-maps in $\mathrm{mod}_D(A\Gamma_A)$.

3.G. Modular theory

The formal transition from $A\Gamma_A$-modules to D-comodules permits the possibility of developing a modular theory.

The D-comodule interpretation permits viewing every right D-comodule as a left module for the A-algebra $D^* = \mathrm{Hom}_A(D, A)$. The super algebra structure in D^* is the dual of the super coalgebra structure on D, i.e., if ξ, $\eta \in D^*$, define their product (convolution) $\xi\eta$ to be the map of D to A which takes $f \in D$ to

$$(\xi\eta)(f) = \sum_h \xi(f_h)\eta(f'_h). \tag{3.4}$$

The identity element of D^* is $\varepsilon : D \to A$. If $M_A = (M_A, \gamma)$ lies in $\mathrm{com}(D)$, make M_A into a D^*-module by the rule

$$\xi v = (I_{M_A} \otimes \xi)(\gamma(v)), \qquad \xi \in D^*, \qquad v \in M_A.$$

Working with a basis $\{v_b\}$ of the free A-module M_A, the rule becomes

$$\xi v_b = \sum_{a\in B} v_a \xi(r_{a,b}), \qquad b \in B. \tag{3.5}$$

There are then three kinds of matrix representations associated with the original free $A\Gamma_A$-module $M_A = (M_A, \tau)$, relative to the basis $\{v_b\}$:
(i) the representation $g \mapsto (r_{a,b}(g))$ of Γ_A;
(ii) the matrix $R = (r_{a,b})$ whose coefficients are functions on Γ_A, satisfying

$$\Delta r_{a,b} = \sum_c r_{a,c} \otimes r_{c,b}, \qquad \varepsilon(r_{a,b}) = \delta_{a,b};$$

it can be viewed as a representation of the super coalgebra D;
(iii) the representation $\xi \mapsto (\xi(r_{a,b}))$ of the super algebra D^*, given by the equations (3.E.).

To recover (i) from (iii), for each $g \in \Gamma_A$ let $e_g : D \to A$ be evaluation at g, thus $e_g(f) = f(g)$, for all $f \in D$. Then $e_g \in D^*$ and the map $e : \Gamma_A \to D^*$ satisfies $e_g e_{g'} = e_{gg'}$, $e_{1_{\Gamma_A}} = \varepsilon$, for g, $g' \in \Gamma_A$. So e can be extended to an A-module map $e : A\Gamma_A \to D^*$, and composing (iii) with e we recover (i): $\Gamma_A \ni g \mapsto e_g \mapsto (e_g(r_{a,b})) = (r_{a,b}(g))$.

3.H. Definitions for modularity

To develop a modular representation theory, we introduce the following definitions, closely following the standard – non-super – case as in [Gr07]. The idea is to give a uniform theory, for all fields and superalgebras,, beginning from the base ring $\mathbb{Z}$. Thus to the same extent that $\mathrm{GL}(n, F)$, n fixed, F varying over some class $\mathfrak{F}$ of commutative rings, is defined over $\mathbb{Z}$, and this makes possible a "modular theory" for the polynomial representations of these groups, we assert that $\mathrm{GL}(m|n, A)$, $m|n$ fixed, A a superalgebra over F, F varying over some class $\mathfrak{F}$ of commutative rings, is defined over $\mathbb{Z}$. The definition we propose is as follows.

Denote by $\mathfrak{F}$ the class of all infinite fields with $2 \neq 0$. Suppose given is a family $\{A_F, \Gamma_{A_F}, D_{A_F}\}$, where for each $F \in \mathfrak{F}$, A_F is an F-superalgebra, Γ_{A_F} is a semi supergroup, and D_{A_F} is an A_F-sub super coalgebra of $F_{A_F}(A_F^{\Gamma_{A_F}})$. Suppose also the following conditions are satisfied.

$\mathbb{Z}$0.1 The $\mathbb{Q}$-superalgebra $A_\mathbb{Q}$ contains a $\mathbb{Z}$-form $A_\mathbb{Z}$, thus $A_\mathbb{Z}$ is a superalgebra, and a lattice in $A_\mathbb{Q}$, which means that $A_\mathbb{Z} = \sum_\nu \mathbb{Z}a_\nu$ for some $\mathbb{Q}$-basis $\{a_\nu\}$ of $A_\mathbb{Q}$.

$\mathbb{Z}$0.2 For each $F \in \mathfrak{F}$ there is an F-superalgebra isomorphism $\alpha_F : A_\mathbb{Z} \otimes F \to A_F$ (here $\otimes$ means $\otimes_\mathbb{Z}$, and $A_\mathbb{Z} \otimes F$ is made into an F-superalgebra by extension of scalars).

$\mathbb{Z}$1. The $\mathbb{Q}$-superalgebra $D_{A_\mathbb{Q}} = (D_{A_\mathbb{Q}}, \Delta_\mathbb{Q}, \varepsilon_\mathbb{Q})$ contains a $\mathbb{Z}$-form $D_{A_\mathbb{Z}}$, i.e.,

(a) $D_{A_\mathbb{Z}}$ is a lattice in $D_{A_\mathbb{Q}}$, thus $D_{A_\mathbb{Z}} = \sum_\nu \mathbb{Z}d_\nu$ for some $\mathbb{Q}$-basis $\{d_\nu\}$ of $D_{A_\mathbb{Q}}$, and

(b) $\Delta_\mathbb{Q}(D_{A_\mathbb{Z}}) \subset D_{A_\mathbb{Z}} \otimes D_{A_\mathbb{Z}}$ and $\varepsilon_\mathbb{Q}(D_{A_\mathbb{Z}}) \subset \mathbb{Z}$.

$\mathbb{Z}$2. For each $F \in \mathfrak{F}$ there is an F-supercoalgebra isomorphism $\beta_F : D_{A_\mathbb{Z}} \otimes F \to D_{A_F}$, $\otimes$ means here $\otimes_\mathbb{Z}$, and $D_{A_\mathbb{Z}} \otimes F$ is made into an F-supercoalgebra by extension of scalars.

The example in which we are interested here is that of $\Gamma_A = \mathrm{GL}(m|n, A)$, where $A = A_F = A_\mathbb{Z} \otimes_\mathbb{Z} F$, and $A_\mathbb{Z}$ is a superalgebra over $\mathbb{Z}$. For D_{A_F} we take either $C_A(m|n)$ or $C_A(m|n, r)$ for some $r \geq 0$. Then the family $\{\Gamma_{A_F}, D_{A_F}\}$ is defined over $\mathbb{Z}$.

Essential for a modular representation theory of any family $\{A_F, \Gamma_{A_F}, D_{A_F}\}$ which is defined over $\mathbb{Z}$ is the process of modular reduction. Put $\mathfrak{M}_{A_F}$ for the category $\mathrm{mod}_{D_{A_F}}(A_F\Gamma_{A_F})$, for any $F \in \mathfrak{F}$. An object in $\mathfrak{M}_{A_Q}$ is a finite rank free $A_\mathbb{Q}$-module on which $\Gamma_{A_\mathbb{Q}}$ acts. If $\{v_{b,\mathbb{Q}};\, b \in B\}$ is a $\mathbb{Q}$-basis of the free $A_\mathbb{Q}$-module $M_{A_\mathbb{Q}}$, then we have equations as in (3.1)

$$gv_{b,\mathbb{Q}} = \sum_{a \in B} v_{a,\mathbb{Q}} r^\mathbb{Q}_{a,b}(g), \qquad g \in \Gamma_{A_\mathbb{Q}}, \quad b \in B. \tag{3.6}$$

The functions $r^\mathbb{Q}_{a,b}$ lie in $D_{A_\mathbb{Q}}$ and satisfy equations as in (3.2).

A subset $M_{A_\mathbb{Z}}$ of $M_{A_\mathbb{Q}}$ is called a *$\mathbb{Z}$-form*, or an *admissible lattice*, of $M_{A_\mathbb{Q}}$, if

(a) $M_{A_\mathbb{Z}}$ is a lattice in $M_{A_\mathbb{Q}}$, namely $M_{A_\mathbb{Z}} = \sum_{b \in B} \mathbb{Z}v_{b,\mathbb{Q}}$ for some $\mathbb{Q}$-basis $\{v_{b,\mathbb{Q}}\}$ of $M_{A_\mathbb{Q}}$, and

(b) all the coefficient functions $r^\mathbb{Q}_{a,b}$ in this basis lie in $D_{A_\mathbb{Z}}$.

Another way of expressing condition (b) is to convert $M_{A_\mathbb{Q}}$ to a $D_{A_\mathbb{Q}}$-super comodule by means of the map $\gamma_\mathbb{Q} : M_{A_\mathbb{Q}} \to M_{A_\mathbb{Q}} \otimes D_{A_\mathbb{Q}}$, using equations as in (3.3). Then (b) is equivalent to

(b$'$) $\gamma_\mathbb{Q}(M_{A_\mathbb{Z}}) \subset M_{A_\mathbb{Z}} \otimes D_{A_\mathbb{Z}}$.

Given $F \in \mathfrak{F}$ we can make the A_F-module $M_{A_F} = M_{A_\mathbb{Z}} \otimes F$ (here $\otimes$ means $\otimes_\mathbb{Z}$) into an object of $\mathfrak{M}_{A_F}$ as follows. Using the F-super coalgebra isomorphism $\beta_F : D_{A_\mathbb{Z}} \otimes F \to D_{A_F}$ of $\mathbb{Z}$2, define $r^F_{a,b} = \beta_F(r^\mathbb{Q}_{a,b} \otimes 1_F) \in M_{A_F}$. These $r^F_{a,b}$ satisfy equations of the form (3.2). So we may define action of Γ_{A_F} on M_{A_F} by

$$gv_{b,F} = \sum_{a \in B} v_{a,F} r^F_{a,b}(g), \qquad g \in \Gamma_{A_F}, \quad b \in B.$$

Here $v_{b,F} = v_{b,\mathbb{Q}} \otimes 1_F$ for $b \in B$. The process of converting M_{A_Q}, via the $\mathbb{Z}$-form $M_{A_\mathbb{Z}}$, into M_{A_F}, is called *modular reduction*. In the non-super case, a general theorem ([Ser68, Lemme 2, p. 43], [Gr76, (2.2d), p. 159]) asserts that each $M_\mathbb{Q} \in \mathfrak{M}_\mathbb{Q}$ has at least one $\mathbb{Z}$-form $M_\mathbb{Q}$; different $\mathbb{Z}$-forms $M_\mathbb{Z}$, $M'_\mathbb{Z}$, … of the same $M_\mathbb{Q}$ may give non-isomorphic $M_F = M_\mathbb{Z} \otimes F$, $M'_F = M'_\mathbb{Z} \otimes F$, … in $\mathfrak{M}_F$, but another general theorem asserts that all these modules M_F, M'_F, … have the same composition factor multiplicities. One defines then composition numbers. It would be interesting to check that these assertions extend to the super case.

4. Super Schur algebra

4.A. Coefficient functions

Let F be an infinite field with $2 \neq 0$, A a superalgebra over F, thus it is $\mathbb{Z}/2$-graded, $= A_0 \oplus A_1$, with parity function p: $p(a) = 0$ if $a \in A_0$, $p(a) = 1$ if $a \in A_1$, on the homogeneous elements. Write E for the functor $A \mapsto E_A = A^{m|n}$, so E_A is the free A-module spanned by $e_1, \dots, e_m, e_{m+1}, \dots, e_{m+n}$, with $p(e_i) = 0$ if $1 \le i \le m$, $= 1$ if $m < i \le m+n$. Then

$$E_A = \left\{ \begin{pmatrix} a_1 \\ * \\ * \\ * \\ a_{m+n} \end{pmatrix} \right\} = E_{A,0} \oplus E_{A,1},$$

where

$$E_{A,0} = \left\{ \sum_i a_i e_i;\ a_i \in A_0\,(1 \le i \le m),\ a_i \in A_1\,(m < i \le m+n) \right\},$$

$$E_{A,1} = \left\{ \sum_i a_i e_i;\ a_i \in A_1\,(1 \le i \le m),\ a_i \in A_0\,(m < i \le m+n) \right\}.$$

Put $\mathrm{GL}(m|n) = \mathrm{Aut}\,E$ for the functor whose set of A-points is the group $\Gamma_A = \mathrm{GL}(m|n, A) = \mathrm{Aut}_A\,A^{m|n}$. These are automorphisms of degree 0, of graded A-modules, presented in the standard basis $e_1, \dots$ by a matrix $\left(\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right)$, where the entries of A', D' are homogeneous of parity 0, thus even, thus in A_0, and those of B', C' are odd. Such $g = \left(\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right)$ maps $E_{A,0}$ and $E_{A,1}$ to themselves. Thus g maps to itself the A_0-module

$$A_0^m \times A_1^n = \left\{ \sum_i a_i e_i;\ a_i \in A_0\,(1 \le i \le m),\ \ a_i \in A_1\,(m < i \le m+n) \right\}.$$

Write $A^\Gamma = A^{\Gamma_A}$ for the A-superalgebra of maps $\Gamma_A = \mathrm{GL}(m|n, A) \to A$. For each $1 \le \mu, \nu \le m+n$, denote the *coefficient function*, which maps $g \in \Gamma_A$ to its (μ,ν)-coefficient $g_{\mu,\nu}$, by $c_{\mu,\nu}$. It lies in A^Γ. Denote by $C_A = C_A(m|n)$ the super F-subalgebra of A^Γ generated by the $c_{\mu,\nu}$ $(1 \le \nu, \mu \le m+n)$. The elements of C_A will be called the *polynomial functions* on Γ_A. The $c_{\mu,\nu}$ are algebraically independent, as F is infinite. This is actually the only use we make of the assumption that F is infinite. It suffices to work with big enough fields. So $C = C_A$ is the superalgebra of all polynomials over F in the $(m+n)^2$ indeterminates $c_{\mu,\nu}$. The parity of $c_{\mu,\nu}$ is 0 if it is a coefficient of A', D', and it is 1 if it is a coefficient of B', C'. The coefficients with parity $p = 1$ anti-commute:

$$c_{\mu_1,\nu_1} c_{\mu_2,\nu_2} = (-1)^{p(c_{\mu_1,\nu_1})p(c_{\mu_2,\nu_2})} c_{\mu_2,\nu_2} c_{\mu_1,\nu_1}.$$

The degree r coefficient superspace, denoted $C_A(m|n,r)$, is the sub superspace of $C_A = C_A(m|n)$ consisting of all polynomials over A in the coefficient functions $c_{\mu,\nu}$, homogeneous – as polynomials – of degree r. It has degree $\binom{(m+n)^2+r-1}{r}$ over A: the number of monomials $x_1^{m_1} \dots x_k^{m_k}$ of degree r can be computed on writing $\bullet \cdots \bullet \,|\, \bullet \cdots \bullet \,|\, \dots \,|\, \bullet \cdots \bullet$, where we put m_1 bullets, then a separator, the first box is filled with x_1's, the 2nd with x_2's, etc. There are $k-1$ separators and $r+k-1$ bullets and separators, thus $\binom{r+k-1}{k-1} = \binom{k+r-1}{r}$ monomials, and we have $k = (m+n)^2$ variables. In particular $C_A(m|n, 0) = A \cdot 1_{C_A}$, where 1_{C_A} is the constant function, which maps $g \in \Gamma_A$ to 1_A for each g in Γ_A. The F-superalgebra C_A has the standard grading $C_A = C_A(m|n) = \oplus_{r \ge 0} C_A(m|n,r)$.

The symmetric group S_r acts on the set $I(m|n,r) = \{i = (i_1, \dots, i_r);\ 1 \le i_j \le m+n\}$ on the right: $i\sigma = (i_{\sigma(1)}, \dots, i_{\sigma(r)})$. It acts on the set $I(m|n,r) \times I(m|n,r)$ by $(i,j)\sigma = (i\sigma, j\sigma)$. Write $i \sim j$

if $j = i\sigma$ for some $\sigma \in S_r$, i.e., i and j are in the same S_r-orbit. Also write $(i,j) \sim (k,l)$ if $k = i\sigma$, $l = j\sigma$, for some $\sigma \in S_r$.

The superspace $C_A(m|n,r)$ is spanned as an A-space by the monomials

$$c_{i,j} = c_{i_1,j_1} c_{i_2,j_2} \dots c_{i_r,j_r},$$

for all i, $j \in I(m|n,r)$. The pair (i,j) is not uniquely determined by the monomials $c_{i,j}$. We have $c_{i,j} = \pm c_{k,l}$ if and only if $(i,j) \sim (k,l)$. The A-superspace $C_A(m|n,r)$ has as an A-basis the set of distinct monomials $c_{i,j}$, up to a sign, and these are in bijective correspondence with the S_r-orbits of $I(m|n,r) \times I(m|n,r)$. The number of these orbits is $\binom{(m+n)^2+r-1}{r}$.

4.B. Comultiplication

The comultiplication

$$\Delta : A^\Gamma \to A^{\Gamma\times\Gamma}, \ \ \Delta f(a,b) = f(ab), \ \text{ and counit } \ \varepsilon : A^\Gamma \to A, \ \ \varepsilon(f) = f(1),$$

act on the coefficient functions $c_{\mu,\nu}$ $(1 \le \mu,\, \nu \le m+n)$ as follows:

$$\Delta c_{\mu,\nu} = \sum_{1\le\lambda\le m+n} c_{\mu,\lambda} \otimes c_{\lambda,\nu}, \qquad \varepsilon(c_{\mu,nu}) = \delta_{\mu,\nu}.$$

Indeed,

$$\Delta c_{i,j}(g,h) = c_{i,j}(gh) = \sum_k c_{i,k}(g) c_{k,j}(h) = \sum_k (c_{i,k} \otimes c_{k,j})(g,h), \text{ and } \varepsilon(c_{i,j}) = c_{i,j}(I) = \delta_{i,j}.$$

Both Δ and ε are multiplicative. Hence for any multi indices p, $q \in I(m|n,r)$ (of length $r \ge 1$) we have

$$\Delta(c_{p,q}) = \sum_{s\in I(m|n,r)} c_{p,s} \otimes c_{s,q}, \qquad \varepsilon(c_{p,q}) = \delta_{p,q}.$$

These formulae show that $C_A(m|n)$ is a super sub-co-algebra, hence also a super sub-bi-algebra, of $F_A(A^\Gamma)$, and that each $C_A(m|n,r)$ is a super sub-co-algebra of $C_A(m|n)$; for $r = 0$ this follows from $\Delta 1_C = 1_C \otimes 1_C$, $C = C_A(m|n)$.

Write $\mathfrak{M}_A(m|n)$, $\mathfrak{M}_A(m|n,r)$, for the categories $\mathrm{mod}_{C_A(m|n)}(A\Gamma_A)$, $\mathrm{mod}_{C_A(m|n,r)}(A\Gamma_A)$. Thus $\mathfrak{M}_A(m|n)$ is the category of finite dimensional left $A\Gamma_A$-modules which afford polynomial representations of $\Gamma_A = \mathrm{GL}(m|n,A)$, and $\mathfrak{M}_A(m|n,r)$ is the subcategory consisting of those affording representations of Γ_A in which all the coefficients are polynomials homogeneous in the coefficient functions $c_{\mu,\nu}$ of degree r.

4.C. Complete super reducibility

It is known that there is no complete reducibility for representations of Lie superalgebras, in contrast with the standard, non-super case, where there is complete reducibility. But we shall consider only a category of representations closed under tensor products, and will see as a result of our super Schur duality that complete reducibility holds in our case.

As an example where indecomposable representation occurs in the super case, consider the Lie superalgebra $\mathrm{gl}(n|n)$. Its adjoint representation has length 3. The maximal submodule (of codimension 1) consists of matrices with zero supertrace. This module is indecomposable. Its socle is the trivial submodule spanned by the Id matrix. This gives an indecomposable representation of $\mathrm{gl}(n|n)$ and of $\mathrm{sl}(n|n)$; $\mathrm{psl}(n|n)$ is simple for $n > 1$.

The only series where all finite-dimensional modules are completely reducible is $\mathrm{osp}(1|2n)$. For other series there is complete reducibility for some central characters (they are called typical). The trivial central character (corresponding to the trivial module) is always atypical.

4.D. Super Schur algebra

Let $r \geq 0$ be fixed, define $S_A(m|n,r)$ to be the space dual to the superspace $C_A(m|n,r)$:

$$S_A(m|n,r) = C_A(m|n,r)^* = \mathrm{Hom}_A(C_A(m|n,r),A).$$

Recall that a basis of the superspace $C_A(m|n,r)$ over A is given by the monomials

$$\{c_{i,j} = c_{i_1,j_1}\dots c_{i_r,j_r};\ i=(i_1,\dots,i_r),\ j=(j_1,\dots,j_r)\in I(m|n,r)\},$$

where the $c_{i_t,j_t} \in A^{\Gamma_A}$. As a free A-module, $S_A(m|n,r)$ has the basis $\{\xi_{i,j};\ i,j\in I(m|n,r)\}$ dual to $\{c_{i,j}\}$. Thus we have

$$\xi_{i,j}(c_{i,j}) = 1; \qquad \xi_{i,j}(c_{p,q}) = 0 \text{ if } (i,j) \nsim (p,q), \quad p,\, q\in I(m|n,r).$$

Since

$$c_{\sigma i,\sigma j} = c(p(c_{i,j}),\sigma)c_{i,j}, \qquad 1 = \xi_{\sigma i,\sigma j}(c_{\sigma i,\sigma j}) = \xi_{\sigma i,\sigma j}(c(p(c_{i,j}),\sigma)c_{i,j}),$$

we deduce that $\xi_{\sigma i,\sigma j}(c_{i,j}) = c(p(c_{i,j}),\sigma)$ and

$$\xi_{i,j}(c_{\sigma^{-1}i,\sigma^{-1}j}) = \xi_{i,j}(c(p(c_{i,j}),\sigma^{-1})c_{i,j}) = c(\sigma p(c_{i,j}),\sigma)$$

or $\xi_{i,j}(c_{\sigma i,\sigma j}) = c(p(c_{i,j}),\sigma)$. The sign rule $\xi_{i,j}(c_{\sigma i,\sigma j}) = c(p(c_{i,j}),\sigma)$ and $\xi_{i,j}(c_{p,q}) = 0$ if $(i,j) \nsim (p,q)$ defines $\{\xi_{i,j}\}$ uniquely, thus $\dim_A S_A(m|n,r) = \dim_A C_A(m|n,r) = \binom{(m+n)^2+r-1}{r}$, where $\dim_A$ indicates the rank of a free module over A.

4.E. Product structure

As $C_A(m|n,r)$ is a super coalgebra, its dual $S_A(m|n,r)$ is an associative super algebra over A. The product $\xi\eta$ of two elements ξ, η of $S_A(m|n,r)$ is defined as follows.

If $c\in C_A(m|n,r)$ and $\Delta(c) = \sum_t c_t\otimes c'_t$ (finite sum; c_t, $c'_t \in C_A(m|n,r)$), then

$$(\xi\eta)(c) = c(\xi\eta) = \Delta c(\xi,\eta) = \sum_t (c_t\otimes c'_t)(\xi,\eta) = \sum_t c_t(\xi)c'_t(\eta) = \sum_t \xi(c_t)\eta(c'_t).$$

The unit element of $S_A(m|n,r)$ is denoted by ε. It is given by $\varepsilon(c) = c(1_{\Gamma_A})$ for all $c\in C_A(m|n,r)$. Applying the last displayed equation to $c = c_{p,q}$ of $C_A(m|n,r)$ we get

$$(\xi\eta)(c_{p,q}) = \sum_{s\in I(m|n,r)} \xi(c_{p,s})\eta(c_{s,q}).$$

For $\xi = \xi_{i,j}$, $\eta = \xi_{k,l}$, basis elements of $S_A(m|n,r)$, we deduce *Multiplication Rule* for $S_A(m|n,r)$.

$$\xi_{i,j}\xi_{k,l} = \sum_{p,q} z(i,j,k,l,p,q)\xi_{p,q},$$

where

$$z(i,j,k,l,p,q) = \sum_s c(p(c_{p,s}),\sigma)c(p(c_{s,q}),\tau);$$

the sum ranges over all $s\in I(m|n,r)$ such that there exist σ, $\tau\in S_r$ with $(i,j) = (\sigma p,\sigma s)$, $(k,l) = (\tau s,\tau q)$.

Noteworthy special cases are: For all i, j, k, $l\in I(m|n,r)$ we have

(i) $\xi_{i,j}\xi_{k,l} = 0$ unless $j\sim k$;

(ii) $\xi_{i,i}\xi_{i,l} = \xi_{i,l} = \xi_{i,l}\xi_{l,l}$.

(i) holds since $\xi_{i,j}\xi_{k,l} \neq 0$ implies that there is s with $j = \sigma s$ and $k = \tau s$ for some σ, $\tau \in S_r$, so $j \sim k$.

For (ii), $\xi_{i,i}\xi_{i,l} = \sum_{p,q}\sum_s c(p(c_{p,s}),\sigma)c(p(c_{s,q}),\tau)\xi_{p,q}$; the sum over s is so that $(i,i) = (\sigma p, \sigma s)$ for some σ, thus $s = p \sim i$, so we take $p = i$; and $(i,l) = (\tau s, \tau q)$ for some τ, which – since s is now i – we take $\tau = 1$, so $q = l$. The signs c are 1 for $\sigma = 1 = \tau$. Hence

(iii) $\xi_{i,i}^2 = \xi_{i,i}$; $\quad \xi_{i,i}\xi_{j,j} = 0$ if $i \nsim j$.

If $(j,j) = (\sigma i, \sigma i)$ then $\xi_{j,j}(c_{i,i}) = \xi_{\sigma i,\sigma i}(c_{i,i}) = c(p(c_{i,i}),\sigma)$ is 1 (since $p(c_{i,i}) = (0,\dots,0)$), hence

(iv) $\xi_{j,j} = \xi_{i,i}$ if $i \nsim j$.

The distinct $\xi_{i,i}$ form a set of mutually orthogonal idempotents. Their sum is the unit element ε of $S_A(m|n,r)$:

$$\varepsilon = \sum_i \xi_{i,i},$$

sum over a set of representatives of the S_r-orbits of $I(m|n,r)$. Indeed, $\varepsilon(c_{p,q}) = \delta_{p,q}$, and $\xi_{i,i}(c_{p,q}) \neq 0$ implies $(p,q) = (\sigma i, \sigma i)$ and $\xi_{i,i}(c_{\sigma i,\sigma i}) = 1$, where the S_r-orbit of i in $I(m|n,r)$ is uniquely determined by (p,q).

To construct a modular theory for GL$(m|n)$ it is important to know that for a fixed triple m, n, r, the family of superalgebras $S_A(m|n,r)$ is defined over $\mathbb{Z}$ in the following sense. Let us use the superscript A to denote the basis elements $\xi_{i,j}^A$ of $S_A(m|n,r)$. It is clear from the multiplication rule that the $\mathbb{Z}$-submodule of $S_A(m|n,r)$, that is spanned by the $\xi_{i,j}^{A_\mathbb{Q}}$ $(i,\, j \in I(m|n,r))$, is multiplicatively closed, so it is a $\mathbb{Z}$-order in $S_{A_\mathbb{Q}}(m|n,r)$. Further, for any field F there is an isomorphism of F-superalgebras $S_\mathbb{Z}(m|n,r) \otimes_\mathbb{Z} A_F \simeq S_{A_F}(m|n,r)$, $A_F = A_\mathbb{Z} \otimes_\mathbb{Z} F$, that takes $\xi_{i,j}^{A_\mathbb{Q}} \otimes 1_F \mapsto \xi_{i,j}^{A_F}$.

4.F. Evaluation map

For each g in Γ_A define e_g in $S_A(m|n,r)$ by $e_g(c) = c(g)$ for all $c \in C_A(m|n,r)$. For all g, g' in Γ_A we have $e_g e_{g'} = e_{gg'}$, since

$$e_{gg'}(c) = c(gg') = \Delta c(g,g') = \sum_t c_t(g)c_t'(g') = \sum_t e_g(c_t)e_{g'}(c_t') = e_g e_{g'}(c),$$

the last equality follows from the first displayed formula in 4.E. Also $e_1 = \varepsilon$ by definition of ε (and e_1). Extend the map $g \mapsto e_g$ linearly to get an evaluation map $e = e_A : A\Gamma_A \to S_A(m|n,r)$, which is a morphism of F-superalgebras.

Any function $f \in A^{\Gamma_A}$ has a unique extension to an A-linear map $f : A\Gamma_A \to A$. With this convention, the image under e of an element $\kappa = \sum_{g\in\Gamma_A}\kappa_g g \in A\Gamma_A$ $(\kappa_g \in A)$, is evaluation at κ, namely

$$e(\kappa) : c \mapsto c(\kappa), \qquad c \in C_A(m|n,r).$$

Proposition 4.1. *(i) The map $e = e_A : A\Gamma_A \to S_A(m|n,r)$ is surjective.*
(ii) Put $Y = \ker(e)$. Let f be an element of A^Γ. Then $f \in C_A(m|n,r)$ iff $f(Y) = 0$.

Proof. (*i*) Suppose Im(e) is a proper subset of $S_A(m|n,r) = C_A(m|n,r)^*$. Then there exists some $c \in C_A(m|n,r)$, $c \neq 0$, with $c(g) = e_g(c) = 0$ for all $g \in \Gamma_A$, but such c is 0 (and $\neq 0$).
(*ii*) If $f \in C_A(m|n,r)$ and $\kappa \in Y$, then $e(\kappa) = 0$ ($e(\kappa) : c \mapsto c(\kappa)$, thus $c(\kappa) = 0$ for all c in $C_A(m|n,r)$). Hence $f(\kappa) = 0$, so $f(Y) = 0$.

Conversely, let $f \in A^\Gamma$ satisfy $f(Y) = 0$. By (i) there is an exact sequence

$$0 \to Y \to A\Gamma_A \xrightarrow{e} S_A(m|n,r) \to 0.$$

Hence there is $y \in S_A(m|n,r)^*$ with $y(e(\kappa)) = f(\kappa)$ for all $\kappa \in A\Gamma_A$. By the natural isomorphism $S_A(m|n,r)^* \simeq C_A(m|n,r)$, there exists $c \in C_A(m|n,r)$ with $y(\xi) = \xi(c)$ for all $\xi \in S_A(m|n,r)$. Put $\xi = e(\kappa)$. Then $f(\kappa) = y(e(\kappa)) = (e(\kappa))(c) = c(\kappa)$ for all $\kappa \in A\Gamma_A$. Hence $f = c$ lies in $C_A(m|n,r)$.

Proposition 4.2. *Let $M_A \in \mathrm{mod}(A\Gamma_A)$. Then $M_A \in \mathfrak{M}_A(m|n,r)$ iff $YM_A = 0$.*

Proof. Let $\{v_b\}$ be a basis of the free A-module M_A. Let $\{r_{a,b}\}$ be the invariant matrix defined by the action of $A\Gamma_A$ on this basis (see (3.1)). Then $YM_A = 0$ iff $r_{a,b}(Y) = 0$ for all a, $b \in B$. By the last proposition this is equivalent to saying that $r_{a,b} \in C_A(m|n,r)$ for all a, b, namely that $\mathrm{cf}(M_A) \subset C_A(m|n,r)$. But this means that M_A lies in $\mathfrak{M}_A(m|n,r)$.

These two propositions show that the categories $\mathfrak{M}_A(m|n,r)$ and $\mathrm{mod}(S_A(m|n,r))$ are equivalent, and in a very elementary way: an object M_A in either category may be transformed into an object of the other, using the rule

$$\kappa v = e(\kappa)v, \qquad \text{all } \kappa \in A\Gamma_A, \quad v \in M_A, \tag{4.7}$$

to relate the action on M_A of the two algebras $A\Gamma_A$ and $S_A(m|n,r)$ $(= e(A\Gamma_A))$. Both actions determine the same algebra of linear transformations on M_A. Hence the concepts of submodule, module homomorphism, etc., coincide in the two categories. If the action of Γ_A on a free basis $\{v_b\}$ of M_A is given by equations (3.5), then the action of $S_A(m|n,r)$ is given by

$$\xi v_b = \sum_{a \in B} v_a \xi(r_{a,b}), \qquad \text{all } \xi \in S_A(m|n,r), \quad b \in B.$$

Indeed, apply the last displayed formula with $\kappa = g$, $v = v_b$. By linearity that formula holds for all $\kappa \in A\Gamma_A$ and $v \in M_A$.

4.G. Modular theory

We next describe the characteristic modular reduction, or decomposition, process.

Let $C_{\mathbb{Z}}(m|n)$ and $C_{\mathbb{Z}}(m|n,r)$ be the $\mathbb{Z}$-modules of $C_{A_{\mathbb{Q}}}(m|n)$ and $C_{A_{\mathbb{Q}}}(m|n,r)$, consisting of those polynomials in the $c_{\mu,\nu}$ whose coefficients all lie in $\mathbb{Z}$. These are $\mathbb{Z}$-forms of $C_{A_{\mathbb{Q}}}(m|n)$ and $C_{A_{\mathbb{Q}}}(m|n,r)$. Indeed, $C_{\mathbb{Z}}(m|n,r)$ is the $\mathbb{Z}$-span of the $A_{\mathbb{Q}}$-basis $\{c_{i,j}^{A_{\mathbb{Q}}}\}$ of $C_{A_{\mathbb{Q}}}(m|n,r)$, and we have $\Delta C_{\mathbb{Z}}(m|n,r) \subset C_{\mathbb{Z}}(m|n,r) \otimes C_{\mathbb{Z}}(m|n,r)$ and $\varepsilon(C_{\mathbb{Z}}(m|n,r)) \subset \mathbb{Z}$ (see the 3rd displayed formula in 4.B.). For any infinite field F, and F-superalgebra A_F, there is an F-super coalgebra isomorphism $C_{\mathbb{Z}}(m|n,r) \otimes_{\mathbb{Z}} A_F \simeq C_{A_F}(m|n,r)$, which takes $c_{i,j}^{A_{\mathbb{Q}}} \otimes 1_F \mapsto c_{i,j}^{A_F}$ for all i, $j \in I(m|n,r)$. The $\mathbb{Z}$-order $S_{\mathbb{Z}}(m|n,r)$ of the end of subsection 4.E. is the set of all $\xi \in S_{A_{\mathbb{Q}}}(m|n,r)$ with $\xi(C_{\mathbb{Z}}(m|n,r)) \subset \mathbb{Z}$

Let $M_{A_{\mathbb{Q}}}$ be an object in $\mathfrak{M}_{A_{\mathbb{Q}}}(m|n,r)$. It can be regarded as a module under $S_{A_{\mathbb{Q}}}(m|n,r)$. By a *$\mathbb{Z}$-form* of $M_{A_{\mathbb{Q}}}$ we mean a subset $M_{\mathbb{Z}}$ which

(i) is the $\mathbb{Z}$-span of some $A_{\mathbb{Q}}$-basis $\{v_b^{A_{\mathbb{Q}}}; b \in B\}$ of $M_{A_{\mathbb{Q}}}$, and

(ii) is closed under the action of $S_{\mathbb{Z}}(m|n,r)$.

Let $R_{\mathbb{Q}} = (r_{a,b})$ be the invariant matrix defined by $\{v_b\}$ (3.1). Then condition (ii) just says that all the $r_{a,b}$ lie in $C_{\mathbb{Z}}(m|n,r)$. Another formulation of (ii) is that if $(M_{\mathbb{Q}}, \tau)$ is the $C_{\mathbb{Z}}(m|n,r)$-comodule determined by $M_{\mathbb{Q}}$, then $\tau(M_{\mathbb{Z}}) \subset M_{\mathbb{Z}} \otimes_{\mathbb{Z}} C_{\mathbb{Z}}(m|n,r)$.

Let F be an infinite field with $2 \neq 0$. It is clear that the A_F-module $M_{A_F} = M_{\mathbb{Z}} \otimes_{\mathbb{Z}} A_F$ can be regarded as a left module for $S_{A_F}(m|n,r) \simeq S_{\mathbb{Z}}(m|n,r) \otimes_{\mathbb{Z}} A_F$, hence as an $A_F\Gamma_{A_F}$-module in $\mathfrak{M}_{A_F}(m|n,r)$. The transition from $M_{A_{\mathbb{Q}}}$ to M_{A_F} can be expressed in terms of invariant matrices. The invariant matrix R_{A_F} defined by the A_F-basis $\{v_b \otimes 1_{A_F}\}$ of M_{A_F} is $(r_{a,b} \otimes 1_{A_F})$, where $(r_{a,b}) = R_{A_{\mathbb{Q}}}$ is the invariant matrix defined by the basis $\{v_b\}$ of $M_{A_{\mathbb{Q}}}$. In the case where F has finite characteristic $p \neq 2$, this amounts to reducing mod p the coefficients of $R_{A_{\mathbb{Q}}}$.

5. Super Schur duality

5.A. The super module $E_A^{\otimes r}$

Fix an infinite field with $2 \neq 0$, m, $n > 0$, and put $\Gamma_A = \mathrm{GL}(m|n, A)$ for a super algebra A over F. Let $E_A = A^{m|n} = Ae_1 \oplus \cdots \oplus Ae_m \oplus Ae_{m+1} \oplus \cdots \oplus Ae_{m+n}$ be the free rank $m|n$ A-module, with free basis $\{e_\nu;\ 1 \leq \nu \leq m+n\}$ having parities $p(e_\nu) = 0$ $(1 \leq \nu \leq m)$, $p(e_\nu) = 1$ $(m < \nu \leq m+n)$. In this standard basis, Γ_A acts naturally:

$$ge_\nu = \sum_{1 \leq \mu \leq m+n} e_\mu g_{\mu,\nu} = \sum_{1 \leq \mu \leq m+n} e_\mu c_{\mu,\nu}(g).$$

The corresponding invariant matrix is $C = (c_{\mu,\nu})$. So we see that the $A\Gamma_A$-module E_A is an object of $C_A(m|n, 1)$.

For $r \geq 1$, Γ_A acts on the r-fold tensor product $E_A^{\otimes r} = E_A \otimes \cdots \otimes E_A$ (here $\otimes = \otimes_A$) diagonally. The free A-module $E_A^{\otimes r}$ has A-basis

$$\{e_i = e_{i_1} \otimes \cdots \otimes e_{i_r};\quad i \in I(m|n, r)\}.$$

Relative to this the action of Γ_A is given by

$$ge_j = ge_{j_1} \otimes \cdots \otimes ge_{j_r} = \sum_i e_{i_1} g_{i_1,j_1} \otimes \cdots \otimes \sum_{i_r} e_{i_r} g_{i_r,j_r} = \sum_i e_i c_{i,j}(g) \iota(g; i, j);$$

on the right $i = (i_1, \ldots, i_r) \in I(m|n, r)$ and $c_{i,j}(g) = c_{i_1,j_1}(g) \ldots c_{i_r,j_r}(g) = g_{i_1,j_1} \ldots g_{i_r,j_r}$, for all $g \in \Gamma_A$, $j = (j_1, \ldots, j_r) \in I(m|n, r)$. Further, $\iota(g; i, j) \in \{\pm 1\}$ is obtained according to the sign rule:

$$ge_{j_1} \otimes ge_{j_2} = \sum_{i_1} e_{i_1} g_{i_1,j_1} \otimes \sum_{i_2} e_{i_2} g_{i_2,j_2} = \sum_{i_1,i_2} e_{i_1} \otimes g_{i_1,j_1} e_{i_2} g_{i_2,j_2}$$

$$= \sum_{i_1,i_2} e_{i_1} \otimes e_{i_2} (-1)^{p(g_{i_1,j_1}) p(i_2)} g_{i_1,j_1} g_{i_2,j_2}.$$

Note that $p(e_{i_t} g_{i_t,j_t}) = p(e_{j_t})$, namely $p(g_{i_t,j_t}) = p(e_{i_t}) + p(e_{j_t})$ for all t $(1 \leq t \leq r)$. Thus $\iota(g; i, j) = (-1)^*$ where $*$ equals

$$p(g_{i_1,j_1})p(i_2) + (p(g_{i_1,j_1}) + p(g_{i_2,j_2}))p(i_3) + (p(g_{i_1,j_1}) + p(g_{i_2,j_2}) + p(g_{i_3,j_3}))p(i_4) + \ldots$$

$$= p(g_{i_1,j_1})(p(i_2) + p(i_3) + \cdots + p(i_r)) + \cdots + p(g_{i_t,j_t})(p(i_{t+1}) + \cdots + p(i_r)) + \ldots,$$

thus

$$\iota(g; i, j) = \prod_{1 \leq t < r} (-1)^{(p(e_{i_t}) + p(e_{j_t}))P(i_t)}, \qquad P(i_t) = p(i_{t+1}) + \cdots + p(i_r).$$

In particular $\iota(g; i, j)$ is independent of g and will be denoted $\iota(i, j)$.

The corresponding invariant matrix is $(c_{i,j}\iota(i, j))$, $c_{i,j}(g) = g_{i,j}$, $\iota(i, j) \in \{\pm 1\}$ is independent of $g \in \Gamma_A$. This matrix is equal to some r-fold product $C \times' C \times' \cdots \times' C$, taking into account the signs $\iota(i, j)$ that appear here. Hence $E_A^{\otimes r} \in \mathfrak{M}_A(m|n, r)$, its coefficients are polynomials in the c_{i_t,j_t} of degree r. Further, $E_A^{\otimes r}$ can be regarded as an $S_A(m|n, r)$-module by the rule (see (4.7))

$$\xi e_j = \sum_i e_i \xi(c_{i,j}) \iota(i, j), \qquad \xi \in S_A(m|n, r), \quad i,\ j \in I(m|n, r).$$

Recall that the group S_r acts on $E_A^{\otimes r}$ by $e_j * \sigma = c(p(e_j), \sigma) e_{\sigma j}$, $\sigma \in S_r$, thus

$$v_1 \otimes \cdots \otimes v_r * \sigma = c(p(v), \sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(r)}, \qquad p(v) = (p(v_1), \ldots, p(v_r)).$$

This action commutes with the diagonal action of $g \in \Gamma_A$ (and thus of $A_0\Gamma_A$)

$$g(v_1 \otimes \cdots \otimes v_r) = gv_1 \otimes \cdots \otimes gv_r,$$

as

$$(g(v_1 \otimes \cdots \otimes v_r)) * \sigma = (gv_1 \otimes \cdots \otimes gv_r) * \sigma = c(p(gv),\sigma) gv_{\sigma(1)} \otimes \cdots \otimes gv_{\sigma(r)}$$

and

$$g(v_1 \otimes \cdots \otimes v_r * \sigma) = c(p(v),\sigma) g(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(r)}) = c(p(v),\sigma) gv_{\sigma(1)} \otimes \cdots \otimes gv_{\sigma(r)},$$

but $c(p(gv),\sigma) = c(p(v),\sigma)$ for all $v \in E_A^{\otimes r}$ and $\sigma \in S_r$ as $p(gv) = p(g)(1,\dots,1) + p(v)$, and $p(g) = 0$.

5.B. Super Schur duality

Since the action of Γ_A and S_r on $E_A^{\otimes r}$ commute, it follows that the action of $S_A(m|n,r)$ and S_r on $E_A^{\otimes r}$ commute: $(\xi x) * \sigma = \xi(x * \sigma)$. We then define a homomorphism $\psi : S_A(m|n,r) \to \mathrm{End}_A(E_A^{\otimes r})$ by the action of $S_A(m|n,r)$ on the module $E_A^{\otimes r}$:

$$\psi(\xi)e_j = \sum_i e_i \xi(c_{i,j})\iota(i,j).$$

In matrix form, with $i = (i_1,\dots,i_r)$, $j = (j_1,\dots,j_r)$, $1 \le i_t,\ j_t \le m+n$:

$$\psi(\xi) = (\xi(c_{i,j})\iota(i,j)) \in M((m+n)^r \times (m+n)^r, A).$$

Theorem 5.1. (**Super Schur Duality**) *We have* $\ker\psi = 0$, $\mathrm{Im}\,\psi = \mathrm{End}_{AS_r}(E_A^{\otimes r})$, *namely* ψ *defines an isomorphism* $\psi : S_A(m|n,r) \xrightarrow{\sim} \mathrm{End}_{AS_r}(E_A^{\otimes r})$, $E_A = A^{m|n}$.

Proof. Each $\theta \in \mathrm{End}_A E_A^{\otimes r}$ is represented by a matrix, say $(T_{i,j})$, in $M((m+n)^r \times (m+n)^r, A)$, in the basis $\{e_i\}$ of the free A-module $E_A^{\otimes r}$, namely we have

$$\theta e_j = \sum_i e_i T_{i,j}, \qquad T_{i,j} \in A, \quad i,\, j \in I(m|n,r).$$

We have

$$c(p(e_j),\sigma)\theta(e_{\sigma j}) = \theta(e_j * \sigma) = (\theta e_j) * \sigma = \left(\sum_i e_i T_{i,j}\right) * \sigma$$

$$= c(p(e_j),\sigma) \sum_i e_{\sigma i} T_{i,j},$$

where the 2nd $=$ follows from $g(e_j * \sigma) = (ge_j) * \sigma$ for all $g \in \Gamma_A$, and for the last $=$ note that $p(e_i T_{i,j}) = p(e_j)$, or $p(T_{i,j}) = p(e_i) + p(e_j)$. Replacing j by $\sigma^{-1}j$, i by $\sigma^{-1}i$, we deduce that

$$\theta(e_j) = \sum_i e_i T_{\sigma^{-1}i,\sigma^{-1}j}.$$

Hence $T_{\sigma i,\sigma j} = T_{i,j}$ for all $\sigma \in S_r$, $i,\, j \in I(m|n,r)$. This means that $\mathrm{End}_{AS_r}(E_A^{\otimes r})$ has a free A-basis in one-to-one correspondence with the set Ω of all S_r-orbits in $I \times I$, $I = I(m|n,r)$. If ω is such an S_r-orbit on $I \times I$, the corresponding basis element θ_ω is that $\theta \in \mathrm{End}_A(E_A^{\otimes r})$ whose matrix has $T_{i,j} = 1$ if $(i,j) \in \omega$ and $T_{i,j} = 0$ if $(i,j) \notin \omega$.

Now recall that for $(p,q) \in I \times I$ we have $\xi_{p,q} \in S_A(m|n,r)$ defined by $\xi_{p,q}(c_{i,j}) = c(p(c_{p,q}),\sigma)$ if $(i,j) = (\sigma p, \sigma q)$, and $\xi_{p,q}(c_{i,j}) = 0$ if not. Then

$$\psi(\xi_{p,q})e_j = \sum_i e_i \xi_{p,q}(c_{i,j})\iota(i,j).$$

The nonzero terms in the sum over i must satisfy $(i,j) = (\sigma p, \sigma q)$ for some $\sigma \in S_r$. So take $j = \sigma q$, $i = \sigma p$, to get

$$\psi(\xi_{p,q})e_{\sigma q} = e_{\sigma p}\xi_{p,q}(c_{\sigma p,\sigma q})\iota(\sigma p,\sigma q) = e_{\sigma p} \cdot c(p(c_{p,q}),\sigma) \cdot \iota(\sigma p,\sigma q).$$

Thus $T = (T_{i,j})$ representing $\psi(\xi_{p,q})$ satisfies $T_{i,j} = \xi_{p,q}(c_{i,j})\iota(i,j) = T_{\sigma i,\sigma j}$, that is,

$$c(p(c_{p,q}),\sigma)\iota(\sigma p,\sigma q) = \iota(p,q)$$

is independent of $\sigma \in S_r$ ($\iota(p,q)$ on the right is the value of the left side at $\sigma = 1$). We conclude

$$\psi(\iota(p,q)\xi_{p,q})e_{\sigma q} = e_{\sigma p} \qquad \text{for all } \sigma \in S_r.$$

Hence for all $(p,q) \in I \times I$, the basis element $\iota(p,q)\xi_{p,q}$ of $S_A(m|n,r)$ is represented on $E_A^{\otimes r}$ by $\psi(\iota(p,q)\xi_{p,q}) = \theta_\omega$, where ω is the S_r-orbit containing (p,q). Hence ψ defines an isomorphism $S_A(m|n,r) \xrightarrow{\sim} \mathrm{End}_{AS_r}(E_A^{\otimes r})$.

In the course of the proof we proved

Corollary 5.2. *For all p, q in $I(m|n,r)$ and $\sigma \in S_r$ we have $c(p(c_{p,q}),\sigma) = \iota(\sigma p,\sigma q)/\iota(p,q)$.*

Example 5.1. *Let us check directly the relation in the corollary in a simple case. Take $m = n = 1$, $r = 2$, $i = (i_1, i_2)$, i_1, $i_2 \in \{1,2\}$. Then $c_{p,q} = c_{p_1,q_1}c_{p_2,q_2}$, $p(c_{p,q}) = (p(c_{p_1,q_1}), p(c_{p_2,q_2}))$ is defined by $x_{\sigma(1)}x_{\sigma(2)} = c((p(x_1),p(x_2)),\sigma)x_1x_2$. Take $(p_1,q_1) = (1,2) = (p_2,q_2)$, so that $p(c_{p_1,q_1}) = 1 = p(c_{p_2,q_2})$, and $c(p(c_{p,q}),\sigma) = -1$ for $\sigma = (12) \in S_2$. Then $(\sigma(p_1),\sigma(q_1)) = (2,1)$, and $\sigma(p_2) = 2$. So*

$$\iota(\sigma p,\sigma q) = (-1)^*, \quad * = (p(e_{\sigma p_1}) + p(e_{\sigma q_1}))p(e_{\sigma p_2}) = (p(e_2) + p(e_1))p(e_2) = (1+0)1 = 1;$$

$$\iota(p,q) = (-1)^{**}, \quad ** = (p(e_{p_1}) + p(e_{q_1}))p(e_{p_2}) = 0 \quad \text{as } p(e_{p_2}) = 0.$$

Hence $c(p(c_{p,q}),\sigma)$ and $\iota(\sigma p,\sigma q)/\iota(p,q)$ both are equal to -1.

5.C. Semisimplicity

The proof of the super Schur duality shows that $S_A(m|n,r)$ has a faithful matrix representation by the algebra of all $(m+n)^r \times (m+n)^r$ matrices $(T_{i,j})$ satisfying the condition $T_{\sigma i,\sigma j} = T_{i,j}$ for all i, $j \in I(m|n,r)$, $\sigma \in S_r$. The basis elements $\iota(p,q)\xi_{p,q}$ is represented by the matrix having $T_{i,j} = 1$ if $(i,j) = (\sigma p,\sigma q)$ for some $\sigma \in S_r$, and $T_{i,j} = 0$ if not. Note that $\iota(i,i) = 1$ as $p(g_{i,i}) = 0$ for all $i \in I(m|n,r)$. The idempotents $\xi_{i,i}$ are represented by diagonal matrices, and the orthogonal decomposition $\varepsilon = \sum_i \xi_{i,i}$ can be deduced from that.

Corollary 5.3. *If* char F *is zero or $p > r$, then $S_A(m|n,r)$ is semisimple. Hence each $M_A \in \mathfrak{M}_A(m|n,r)$ is completely reducible.*

Proof. Since char F does not divide $|S_r| = r!$, the group algebra AS_r is semisimple. Hence every AS_r-module, in particular $E_A^{\otimes r}$, is completely reducible. The endomorphism algebra of a completely reducible module is semisimple. So by the theorem, $S_A(m|n,r)$ is semisimple. The equivalence of the categories $\mathfrak{M}_A(m|n,r)$ and mod $S_A(m|n,r)$ completes the proof.

The family of modules $(E_A^{\otimes r})$, with fixed r and varying F and A, is defined over $\mathbb{Z}$, in the sense of the following definition, which is a version of the definition of a GL(n)-module, where GL(n) is regarded as an affine group scheme over $\mathbb{Z}$.

Definition 5.1. *Suppose that for each infinite field F with $2 \neq 0$ and an F-superalgebra A we have an $A\Gamma_A$-module $M_A \in \mathfrak{M}_A(m|n,r)$. We say that the family $\{M_A\}$ is* defined over $\mathbb{Z}$ *if there is a $\mathbb{Z}$-form $M_{\mathbb{Z}}$ of M_A, and for each F and A isomorphisms $A_{\mathbb{Q}} \otimes F \simeq A_F$ and $\delta_A : M_{\mathbb{Z}} \otimes A \xrightarrow{\sim} M_A$ in the category $\mathfrak{M}_A(m|n,r)$. More precisely we say the family $\{M_A\}$ is* $\mathbb{Z}$-defined by *$M_{\mathbb{Z}}$ and δ_A.*

Example 5.2. *Take* $M_A = E_A^{\otimes r}$. *The module* $M_{\mathbb{Z}} = \sum_{i \in I(m|n,r)} \mathbb{Z} \cdot e_i$ *is a* $\mathbb{Z}$*-form of* $M_{A_{\mathbb{Q}}}$*; write* $e_{\mu,A}$ *for the basis elements of* $E_A = A^{m|n}$*, and* $e_{i,A} = e_{i_1,A} \otimes \cdots \otimes e_{i_r,A}$ *for the basis elements of* $E_A^{\otimes r}$. *For each* A *the* A*-map* $\delta_A : M_{\mathbb{Z}} \otimes A \to M_A$ *taking* $e_i \otimes 1_A \mapsto e_{i,A}$ *for all* $i \in I(m|n,r)$*, is an isomorphism in* $\mathfrak{M}_A(m|n,r)$*, so* $\{E_A^{\otimes r}\}$ *is defined over* $\mathbb{Z}$.

Definition 5.2. *Suppose* $\{M_A\}$, $\{N_A\}$ *are families of* A*-modules in* $\mathfrak{M}_A(m|n,r)$*, both defined over* $\mathbb{Z}$*, by* $M_{\mathbb{Z}}$ *and* $\{\delta_A\}$*, and* $N_{\mathbb{Z}}$ *and* $\{\eta_A\}$. *Suppose we have for each* A *a morphism* $\theta_A : M_A \to N_A$ *in* $\mathfrak{M}_A(m|n,r)$*, and* $A_{\mathbb{Q}} \otimes F \simeq A_F$. *We say the family* $\{\theta_A\}$ *is* defined over $\mathbb{Z}$ *if* $\theta_{\mathbb{Q}}$ *maps* $M_{\mathbb{Z}}$ *to* $N_{\mathbb{Z}}$*, and for each* A *the following diagram commutes:*

$$\begin{array}{ccc} M_{\mathbb{Z}}\otimes A & \xrightarrow{\theta_{\mathbb{Q}}\otimes 1_A} & N_{\mathbb{Z}}\otimes A \\ \delta_A\downarrow & & \eta_A\downarrow \\ M_A & \xrightarrow{\theta_A} & N_A \end{array}$$

Example 5.3. *Define the* r*th* symmetric power $D_{r,A} = D_r(E_A)$ *of* E_A *to be the* r*th homogeneous subspace of the polynomial ring* $A[e_1, \ldots, e_{m+n}]$. *The elements* $e_1 = e_{1,A}, \ldots, e_{m+n} = e_{m+n,A}$ *are supercommuting indeterminates, according to the rule* $e_b e_a = (-1)^{p(e_a)p(e_b)} e_a e_b$, $p(e_i) = 0$ $(1 \le i \le m)$, $p(e_i) = 1$ $(m < i \le m+n)$. *There is a surjective* A*-map* $\theta_A : E_A^{\otimes r} \to D_r(E_A)$*, taking* $e_i = e_{i_1} \otimes \cdots \otimes e_{i_r}$ *to the monomial* $e_{(i)} = e_{i_1} \ldots e_{i_r}$*, for all* $i \in I(m|n,r)$. *Now* $D_{r,A}$ *has a unique structure as an* $A\Gamma_A$*-module, such that* θ_A *becomes an* $A\Gamma_A$*-map. In fact the action on* $D_{r,A}$ *of a given* $g \in \Gamma_A$ *is the restriction to* $D_{r,A}$ *of the unique* A*-superalgebra automorphism of* $A[e_1, \ldots, e_{m+n}]$ *which maps* $e_\mu \mapsto g e_\mu$ *for all* μ $(1 \le \mu \le m+n)$. *The family* $\{D_{r,A}\}$ *is defined over* $\mathbb{Z}$. *Indeed the* $\mathbb{Z}$*-form* $D_{r,\mathbb{Z}}$ *in* $D_{r,\mathbb{Q}}$ *is the set of all homogeneous polynomials of degree* r *in the variables* $e_1 = e_{1,\mathbb{Q}}, \ldots, e_{m+n} = e_{m+n,\mathbb{Q}}$*, which have coefficients in* $\mathbb{Z}$. *The isomorphism* $\eta_A : D_{r,\mathbb{Z}} \otimes A \to D_{r,A}$ *takes* $e_{(i),\mathbb{Q}} \otimes 1_A \mapsto e_{(i),A}$*, for all* $i \in I(m|n,r)$. *The family* $\{\theta_A\}$ *of morphisms is defined over* $\mathbb{Z}$ *in the sense of the definitions.*

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