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Abstract: We consider a kinetic stochastic model with a non-linear time-inhomogeneous drag force and a Brownian random force. More precisely, we study the couple position X_t of a particle and its velocity which is a solution of a stochastic differential equation driven by a one-dimensional Brownian motion, with the drift of the form $t^{-\beta}F(v)$, F satisfying some homogeneity condition and $\beta > 0$. The behaviour of (V, X) in large time is proven and the precise rate of convergence is pointed out by using stochastic analysis tools.

Key words: kinetic stochastic equation; time-inhomogeneous diffusions; explosion times; scaling transformations; asymptotic distributions.

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1 Introduction

It is classical that the kinetic Fokker-Planck equation is a partial differential equation that describes the time evolution of the probability density function of the velocity of a particle submitted to some drag and random forces. Moreover the Feynman-Kac formula allows to make the link between the kinetic Fokker-Planck equation and stochastic differential equation driven by a Brownian motion called Langevin equation. In the simple linear case the solution of the Langevin equation is a well-known Gaussian process, the Ornstein-Uhlenbeck process.

Some models in several domains as fluids dynamics, statistical mechanics, biology, are based on the Fokker-Planck and Langevin equations in their classical form or on generalisations, for instance non-linear or driven by other random noises than Brownian motion. The behaviour in large time of the solution to the corresponding stochastic differential equation is one of the usual questions when studying these models. Although the tools of partial differential equations allowed to ask of this kind of questions, since these models are probabilistic, tools based on stochastic processes could be used. For instance, in [CCM10] the persistent turning walker model was introduced, inspired from the modelling of fish motion. An associated two-component Kolmogorov type diffusion solves a kinetic Fokker-Planck equation based on an Ornstein-Uhlenbeck Gaussian process and the authors studied the large time behaviour of this model by using appropriate tools from stochastic analysis.

In the last decade the asymptotic study of solutions of non-linear Langevin's type was the subject of an important number of papers, see for instance [CNP19], [EG15], [FT18]. For instance in [FT18] the following system is studied

$$V_t = v_0 + B_t - \frac{\rho}{2}\int_0^t F(V_s)\,\mathrm{d}s \quad \text{and} \quad X_t = x_0 + \int_0^t V_s\,\mathrm{d}s.$$

In other words one considers a particle moving such that its velocity is a diffusion with an invariant measure behaving like $(1+|v|^2)^{-\rho/2}$, as $|v| \to \infty$. The authors prove that for large time, after a suitable rescaling, the position process behaves as a Brownian motion or other stable processes, following the values of ρ. It should be noted that in these cited papers the standard tools associated to time-homogeneous equations are used: invariant measure, scale function, speed measure and so on. Several of these tools will not be available when the drag force is depending explicitly on time.

Let us describe our problem: consider a one-dimensional time-inhomogeneous stochastic kinetic model driven by a Brownian motion. We denote by $(X_t)_{t\geq 0}$ the one-dimensional process describing the position of a particle at time t having the velocity V_t. The velocity process $(V_t)_{t\geq 0}$ is supposed to follow a Brownian dynamics in a potential $U(t,v)$, varying in time :

$$\mathrm{d}V_t = \mathrm{d}B_t - \frac{1}{2}\partial_v U(t, V_t)\,\mathrm{d}t \quad \text{and} \quad X_t = X_0 + \int_0^t V_s\,\mathrm{d}s. \tag{1}$$

It can be viewed as the perturbation of the classical two-component Kolmogorov diffusion

$$\mathrm{d}V_t = \mathrm{d}B_t \quad \text{and} \quad X_t = X_0 + \int_0^t V_s\,\mathrm{d}s.$$

In the present paper the potential is supposed to be of the form $t^{-\beta}\int_0^v F(u)\,\mathrm{d}u$, with $\beta > 0$ and F satisfying some homogeneity condition. It describes a one dimensional particle evolving in a force field $Ft^{-\beta}$ with a Brownian noise. A natural question is to understand the behaviour of the position process in large time. More precisely we look for the limit in distribution of $v(\varepsilon)X_{t/\varepsilon}$, as $\varepsilon \to 0$, where $v(\varepsilon)$ is some rate of convergence.

When $F = 0$, it is not difficult to see that the rescaled position process $\varepsilon^{3/2}X_{t/\epsilon}$ converges in distribution towards the non-Markov but Gaussian process $\int_0^t B_s\,\mathrm{d}s$. We will prove that this anomalous diffusion behaviour still holds for sufficiently "small at infinity" potential.

Our paper is organised as follows: in the next section we introduce notations and we state our main result. Existence and non-explosion of solutions are studied in Section 3 and the proof of our main result is given in Section 4. Some results on growth rate are collected in Section 5.

2 Notations and main result

Assume first that the system velocity - position is given, for $t \geq t_0 > 0$, by

$$\mathrm{d}V_t = \mathrm{d}B_t - t^{-\beta}F(V_t)\,\mathrm{d}t,\ V_{t_0} = v_0 > 0, \quad \text{and} \quad \mathrm{d}X_t = V_t\,\mathrm{d}t,\ X_{t_0} = x_0 \in \mathbb{R}. \tag{SKE}$$

$\beta > 0$ and $(B_t)_{t\geq 0}$ is a standard Brownian motion. F is supposed to satisfy either

$$\textit{for some } \alpha \in \mathbb{R},\ \forall v \in \mathbb{R},\ \lambda > 0,\ F(\lambda v) = \lambda^\alpha F(v), \tag{$H_1\alpha$}$$

or

$$|F| \leq G \text{ where } G \text{ is a positive function satisfying } (H_1\alpha). \tag{$H_2\alpha$}$$

That is, there exist a positive constant K such that, for all $v \in \mathbb{R}$, $|F(v)| \leq K\,|v|^\alpha$.
In the following, sgn is the sign function with convention $\mathrm{sgn}(0) = 0$. Obviously $(H_2\alpha)$ is a generalization of $(H_1\alpha)$. Nevertheless, we keep both assumptions since some proofs are simpler written under $(H_1\alpha)$ and are similar under $(H_2\alpha)$. As an example of function satisfying $(H_1\alpha)$ one can keep in mind $F : v \mapsto \mathrm{sgn}(v)\,|v|^\alpha$ (see also [GO13]), and as an example of function satisfying $(H_2\alpha)$ (with $\alpha = 0$) $F : v \mapsto {}^{v}/_{(1+v^2)}$ (see also [FT18]).

Remark 2.1. If a function π satisfies $(H_1\alpha)$, then

$$\pi(x) = \begin{cases} \pi(1)x^\alpha & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ \pi(-1)\,|x|^\alpha & \text{if } x < 0. \end{cases} \tag{2}$$

Let us state our main result which holds under both assumptions.

Theorem 2.2. *Consider $\alpha \geq 0$, and $\beta > \frac{\alpha+1}{2}$. Let $(V_t, X_t)_{t\geq t_0}$ be a solution to* (SKE). *When $\alpha > 1$, we suppose also that for all $v \in \mathbb{R}$, $\operatorname{sgn}(F(v)) = \operatorname{sgn}(v)$.*
Then, as $\varepsilon \to 0$,

$$(\sqrt{\varepsilon} V_{t/\varepsilon}, \varepsilon^{3/2} X_{t/\varepsilon})_{t\geq \varepsilon t_0} \Longrightarrow (\mathcal{B}_t, \int_0^t \mathcal{B}_s \,\mathrm{d}s)_{t>0}, \tag{3}$$

in the space of continuous functions $\mathcal{C}(]0,\infty))$ endowed by the uniform topology, where $(\mathcal{B}_t)_{t\geq 0}$ is a standard Brownian motion.

Remark 2.3. Solving a Poisson equation is one method among usual methods to study asymptotic behaviour of integrated processes (see for example [CCM10], [EG15], [FT18]). For instance trying to adapt naively the proof of Theorem 1a), p. 2 in [FT18], we are led to find a solution to the Poisson equation $\frac{1}{2}\partial^2_{xx} g(s,x) + \partial_s g(s,x) - F(x)s^{-\beta}\partial_x g(s,x) = -x$. This PDE does not admit an evident solution and seems to be ill-posed. Thus, due to the time-dependence of the stochastic differential equation satisfied by the velocity process, one has to proceed quite differently.

3 Existence and non-explosion of solution

In the following, suppose $\alpha > -1$ and $\beta > \frac{\alpha+1}{2}$ and define $\Omega = \overline{\mathcal{C}}([t_0,\infty))$, the set of continuous functions $\omega : [t_0,\infty) \to \mathbb{R}\cup\{\infty\}$ which equals ∞ after their explosion time (possibly infinite). Following the idea used in [GO13], we first perform a change of time in (SKE) in order to produce at least one time-homogeneous coefficient in the transformed equation. For every $\mathcal{C}^2$-diffeomorphism $\phi : [0,t_1) \to [t_0,\infty)$, let introduce the scaling transformation Φ_ϕ given, for $\omega \in \Omega$, by

$$\Phi_\phi(\omega)(s) := \frac{\omega(\phi(s))}{\sqrt{\phi'(s)}}, \text{ with } s \in [0,t_1).$$

The result containing the change of time transformation is given in [GO13], Proposition 2.1, p. 187. For the sake of completeness we state and sketch the proof in our context.

Proposition 3.1. *If V is a solution to equation* (SKE)*, then $V^{(\phi)}$ is a solution to*

$$\mathrm{d}V_s^{(\phi)} = \mathrm{d}W_s - \frac{\sqrt{\phi'(s)}}{\phi(s)^\beta} F(\sqrt{\phi'(s)} V_s^{(\phi)})\,\mathrm{d}s - \frac{\phi''(s)}{\phi'(s)} \frac{V_s^{(\phi)}}{2}\,\mathrm{d}s,\ V_0^{(\phi)} = \frac{V_{\phi(0)}}{\sqrt{\phi'(0)}}, \tag{4}$$

where $V^{(\phi)} = \Phi_\phi(V)$ and $W_t := \int_0^t \frac{\mathrm{d}B_{\phi(s)}}{\sqrt{\phi'(s)}}$.
If $V^{(\phi)}$ is a solution to (4)*, then V is a solution to equation* (SKE)*, where $V = \Phi_\phi^{-1}(V^{(\phi)})$ and $B_t - B_{t_0} := \int_{t_0}^t \sqrt{(\phi' \circ \phi^{-1})(s)}\,\mathrm{d}W_{\phi^{-1}(s)}$.*
Furthermore uniqueness in law, pathwise uniqueness or strong existence hold for equation (SKE) *if and only if they hold for equation* (4)*.*

Proof. Let V be a solution to equation (SKE). Thanks to Lévy's characterization theorem of the Brownian motion, W is a standard Brownian motion. Then, by a change of variable $t = \phi(s)$, one gets

$$V_{\phi(t)} - V_{\phi(0)} = \int_0^t \sqrt{\phi'(s)}\,\mathrm{d}W_s - \int_0^t \frac{F(V_{\phi(s)})}{\phi(s)^\beta}\phi'(s)\,\mathrm{d}s.$$

The integration by parts formula yields

$$\mathrm{d}\left(\frac{V_{\phi(s)}}{\sqrt{\phi'(s)}}\right) = \mathrm{d}W_s - \frac{\sqrt{\phi'(s)}}{\phi(s)^\beta}F(V_{\phi(s)})\,\mathrm{d}s - \frac{\phi''(s)}{2\phi'(s)}\frac{V_{\phi(s)}}{\sqrt{\phi'(s)}}\,\mathrm{d}s.$$

From which follows (4). The proof of the second part is similar. $\square$

In the following, we will use two particular changes of time, depending on which term of (4) should become time-homogeneous:

- *exponential change of time*: denoting $\phi_e : t \mapsto t_0 e^t$, the exponential scaling transformation is given by $\Phi_e(\omega) : s \in \mathbb{R}^+ \mapsto \dfrac{\omega_{t_0 e^s}}{\sqrt{t_0}e^{s/2}}$, for $\omega \in \Omega$. Set $V^{(e)} := \Phi_e(V)$. Thanks to Proposition 3.1, the process $(V_t^{(e)})_{t\geq 0}$ satisfies the equation

$$\mathrm{d}V_s^{(e)} = \mathrm{d}W_s - \frac{V_s^{(e)}}{2}\,\mathrm{d}s - t_0^{1/2-\beta}e^{(1/2-\beta)s}F\left(\sqrt{t_0}e^{s/2}V_s^{(e)}\right)\mathrm{d}s, \tag{5}$$

  where $(W_t)_{t\geq 0}$ is a standard Brownian motion.

- *power change of time*: setting $\gamma := \frac{2\beta}{\alpha+1} > 1$, consider $\phi_\gamma \in \mathcal{C}^2([0,t_1))$ the solution to the Cauchy problem

$$\phi_\gamma' = \phi_\gamma^\gamma,\ \phi_\gamma(0) = t_0.$$

  Clearly $\phi_\gamma(t) = \left(t_0^{1-\gamma} + (1-\gamma)t\right)^{1/(1-\gamma)}$: the maximal time t_1 satisfies $(\gamma-1)t_1 = t_0^{1-\gamma}$ and the power scaling transformation is given by $\Phi_\gamma(\omega) : s \in \mathbb{R}^+ \mapsto \dfrac{\omega(\phi_\gamma(s))}{\phi_\gamma(s)^{\gamma/2}}$. The process $V^{(\gamma)} := V^{(\phi_\gamma)}$ satisfies the equation

$$\mathrm{d}V_s^{(\gamma)} = \mathrm{d}W_s - \rho\phi_\gamma^{-\alpha\beta/(\alpha+1)}(s)F\left(\sqrt{\phi_\gamma'(s)}V_s^{(\gamma)}\right)\mathrm{d}s - \gamma\phi_\gamma^{\gamma-1}(s)\frac{V_s^{(\gamma)}}{2}\,\mathrm{d}s, \tag{6}$$

  where $(W_t)_{t\geq 0}$ is a standard Brownian motion.

In the following we will study the existence and the behaviour of the solution to (SKE), first under the homogeneity assumption $(H_1\alpha)$ and then under the domination assumption $(H_2\alpha)$.

3.1 Study under $(H_1\alpha)$

In the following we assume $(H_1\alpha)$. Then the process $V^{(\gamma)}$ satisfies the equation

$$\mathrm{d}V_s^{(\gamma)} = \mathrm{d}W_s - F(V_s^{(\gamma)})\,\mathrm{d}s - \gamma\phi_\gamma^{\gamma-1}(s)\frac{V_s^{(\gamma)}}{2}\,\mathrm{d}s,\ s \in [0,t_1), \tag{7}$$

which can be written, by using the expression of ϕ_γ, as

$$\mathrm{d}V_s^{(\gamma)} = \mathrm{d}W_s - F(V_s^{(\gamma)})\,\mathrm{d}s - \delta\frac{V_s^{(\gamma)}}{t_1 - s}\,\mathrm{d}s,\ s \in [0,t_1), \tag{8}$$

where $\delta = \dfrac{\gamma}{2(\gamma-1)}$. Proposition 3.2, p. 188, in [GO13] can be stated in the present situation:

Proposition 3.2. *For $\alpha > -1$, there exists a pathwise unique strong solution to* (SKE)*, defined up to the explosion time.*

Proof. We sketch the proof in our context. Note that, since $\alpha > -1$, $x \mapsto |x|^\alpha$ is locally integrable. Leaving out the third term on the right-hand side of (7), one gets

$$\mathrm{d}H_s = \mathrm{d}W_s - F(H_s)\,\mathrm{d}s,\ \ s \in [0, t_1). \tag{9}$$

By using Proposition 2.2, p. 28, in [CE05], there exists a unique weak solution H to the time-homogeneous equation (9) defined up to the explosion time. Moreover, the Girsanov transformation induces a linear bijection between weak solutions defined up to the explosion time to equations (7) and (9). It follows that there exists a unique weak solution $V^{(\gamma)}$ to equation (7). Therefore, by using Proposition 3.1, there exists a unique weak solution V to equation (SKE). Besides, by using Corollary 3.4 and Proposition 3.2, pp. 389-390, in [RY99], pathwise uniqueness holds for the equation (SKE). The conclusion follows from Theorem 1.7, p. 368, in [RY99]. $\square$

Remark 3.3. When $\alpha = 1$, drift and diffusion are Lipschitz and satisfy locally linear growth. The existence and non-explosion of V follow from Theorem 2.9, p. 289, in [KS98].

Proposition 3.4.

- *When $\alpha \leq 1$ or for all $v \in \mathbb{R}$, $\operatorname{sgn}(F(v)) = \operatorname{sgn}(v)$, the explosion time of V is a.s. infinite.*
- *Else, i.e. if $\alpha > 1$ and $(F(-1), F(1)) \in ((0,\infty)) \times [0,\infty)) \cup (\mathbb{R} \times (-\infty, 0])$, $\mathbb{P}(\tau_\infty = \infty) \in (0,1)$, where τ_∞ denotes the explosion time of V.*

Proof. We split the proof in several steps.

STEP 1. Assume first that $\alpha \leq 1$ or $\operatorname{sgn}(F(v)) = \operatorname{sgn}(v)$. We will use Theorem 10.2.1, p. 254, in [SV06]. Call $\mathcal{L}_t$ the time-inhomogeneous infinitesimal generator of V, given by

$$\mathcal{L}_t := \frac{1}{2}\frac{\partial^2}{\partial x^2} - \frac{F(x)}{t^\beta}\frac{\partial}{\partial x}. \tag{10}$$

Let φ be a twice continuous differentiable positive function such that $\varphi(x) = 1 + x^2$ for all $|x| \geq 1$, $\varphi(x) = 1$ for all $|x| \leq \frac{1}{2}$ and $\varphi \geq 1$. Note that φ does not depend on time. Hence $\big(\partial_t + \mathcal{L}_t\big)\varphi = \mathcal{L}_t\varphi$.

Let $T \geq t_0$ and call c_T the supremum of $\mathcal{L}_t\varphi$ on $[t_0, T] \times [-1, 1]$. Then, for all $|x| \leq 1$ and $t \in [t_0, T]$,

$$\mathcal{L}_t\varphi(x) \leq c_T \leq c_T\varphi(x).$$

Moreover, for all $|x| > 1$ and $t \in [t_0, T]$, for C a positive constant,

$$\mathcal{L}_t\varphi(x) = -2x\frac{F(x)}{t^\beta} + 1 \leq \begin{cases} 1 \leq \varphi(x), & \text{if for all } v \in \mathbb{R},\ \operatorname{sgn}(F(v)) = \operatorname{sgn}(v), \\ 2\max(|F(1)|, |F(-1)|)x^2 + 1 \leq C\varphi(x), & \text{if } \alpha \leq 1. \end{cases}$$

So, by using Theorem 10.2.1, p. 254, in [SV06], we deduce that τ_∞ is infinite a.s.

STEP 2. Assume now the contrary, that is $\alpha > 1$ and $(F(-1), F(1)) \in ((0,\infty)) \times [0,\infty)) \cup (\mathbb{R} \times (-\infty, 0])$. We follow the ideas of the proof of Proposition 3.7, pp. 191-192, in [GO13]. We first show that $\mathbb{P}(\tau_\infty = \infty) > 0$. Let $V^{(\gamma)}$ be the pathwise unique strong solution to equation (8). Also denote by b, the δ-Brownian bridge, the pathwise unique strong solution to equation

$$\mathrm{d}b_s = \mathrm{d}W_s - \delta\frac{b_s}{t_1 - s}\,\mathrm{d}s,\ \ b_0 = x_0,\ \ s \in [0, t_1). \tag{11}$$

Note that the equation (11) is obtained from (8) by omitting the second term on the right-hand side. Denote by $\tau_\infty^{(\gamma)}$ the explosion time of $V^{(\gamma)}$, clearly, $\tau_\infty^{(\gamma)} \in [0,t_1] \cup \{\infty\}$ a.s. and $\{\tau_\infty^{(\gamma)} \geq t_1\} = \{\tau_\infty = \infty\}$. Note that b becomes continuous on $[0,t_1]$, with $b_{t_1} = 0$ a.s.

Fix $n \geq 1$, for all $s \in [0,t_1]$, define

$$T_n := \inf\left\{s \in [0,t_1), \left|V_s^{(\gamma)}\right| \geq n\right\}, \ \sigma_n := \inf\{s \in [0,t_1], |b_s| \geq n\},$$

and

$$\mathcal{E}(s) := \exp\left(\int_0^s -F(b_u)\,\mathrm{d}W_u - \frac{1}{2}\int_0^s F(b_u)^2\,\mathrm{d}u\right).$$

Then, one has, since $\alpha > 1 \geq 0$,

$$\begin{aligned}\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^{s\wedge\sigma_n} F(b_u)^2\,\mathrm{d}u\right)\right] &\leq \mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^{s\wedge\sigma_n} n^{2\alpha}\max(F(1)^2, F(-1)^2)\,\mathrm{d}u\right)\right]\\ &\leq \exp\left(\frac{t_1}{2}n^{2\alpha}\max(F(1)^2, F(-1)^2)\right),\end{aligned}$$

so Novikov's condition applies to $(\mathcal{E}_{s\wedge\sigma_n})_{s\geq 0}$. By using the Girsanov transformation between b and $V^{(\gamma)}$, we can write for every integer $n \geq 1$, $s \in [0,t_1]$ and $A \in \mathcal{F}_s$,

$$\mathbb{E}\left[\mathbb{1}_A\left(V_{\bullet\wedge T_n}^{(\gamma)}\right)\mathbb{1}_{T_n > s}\right] = \mathbb{E}\big[\mathbb{1}_A\left(b_{\bullet\wedge\sigma_n}\right)\mathcal{E}(s\wedge\sigma_n)\mathbb{1}_{\sigma_n > s}\big].$$

Letting $n \to \infty$, we obtain by the dominated convergence theorem and Fatou's lemma,

$$\mathbb{E}\left[\mathbb{1}_A\left(V^{(\gamma)}\right)\mathbb{1}_{\tau_\infty^{(\gamma)} > s}\right] \geq \mathbb{E}\left[\mathbb{1}_A\left(b\right)\mathcal{E}(s)\right].$$

Hence, $\mathbb{P}(\tau_\infty = \infty) = \mathbb{P}(\tau_\infty^{(\gamma)} \geq t_1) \geq \mathbb{E}[\mathcal{E}(t_1)] > 0$.

Step 3. We will show that $\mathbb{P}(\tau_\infty = \infty) < 1$ when $F(1) > 0$ and $F(-1) > 0$. Our strategy is to apply Theorem 10.2.1, p. 254, in [SV06]. Let $T > t_0$ and choose $a \in (1,\alpha)$. Also, one can choose $k \geq 1$ such that $a(a-1)^{-1} < k(T-t_0)$. Introduce the continuous differentiable negative function $f_1 : x \mapsto \dfrac{-1/2}{1+|x|^a}$, and, for $\mu > 0$, the bounded twice continuous differentiable function

$$G_{1,\mu}(x) = \exp\left(\mu\int_{-\infty}^x f_1(y)\,\mathrm{d}y\right), \ x \in \mathbb{R}.$$

For all $t \in [t_0,T]$ and $x \in \mathbb{R}$,

$$\begin{aligned}(\partial_t + \mathcal{L}_t)G_{1,\mu}(x) = \mathcal{L}_t G_{1,\mu}(x) &= \mu G_{1,\mu}(x)\left[F(x)t^{-\beta}\left|f_1(x)\right| + \frac{1}{2}f_1'(x) + \frac{\mu}{2}f_1^2(x)\right]\\ &\geq \mu G_{1,\mu}(x)\left[F(x)T^{-\beta}\left|f_1(x)\right| + \frac{1}{2}f_1'(x) + \frac{\mu}{2}f_1^2(x)\right].\end{aligned}$$

Since $|f_1(x)| \underset{|x|\to\infty}{\sim} \frac{1}{2}|x|^{-a}$, $\lim\limits_{|x|\to\infty} F(x)\left|f_1(x)\right| = +\infty$, and using that $\lim\limits_{|x|\to\infty} f_1'(x) = 0$, there exists $r \geq 1$ such that, for all $\mu > 0$,

$$(\partial_t + \mathcal{L}_t)G_{1,\mu}(x) \geq \mu G_{1,\mu}(x)\left[F(x)T^{-\beta}\left|f_1(x)\right| + \frac{1}{2}f_1'(x)\right] \geq k\mu G_{1,\mu}(x) \text{ on } [t_0,T]\times[-r,r]^c.$$

Moreover, since f_1^2 is bounded away from zero, while $|f_1'|$ is bounded on $[-r,r]$, there exists μ_0, such that, since F is non-negative,

$$\big(\partial_t + \mathcal{L}_t\big)G_{1,\mu_0}(x) \geq \mu_0 G_{1,\mu_0}(x)\left[\frac{1}{2}f_1'(x) + \frac{\mu_0}{2}f_1^2(x)\right] \geq k\mu_0 G_{1,\mu_0}(x) \text{ on } [t_0,T]\times[-r,r].$$

Hence, for all $t \in [t_0,T]$ and $x \in \mathbb{R}$, $\big(\partial_t + \mathcal{L}_t\big)G_{1,\mu_0}(x) \geq k\mu_0 G_{1,\mu_0}(x)$. Besides, since $|f_1(x)| \leq 1 \wedge |x|^{-a}$,

$$\int_{-\infty}^{x_0}\big(-f_1(x)\big)\,\mathrm{d}x \leq \int_{\mathbb{R}}\big(1 \wedge |x|^{-a}\big)\,\mathrm{d}x = a(a-1)^{-1} < k(T-t_0).$$

Thus, $G_{1,\mu_0}(x_0) > e^{-k\mu_0(T-t_0)} \geq e^{-k\mu_0(T-t_0)}\sup_{x\in\mathbb{R}} G_{1,\mu_0}(x)$. Therefore, Theorem 10.2.1, p. 254, in [SV06] applies and V explodes in finite time with positive probability.

When $F(-1) < 0$ and $F(1) < 0$, one can work in the same way, using instead $G_{1,\mu}$ the function $x \mapsto \exp\Big(\mu\int_x^{+\infty} f_1(y)\,\mathrm{d}y\Big)$, in order to get that $\mathbb{P}(\tau_\infty = \infty) < 1$.

Step 4. It remains to show that $\mathbb{P}(\tau_\infty = \infty) < 1$ when $F(1) < 0$ and $F(-1) > 0$. As in the previous step, we choose $a \in (1,\alpha)$ and for any $T > t_0$, one can choose again $k \geq 1$ such that $a(a-1)^{-1} < k(T-t_0)$. Moreover, one can see that there exists a continuous differentiable odd function f_2, defined on $\mathbb{R}$, vanishing only at $x = 0$, such that $|f_2(x)| \leq 1 \wedge |x|^{-a}$, and

$$f_2(x) := kx,\ x \in \left[-\frac{1}{2k},\frac{1}{2k}\right],\ \lim_{|x|\to\infty}|x|^\alpha\,|f_2(x)| = \infty \text{ and } \lim_{|x|\to\infty} f_2'(x) = 0.$$

For $\mu > 0$, we introduce the bounded twice continuous differentiable function

$$G_{2,\mu}(x) := \exp\left(\mu\int_0^x f_2(y)\,\mathrm{d}y\right),\ x \in \mathbb{R}.$$

Note that for all $x \in \mathbb{R}$ and $t \in [t_0,T]$,

$$\begin{aligned}\big(\partial_t + \mathcal{L}_t\big)G_{2,\mu}(x) = \mathcal{L}_t G_{2,\mu}(x) &= \mu G_{2,\mu}(x)\left[\frac{|F(x)f_2(x)|}{t^\beta} + \frac{1}{2}f_2'(x) + \frac{\mu}{2}f_2^2(x)\right]\\ &\geq \mu G_{2,\mu}(x)\left[\rho\frac{|x|^\alpha\,|f_2(x)|}{t^\beta} + \frac{1}{2}f_2'(x) + \frac{\mu}{2}f_2^2(x)\right],\end{aligned}$$

where $\rho = \min\big\{\,|F(1)|,|F(-1)|\,\big\} > 0$. One can conclude, using the same argument as in the proof of Proposition 3.7, p. 13, in [GO13]. ☐

3.2 Study under $(H_2\alpha)$

We assume now $(H_2\alpha)$. Since, the equation (6) doesn't have any time-homogeneous term, the previous method cannot be used to conclude to the existence up to explosion. Instead, one will use the exponential change of time process to get

Proposition 3.5. *If $\alpha \geq 0$, there exists a pathwise unique strong solution to* (SKE)*, defined up to the explosion time.*

Proof. The proof is identical to that of Proposition 3.2, by considering $V^{(e)}$ instead of $V^{(\gamma)}$. ☐

Proposition 3.6.

- *When $\alpha \leq 1$ or $\operatorname{sgn}(F(v)) = \operatorname{sgn}(v)$, the explosion time of V is a.s. infinite.*
- *Else, $\mathbb{P}(\tau_\infty = \infty) > 0$, where τ_∞ denotes the explosion time of V.*

Proof. The proof is identical to that of Proposition 3.4 by considering G instead of $|F|$.

☐

4 Asymptotic behaviour of the solution

Proposition 4.1. *Assume* $\alpha \geq 0$*. If* $\alpha \leq 1$ *or* $\operatorname{sgn}(F(v)) = \operatorname{sgn}(v)$*, then*

$$\forall t \geq t_0,\ \mathbb{E}\left[|V_t|\right] \leq C_{\alpha,\beta,t_0}\sqrt{t} \text{ and } \mathbb{E}\left[|V_t|^\alpha\right] \leq C_{\alpha,\beta,t_0} t^{\frac{\alpha}{2}}.$$

Proof.
STEP 1. Assume first that $\alpha > 1$, hence we have supposed that $\operatorname{sgn}(F(v)) = \operatorname{sgn}(v)$. By Itô's formula, for all $t \geq t_0$,

$$(V_t)^2 = v_0^2 + \int_{t_0}^t 2V_s \,\mathrm{d}B_s - \int_{t_0}^t 2s^{-\beta} V_s F(V_s)\,\mathrm{d}s + (t - t_0).$$

By using the well known Lévy's characterization, the local martingale $\mathrm{d}\beta_t = \frac{V_t}{|V_t|}\mathbb{1}_{V_t \neq 0}\,\mathrm{d}B_t$ is a Brownian motion. Thus, one can write,

$$|V_t|^2 = (V_t)^2 = v_0^2 + \int_{t_0}^t 2\sqrt{|V_s|^2}\,\mathrm{d}\beta_s \underbrace{- \int_{t_0}^t 2s^{-\beta} V_s F(V_s)\,\mathrm{d}s}_{\leq 0} + (t - t_0).$$

Defining the square of the one-dimensional Bessel process started at v_0^2 (see p 439 in [RY99]) by

$$Z_t = v_0^2 + \int_{t_0}^t 2\sqrt{|Z_s|}\,\mathrm{d}\beta_s + (t - t_0), \tag{12}$$

and using the comparison theorem (see Theorem 3.7 p 394, in [RY99]), one deduces that for all $t \geq t_0$, $|V_t|^2 \leq Z_t$. Since the square of a Brownian motion is a solution, unique in law, of (12), putting to the power $\frac{\kappa}{2}$ with $\kappa \in \{1, \alpha\}$, and taking the expectation one gets, for all $t \geq t_0$

$$\mathbb{E}\left[|V_t|^\kappa\right] \leq \mathbb{E}\left[Z_t^{\frac{\kappa}{2}}\right] = \mathbb{E}\left[|v_0 + B_t|^\kappa\right] \leq c_\kappa |v_0|^\kappa + c_\kappa t^{\frac{\kappa}{2}} \mathbb{E}\left[|B_1|^\kappa\right] = C_\kappa t^{\frac{\kappa}{2}}.$$

STEP 2. Assume now that $\alpha \in [0, 1]$. Then Jensen's inequality yields, for all $t \geq t_0$, $\mathbb{E}\left[|V_t|^\alpha\right] \leq \mathbb{E}\left[|V_t|\right]^\alpha$, hence it suffices to verify only the first inequality.

Defining, for all $n \geq 0$, the stopping times $T_n := \inf\{t \geq t_0,\ |V_t| \geq n\}$, one can write, for $t \geq t_0$ and $n \geq 0$,

$$V_{t\wedge T_n} = v_0 - B_{t_0} + B_{t\wedge T_n} - \int_{t_0}^{t\wedge T_n} s^{-\beta} F(V_{s\wedge T_n})\,\mathrm{d}s.$$

Note that under both hypotheses ($(H_1\alpha)$ or $(H_2\alpha)$), there exists a positive constant K, such that $|F(v)| \leq K|v|^\alpha$, so

$$\begin{aligned} |V_{t\wedge T_n}| &\leq |v_0 - B_{t_0}| + |B_{t\wedge T_n}| + \int_{t_0}^{t\wedge T_n} s^{-\beta} |F(V_{s\wedge T_n})|\,\mathrm{d}s \\ &\leq |v_0 - B_{t_0}| + |B_{t\wedge T_n}| + \int_{t_0}^{t\wedge T_n} s^{-\beta} K |V_{s\wedge T_n}|^\alpha\,\mathrm{d}s. \end{aligned}$$

Taking expectation, it becomes since $(B_t^2 - t)_{t\geq 0}$ is a martingale,

$$\begin{aligned}
\mathbb{E}\left[|V_{t\wedge T_n}|\right] &\leq \mathbb{E}\left[|v_0 - B_{t_0}|\right] + \mathbb{E}\left[|B_{t\wedge T_n}|\right] + \int_{t_0}^t s^{-\beta} K \mathbb{E}\left[|V_{s\wedge T_n}|^\alpha\right] \mathrm{d}s \\
&\leq \mathbb{E}\left[|v_0 - B_{t_0}|\right] + \sqrt{\mathbb{E}\left[B_{t\wedge T_n}^2\right]} + \int_{t_0}^t s^{-\beta} K \mathbb{E}\left[|V_{s\wedge T_n}|\right]^\alpha \mathrm{d}s \\
&\leq \mathbb{E}\left[|v_0 - B_{t_0}|\right] + \sqrt{\mathbb{E}\left[t\wedge T_n\right]} + \int_{t_0}^t s^{-\beta} K \mathbb{E}\left[|V_{s\wedge T_n}|\right]^\alpha \mathrm{d}s \\
&\leq C_{t_0}\sqrt{t} + \int_{t_0}^t s^{-\beta} K \mathbb{E}\left[|V_{s\wedge T_n}|\right]^\alpha \mathrm{d}s.
\end{aligned}$$

At this level we need a slightly modified Gronwall's lemma:

Lemma 4.2 (Gronwall-type lemma)**.** *Fix $\alpha \in [0,1)$, $t_0 \in \mathbb{R}$. Assume that g is a non-negative real-valued function, that b is a positive function and a is a differentiable real-valued function. Moreover suppose that bg^α is a continuous function. If*

$$\forall t \geq t_0,\ g(t) \leq a(t) + \int_{t_0}^t b(s) g(s)^\alpha \,\mathrm{d}s < \infty. \tag{13}$$

Then,

$$\forall t \geq t_0,\ g(t) \leq C_\alpha \left[a(t) + \left((1-\alpha) \int_{t_0}^t b(s)\,\mathrm{d}s \right)^{\frac{1}{1-\alpha}} \right],$$

where $C_\alpha := 2^{\frac{1}{1-\alpha}}$.

We postpone the proof of this lemma and we apply it with $g_n : t \mapsto \mathbb{E}\left[|V_{t\wedge T_n}|\right]$ which is bounded by n. By Fatou's lemma one gets,

$$\begin{aligned}
\forall t \geq t_0,\ \mathbb{E}\left[|V_t|\right] \leq \liminf_{n\to\infty} \mathbb{E}\left[|V_{t\wedge T_n}|\right] &\leq C_\alpha \left[C_{t_0}\sqrt{t} + \left(\frac{1-\alpha}{1-\beta} K (t^{1-\beta} - t_0^{1-\beta}) \right)^{\frac{1}{1-\alpha}} \right] \\
&\leq C_{\alpha,\beta,t_0}\sqrt{t}.
\end{aligned}$$

This ends the proof of the proposition except for the proof of Lemma 4.2. ◻

Proof of Lemma 4.2. For $t \geq t_0$ we set $H(t)$ the right-hand side of (13) so the hypothesis can be written $g(t) \leq H(t)$. Since $\alpha \geq 0$, $g(t)^\alpha \leq H(t)^\alpha$, then, multiplying by $b(t) > 0$,

$$b(t)g(t)^\alpha \leq b(t)H(t)^\alpha.$$

Now, let's bring in the derivative of H

$$H'(t) = a'(t) + b(t)g(t)^\alpha \leq a'(t) + b(t)H(t)^\alpha.$$

We deduce that

$$\frac{H'(t)}{H(t)^\alpha} = \frac{a'(t) + b(t)g(t)^\alpha}{H(t)^\alpha} \leq \frac{a'(t)}{\left(a(t) + \int_{t_0}^t b(s)g(s)^\alpha\,\mathrm{d}s\right)^\alpha} + b(t) \leq \frac{a'(t)}{a(t)^\alpha} + b(t)$$

where we used again (13) and the non-negativity of $\displaystyle\int_{t_0}^t b(s)g(s)^\alpha \,\mathrm{d}s$. By integrating, since $\alpha < 1$,

$$(1-\alpha)^{-1}\left[H(t)^{1-\alpha} - a(t_0)^{1-\alpha}\right] \leq (1-\alpha)^{-1}\left[a(t)^{1-\alpha} - a(t_0)^{1-\alpha}\right] + \int_{t_0}^t b(s)\,\mathrm{d}s$$

or equivalently,

$$H(t)^{1-\alpha} \leq a(t)^{1-\alpha} + (1-\alpha)\int_{t_0}^t b(s)\,\mathrm{d}s.$$

By using again (13), since $\frac{1}{1-\alpha} > 0$,

$$g(t) \leq \left(a(t)^{1-\alpha} + (1-\alpha)\int_{t_0}^t b(s)\,\mathrm{d}s\right)^{\frac{1}{1-\alpha}} \leq C_\alpha \left[a(t) + \left((1-\alpha)\int_{t_0}^t b(s)\,\mathrm{d}s\right)^{\frac{1}{1-\alpha}}\right].$$

□

Remark 4.3. If g is not continuous, note that the function H is continuous and satisfies (13), since b is positive and $g \leq H$. So, one can apply the lemma to H and then use the inequality $g \leq H$. [1]

Proof of Theorem 2.2. We split the proof in three steps.

Step 1. We note that it is enough to prove that $(V_t^{(\varepsilon)})_{t>0} := (\sqrt{\varepsilon}V_{t/\varepsilon})_{t>0}$ converges in distribution to a Brownian motion in the space of continuous functions $\mathcal{C}(]0,\infty))$ endowed by the uniform topology

$$\mathrm{d} : f, g \in \mathcal{C}(]0,+\infty)) \mapsto \sum_{n=1}^{+\infty} \frac{1}{2^n}\min\Big(1, \sup_{[\frac{1}{n},n]} |f(t)-g(t)|\Big).$$

Indeed, assume that the convergence of the rescaled velocity process is proved. For $\epsilon \in (0,1]$ and $t \geq \epsilon t_0$ one can write

$$\varepsilon^{3/2}X_{t/\varepsilon} = \varepsilon^{3/2}x_0 + \int_{\varepsilon t_0}^t V_s^{(\varepsilon)}\,\mathrm{d}s.$$

Clearly the theorem will be proved once we show that $(V_\bullet^{(\varepsilon)}, \int_{\varepsilon t_0}^\bullet V_s^{(\varepsilon)}\,\mathrm{d}s) =: g_\varepsilon(V_\bullet^{(\varepsilon)})$ converges weakly in $\mathcal{C}(]0,\infty))$ endowed by the uniform topology. Here the mapping $g_\varepsilon : v \mapsto \left(v_t, \int_{\varepsilon t_0}^t v_s\,\mathrm{d}s\right)_{t>0}$ is defined and valued on $\mathcal{C}(]0,\infty)) \times \mathcal{C}(]0,\infty))$. This mapping is converging, as $\varepsilon \to 0$, to the continuous mapping $g : v \mapsto \left(v_t, \int_0^t v_s\,\mathrm{d}s\right)_{t>0}$.

Let $h : \mathcal{C}(]0,\infty)) \times \mathcal{C}(]0,\infty)) \to \mathbb{R}$ be a bounded and uniformly continuous function. We will show that $\mathbb{E}[(h\circ g_\epsilon)(V^{(\epsilon)})] \underset{\epsilon\to 0}{\longrightarrow} \mathbb{E}[(h\circ g)(\mathcal{B})]$. One can write

$$\mathbb{E}\left[(h\circ g_\epsilon)(V^{(\epsilon)})\right] = \mathbb{E}\left[(h\circ g_\epsilon)(V^{(\epsilon)}) - (h\circ g)(V^{(\epsilon)})\right] + \mathbb{E}\left[(h\circ g)(V^{(\epsilon)})\right].$$

The second term converges to $\mathbb{E}[(h\circ g)(\mathcal{B})]$ by the continuous mapping theorem (see Theorem 2.7 p. 21, in [Bil99]), since $(V^{(\epsilon)})_{\epsilon>0}$ converges in distribution towards a Brownian motion. Moreover, since h is uniformly continuous, for any $\eta > 0$, there exists $\delta > 0$ such that,

$$\forall (f,g) \in \mathcal{C}(]0,+\infty)),\ \mathrm{d}(f,g) \leq \delta \Rightarrow |h\circ f - h\circ g| \leq \eta. \tag{14}$$

[1] We thank Thomas Cavallazzi for this remark.

Hence, since h is bounded, one gets with C a positive constant,

$$\begin{aligned}\mathbb{E}\left[\left|h\circ g_\epsilon(V^{(\epsilon)})-h\circ g(V^{(\epsilon)})\right|\right] &\le \eta+2\|h\|_\infty\mathbb{P}\left(\mathrm{d}\left(g_\varepsilon(V^\varepsilon),g(V^{(\varepsilon)})\right)>\delta\right)\\ &\le \eta+2\|h\|_\infty\mathbb{P}\left(C\int_0^{\varepsilon t_0}\left|V_s^{(\varepsilon)}\right|\mathrm{d}s>\delta\right)\underset{\epsilon\to 0}{\longrightarrow}\eta,\end{aligned}$$

because $\int_0^{\varepsilon t_0}\left|V_s^{(\varepsilon)}\right|\mathrm{d}s$ converges almost surely towards 0. One concludes using the Portmanteau theorem (see Theorem 2.1. p. 16, in [Bil99]).

Step 2. Let us prove now the convergence of the rescaled velocity process.
Let $\epsilon\in(0,1]$ and $t\ge\epsilon t_0$. One can write

$$\begin{aligned}\sqrt{\varepsilon}V_{t/\varepsilon}=&\sqrt{\varepsilon}(v_0-B_{t_0})+\sqrt{\varepsilon}B_{t/\varepsilon}-\sqrt{\varepsilon}\int_{t_0}^{t/\varepsilon}F(V_s)s^{-\beta}\,\mathrm{d}s\\ =&\sqrt{\varepsilon}(v_0-B_{t_0})+\sqrt{\varepsilon}B_{t/\varepsilon}-\varepsilon^{\beta-1/2}\int_{\varepsilon t_0}^{t}F(V_{u/\varepsilon})u^{-\beta}\,\mathrm{d}u.\end{aligned}$$

It becomes

$$V_t^{(\varepsilon)}=\sqrt{\varepsilon}(v_0-B_{t_0})+\sqrt{\varepsilon}B_{t/\varepsilon}-\varepsilon^{\beta-(\alpha+1)/2}\int_{\varepsilon t_0}^{t}(\sqrt{\varepsilon})^\alpha F\left(\frac{V_u^{(\varepsilon)}}{\sqrt{\varepsilon}}\right)u^{-\beta}\,\mathrm{d}u. \tag{15}$$

By self-similarity, $B^{(\varepsilon)}:=(\sqrt{\varepsilon}B_{t/\varepsilon})_{t\ge 0}$ has the same distribution as a Brownian motion $\mathcal{B}$. The proof will be complete once we prove that

$$\forall T\ge\varepsilon t_0,\quad \sup_{\varepsilon t_0\le t\le T}\left|V_t^{(\varepsilon)}-B_t^{(\varepsilon)}\right|\xrightarrow{\mathbb{P}}0,\quad\text{as }\varepsilon\to 0. \tag{16}$$

Indeed, fix $a>0$ and choose $N>0$ such that $\sum_{n=N+1}^{+\infty}\frac{1}{2^n}\le\frac{a}{2}$. Then,

$$\mathrm{d}\left(V^{(\varepsilon)},B^{(\varepsilon)}\right)\le\frac{a}{2}+\sum_{n=1}^{N}\frac{1}{2^n}\sup_{[\frac{1}{n},n]}\left|V_t^{(\varepsilon)}-B_t^{(\varepsilon)}\right|.$$

It follows that

$$\mathbb{P}\left(\mathrm{d}\left(V^{(\varepsilon)},B^{(\varepsilon)}\right)>a\right)\le\sum_{n=1}^{N}\mathbb{P}\left(\sup_{[\frac{1}{n},n]}\left|V_t^{(\varepsilon)}-B_t^{(\varepsilon)}\right|>a'\right)\underset{\epsilon\to 0}{\longrightarrow}0,$$

where $a'=a(\sum_{n\ge 1}^{+\infty}1/2^n)^{-1}$. It remains to apply Theorem 3.1, p. 27, in [Bil99].

Step 3. Let us prove now (16). Recall that under both hypothesis ($(H_1\alpha)$ and $(H_2\alpha)$), there exists a positive constant K, such that $(\sqrt{\varepsilon})^\alpha\left|F\left(\frac{V_u^{(\varepsilon)}}{\sqrt{\varepsilon}}\right)\right|\le K\left|V_u^{(\varepsilon)}\right|^\alpha$. It follows from (15) that, for all $T\ge\varepsilon t_0$,

$$\begin{aligned}\sup_{\varepsilon t_0\le t\le T}\left|V_t^{(\varepsilon)}-B_t^{(\varepsilon)}\right|\le&\sqrt{\varepsilon}\,|v_0-B_{t_0}|+\varepsilon^{\beta-(\alpha+1)/2}\sup_{\varepsilon t_0\le t\le T}\left|\int_{\varepsilon t_0}^{t}(\sqrt{\varepsilon})^\alpha F\left(\frac{V_u^{(\varepsilon)}}{\sqrt{\varepsilon}}\right)u^{-\beta}\,\mathrm{d}u\right|\\ \le&\sqrt{\varepsilon}\,|v_0-B_{t_0}|+\varepsilon^{\beta-(\alpha+1)/2}\int_{\varepsilon t_0}^{T}K\left|V_u^{(\varepsilon)}\right|^\alpha u^{-\beta}\,\mathrm{d}u.\end{aligned}$$

Taking the expectation and using Proposition 4.1,

$$\begin{aligned}\varepsilon^{\beta-(\alpha+1)/2}\mathbb{E}\left[\int_{\varepsilon t_0}^{T} K\left|V_u^{(\varepsilon)}\right|^{\alpha} u^{-\beta}\,\mathrm{d}u\right] &= \varepsilon^{\beta-(\alpha+1)/2}\int_{\varepsilon t_0}^{T} K\mathbb{E}\left[\left|V_u^{(\varepsilon)}\right|^{\alpha}\right] u^{-\beta}\,\mathrm{d}u\\ &\leq \varepsilon^{\beta-(\alpha+1)/2}\int_{\varepsilon t_0}^{T} KC_{\alpha,\beta,t_0}u^{\frac{\alpha}{2}-\beta}\,\mathrm{d}u\\ &\leq C\left(\varepsilon^{\beta-(\alpha+1)/2}T^{\frac{\alpha}{2}-\beta+1}-t_0^{\frac{\alpha}{2}-\beta+1}\sqrt{\varepsilon}\right).\end{aligned}$$

Hence

$$\mathbb{E}\left[\sup_{\varepsilon t_0\leq t\leq T}\left|V_t^{(\varepsilon)}-B_t^{(\varepsilon)}\right|\right]=O(\varepsilon^{\gamma}),$$

where $\gamma=\min(\frac{1}{2},\beta-{}^{(\alpha+1)}/_{2})>0$. This concludes the proof. $\square$

5 Growth rate of the velocity process V

We turn now to the study of the growth rate of the velocity process V. In this section, we assume also that

$$\forall x\in\mathbb{R},\ \forall\lambda>0,\ \operatorname{sgn}(F(\lambda x))=\operatorname{sgn}(F(x)). \tag{H_3}$$

Proposition 5.1. *When $\alpha<1$ or $\operatorname{sgn}(F(v))=\operatorname{sgn}(v)$,*

$$\limsup_{t\to\infty}\frac{|V_t|}{\sqrt{2\ln\ln t}}\leq 1\ \textit{a.s. and}\ \limsup_{t\to\infty}\frac{V_t}{\sqrt{2\ln\ln t}}=1\ \textit{a.s.} \tag{17}$$

Besides

$$\limsup_{t\to\infty}\frac{|X_t|}{\int_1^t\sqrt{2\ln\ln s}\,\mathrm{d}s}\leq 1\ \textit{a.s.} \tag{18}$$

We discuss first results of existence and behaviour of some time-homogeneous processes $V^{\pm}$ such that $V^{-}\leq V^{(e)}\leq V^{+}$ almost surely. For $\alpha>-1$, let π be a non-negative function satisfying

$$\forall x\in\mathbb{R},\lambda>0,\ \pi(\lambda x)=\lambda^{\alpha}\pi(x).$$

Recall that π satisfies (2). Under $(H_1\alpha)$ we take $\pi=|F|$ and under $(H_2\alpha)$, we take $\pi=G$.

Define the pathwise unique strong solution (up to the explosion time) to the time-homogeneous equation

$$\mathrm{d}V_s^{\pm}=\mathrm{d}W_s-\frac{V_s^{\pm}}{2}\,\mathrm{d}s\pm t_0^{\frac{\alpha+1}{2}-\beta}\pi(V_s^{\pm})\mathbb{1}_{\{\pm F(V_s^{\pm})<0\}}\,\mathrm{d}s. \tag{19}$$

Lemma 5.2. *Set $\tau_\infty^{\pm}$, τ_∞, respectively the explosion time of $V^{\pm}$ and V.*

(i) If $\alpha\leq 1$ or $F(1)\geq 0$, then $\tau_\infty^{+}=\infty$ a.s.

(ii) If $\alpha\leq 1$ or $F(-1)\leq 0$, then $\tau_\infty^{-}=\infty$ a.s.

(iii) If $\alpha>1$ and $F(1)<0$, then $\mathbb{P}(\tau_\infty^{+}=\infty)=0$.

(iv) If $\alpha>1$ and $F(-1)>0$, then $\mathbb{P}(\tau_\infty^{-}=\infty)=0$.

Proof. STEP 1. Firstly let us prove that $V^- \le V^{(e)} \le V^+$ almost surely. Indeed, if we denote

$$\mathtt{b}(t,x) = -\frac{x}{2} - t_0^{1/2-\beta} e^{(1/2-\beta)t} F(\sqrt{t_0} e^{t/2} x) \text{ and } \mathtt{b}^+(x) = -\frac{x}{2} + t_0^{(\alpha+1)/2-\beta} \pi(x) \mathbb{1}_{\{F(x)\le 0\}},$$

thanks to (H_3), we can write, for all $t \ge 0$ and all $x \in \mathbb{R}$,

$$\mathtt{b}(t,x) \le \mathtt{b}^+(x) \iff -e^{(1/2-\beta)t} F(\sqrt{t_0} e^{t/2} x) \le t_0^{\frac{\alpha}{2}} \pi(x) \mathbb{1}_{\{F(x)\le 0\}}.$$

This inequality holds true by the choice of π and the assumption $(\alpha+1)/2 - \beta < 0$. By using the comparison theorem (see Theorem 3.7 p 394, in [RY99]) one gets, $V^{(e)} \le V^+$, almost surely. The other inequality can be obtained in the same way.

STEP 2. Call $\tau_\infty^{(e)}$ the explosion time of $V^{(e)}$, then $\{\tau_\infty^{(e)} = \infty\} = \{\tau_\infty = \infty\}$. So, $\{\tau_\infty^- = \infty\} \cap \{\tau_\infty^+ = \infty\} \subset \{\tau_\infty = \infty\}$.
We give the detailed proof for (i) and (iii), the other parts could be obtained by changing "+" and "$-$" in the reasoning. First, we prove (i). The scale function of V^+ is given, for $x \in \mathbb{R}$, by

$$\mathtt{p}^+(x) := \int_0^x \exp\left(\frac{y^2}{2} - 2t_0^{\frac{(\alpha+1)}{2}-\beta} \int_0^y \pi(z) \mathbb{1}_{\{F(z)\le 0\}}\, \mathrm{d}z\right) \mathrm{d}y.$$

Note that, if $x < 0$, $-\mathtt{p}^+(x) \ge \int_x^0 e^{y^2/2}\, \mathrm{d}y$. Thus $\mathtt{p}^+(-\infty) = -\infty$. Suppose that $F(1) \ge 0$, then for $x \ge 0$, $\mathtt{p}^+(x) = \int_0^x e^{y^2/2}\, \mathrm{d}y$, so $\mathtt{p}^+(\infty) = \infty$. By Proposition 5.22, p. 345, in [KS98], the conclusion follows. Assume now that $\alpha < 1$ and $F(1) < 0$. Then, for $x \ge 0$, $\mathtt{p}^+(x) = \int_0^x \exp\left(\frac{y^2}{2} - 2t_0^{\frac{\alpha+1}{2}-\beta} \pi(1) \frac{y^{\alpha+1}}{\alpha+1}\right) \mathrm{d}y$, so $\mathtt{p}^+(\infty) = \infty$. Using the same result in [KS98], the conclusion follows. If $\alpha = 1$, the drift has linear growth and the conclusion is clear.

STEP 3. We proceed with the proof of (iii). Assume $\alpha > 1$ and $F(1) < 0$. As previously, $\mathtt{p}^+(-\infty) = -\infty$. Besides, $\mathtt{p}^+(\infty) < \infty$. Denote $\mathtt{m}^+ : y \mapsto 2/(\mathtt{p}^+)'(y)$ the speed measure of V^+. Fix $y > 0$, then, setting $\mathtt{c} = 2t_0^{\frac{\alpha+1}{2}-\beta} \pi(1) > 0$, one can apply integration by parts to get:

$$\begin{aligned}
\left(\mathtt{p}^+(\infty) - \mathtt{p}^+(\mathtt{y})\right) \mathtt{m}^+(y) =& 2\exp\left(-\frac{y^2}{2} + \mathtt{c}\frac{y^{\alpha+1}}{\alpha+1}\right) \int_y^{+\infty} \exp\left(\frac{z^2}{2} - \mathtt{c}\frac{z^{\alpha+1}}{\alpha+1}\right) \mathrm{d}z \\
=& \frac{2}{\mathtt{c}y^\alpha - y} + 2\exp\left(-\frac{y^2}{2} + \mathtt{c}\frac{y^{\alpha+1}}{\alpha+1}\right) \int_y^\infty e^{\frac{z^2}{2} - \mathtt{c}\frac{z^{\alpha+1}}{\alpha+1}} \frac{1 - \mathtt{c}\alpha z^{\alpha-1}}{(z - \mathtt{c}z^\alpha)^2}\, \mathrm{d}z.
\end{aligned}$$

One can deduce, by integrating small o, that

$$\left(\mathtt{p}^+(\infty) - \mathtt{p}^+(\mathtt{y})\right) \mathtt{m}^+(y) \underset{y\to\infty}{\sim} \frac{2}{\mathtt{c}y^\alpha - y}$$

which is an integrable function at ∞. The conclusion follows from Theorem 5.29, p. 348, in [KS98]. $\square$

Proof of Proposition 5.1.

STEP 1. Remark first that the first inequality of (17) is equivalent to $\limsup_{t\to\infty} \frac{\left|V_t^{(e)}\right|}{\sqrt{2\ln t}} \le 1$. Assuming

$$\limsup_{t\to\infty} \frac{V_t^+}{\sqrt{2\ln t}} \le 1 \text{ a.s.,} \quad \text{and} \quad \limsup_{t\to\infty} \frac{-V_t^-}{\sqrt{2\ln t}} \le 1 \text{ a.s,} \tag{20}$$

one gets the first inequality of (17) writting

$$\limsup_{t\to\infty}\frac{V_t^{(e)}}{\sqrt{2\ln t}}\leq\limsup_{t\to\infty}\frac{V_t^{+}}{\sqrt{2\ln t}}\leq 1 \text{ a.s. and } \limsup_{t\to\infty}\frac{-V_t^{(e)}}{\sqrt{2\ln t}}\leq\limsup_{t\to\infty}\frac{-V_t^{-}}{\sqrt{2\ln t}}\leq 1 \text{ a.s.}$$

Before, proving (20) let us state Motoo's theorem which proof is given in [Mot59].

Theorem 5.3 (Motoo). *Let Z be a regular continuous strong Markov process in (a,∞), $a\in[-\infty,\infty)$. Assume also that Z is time-homogeneous, with scale function $\mathtt{s}$. For every real positive increasing function h,*

$$\mathbb{P}\left(\limsup_{t\to\infty}\frac{Z_t}{h(t)}\geq 1\right)=0 \textit{ or } 1 \textit{ according to whether } \int^{\infty}\frac{\mathrm{d}t}{\mathtt{s}(h(t))}<\infty \textit{ or } =\infty.$$

Motoo's theorem yields for all $\varepsilon>0$,

$$\mathbb{P}\Big(\limsup_{t\to\infty}\frac{V_t^{+}}{\sqrt{2\ln t}}\geq 1+\varepsilon\Big)=0 \text{ and } \mathbb{P}\Big(\limsup_{t\to\infty}\frac{-V_t^{-}}{\sqrt{2\ln t}}\geq 1+\varepsilon\Big)=0.$$

Indeed, define $\tilde{V}^-:=-V^-$, then

$$\mathrm{d}\tilde{V_s}^- = -\mathrm{d}W_s-\frac{\tilde{V_s}^-}{2}\,\mathrm{d}s+t_0^{(\alpha+1)/2-\beta}\pi(-\tilde{V_s}^-)\mathbb{1}_{\{F(-\tilde{V_s}^-)>0\}}\,\mathrm{d}s.$$

Fix $y_0>0$. The scale function of V^+ and $\tilde{V}^-$ is given, for $y\geq y_0$, by

$$\mathtt{s}^{\pm}(y):=\kappa\int_{y_0}^{y}\exp\Big(\frac{z^2}{2}-2C^{\pm}\frac{z^{\alpha+1}}{\alpha+1}\Big)\,\mathrm{d}z.$$

Here and elsewhere κ denotes positive constants which can change of value from line to line, and

$$C^+:=t_0^{\frac{\alpha+1}{2}-\beta}\pi(1)\mathbb{1}_{\{F(1)<0\}}\quad\text{for } V^+\quad\text{ and }\quad C^-:=t_0^{\frac{\alpha+1}{2}-\beta}\pi(-1)\mathbb{1}_{\{F(-1)>0\}}\quad\text{for } \tilde{V}^-.$$

Let $\epsilon>0$. Define the positive increasing function $h: t\mapsto(1+\epsilon)\sqrt{2\ln t}$. We will show that ${}^1\!/\!_{\mathtt{s}(h)}$ is integrable at infinity. Firstly, remark that

$$\int_{y_0}^{+\infty}\frac{1}{\mathtt{s}(h(t))}\,\mathrm{d}t=\int_{h(y_0)}^{+\infty}\frac{1}{\mathtt{s}(y)}\frac{\mathrm{d}y}{h'(h^{-1}(y))}=\int_{h(y_0)}^{+\infty}\frac{1}{\mathtt{s}(y)}\frac{y\exp\big(y^2/\big(2(1+\epsilon)^2\big)\big)}{(1+\varepsilon)^2}\,\mathrm{d}y.$$

It remains to find an equivalent of $\mathtt{s}$ at infinity. In the following "$\asymp$" means equality up to a multiplicative positive constant. Fix $y>y_0$, integrating by parts, one gets,

$$\begin{aligned}\mathtt{s}(y)&\asymp\int_{y_0}^{y}\exp\Big(\frac{z^2}{2}-2C^{\pm}\frac{z^{\alpha+1}}{\alpha+1}\Big)\big(z-2C^{\pm}z^{\alpha}\big)\cdot\frac{1}{z-2C^{\pm}z^{\alpha}}\,\mathrm{d}z\\&\asymp\frac{\exp\big(\frac{y^2}{2}-2C^{\pm}\frac{y^{\alpha+1}}{\alpha+1}\big)}{y-2C^{\pm}y^{\alpha}}-\kappa+\int_{y_0}^{y}\frac{1-2\alpha C^{\pm}z^{\alpha-1}}{(z-2C^{\pm}z^{\alpha})^2}\exp\Big(\frac{z^2}{2}-2C^{\pm}\frac{z^{\alpha+1}}{\alpha+1}\Big)\,\mathrm{d}z.\end{aligned}$$

Since $\alpha>-1$, $\displaystyle\lim_{y\to\infty}\frac{1-2\alpha C^{\pm}y^{\alpha-1}}{(y-2C^{\pm}y^{\alpha})^2}=0$, except when $\alpha=1$ and $C^{\pm}=\frac{1}{2}$. Moreover the function $y\mapsto\exp\Big(\frac{y^2}{2}-2C^{\pm}\frac{y^{\alpha+1}}{\alpha+1}\Big)$ is not integrable at infinity, when $C^{\pm}=0$, or $\alpha<1$, or $\alpha=1$ and $C^{\pm}<\frac{1}{2}$. In these cases one gets, by integration,

$$\mathtt{s}(y)\underset{y\to\infty}{\sim}\kappa\frac{\exp\big(\frac{y^2}{2}-2C^{\pm}\frac{y^{\alpha+1}}{\alpha+1}\big)}{y-2C^{\pm}y^{\alpha}}.$$

Hence,

$$\frac{1}{\mathtt{s}(y)}\frac{y\exp\left(\frac{y^2}{2(1+\varepsilon)^2}\right)}{(1+\varepsilon)^2} \underset{y\to\infty}{\sim} \kappa\big(y^2-2C^{\pm}\,y^{\alpha+1}\big)\exp\Big(-\frac{y^2}{2}\Big(1-\frac{1}{(1+\varepsilon^2)}\Big)+2C^{\pm}\,\frac{y^{\alpha+1}}{\alpha+1}\Big).$$

which is integrable if $C^{\pm}=0$ or $\alpha<1$, or $\alpha=1$ and $C^{\pm}<\frac{1}{2}$. One can conclude using Motoo's theorem.

STEP 2. The second inequality of (17) is equivalent to $\limsup_{t\to\infty}\dfrac{V_t^{(e)}}{\sqrt{2\ln t}}=1$ a.s. From the first step, we have $\limsup_{t\to\infty}\dfrac{V_t^{(e)}}{\sqrt{2\ln t}}\le 1$. We need to prove the opposite inequality. Note that the first inequality of (17) implies

$$\lim_{t\to\infty} t_0^{1/2-\beta}e^{(1/2-\beta)t}F\big(\sqrt{t_0}e^{t/2}V_t^{(e)}\big)=0 \text{ a.s.} \tag{21}$$

For $u\ge 0$, introduce the pathwise unique strong solution of

$$\mathrm{d}V_s(u)=\mathrm{d}W_s-\left(\frac{V_s(u)}{2}+1\right)\mathrm{d}s,\ V_u(u)=V_u^{(e)},$$

and define the stopping time $\tau_u:=\inf\left\{t\ge u,\ t_0^{1/2-\beta}e^{(1/2-\beta)t}\left|F\big(\sqrt{t_0}e^{t/2}V_t^{(e)}\big)\right|>1\right\}$. By using the comparison theorem (see Theorem 3.7 p 394, in [RY99]) and a classical argument of localisation one gets, $V_{\bullet\wedge\tau_u}(u)\le V^{(e)}_{\bullet\wedge\tau_u}$, almost surely. Hence

$$\forall t\ge u,\ V_t(u)\le V_t^{(e)} \text{ a.s. on } \{\tau_u=\infty\}=\left\{\sup_{t\ge u} t_0^{1/2-\beta}e^{(1/2-\beta)t}\left|F\big(\sqrt{t_0}e^{t/2}V_t^{(e)}\big)\right|\le 1\right\}:=\Omega_u. \tag{22}$$

The scale function of $V(u)$ is given, for $y\ge 1$, by

$$\mathtt{s_u}(y):=\kappa\int_1^y\exp\left(\frac{z^2}{2}+2z\right)\mathrm{d}z.$$

Hence, with $g:t\mapsto(1-\varepsilon)\sqrt{2\ln t}$, for t big enough,

$$\begin{aligned}\frac{1}{\mathtt{s_u}(g(t))}&=\left(\kappa\int_1^{g(t)}\exp\left(\frac{z^2}{2}+2z\right)\mathrm{d}z\right)^{-1}\ge\left(\kappa g(t)\exp\left(\frac{g(t)^2}{2}+2g(t)\right)\right)^{-1}\\&\ge\frac{\exp\left(-2\sqrt{2\ln t}(1-\varepsilon)\right)}{t^{(1-\varepsilon)^2}(1-\varepsilon)\sqrt{2\ln t}},\quad\text{which is not integrable at infinity.}\end{aligned}$$

By applying Motoo's theorem to $V(u)$ and using (21) we get

$$1\le\limsup_{t\to\infty}\frac{V_t(u)}{\sqrt{2\ln t}}\le\limsup_{t\to\infty}\frac{V_t^{(e)}}{\sqrt{2\ln t}} \text{ a.s. on } \Omega_u, \text{ and } \mathbb{P}\left(\cup_{u\ge 0}\Omega_u\right)=1.$$

This concludes the proof of (17).

STEP 3. We prove now (18). Fix $\eta>0$. thanks to (17), there exists $T_0>1$ such that

$$\forall T\ge T_0,\ \sup_{t\ge T}\frac{|V_t|}{\sqrt{2\ln\ln t}}\le 1+\eta. \tag{23}$$

Fix $T \geq T_0$, then, denoting $H : t \in (1, \infty) \mapsto \int_1^t \sqrt{2 \ln \ln s}\, \mathrm{d}s$,

$$\begin{aligned}\sup_{t \geq T} \frac{|X_t|}{H(t)} &\leq (T_0 - t_0) \sup_{s \in [t_0, T_0]} |V_s| \sup_{t \geq T} \frac{1}{H(t)} + \sup_{t \geq T} \sup_{s \in [T_0, t]} \frac{|V_s|}{\sqrt{2 \ln \ln s}} \frac{1}{H(t)} \int_{T_0}^t \sqrt{2 \ln \ln s}\, \mathrm{d}s \\ &\leq (T_0 - t_0) \sup_{s \in [t_0, T_0]} |V_s| \sup_{t \geq T} \frac{1}{H(t)} + (1 + \eta) \sup_{t \geq T} \frac{1}{H(t)} \int_{T_0}^t \sqrt{2 \ln \ln s}\, \mathrm{d}s \underset{T \to \infty}{\longrightarrow} 1 + \eta.\end{aligned}$$

□

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