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GENERALIZED BOUNDS FOR HYPERBOLIC SINE AND HYPERBOLIC COSINE FUNCTIONS

RAMKRISHNA M. DHAIGUDE, YOGESH J. BAGUL, AND VINAY M. RAUT

ABSTRACT. The main objective of this paper is to establish several new lower and upper bounds for the functions $\sinh x/x$ and $\cosh x$. Following the simple approach, our results give refinements and generalizations of some known inequalities involving these functions.

1. INTRODUCTION

Inequalities involving hyperbolic functions are as much important as inequalities involving trigonometric functions. Recently many searchers established hyperbolic inequalities(see e.g. [3], [4], [5], [6], [7], [8], [11], [12], [13], [14], [15], [16] and references therein). We start by giving brief summary of already proved results pertaining the main results of this paper.

The inequalities

$$1 + \frac{x^2}{6} < \frac{\sinh x}{x} < 1 + \frac{x^2}{k_1}; 0 < x < 1 \quad (1.1)$$

where $k_1 \approx 5.707724$ and

$$1 + \frac{x^2}{2} < \cosh x < 1 + \frac{x^2}{k_2}; 0 < x < 1 \quad (1.2)$$

where $k_2 \approx 1.841348$ are proved in [3] and [6] respectively. Recently, Christophe Chesneau and Yogesh J. Bagul [8] established the following results:

$$\left(1 + \frac{x^2}{\pi^2}\right)^{\frac{\pi^2}{6}} < \frac{\sinh x}{x}; x > 0 \quad (1.3)$$

and

$$\left(1 + \frac{4x^2}{\pi^2}\right)^{\frac{\pi^2}{8}} < \cosh x; x > 0. \quad (1.4)$$

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Before the establishment of above inequalities, Ling Zhu [16] in 2008, discovered some inequalities having similarity with these inequalities. Zhu's inequalities are stated as

Statement 1. ([16]) Let $0 < x < r$. Then

$$\left(\frac{r^2 + x^2}{r^2 - x^2}\right)^\alpha < \frac{\sinh x}{x} < \left(\frac{r^2 + x^2}{r^2 - x^2}\right)^\beta \quad (1.5)$$

holds if and only if $\alpha \leq 0$ and $\beta \geq \frac{r^2}{12}$.

Statement 2. ([16]) Let $0 < x < r$. Then

$$\left(\frac{r^2 + x^2}{r^2 - x^2}\right)^\alpha < \cosh x < \left(\frac{r^2 + x^2}{r^2 - x^2}\right)^\beta \quad (1.6)$$

holds if and only if $\alpha \leq 0$ and $\beta \geq \frac{r^2}{4}$.

In this paper, we aim to refine and generalize all the inequalities listed above.

2. PRELIMINARIES AND LEMMAS

We are familiar with the following power series expansions [9, 1.411]:

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}, \text{ and } \cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}. \quad (2.1)$$

The series expansions in (2.1) are useful for our main results. We also need the following two known results.

Lemma 1. ([2, p. 10]) Let $f, g : [a, b] \rightarrow \mathbb{R}$ be continuous. Moreover, let f, g be differentiable on (a, b) and $g'(x) \neq 0$, on (a, b) . Let,

$$A_1(x) = \frac{f(x) - f(a)}{g(x) - g(a)}, A_2(x) = \frac{f(x) - f(b)}{g(x) - g(b)}, \quad x \in (a, b).$$

- (i) $A_1(\cdot)$ and $A_2(\cdot)$ are increasing(strictly increasing) on (a, b) if $f'(\cdot)/g'(\cdot)$ is increasing(strictly increasing) on (a, b) .
- (ii) $A_1(\cdot)$ and $A_2(\cdot)$ are decreasing(strictly decreasing) on (a, b) if $f'(\cdot)/g'(\cdot)$ is decreasing(strictly decreasing) on (a, b) .

The Lemma 1 is known in the literature as l'Hôpital's rule of monotonicity. For Lemma 2 refer [1, 10].

Lemma 2. ([1, 10]) Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ be convergent for $|x| < R$, where a_n and b_n are real numbers for $n = 0, 1, 2, \dots$ such that $b_n > 0$. If the sequence a_n/b_n is strictly increasing(or decreasing), then the function $A(x)/B(x)$ is also strictly increasing(or decreasing) on $(0, R)$.

Lemma 1 and Lemma 2 are proved to be important tools in the field of inequalities.

3. MAIN RESULTS

The first result of the paper states

Theorem 1. *Let $a \geq \frac{1}{15}$. Then the function $F(x) = \frac{\log(\sinh x/x)}{\log(1+ax^2)}$ is strictly increasing on $(0, r)$ where $r \in (0, \infty)$. In particular, with this fixed value of a , the best possible constants α and β such that*

$$(1 + ax^2)^\alpha < \frac{\sinh x}{x} < (1 + ax^2)^\beta \quad (3.1)$$

are $\frac{1}{6a}$ and $\frac{\log(\sinh r/r)}{\log(1+ar^2)}$ respectively. i.e. $\alpha \leq \frac{5}{2}$ and $\beta \leq \frac{\log(\sinh r/r)}{\log(1+r^2/15)}$.

Proof. Consider

$$F(x) = \frac{\log\left(\frac{\sinh x}{x}\right)}{\log(1+ax^2)} = \frac{F_1(x)}{F_2(x)}$$

where $F_1(x) = \log\left(\frac{\sinh x}{x}\right)$ and $F_2(x) = \log(1+ax^2)$ with $F_1(0+) = 0 = f_2(0)$. After differentiating

$$\begin{aligned} \frac{F_1'(x)}{F_2'(x)} &= \frac{1}{2a}(1+ax^2) \left(\frac{x \cosh x - \sinh x}{x^2 \sinh x} \right) \\ &= \frac{1}{2a} \left(\frac{\coth x}{x} - \frac{1}{x^2} + ax \coth x - a \right) \\ &= \frac{1}{2a} F_3(x). \end{aligned}$$

Now $F_3(x)$ is increasing if and only if $F_3'(x) > 0$. Consequently, by using Lemma 1 we can conclude that $F(x)$ will be increasing if $F_3'(x) > 0$. This means

$$\left[2 - \left(\frac{x}{\sinh x} \right)^2 - \frac{x}{\tanh x} \right] > ax^2 \left[\left(\frac{x}{\sinh x} \right)^2 - \frac{x}{\tanh x} \right]$$

which is equivalent to

$$\frac{\left[2 - \left(\frac{x}{\sinh x} \right)^2 - x \coth x \right]}{x^2 \left[\left(\frac{x}{\sinh x} \right)^2 - x \coth x \right]} < a,$$

due to well known relation $\left(\frac{x}{\sinh x}\right)^2 < 1 < \frac{x}{\tanh x}$. Thus $F_4(x) < a$, where

$$\begin{aligned} F_4(x) &= \frac{\left[2 - \left(\frac{x}{\sinh x}\right)^2 - x \coth x\right]}{x^2 \left[\left(\frac{x}{\sinh x}\right)^2 - x \coth x\right]} = \frac{2 \sinh^2 x - x^2 - x \sinh x \cosh x}{x^2(x^2 - x \sinh x \cosh x)} \\ &= \frac{2 \cosh 2x - 2 - 2x^2 - x \sinh 2x}{2x^4 - x^3 \sinh 2x}. \end{aligned}$$

Utilizing (2.1) we get

$$\begin{aligned} F_4(x) &= \frac{\sum_{n=0}^{\infty} \frac{2^{2n+1}}{(2n)!} x^{2n} - 2 - 2x^2 - \sum_{n=0}^{\infty} \frac{2^{2n+1}}{(2n+1)!} x^{2n+2}}{2x^4 - \sum_{n=0}^{\infty} \frac{2^{2n+1}}{(2n+1)!} x^{2n+4}} \\ &= \frac{\sum_{n=2}^{\infty} \frac{2^{2n+1}}{(2n)!} x^{2n} - \sum_{n=2}^{\infty} \frac{2^{2n-1}}{(2n-1)!} x^{2n}}{-\sum_{n=3}^{\infty} \frac{2^{2n-3}}{(2n-3)!} x^{2n}} \\ &= \frac{\sum_{n=3}^{\infty} \left[\frac{2^{2n-1}}{(2n-1)!} - \frac{2^{2n+1}}{(2n)!} \right] x^{2n}}{\sum_{n=3}^{\infty} \frac{2^{2n-3}}{(2n-3)!} x^{2n}} = \frac{\sum_{n=0}^{\infty} a_n x^{2n}}{\sum_{n=0}^{\infty} b_n x^{2n}} = \frac{A(x)}{B(x)}. \end{aligned}$$

Clearly $A(x)$ and $B(x)$ are convergent by ratio test and

$$\frac{a_n}{b_n} = \frac{4(n-2)}{n(2n-1)(2n-2)} = c_n(\text{say}).$$

We claim that $c_n > c_{n+1}$. For if $c_n \leq c_{n+1}$, then it implies $2(2n-2)(n+1)(2n+1) \leq (n-1)(2n-1)(2n-2)$, i.e. $4n^2 - 9n - 1 \leq 0$ or $n \leq \frac{9+\sqrt{97}}{8} < \frac{19}{8} < 3$, which contradicts to $n \geq 3$. Therefore, a sequence $\left\{\frac{a_n}{b_n}\right\}$ is strictly decreasing and $b_n > 0, \forall n \geq 3$. By Lemma 2, $F_4(x)$ is strictly decreasing on $(0, r)$. So, $\sup\{F_4(x) : x > 0\} \leq a$ and $\lim_{x \rightarrow 0+} F_4(x) = \frac{1}{15}$ gives $a \geq \frac{1}{15}$. Now $F(x)$ is increasing and we have

$$\lim_{x \rightarrow 0+} F(x) \leq F(x) \leq \lim_{x \rightarrow r-} F(x).$$

Lastly, $\lim_{x \rightarrow 0+} F(x) = \frac{1}{6a}$ by l'Hôpital's rule and $\lim_{x \rightarrow r-} F(x) = \frac{\log(\sinh r/r)}{\log(1+ar^2)}$ finish the proof. $\square$

We prove our second result without making use of power series expansions in (2.1).

Theorem 2. *Let $a \geq \frac{1}{3}$. Then the function $G(x) = \frac{\log(\cosh x)}{\log(1+ax^2)}$ is strictly increasing in $(0, r)$ where $r \in (0, \infty)$. In particular, the best positive constants*

γ and δ such that

$$(1 + ax^2)^\gamma < \cosh x < (1 + ax^2)^\delta \quad (3.2)$$

are $\frac{1}{2a}$ and $\frac{\log(\cosh r)}{\log(1+ar^2)}$. i.e. $\gamma \leq \frac{3}{2}$ and $\delta \leq \frac{\log(\cosh r)}{\log(1+r^2/3)}$.

Proof. Consider

$$G(x) = \frac{\log(\cosh x)}{\log(1 + ax^2)} = \frac{G_1(x)}{G_2(x)}$$

where $G_1(x) = \log(\cosh x)$ and $G_2(x) = \log(1 + ax^2)$ with $G_1(0) = G_2(0) = 0$. Differentiating

$$\frac{G'_1(x)}{G'_2(x)} = \frac{1}{2a} \frac{(1 + ax^2) \sinh x}{x \cosh x} = \frac{1}{2a} G_3(x)$$

where $G_3(x) = \frac{(1+ax^2)\sinh x}{x \cosh x}$. Obviously, $\frac{G'_1(x)}{G'_2(x)}$ is increasing if and only if $G'_3(x) > 0$. By Lemma 1, $G(x)$ will be increasing if $G'_3(x) > 0$. That means $x \cosh^2 x - \sinh x \cosh x - x \sinh^2 x > ax^2 (-\sinh x \cosh x + x \sinh^2 x - x \cosh^2 x)$.

Or

$$\frac{(x \cosh^2 x - \sinh x \cosh x - x \sinh^2 x)}{x^2 (-\sinh x \cosh x + x \sinh^2 x - x \cosh^2 x)} = G_4(x) < a$$

due to the fact that $\tanh x < \coth x$. $G_4(x)$ can be written as

$$G_4(x) = \frac{2x - \sinh 2x}{-x^2(2x + \sinh 2x)} = \frac{G_5(x)}{G_6(x)}$$

where $G_5(x) = 2x - \sinh 2x$ and $G_6(x) = -2x^3 - x^2 \sinh 2x$ satisfying $G_5(0) = G_6(0) = 0$. Differentiation gives

$$\frac{G'_5(x)}{G'_6(x)} = \frac{2 - 2 \cosh 2x}{-6x62 - 2x^2 \cosh 2x - 2x \sinh 2x} = \frac{G_7(x)}{G_8(x)}$$

with $G_7(0) = 0 = G_8(0)$. Continuing the argument

$$\frac{G'_7(x)}{G'_8(x)} = \frac{2 \sinh 2x}{6x + 2x^2 \sinh 2x + 4x \cosh 2x + \sinh 2x} = \frac{G_9(x)}{G_{10}(x)}.$$

Further

$$\begin{aligned} \frac{G'_9(x)}{G'_{10}(x)} &= \frac{4 \cosh 2x}{6 + 12x \sinh 2x + 4x^2 \cosh 2x + 6 \cosh 2x} \\ &= \frac{4}{6 \operatorname{sech} 2x + 12x \tanh 2x + 4x^2 + 6} = \frac{2}{G_{11}(x)} \end{aligned}$$

Now $G'_{11}(x) = 6 \tanh 2x(1 - \operatorname{sech} 2x) + 12x \operatorname{sech}^2 x + 4x > 0$ implies $G_{11}(x)$ is strictly increasing on $(0, r)$. By Lemma 1, $G_4(x)$ is also decreasing on $(0, r)$.

Therefore, $\sup \{G_4(x) : x \in (0, r)\} \leq a$ and $\lim_{x \rightarrow 0+} G_4(x) = \frac{1}{3}$ gives $a \geq \frac{1}{3}$. Now $G(x)$ being increasing in $(0, r)$ for specified values of a we have that

$$\lim_{x \rightarrow 0+} G(x) < G(x) < \lim_{x \rightarrow r-} G(x).$$

Finally, $\lim_{x \rightarrow 0+} G(x) = \frac{1}{2a}$ and $\frac{\log(\cosh r)}{\log(1+ar^2)}$ prove the desired result. $\square$

4. APPLICATIONS

Double inequalities in (1.1) is a particular case of Theorem 1 where $a = 1/6$ and $r = 1$. On the similar line inequality (1.3) can be obtained by taking $a = 1/\pi^2$ in (3.1). To get the sharpest inequality of this kind we put $a = 1/15$ in Theorem 1. It is stated as follows:

$$\left(1 + \frac{x^2}{15}\right)^{\frac{5}{2}} < \frac{\sinh x}{x} < \left(1 + \frac{x^2}{15}\right)^{\beta_1}; x \in (0, r) \text{ where } r \in (0, \infty) \quad (4.1)$$

and $\beta_1 = \frac{\log(\sinh r/r)}{\log(1+r^2/15)}$. Two sided inequality in (4.1) is a refinement and (or) generalization of inequalities in (1.1), (1.3) and (1.5).

On the other hand, inequalities in (1.2) and (1.4) are particular cases of Theorem 2 where $a = 1/2$ ($r = 1$) and $a = 4/\pi^2$ respectively. The sharpest inequality of this kind is obtained by putting $a = 1/3$ in Theorem 2 as follows:

$$\left(1 + \frac{x^2}{3}\right)^{\frac{3}{2}} < \cosh x < \left(1 + \frac{x^2}{3}\right)^{\delta_1}; x \in (0, r) \text{ where } r \in (0, \infty) \quad (4.2)$$

and $\delta_1 = \frac{\log(\cosh x)}{\log(1+r^2/3)}$. Again two sided inequality in (4.2) is a refinement and (or) generalization of inequalities in (1.2), (1.4) and (1.6).

At the end we state and prove the following proposition:

Proposition 1. *Let $x > 0$. Then*

$$1 + \frac{x^2}{6} < \left(\frac{1 + \cosh x}{2}\right)^{\frac{2}{3}}. \quad (4.3)$$

Proof. Set

$$H(x) = 2 \left(1 + \frac{x^2}{6}\right)^{\frac{3}{2}} - \cosh x.$$

On differentiating

$$H'(x) = x \left[\left(1 + \frac{x^2}{6}\right)^{\frac{1}{2}} - \frac{\sinh x}{x} \right] < 0,$$

as $\left(1 + \frac{x^2}{6}\right)^{\frac{1}{2}} < 1 + \frac{x^2}{6} < \frac{\sinh x}{x}$ by Theorem 1. Hence $H(x)$ is strictly decreasing for $x > 0$. So $\lim_{x \rightarrow 0} H(x) = 1 > H(x)$ gives the desired inequality. $\square$

Remark 1. Combining inequality (4.3) with the inequality [14, Thm. 2.3]

$$\left(\frac{1 + \cosh x}{2}\right)^{\frac{2}{3}} < \frac{\sinh x}{x}$$

we get

$$1 + \frac{x^2}{6} < \left(\frac{1 + \cosh x}{2}\right)^{\frac{2}{3}} < \frac{\sinh x}{x}.$$

5. CONCLUSION

We obtained several inequalities involving $\sinh x/x$ and $\cosh x$, thus refined some known inequalities in the literature with complete new approach.

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