OPTIMAL CONTROL WITH FINAL OBSERVATION OF A FRACTIONAL DIFFUSION WAVE EQUATION

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Abstract. We consider a controlled fractional diffusion wave equation involving Riemann-Liouville fractional derivative or order $\alpha \in (1,2)$. First we prove by means of eigenfunction expansions the existence of solutions to such equations. Then we show that we can approach the fractional integral of order $2-\alpha$ of the state at final time by a desired state by acting on the control. Using the first order Euler-Lagrange optimality, we obtain the characterization of the optimal control.

1. Introduction

Let $n \in \mathbb{N}^*$ and Ω be a bounded open subset of $\mathbb{R}^n$ with boundary $\partial\Omega$ of class $\mathcal{C}^2$. For the time $T > 0$, we set $Q = \Omega\times]0,T[$ and $\Sigma = \partial\Omega\times]0,T[$, and we consider the following fractional diffusion wave equation:

$$
\left\{
\begin{array}{rcll}
D_{RL}^{\alpha} y(x,t) - \Delta y(x,t) & = & v(x,t) & (x,t) \in Q \\
y(\sigma,t) & = & 0 & (\sigma,t) \in \Sigma \\
I^{2-\alpha} y(x,0^+) & = & y^0 & x \in \Omega \\
\dfrac{d}{dt} I^{2-\alpha} y(x,0^+) & = & y^1 & x \in \Omega
\end{array}
\right. \tag{1}
$$

where $1 < \alpha < 2$, $y^0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $y^1 \in L^2(\Omega)$ and $v \in L^2(Q)$. $I^{2-\alpha} y(x,0^+) = \lim_{t\to 0} I^{2-\alpha} y(x,t)$ and $\dfrac{d}{dt} I^{2-\alpha} y(x,0^+) = \lim_{t\to 0} \dfrac{d}{dt} I^{2-\alpha} y(x,t)$ where the fractional integral I^{α} of order α and the fractional derivative D_{RL}^{α} of order α are to be understood in the Riemann-Liouville sense.

There are many works on fractional diffusion wave equation. For instance, Mainardi et al. [10, 12] generalized the diffusion equation by replacing the first time derivative with a fractional derivative of order α. These authors proved that the process changes from slow diffusion to classical diffusion, then to diffusion-wave and finally to classical wave when α increases from 0 to 2. The fundamental solutions of the Cauchy problems associated to these generalized diffusion equation ($0 < \alpha < 2$) are studied in [12, 13]. By means of Fourier-Laplace transforms, the

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authors expressed these solutions in term of Wright-type functions that can be interpreted as spatial probability density functions evolving in time with similarity properties. Agrawal [14] studied the solutions for a fractional diffusion-wave equation defined in a bounded domain when the fractional time derivative is described in the Caputo sense. Using Laplace transform and finite sine transform technique, the author obtained the general solution in terms of Mittag-Leffler functions. Note also that the formulation of Mainardi et al. is extended to a fractional wave equation that contains a fourth order space derivative term by Agrawal [15]. In [17], Yamamoto et al. studied by means of eigenfunctions the initial value/boundary value problems for fractional diffusion equation and apply the results to some inverse problems. We also refer to [11, 28, 16, 30, 31, 32, 34] and the reference therein for more literature on fractional diffusion equations.
Optimal control of fractional diffusion equations has also been studied by several authors. In [19], Agrawal considered two problems, the simplest fractional variation problem and fractional variational problem of Lagrange. For both problems, the author developed the Euler-Lagrange type necessary conditions which must be satisfied for the given functional to be extremum. In [22], the Euler-Lagrange equations and the transversality conditions for fractional variational problems is presented by the same author when the fractional derivatives are defined in sense of Riemann-Liouville and Caputo. Frederico Gastao et al.[20] used Agrawal's Euler-Lagrange equation and the Lagrange multiplier technique to obtain a Noether-like theorem for the fractional optimal control problem in the sense of Caputo. In [23] Özdemir et al. investigated the fractional optimal control problem of a distributed system in cylindrical coordinates whose dynamics are defined in the Riemann-Liouville sense. The authors used the method of separation of variables to find the solution of the problem and the eigenfunctions to eliminate the terms containing space parameters in the one hand and, on the other hand to define the problem in terms of a set generalized state and controls variables. Following the same technique, Karadeniz [29] et al. presented the formulation of an axis-symmetric fractional optimal control problem when the dynamic constraints of the system are given by a fractional diffusion-wave equation and the performance index is described with a state and a control function.

In [26], Baleanu et al. gave formulation for a fractional optimal control problems when the dimensions of the state and control variables are different from each other. In [24], Jeličić et al. proposed necessary conditions for optimality in optimal control problems with dynamics by differential equations of fractional order. In Mophou [7] applied the classical control theory to a fractional diffusion equation involving Riemann-Liouville fractional derivative in a bounded domain. The author showed that the considered optimal control problem has a unique solution. In [8], Dorville et al. showed that existence and uniqueness of a boundary the following boundary fractional optimal control when the dynamic constraints is described by a fractional diffusion equation involving Riemann-Liouville fractional derivative. We also refer to [21, 5, 25, 6, 33] and references therein for more literature on optimal control of fractional evolution equations.

In this paper, we are concerned with the following optimal control problem: find the control $u \in \mathcal{U}_{ad}$ such that

$$J(u) = \inf_{v\in\mathcal{U}_{ad}} \frac{1}{2}\|I^{2-\alpha}y(v,T) - z_d\|^2_{L^2(\Omega)} + \frac{N}{2}\|v\|^2_{L^2(Q)}$$

where $N > 0$, $z_d \in L^2(\Omega)$, $I^{2-\alpha}$ is the Riemann-Liouville fractional integral of order $2-\alpha$, $0 < \alpha < 2$ and $\mathcal{U}_{ad}$ is a closed convex subset of $L^2(Q)$. To solve this problem, we first prove by means of eigenfunction expansions that the controlled system (1) has a solution. Then we show that the optimal control problem has also a unique solution that we characterize by means of first order Euler-Lagrange optimality condition and adjoint state which dynamic is described by the right Caputo fractional derivative of order α.

The rest of this paper is organized as follows. Section 2 is devoted to some definitions and preliminary results. In Section 3, we prove the existence and uniqueness of the solution of (1) using the eigenfunction expansion . In section 4, we show that the considered optimal control problem holds and give the optimality systems that characterize the optimal control.

2. Preliminaries

Definition 2.1. [9] Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a continuous function on $\mathbb{R}_+$, and $\alpha > 0$. Then the expression

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \quad t > 0$$

is called the Riemann-Liouville integral of order α of the function f.

Definition 2.2. [27]

Let $f : \mathbb{R}_+ \to \mathbb{R}$. The left Riemann-Liouville fractional derivative of order $\alpha \in (1,2)$ of f is defined by

$$D^\alpha_{RL} f(t) = \frac{1}{\Gamma(2-\alpha)} . \frac{d^2}{dt^2} \int_0^t (t-s)^{1-\alpha} f(s)ds, \quad t > 0,$$

provided that the integral exists.

Definition 2.3. [2] Let $f : \mathbb{R}_+ \to \mathbb{R}$. The left Caputo fractional derivative of order $\alpha \in (1,2)$ of f is defined by

$$D^\alpha_C f(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} f''(s)ds, \quad t > 0, \tag{2}$$

provided that the integral exists.

Definition 2.4. [1, 27, 2] Let $f:\mathbb{R}_+\to\mathbb{R}$, $1<\alpha<2$. The right Caputo fractional derivative of order α of f is defined by

$$D_C^{\alpha} f(t)=\frac{1}{\Gamma(2-\alpha)}\int_t^T (s-t)^{1-\alpha} f''(s)ds, \quad 0<t<T, \tag{3}$$

provided that the integral exists.

Lemma 2.5. [1, 27, 2] *Let $T>0$, $v\in\mathcal{C}^2([0,T])$ and $\alpha\in(1;2), m\in\mathbb{N}$. Then for $t\in[0,T]$, we have the following properties:*

$$D_{RL}^{\alpha} v(t)=\frac{d^2}{dt^2} I^{2-\alpha} v(t) \tag{4a}$$

$$D_{RL}^{\alpha} I^p \alpha v(t)=v(t); \tag{4b}$$

$$D_C^p I^{\alpha} v(t)=v(t); \tag{4c}$$

$$I^{\alpha} D_C^p u(t)=u(t)-u(0)-tu'(0); \tag{4d}$$

$$I^{\alpha} D_{RL}^{\alpha} u(t)=u(t)-\frac{t^{\alpha-1}}{\Gamma(\alpha)}(I^{2-\alpha}u)'(0)-\frac{t^{\alpha-2}}{\Gamma(\alpha-1)} I^{2-\alpha} u(0). \tag{4e}$$

Definition 2.6. [1]
For $t\in\mathbb{R}^+,\alpha>0$ and $\beta>0$ we denote by,

$$E_{\alpha,\beta}(t)=\sum_{k=0}^{\infty}\frac{t^k}{\Gamma(\alpha k+\beta)}, \tag{5}$$

the two-parameters Mittag-Leffler function and we set

$$E_{\alpha,1}(t)=E_{\alpha}(t).$$

Lemma 2.7. [1] *For a positive integer n, $\lambda>0$ and $\alpha>0$, we have,*

$$\frac{d^n}{dt^n} E_{\alpha}(-\lambda t^{\alpha})=-\lambda t^{\alpha-n} E_{\alpha,\alpha-n+1}(-\lambda t^{\alpha}), t>0. \tag{6}$$

and

$$\frac{d}{dt}(tE_{\alpha,2}(-\lambda t^{\alpha}))=E_{\alpha}(-\lambda t^{\alpha}), t>0. \tag{7}$$

Theorem 2.8. [1] *If $\alpha<2$, β is an arbitrary real number, μ is such that*

$$\frac{\pi\alpha}{2}<\mu<\min\{\pi,\pi\alpha\}$$

and C is a real constant, then

$$|E_{\alpha,\beta}(z)|\le\frac{C}{1+|z|}, \tag{8}$$

$$(\mu\le|arg(z)|\le\pi), \quad |z|\ge 0.$$

One can prove as in [7] the following result obtained by a simple integration by part.

Lemma 2.9. *For any $\varphi \in \mathcal{C}^\infty(\overline{Q})$, we have*

$$
\begin{aligned}
&\int_0^T \int_\Omega (D^\alpha_{RL} y(x,t) - \Delta y(x,t))\varphi(x,t)dxdt = \\
&\int_\Omega \varphi(x,T)\frac{d}{dt}(I^{2-\alpha}y(x,T))dx - \int_\Omega \varphi(x,0)\frac{d}{dt}(I^{2-\alpha}y(x,0^+))dx - \\
&\int_\Omega I^{2-\alpha}y(x,T)\varphi'(x,T)dx + \int_\Omega I^{2-\alpha}y(x,0)\varphi'(x,0)dx + \\
&\int_0^T \int_{\partial\Omega} y(\sigma,s)\frac{\partial\varphi}{\partial\nu}(\sigma,s)d\sigma dt - \int_0^T \int_{\partial\Omega} \frac{\partial y}{\partial\nu}(\sigma,s)\varphi(\sigma,s)d\sigma dt + \\
&\int_\Omega \int_0^T y(x,t)(\mathcal{D}^\alpha_C \varphi(x,t) - \Delta\varphi(x,t))dxdt.
\end{aligned}
\tag{9}
$$

From Lemma 2.9 we have the following result:

Corollary 2.10. Let $\mathbb{D}(0,T)$ be the set of $\mathcal{C}^\infty$ functions on $(0,T)$ with compact support. Then for all $\varphi \in \mathbb{D}(0,T)$,

$$\int_0^T D^\alpha_{RL} y(t)\varphi(t)dt = \int_0^T y(t)\mathcal{D}^\alpha_C \varphi(t)dt$$

where $\mathcal{D}^\alpha_C$ is right fractional Caputo derivative.

3. Existence and uniqueness results

In this section, we prove the existence and uniqueness of a weak solution of the following fractional diffusion wave equation:

$$
\left\{
\begin{array}{rcll}
D^\alpha_{RL} y(x,t) - \Delta y(x,t) & = & v(x,t), & (x,t) \in Q \\
y(\sigma,t) & = & 0, & (\sigma,t) \in \Sigma \\
I^{2-\alpha}y(0) & = & y^0(x), & x \in \Omega \\
\dfrac{d}{dt}I^{2-\alpha}y(0) & = & y^1(x), & x \in \Omega
\end{array}
\right.
\tag{10}
$$

where $1 < \alpha < 2, y^0 \in H^2(\Omega) \cap H^1_0(\Omega), y^1 \in L^2(\Omega)$ and $v \in L^2(Q)$.

Set

$$(\varphi,\psi)_{L^2(\Omega)} = \int_\Omega \varphi(x)\psi(x)dx, \quad \forall \varphi,\psi \in L^2(\Omega)$$

the inner scalar product on $L^2(\Omega)$ and denote by $\|.\|_{L^2(\Omega)}$ the associate norm. Set also

$$a(\varphi,\psi)=\int_\Omega \nabla\varphi(x)\nabla\psi(x)dx, \quad \forall\varphi,\psi\in H_0^1(\Omega). \tag{11}$$

Then the bilinear form $a(.,.)$ defines an inner scalar product on $H_0^1(\Omega)$. We denote by

$$\|\varphi\|^2_{H_0^1(\Omega)}=a(\varphi,\varphi), \tag{12}$$

the associate norm.

On the other hand, we know that it admits real eigenvalues $0<\lambda_1\leq\lambda_2\leq\lambda_3\leq\cdots$ with $\lambda_k\to\infty$ as $k\to\infty$ since $(-\Delta)$ is a compact self-adjoint operator on $L^2(\Omega)$. Moreover there exists an orthonormal basis $\{w_k\}_{k=1}^\infty$ of $L^2(\Omega)$, where $w_k\in H_0^1(\Omega)$ is an eigenfunction corresponding to λ_k: $-\Delta w_k=\lambda_k w_k$. This means that

$$a(w_k,p)=\lambda_k(w_k,p)_{L^2(\Omega)}, \quad \forall p\in H_0^1(\Omega). \tag{13}$$

Furthermore, the sequence $\left\{\dfrac{w_k}{\sqrt{\lambda_k}}\right\}_{k=1}^\infty$ being an orthonormal basis of $H_0^1(\Omega)$ for the scalar product $a(.,.)$, we have

$$\forall\psi\in H_0^1(\Omega), \quad \|\psi\|_{H_0^1(\Omega)}=\sum_{i=1}^{+\infty}\lambda_i(\psi,\ w_i)^2_{L^2(\Omega)}. \tag{14}$$

Now, we assume that $y\in\mathcal{C}^\infty(\bar{Q})$ and we introduce, for all $t\in(0,T)$, the functions $y(t):x\in\Omega\to y(x,t)$ and $v(t):x\in\Omega\to v(x,t)$. Then multiplying the first equation of (10) by a function $u\in H_0^1(\Omega)$ and integrating by part over Ω, we obtain

$$\int_\Omega D^\alpha_{RL}y(t)\,u\,dx+\int_\Omega\nabla y(t)\nabla u dx=\int_\Omega v(t)\,u\,dx, \tag{15}$$

which can be rewritten as

$$D^\alpha_{RL}(y(t),u)_{L^2(\Omega)}+a(y(t),u)=(v(t),u)_{L^2(\Omega)}.$$

Hence for all $t\in(0,T)$, Problem (10) becomes

$$\left\{\begin{array}{rcllll} D^\alpha_{RL}(y(t),u)_{L^2(\Omega)}+a(y(t),u) &=& (v(t),u)_{L^2(\Omega)} & \text{in} & \Omega, & \forall u\in H_0^1(\Omega),\\ y(t) &=& 0 & \text{on} & \partial\Omega & \\ I^{2-\alpha}y(0) &=& y^0 & \text{in} & \Omega & \\ \dfrac{d}{dt}I^{2-\alpha}y(0) &=& y^1 & \text{in} & \Omega & \end{array}\right. \tag{16}$$

We then consider the following problem: *Given* $1 < \alpha < 2$, $y^0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $y^1 \in L^2(\Omega)$ *and* $v \in L^2(Q)$, *find*

$$y \in L^2((0,T), H_0^1(\Omega)), \tag{17a}$$

$$I^{2-\alpha} y \in \mathcal{C}([0,T]; H_0^1(\Omega)), \tag{17b}$$

$$\frac{d}{dt}(I^{2-\alpha} y) \in \mathcal{C}([0,T]; L^2(\Omega)) \tag{17c}$$

such that

$$\forall u \in H_0^1(\Omega), D_{RL}^{\alpha}(y(t), u)_{L^2(\Omega)} + a(y(t), u) = (v(t), u)_{L^2(\Omega)} \quad \forall t \in (0,T), \tag{18a}$$

$$I^{2-\alpha} y(0) = y^0 \ \textit{in}\ \Omega \ \textit{and}\ \frac{d}{dt} I^{2-\alpha} y(0) = y^1 \ \textit{in}\ \Omega. \tag{18b}$$

Theorem 3.1. *Let* $1 < \alpha < 2$. *Let also* $a(.,.)$ *be the bilinear form defined by* (11). *Then the problem* (17) − (18) *has a weak solution* y *given by:*

$$\begin{array}{rcl} y(t) & = & \displaystyle\sum_{i=1}^{+\infty} \left\{ t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_i t^{\alpha}) y_i^0 + t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i t^{\alpha}) y_i^1 \right\} w_i \\ & + & \displaystyle\sum_{i=1}^{+\infty} \left\{ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i (t-s)^{\alpha}) v_i(s) ds \right\} w_i \end{array} . \tag{19}$$

where λ_i *is the eigenvalue of the operator* $-\Delta$ *corresponding to the eigenfunction* w_i, $y_i^0 = (y^0, w_i)_{L^2(\Omega)}, y_i^1 = (y^1, w_i)_{L^2(\Omega)}$ *and* $v_i(t) = (v(t), w_i)_{L^2(\Omega)}$ *are respectively the* i*-th component of* y^0, y^1 *and* $v(t)$ *in the orthonormal basis* $\{w_i\}_{i=1}^{\infty}$ *of* $L^2(\Omega)$.

Proof. Replacing u by w_i in (18a) and using the fact that $a(y(t), w_i) = \lambda_i (y(t), w_i)_{L^2(\Omega)} = \lambda_i y_i$, we deduce from (18) that $y_i = (y(t), w_i)_{L^2(\Omega)}$ is solution of the fractional ordinary differential equation:

$$\left\{ \begin{array}{lcl} D_{RL}^{\alpha} y_i(t) + \lambda_i y_i(t) & = & v_i(t), \forall t \in (0,T), \\ I^{2-\alpha} y_i(0) & = & y_i^0, \\ \dfrac{d}{dt} I^{2-\alpha} y_i(0) & = & y_i^1. \end{array} \right. \tag{20}$$

Using the Laplace transform, one can easily prove that the solution of (20) is given by:

$$\begin{array}{rcl} y_i(t) & = & t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_i t^{\alpha}) y_i^0 + t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i t^{\alpha}) y_i^0 \\ & + & \displaystyle\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i (t-s)^{\alpha}) v_i(s) ds. \end{array}$$

Hence, we obtain (19) since $y(t) = \displaystyle\sum_{i=1}^{+\infty} y_i(t) w_i$.

□

Theorem 3.2. *Let $3/2 < \alpha < 2$. Let also $a(.,.)$ be the bilinear form defined by* (11)*. Then the problem (17)-(18) has a unique solution.*

Proof. Let V_m be a subspace of $H_0^1(\Omega)$ generated by $w_1, w_2, \ldots, w_m$. Consider the following approximate problem associated to $(17)-(18)$: find $y_m : t \in [0,T] \to y_m(t) \in V_m$ solution to

$$D_{RL}^{\alpha}(y_m(t), u)_{L^2(\Omega)} + a(y_m(t), u) = (v(t), u)_{L^2(\Omega)}, \forall u \in V_m, \tag{21a}$$

$$I^{2-\alpha} y_m(0) = y_m^0, \qquad \frac{d}{dt} I^{2-\alpha} y_m(0) = y_m^1, \tag{21b}$$

where

$$y_m^0 = \sum_{i=1}^{m} y_i^0 w_i \text{ and } y_m^1 = \sum_{i=1}^{m} y_i^1 w_i.$$

Then proceeding as for the proof of Theorem 3.1, we show that

$$\begin{aligned} y_m(t) &= \sum_{i=1}^{m} \left\{ t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_i t^\alpha) y_i^0 + t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i t^\alpha) y_i^1 \right\} w_i. \\ &+ \sum_{i=1}^{m} \left\{ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i (t-s)^\alpha) v_i(s) ds \right\} w_i \\ &= \sum_{i=1}^{m} y_i(t) w_i. \end{aligned} \tag{22}$$

is solution of the problem (21). To complete the proof of Theorem 3.2, we proceed in two steps.

Step 1: We show that the sequences (y_m), $(I^{2-\alpha} y_m)$ and $\left(\frac{d}{dt}(I^{2-\alpha} y_m) \right)$ are respectively Cauchy sequences in $L^2((0,T); H_0^1(\Omega))$,$\mathcal{C}([0,T]; H_0^1(\Omega))$ and $\mathcal{C}([0,T]; L^2(\Omega))$.

Let m and p be two integers such that $p > m \geq 1$. We have

$$y_p(t) - y_m(t) = \sum_{i=m+1}^{p} y_i(t) w_i.$$

Therefore, we can write,

$$\begin{aligned} a(y_p(t) - y_m(t), y_p(t) - y_m(t)) &= \sum_{i=m+1}^{p} \lambda_i [y_i(t)]^2 \\ &\leq 2 \sum_{i=m+1}^{p} \lambda_i t^{2\alpha-4} E_{\alpha,\alpha-1}^2(-\lambda_i t^\alpha) |y_i^0|^2 \\ &+ 2 \sum_{i=m+1}^{p} \lambda_i t^{2\alpha-2} E_{\alpha,\alpha}^2(-\lambda_i t^\alpha) |y_i^1|^2 \\ &+ 2 \sum_{i=m+1}^{p} \lambda_i \left\{ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i (t-s)^\alpha) v_i(s) ds \right\}^2 \end{aligned}$$

Then, we have,

$$\begin{aligned}||y_p(t)-y_m(t)||^2_{L^2((0,T);H^1_0(\Omega))} &= \int_0^T a(y_p(t)-y_m(t),y_p(t)-y_m(t))dt\\ &\le A_p+B_p+C_p\end{aligned}$$

where

$$\begin{aligned}A_P &= 2\sum_{i=m+1}^{p}\lambda_i|y_i^0|^2\int_0^T t^{2\alpha-4}E^2_{\alpha,\alpha-1}(-\lambda_i t^\alpha)dt\\ B_P &= 2\sum_{i=m+1}^{p}\lambda_i|y_i^1|^2\int_0^T t^{2\alpha-2}E^2_{\alpha,\alpha}(-\lambda_i t^\alpha)dt\\ C_P &= 2\sum_{i=m+1}^{p}\lambda_i\int_0^T\left\{\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i(t-s)^\alpha)v_i(s)ds\right\}^2 dt\end{aligned}$$

Using (8) and the fact that $3/2<\alpha<2$, we obtain,

$$A_p \le 2C^2\frac{T^{2\alpha-3}}{2\alpha-3}\sum_{i=m+1}^{p}\lambda_i|y_i^0|^2,$$

$$B_p \le 2C^2\frac{T^{\alpha-1}}{\alpha-1}\sum_{i=m+1}^{p}|y_i^1|^2$$

And

$$\begin{aligned}C_P &\le 2\sum_{i=m+1}^{p}\lambda_i\int_0^T\left(\int_0^t|v_i(s)|^2ds\right)\left(\int_0^t(t-s)^{2\alpha-2}E^2_{\alpha,\alpha}(-\lambda_i(t-s)^\alpha)ds\right)dt\\ &\le 2\sum_{i=m+1}^{p}\lambda_i\int_0^T\frac{C^2}{\lambda_i}\left(\int_0^t|v_i(s)|^2ds\right)\left(\int_0^t(t-s)^{\alpha-2}ds\right)dt\\ &\le 2\int_0^T\frac{C^2t^{\alpha-1}}{(\alpha-1)}dt\sum_{i=m+1}^{p}\left(\int_0^t|v_i(s)|^2ds\right)\\ &\le 2\frac{C^2T^\alpha}{(\alpha-1)}\sum_{i=m+1}^{p}\left(\int_0^T|v_i(s)|^2ds\right).\end{aligned}$$

Thus

$$\begin{aligned}||y_p(t)-y_m(t)||^2_{L^2((0,T);H^1_0(\Omega))} &\le 2C^2\frac{T^{2\alpha-3}}{(2\alpha-3)}\sum_{i=m+1}^{p}\lambda_i|y_i^0|^2\\ &+ 2C^2\frac{T^{\alpha-1}}{(\alpha-1)}\sum_{i=m+1}^{p}|y_i^1|^2\\ &+ 2\frac{C^2T^\alpha}{\alpha(\alpha-1)}\sum_{i=m+1}^{p}\left(\int_0^T|v_i(t)|^2dt\right).\end{aligned}\tag{23}$$

We can write,

(24)
$$I^{2-\alpha}(y_p(t)-y_m(t)) = \sum_{i=m+1}^{p} Z_{1i}(t)y_i^0 w_i + \sum_{i=m+1}^{p} Z_{2i}(t)y_i^1 w_i + \sum_{i=m+1}^{p} Z_{2i}(t)w_i,$$

where

$$\begin{aligned} Z_{1i}(t) &= \tfrac{1}{\Gamma(2-\alpha)} \textstyle\int_0^t (t-s)^{1-\alpha}s^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda_i s^\alpha)ds, \\ Z_{2i}(t) &= \frac{1}{\Gamma(2-\alpha)}\int_0^t (t-s)^{1-\alpha}s^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i s^\alpha)ds, \\ Z_{3i}(t) &= \frac{1}{\Gamma(2-\alpha)}\int_0^t (t-s)^{1-\alpha}\left[\int_0^s (s-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i(s-\tau)^\alpha)v_i(\tau)d\tau\right]ds. \end{aligned}$$

Observing that

$$\begin{aligned} \int_0^t (t-s)^{1-\alpha}s^{\alpha k+\alpha-2}ds &= t^{\alpha k}\frac{\Gamma(2-\alpha)\Gamma(\alpha k+\alpha-1)}{\Gamma(\alpha k+1)}, \\ \int_0^t (t-s)^{1-\alpha}s^{\alpha k+\alpha-1}ds &= t^{\alpha k+1}\frac{\Gamma(2-\alpha)\Gamma(\alpha k+\alpha)}{\Gamma(\alpha k+2)}, \end{aligned}$$

$$\int_\tau^t (t-s)^{1-\alpha}(s-\tau)^{\alpha k+\alpha-1}ds = (t-\tau)^{\alpha k+1}\frac{\Gamma(2-\alpha)\Gamma(\alpha k+\alpha)}{\Gamma(\alpha k+2)},$$

and using the definition of the Mittag-Leffler function given by (5), we obtain that

(25a) $$Z_{1i}(t) = E_\alpha(-\lambda_i t^\alpha),$$

(25b) $$Z_{2i}(t) = tE_{\alpha,2}(-\lambda_i t^\alpha),$$

(25c) $$Z_{3i}(t) = \int_0^t (t-s)E_{\alpha,2}(-\lambda_i(t-s)^\alpha)v_i(s)ds.$$

Hence using (6) and (7), we get

(26a) $$\frac{d}{dt}Z_{1i}(t) = -\lambda_i t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i t^\alpha),$$

(26b) $$\frac{d}{dt}Z_{2i}(t) = E_\alpha(-\lambda_i t^\alpha),$$

(26c) $$\frac{d}{dt}Z_{3i}(t) = \int_0^t E_\alpha(-\lambda_i(t-s)^\alpha)v_i(s)ds.$$

From (24) and (25),

$$\begin{aligned} I^{2-\alpha}(y_p(t)-y_m(t)) &= \sum_{i=m+1}^{p} E_\alpha(-\lambda_i t^\alpha)y_i^0 w_i \\ &+ \sum_{i=m+1}^{p} tE_{\alpha,2}(-\lambda_i t^\alpha)y_i^1 w_i \\ &+ \sum_{i=m+1}^{p}\left\{\int_0^t v_i(\tau)(t-\tau)E_{\alpha,2}(-\lambda_i(t-\tau)^\alpha)d\tau\right\}w_i, \end{aligned}$$

which in view of (8), (12) and the Cauchy-Schwartz inequality gives

$$\begin{aligned}
||I^{2-\alpha}(y_p(t)-y_m(t))||^2_{H_0^1(\Omega)} &= a(I^{2-\alpha}(y_p(t)-y_m(t)), I^{2-\alpha}(y_p(t),y_m(t)))\\
&\leq 2C^2\sum_{i=m+1}^{p}\lambda_i|y_i^0|^2+2C^2T^{2-\alpha}\sum_{i=m+1}^{p}|y_i^1|^2\\
&+ 2C^2\frac{T^{3-\alpha}}{3-\alpha}\sum_{i=m+1}^{p}\int_0^T|v_i(\tau)|^2d\tau
\end{aligned}$$

and

$$\begin{aligned}
\sup_{t\in[0,T]}||I^{2-\alpha}(y_p(t)-y_m(t))||_{H_0^1(\Omega)} &\leq \Pi\left(\sum_{i=m+1}^{p}\lambda_i|y_i^0|^2\right)^{1/2}\\
&+ \Pi\left(\sum_{i=m+1}^{p}|y_i^1|^2\right)^{1/2}\\
&+ \Pi\sum_{i=m+1}^{p}\left(\int_0^T|v_i(\tau)|^2d\tau\right)^{1/2},
\end{aligned}\tag{27}$$

where

$$\Pi=\sup\left(C\sqrt{2}, C\sqrt{2T^{2-\alpha}}, C\sqrt{\frac{2T^{3-\alpha}}{(3-\alpha)}}\right).$$

Using (24) and (26), we obtain that

$$\begin{aligned}
\left\|\frac{d}{dt}I^{2-\alpha}(y_p(t)-y_m(t))\right\|^2_{L^2(\Omega)} &\leq 2\sum_{i=m+1}^{p}\lambda_i^2|y_i^0|^2[t^{2\alpha-2}E^2_{\alpha,\alpha}(-\lambda_it^\alpha)]\\
&+ 2\sum_{i=m+1}^{p}|y_i^1|^2E^2_\alpha(-\lambda_it^\alpha)\\
&+ 2\sum_{i=m+1}^{p}\left|\int_0^t v_i(s)E_\alpha(-\lambda_i(t-s)^\alpha)ds\right|^2,
\end{aligned}$$

which in view of (8) gives

$$\begin{aligned}
\left\|\frac{d}{dt}I^{2-\alpha}(y_p(t)-y_m(t))\right\|^2_{L^2(\Omega)} &\leq 2C^2T^{2\alpha-2}\sum_{i=m+1}^{p}\lambda_i^2|y_i^0|^2\\
&+ 2C^2\sum_{i=m+1}^{p}|y_i^1|^2\\
&+ 2C^2\sum_{i=m+1}^{p}\int_0^T|v_i(s)|^2ds.
\end{aligned}$$

Consequently,

(28)
$$\begin{aligned}\sup_{t\in[0,T]}\left\|\frac{d}{dt}I^{2-\alpha}(y_p(t)-y_m(t))\right\|_{L^2(\Omega)} &\leq \sqrt{2}CT^{\alpha-1}\left(\sum_{i=m+1}^{p}\lambda_i^2|y_i^0|^2\right)^{1/2}\\ &+ \sqrt{2}C\left(\sum_{i=m+1}^{p}|y_i^1|^2\right)^{1/2}\\ &+ \sqrt{2}C\left(\sum_{i=m+1}^{p}\int_0^T|v_i(s)|^2ds\right)^{1/2}.\end{aligned}$$

As $y^0\in H^2(\Omega)\cap H_0^1(\Omega)$, $y^1\in L^2(\Omega)$ and $v\in L^2(Q)$, we have

$$\begin{aligned}\lim_{m,p\to\infty}\left(\sum_{i=m+1}^{p}\int_0^T|v_i(s)|^2ds\right)^{1/2} &= 0,\\ \lim_{m,p\to\infty}\left(\sum_{i=m+1}^{p}\lambda_i|y_i^0|^2\right)^{1/2} &= 0,\\ \lim_{m,p\to\infty}\left(\sum_{i=m+1}^{p}\lambda_i^2|y_i^0|^2\right)^{1/2} &= 0,\\ \lim_{m,p\to\infty}\left(\sum_{i=m+1}^{p}|y_i^1|^2\right)^{1/2} &= 0.\end{aligned}$$

Then, it follows from (23),(27) and (28) that (y_m), $(I^{2-\alpha}y_m)$ and $\left(\dfrac{d}{dt}I^{2-\alpha}y_m\right)$ are Cauchy in $L^2((0,T);H_0^1(\Omega))$, $\mathcal{C}([0,T];H_0^1(\Omega))$ and $\mathcal{C}([0,T];L^2(\Omega))$ respectively. We deduce that

$$y_m \to y \quad \text{in} \quad L^2((0,T);H_0^1(\Omega)), \tag{29}$$

$$I^{2-\alpha}y_m \to I^{2-\alpha}y \quad \text{in} \quad \mathcal{C}([0,T];H_0^1(\Omega)), \tag{30}$$

$$\frac{d}{dt}I^{2-\alpha}y_m \to \frac{d}{dt}I^{2-\alpha}y \quad \text{in} \quad \mathcal{C}([0,T],L^2(\Omega)) \tag{31}$$

Step 2: We prove that y is solution of the problem (17)-(18).

Let $\varphi\in\mathbb{D}(0,T)$. Let also $\mu\geq 1$ be an integer. Then from (21a), we have for all $m\geq\mu$,

$$\begin{aligned}\int_0^T(v(t),u)_{L^2(\Omega)}\varphi(t)dt &= \int_0^T D_{RL}^{\alpha}(y_m(t),u)_{L^2(\Omega)}\varphi(t)dt\\ &+ \int_0^T a(y_m(t),u)\varphi(t)dt, \quad \forall u\in V_\mu,\end{aligned}$$

which according to the Corollary 2.10 , gives

$$\int_0^T (v(t),u)_{L^2(\Omega)}\varphi(t)dt = \int_0^T (y_m(t),u)_{L^2(\Omega)}\mathcal{D}_C^\alpha\varphi(t)dt + \int_0^T a(y_m(t),u)\varphi(t)dt, \quad \forall u\in V_\mu.$$

Therefore passing to the limit while using (29), we get

$$\int_0^T (v(t),u)_{L^2(\Omega)}\varphi(t)dt = \int_0^T (y(t),u)_{L^2(\Omega)}\mathcal{D}_C^\alpha\varphi(t)dt + \int_0^T a(y(t),u)\varphi(t)dt, \quad \forall u\in V_\mu,$$

since $\bigcup\limits_{\mu\geq 1} V_\mu$ is dense in $H_0^1(\Omega)$, we can write

$$\int_0^T (v(t),u)_{L^2(\Omega)}\varphi(t)dt = \int_0^T (y(t),u)_{L^2(\Omega)}\mathcal{D}_C^\alpha\varphi(t)dt + \int_0^T a(y(t),u)\varphi(t)dt, \quad \forall u\in H_0^1(\Omega),$$

Hence, using again the Corollary 2.10, we obtain that,

$$\int_0^T (v(t),u)_{L^2(\Omega)}\varphi(t)dt = \int_0^T D_{RL}^\alpha(y(t),u)_{L^2(\Omega)}\varphi(t)dt + \int_0^T a(y(t),u)\varphi(t)dt, \quad \forall u\in H_0^1(\Omega),$$

This means that,

$$\forall u\in H_0^1(\Omega),\ (v(t),u)_{L^2(\Omega)} = D_{RL}^\alpha(y(t),u)_{L^2(\Omega)} + a(y(t),u), \quad \forall t\in(0,T).$$

From (30) and (31), we get that

$$I^{2-\alpha}y_m(0) = \sum_{i=1}^m y_i^0 w_i \to \sum_{i=1}^{+\infty} y_i^0 w_i = I^{2-\alpha}y(0) \quad \text{in } H_0^1(\Omega)$$

and

$$\frac{d}{dt}I^{2-\alpha}y_m(0) = \sum_{i=1}^m y_i^1 w_i \to \sum_{i=1}^{+\infty} y_i^1 w_i = \frac{d}{dt}I^{2-\alpha}y(0) \quad \text{in } L^2(\Omega).$$

This implies that $I^{2-\alpha}y(0)=y^0$ and $\dfrac{d}{dt}I^{2-\alpha}y(0)=y^1$.

□

Theorem 3.3. *Let $3/2<\alpha<2$. Then the solution y of problem (17)-(18) verifies the following estimates*

$$\|y\|_{L^2((0,T);H_0^1(\Omega))} \leq \Delta\left(\|y^0\|_{H_0^1(\Omega)} + \|y^1\|_{L^2(\Omega)} + \|v\|_{L^2(Q)}\right) \tag{32}$$

$$\|I^{2-\alpha}y\|_{\mathcal{C}([0,T];H_0^1(\Omega))} \leq \Pi\left(\|y^0\|_{H_0^1(\Omega)} + \|y^1\|_{L^2(\Omega)} + \|v\|_{L^2(Q)}\right) \tag{33}$$

$$\left\|\frac{d}{dt}I^{2-\alpha}y\right\|_{\mathcal{C}([0,T];L^2(\Omega))} \leq \Theta\left(\|y^0\|_{H^2(\Omega)} + \|y^1\|_{L^2(\Omega)} + \|v\|_{L^2(Q)}\right) \tag{34}$$

where

$$\Delta = \max\left(C\sqrt{\frac{2T^{2\alpha-3}}{(2\alpha-3)}}, C\sqrt{\frac{T^{\alpha-1}}{(\alpha-1)}}, C\sqrt{\frac{2T^{\alpha}}{(\alpha-1)}}\right),$$

$$\Pi = \sup\left(C\sqrt{2}, C\sqrt{2T^{2-\alpha}}, C\sqrt{\frac{2T^{3-\alpha}}{(3-\alpha)}}\right)$$

and

$$\Theta = \max\left(\sqrt{2}CT^{\alpha-1}, \sqrt{2}C\right)$$

Proof. In view of Theorem 3.2, the solution of problem (17)-(18) is given by

$$\begin{aligned} y(t) &= \sum_{i=1}^{+\infty}\left\{t^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda_i t^{\alpha})y_i^0 + t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i t^{\alpha})y_i^1\right\}w_i \\ &+ \sum_{i=1}^{+\infty}\left\{\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i(t-s)^{\alpha})v_i(s)ds\right\}w_i \end{aligned}$$

Thus using the calculations obtained in the proof of Theorem 3.2, pages 9-12, we have

$$\begin{aligned} \int_0^T ||y(t)||^2_{H_0^1(\Omega)}dt &= \int_0^T a(y(t),y(t))dt \\ &\leq 2C^2\frac{T^{2\alpha-3}}{(2\alpha-3)}\sum_{i=1}^{+\infty}\lambda_i|y_i^0|^2 \\ &+ 2C^2\frac{T^{\alpha-1}}{(\alpha-1)}\sum_{i=1}^{+\infty}|y_i^1|^2 \\ &+ 2\frac{C^2T^{\alpha}}{\alpha(\alpha-1)}\sum_{i=1}^{+\infty}\left(\int_0^T|v_i(t)|^2dt\right). \end{aligned}$$

This means that

$$\begin{aligned} ||y_p(t)-y_m(t)||^2_{L^2((0,T);H_0^1(\Omega))} &\leq 2C^2\frac{T^{2\alpha-3}}{(2\alpha-3)}\|y^0\|^2_{H_0^1(\Omega)} \\ &+ 2C^2\frac{T^{\alpha-1}}{(\alpha-1)}\|y^1\|^2_{L^2(\Omega)} \\ &+ 2\frac{C^2T^{\alpha}}{\alpha(\alpha-1)}\|v\|^2_{L^2(Q)}. \end{aligned}$$

Hence we deduce (32).

Using (24), (25), (8), (12) and the Cauchy-Schwartz inequality, we get

$$
\begin{aligned}
||I^{2-\alpha}y(t)||^2_{H^1_0(\Omega)} &= a(I^{2-\alpha}y(t), I^{2-\alpha}y(t)) \\
&\leq 2C^2\sum_{i=1}^{+\infty}\lambda_i|y_i^0|^2 + 2C^2T^{2-\alpha}\sum_{i=1}^{+\infty}|y_i^1|^2 \\
&+ 2C^2\frac{T^{3-\alpha}}{3-\alpha}\sum_{i=1}^{+\infty}\int_0^T |v_i(\tau)|^2 d\tau
\end{aligned}
$$

and

$$
\sup_{t\in[0,T]} ||I^{2-\alpha}y(t)||_{H^1_0(\Omega)} \leq \Pi\left(\|y^0\|_{H^1_0(\Omega)} + \|y^1\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(Q))}\right)
$$

where

$$
\Pi = \sup\left(C\sqrt{2}, C\sqrt{2T^{2-\alpha}}, C\sqrt{\frac{2T^{3-\alpha}}{(3-\alpha)}}\right).
$$

Finally, using (24), (26) and (8), we obtain that

$$
\begin{aligned}
\left\|\frac{d}{dt}I^{2-\alpha}y(t)\right\|^2_{L^2(\Omega)} &\leq 2C^2T^{2\alpha-2}\sum_{i=1}^{+\infty}\lambda_i^2|y_i^0|^2 \\
&+ 2C^2\sum_{i=1}^{+\infty}|y_i^1|^2 \\
&+ 2C^2\sum_{i=1}^{+\infty}\int_0^T |v_i(s)|^2 ds.
\end{aligned}
$$

Consequently,

$$
\sup_{t\in[0,T]}\left\|\frac{d}{dt}I^{2-\alpha}y(t)\right\|_{L^2(\Omega)} \leq \Theta\left(\|y^0\|_{H^2(\Omega)} + \|y^1\|_{L^2(\Omega)} + \|v\|_{L^2(Q))}\right)
$$

where

$$
\Theta = \max\left(\sqrt{2}CT^{\alpha-1}, \sqrt{2}C\right)
$$

□

We need the following results to solve the optimal control problem.

Proposition 3.4. Let $1 < \alpha < 2$ and $p^1 \in L^2(\Omega)$. Denote by $\mathcal{D}^\alpha_C p$, the right Caputo fractional derivative of the function p. Then problem:

$$
(35) \qquad \left\{\begin{array}{rcll}
\mathcal{D}^\alpha_C p(x,t) - \Delta p(x,t) &=& 0 & (x,t)\in Q \\
p(\sigma,t) &=& 0 & (x,t)\in\Sigma \\
p(x,0) &=& 0 & x\in\Omega \\
p'(x,0) &=& p^1 & x\in\Omega
\end{array}\right.
$$

has a unique solution $p \in \mathcal{C}([0,T]; L^2(\Omega))$. Moreover, there exists a positive constant C such that

$$\|p\|_{\mathcal{C}([0,T];L^2(\Omega))} + \left\|\frac{\partial p}{\partial t}\right\|_{\mathcal{C}([0,T];L^2(\Omega))} \le C\|p^1\|^2_{L^2(\Omega)}. \tag{36}$$

Proof. Set

$$\mathcal{T}_T p(t) = p(T-t), \quad t \in (0,T).$$

Then

$$\frac{d^2}{dt^2}\mathcal{T}_T p(t) = p''(T-t) = \mathcal{T}_T p''(t).$$

The left fractional Caputo derivative is defined by,

$$\mathcal{D}^\alpha_C p(t) = \frac{1}{\Gamma(2-\alpha)}\int_t^T (s-t)^{1-\alpha} p''(s)ds.$$

Making the change of variable $t \to T-t$, we obtain,

$$\begin{aligned}
\mathcal{D}^\alpha_C p(T-t) &= \frac{1}{\Gamma(2-\alpha)}\int_{T-t}^T (t-(T-s))^{1-\alpha} p''(s)ds \\
&= \frac{1}{\Gamma(2-\alpha)}\int_0^t (t-u)^{1-\alpha} p''(T-u)du.
\end{aligned}$$

This is implies that

$$D^\alpha_C \mathcal{T}_T p(t) = \mathcal{D}^\alpha_C \mathcal{T}_T p(t). \tag{37}$$

Now, making the change of variable $t \to T-t$, in (35) and using (37), we deduce that

$$\left\{\begin{aligned}
D^\alpha_C p(x,\tau) - \Delta p(x,\tau) &= 0 && (x,\tau) \in Q \\
p(\sigma,\tau) &= 0 && (\sigma,\tau) \in \Sigma \\
p(x,0) &= 0 && x \in \Omega \\
p'(x,0) &= p^1 && x \in \Omega
\end{aligned}\right. \tag{38}$$

since $\tau = T-t \in (0,T)$. Therefore using Theorem 2.3 in [17] we deduce that there exists a unique $p \in \mathcal{C}([0,T];L^2(\Omega))$ solution of (38) such that

$$\|p\|_{\mathcal{C}([0,T];L^2(\Omega))} + \left\|\frac{\partial p}{\partial t}\right\|_{\mathcal{C}([0,T];L^2(\Omega))} \le C\|p^1\|^2_{L^2(\Omega)}.$$

Since (38) is equivalent to (35), there exists a unique $p \in \mathcal{C}([0,T];L^2(\Omega))$ solution of (35) such that (36) holds.

Remark 3.5. Since $[0,T]$ is bounded, the solution p of (35) belongs to $L^2(Q)$ and satisfies

$$\|p\|_{L^2(Q)} \le C\|p^1\|^2_{L^2(\Omega)}. \tag{39}$$

□

4. Optimal Control

In this section, we consider the following problem:

$$
\left\{
\begin{array}{rcl}
D^{\alpha}_{RL}y(x,t) - \Delta y(x,t) & = & v(x,t) \quad (x,t) \in Q \\
y(x,t) & = & 0 \quad (x,t) \in \Sigma \\
I^{2-\alpha}y(x,0) & = & y^0 \quad x \in \Omega \\
\dfrac{d}{dt}I^{2-\alpha}y(x,0) & = & y^1 \quad x \in \Omega
\end{array}
\right.
\tag{40}
$$

where $y^0 \in H^2(\Omega) \cap H^1_0(\Omega)$, $y^1 \in L^2(\Omega)$ and the control $v \in \mathcal{U}_{ad}$, a closed convex set of $L^2(Q)$. In view of Theorem 3.2, we know that $y = y(v,x,t) \in L^2((0,T); H^1_0(\Omega))$ and $I^{2-\alpha}y \in \mathcal{C}([0,T]; H^1_0(\Omega)$. Hence $I^{2-\alpha}y(v,T) \in H^1_0(\Omega) \subset L^2(\Omega)$ and we define the functional

$$
J(v) = \frac{1}{2}\|I^{2-\alpha}y(v,T) - z_d\|^2_{L^2(\Omega)} + \frac{N}{2}\|v\|^2_{L^2(Q)}
\tag{41}
$$

where $z_d \in L^2(\Omega)$ and $N > 0$.

We are concerned with the optimal control problem: find $u \in \mathcal{U}_{ad}$ such that,

$$
J(u) = \inf_{v \in \mathcal{U}_{ad}} J(v).
\tag{42}
$$

Theorem 4.1. *Assume that the state $y = y(v,x,t)$ is solution of the system (40). Then there exists a unique optimal control u in $\mathcal{U}_{ad}$ such that (42) holds.*

Proof. Let $(v_n) \in \mathcal{U}_{ad}$ be a minimizing sequence such that,

$$
\lim_{n \to +\infty} J(v_n) = \inf_{v \in \mathcal{U}_{ad}} J(v).
\tag{43}
$$

Then, there exists $C > 0$ such that

$$
J(v_n) \leq C.
$$

It then follows from the structure of J given by (41) that

$$
\|v_n\|_{L^2(Q)} \leq C
\tag{44a}
$$

$$
\|I^{2-\alpha}y(v_n,T)\|_{L^2(\Omega)} \leq C
\tag{44b}
$$

Moreover $y_n = y(v_n,x,t)$ being solution of (40), y_n satisfies:

$$
D^{\alpha}_{RL}y_n(x,t) - \Delta y_n(x,t) = v_n(x,t) \quad (x,t) \in Q,
\tag{45a}
$$

$$
y_n(\sigma,t) = 0 \quad (\sigma,t) \in \Sigma,
\tag{45b}
$$

$$
I^{2-\alpha}y_n(x,0) = y^0 \quad x \in \Omega,
\tag{45c}
$$

$$
\frac{d}{dt}I^{2-\alpha}y_n(x,0) = y^1 \quad x \in \Omega
\tag{45d}
$$

and in view of Theorem 3.3, we have

$$\|y_n\|_{L^2((0,T);H_0^1(\Omega))} < C, \tag{46a}$$

$$\|I^{2-\alpha}y_n\|_{L^2((0,T);H_0^1(\Omega))} < C, \tag{46b}$$

$$\|\frac{d}{dt}I^{2-\alpha}y_n\|_{L^2((0,T);L^2(\Omega))} < C. \tag{46c}$$

Now using $(44a)$ and $(45a)$, we deduce that,

$$\|D_{RL}^{\alpha}y_n - \Delta y_n\|_{L^2(Q)} \le C. \tag{47}$$

Hence, from (44a), (47) and (46), we can extract subsequences of (v_n) and (y_n) (still called (v_n) and (y_n)) such that

$$v_n \rightharpoonup u \text{ weakly in } L^2(Q), \tag{48a}$$

$$D_{RL}^{\alpha}y_n - \Delta y_n \rightharpoonup \delta \text{ weakly in } L^2(Q), \tag{48b}$$

$$y_n \rightharpoonup y \text{ weakly in } L^2(Q), \tag{48c}$$

$$I^{2-\alpha}y_n \rightharpoonup \gamma \text{ weakly in } L^2([0,T], H_0^1(\Omega)), \tag{48d}$$

$$\frac{d}{dt}I^{2-\alpha}y_n \rightharpoonup \eta \text{ weakly in} L^2(Q). \tag{48e}$$

$\mathcal{U}_{ad}$ being convex closed subset of $L^2(Q)$, we have,

$$u \in \mathcal{U}_{ad}$$

Set $\mathbb{D}(Q)$, the set of $\mathcal{C}^\infty$ function on Q with compact support and denote by $\mathbb{D}'(Q)$ its dual. Then multiplying (45a) by $\varphi \in \mathbb{D}(Q)$ and integrating by part over Q, we obtain

$$\int_0^T\int_\Omega (D_{RL}^{\alpha}y_n(x,t) - \Delta y_n(x,t))\varphi(x,t)dxdt = \int_0^T\int_\Omega v_n(x,t)\,\varphi(x,t)dxdt \quad \forall \varphi \in \mathbb{D}(Q).$$

Therefore using Lemma 2.9, we get

$$\int_0^T\int_\Omega y_n(x,t)(\mathcal{D}_C^{\alpha}\varphi(x,t) - \Delta\varphi(x,t))dxdt = \int_0^T\int_\Omega v_n(x,t)\,\varphi(x,t)dxdt \quad \forall \varphi \in \mathbb{D}(Q).$$

Passing to the limit in this latter identity while using (48c) and (48a), we obtain that

$$\int_0^T\int_\Omega y(x,t)(\mathcal{D}_C^{\alpha}\varphi(x,t) - \Delta\varphi(x,t))dxdt = \int_0^T\int_\Omega u(x,t)\,\varphi(x,t)dxdt \quad \forall \varphi \in \mathbb{D}(Q),$$

which by using again Lemma 2.9 gives

$$\int_0^T\int_\Omega (D_{RL}^{\alpha}y(x,t) - \Delta y(x,t))\varphi(x,t)dxdt = \int_0^T\int_\Omega u(x,t)\,\varphi(x,t)dxdt \quad \forall \varphi \in \mathbb{D}(Q).$$

This implies that

$$D^{\alpha}_{RL}y(x,t) - \Delta y(x,t) = u(x,t) \quad (x,t) \in Q. \tag{49}$$

On the other hand, we have

$$\int_{\Omega}\int_0^T I^{2-\alpha}y_n(x,t)\varphi(x,t)dtdx =$$
$$\int_{\Omega}\int_0^T y_n(x,s)\left(\frac{1}{\Gamma(2-\alpha)}\int_s^T (t-s)^{1-\alpha}\varphi(x,t)dt\right)dsdx \quad \forall\varphi \in \mathbb{D}(Q).$$

Passing to the limit in this latter identity while using (48c) and (48d), we get

$$\begin{aligned}\int_0^T \gamma\varphi(x,t)dtdx &= \int_{\Omega}\int_0^T y(x,s)\left(\frac{1}{\Gamma(2-\alpha)}\int_s^T (t-s)^{1-\alpha}\varphi(x,t)dt\right)dsdx \\ &= \int_{\Omega}\int_0^T I^{2-\alpha}y(x,t)\varphi(x,t)dtdx \quad \forall\varphi \in \mathbb{D}(Q).\end{aligned}$$

This implies that

$$I^{2-\alpha}y(x,t) = \gamma \text{ in } Q.$$

Hence (48d) can be rewritten as

$$I^{2-\alpha}y_n \rightharpoonup I^{2-\alpha}y \text{ weakly in } L^2([0,T], H^1_0(\Omega)). \tag{50}$$

From (50), we have that

$$\frac{d}{dt}I^{2-\alpha}y_n \rightharpoonup \frac{d}{dt}I^{2-\alpha}y \text{ weakly in } \mathbb{D}'(Q)$$

and because of (48e),

$$\frac{d}{dt}I^{2-\alpha}y_n \rightharpoonup \frac{d}{dt}I^{2-\alpha}y = \eta \text{ weakly in } L^2(Q) \tag{51}$$

We have $y \in L^2((0,T); H^1_0(\Omega))$ and $I^{2-\alpha}y \in L^2((0,T); H^1_0(\Omega))$. Therefore, $D^{\alpha}_{RL}y = \dfrac{d^2}{dt^2}I^{2-\alpha}y \in H^{-2}((0,T); H^1_0(\Omega)) \subset H^{-2}((0,T); L^2(\Omega))$ on the one hand, and on the other hand $\Delta y \in H^{-2}((0,T); L^2(\Omega))$ because we have $\Delta y = D^{\alpha}_{RL}y - u$. Thus for almost all $t \in (0,T))$, $y(t) \in L^2(\Omega)$ and $\Delta y \in L^2(\Omega)$. Hence we deduce that $y|\partial\Omega$ and $\dfrac{\partial y}{\partial v}_{|\partial\Omega}$ exist and belong respectively to $H^{-1/2}(\partial\Omega)$ and to $\dfrac{\partial y}{\partial v}(t) \in H^{-3/2}(\partial\Omega)$. (*see* [3]).

On the other hand, y being in $L^2((0,T); H^1_0(\Omega))$ and u in $L^2((0,T); L^2(\Omega))$, we deduce that $\Delta y \in L^2((0,T); H^{-1}(\Omega))$ and consequently $D^{\alpha}_{RL}y = \dfrac{d^2}{dt^2}I^{2-\alpha}y = \Delta y + u \in L^2((0,T); H^{-2}(\Omega))$. Hence in view of (51), we have that $I^{2-\alpha} \in C([0,T]; L^2(\Omega))$ and $\dfrac{d}{dt}I^{2-\alpha}y \in C([0,T]; H^{-1}(\Omega))$ (*see* [4]).

Now multiplying $(45a)$ by a function $\varphi \in C^{\infty}(\overline{Q})$ with $\varphi_{|\partial\Omega} = 0$ and $\varphi(x,T) = \varphi'(x,T) = 0$ in Ω, and integrating by part over Q, we obtain

$$\begin{aligned}\int_0^T\int_\Omega v_n(x,t)\varphi(x,t)dxdt &= \int_0^T\int_\Omega (D^\alpha_{RL}y_n(x,t)-\Delta y_n(x,t))\varphi(x,t)dxdt\\ &= -\int_\Omega y^1\varphi(x,0)dx+\int_\Omega y^0\varphi'(x,0)dx\\ &+ \int_\Omega\int_0^T y_n(x,t)(\mathcal{D}^\alpha_C\varphi(x,t)-\Delta\varphi(x,t))dxdt\end{aligned}$$

since (45c) and (45d) hold. Thus, passing to the limit this latter identity while using (48a) and (48c), we have that

$$\begin{aligned}\int_0^T\int_\Omega u(x,t)\varphi(x,t)dxdt &= -\int_\Omega y^1\varphi(x,0)dx+\int_\Omega y^0\varphi'(x,0)dx\\ &+ \int_\Omega\int_0^T y(x,t)(\mathcal{D}^\alpha_C\varphi(x,t)-\Delta\varphi(x,t))dxdt,\\ &\forall\varphi\in C^\infty(\overline{Q})\text{ such that }\varphi_{|\partial\Omega}=0,\ \varphi(T)=\varphi'(T)=0\text{ in }\Omega,\end{aligned}$$

which in view of Lemma 2.9 gives

$$\begin{aligned}\int_0^T\int_\Omega u(x,t)\varphi(x,t)dxdt &= -\int_\Omega y^1\varphi(x,0)dx+\int_\Omega y^0\varphi'(x,0)dx\\ &+ \int_0^T\int_\Omega (D^\alpha_{RL}y(x,t)-\Delta y(x,t))\varphi(x,t)dxdt\\ &+ \langle\varphi(x,0),\frac{d}{dt}I^{2-\alpha}y(x,0)\rangle_{H^1_0(\Omega),H^{-1}(\Omega)}\\ &- (\varphi'(x,0),I^{2-\alpha}y(x,0))_{L^2(\Omega)}\\ &- \int_0^T\langle y(\sigma,t),\frac{\partial\varphi}{\partial\nu}(\sigma,t)\rangle_{H^{-1/2}(\partial\Omega),H^{1/2}(\partial\Omega)}dt\\ &\forall\varphi\in C^\infty(\overline{Q})\text{ such that }\varphi_{|\partial\Omega}=0,\ \varphi(T)=\varphi'(T)=0\text{ in }\Omega,.\end{aligned}$$

Using (49), this latter identity is reduced to

$$\begin{aligned}0 &= -\int_\Omega y^1\varphi(x,0)dx+\int_\Omega y^0\varphi'(x,0)dx\\ &+ \langle\varphi(x,0),\frac{d}{dt}I^{2-\alpha}y(x,0)\rangle_{H^1_0(\Omega),H^{-1}(\Omega)}\\ &- (\varphi'(x,0),I^{2-\alpha}y(x,0))_{L^2(\Omega)}\\ &- \int_0^T\langle y(\sigma,t),\frac{\partial\varphi}{\partial\nu}(\sigma,t)\rangle_{H^{-1/2}(\partial\Omega),H^{1/2}(\partial\Omega)}dt\\ &\forall\varphi\in C^\infty(\overline{Q})\text{ such that }\varphi_{|\partial\Omega}=0,\ \varphi(T)=\varphi'(T)=0\text{ in }\Omega.\end{aligned}\tag{52}$$

Choosing successively in (52) φ such that $\varphi(x,0)=\varphi'(x,0)=0$ and $\varphi(x,0)=0$, we deduce that

$$y(x,t)=0\quad (x,t)\in\Sigma,\tag{53}$$

$$I^{2-\alpha}y(x,0)=y^0\quad x\in\Omega\tag{54}$$

and

$$\frac{d}{dt}I^{2-\alpha}y(x,0) = y^1 \quad x \in \Omega. \tag{55}$$

From (49), (53), (54) and (55), we have $y = y(u,x,t)$ is solution of the system (40). It then follows from the lower semi-continuity of the functional J and

$$J(u) \leq \liminf J(v_n).$$

Hence in view of (43), we get

$$J(u) = \inf_{v \in \mathcal{U}_{ad}} J(v).$$

From the strict convexity of J, we obtain the uniqueness of the optimal control u. □

Theorem 4.2. *Let u be solution of (42). Then there exists $p \in L^2(Q)$ such that (u, y, p) verifies the following optimality systems:*

$$\left\{\begin{array}{rcll} D^{\alpha}_{RL}y(x,t) - \Delta y(x,t) &=& u(x,t) & (x,t) \in Q, \\ y(\sigma,t) &=& 0 & (\sigma,t) \in \Sigma, \\ I^{2-\alpha}y(x,0) &=& y^0(x) & x \in \Omega, \\ \dfrac{d}{dt}I^{2-\alpha}y(x,0) &=& y^1(x) & x \in \Omega, \end{array}\right. \tag{56}$$

$$\left\{\begin{array}{rcll} \mathcal{D}^{\alpha}_{C}p(x,t) - \Delta p(x,t) &=& 0 & (x,t) \in Q, \\ p(\sigma,t) &=& 0 & (\sigma,t) \in \Sigma, \\ p(x,T) &=& 0 & x \in \Omega, \\ p'(x,T) &=& I^{2-\alpha}y(u,T) - z_d & x \in \Omega, \end{array}\right. \tag{57}$$

and

$$(Nu - p, v - u)_{L^2(Q)} \geq 0 \quad \forall v \in \mathcal{U}_{ad}. \tag{58}$$

Proof. We express the Euler-Lagrange optimality conditions which characterize u:

$$\lim_{k \to 0} \frac{J(u + k(v-u)) - J(u)}{k} \geq 0 \quad \forall v \in \mathcal{U}_{ad}$$

After some calculations, we obtain

$$\begin{array}{c} \displaystyle\int_\Omega (I^{2-\alpha}y(u,T) - z_d) I^{2-\alpha}z(w,T)dx + \\ \displaystyle\int_0^T \int_\Omega Nu(x,t)\, w(x,t) dt dx \geq 0 \quad \forall v \in \mathcal{U}_{ad}, \end{array} \tag{59}$$

where $w = v - u$ and the state $z(w)$ associated to $w \in L^2(Q)$ is solution of ,

$$
(60) \qquad \begin{cases} D^{\alpha}_{RL} z(x,t) - \Delta z(x,t) &= w(x,t) \quad (x,t) \in Q \\ z(\sigma,t) &= 0 \qquad (\sigma,t) \in \Sigma \\ I^{2-\alpha} z(x,0) &= 0 \qquad x \in \Omega \\ \dfrac{d}{dt} I^{2-\alpha} z(x,0) &= 0 \qquad x \in \Omega. \end{cases}
$$

To interpret (59), we consider the following adjoint system:

$$
(61) \qquad \begin{cases} \mathcal{D}^{\alpha}_{C} p(x,t) - \Delta p(x,t) &= 0 \qquad (x,t) \in Q, \\ p(\sigma,t) &= 0 \qquad (\sigma,t) \in \Sigma, \\ p(x,T) &= 0 \qquad x \in \Omega, \\ p'(x,T) &= I^{2-\alpha} y(u,T) - z_d \quad x \in \Omega. \end{cases}
$$

In view of Proposition 3.4 and Remark 3.5, we have that $p \in \mathcal{C}([0,T], L^2(\Omega)) \subset L^2(Q)$ since $I^{2-\alpha} y(u,T) - z_d \in L^2(Q)$. Multiplying (60) by the solution p of (61), we obtain that

$$
\begin{aligned} \int_0^T \int_\Omega w(x,t) p(x,t)\,dxdt &= \int_0^T \int_\Omega (D^{\alpha}_{RL} z(x,t) - \Delta z(x,t)) p(x,t)\,dxdt \\ &= -\int_\Omega p'(x,T)(I^{2-\alpha} y(u,T) - z_d) I^{2-\alpha} z(w,T)\,dx. \end{aligned}
$$

This means that

$$
(62) \qquad \int_0^T \int_\Omega w(x,t) p(x,t)\,dxdt = -\int_\Omega p'(x,T)(I^{2-\alpha} y(u,T) - z_d) I^{2-\alpha} z(w,T)\,dx.
$$

Combining (59) with (62), we (58).

□

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