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On the existence of $C^{(n)}$-almost automorphic mild solutions of certain differential equations in Banach spaces

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Abstract. In this paper we study the existence and uniqueness of almost automorphic mild solutions to the semilinear equation $u'(t) = A(t)u(t) + f(t, u(t)),\ t \in \mathbb{R}$ where $A(t)$ satisfies the Acquistapace-Terreni conditions and generates an exponentially stable family of operators $(U(t,s)_{t\geq s}$ and $f(t,u)$ is almost automorphic in $t \in \mathbb{R}$ for any $u \in X$. This extends a well-known result by G.M. N'Guérékata in the case where $A(t) = A$ does not depend on t. We also investigate the existence of mild C^n-almost automorphic solutions of the linear version of the above equation when A is an operator of simplest type.

1. Introduction

The concept of almost automorphic functions was introduced in the literature by S. Bochner [**4, 5**] while studying some problems in differential geometry. It was then developed by W.A. Veech [**17**] for functions on groups and M. Zaki [**18**] for functions on the real line. Further developments were boosted by G.M. N'Guérékata's books [**13, 15**]. The concept turns out to generalize the concept of almost periodic functions in the sense of Bohr and has applications in several problems in mathematics and the sciences (cf. for instance [**6, 8, 9, 10, 11, 12, 19, 21**] and references therein.

In [**14**], the author studied the semilinear equation in a Banach space X

$$u'(t) = Au(t) + f(t, u(t)),\quad t \in \mathbb{R} \tag{1.1}$$

where $A : D(A) \subset X \to X$ is the infinitesimal generator of an exponentially stable C_0-semigroup of bounded operators, and $f(t,u)$ is almost automorphic in t for any $u \in X$ and is lipschitzian in t uniformly in u. He proved that the equation possesses a unique almost automorphic mild solution. This result has been extended to general cases where f is not necessarily lipschitzian and the operator A not necessarily a generator of a C_0-semigroup .

2000 *Mathematics Subject Classification.* 34G10, 47D06, 45M05.

This work has been conducted during the first author's visit at AIMS-Cameroon Research Center in Limbe, Cameroon in March 2019. He is grateful for the invitation by the second author, supported by the Alexander von Humboldt foundation, under the program financed by the BMBF entitled "German research Chairs".

In [**3**], the authors studied the existence of S-asymptotically ω-periodic mild solution of the nonautonomous semilinear equation

$$u'(t) = A(t)u(t) + f(t, u(t)), \quad t \in \mathbb{R}^+ \tag{1.2}$$

where $A(t)$ generates a ω-periodic exponentially stable evolutionary process $(U(t,s)_{t\geq s}$.

Motivated by the above results, we prove in the present paper that Equation(1.2) considering $A(t)$ generates an evolutionary process $(U(t,s)_{t\geq s}$ which is not necessarily ω-periodic, admits a unique almost automorphic mild solution assuming appropriate conditions on the function $f(t,u)$. This is our main result Theorem 4.3. We also discuss C^n-almost automorphic mild solutions to the linear version of the above Equation (1.1). These results extend the ones obtained in [**2**] to the case of this types of functions and the operator A being of simplest type.

2. $C^{(n)}$-almost automorphic functions

Definition 2.1. [**4, 5, 16, 18**] A continuous function $f : \mathbb{R} \to X$ is said to be almost automorphic if for every sequence or real numbers (s'_n) there exists a subsequence (s_n) such that

$$\lim_{n\to\infty} f(t+s_n) = g(t)$$

exists for each $t \in \mathbb{R}$ and

$$\lim_{n\to\infty} g(t-s_n) = f(t)$$

for each $t \in \mathbb{R}$.

Remark 2.2. (i) The function g in the definition above is measurable, but not necessarily continuous.
(ii) If the convergence above is uniform in $t \in \mathbb{R}$, then f is almost periodic in the sense of Bohr.

Proposition 2.3. [**13**] If f, f_1, f_2 are almost automorphic functions $\mathbb{R} \to X$, λ a scalar, then the following are true:

i) $f_1 + f_2, \ \ \lambda f$ are almost automorhic.
ii) the translation $f_\tau(t) := f(t+\tau), \ \ t \in \mathbb{R}$ is almost automorphic.
iii) the function $\tilde{f}(t) := f(-t), \ \ t \in \mathbb{R}$ is almost automorphic.
iv) the range of f $\mathcal{R}_f := \{f(t) : \ t \in \mathbb{R}\}$ is relatively compact in X.

Theorem 2.4. [**13**] *If (f_n) is a sequence of almost automorphic functions which convergent uniformly in $t \in \mathbb{R}$ to f, then f is almost automorphic.*

Remark 2.5. Denote $AA(X)$ the space of all almost automorphic functions $\mathbb{R} \to X$. Equipped with the norm $\|f\| = \sup_{\mathbb{R}} \|f(t)\|$, $AA(X)$ turns out to be a Banach space.
If we denote by $AP(X)$ the Banach space of all almost periodic functions in the sens of Bohr, then we have

$$AP(X) \subset AA(X).$$

The inclusion is strict. Indeed the function

$$f(t) = sin(\frac{1}{2 + cost + cos\sqrt{2}t}) \tag{2.1}$$

is almost automorphic but not almost periodic

We recall the following Zheng-Ding-N'Guérékata theorem in [**21**] which provides an information on the "amount" of almost automorphic functions which are not almost periodic. We also give a different and more elegant proof than in [**21**] suggested by Marek Balcerzak in his review MR3036705.

THEOREM 2.6. *$AP(X)$ is a set of first category in $AA(X)$.*

PROOF. It suffices to note from the above that $AP(X)$ is a proper closed subspace of $AA(X)$ equipped with the supnorm. Therefore it is of first category in $AA(X)$. □

DEFINITION 2.7. [**9**] A continuous function $f : \mathbb{R} \to X$ is said to be C^n-almost automorphic for $n \geq 1$ if for $i = 1, 2, ..., n$, the i-th derivative $f^{(i)}$ of f is almost automorphic.

PROPOSITION 2.8. [**10, 14**] The set $AA^{(n)}(X)$ of all C^n-almost automorphic functions $f : \mathbb{R} \to X$ is a Banach space under the norm

$$\|f\|_n = \sup_{t\in\mathbb{R}}(\sum_{i=1}^{n} \|f^{(i)}(t)\|).$$

THEOREM 2.9. *Suppose that X does not contain a subspace isomorphic to c_o and $f \in AA^{(n)}(X)$. Then*

$$F(t) := \int_0^t f(\xi)d\xi \in AA^{(n+1)}(X) \text{ iff the } \mathcal{R}_f := \{f(t) : t \in \mathbb{R}\} \text{ is bounded in} X.$$

PROOF. The "if" part is immediate. Let's prove the "only" part. We use the induction principle. The case $n = 0$ is well-known (cf. for instance [**14**]). Assume that $f \in AA^{(k-1)}(X)$ for some k. That is $F' = f \in AA^{(k-1)}(X)$. But by assumption f is bounded. Therefore $F \in AA^{(k)}(X)$. The proof is complete. □

3. Linear Equations

We will consider in this section, the following equation in a complex Banach space X which does not contain a subspace isomorphic to c_o.

$$u'(t) + Au(t) + f(t), \quad t \in \mathbb{R}, \tag{3.1}$$

where $A : D(A) \subset X \to X$ is a linear operator and $f \in C(\mathbb{R}, X)$.
In what follows we will use the notation $\Pi := \{z \in \mathbb{C} : \mathcal{R}ez \neq 0\}$

DEFINITION 3.1. [**7, 20**] A linear operator $A : D(A) \subset X \to X$ is said to be of simplest type if $A \in B(X)$ and

$$A = \sum_{i=1}^{n} \lambda_i P_i,$$

where $\lambda_i \in \mathbb{C}, i = 1, 2, ...n$ and $(P_i)_{1\leq i\leq n}$ forms a complete system $\sum_{i=1}^{n} P_i = I$ of mutually disjoint operators on X, i.e. $P_iP_j = \delta_{ij}P_j$

LEMMA 3.2. *Let's consider Equation(3.1) with $A = \lambda \in \mathbb{C}$ and $f \in AA^{(n)}(X)$. Then every bounded solution u satisfies*

$$u \in AA^{(n)}(X), \quad \lambda \in \Pi$$

and

$$u \in AA^{(n+1)}(X), \quad \lambda \notin \Pi.$$

PROOF. The proof is an immediate consequence of Lemma 4.1 in [**9**] □

THEOREM 3.3. *Assume that $f \in AA^{(n)}(X)$ and A is of simplest type.*

Then every bounded solution u of Eq.(3.1) satisfies

$$u \in AA^{(n)}(X), \ \ \lambda \in \Pi$$

and

$$u \in AA^{(n+1)}(X), \ \ \lambda \notin \Pi.$$

PROOF. Applying the projection P_k to Eq.(3.1) gives

$$P_k u'(t) = \frac{d}{dt}(P_k u)(t) = P_k(\sum_{i=1}^{n} \lambda_i P_i)u(t) + P_k f(t) = \lambda_k (P_k u)(t) + (P_k f)(t)$$

Observe that $P_k f \in AA^{(n)}$ since P_k is a bounded linear operator. Therefore by the Lemma 3.2 above,

$$P_k u \in AA^{(n)}(X), \ \ \lambda \in \Pi$$

and

$$P_k u \in AA^{(n+1)}(X), \ \ \lambda \notin \Pi.$$

Now since $u(t) = \sum_{i=1}^{n} (P_i u)(t)$, the conclusion follows immediately. □

4. Nonautonomous case

Consider now in a complex Banach space X the equation

$$u'(t) = A(t)u(u) + f(t), \ \ t \in \mathbb{R} \tag{4.1}$$

And assume that $A(t), \ t \in \mathbb{R}$, satisfies the Acquistapace-Terreni conditions [**1**]. That means there exist constants $\lambda_0 \geq 0, \ \theta \in (\frac{\pi}{2}, \pi), \ L, K \geq 0$ and $\alpha, \beta \in (0,1]$ with $\alpha + \beta > 1$ such that

$$\sum_{\theta} \cup \{0\} \subset \rho(A(t) - \lambda_0), \ \ \|R(\lambda, A(t) - \lambda_0)\| \leq \frac{K}{1 + |\lambda|} \tag{4.2}$$

and

$$\|(A(t) - \lambda_0)R(\lambda, A(t) - \lambda_0)[R(\lambda_0, A(t) - R(\lambda_0, A(s))]\| \leq L|t - s|^{\alpha}|\lambda|^{\beta} \tag{4.3}$$

for $t, s \in \mathbb{R}, \ \ \lambda \in \sum_{\theta} := \{\lambda \in \mathbb{C}^{\star} : |arg\lambda| \leq \theta\}$. Then from [**1**] Theorem 2.3, there exists a unique evolution family $\{U(t,s)\}_{-\infty < s \leq t < \infty}$ on X which governs the linear version of (4.1).

The family $\{U(t,s)\}_{-\infty < s \leq t < \infty}$ satisfies the following properties

(i) $U(t,t) = I, \ \ \forall t \in \mathbb{R}$,
(ii) $U(t,r)T(r,s) = U(t,s), \ \ \forall t \geq r \geq s$,
(iii) The map $(t,s) \to U(t,s)\xi$ is continuous for every $\xi \in X$.

DEFINITION 4.1. A function $u \in C(\mathbb{R}, C)$ is said to be a mild solution to Eq.(4.1) if it has the integral representation

$$u(t) = U(t,s)u(s) + \int_{s}^{t} U(t,s)f(s)ds, \ \ t, s \in \mathbb{R}, \ t \geq s. \tag{4.4}$$

Under the above assumptions, it can be proved as in G. N'Guérékata et *al.* [**2**], Theorem 3.6 that any mild solution can be written as

$$u(t) = \int_{-\infty}^{t} U(t,s)f(s)ds, \quad t \in \mathbb{R} \tag{4.5}$$

Let make the following assumptions:

H1. $f \in AA(X)$

H2. There exists a function $\varphi \in L^1(\mathbb{R}, \mathbb{R}^+)$ such that

$$\|U(t,s)\| \leq \varphi(t-s).$$

THEOREM 4.2. *Assume that assumptions (H1)-(H2) hold true. Assume also that that $A(t)$, $t \in \mathbb{R}$, satisfies the Acquistapace-Terreni conditions.*

Then Eq.(4.1) admits a unique mild solution in $AA(X)$.

PROOF. Observing that $\lim_{t\to\infty} \varphi(t) = 0$, it can be shown as in [**2**] that the function $\Lambda : \mathbb{R} \to BC(\mathbb{R}, X)$ defined by

$$\Lambda(t) := \int_{-\infty}^{t} U(t,s)f(s)ds$$

is the unique solution to Eq.(4.1). Now let's prove it is almost automorphic.

Let (s_n) be an arbitrary sequence of real numbers. Since f is almost automorphic, there exists a subsequence (s_n) of (s'_n) such that

$$\lim_{n\to\infty} f(t+s_n) = g(t)$$

exists for each $t \in \mathbb{R}$ and

$$\lim_{n\to\infty} g(t-s_n) = f(t)$$

for each $t \in \mathbb{R}$.

Consider $\tilde{\Lambda} : \mathbb{R} \to BC(\mathbb{R}, X)$ such that

$$\tilde{\Lambda}(t) := \int_{-\infty}^{t} U(t,s)g(s)ds.$$

In view of the properties of U and g, this functions is well defined. Now we have

$$\begin{aligned}
\Lambda(t+s_n) - \tilde{\Lambda}(t) &= \int_{-\infty}^{t+s_n} U(t+s_n,s)f(s)ds - \int_{-\infty}^{t} U(t,s)g(s)ds \\
&= \int_{-\infty}^{t} U(t+s_n, s+s_n)f(s+s_n)ds - \int_{-\infty}^{t} U(t,s)g(s)ds.
\end{aligned}$$

Now observe that for any $n = 1, 2, ...$ we have

$$\|U(t+s_n, s+s_n)f(s+s_n)\| \leq M\varphi(t-s)\|f\|_\infty$$

and the right hand member of the inequality belongs to $L^1(\mathbb{R}, \mathbb{R}^+)$. Therefore, in virtue of the Lebesgue Dominated Convergence theorem,

$$\int_{-\infty}^{t} U(t+s_n, s+s_n)f(s+s_n)ds \to \int_{-\infty}^{t} U(t,s)g(s)ds, \text{ as } n \to \infty.$$

In other words,

$$\Lambda(t+s_n) \to \tilde{\Lambda}(t), \text{ as } n \to \infty.$$

Analogously we can prove that

$$\tilde{\Lambda}(t-s_n) \to \Lambda(t), \text{ as } n \to \infty.$$

That means $\Lambda(t) \in AA(X)$.

□

Consider now in a complex Banach space X the semilinear equation

$$u'(t) = A(t)u(u) + F(t,u(t)), \quad t \in \mathbb{R}. \tag{4.6}$$

THEOREM 4.3. *And assume that $A(t)$, $t \in \mathbb{R}$, satisfies the Acquistapace-Terreni conditions and (H2) holds. Assume also that $F(t,x)$ is almost automorphic in $t \in \mathbb{R}$ uniformly in $x \in K$ for any bounded set $K \subset X$ and*

$$\|F(t,x) - F(t,y)\| \le L\|x-y\|, \quad t \in \mathbb{R}, \ x,y \in X.$$

Then Eq.(4.6) has a unique mild solution in $AA(X)$ if in addition we have $L\|\varphi\|_{L^1(\mathbb{R},\mathbb{R}^+)} < 1$.

PROOF. Consider the operator $\Upsilon : AA(X) \to AA(X)$ defined by

$$(\Upsilon u)(t) := \int_{-\infty}^{t} U(t,s)F(s,u(s))ds.$$

In view of Theorem 2.2.5 [**13**] and 4.2 above this operator is well-defined.

Now take $u, v \in AA(X)$. Then we get

$$\begin{aligned} \|(\Upsilon u)(t) - (\Upsilon v)(t)\| &= \left\| \int_{-\infty}^{t} U(t,s)[F(s,u(s)) - F(s,v(s))ds \right\| \\ &\le \int_{-\infty}^{t} \|U(t,s)\| \|F(s,u(s)) - F(s,v(s))ds \\ &\le L\|\varphi\|_{L^1(\mathbb{R},\mathbb{R}^+)} \|u-v\|_\infty. \end{aligned}$$

Therefore

$$\|(\Upsilon u) - (\Upsilon v)\|_\infty \le L\|\varphi\|_{L^1(\mathbb{R},\mathbb{R}^+)} \|u-v\|_\infty$$

and the conclusion follows in virtue of the Banach fixed point theorem. □

References

1. P. Acquistapace, *Evolution operators and strong solution of abstract linear parabolic equations*, Differential Integral Equations **1** (1988), 433-457.
2. J.B. Baillon, J. Blot, G.M. N'Guérékata, D. Pennequin, *On $C^{(n)}$-almost periodic solutions to some nonautonomous differential equations in Banach spaces*, Annales Societatis Mathematicae Polonae, Series 1, XLVI (2), (2006), 263-273.
3. J. Blot, P. Cieutat, G.M. N'Guérékata, *S-asymptotically ω-periodic fucntions and applications to evolution equations*, Afr. Diaspora J. Math. **12**, No.2 (2011), 113-121
4. S. Bochner, *Uniform convergence of monotone sequences of functions*, Proc. Nat. Acad. Sci. **47**, (1961), 582-585.
5. S. Bochner, *A new approach to almost periodicity*, Proc. Nat. Acad. Sci., **48**, (1962), 2039-2043.
6. D. Bugakewski, T. Diagana, *Almost automorphic of the convolution operator and applications to differential and functional differential equations*, Nonlinear Studies, **13**, No.2 (2006), 129-140.
7. J.L. Daleckii, M.G. Krein, *Stability of Solutions of Differential Equations in Banach Spaces*, American Mathematical Society Translations, **43** Providence Rhode Island, 1974.

8. B. Es-sebbar, K. Ezzinbi, G.M. N'Guérékata, *Bohr-Neugebauer property for almost automorphic partial fucntional differential equations*, Appl. Anal. **98** (2019), 381-407.
9. K. Ezzinbi, S. Fatajou, G.M. N'Guérékata, *$C^{(n)}$-almost automorphic solutions for partial neutral functional differential equations*, Applicable Analysis, Vol. 86 Issue 9 (2007), 1127-1146.
10. K. Ezzinbi, V. Nelson, G.M. N'Guérékata, *$C^{(n)}$-almost automorphic solutions of nonautonomous differential equations*, Cubo A Mathematical Journal, vol.10, no.02 (2009), 62-74.
11. M. Kostić, *Generalized almost automorphic and asymptotically almost automorphic solutions of abstract Volterra integro-differential inclusions*, Fract. Differ. Calc. **8** (2018), no.2 255-284.
12. M. Kostić, *On generalized $C^{(n)}$-almost periodic solutions of abstract Volterra integro-differential equations*, Novi Sad J. Math. **48** (2018), no.1, 73-91.
13. G.M. N'Guérékata, *Almost Automorphic and Almost Periodic Functions in Abstract Spaces*, Kluwer Academic Plenum, 2001, New York.
14. G.M. N'Guérékata, *Existence and uniqueness of mild solutions of some abstract semilinear differential equations in banach spaces*, Semigroup Forum, **69(1)** (2004), 80-86.
15. G.M. N'Guérékata, *Topics in Almost Automorphy*, Springer, New York, 2005.
16. G.M. N'Guérékata, *Spectral Theory of Bounded Functions and Applications to Evolution Equations*, Nova Science Publ., New York 2018.
17. W.A. Veech, *Almost automorphic functions on groups*, Amer. J. Math., **87** (July 1965), 719-752.
18. M. Zaki, *Almost automorphic solutions of certain abstract differential equations*, Annali di Mat. Pura ed Appl., series 4 **101** (1974), 91-114.
19. S. Zaidman, *Almost periodic functions in Abstract Spaces*, **126**, Pitman Publishing, Boston London 1986.
20. S. Zaidman, *Topics in Abstract Differential Equations II*, Pitman Research Notes in Mathematics Series,Longman Scientific and Technical, Harlow Essex, 1995.
21. Z-M. Zheng, H-S. Ding, G.M. N'Guérékata, *The space of continuous periodic functions is a set of first category in $AP(X)$*, J. Funct. Spaces Appl.. 2013 Art. ID 275702, 3pp.

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