The bogomolov-beauville-yau decomposition for klt projective varieties with trivial first chern class -without tears-

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Contents

ABSTRACT. We give a simplified proof (in characteristic zero) of the decomposition theorem for complex projective varieties with klt singularities and numerically trivial canonical bundle. The proof mainly consists in reorganizing some the partial results obtained by many authors and used in the previous proof, but avoids those in positive characteristic by S. Druel. The single, to some extent new, contribution is an algebraicity and bimeromorphic splitting result for generically locally trivial fibrations with fibres without holomorphic vector fields. We give first the proof in the easier smooth case, following the same steps as in the general case, treated next.

The last two words of the title are plagiarized fom [4]

1. INTRODUCTION

When X is smooth, connected, compact Kähler, with $c_1(X) = 0$, the classical, metric, proof of the 'Bogomolov-Beauville-(Yau) decomposition theorem', given in [2] (the arguments of [6] being Hodge-theoretic), starts with a Ricci-flat Kähler metric ([24]), and then decomposes the universal cover X' of X according to De Rham theorem, in its holonomy factors. The Cheeger-Gromoll theorem then distinguishes the flat euclidian factor $\mathbb{C}^s$ of X' from the (simply-connected) product P of the others (which are compact and with holonomy either $SU(m)$ or $Sp(k)$). The compactness of P combined with Bieberbach's theorem now imply that a finite étale cover of X is the product of a complex torus $\mathbb{C}^s/\Gamma$ with P.

We shall first give a different proof, but only for X smooth projective, of this product decomposition, weaker in the sense that P is not showed to be simply connrected (see Theorem 2.1 below). The proof indeed does not go through the universal cover, and uses neither the De Rham, nor the Cheeger-Gromoll theorems.

This allows for its extension (given next) to the singular case obtained in [20], which uses many other other partial results, among which are those of [17] and [13] (which plays to a certain extent the rôle of the Cheeger-Gromoll theorem). Our proof makes the positive characteristic step in [13] superfluous, by deducing directly the algebraicity of the foliation given by the flat factor of the holonomy, independently of the Albanese map, from the splitting result (see Theorem 3.3) below, once the algebraicity of the leaves of the foliations given by the non-flat factors of the holonomy have been first shown to be algebraic.

I thank Benoit Claudon and Mihai Păun for their help in reading the text, and several discussions. After this text was posted on arXiv, I received useful comments by S. Druel, and H. Guenancia. I thank both of them too.

2. The smooth case

We tret this case first, in order to show in a simpler context the steps in the general case.

Theorem 2.1. *Let X be a smooth connected complex projective manifold with $c_1(X) = 0$. There exists a finite étale cover of X which is a product of an abelian variety with projective manifolds either irreducible symplectic, or Calabi-Yau.*

Remark 2.2. *The notions of irreducible symplectic and Calabi-Yau manifolds are defined as in [2]: either by the values of $h^{p,0}$, or by the holonomy of any Ricci-flat Kähler metric. We need the projectivity of X because the Kähler version of [11] is not known. Our proof also does not show the finiteness of the fundamental groups of symplectic or Calabi-Yau manifolds. A partial solution is given to this finiteness property in Proposition 2.7 below, based on more general L^2-methods.*

Proof. (of theorem 2.1) We equip X with any Ricci-flat Kähler metric ([24]). Let Hol^0 (resp. Hol) be its restricted holonomy (resp. holonomy) representation, and $T_X = F \oplus (\oplus_i T_i)$ be a (local near any given point of X) splitting of the tangent bundle of X into factors which are irreducible for the action of Hol^0. These local factors correspond also to a local splitting of X into a direct product of Kähler submanifolds. In particular, these local products are regular holomorphic foliations. Here F is the 'flat' factor consisting of the holonomy-invariant tangent vectors. Now Hol^0 is a normal subgroup of Hol, and Hol/Hol^0 acts by permutation on the factors of the restricted holonomy decomposition. Because the action of Hol/Hol^0 is induced by a representation $\pi_1(X) \to Hol/Hol^0$, the local holonomy decomposition of T_X above holds globally on a suitable finite étale cover of X.

We now replace X by such a finite étale cover, and obtain a global product decomposition $T_X = F \oplus (\oplus_i T_i)$ by regular holomorphic foliations, the restricted holonomy of F being trivial, while the ones of the T_i are irreducible and of the form either $SU(m_i)$, or $Sp(k_i)$.

Lemma 2.3. *Let $T_X = \oplus_j E_j$ be a direct sum decomposition by foliations E_j, with $c_1(X) = 0$. Then $c_1(E_j) = 0, \forall j$.*

Proof. Assume not, and let H be a polarisation on X, with $n := dim(X)$. Then $c_1(E_j).H^{n-1} \neq 0$, for some j. Since $\sum_j c_1(E_j).H^{n-1} = 0$, we get: $c_1(E_h).H^{n-1} > 0$ for some h. It then follows from [11], Lemma 4.10, that E_h contains a subfoliation G with $\mu_{H,min}(G) > 0$, and by [11], Theorem 4.1, that K_X is not pseudo-effective, contrary to the hypothesis $c_1(T_X) = 0$.   □

From the preceding lemma 2.3, if $T_X = F \oplus (\oplus_i T_i)$ is the holonomy decomposition of T_X considered above for X smooth projective with $c_1(X) = 0$, we get that $c_1(F) = c_1(T_i) = 0, \forall i$.

Lemma 2.4. *The dual $T_i^\star$ of each T_i is not pseudo-effective (which means that, for any polarisation H, and any given $k > 0$, $h^0(X, Sym^m(T_i^\star) \otimes H^k) = \{0\}$ for $m \geq m(k)$).*

Proof. We proceed in two steps. From [16], §15.3 and Proposition 24.22 follows that $Sym^m(T_i), \forall i, \forall m > 0$ is an irreducible representation, hence stable. Next, [12], Theorem 1.3 (or alternatively [20], Theorem 1.1) implies that $T_i^\star$ is not pseudo-effective for each i. $\square$

From [11], Theorem 4.2, Lemma 4.6, we now get the first claim of the next result[1]

Lemma 2.5. *Each of the foliations T_i has algebraic leaves, which are compact[2] since T_i is everywhere regular and X is smooth. Thus T_i defines a smooth (proper) fibration $f_i : X \to B_i$ on a smooth projective base B_i. Each of these fibrations is locally trivial with fibre F_i, and becomes a product $X' = F_i \times B_i'$ after a suitable finite étale base-change $B_i' \to B_i$.*

Proof. Second claim: let $C_i := F \oplus (\oplus_{\neq i} T_i)$ be the complement in T_X of T_i. This defines locally over B_i a regular holomorphic foliation which is transversal to f_i, and thus shows that f_i is locally isotrivial over B_i. Third claim: it is sufficient to know that $Aut(F_i)$ is dicrete, or that $h^0(F_i, T_{F_i}) = 0$. But this is easy, since F_i is a projective manifold with $c_1 = 0$ and irreducible non-trivial holonomy, which thus does not leave any tangent vector invariant, which implies the claimed vanishing by the Bochner principle. $\square$

Consider any one of the projections $f_i : F_i \times B_i \to B_i$ (after a suitable finite étale cover). Then $c_1(B_i) = 0$, and its holonomy decomposition is $F \oplus (\oplus_{j \neq i} T_i)$. Proceeding inductively on $dim(X)$, we obtain a decomposition in a product $X = (\times_i F_i) \times B$, where B is smooth projective with $c_1(B) = 0$ and trivial holonomy F.

The next lemma then concludes the proof of Theorem 2.1. $\square$

Lemma 2.6. *([5]) Let X be a connected compact Kähler manifold with $c_1(X) = 0$, and with trivial restricted holonomy representation (relative to some Ricci-flat Kähler metric). Then X is covered by a torus.*

The Symplectic and the even-dimensional Calabi-Yau manifolds can be shown to have a finite fundamental group by L^2-methods which extend to the singular case. Another approach is given right after, which works more generally, for compact Riemannian manifolds with nonnegative Ricci curvature and vanishing 'maximal b_1', but does not extend in an obvious way to the singular case.

Proposition 2.7. *Let X be a connected compact Kähler manifold with $c_1(X) = 0$ with $\chi(\mathcal{O}_X) \neq 0$. Then $\pi_1(X)$ is finite.*

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[1] Although not explicitely stated in [11], this is a main step of the proof of 4.2, and suggested by the proof of Lemma 4.6 there. The explicit formulation was first given in [13], §8. Since only the particular case of a polarisation H^{n-1} is used here, one could even alternatively apply [7].

[2] By contradiction: if not, the leaf through a regular point of the boundary of the closure of a leaf should be contained in this boundary, and of the same dimension. In the singular case, this compactness fails, and more delicate arguments are required.

Proof. We give two proofs, both relying on [1]

First proof. This is the proof given in [9], Corollary 5.3, and Remark 5.5. By [9], Theorem 4.1, it is sufficient to show that $\kappa^+(X) \leq 0$, that is: $\kappa(X, det(F)) \leq 0$, for any subsheaf $F \subset \Omega_X^p, \forall p > 0$, which follows from the semi-stability of the symmetric powers of Ω_X^p.

Second proof: if $X' \to X$ is the universal cover, and h an L^2-holomorphic p-form on X', then h is parallel (because the Laplacian of its squared norm equals the square norm of its covariant derivative, and so is nonnegative everywhere. Gaffney's integration trick implies that the Laplacian identically vanishes, since h is L^2 and X' is complete). Thus h comes from X, and so vanishes if X' is noncompact. By [1], one gets: $0 = \sum_{p \in \{0,n\}} (-1)^p h_{(2)}^0(X', \Omega_{X'}^p) = \chi_{(2)}(X', \mathcal{O}_{X'}) = \chi(X, \mathcal{O}_{X'}) \neq 0$, a contradiction. □

Corollary 2.8. *If X is a compact Kähler manifold of dimension n, and irreducible symplectic (resp. Calabi-Yau of even dimension), then $\pi_1(X)$ is finite of cardinality dividing $(\frac{n}{2} + 1)$ (resp. 2).*

Proof. Let $X' \to X$ be the (compact) universal cover of X, of degree d. We have then $\chi(\mathcal{O}_{X'}) = d.\chi(\mathcal{O}_X)$. On the other hand, X' is still irreducible symplectic (resp. Calabi-Yau), and so we have: $\chi(\mathcal{O}_{X'}) = \sum_{p=0}^{p=\frac{n}{2}} (-1)^{2p} h^0(X', \Omega_{X'}^{2p}) = \frac{n}{2} + 1$ (resp. $\chi(\mathcal{O}_{X'}) = \sum_{p \in \{0,n\}} (-1)^p h^0(X', \Omega_{X'}^p) = 2$). □

The following argument works for any odd dimensional Calabi-Yau manifolds, but does not immediately extends to the singular case.

Proposition 2.9. *Let M be a compact connected Riemannian manifold with nonegative Ricci curvature such that $b_1(M') = 0$ for any finite étale cover M' of M. The fundamental group of M is finite.*

Proof. By [22], the growth of $\pi_1(M)$ is polynomial (of degree bounded by the dimension of M). From [19], $\pi_1(M)$ is virtually nilpotent. Thus $\pi_1(M')$ is nilpotent and torsionfree for some finite étale cover M' of M. Thus $\pi_1(M')$ is either trivial, or has an abelianisation of positive rank. Since $b_1(M') = 0$, $\pi_1(M') = \{1\}$, hence the claim. □

3. The singular version

Let X be a complex projective variety with klt singularities whose first Chern class is zero, i.e. $c_1(X) = 0$. By [23], Chap. V, Corollary 4.9, the condition $K_X \equiv 0$ implies that K_X is $\mathbb{Q}$-trivial. We may, and shall, assume, by passing to an index-one cover, that the singularities of X are canonical and that K_X is trivial. (Instead of [23], one could use when the singularities are canonical, either [21], Thm 8.2, or [10], Thm 3.1 applied to a resolution of X).

We denote by ω the unique Ricci-flat metric of X which belongs to a given Kähler class ([15]). We will see now that the steps of the previous proof extend to the singular context, using the results from [17], §8,9 and [13], Prop. 4.10 and Prop. 3.13. The single new input here is the algebraicity criterion for foliations 3.3 below, which makes superfluous the characteristic $p > 0$ methods and results used in [13]. The results of [18] and [21] used in [13] are also no longer needed.

Theorem 3.1. *([20]) Let X be a normal complex variety with klt singularities and with $c_1(T_X) = 0$. There exists a quasi-étale cover $f : \widetilde{X} \to X$ with canonical*

singularities which is a product $\widetilde{X} = \Pi_j Y_j \times A$, where A is an Abelian variety, and the $Y_j's$ are varieties with canonical singularities, trivial canonical bundle, and irreducible restricted holonomy either $Sp(k_j)$, or $SU(m_j)$ (see §3.1 below). The Y_j' are respectively said to be irreducible symplectic (resp. Calabi-Yau).

3.1. **Restricted holonomy cover.** We consider ω the 'EGZ' Ricci-flat metric on X constructed in [15]. As showed[3] in [17], Prop. 7.3, after a quasi-étale cover, obtained from the permutation representation of the holonomy on the factors of the restricted holonomy, the tangent sheaf T_X of X decomposes as follows:

$$(1) \qquad T_{X_{\mathrm{reg}}} = \mathcal{F} \oplus (\oplus_i \mathcal{E}_i)$$

where the restricted holonomy of $\mathcal{F}$ is trivial, and the other ones are either $SU(n_i)$ or $Sp(k_i)$. The other properties of $\mathcal{E}_i$ used here are:

(i) *The sheaf $\mathcal{E}_i$ defines a non-singular foliation on X_{reg}.*

(ii) *The first Chern classes of $\mathcal{E}_i, \mathcal{F}$ are zero.*

(iii) *All the symmetric powers of $\mathcal{E}_i$ and their duals are irreducible representations of the holonomy factors, and stable, for any polarisation on X. The first property follows from standard representation theory, given the structure of the holonomy group. The stability is [17], Theorem 8.1, see also claim 9.17.*

(iv) *We have $h^0(X, \mathcal{E}_i) = 0, \forall i$. This is again due to [17], Theorem 8.1: the global sections of E_i are parallel, but the sheaf $\mathcal{E}_i$ has no non-zero parallel section over X_{reg}.*

(v) *The preceding properties still hold true for any finite quasi-étale cover of X. Indeed, the Ricci-flat metric on X lifts to such covers, and the restricted holonomy decomposition lifts there too.*

(vi) *The holonomy factors and their holonomy groups do not depend on the Ricci-flat Kähler metric chosen.*

3.2. **Algebraic foliations.** Recall that a foliation on X is said to be algebraic if so are its leaves.

In the decomposition (1) the foliations $\mathcal{E}_i$ are algebraic. Indeed, by either [20] (or [12], Theorem 3.1) none of the $\mathcal{E}_i's$ is pseudo-effective[4]. We can thus apply [11], Theorem 4.2, Lemma 4.6, which implies that they are algebraic.

Our goal now is to show that $\mathcal{F}$ too is algebraic. The claim is true unless some non-zero factor $\mathcal{E}_i$ appears in (1). We thus assume that

$$(2) \qquad T_{X_{\mathrm{reg}}} = \mathcal{G} \oplus \mathcal{E}$$

where $\mathcal{E}$ has positive rank and the properties (i)-(v) are satisfied. So here we assume implicitly that $\mathcal{E}$ is one of the factors $\mathcal{E}_i$ in (1) and $\mathcal{G}$ is the sum of the rest of them. Observe that $\mathcal{G}$ is a foliation, since the decomposition (1) is induced by the local holonomy splitting of X_{reg} (in general, the sum of two foliations need not be inegrable).

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[3]In the first version, Prop. 7.9 was quoted, instead of Prop. 7.3 which is sufficient for our purposes, as pointed out by S. Druel and H. Guenancia, whom I thank for this observation.

[4]The result of [12] can indeed be applied on a resolution of the singularities of X, by lifting both the foliation and an ample class, since its argument deals with the general point of X only.

Lemma 3.2. *Let X be an algebraic variety with canonical singularities and trivial first Chern class. Let ω be the* Ricci*-flat metric in some Kähler class on X, and $T_{X_{\text{reg}}} = \mathcal{G} \oplus \mathcal{E}$ a corresponding decomposition as in the preceding lines. Then:*

(1) The foliation $\mathcal{G}$ is algebraic.

(2) There exists a quasi-étale cover $f : \widetilde{X} \to X$, where $\widetilde{X}$ has canonical singularities, and a product decomposition $\widetilde{X} = F \times Y$ which coincides at the tangent level with the decomposition $T\widetilde{X} = f^{[*]}\mathcal{E} \oplus f^{[*]}\mathcal{G}$.

Proof. Assume that $\mathcal{G}$ is algebraic, this will be shown next. The second claim then directly follows from [13], Prop. 4.10 (notice that the assumption $\tilde{q}(X) = 0$ there can be weakened to: $\tilde{q}(F) = 0$, if F is the closure of a generic leaf of $\mathcal{E}$. The property $\tilde{q}(X) = 0$ is indeed used only to apply Prop. 4.8 of loc. cit., but 4.8 requires only the vanishing of $\tilde{q}$ for the fibres of $\mathcal{E}$). Now $\tilde{q}(F) = 0$ follows from the properties (iii), (v) of the holonomy factors quoted above. $\qquad\square$

The algebraicity of $\mathcal{G}$ follows from Theorem 3.3, which in fact implies more: the bimeromorphic decomposition of X as a product, birationally, after a finite cover[5]. We may, and shall, assume that X has $\mathbb{Q}$-factorial terminal singularities by step 1 of the proof of Prop. 4.10 of [13]. By Prop. 3.13 of loc. cit, there is a Zariski open subset[6] X^0 of X, and a projective morphism $\varphi^0 : X^0 \to Y^0$ which is a locally trivial fibration in the analytic topology, its fibres being isomorphic to some F with $\tilde{q}(F) = 0$, by the properties (iv), (v) of the holonomy factors quoted in §3.1 above. The conclusion then follows from the next algebraicity criterion for foliations.

Theorem 3.3. *Let X and Y be two Kähler[7] normal spaces and let $f : X \to Y$ be a surjective holomorphic map with connected fibres. We denote by $\mathcal{E} := \mathcal{F}_{X/Y}$ the foliation on X induced by f. We assume that:*

(1) f is a trivial fibration, locally in the analytic topology, with fibre F over some nonempty Zariski open set Y_0 of Y

(2) $h^0(F, T_F) = 0$, and so the automorphism group of F is discrete.

Then, there is a finite map $\vartheta : V \to Y$, étale over Y_0, such that base-changing $f : X \to Y$ and normalising the fibre-product $X_V := X \times_Y V$, we have a birational decomposition $\delta : X_V \dashrightarrow F \times V$, isomorphic over Y_0.

Moreover, if $\mathcal{G}$ is any distribution on X such that $T_X = \mathcal{F}_{X/Y} \oplus \mathcal{G}$ over $f^{-1}(U)$, for some nonempty analytically open $U \subset Y_0$, then: $\delta_*((id_X \times \vartheta)^*(\mathcal{G})) = \mathcal{H}$, where $\mathcal{H} := T_{X_V/F} \subset T_{X_V}$ is the horizontal foliation defined by the product decomposition of $T_{F \times V}$. In particular, $\mathcal{G}$ is an algebraic foliation, and is the unique distribution on X which is everywhere transversal to $T_{X/Y}$ over some open subset $U \subset Y_0$ as above.

Remark 3.4. *1. The birational splitting after a generically finite base-change $V \to Y$ (but not necessarily étale over Y_0) always exists if X is projective (or Moishezon) under the single hypothesis (1) of Theorem 3.3. But the algebraicity of $\mathcal{G}$ requires the hypothesis (2) as seen, for example, when $f : X \to Y$ is a morphism of Abelian varieties with positive-dimensional fibres, which has many horizontal non-algebraic foliations.*

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[5]S. Druel informed me that one could also apply his Theorem 1.5 in [14]. Since he hypothesis, scope and proofs of both results are different, it seems worth stating and proving Theorem 3.3.

[6]Up to a finite étale cover of X^0, by shrinking the open set Y^0 of the proof.

[7]Or in the class $\mathcal{C}$.

2. There is certainly a bimeromorphic version of Theorem 3.3, where the generic fibres of f are assumed to be bimeromorphically equivalent, by similar arguments.

Proof. Let $\varphi : Z := F \times Y \to Y, \psi : F \times Y \to F$ be the projections onto the second (resp. first) factor. As in [8], §8, we define:

$$(3) \qquad\qquad \mathrm{Iso}^*(Z, X/Y) \subset \mathcal{C}(Z \times_Y X/Y)$$

to be the subset of the relative Chow variety of $Z \times_Y X$ over Y parametrising the graphs of isomorphisms of F-seen as a fiber of φ over a point $y \in Y_0$- to X_y, the fiber of f over y. According to [8], §8, the set $\mathrm{Iso}^*(Z, F)$ is a Zariski open subset (with possibly infinitely many components) of the relative Chow scheme of $(Z \times_Y X/Y)$, which consists of cycles contained in one of the fibres of the fibre-product over Y. Let $\mathrm{Iso}(Z, X/Y)$ be the closure of $\mathrm{Iso}^\star(Z, X/Y)$ in $\mathcal{C}(Z \times_Y X/Y)$. It is equipped with a projection to Y, and it is a union of irreducible components of the Chow-Barlet scheme, according to general results in *loc. cit.* Since f is locally trivial over a Zariski open subset of Y, the projection $\mathrm{Iso}^*(Z, X/Y) \to Y$ is open over Y_0. This projection is proper since the irreducible components of $\mathrm{Iso}(Z, X/Y)$ are compact (essentially by a result of D. Lieberman, based on E. Bishop's theorem). Moreover, by the assumption (2) of Theorem 3.3, the fibres of $\mathrm{Iso}(Z, X/Y)$ to Y are discrete over Y_0. If V is an irreducible component of $\mathrm{Iso}(Z, X/Y)$, its projection $\vartheta : V \to Y$ is thus onto, and finite étale over Y_0.

We thus get a fibre product $X_V := X \times_Y V$, with the obvious projections $f_V : X_V \to V, g : X_V \to X$. Let $V_0 := \vartheta^{-1}(Y_0)$. Any $v \in V_0$ is thus equipped naturally with an isomorphism $ev_v : F \cong X_y, y := \vartheta(v)$. This evaluation map extends (see [8], §8, Prop. 1) meromorphically to: $ev : F \times V \to X$ which is thus bimeromorphic, and isomorphic over V_0.

In order to simplify notations, we replace X, Y, f by X_V, V, f_V respectively, and identify via ev X_V with $F \times V = F \times Y$ (recall ev is isomorphic over $V_0 = Y_0$), and all the assumptions of Theorem 3.3 are preserved. The projections of $X = F \times Y$ onto its second (resp. first) factor are denoted $f = \varphi$ and ψ.

To establish the last claim of Theorem 3.3, we only have to check that $\mathcal{G}$ coincides over Y_0 with the sheaf $T_{X/F} := \mathcal{H}$, which will also prove the algebraicity of $\mathcal{G}$.

We restrict everything over the open set $U \subset Y_0$ appearing in the last assumption of Theorem 3.3, so we assume that $X_U := f^{-1}(U) = F \times U$, and we have thus a first decomposition: $T_{X_U} = \psi^*(T_F) \oplus \mathcal{H}$, where $\mathcal{H}$ is the kernel of the map $d\psi : T_{X_U} \to \psi^*(T_F)$.

The second decomposition $T_{X_U} = \psi^*(T_F) \oplus \mathcal{G}$ gives equivalently an isomorphism $df_{|\mathcal{G}} : \mathcal{G} \to f^*(T_U)$ over X_U. Let $(df_{|\mathcal{G}})^{-1} : f^*(T_U) \to \mathcal{G}$ be its inverse. Let $\gamma := d\psi \circ (df_{|\mathcal{G}})^{-1} : f^*(T_U) \to \psi^*(T_F)$ be the composite map, seen as an element $\gamma \in H^0(X_U, f^*(\Omega_U^1) \otimes \psi^*(T_F))$. We have the following equalities:

$$H^0(X_U, f^*(\Omega_U^1) \otimes \psi^*(T_F)) = H^0(U, \Omega_U^1 \otimes f_*(\psi^*(T_F))) = H^0(U, \Omega_U^1 \otimes \{0\}) = \{0\},$$

the last two equalities because of assumption (2), which implies that $f_*(\psi^*(T_F)) = \{0\}$. This shows that $\mathcal{G} = \mathcal{H} = T_{Z/F}$ over X_U, and so everywhere by analytic continuation. $\square$

We can now conclude the proof of Theorem 3.1 by induction on $\dim(X)$, since we now know that (up to quasi-étale covers) $X = Y \times Z$, in which Y is a product of varieties with canonical singularities, $c_1 = 0$, restricted holonomy either SU or Sp,

and Z is in the same class of varieties, but with trivial restricted holonomy (i.e: $TZ = \mathcal{F}$). Theorem 3.1 then follows from:

3.3. **A singular Bieberbach theorem.** Assume now that only the factor $\mathcal{F}$ appears in the decomposition (1). We are reduced to show that if Z has canonical singularities, trivial first Chern class and trivial restricted holonomy group, it is covered by an abelian variety. But this is just Corollary 1.16 in [18].

3.4. **The fundamental group.** Let X be a complex projective variety with klt singularities and $K_X \equiv 0$. Recall that X is said to be irreducible symplectic (resp. Calabi-Yau) if its restricted holonomy representation for any, or some EGZ Ricci-flat Kähler metric is irreducible and of the form $Sp(m)$ (resp. $SU(n)$). In this situation, we have the following result, entirely similar to the smooth case:

Theorem 3.5. *If $\chi(\mathcal{O}_X) \neq 0$, then $\pi_1(X)$ is finite, and its cardinality divides $\chi(\mathcal{O}_X)$, which is positive. This applies to irreducible symplectic varieties and to even-dimensional Calabi-Yau varieties, with the same bounds as in Corollary 2.8.*

Proof. Let $\rho : X' \to X$ be some resolution. It is sufficient to prove the finiteness statement for X', since the map $\rho_*(\pi_1(X')) \to \pi_1(X)$ is surjective. We can then apply [9], which says that $\pi_1(X')$ is finite if $\kappa(X', det(\mathcal{F})) \leq 0$ for any $\mathcal{F} \subset \Omega^p_{X'}), \forall p > 0$. Since the sections of $det(\mathcal{F})^{\otimes m}$ are sections of $Sym^m(\Omega^p_{X'})$, and the restrictions of these are reflexive sections, hence parallel over the regular locus of X by [17], Theorem 8.2, these sections are determined by their value in one single point of X_{reg}. Thus $\kappa(X', det(\mathcal{F})) \leq 0$.

The invariant $\chi(\mathcal{O}_X)$ behaves as in the smooth case when X has klt and thus rational singularities. It is, in particular, multiplicative under finite étale covers.

If X is irreducible symplectic (resp. even-dimensional Calabi-Yau) and n-dimensional, we have: $h^0(X, \Omega_X^{[p]}) \leq h_{n,p}$, where $h_{n,p} = 0$ for p odd, and $h_{n,p} = 1$ for $p \leq n$ even (resp. $h_{n,p} = 0$ for $p \neq 0, n$, and $h_{n,p} = 1$ for $p = 0, n$), which gives the claim, since we have equality on the universal cover. $\qed$

Remark 3.6. *Our proof of Theorem 3.5 differs from the one in [17], 13.1. We refer to [9], §5 for further remarks on this topic.*

REFERENCES

[1] M. Atiyah. Elliptic operators, discrete groups, and Von Neumann algebras. Astérisque 32-33, 43-72. SMF (1976)
[2] A. Beauville. Variétés Kählériennes dont la première classe de Chern est nulle. J. Diff. Geom. 18, 755-782 (1983)
[3] A.L. Besse. Einstein manifolds. Ergeb. Der Math.u.i.Grenz.Geb.10. Springer Verlag (1987)
[4] F. Beukers, AJ. van der Poorten, R. Yager. Dyson's lemma without tears. Indagationes Mathematicae, 2(1), 19-28 (1991)
[5] L. Bieberbach. Über die Bewegungsgruppen des Euklidischen Raüme. I,II. Math. Ann. 70, 297-336 (1911), 72, 400-412 (1912)
[6] F. Bogomolov. Hamiltonian Kähler manifolds. Sov. Math. Dokl. 19, 1462-1465 (1978).
[7] F. Bogomolov-M. McQuillan. Rational curves on foliated varieties. IHES/M/01/07 (2001).
[8] F. Campana. Réduction d'Albanese d'un morphisme propre et faiblement kählérien, II. Comp. Math. 54, 373-398 (1985).
[9] F. Campana. Fundamental group and positivity of the cotangent bundles of compact Kähler manifolds. J. Alg. Geom. 4, 487-502 (1995)
[10] F. Campana-T. Peternell. Geometric stability of the cotangent bundle and the universal cover of a projective manifold. Bull. SMF (2011)
[11] F. Campana-M. Păun. Foliations with positive movable slope. Publ. Math. IHÉS. (2019)

[12] F. Campana-J. Cao-M. Păun. Subharmonicity of direct images and applications. ArXiv.

[13] S. Druel. A decomposition theorem for singular spaces with trivial canonical class of dimension at most five. Inv. math. 211, 245-296 (2018)

[14] S. Druel. Codimension one foliations with numerically trivial canonical class on singular spaces (mars 2020). arXiv:1809.06905 (2018).

[15] P. Eyssidieux-V. Guej-A. Zeriahi. Singular Kähler-Einstein metrics. J. AMS. 22 (2009), 607-639.

[16] W. Fulton-J. Harris. Representation theory. A first course. Springer-Verlag (1991).

[17] D. Greb-H. Guenancia-S. Kebekus. Klt varieties with trivial canonical class, holonomy, differential forms, and fundamental groups. Geometry and Topology 23-4 (2019), 2051-2124.

[18] D. Greb-S. Kebekus-T. Peternell. Etale fundamental group of Kawamata log-terminal spaces, flat sheaves and quotients of abelian varieties. Duke Math. J. Volume 165, Number 10 (2016), 1965-2004.

[19] M. Gromov. Groups of polynomial growth and expanding maps. Publ. IHES. Vol. 53 (1981), 53-78.

[20] A. Höring-T. Peternell. Algebraic integrability of foliations with numerically trivial canonical bundle. Inv. Math. 216, 395-419 (2019)

[21] Y. Kawamata. Minimal models and the Kodaira dimension of algebraic fiber spaces. Journal für die reine und angewandte Mathematik, vol. 363 (1985), 1-46.

[22] J. Milnor. A note on curvature and fundamental group. J. Diff. Geom. vol 2. (1968), 1-7.

[23] N. Nakayama. Zariski decomposition and Abundance. MSJ Memoirs, Volume 14 Tokyo,The Mathematical Society of Japan, 2004

[24] S.T. Yau. On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation I, Comm. Pure and Appl. Math. 31, 339-411 (1978)

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