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SOME CARLITZ TYPE IDENTITIES FOR THE BERNOULLI NUMBERS OF THE SECOND KIND

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Abstract. In this short note, we obtain some identities for the Bernoulli numbers of the second kind. They are similar to the Carlitz's type recurrence relations for the (ordinary) Bernoulli numbers. The proof is quite simple and uses the method of generating functions.

The Carlitz's identities for the Bernoulli numbers B_n, write in convolution symbolic notation, $(-1)^m(B_n+1)^m = (-1)^n(B_m+1)^n$, and seem to appear first in the paper [1]. For recent papers, we cite [3] and the references therein.

The Bernoulli numbers of the second kind b_n can be defined by the way of the generating function, see [2],

$$f(t) := \frac{t}{\log(1+t)} = \sum_{n=0}^{\infty} b_n t^n \quad (|t| < 1)\ . \tag{1}$$

Here is our main result.

Proposition. For nonnegative integers m and k, we have

$$(-1)^m \sum_{n=0}^{m} < n-1 >_{m-n} \binom{n+k}{n} b_{n+k} = (-1)^k \sum_{n=0}^{k} < n-1 >_{k-n} \binom{n+m}{n} b_{n+m}\ ,$$

where $< n-1 >_{m-n}$ and $< n-1 >_{k-n}$ are rising factorials : $< x >_0 = 1$ and for $l \geq 1$, $< x >_l = x(x+1)\ldots(x+l-1)$.

Making $m = 0$ in the proposition above, we get

Corollary. For nonnegative integer k, we have

$$b_k = (-1)^k \sum_{n=0}^{k} < n-1 >_{k-n} b_n\ .$$

Proof. Let us define the *symetric* two variables function

$$F(t,s) := \frac{t-s}{\log(1+t) - \log(1+s)}\ .$$

We begin by connecting both functions F and f,

$$F(t,s) = \frac{t-s}{\log(1+t) - \log(1+s)} = (1+s)\frac{(t-s)/(1+s)}{\log(1+\frac{t-s}{1+s})} = (1+s)f(\frac{t-s}{1+s})\ .$$

then, we expand thanks to (1) ,

$$\begin{aligned}(1+s)f(\frac{t-s}{1+s}) &= (1+s)\sum_{n=0}^{\infty} b_n \left(\frac{t-s}{1+s}\right)^n = \sum_{n=0}^{\infty} b_n (t-s)^n (1+s)^{1-n} \\ &= \sum_{n=0}^{\infty} b_n \left(\sum_{k=0}^{n} \binom{n}{k} t^k (-s)^{n-k}\right)\left(\sum_{m=0}^{\infty} < n-1 >_m (-s)^m\right)\ .\end{aligned}$$

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Hence we have,

$$\begin{aligned}
F(t,s) &= \sum_{n=0}^{\infty} b_n \left(\sum_{k=0}^{n} \binom{n}{k} t^k \right) \left(\sum_{m=0}^{\infty} < n-1 >_m (-s)^{m+n-k} \right) \\
&= \sum_{n=0}^{\infty} b_n \left(\sum_{k=0}^{n} \binom{n}{k} t^k \right) \left(\sum_{m=n-k}^{\infty} < n-1 >_{m-n+k} (-s)^{m} \right) \\
&= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} (-1)^m \left(\sum_{n=k}^{m+k} \binom{n}{k} b_n < n-1 >_{m-n+k} \right) t^k s^m \\
&= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} (-1)^m \left(\sum_{n=0}^{m} < n-1 >_{m-n} \binom{n+k}{k} b_{n+k} \right) t^k s^m \, .
\end{aligned}$$

As $F(s,t) = F(t,s)$, the proof is complete. □

Remark. The same method applies to the general function

$$F(t,s,x,y) = \frac{t-s}{\log(1+t) - \log(1+s)} (1+t)^x (1+s)^y \, .$$

The computations are longuer but provide identities for more different known families of numbers and polynomials.

References

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