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SOME CARLITZ TYPE IDENTITIES FOR THE BERNOULLI NUMBERS OF THE SECOND KIND

J. CHIKHI

Abstract. In this short note, we obtain some identities for the Bernoulli numbers of the second kind. They are similar to the Carlitz's type recurrence relations for the (ordinary) Bernoulli numbers. The proof is quite simple and uses the method of generating functions.

The Carlitz's identities for the Bernoulli numbers B_n , write in convolution symbolic notation, $(-1)^m(B_n + 1)^m = (-1)^n(B_m + 1)^n$, and seem to appear first in the paper [1]. For recent papers, we cite [3] and the references therein.

The Bernoulli numbers of the second kind b_n can be defined by the way of the generating function, see [2],

$$(1) \quad f(t) := \frac{t}{\log(1+t)} = \sum_{n=0}^{\infty} b_n t^n \quad (|t| < 1) .$$

Here is our main result.

Proposition. For nonnegative integers m and k , we have

$$(-1)^m \sum_{n=0}^m \langle n-1 \rangle_{m-n} \binom{n+k}{n} b_{n+k} = (-1)^k \sum_{n=0}^k \langle n-1 \rangle_{k-n} \binom{n+m}{n} b_{n+m} ,$$

where $\langle n-1 \rangle_{m-n}$ and $\langle n-1 \rangle_{k-n}$ are rising factorials : $\langle x \rangle_0 = 1$ and for $l \geq 1$, $\langle x \rangle_l = x(x+1) \dots (x+l-1)$.

Making $m = 0$ in the proposition above, we get

Corollary. For nonnegative integer k , we have

$$b_k = (-1)^k \sum_{n=0}^k \langle n-1 \rangle_{k-n} b_n .$$

Proof. Let us define the symmetric two variables function

$$F(t, s) := \frac{t-s}{\log(1+t) - \log(1+s)} .$$

We begin by connecting both functions F and f ,

$$F(t, s) = \frac{t-s}{\log(1+t) - \log(1+s)} = (1+s) \frac{(t-s)/(1+s)}{\log(1 + \frac{t-s}{1+s})} = (1+s) f\left(\frac{t-s}{1+s}\right) .$$

then, we expand thanks to (1) ,

$$\begin{aligned} (1+s) f\left(\frac{t-s}{1+s}\right) &= (1+s) \sum_{n=0}^{\infty} b_n \left(\frac{t-s}{1+s}\right)^n = \sum_{n=0}^{\infty} b_n (t-s)^n (1+s)^{1-n} \\ &= \sum_{n=0}^{\infty} b_n \left(\sum_{k=0}^n \binom{n}{k} t^k (-s)^{n-k} \right) \left(\sum_{m=0}^{\infty} \langle n-1 \rangle_m (-s)^m \right) . \end{aligned}$$

Key words and phrases : Bernoulli numbers of the first and second kind, Carlitz identities, falling factorials.

Hence we have,

$$\begin{aligned}
F(t, s) &= \sum_{n=0}^{\infty} b_n \left(\sum_{k=0}^n \binom{n}{k} t^k \right) \left(\sum_{m=0}^{\infty} \langle n-1 \rangle_m (-s)^{m+n-k} \right) \\
&= \sum_{n=0}^{\infty} b_n \left(\sum_{k=0}^n \binom{n}{k} t^k \right) \left(\sum_{m=n-k}^{\infty} \langle n-1 \rangle_{m-n+k} (-s)^m \right) \\
&= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} (-1)^m \left(\sum_{n=k}^{m+k} \binom{n}{k} b_n \langle n-1 \rangle_{m-n+k} \right) t^k s^m \\
&= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} (-1)^m \left(\sum_{n=0}^m \langle n-1 \rangle_{m-n} \binom{n+k}{k} b_{n+k} \right) t^k s^m.
\end{aligned}$$

As $F(s, t) = F(t, s)$, the proof is complete. □

Remark. The same method applies to the general function

$$F(t, s, x, y) = \frac{t-s}{\log(1+t) - \log(1+s)} (1+t)^x (1+s)^y.$$

The computations are longer but provide identities for more different known families of numbers and polynomials.

REFERENCES

1. L. Carlitz, Problem 795. Math. Mag. 44 (1971), 107.
2. C. Jordan, Calculus of Finite Differences, 2nd ed., Chelsea Publ. Co., New York, 1950.
3. H. Prodinger, A Short Proof of Carlitz's Bernoulli Number Identity, J. Integer Sequences 17.2 (2014)

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