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Gallai's path decomposition conjecture for graphs of small maximum degree

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Abstract

Gallai's path decomposition conjecture states that the edges of any connected graph on n vertices can be decomposed into at most $\frac{n+1}{2}$ paths. We confirm that conjecture for all graphs with maximum degree at most five.

1 Introduction

A *decomposition* $\mathcal{D}$ of a graph G is a collection of subgraphs of G such that each edge belongs to precisely one graph in $\mathcal{D}$. A *path decomposition* is a decomposition $\mathcal{D}$ such that every subgraph in $\mathcal{D}$ is a path. If G has a path decomposition $\mathcal{D}$ such that $|\mathcal{D}| = k$, then we say that G can be *decomposed* into k paths. In answer to a question of Erdős, Gallai conjectured the following, see [4].

Conjecture 1.1. [4] *Every connected graph on n vertices can be decomposed into $\lceil \frac{n}{2} \rceil$ paths.*

Gallai's conjecture is easily seen to be sharp: If G is a graph in which every vertex has odd degree, then in any path decomposition of G each vertex must be the endpoint of some path, and so at least $\lceil \frac{n}{2} \rceil$ paths are required. Lovász [4] proved that every graph on n vertices has a decomposition $\mathcal{D}$ consisting of paths and cycles, and such that ~~$|\mathcal{D}| = \lfloor \frac{n}{2} \rfloor$~~$|\mathcal{D}| \leq \lfloor \frac{n}{2} \rfloor$. By an argument similar to the above, it follows that in a graph with at most one vertex of even degree, such a decomposition must be a path decomposition. Thus, Gallai's conjecture holds for all graphs with at most one vertex of even degree.

Let G_E denote the subgraph of G induced by the vertices of even degree. Building on Lovász's result, Conjecture 1.1 has been proved for several classes of graphs defined by imposing some structure on G_E. The first result of this kind was obtained by Pyber [1].

Theorem 1.1. [1] *If G is a graph on n vertices such that G_E is a forest, then G can be decomposed into $\lfloor \frac{n}{2} \rfloor$ paths.*

Later, Theorem 1.1 was strengthened by Fan [2], who proved the following.

Theorem 1.2. [2] *If G is a graph on n vertices such that each block of G_E is a ~~triangle-free~~ triangle-free graph of maximum degree at most 3, then G can be decomposed into $\lfloor \frac{n}{2} \rfloor$ paths.*

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Gallai's conjecture is also known to hold for a variety of other graph classes. In 1988, Favaron and Koudier [6] proved that the conjecture holds for graphs where the degree of every vertex is either 2 or 4. More recently, Botler and Jiménez [3] proved that the conjecture holds for $2k$-regular graphs of large girth and admitting a pair of disjoint perfect matchings. Jiménez and Wakabayashi [7] showed that the conjecture holds for a subclass of planar, triangle-free graphs satisfying a distance condition on the vertices of odd degree. Finally, it was shown by Geng, Fang and Li [5], that the conjecture holds for maximal outerplanar graphs. In this article, we prove that Gallai's conjecture holds for the class of graphs with maximum degree at most 5.

Theorem 1.3. *Let G be a connected graph on n vertices. If $\Delta(G) \leq 5$, then G admits a path decomposition into $\lceil \frac{n}{2} \rceil$ paths.*

To prove Theorem 1.3, we show that if G is a smallest counterexample, then G cannot contain ~~one~~ any of 5 ~~configurations~~given configurations (Lemma 3.1). This restriction is enough to show that G_E is a forest (Lemma 3.2), whence the result follows by Theorem 1.1. It seems that proving Theorem 1.3 for graphs of maximum degree 6 will require some new ideas. However, we think the approach of considering graphs of bounded maximum degree allows step-by-step improvements which ~~could~~ may eventually lead to a general solution.

In proving special cases of Conjecture 1.1, the presence of a ceiling in the bound brings with it a number of technical complications. It is therefore tempting to explore ways of proving a stronger, ceiling-free version except in a few special cases. We say a graph is an *odd semi-clique* if it is obtained from a clique on $2k+1$ vertices by deleting at most $k-1$ edges. By a simple counting argument, we can see that an odd semi-clique on $2k+1$ vertices does not admit a path decomposition into k paths. It is natural to ask if these are the only obstructions:

Question 1.1. *Does every connected graph G that is not an odd semi-clique admit a path decomposition into $\lfloor \frac{|V(G)|}{2} \rfloor$ paths?*

2 ~~Definitions and notation~~Background

All graphs in this article are finite and simple, that is, they contain no loops or multiple edges. We say that a path decomposition $\mathcal{D}$ of a graph G is *good* if $|\mathcal{D}| \leq \lceil \frac{|V(G)|}{2} \rceil$.

In figures we make use of the following conventions: Solid black circles denote vertices for which all incident edges are depicted. White hollow circles denote vertices which may have other, undepicted incident edges. Vertices containing a number indicate a vertex of that specific degree. A dotted line between two vertices indicates that those vertices are non-adjacent. We often use *component* as a shortcut for connected component.

We will often modify a path decomposition of a graph G to give a path decomposition of another graph G'. To describe these modifications we use a number of fixed expressions, which we formally define here. Let $\mathcal{D}$ be a path decomposition of G. Let $P \in \mathcal{D}$ be a path and Q be a subpath of P. If R is a path in G' with the same end vertices as ~~P~~Q, we say that we *replace* Q with R to mean that we define a new path $P' = P - Q + R$ and redefine $\mathcal{D}$ to be the collection $\mathcal{D} - P + P'$. If R is a path in G' with an endpoint in common with P, we say that we *extend* P with R to mean that we define a new path $P' = P + R$ and redefine $\mathcal{D}$ to be the collection $\mathcal{D} - P + P'$. For a vertex u on

P, we say that we *split* P *at* u to mean that we define paths P_1 and P_2 such that $P_1 \cup P_2 = P$ and $P_1 \cap P_2 = u$, and redefine $\mathcal{D}$ to be the collection $\mathcal{D} - P + P_1 + P_2$. Finally, for a path R in G', we say that we *add* the path R to mean that we redefine $\mathcal{D}$ to be the collection $\mathcal{D} + R$. Given a graph G and two disjoint sets A and B of vertices of G, we denote by $E(A, B)$ the set of edges with one endpoint in A and the other in B.

By simple arithmetic, we can obtain the three propositions below.

Proposition 2.1. *Let G and G' be two graphs such that $|V(G)| \geq |V(G')| + 2$, and let $\mathcal{D}$ be a path decomposition of G. If there is a good path decomposition $\mathcal{D}'$ of G' and $|\mathcal{D}| \leq |\mathcal{D}'| + 1$, then $\mathcal{D}$ is a good path decomposition of G.*

~~Let $|V(G)| = n$. We have $|\mathcal{D}| \leq |\mathcal{D}'| + 1 \leq \lceil \frac{n-2}{2} \rceil + 1 = \lceil \frac{n}{2} \rceil$. Thus $\mathcal{D}$ is a good path decomposition of G.~~

Proposition 2.2. *Let G, G_1 and G_2 be three graphs such that $|V(G)| \geq |V(G_1)| + |V(G_2)|$, and let $\mathcal{D}$ be a path decomposition of G. If there are good path decompositions $\mathcal{D}_1$ and $\mathcal{D}_2$ of G_1 and G_2 (respectively) and $|\mathcal{D}| \leq |\mathcal{D}_1| + |\mathcal{D}_2| - 1$, then $\mathcal{D}$ is a good path decomposition of G.*

~~Let G, G_1 and G_2 have n, n_1 and n_2 vertices respectively. We have $|\mathcal{D}| \leq |\mathcal{D}_1| + |\mathcal{D}_2| - 1 \leq \lceil \frac{n_1}{2} \rceil + \lceil \frac{n_2}{2} \rceil - 1 \leq \lceil \frac{n}{2} \rceil$. Thus $\mathcal{D}$ is a good path decomposition of G.~~

Proposition 2.3. *Let G, G_1 and G_2 be three graphs such that $|V(G)| \geq |V(G_1)| + |V(G_2)| + 1$, and let $\mathcal{D}$ be a path decomposition of G. If there are good path decompositions $\mathcal{D}_1$ and $\mathcal{D}_2$ of G_1 and G_2 (respectively) and $|\mathcal{D}| \leq |\mathcal{D}_1| + |\mathcal{D}_2|$, then $\mathcal{D}$ is a good path decomposition of G.*

~~Let G, G_1 and G_2 have n, n_1 and n_2 vertices respectively. We have $|\mathcal{D}| \leq |\mathcal{D}_1| + |\mathcal{D}_2| \leq \lceil \frac{n_1}{2} \rceil + \lceil \frac{n_2}{2} \rceil \leq \lceil \frac{n}{2} \rceil$. Thus $\mathcal{D}$ is a good path decomposition of G.~~

3 Main Result

Let G be a graph with $\Delta(G) \leq k$. We first prove that a number of configurations are reducible in G~~, if~~. That is, if G contains one of them and Gallai's conjecture holds for all smaller graphs of maximum degree k, then Gallai's conjecture holds for G.

Lemma 3.1. *Let $k \in \mathbb{N}$. Let G be a connected graph with maximum degree $\Delta(G) \leq k$, and suppose that G does not admit a good path decomposition. If G is vertex minimal with these properties, then G does not contain any of the following configurations (see Figure 1):*

C_1: *A vertex of degree 2 whose neighbours are not adjacent.*

C_2: *A cut-edge uv such that $d(u)$ and $d(v)$ are even.*

C_3: *An edge uv such that $d(u) = d(v) = 4$, and u and v have precisely 2 common neighbours~~, and $d(u) = d(v) = 4$~~.*

C_4: *An edge uv such that $d(u) = d(v) = 4$, and for ~~t_1, t_2, t_3~~ t_1, t_2, t_3 (resp. ~~w_1, w_2, w_3~~ w_1, w_2, w_3) the three other ~~neighbors~~ neighbours of u (resp. v), the pairs t_1t_2 and w_1w_2 are not edges and $t_3 \neq w_3$.*

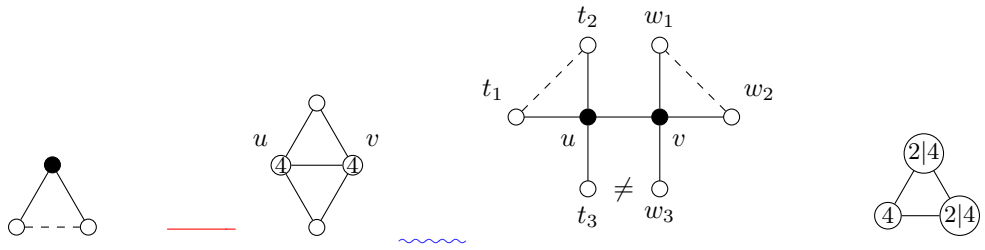


Figure 1: Configurations C_1, C_3, C_4 and C_5 from Lemma 3.1.

C_5: *A triangle uvw such that $d(u) = 4$ and ~~$d(v), d(w) \in \{2, 4\}$~~ $d(v), d(w) \in \{2, 4\}$.*

Proof.

Claim 1. *G does not contain the configuration C_1.*

Proof. Suppose that the claim is false. Let u be the vertex of degree 2 with $N(u) = \{v, w\}$, and let G' be the graph $G - u + vw$. Since v and w are non-adjacent in G, G' is a simple graph. By the minimality of G, we have that G' admits a good path decomposition $\mathcal{D}'$. By Proposition 2.3, we obtain a good path decomposition of G by replacing the edge vw with the path vuw (see Figure 2).

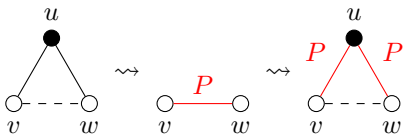


Figure 2: The reduction of C_1.

This contradicts the assumption that G has no such decomposition. ◊

Claim 2. *G does not contain the configuration C_2.*

Proof. Suppose that the claim is false. Deleting uv from G results in two connected graphs G_1 and G_2, containing u and v respectively. By the minimality of G, both G_1 and G_2 admit good path decompositions $\mathcal{D}_1$ and $\mathcal{D}_2$. To obtain a path decomposition of G, note that, since u has odd degree in G_1, there is a path $P_u \in \mathcal{D}_1$ ending at u. Similarly, there is a path $P_v \in \mathcal{D}_2$ ending at v. Now let $\mathcal{D}$ be the path decomposition of G formed by taking the union $\mathcal{D}_1 \cup \mathcal{D}_2$, deleting P_u and P_v, and adding a new path $P = P_u + uv + P_v$ (see Figure 3). By Proposition 2.2, $\mathcal{D}$ is a good path decomposition of G, a contradiction.

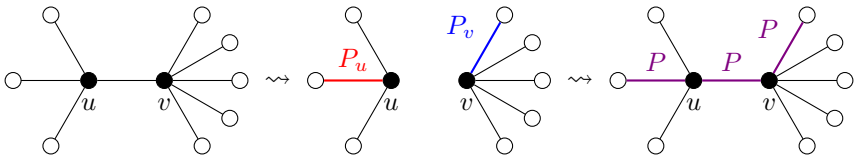


Figure 3: The reduction of C_2.

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Claim 3. *G does not contain the configuration C_3.*

Proof. Suppose that the claim is false. Let uv be the edge stated in Configuration C_3, and let x and y be the two common neighbours of u and v. Since $d(u) = d(v) = 4$, both ~~u and v both~~ u and v have precisely one ~~other neighbour. Let these vertices be~~ more neighbour, say u' and v' ~~respectively. Since u and v have precisely two common neighbours, we have that~~ , respectively, where $u' \neq v'$.

Suppose first that at most one of the edges ~~$xu', u'y, yv', v'x$~~ xu', $u'y$, yv', $v'x$ is present in G, say xu'. If $G - u - v$ is connected, then let G' be the graph $G - u - v + u'y + v'x$. Otherwise, let $G' = G - u - v + u'y + v'x + xy$. Note that G' is connected, and so by the minimality of G, it admits a good path decomposition. Now, replace $v'x$ by ~~xvv'~~ $v'vx$ and replace $u'y$ by $u'uy$. Furthermore, if $xy \in E(G') \setminus E(G)$, then replace xy by $xuvy$. Otherwise, add a new path $xuvy$ to the decomposition. By Proposition 2.1, and since we ~~add~~ added at most one new path, the resulting decomposition is a good path decomposition of G. This contradicts the assumption that G has no such decomposition.

Next, suppose that ~~$xu', u'y, yv', v'x \in E(G)$~~ xu', $u'y$, yv', $v'x \in E(G)$, so the graph $G' = G - u - v$ is connected. By the minimality of G, the graph G' has a good path decomposition. Now replace the edge xu' with the path $xvuu'$, and add a new path $u'xuyvv'$ to the decomposition. By Proposition 2.1, and since we ~~add~~ added at most one new path, the resulting decomposition is a good path decomposition of G, contradicting the assumption.

Finally, suppose that precisely two or three of the edges ~~$xu', u'y, yv', v'x$~~ xu', $u'y$, yv', $v'x$ are present in G. As a consequence, from the set $\{xu', u'y, yv', v'x\} \setminus E(G)$, we may choose an edge, xu' say, such that the graph $G' = G - u - v + xu'$ is connected. By the minimality of G, the graph G' has a good path decomposition. Now replace xu' by $xvuu'$, and add a new path $xuyvv'$ to the decomposition. Again, by Proposition 2.1, and since we ~~add~~ added at most one new path, the resulting decomposition is a good path decomposition of G, contradicting the assumption.

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Claim 4. *G does not contain the configuration C_4.*

Proof. Suppose that the claim is false. Since G does not contain Configuration C_3, the vertices u and v do not have precisely two common neighbours. First suppose that u and v have 3 common neighbours x, y and z. In this case, since there is a pair of non-adjacent vertices amongst $N(u) \setminus \{v\}$, we may assume $xy \notin E(G)$. Furthermore, by the definition of Configuration C_4, the third vertex z is non-adjacent to at least one of x or y. ~~We conclude that there are two non-edges amongst x, y and z, say these are xy and yz~~; say $yz \notin E(G)$. Let G' be the graph $G - u - v + xy + yz$. It is easy to see that G' is connected. By the minimality of G, the graph G' has a good path decomposition. In this decomposition, replace xy by xuy and replace yz by yvz. Finally, add a new path $xvuz$ (see Figure 4). This gives a good path decomposition of G, a contradiction. We may thus assume that u and v have at most one common neighbour.

We now consider three cases depending on the structure of $G - \{u, v\}$. In each case we assume the previous ones do not apply (up to symmetry).

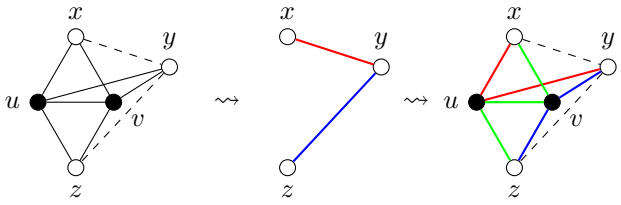


Figure 4: The reduction of C_4 in the case where u and v have three common ~~neighbors~~neighbours.

1. *Assume that $G-u$ has at least three connected components.* Because uv is not a cut-edge, the component of $G-u$ containing v contains at least one other ~~neighbor~~ neighbour of u. Thus $G-u$ has precisely three components, and t_1 and t_2 lie in different components of $G-u$. Let G' be the graph formed from $G-u$ by adding the edge t_1t_2. Thus G' has two components G_1 and G_2, and by the minimality of G, both have good path decompositions $\mathcal{D}_1$ and $\mathcal{D}_2$. Without loss of generality we suppose G_2 contains v. Let $P \in \mathcal{D}_1$ be the path containing the edge t_1t_2. Furthermore, let P_1 and P_2 be the possibly empty subpaths of $P - t_1t_2$ containing t_1 and t_2 respectively. Note that since v has degree 3 in G', there is some path $Q \in \mathcal{D}_2$ which ends at v. We construct a path decomposition of G by taking the union $\mathcal{D}_1 \cup \mathcal{D}_2$ and replacing P and Q with the paths $P_1 + t_1uv + Q$ and $P_2 + t_2ut_3$. By Proposition 2.3, and since we ~~introduced no new~~ did not increase the number of paths, the resulting path decomposition is good, a contradiction.

2. *Assume that $G-\{u,v\}$ has at least four connected components.* Since both $G-u$ and $G-v$ have at most two connected components, there are precisely four connected components ~~C_1, C_2, C_3 and C_4~~H_1, H_2, H_3 and H_4. Furthermore, two of these components contain both a neighbour of u and a neighbour of v, one component contains only a neighbour of u, and one component contains only a neighbour of v. Relabeling if necessary, we may suppose that ~~$t_1, w_1 \in C_1$, $t_2, w_2 \in C_2$, $t_3 \in C_3$ and $w_3 \in C_4$~~$t_1, w_1 \in H_1$, $t_2, w_2 \in H_2$, $t_3 \in H_3$ and $w_3 \in H_4$. This relabelling preserves the fact that ~~$t_1t_2, w_1w_2 \notin E(G)$~~ t_1t_2, $w_1w_2 \notin E(G)$ and $t_3 \neq w_3$. Consider the graph G_1 obtained from ~~C_1 and C_2~~ H_1 and H_2 by adding the edges t_1t_2 and w_1w_2. Similarly, consider the graph G_2 obtained from ~~C_3 and C_4~~ H_3 and H_4 by adding the edge t_3w_3. By the minimality of G, we obtain good path decompositions of G_1 and G_2, which we merge in the obvious way. The edge t_1t_2 is replaced with t_1ut_2, w_1w_2 with w_1vw_3, and t_3w_3 with t_3uvw_3) to obtain a path decomposition of G. By Proposition 2.3, this yields a good path decomposition of G.

3. Now $G-\{u,v\}$ has at most three connected components, and each of $G-u$ and $G-v$ has at most two connected components. Let $T = \{t_1, t_2, t_3\}$ and $W = \{w_1, w_2, w_3\}$. We claim that we can relabel the vertices in T and W such that the graph $G - u - v + t_1t_2 + w_1w_2$ is connected and the properties that ~~$t_1t_2, w_1w_2 \notin E(G)$~~ t_1t_2, $w_1w_2 \notin E(G)$ and $t_3 \neq w_3$ are preserved. Indeed if u and v have a common neighbour, let $t \in T$ and $w \in W$ be such that $t = w$. Otherwise, let $t = t_1$ and $w = w_1$. Suppose first that t and w lie in the same component of $G-u-v$. Since $G-u-v$ has at most 3 components, and $G-u$ and $G-v$ have at most 2 components, there are non edges tt' and ww' for some $t' \in T$ and $w' \in W$

such that $G-u-v+tt'+ww'$ is connected. Furthermore, since t and w are the only possible common neighbours of u and v, we have that the single vertices in $T \setminus \{t, t'\}$ and $W \setminus \{w, w'\}$ are not equal. Thus, letting $t_1 = t$, $t_2 = t'$, $w_1 = w$, $w_2 = w'$ and setting t_3 and w_3 to be the remaining vertices gives the desired relabeling.

Suppose now that t and w lie in different components of $G-u-v$. In particular this implies that $T \cap W = \emptyset$. Again, since $G-u-v$ has at most 3 components, and $G-u$ and $G-v$ have at most 2 components, there are non-edges e_T and e_W amongst the vertices of T and W respectively, such that $G-u-v+e_T+e_W$ is connected. We relabel the vertices in T and W such that t_1 and t_2 are the endpoints of e_T, w_1 and w_2 are the endpoints of e_W, and t_3 and w_3 are the remaining vertices. Since $T \cap W = \emptyset$, we have that $t_3 \neq w_3$ ~~are~~ as required.

Let G' be the graph obtained from $G - \{u, v\}$ by adding the edges t_1t_2 and w_1w_2. By the argument above, G' is connected, and so by the minimality of G, there is a good path decomposition of G'. We obtain a path decomposition of G by replacing t_1t_2 with t_1ut_2 and w_1w_2 with w_1vw_2, and adding the path t_3uvw_3. Note that since $t_3 \neq w_3$ the latter is really a path. By Proposition 2.1, and since we ~~add~~ added at most one new path, this yields a good path decomposition of G.

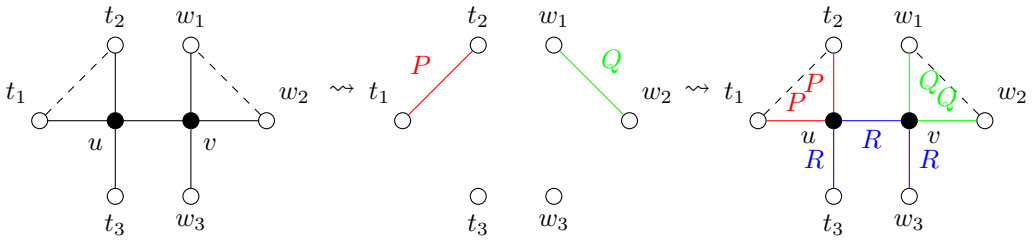


Figure 5: The reduction of C_4 in the connected case.

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Claim 5. *G does not contain the configuration C_5.*

Proof. We first consider the case where a pair in $\{u, v, w\}$, say $\{u, v\}$, has three common ~~neighbors~~neighbours. Let x and y be the two ~~neighbors~~ neighbours of $\{u, v\}$ besides w. We argue that wxy induces a triangle. Indeed, first assume there are at least two edges missing, say $xw, wy \notin E(G)$. Consider the graph ~~$G'' = G + xw + wy$~~$G'' = G - u - v + xw + wy$, note that it is connected, and consider a good path decomposition of it. We obtain a path decomposition of G by replacing the edge xw with xuw, replacing the edge wy with wvy, and adding the path $xvuy$, see Figure 6. By Proposition 2.1, this yields a good path decomposition of G.

Assume now that there is precisely one edge missing, say the edge xy. Consider G', the graph obtained from $G - \{u, v\}$ by adding the edge xy. If G' is connected, then by the minimality of G, it has a good path decomposition. From this, we obtain a path decomposition of G by replacing the edge xy with $xuvy$ and adding the path $xvwuy$, see Figure 7. By Proposition 2.1, this yields a good path decomposition of G.

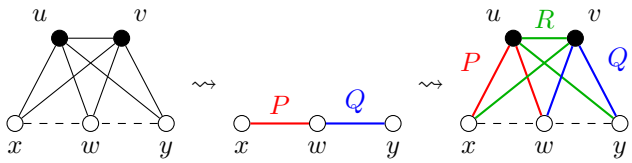


Figure 6: The reduction of C_5 when u and v have three common ~~neighbors~~ neighbours that induce at least two non-edges.

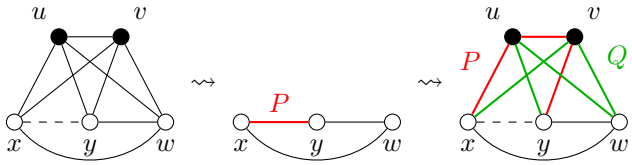


Figure 7: The reduction of C_5 when u and v have three common ~~neighbors~~ neighbours that induce precisely one non-edge.

Therefore x, y and w induce a triangle. Let $G' = G - \{u, v\}$, and note that G' is connected. Thus, by the minimality of G, the graph G' admits a good path decomposition $\mathcal{D}'$. We obtain a path decomposition of G as follows: First assume without loss of generality that xy and wy do not belong to the same path of $\mathcal{D}'$. Let Q' be the path of $\mathcal{D}'$ containing the edge xw, and $Q = Q' - xw + xu$. We consider $\mathcal{D}'' = \mathcal{D}' - Q' + Q$. Let P' be the path of $\mathcal{D}''$ containing the edge xy. We write $P' = P'_1 + xy + P'_2$, where P'_1 and P'_2 may be empty paths. Set $P_1 = P'_1 + xyvw$ and $P_2 = P'_2 + yuvxw$. Note that P_1 and P_2 are paths even if P'_1 also contains the edge xu or in other words $P'_1 = Q$. Finally, let R' be the path of $\mathcal{D}''$ containing the edge ~~yw~~wy, and set $R = R' + wu$. Remember that we assumed $P' \neq R'$. We note that $\mathcal{D} = \mathcal{D}'' - P' - R' + P_1 + P_2 + R$ is a path decomposition of G, with precisely one more path than $\mathcal{D}'$, see Figure 8. Thus $\mathcal{D}$ is a good path decomposition by Proposition 2.1, a contradiction. Therefore no pair in $\{u, v, w\}$ has three common ~~neighbors~~neighbours.

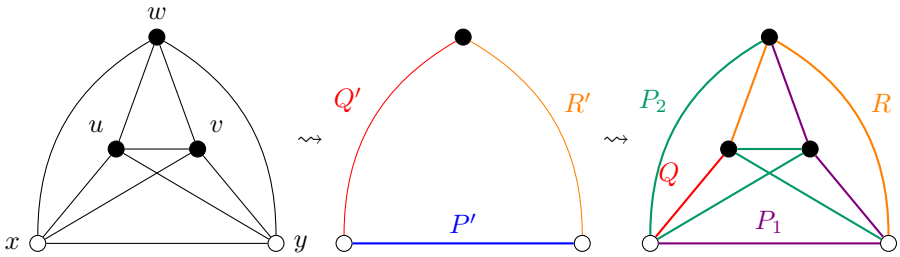


Figure 8: The reduction of C_5 when u and v have three common ~~neighbors~~ neighbours that induce a triangle. We assume P' and R' are distinct, though Q' might be the same as R' or P' or be altogether distinct from both.

Now, since ~~$d(u), d(v), d(w) \leq 4$~~ $d(u), d(v), d(w) \leq 4$ and by Claim 3, we conclude that no pair of vertices in $\{u, v, w\}$ has a common ~~neighbor~~ neighbour other than the third vertex. If they exist, let $\{x_1, x_2\}$, $\{y_1, y_2\}$ and $\{z_1, z_2\}$ be the two other ~~neighbors~~ neighbours of u, v and w respectively. We consider three cases.

1. *Assume first that one of v and w has degree* 2, say $d(v) = 2$. Let G' be the graph obtained from $G - v$ by contracting the edge uw. Note that G' is connected and $|V(G')| = |V(G)| - 2$. By the minimality of G, there is a good path decomposition $\mathcal{D}'$ of G'. To obtain a path decomposition of G, we consider two cases depending on whether ux_1 and ux_2 belong to the same path in $\mathcal{D}'$, see Figures 9 and 10. If they do not, then replace ux_1 with the path wux_1, and replace ux_2 with the path $wvux_2$. However, if ux_1 and ux_2 belong to the same path $P \in \mathcal{D}'$, then split P at u into two paths P_1 and P_2. Extend P_1 with the edge uw and extend P_2 with the path uvw. Note that no edge incident to w is in P_1 or P_2. By Proposition 2.1, and since we created at most one new path, this yields a good path decomposition of G.

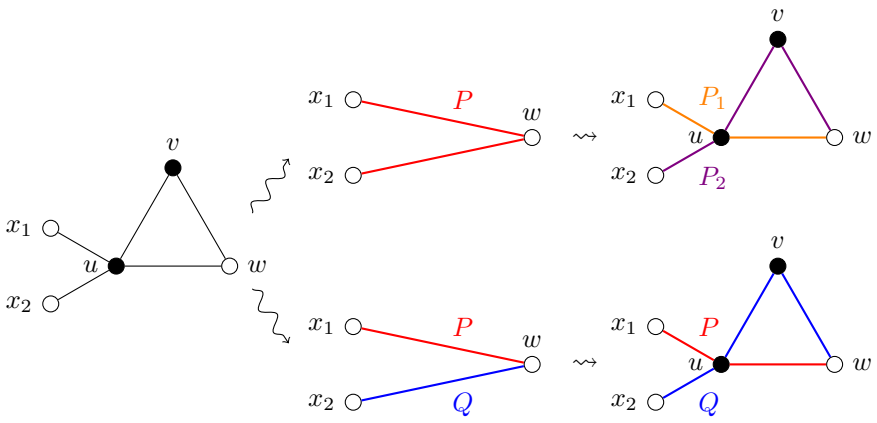


Figure 9: The reduction of C_5 when u and w have precisely one common ~~neighbor~~ neighbour and $d(v) = 2$.

2. *Assume that one of the edges* ~~$ux_1, ux_2, vy_1, vy_2, wz_1, wz_2$~~ $ux_1, ux_2, vy_1, vy_2, wz_1, wz_2$ *is not a cut-edge.* Assume without loss of generality that ux_1 is such an edge. Let G' be the graph obtained from $G - u$ by contracting the edge vw to a vertex s, and adding the edge sx_2. Note that G' is connected and $|V(G')| = |V(G)| - 2$, so by the minimality of G, there is a good path decomposition $\mathcal{D}'$ of G'.

   We obtain a path decomposition of G as follows. We first replace any subpath of the form ysz, $y \in \{y_1, y_2\}$, $z \in \{z_1, z_2\}$ with $yvwz$ (preferably) or with $yvuwz$ (if there are two such subpaths). We then replace any subpath of the form x_2st, $t \in \{y_1, y_2, z_1, z_2\}$, with x_2urt where r is the vertex of $\{v, w\}$ adjacent to t. We replace any remaining edge of the form ts, $t \in \{x_2, y_1, y_2, z_1, z_2\}$ with tr, where r is the vertex of $\{u, v, w\}$ adjacent to t. Let $\mathcal{D}''$ be the resulting collection of disjoint paths in G. Note that since $d(s) = 5$, there is a path P in $\mathcal{D}'$ that ends in s, thus a path P' in $\mathcal{D}''$ that ends in $r \in \{u, v, w\}$. We consider the set of edges of G that do not belong to a path in $\mathcal{D}''$. If that set does not induce a path, then we extend P' to wu or wv. Note that this guarantees the only remaining edges induce a path Q, which we add to the path collection. By Proposition 2.1, and since we added at most one new path, this yields a good path decomposition of G.

3. Now $d(u) = d(v) = d(w) = 4$ and every edge with precisely one endpoint in $\{u, v, w\}$ is a cut-edge. Consider the graph G' obtained from $G - \{u, v, w\}$ by adding the three edges

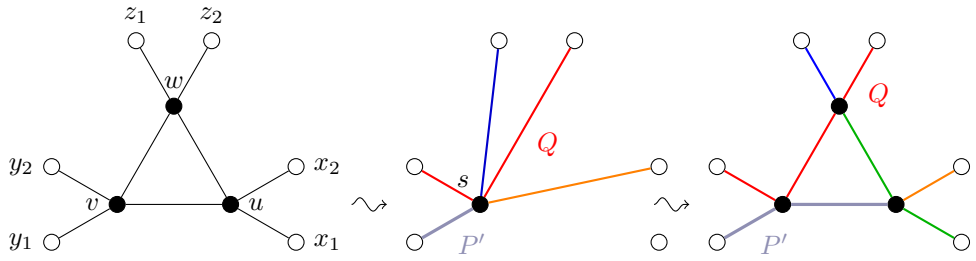


Figure 10: An example of the reduction of C_5 when $d(u) = d(v) = d(w) = 4$, the triangle (u, v, w) is adjacent to no other triangle and some edge in $E(\{u, v, w\}, \{x_1, x_2, y_1, y_2, z_1, z_2\})$ is not a cut-edge.

x_1y_1, x_2y_2 and z_1z_2. Note that G' has precisely three connected components G_1, G_2, and G_3. By the minimality of G, there are good path decompositions of G_1, G_2 and G_3. We obtain a path decomposition of G by replacing x_1y_1 with the path x_1uvy_1, replacing x_2y_2 with x_2uwvy_2, and replacing z_1z_2 with the path z_1wz_2 (see Figure 11). These paths are all distinct since the edges x_1y_1, x_2y_2 and z_1z_2 belong to different components of G'. Note that the total number of paths involved in the resulting path decomposition of G is at most $\frac{|V(G_1)|+1}{2} + \frac{|V(G_2)|+1}{2} + \frac{|V(G_3)|+1}{2} = \frac{|V(G)|}{2}$, thus it is a good path decomposition.

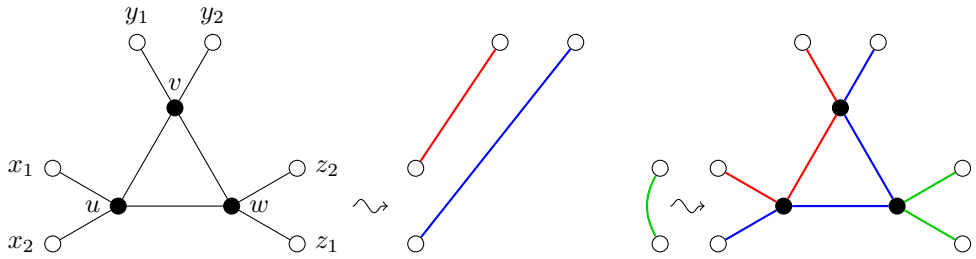


Figure 11: The reduction of C_5 when $d(u) = d(v) = d(w) = 4$, the triangle (u, v, w) is adjacent to no other triangle and every edge in $E(\{u, v, w\}, \{x_1, x_2, y_1, y_2, z_1, z_2\})$ is a cut-edge.

$\Diamond$

By Claims 1, 2, 3, 4 and 5, the ~~lemma statement holds~~proof of Lemma 3.1 is complete. $\square$

Recall that G_E denotes the graph induced ~~on~~ by the vertices of even degree in G.

Lemma 3.2. *Let G be a connected graph such that $G \notin \{K_3, K_5\}$. If $\Delta(G) \leq 5$ and G does not contain ~~configurations~~ Configurations $C_1, \ldots, C_5$, then the graph G_E is a forest.*

Proof. Let $H = G_E$ and suppose for a contradiction that H contains a cycle C. Suppose further that there is $v \in V(C)$ with ~~$d(v) = 2$~~$d_G(v) = 2$, and let $N(v) = \{u, w\}$. Since C is a cycle in H, we have that ~~$d(u), d(w) \in \{2, 4\}$~~$d_G(u), d_G(w) \in \{2, 4\}$. Furthermore, since G does not contain ~~configuration~~ Configuration C_1, we have that $uw \in E(G)$. Now $G \neq K_3$, so at least one of u and w has degree 4 in G. It follows that ~~u, v~~ u, v and w form ~~configuration~~ Configuration C_5, a contradiction. Thus, if C is a cycle in H, then $d_G(v) = 4$ for all vertices $v \in V(C)$. Since G does not contain ~~configuration~~ Configuration C_5, it immediately follows that $|C| > 3$.

Let uv be an edge of C. Let ~~t_1, t_2, t_3~~ t_1, t_2, t_3 be the neighbours of u apart from v and let ~~w_1, w_2, w_3~~ w_1, w_2, w_3 be the neighbours of v apart from u. Note that, since uv is an edge of C, at least one of ~~t_1, t_2, t_3~~ t_1, t_2, t_3 has degree 4. Similarly, at least one of ~~w_1, w_2, w_3~~ w_1, w_2, w_3 has degree 4. Now, u and v do not have 3 common neighbours, since otherwise G contains ~~configuration~~ Configuration C_5, a contradiction. Furthermore, since G does not contain ~~configuration~~ Configuration C_3, the vertices u and v have at most one common neighbour. Thus, in what follows, we allow the possibility that $t_1 = w_1$, but always assume that ~~$t_2, t_3 \notin \{w_1, w_2, w_3\}$ and $w_2, w_3 \notin \{t_1, t_2, t_3\}$~~$t_2, t_3 \notin \{w_1, w_2, w_3\}$ and $w_2, w_3 \notin \{t_1, t_2, t_3\}$.

~~Suppose first that $t_1t_2 \notin E(G)$. Since~~ We first argue that $\{t_1, t_2, t_3\}$ does not induce a clique. Assume by contradiction that $\{t_1, t_2, t_3\}$ induces a clique. One of them has degree 4. Note that since G does not contain ~~configuration C_4, we must have that $w_1w_2, w_2w_3, w_1w_3 \in E(G)$. Otherwise, since $t_3 \notin \{w_1, w_2, w_3\}$, we have that G contains configuration C_4, a contradiction. But now the vertices w_1, w_2, w_3 form a clique, and at least one of them has~~ Configuration C_5, we can assume without loss of generality that $d_G(t_2) = 4$. Note that t_2 and u form two adjacent vertices of degree 4 ~~. It follows~~ with exactly two common neighbors, since u is adjacent to v and t_2 is not. This is in contradiction with the fact that G ~~contains configuration~~ does not contain Configuration C_3~~, a contradiction~~.

~~It follows that~~ By symmetry, we obtain that $\{w_1, w_2, w_3\}$ does not induce a clique.

Suppose now that $t_1t_2 \notin E(G)$. Since G does not contain Configuration C_4, we must have that $w_1w_2, w_2w_3, w_1w_3 \in E(G)$. This contradicts the fact that $\{w_1, w_2, w_3\}$ does not induce a clique.

By symmetry, all of the edges ~~$t_1t_2, t_1t_3, w_1w_2, w_1w_3 \in E(G)$~~$t_1t_2$, t_1t_3, w_1w_2, $w_1w_3 \in E(G)$. As a consequence, $t_1 \neq w_1$, otherwise this vertex would have degree 6, which is larger than $\Delta(G)$. Thus $\{t_1, t_2, t_3\} \cap \{w_1, w_2, w_3\} = \emptyset$. With this extra information, we can repeat the argument above shows that, in fact, if any ~~edge amongst t_1, t_2, t_3 is not in $E(G)$, then w_1, w_2, w_3 induce~~ pair of vertices amongst t_1, t_2, t_3 is not adjacent in G, then $\{w_1, w_2, w_3\}$ induces a clique. ~~Thus~~Therefore, either $\{t_1, t_2, t_3\}$ or $\{w_1, w_2, w_3\}$ ~~induce~~ induces a clique, ~~which again gives a contradictionsince G does not contain configuration C_3.~~

a contradiction. □

The proof of Theorem 1.3 now follows easily.

Proof of Theorem 1.3. Let G be a smallest counterexample to the theorem. By Lemma 3.1, the graph G does not contain ~~configurations~~ Configurations $C_1, \ldots, C_5$. Thus, by Lemma 3.2, the graph G_E is a forest. But now G admits a good path decomposition by Theorem 1.1, a contradiction. □

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