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Abstract

In this paper, we study a model of magnetohydrodynamics problem and prove the existence of weak solution to the stationary magnethohydrodynamic system in a three dimensional bounded domain Ω of class $C^{1,1}$. To our knowledge, all previous works consider the domain Ω simply-connected. Our proof is based on some weak estimates concerning vectors potential in negative Sobolev spaces. We also give some regularity results in L^p-theory.
Keywords: Magnetohydrodynamics, Stokes equations, Navier-Stokes equations, vector potentials, weak solutions, strong solutions.

1. Introduction

Magnetohydrodynamics (MHD) is the theory of macroscopic interaction of electrically conducting fluids and electromagnetic fields. (MHD) flow is governed by the Navier-Stokes equations for the fluid velocity and Maxwell's equations for the magnetic field. The equations are non-linearly coupled via Ohm's law and the Lorentz force.

Studying this coupled system is of interest since they have many applications in engineering problems, such as sustained plasma confinement for controlled thermonuclear fusion, liquid-metal cooling of nuclear reactors and electromagnetic casting of metals. They

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1

are also used in fusion technology and submarine propulsion devices. Other applications and uses of micro-polar fluids can be found in Lukaszewicz [13].

The present work is concerned with the existence and the regularity of the solution for the stationary magnetohydrodynamic equations which describe the steady state flow of a viscous, incompressible, electrically conducting fluid in three dimensional bounded domain Ω.

We consider here the following system denoted by (MHD):

$$\begin{cases} -\nu\Delta \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u} - \dfrac{1}{\rho\mu}\,(\boldsymbol{B}\cdot\nabla)\,\boldsymbol{B} + \dfrac{1}{2\rho\mu}\nabla\left(|\boldsymbol{B}|^2\right) + \dfrac{1}{\rho}\nabla\pi = \boldsymbol{f} & \text{in } \Omega,\\ -\lambda\Delta\boldsymbol{B} + (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{B} - (\boldsymbol{B}\cdot\nabla)\,\boldsymbol{u} = \boldsymbol{k} & \text{in } \Omega,\\ \operatorname{div}\,\boldsymbol{u} = \operatorname{div}\,\boldsymbol{B} = 0 & \text{in } \Omega,\\ \boldsymbol{u} = \boldsymbol{0}, \quad \boldsymbol{B}\cdot\boldsymbol{n} = 0, \quad \mathbf{curl}\,\boldsymbol{B}\times\boldsymbol{n} = \boldsymbol{0} & \text{on } \Gamma,\\ \displaystyle\int_{\Sigma_j} \boldsymbol{B}\cdot\boldsymbol{n} = 0, \ 1 \leqslant j \leqslant J, & \end{cases}$$

where Ω is a bounded open connected set of $\mathbb{R}^3$ of class $C^{1,1}$, possibly multiply-connected, with boundary Γ such that $\Gamma = \bigcup_{i=0}^{I} \Gamma_i$ where Γ_i are the connected components of Γ. When Ω is not simply-connected, we suppose that there exists J connected open surfaces, called 'cuts', contained in Ω, such that each surface Σ_j is an open part of a smooth manifold and the boundary of each Σ_j is contained in Γ. The intersection $\overline{\Sigma_i}\cap\overline{\Sigma_j}$ is empty for $i \neq j$ and the open set $\Omega^\circ = \Omega \setminus \bigcup_{j=1}^{J} \Sigma_j$ is simply-connected. See Figure 1, for $J = 1$ with $I = 3$

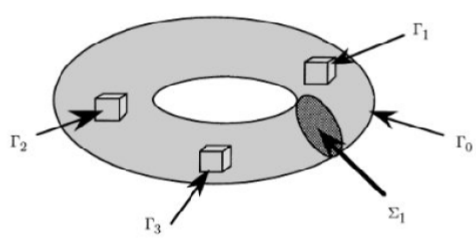

Figure 1

If Ω is simply-connected, the last condition on the flow of $\boldsymbol{B}$ through the cuts is not required any more.

The unknown variables are $\boldsymbol{u}, \pi$ and $\boldsymbol{B}$ which represent the velocity field, the pressure and the magnetic field respectively, while $\boldsymbol{f}$ and $\boldsymbol{k}$ are given external forces, ν, μ and ρ are the constants of kinematic viscosity, magnetic permeability and density of Eulerian flow respectively and $\lambda = \dfrac{\eta}{\mu}$ with electrical resistivity η.

To our knowledge in all previous works, the domain Ω is supposed to be simply-connected. In our work, the domain may be simply-connected or not simply-connected. In this last case, it is necessary to add the condition concerning the flows of $\boldsymbol{B}$ through the cuts Σ_j. We will see later the justification of this condition when the domain is not simply-connected.

There are quite vast literature available concerning the solvability of (MHD) under different types of boundary conditions though most of these works are done with the time-dependent problem. Sermange and Temam [17] proved in two-dimension, the global existence and uniqueness of weak solution that is strong for regular data. They also obtained as for the Navier-Stokes equations, a global weak solution in three-dimension and for more regular data, they showed that a strong solution exists and is unique locally in time. By using the spectral Galerkin method, Rojas-Medar and Boldrini [15] proved, under smallness of data, global in time existence of strong solutions and gave several estimates for the solution and their approximations. The (MHD) flow of a second grad fluid has been studied by Hamdache and Jaffal-Mourtada [11] where they showed that a unique solution exists for small time and it is actually global in time for small initial data.

Concerning the stationary (MHD) problem, Gunzburger, Meir and Peterson [10] studied the system in a bounded, three-dimensional simply-connected domain, either of class $\mathcal{C}^{1,1}$ or convex, with inhomogeneous Dirichlet boundary condition for the velocity, satisfying naturally some smallness condition and with the normal and the tangential component of the vorticity of the magnetic field given. They proved the existence and uniqueness of weak solution $(\boldsymbol{u}, \boldsymbol{B}) \in \boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^1(\Omega)$ under smallness assumption on boundary data for the velocity. On the other hand, Bermudez, Munoz-Sola and Vazquez [5] considered a coupling between the equations of magnetohydrodynamics and the heat equation in a simply connected domain of class $\mathcal{C}^{1,1}$ or a bounded Lipschitz polyedron and gave existence results of weak solution under certain conditions. Using a Faedo-Galerkin approximation combined with Schaefer's fixed point theorem, C. Zhao and K. Li [19] proved the existence of weak solution in three dimensional bounded domain with homogeneous Dirichlet boundary conditions for the velocity and for the magnetic field. The uniqueness result proved in Theorem 3.1 of [19] means that in fact the magnetic field is trivial and then the coupled problem is actually reduced to the Navier-Stokes equation. For further references, we mention [6], [8], [14], [16].

In the present work we study the existence and the regularity of weak solution for the (MHD) problem with the same boundary conditions as in [10] or in [5]. But here we consider the more general case where the domain Ω is not necessarily simply connected. To prove the existence result, we use the Leray-Schauder fixed point theorem. And to obtain the compactness properties of the operator, one main tool is given by some estimates for very weak vector potentials corresponding to vector fields belonging to some negative Sobolev spaces. We also investigate the L^p-theory for the solution. More precisely, we will prove the existence of generalized in $\boldsymbol{W}^{1,p}(\Omega)$ for $p \geq 2$ and strong solution in $\boldsymbol{W}^{2,p}(\Omega)$ for $p \geq \frac{6}{5}$.

The first main result of our work, stated in Theorem 3.4, concerns the existence and uniqueness of a vector potential, in particular the estimate (3.9) which is important to obtain some further estimates for the magnetic field.

Theorem 4.1 gives existence of weak solutions for the magnetohydrodynamic problem (MHD) and some estimates.

We end the introduction giving an outline of the paper. In Section 2, notations, some basic assumptions and preliminary results are stated. In Section 3, some results concerning weak vector potentials are given and we study the existence of very weak vector potentials for vector fields belonging to negative Sobolev spaces. In Section 4, the existence of weak solution for the (MHD) problem is established. Finally Section 5 is devoted to study the regularity of the weak solution.

Unless otherwise stated, we follow the convention that C is an unspecified positive constant that may vary from expression to expression, even across an inequality (but not across an equality) and depends only on the data of the problem (ν, ρ, μ, λ and Ω).

2. Notations and preliminary results

For $1 < p < \infty$ and $m \in \mathbb{R}$, let $L^p(\Omega)$ and $W^{m,p}(\Omega)$ be the usual Lebesgue and Sobolev spaces respectively. We denote by $\boldsymbol{L}^p(\Omega) = [L^p(\Omega)]^3$, $\boldsymbol{W}^{m,p}(\Omega) = [W^{m,p}(\Omega)]^3$ and we use the bold notation for vector fields. Then we define the following Banach spaces:

$$\boldsymbol{H}^p(\mathbf{curl}, \Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{L}^p(\Omega)\},$$

$$\boldsymbol{H}^p(\mathrm{div}, \Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \mathrm{div}\,\boldsymbol{v} \in L^p(\Omega)\}$$

equiped with the norms

$$\|\boldsymbol{v}\|_{\boldsymbol{H}^p(\mathbf{curl},\Omega)} = \left(\|\boldsymbol{v}\|^p_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|^p_{\boldsymbol{L}^p(\Omega)}\right)^{1/p}$$

and

$$\|\boldsymbol{v}\|_{\boldsymbol{H}^p(\mathrm{div},\Omega)} = \left(\|\boldsymbol{v}\|^p_{\boldsymbol{L}^p(\Omega)} + \|\mathrm{div}\,\boldsymbol{v}\|^p_{L^p(\Omega)}\right)^{1/p}$$

We also define the space

$$\boldsymbol{X}^p(\Omega) = \boldsymbol{H}^p(\mathbf{curl}, \Omega) \cap \boldsymbol{H}^p(\mathrm{div}, \Omega),$$

and the subspaces

$$\boldsymbol{X}^p_N(\Omega) = \{\boldsymbol{v} \in \boldsymbol{X}^p(\Omega);\ \boldsymbol{v} \times \boldsymbol{n} = \mathbf{0} \text{ on } \Gamma\},$$

$$\boldsymbol{X}^p_T(\Omega) = \{\boldsymbol{v} \in \boldsymbol{X}^p(\Omega);\ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\}.$$

Note that any function $\boldsymbol{v}$ in $\boldsymbol{H}^p(\mathbf{curl}, \Omega)$ has a tangential trace $\boldsymbol{v} \times \boldsymbol{n}$ in $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$, defined by

$$\forall \boldsymbol{\varphi} \in \boldsymbol{W}^{1,p'}(\Omega), \quad \langle \boldsymbol{v} \times \boldsymbol{n}, \boldsymbol{\varphi}\rangle_\Gamma = \int_\Omega \boldsymbol{v} \cdot \mathbf{curl}\,\boldsymbol{\varphi}\, dx \ - \int_\Omega \boldsymbol{\varphi} \cdot \mathbf{curl}\,\boldsymbol{v}\, dx \qquad (2.1)$$

where $\langle \cdot, \cdot \rangle_\Gamma$ denotes the duality bracket between $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ and $\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)$, p and p' are conjugate exponents. In fact, (2.1) holds also for $\boldsymbol{v} \in \boldsymbol{L}^p(\Omega)$ and $\mathbf{curl}\ \boldsymbol{v} \in \boldsymbol{L}^{r(p)}(\Omega)$, (see (2.10) for the definition of $r(p)$).

And any function $\boldsymbol{v}$ in $\boldsymbol{H}^p(\mathrm{div},\Omega)$ has a normal trace $\boldsymbol{v}\cdot\boldsymbol{n}$ in $W^{-\frac{1}{p},p}(\Gamma)$, defined by

$$\forall\varphi\in\boldsymbol{W}^{1,p'}(\Omega),\quad \langle\boldsymbol{v}\cdot\boldsymbol{n},\varphi\rangle_\Gamma=\int_\Omega \boldsymbol{v}\cdot\mathbf{grad}\,\varphi\,dx\;+\;\int_\Omega(\mathrm{div}\,\boldsymbol{v})\,\varphi\,dx. \tag{2.2}$$

Theorem 2.1. *i) The space* $\boldsymbol{X}^p_N(\Omega)$ *is continuously imbedded in* $\boldsymbol{W}^{1,p}(\Omega)$, *and we have the following inequality: for every function* $\boldsymbol{v}\in\boldsymbol{W}^{1,p}(\Omega)$ *with* $\boldsymbol{v}\times\boldsymbol{n}=\mathbf{0}$ on Γ,

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)}\le C(\|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}+\|\mathrm{div}\,\boldsymbol{v}\|_{\mathrm{L^p(\Omega)}}+\sum_{\mathrm{i=1}}^{\mathrm{I}}|\langle\boldsymbol{v}\cdot\boldsymbol{n},\,1\rangle_{\Gamma_\mathrm{i}}|)$$

ii) The space $\boldsymbol{X}^p_T(\Omega)$ *is continuously imbedded in* $\boldsymbol{W}^{1,p}(\Omega)$, *and we have the following inequality: for every function* $\boldsymbol{v}\in\boldsymbol{W}^{1,p}(\Omega)$ *with* $\boldsymbol{v}\cdot\boldsymbol{n}=\mathbf{0}$ on Γ,

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)}\le C(\|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}+\|\mathrm{div}\,\boldsymbol{v}\|_{\mathrm{L^p(\Omega)}}+\sum_{\mathrm{j=1}}^{\mathrm{J}}|\langle\boldsymbol{v}\cdot\boldsymbol{n},\,1\rangle_{\sum_\mathrm{j}}|)$$

Furthermore, we give the following theorem which extends Theorem 2.1 in the case where the boundary conditions $\boldsymbol{v}\cdot\boldsymbol{n}$ and $\boldsymbol{v}\times\boldsymbol{n}$ are replaced by inhomogeneous one, see [4], Theorem 3.5 and Corollary 5.2. For that, we introduce the following spaces:

$$\boldsymbol{X}^{1,p}(\Omega)=\left\{\boldsymbol{v}\in\boldsymbol{X}^p(\Omega),\ \boldsymbol{v}\cdot\boldsymbol{n}\in W^{1-\frac{1}{p},p}(\Gamma)\right\},$$

$$\boldsymbol{Y}^{1,p}(\Omega)=\left\{\boldsymbol{v}\in\boldsymbol{X}^p(\Omega),\ \boldsymbol{v}\times\boldsymbol{n}\in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)\right\}.$$

Theorem 2.2. *i) The space* $\boldsymbol{X}^{1,p}(\Omega)$ *is continuously imbedded in* $\boldsymbol{W}^{1,p}(\Omega)$ *and we have the following estimate for any* $\boldsymbol{v}$ *in* $\boldsymbol{X}^{1,p}(\Omega)$,

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)}\leqslant C(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}+\|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}+\|\mathrm{div}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}+\|\boldsymbol{v}\cdot\boldsymbol{n}\|_{W^{1-\frac{1}{p},p}(\Gamma)}).$$

ii) The space $\boldsymbol{Y}^{1,p}(\Omega)$ *is continuously imbedded in* $\boldsymbol{W}^{1,p}(\Omega)$ *and we have the following estimate for any* $\boldsymbol{v}$ *in* $\boldsymbol{Y}^{1,p}(\Omega)$,

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)}\leqslant C(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}+\|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}+\|\mathrm{div}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}+\|\boldsymbol{v}\times\boldsymbol{n}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)}).$$

To study the existence and the uniqueness of a weak vector potential, we shall need to introduce the following spaces:

$$\boldsymbol{K}_N(\Omega)=\left\{\ \boldsymbol{v}\in\boldsymbol{L}^2(\Omega),\ \mathrm{div}\,\boldsymbol{v}\ =\ 0,\ \ \mathbf{curl}\,\boldsymbol{v}=\mathbf{0}\ \text{ in }\Omega,\ \ \boldsymbol{v}\times\boldsymbol{n}=\mathbf{0}\ \text{ on }\Gamma\ \right\}, \tag{2.3}$$

$$\boldsymbol{H} = \big\{ \boldsymbol{v} \in \boldsymbol{L}^2(\Omega);\ \operatorname{div} \boldsymbol{v} = 0 \ \text{ in } \ \Omega, \quad \boldsymbol{v} \cdot \boldsymbol{n} = 0 \ \text{ on } \ \Gamma \big\},$$

and

$$\boldsymbol{K}_T(\Omega) = \{ \boldsymbol{v} \in \boldsymbol{H};\ \mathbf{curl}\, \boldsymbol{v} = \boldsymbol{0} \text{ in } \ \Omega \}. \tag{2.4}$$

As given in [2], Proposition 3.18 and Proposition 3.14, we recall that:

i) The space $\boldsymbol{K}_N(\Omega)$ is spanned by the functions $\nabla q_i^N, 1 \leq i \leq I$, where each q_i^N is the unique solution in $\boldsymbol{H}^1(\Omega)$ of the problem

$$\begin{cases} -\Delta q_i^N = 0 \qquad \text{in } \Omega, \\ q_i^N|_{\Gamma_0} = 0 \quad \text{and} \quad q_i|_{\Gamma_k} = \text{constant}, \ \ 1 \leq k \leq I, \\ \big\langle \partial_n q_i^N,\, 1 \big\rangle_{\Gamma_k} = \delta_{i\,k}, \ \ 1 \leq k \leq I, \quad \text{and} \quad \big\langle \partial_n q_i^N,\, 1 \big\rangle_{\Gamma_0} = -1, \end{cases} \tag{2.5}$$

ii) The space $\boldsymbol{K}_T(\Omega)$ is spanned by the functions $\widetilde{\nabla q_j^T}, 1 \leq j \leq J$, where each $q_j^T \in \boldsymbol{H}^1(\Omega^\circ)$ is unique up to an additive constant and satisfies:

$$\begin{cases} -\Delta q_j^T = 0 \qquad \text{in } \Omega^\circ, \\ \partial_n q_j^T = 0 \qquad \text{on } \Gamma, \\ \Big[q_j^T \Big]_k = \text{constant} \quad \text{and} \quad [\, \partial_n q_j^T \,]_k = 0, \ \ 1 \leq k \leq J, \\ \Big\langle \partial_n q_j^T,\, 1 \Big\rangle_{\Sigma_k} = \delta_{j\,k}, \ \ 1 \leq k \leq J. \end{cases} \tag{2.6}$$

We note that $\boldsymbol{K}_T(\Omega) = \{\boldsymbol{0}\}$ if and only if Ω is simply-connected. Likewise $\boldsymbol{K}_N(\Omega) = \{\boldsymbol{0}\}$ if and only if Γ is connected.

We recall now a basic theorem about a vector potential given in [2], Theorem 3.20:

Theorem 2.3. *For any function $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ which satisfies:*

$$\operatorname{div} \boldsymbol{f} = 0 \ \textit{ in } \Omega, \quad \boldsymbol{f} \cdot \boldsymbol{n} = 0 \ \textit{ on } \ \Gamma, \quad \langle\, \boldsymbol{f} \cdot \boldsymbol{n}\,, 1 \,\rangle_{\Sigma_j} = 0, \quad 1 \leqslant j \leqslant J, \tag{2.7}$$

there exists a unique vector potential $\boldsymbol{\psi} \in \boldsymbol{H}_0^1(\Omega)$ such that

$$\begin{aligned} \boldsymbol{f} \ &= \ \mathbf{curl}\, \boldsymbol{\psi} \quad \textit{in} \quad \Omega, \qquad \textit{with} \quad \operatorname{div}(\Delta \boldsymbol{\psi}) \ = \ 0 \ \ \textit{in} \ \ \Omega, \\ \Big\langle \ \frac{\partial}{\partial \boldsymbol{n}} (\operatorname{div} \boldsymbol{\psi})\,, 1 \ \Big\rangle_{\Gamma_i} \ &= \ 0, \quad 0 \leqslant i \leqslant I \end{aligned} \tag{2.8}$$

and satisfying the estimate

$$\|\boldsymbol{\psi}\|_{\boldsymbol{H}^1(\Omega)} \leqslant C(\Omega)\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}. \tag{2.9}$$

Remark 1. i) Note that the condition (2.7) is necessary to the above vector potential.
ii) The condition div $(\Delta\boldsymbol{\psi}) = 0$ in Ω implies that the quantity

$$\left\langle \frac{\partial}{\partial \boldsymbol{n}} (\operatorname{div} \boldsymbol{\psi}), 1 \right\rangle_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma_i)\times \boldsymbol{H}^{\frac{1}{2}}(\Gamma_i)}$$

makes sense. The uniqueness of the function $\boldsymbol{\psi}$ is given by the two last conditions of (2.8) and follows from the characterization of the kernel

$$\boldsymbol{K}_0(\Omega) = \left\{\boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega);\ \ \mathbf{curl}\,\boldsymbol{w} = \mathbf{0} \text{ and } \operatorname{div}(\Delta\boldsymbol{w}) = 0 \text{ in } \Omega\right\},$$

which is of dimension I and spanned by the functions ∇q_i with $1 \leqslant i \leqslant I$ and where each q_i is the unique solution in $\boldsymbol{H}^2(\Omega)$ of the problem

$$\begin{cases} \Delta^2 q_i = 0 \quad \text{in } \Omega \\ q_i|_{\Gamma_0} = 0 \quad q_i|_{\Gamma_k} = \text{cst}, \quad 1 \leqslant k \leqslant I \\ \frac{\partial q_i}{\partial \boldsymbol{n}} = 0 \quad \text{on } \Gamma \\ \left\langle \frac{\partial}{\partial \boldsymbol{n}}(\Delta q_i),\, 1\right\rangle_{\Gamma_k} = \delta_{ik}, \quad 1 \leqslant k \leqslant I, \quad \left\langle \frac{\partial}{\partial \boldsymbol{n}}(\Delta q_i), 1\right\rangle_{\Gamma_0} = -1. \end{cases}$$

Note that $\nabla q_i = \mathbf{0}$ on Γ because $\nabla q_i \cdot \boldsymbol{n} = 0$ on Γ and since q_i is constant on each connected component of Γ we have also $\nabla q_i \times \boldsymbol{n} = \mathbf{0}$ on Γ. For more details, we can see [2], Proposition 3.21.

To study the regularity of the week solution for the problem (MHD), we give the following theorem. Before that, we define, for any $1 < p < \infty$:

$$\begin{cases} r(p) = \max\left\{1, \frac{3p}{p+3}\right\} & \text{if} \quad p \neq \frac{3}{2}, \\ r(p) > 1 & \text{if} \quad p = \frac{3}{2}. \end{cases} \tag{2.10}$$

This definition of $r(p)$ makes sense to the RHS below, see (2.15).

Theorem 2.4. ***i)*** *Let* $\boldsymbol{f} \in \boldsymbol{L}^{r(p)}(\Omega)$ *with* $\operatorname{div} \boldsymbol{f} = 0$ *in* Ω *and verifying the following compatibility conditions:*

$$\text{for any} \quad \boldsymbol{v} \in \boldsymbol{K}_T(\Omega), \quad \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} = 0, \tag{2.11}$$

$$\boldsymbol{f} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma. \tag{2.12}$$

Then, the problem

$$(E_T) \qquad \begin{cases} -\Delta \boldsymbol{\xi} = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{\xi} = 0 & \text{in } \Omega, \\ \boldsymbol{\xi} \cdot \boldsymbol{n} = 0 \quad \text{and} \quad \mathbf{curl}\, \boldsymbol{\xi} \times \boldsymbol{n} = \mathbf{0} & \text{on } \Gamma, \\ \langle \boldsymbol{\xi} \cdot \boldsymbol{n}, 1 \rangle_{\Sigma_j} = 0, \quad 1 \leq j \leq J. \end{cases}$$

has a unique solution $\boldsymbol{\xi}$ *in* $\boldsymbol{W}^{1,p}(\Omega)$ *satisfying the estimate:*

$$\| \boldsymbol{\xi} \|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C(\Omega) \| \boldsymbol{f} \|_{\boldsymbol{L}^{r(p)}(\Omega)}. \tag{2.13}$$

ii) *Moreover if* $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$ *and* Ω *is of class* $C^{2,1}$, *then the solution* $\boldsymbol{\xi}$ *is in* $\boldsymbol{W}^{2,p}(\Omega)$ *and satisfies the estimate:*

$$\|\boldsymbol{\xi}\|_{\boldsymbol{W}^{2,p}(\Omega)} \leqslant C(\Omega) \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \tag{2.14}$$

Proof. **i)** The existence result of the solution $\boldsymbol{\xi}$ was proved in [3], Proposition 4.3 with $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$. We will see that we can obtain the same result with $\boldsymbol{f}$ only in $\boldsymbol{L}^{r(p)}(\Omega)$. Observe that Problem (E_T) is equivalent to the following:

$$\begin{cases} \text{Find } \ \boldsymbol{\xi} \in \boldsymbol{V}^p_\Sigma(\Omega) \qquad \text{such that} \\ \forall \boldsymbol{\varphi} \in \boldsymbol{V}^{p'}_\Sigma(\Omega), \quad \displaystyle\int_\Omega \mathbf{curl}\, \boldsymbol{\xi} \cdot \mathbf{curl}\, \boldsymbol{\varphi} \ dx \ = \ \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} \ dx, \end{cases} \tag{2.15}$$

where

$$\boldsymbol{V}^p_\Sigma(\Omega) = \Big\{ \boldsymbol{\xi} \in \boldsymbol{W}^{1,p}(\Omega); \ \operatorname{div} \boldsymbol{\xi} = 0 \ \text{ in } \ \Omega, \ \boldsymbol{\xi} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma \text{ and } \langle \boldsymbol{\xi} \cdot \boldsymbol{n}, 1 \rangle_{\Sigma_j} = 0, \ 1 \leq j \leq J \Big\}. \tag{2.16}$$

And note that for $\boldsymbol{f} \in \boldsymbol{L}^{r(p)}(\Omega)$, the integral in (2.15) is well defined because $\boldsymbol{W}^{1,p'}(\Omega) \hookrightarrow \boldsymbol{L}^{[r(p)]'}$ for any $1 < p < \infty$. In particular, for $p = \frac{3}{2}$, $\boldsymbol{W}^{1,3}(\Omega) \hookrightarrow \boldsymbol{L}^q(\Omega)$ for any $q < \infty$

and then the RHS of (2.15) is well defined for $\boldsymbol{\varphi} \in \boldsymbol{W}^{1,3}(\Omega)$ provided that $\boldsymbol{f} \in \boldsymbol{L}^{r(p)}(\Omega)$ with $r(p) > 1$.

As proved in [3], Proposition 4.3, Problem (2.15) has a unique solution $\boldsymbol{\xi} \in \boldsymbol{V}^p_\Sigma(\Omega)$ by using the inf-sup condition given in [4], Lemma 4.4. So that, for any $\ell \in \left[\boldsymbol{V}^p_\Sigma(\Omega)\right]'$ there exists a unique $\boldsymbol{\xi}$ solution of (2.15). In particular, if $\ell(\varphi) = \displaystyle\int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi}\, dx$ with $\boldsymbol{f} \in \boldsymbol{L}^{r(p)}(\Omega)$. In order to interpret the above variational formulation, we need to extend (2.15) to any test function $\boldsymbol{\varphi}$ without condition on the fluxes over Σ. The variational formulation (2.15) is equivalent to the following problem :

$$\begin{cases} \text{Find } \ \boldsymbol{\xi} \in \boldsymbol{V}^p_\Sigma(\Omega) \quad \text{ such that} \\ \forall \boldsymbol{\varphi} \in \boldsymbol{X}^{p'}_T(\Omega), \ \displaystyle\int_\Omega \mathbf{curl}\, \boldsymbol{\xi} \cdot \mathbf{curl}\, \boldsymbol{\varphi} \ dx \ = \ \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi}\, dx \end{cases} \tag{2.17}$$

Clearly, (2.17) implies (2.15). Conversely, let be $\boldsymbol{\xi} \in \boldsymbol{V}^p_\Sigma(\Omega)$ solution of (2.17) and $\boldsymbol{\varphi} \in \boldsymbol{X}^{p'}_T(\Omega)$. Setting

$$\widetilde{\boldsymbol{\varphi}} = \boldsymbol{\varphi} - \nabla\chi - \sum_{j=1}^J \langle (\boldsymbol{\varphi} - \nabla\chi)\cdot \boldsymbol{n}, 1\rangle_{\Sigma_j} \widetilde{\nabla q^T_j} \tag{2.18}$$

where $\chi \in H^2(\Omega)$ is the unique solution, up to an additive constant, satisfying

$$-\Delta\chi = \operatorname{div}\, \boldsymbol{\varphi} \ \text{ in } \ \Omega, \quad \frac{\partial\chi}{\partial\boldsymbol{n}} = 0 \ \text{ on } \ \Gamma. \tag{2.19}$$

Then $\widetilde{\boldsymbol{\varphi}} \in \boldsymbol{X}^{p'}_T(\Omega)$ and for any $1 \leq k \leq J$, we have $\langle \widetilde{\boldsymbol{\varphi}} \cdot \boldsymbol{n}, 1\rangle_{\Sigma_k} = 0$, because $\left\langle \widetilde{\nabla q^T_j} \cdot \boldsymbol{n}, 1\right\rangle_{\Sigma_j} = \delta_{jk}$. That means that $\widetilde{\boldsymbol{\varphi}} \in \boldsymbol{V}^{p'}_\Sigma(\Omega)$ and then

$$\int_\Omega \mathbf{curl}\, \boldsymbol{\xi} \ \cdot \ \mathbf{curl}\, \boldsymbol{\varphi}\, dx \ = \int_\Omega \mathbf{curl}\, \boldsymbol{\xi} \ \cdot \ \mathbf{curl}\, \widetilde{\boldsymbol{\varphi}}\, dx \ = \int_\Omega \boldsymbol{f} \cdot \widetilde{\boldsymbol{\varphi}}\, dx.$$

Now, because $\operatorname{div}\, \boldsymbol{f} = 0$ in Ω, (2.11) and (2.12), we have

$$\int_\Omega \boldsymbol{f} \cdot \nabla\chi dx = \int_\Omega \boldsymbol{f} \cdot \widetilde{\nabla q^T_j} dx = 0$$

and then

$$\int_\Omega \mathbf{curl}\, \boldsymbol{\xi} \cdot \mathbf{curl}\, \boldsymbol{\varphi}\, dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi}\, dx.$$

Now, we claim that

$$\mathbf{curl}\ \mathbf{curl}\, \boldsymbol{\xi} = \boldsymbol{f} \ \text{ in } \ \Omega \quad \text{ and } \quad \mathbf{curl}\, \boldsymbol{\xi} \times \boldsymbol{n} = \mathbf{0} \ \text{ on } \ \Gamma.$$

Indeed, taking $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$ in (2.17), we deduce immediately the first property. Setting $\boldsymbol{z} = \textbf{curl}\ \boldsymbol{\xi}$, we deduce, in particular that $\boldsymbol{z} \in \boldsymbol{L}^p(\Omega)$ and $\textbf{curl}\ \boldsymbol{z} \in \boldsymbol{L}^{r(p)}(\Omega)$. So, for any $\boldsymbol{\varphi} \in \boldsymbol{X}_T^{p'}(\Omega)$, we have

$$\int_\Omega \textbf{curl}\ \textbf{curl}\ \boldsymbol{\xi} \cdot \boldsymbol{\varphi}\, dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi}\, dx.$$

Using Green formula (2.1) with $\boldsymbol{z} \in \boldsymbol{L}^p(\Omega)$, $\textbf{curl}\ \boldsymbol{z} \in \boldsymbol{L}^{r(p)}(\Omega)$ and $\boldsymbol{\varphi} \in \boldsymbol{W}^{1,p'}(\Omega)$, we deduce that for any $\boldsymbol{\varphi} \in \boldsymbol{X}_T^{p'}(\Omega)$,

$$\langle \textbf{curl}\ \boldsymbol{\xi} \times \boldsymbol{n}, \boldsymbol{\varphi} \rangle_\Gamma = 0.$$

Now, for any element $\boldsymbol{\mu} \in \boldsymbol{W}^{1-\frac{1}{p'},p'}(\Gamma)$, there exists $\boldsymbol{\varphi} \in \boldsymbol{W}^{1,p'}(\Omega)$ such that $\boldsymbol{\varphi} = \boldsymbol{\mu}_\tau$ on Γ, where $\boldsymbol{\mu}_\tau$ is the tangential component of $\boldsymbol{\mu}$ on Γ. Then $\boldsymbol{\varphi}$ belongs to $\boldsymbol{X}_T^{p'}(\Omega)$ and

$$0 = \langle \textbf{curl}\ \boldsymbol{\xi} \times \boldsymbol{n}, \boldsymbol{\varphi} \rangle_\Gamma = \langle \textbf{curl}\ \boldsymbol{\xi} \times \boldsymbol{n}, \boldsymbol{\mu}_\tau \rangle_\Gamma = \langle \textbf{curl}\ \boldsymbol{\xi} \times \boldsymbol{n}, \boldsymbol{\mu} \rangle_\Gamma$$

which implies that

$$\textbf{curl}\ \boldsymbol{\xi} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{in} \quad \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma).$$

ii) We suppose now that $\boldsymbol{f}$ belongs to $\boldsymbol{L}^p(\Omega)$ and let $\boldsymbol{\xi} \in \boldsymbol{W}^{1,p}(\Omega)$ the solution given in i). Then $\boldsymbol{z} = \textbf{curl}\ \boldsymbol{\xi}$ satisfies

$$\boldsymbol{z} \in \boldsymbol{L}^p(\Omega), \quad \text{div}\ \boldsymbol{z} = 0, \quad \textbf{curl}\ \boldsymbol{z} = \boldsymbol{f} \in \boldsymbol{L}^p(\Omega) \quad \text{and} \quad \boldsymbol{z} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{on} \quad \Gamma.$$

Applying Theorem 2.2, we obtain that $\boldsymbol{z} \in \boldsymbol{W}^{1,p}(\Omega)$. As Ω is of class $C^{2,1}$, then we get, by Corollary 3.5 in [4], that $\boldsymbol{\xi}$ belongs to $\boldsymbol{W}^{2,p}(\Omega)$ and satisfies the estimate (2.14). ☐

Remark 2. Using the theory of vector potentials developed in [4], we obtain immediately the regularity (2.13) when Ω is $\mathcal{C}^{1,1}$ and (2.14) in the case Ω is $\mathcal{C}^{2,1}$. But it is possible to prove the regularity $W^{2,p}(\Omega)$ only for Ω of class $\mathcal{C}^{1,1}$ since the problem (E_T) takes the form of an uniformly elliptic operator with complementing boundary conditions in the sense of Agmon-Douglis-Nirenberg [1].

3. Very weak vector potentials

In this section we are interested to study the existence of potential vectors ψ for vector fieled $\boldsymbol{u}$ belonging to negative Sobolev spaces.

As consequence of Theorem 2.3, we have the following remark.

Remark 3. i) Let $\boldsymbol{u} \in \boldsymbol{H}$. Then we have the following equivalence:
$\langle\, \boldsymbol{u} \cdot \boldsymbol{n}\ ,\ 1\, \rangle_{\Sigma_j} = 0$, for any $1 \leqslant j \leqslant J$ if and only if for any $\boldsymbol{\varphi} \in \boldsymbol{K}_T(\Omega)$, $\displaystyle\int_\Omega \boldsymbol{u} \cdot \boldsymbol{\varphi}\, dx = 0$.
ii) Let us define the following spaces

$$\boldsymbol{E} = \left\{ \boldsymbol{\psi} \in \boldsymbol{H}_0^1(\Omega);\ \operatorname{div}(\Delta \boldsymbol{\psi}) = 0 \ \text{ in}\, \Omega,\ \left\langle \frac{\partial}{\partial \boldsymbol{n}} (\operatorname{div} \boldsymbol{\psi}),\ 1 \right\rangle_{\Gamma_i} = 0,\ 1 \leqslant i \leqslant I \right\},$$

and

$$[\boldsymbol{K}_T(\Omega)]^\perp = \left\{ \boldsymbol{v} \in \boldsymbol{H}, \int_\Omega \boldsymbol{v} \cdot \boldsymbol{\varphi} = 0, \quad \forall \boldsymbol{\varphi} \in \boldsymbol{K}_T(\Omega) \right\}. \tag{3.1}$$

It is then clear, by Theorem 2.3, that the following operator

$$\mathbf{curl} :\ \boldsymbol{E} \ \longrightarrow [\boldsymbol{K}_T(\Omega)]^\perp$$

is an isomorphism.

We can then rewrite Theorem (2.3) as follow:

Theorem 3.1. *For any $\boldsymbol{f} \in [\boldsymbol{K}_T(\Omega)]^\perp$ there exists a unique $\boldsymbol{\psi} \in \boldsymbol{E}$ such that*

$$\mathbf{curl}\, \boldsymbol{\psi} = \boldsymbol{f} \quad \text{in } \Omega$$

with the following estimate:

$$\|\boldsymbol{\psi}\|_{\boldsymbol{H}^1(\Omega)} \leqslant C(\Omega) \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}. \tag{3.2}$$

Remark 4. In general, to study the existence of vector potentials, we consider vectors fields in some Lebesgue spaces, for example in $\boldsymbol{L}^2(\Omega)$, with the following compatibility condition:

$$\operatorname{div} \boldsymbol{f} = 0 \ \text{ in } \ \Omega, \quad \text{ and } \quad \langle \boldsymbol{f} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0, \ \ \forall\, 1 \leqslant i \leqslant I.$$

In this case, the vector potential solution $\boldsymbol{\psi}$ belongs to $\boldsymbol{H}^1(\Omega)$. What about if now, the vector field $\boldsymbol{f}$ belongs only to $\boldsymbol{H}^{-1}(\Omega)$?

Let then $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$ with $\operatorname{div} \boldsymbol{f} = 0$ in Ω. We want to study the following problem:

$$(\mathcal{P}) \begin{cases} \text{Find } \ \boldsymbol{\psi} \in [\boldsymbol{K}_T(\Omega)]^\perp \ \text{ such that} \\ \qquad \mathbf{curl}\, \boldsymbol{\psi} = \ \boldsymbol{f} \qquad \text{in } \ \Omega. \end{cases}$$

We remark that if $\boldsymbol{\psi} \in [\boldsymbol{K}_T(\Omega)]^\perp$ is solution of problem $(\mathcal{P})$, then the condition $\operatorname{div} \boldsymbol{f} = 0$ in Ω is clearly necessary. On the other hand, for any $1 \leqslant i \leqslant I$, we have

$$\langle \mathbf{curl}\, \boldsymbol{\psi},\ \nabla q_i \rangle_{\boldsymbol{H}^{-1}(\Omega)\times \boldsymbol{H}_0^1(\Omega)} = 0,$$

because the space

$$\boldsymbol{\mathcal{V}}(\Omega) = \{\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega);\ \operatorname{div} \boldsymbol{\varphi} = 0 \ \text{ in } \Omega\}$$

is dense in the space

$$\left\{\boldsymbol{\varphi} \in \boldsymbol{H}_0^1(\Omega);\ \ \operatorname{div} \boldsymbol{\varphi} \ = 0 \ \text{ in } \Omega\right\}.$$

As consequence, to solve problem $(\mathcal{P})$ we need to suppose that $\boldsymbol{f}$ satisfies the condition

$$\langle \ \boldsymbol{f} \ ,\ \nabla q_i \rangle_{\boldsymbol{H}^{-1}(\Omega)\times \boldsymbol{H}_0^1(\Omega)} = 0, \quad \forall\, 1 \leqslant i \leqslant I. \tag{3.3}$$

Proposition 3.2. *Let* $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$ *with* $\operatorname{div} \boldsymbol{f} = 0$ *in* Ω *and satisfying the condition* (3.3). *Then the problem* $(\mathcal{P})$ *is equivalent to the following very weak variational formulation*

$$(\mathcal{Q}) \begin{cases} \textit{Find } \boldsymbol{\psi} \in [\boldsymbol{K}_T(\Omega)]^\perp \textit{ such that} \\ \forall \boldsymbol{\varphi} \in \boldsymbol{E},\ \displaystyle\int_\Omega \boldsymbol{\psi} \cdot \mathbf{curl}\, \boldsymbol{\varphi} \ \boldsymbol{d}x \ = \ \langle \ \boldsymbol{f} \ ,\ \boldsymbol{\varphi} \ \rangle_{\boldsymbol{H}^{-1}(\Omega)\times \boldsymbol{H}_0^1(\Omega)} \end{cases}$$

Proof. **i)** The implication $(\mathcal{P}) \implies (\mathcal{Q})$ is trivial, because for any $\boldsymbol{\psi} \in [\boldsymbol{K}_T(\Omega)]^\perp$ satisfying $(\mathcal{P})$, and using the density of $\boldsymbol{\mathcal{D}}(\Omega)$ in $\boldsymbol{H}_0^1(\Omega)$, we have

$$\langle \ \mathbf{curl}\, \boldsymbol{\psi},\ \boldsymbol{\varphi} \ \rangle_{\boldsymbol{H}^{-1}(\Omega)\times \boldsymbol{H}_0^1(\Omega)} \ = \int_\Omega \boldsymbol{\psi} \cdot \mathbf{curl}\, \boldsymbol{\varphi}\, dx$$

ii) Conversely, suppose that $\boldsymbol{\psi} \in [\boldsymbol{K}_T(\Omega)]^\perp$ is solution of $(\mathcal{Q})$. Given $\boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega)$, let $\chi \in H_0^2(\Omega)$ the unique solution satisfying

$$\Delta^2 \chi = \operatorname{div}(\Delta \boldsymbol{w}) \quad \text{in } \ \Omega.$$

Setting

$$\boldsymbol{z} = \boldsymbol{w} - \nabla \chi - \sum_{i=1}^{I} \left\langle \frac{\partial}{\partial \boldsymbol{n}} (\operatorname{div} \boldsymbol{w} - \Delta \chi)\ ,\ 1 \right\rangle_{\Gamma_i} \nabla q_i. \tag{3.4}$$

We easily verify that $\boldsymbol{z} \in E$ and

$$\begin{aligned} \langle \mathbf{curl}\, \boldsymbol{\psi} \,,\, \boldsymbol{w} \rangle_{\boldsymbol{H}^{-1}(\Omega)\times \boldsymbol{H}_0^1(\Omega)} &= \int_\Omega \boldsymbol{\psi} \cdot \mathbf{curl}\, \boldsymbol{w}\, dx \\ &= \int_\Omega \boldsymbol{\psi} \cdot \mathbf{curl}\, \boldsymbol{z}\, dx \\ &= \langle\, \boldsymbol{f} \,,\, \boldsymbol{z} \,\rangle_{\boldsymbol{H}^{-1}(\Omega)\times \boldsymbol{H}_0^1(\Omega)} \\ &= \langle\, \boldsymbol{f} \,,\, \boldsymbol{w} \,\rangle_{\boldsymbol{H}^{-1}(\Omega)\times \boldsymbol{H}_0^1(\Omega)}\,. \end{aligned}$$

because $\operatorname{div} \boldsymbol{f} = 0$ in Ω implies that $\langle\, \boldsymbol{f} \,,\, \nabla \chi \,\rangle_{\boldsymbol{H}^{-1}(\Omega)\times \boldsymbol{H}_0^1(\Omega)} = 0$ since we have $\chi \in H_0^2(\Omega)$ and by assumption

$$\langle\, \boldsymbol{f} \,,\, \nabla q_i \,\rangle_{\boldsymbol{H}^{-1}(\Omega)\times \boldsymbol{H}_0^1(\Omega)} = 0, \quad \forall\, 1 \leqslant i \leqslant I.$$

We proved that $\boldsymbol{f} = \mathbf{curl}\, \boldsymbol{\psi}$ and then $\boldsymbol{\psi}$ is solution of Problem $(\mathcal{P})$. □

The following theorem gives an existence and uniqueness result of very weak vector potential and ensures a positive answer to the above question.

Theorem 3.3. *Let* $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$ with $\operatorname{div} \boldsymbol{f} = 0$ in Ω *and satisfying the condition* (3.3)*. Then Problem* $(\mathcal{P})$ *has a unique solution* $\boldsymbol{\psi} \in [\boldsymbol{K}_T(\Omega)]^\perp$ *satisfying*

$$\|\boldsymbol{\psi}\|_{\boldsymbol{L}^2(\Omega)} \leqslant C(\Omega) \|\boldsymbol{f}\|_{\boldsymbol{H}^{-1}(\Omega)}. \tag{3.5}$$

Proof. In fact, we will solve problem $(\mathcal{Q})$ by using a duality argument. For that, let $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$ with $\operatorname{div} \boldsymbol{f} = 0$ in Ω be fixed and let $\mathbf{F} \in [\boldsymbol{K}_T(\Omega)]^\perp$. Theorem 3.1 implies that there exists a unique $\boldsymbol{v} \in \boldsymbol{E}$ such that

$$\mathbf{curl}\, \boldsymbol{v} = \mathbf{F} \quad \text{in } \Omega,$$

satisfying the following estimate

$$\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} \leqslant C(\Omega) \|\mathbf{F}\|_{\boldsymbol{L}^2(\Omega)}\,. \tag{3.6}$$

Considering the linear mapping

$$\begin{aligned} \ell \,:\, [\boldsymbol{K}_T(\Omega)]^\perp &\longrightarrow \mathbb{R} \\ \boldsymbol{F} &\longrightarrow \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{\boldsymbol{H}^{-1}(\Omega)\times \boldsymbol{H}_0^1(\Omega)}\,. \end{aligned}$$

We have

$$| \langle \, \boldsymbol{f} \, , \, \boldsymbol{v} \, \rangle | \leqslant \|\boldsymbol{f}\|_{\boldsymbol{H}^{-1}(\Omega)} \|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} \leqslant C(\Omega) \|\boldsymbol{f}\|_{\boldsymbol{H}^{-1}(\Omega)} \|\mathbf{F}\|_{\boldsymbol{L}^2(\Omega)}.$$

So the linear form ℓ is continuous on the Hilbert space $[\boldsymbol{K}_T(\Omega)]^{\perp}$. By Riesz theorem, we deduce the existence of a unique $\boldsymbol{\psi} \in [\boldsymbol{K}_T(\Omega)]^{\perp}$ such that for any $\boldsymbol{v} \in \boldsymbol{E}$,

$$\int_\Omega \mathbf{F} \cdot \boldsymbol{\psi} \, dx \; = \langle \, \boldsymbol{f} \, , \, \boldsymbol{v} \, \rangle$$

and satisfying the estimate (3.5). □

Now, if the data $\boldsymbol{f}$ belongs to $\boldsymbol{L}^{\frac{6}{5}}(\Omega)$ which is a subspace of $\boldsymbol{H}^{-1}(\Omega)$, with additional condition on her normal trace, we will prove

Theorem 3.4. *Let* $\boldsymbol{f} \in \boldsymbol{L}^{\frac{6}{5}}(\Omega)$, *and satisfying*

$$\mathrm{div} \boldsymbol{f} = 0 \;\; in \;\; \Omega, \quad \boldsymbol{f} \cdot \boldsymbol{n} = 0 \;\; on \;\; \Gamma \;\; and \quad \langle \boldsymbol{f} \cdot \boldsymbol{n}, 1 \rangle_{\sum_j} = 0, \;\; \forall \, 1 \leqslant j \leqslant J. \tag{3.7}$$

i) *There exists a unique* $\boldsymbol{v} \in \boldsymbol{W}^{1,\frac{6}{5}}(\Omega)$ *with* $\mathrm{div}\, \boldsymbol{v} = 0$ *in* Ω *and such that*

$$\mathbf{curl}\, \boldsymbol{v} = \boldsymbol{f} \;\; in \;\; \Omega, \quad \boldsymbol{v} \times \boldsymbol{n} = \mathbf{0} \;\; on \;\; \Gamma \;\; and \;\; \langle \boldsymbol{v} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0 \;\; \forall \; 1 \leqslant i \leqslant I.$$

and satisfying the following estimate:

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,\frac{6}{5}}(\Omega)} \; \leqslant \; C(\Omega) \|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)}. \tag{3.8}$$

ii) *Moreover, we have*

$$\|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} \leqslant C(\Omega) \|\boldsymbol{f}\|_{[\boldsymbol{H}^1_\tau(\Omega)]'}, \tag{3.9}$$

where $\left[\boldsymbol{H}^1_\tau(\Omega)\right]'$ *is the dual space of*

$$\boldsymbol{H}^1_\tau(\Omega) = \left\{ \boldsymbol{w} \in \boldsymbol{H}^1(\Omega) \;\; ; \; \boldsymbol{w} \cdot \boldsymbol{n} = 0 \;\; \text{on} \;\; \Gamma \right\}.$$

Proof. The proof is divided into five steps.

Step 1. Let $\boldsymbol{f}$ be satisfying the above hypothesis. Then $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$ and

$$\langle \boldsymbol{f}, \nabla q_i \rangle_{\boldsymbol{H}^{-1}(\Omega) \times \boldsymbol{H}^1_0(\Omega)} = \int_\Omega \boldsymbol{f} \cdot \nabla q_i \; dx = 0.$$

By Theorem 3.3 , there exists a unique $\boldsymbol{\psi}_0 \in [\boldsymbol{K}_T(\Omega)]^{\perp}$ such that

$$\mathbf{curl}\,\boldsymbol{\psi}_0 = \boldsymbol{f} \quad \text{in} \quad \Omega \tag{3.10}$$

and

$$\|\boldsymbol{\psi}_0\|_{\boldsymbol{L}^2(\Omega)} \leqslant C(\Omega)\|\boldsymbol{f}\|_{\boldsymbol{H}^{-1}(\Omega)} \leqslant C(\Omega)\|\boldsymbol{f}\|_{[\boldsymbol{H}^1_\tau(\Omega)]'} \,. \tag{3.11}$$

As $\boldsymbol{f} \in \boldsymbol{L}^{\frac{6}{5}}(\Omega)$ and Ω is of class $C^{1,1}$, then by Theorem 2.2 point i), $\boldsymbol{\psi}_0 \in \boldsymbol{W}^{1,\frac{6}{5}}(\Omega)$ and we have the following estimate

$$\|\boldsymbol{\psi}_0\|_{\boldsymbol{W}^{1,\frac{6}{5}}(\Omega)} \leqslant C\left(\|\boldsymbol{\psi}_0\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)}\right) \leqslant C\,\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)}. \tag{3.12}$$

Step 2. Using Lax-Milgram theorem, we know that the following problem:

$$\begin{cases} \text{Find } \boldsymbol{\xi} \in \boldsymbol{V}^2_\Sigma(\Omega) \quad \text{such that,} \\ \forall \boldsymbol{\varphi} \in \boldsymbol{V}^2_\Sigma(\Omega),\ \displaystyle\int_\Omega \mathbf{curl}\,\boldsymbol{\xi} \cdot \mathbf{curl}\,\boldsymbol{\varphi}\,dx = \int_\Omega \boldsymbol{\psi}_0 \cdot \mathbf{curl}\,\boldsymbol{\varphi}\,dx - \int_\Omega \mathbf{curl}\,\boldsymbol{\psi}_0 \cdot \boldsymbol{\varphi}\,dx \end{cases} \tag{3.13}$$

has a unique solution. The coercivity in $\boldsymbol{V}^2_\Sigma(\Omega)$ of the above bilinear form is due to the equivalence

$$\|\boldsymbol{\xi}\|_{\boldsymbol{H}^1(\Omega)} \ \simeq\ \|\mathbf{curl}\,\boldsymbol{\xi}\|_{\boldsymbol{L}^2(\Omega)}, \tag{3.14}$$

(see Theorem 2.1 point **ii)**).

The variational formulation (3.13) in fact is equivalent to the following:

$$\begin{cases} \text{Find } \boldsymbol{\xi} \in \boldsymbol{V}^2_\Sigma(\Omega) \quad \text{such that,} \\ \forall \boldsymbol{\varphi} \in \boldsymbol{X}^2_T(\Omega),\ \displaystyle\int_\Omega \mathbf{curl}\,\boldsymbol{\xi} \cdot \mathbf{curl}\,\boldsymbol{\varphi}\,dx = \int_\Omega \boldsymbol{\psi}_0 \cdot \mathbf{curl}\,\boldsymbol{\varphi}\,dx - \int_\Omega \mathbf{curl}\,\boldsymbol{\psi}_0 \cdot \boldsymbol{\varphi}\,dx \end{cases} \tag{3.15}$$

Clearly (3.15) implies (3.13).

Conversely, let be $\boldsymbol{\xi} \in \boldsymbol{V}^2_\Sigma(\Omega)$ solution of (3.13) and $\boldsymbol{\varphi} \in \boldsymbol{X}^2_T(\Omega)$. As in the proof of Theorem 2.4, with the same notations, we get

$$\begin{aligned} \int_\Omega \mathbf{curl}\,\boldsymbol{\xi} \cdot \mathbf{curl}\,\boldsymbol{\varphi}\,dx \quad &= \quad \int_\Omega \mathbf{curl}\,\boldsymbol{\xi} \cdot \mathbf{curl}\,\widetilde{\boldsymbol{\varphi}}\,dx \\ &= \quad \int_\Omega \boldsymbol{\psi}_0 \cdot \mathbf{curl}\,\boldsymbol{\varphi}\,dx \ - \int_\Omega \mathbf{curl}\,\boldsymbol{\psi}_0 \cdot \boldsymbol{\varphi}\,dx \end{aligned}$$

because

$$\int_\Omega \textbf{curl}\,\boldsymbol{\psi}_0 \cdot \nabla\chi \, dx = \int_\Omega \textbf{curl}\,\boldsymbol{\psi}_0 \cdot \widetilde{\nabla q_j^T} \, dx = 0,$$

where the second identity holds thanks to (3.7), since from Lemma 3.10 in [2] we have

$$\int_\Omega \textbf{curl}\,\boldsymbol{\psi}_0 \cdot \widetilde{\nabla q_j^T} \, dx = \int_\Omega \boldsymbol{f} \cdot \nabla q_j^T \, dx = \sum_j \left\langle \boldsymbol{f}\cdot\boldsymbol{n}, \left[q_j^T\right]_j \right\rangle_{\Sigma_j} = \sum_j \left[q_j^T\right]_j \langle \boldsymbol{f}\cdot\boldsymbol{n}, 1\rangle_{\Sigma_j} = 0.$$

Step 3. Now, we will prove that

$$\textbf{curl}\,\textbf{curl}\,\boldsymbol{\xi} = \boldsymbol{0} \quad \text{in} \quad \Omega \quad \text{and} \quad \textbf{curl}\,\boldsymbol{\xi}\times\boldsymbol{n} = \boldsymbol{\psi}_0\times\boldsymbol{n} \quad \text{on} \quad \Gamma$$

Indeed, taking $\boldsymbol{\varphi}\in\mathcal{D}(\Omega)$ in (3.15), we deduce immediately the first property. Now taking $\boldsymbol{\varphi}\in \boldsymbol{X}_T^2(\Omega)$, we get after using Green formula (2.1)

$$\langle(\textbf{curl}\,\boldsymbol{\xi}-\boldsymbol{\psi}_0)\times\boldsymbol{n}, \boldsymbol{\varphi}\rangle_\Gamma = 0.$$

Now, for any $\boldsymbol{\mu}\in \boldsymbol{H}^{\frac{1}{2}}(\Omega)$, there exists $\boldsymbol{\varphi}\in\boldsymbol{X}_T^2(\Omega)$ such that $\boldsymbol{\varphi}=\boldsymbol{\mu}_\tau$ on Γ. So,

$$0 = \langle(\textbf{curl}\,\boldsymbol{\xi}-\boldsymbol{\psi}_0)\times\boldsymbol{n}, \boldsymbol{\varphi}\rangle_\Gamma = \langle(\textbf{curl}\,\boldsymbol{\xi}-\boldsymbol{\psi}_0)\times\boldsymbol{n}, \boldsymbol{\mu}_\tau\rangle_\Gamma = \langle(\textbf{curl}\,\boldsymbol{\xi}-\boldsymbol{\psi}_0)\times\boldsymbol{n}, \boldsymbol{\mu}\rangle_\Gamma$$

which means that

$$(\textbf{curl}\,\boldsymbol{\xi}-\boldsymbol{\psi}_0)\times\boldsymbol{n} = \boldsymbol{0} \quad \text{in} \quad \boldsymbol{H}^{-\frac{1}{2}}(\Gamma). \tag{3.16}$$

Step 4. As in [2], we define the following vector field

$$\boldsymbol{v} = \boldsymbol{\psi}_0 - \textbf{curl}\,\boldsymbol{\xi} - \sum_{i=1}^{I} \langle\, (\boldsymbol{\psi}_0 - \textbf{curl}\,\boldsymbol{\xi})\cdot\boldsymbol{n}\ ,\ 1\,\rangle_{\Gamma_i} \nabla q_i^N, \tag{3.17}$$

where $\left\{\nabla q_i^N\right\}_{1\leqslant i\leqslant I}$ is the basis of the space $\boldsymbol{K}_N(\Omega)$ given in (2.3).

Now, we will verify that $\boldsymbol{v}$ satisfies the statements required in point i) of Theorem 3.4. Setting $\boldsymbol{z}=\textbf{curl}\,\boldsymbol{\xi}$, we will firstly show that $\boldsymbol{z}\in\boldsymbol{W}^{1,\frac{6}{5}}(\Omega)$. Note that we have

$$\textbf{curl}\,\boldsymbol{z} = -\Delta\boldsymbol{\xi} \quad \text{in} \quad \Omega \quad \text{and} \quad \boldsymbol{z}\times\boldsymbol{n} = \boldsymbol{\psi}_0\times\boldsymbol{n} \quad \text{on} \quad \Gamma.$$

Then $\boldsymbol{z}\in\boldsymbol{L}^2(\Omega)$ and verifies

$$\operatorname{div}\boldsymbol{z} = 0, \quad \textbf{curl}\,\boldsymbol{z} = \boldsymbol{0} \ \text{in}\ \Omega \ \text{and} \ \boldsymbol{z}\times\boldsymbol{n}\in\boldsymbol{W}^{\frac{1}{6},\frac{5}{6}}(\Gamma),$$

since $\boldsymbol{\psi}_0 \in \boldsymbol{W}^{1,\frac{6}{5}}(\Omega)$ and Ω is of class $C^{1,1}$. By Theorem 2.2 point ii), we deduce that $\boldsymbol{z} \in \boldsymbol{W}^{1,\frac{6}{5}}(\Omega)$. Moreover $\boldsymbol{z}$ satisfies the following estimate

$$
\begin{aligned}
\|\boldsymbol{z}\|_{\boldsymbol{W}^{1,\frac{6}{5}}(\Omega)} &\leq C\left(\|\boldsymbol{z}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbf{curl}\,\boldsymbol{z}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\boldsymbol{z}\times\boldsymbol{n}\|_{\boldsymbol{W}^{\frac{1}{6},\frac{6}{5}}(\Omega)}\right)\\
&\leq C\left(\|\boldsymbol{z}\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\boldsymbol{\psi}_0\|_{\boldsymbol{W}^{1,\frac{6}{5}}(\Omega)}\right) \qquad (3.18)
\end{aligned}
$$

Step 5. We are in position to prove the estimates (3.8) and (3.9). For that, taking $\boldsymbol{\varphi} = \boldsymbol{\xi}$ in (3.13), we get thanks to (3.10)

$$
\begin{aligned}
\|\mathbf{curl}\,\boldsymbol{\xi}\|^2_{\boldsymbol{L}^2(\Omega)} &= \int_\Omega \boldsymbol{\psi}_0 \cdot \mathbf{curl}\,\boldsymbol{\xi}\,dx \;-\; \int_\Omega \boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{\psi}_0\,dx\\
&= \int_\Omega \boldsymbol{\psi}_0 \cdot \mathbf{curl}\,\boldsymbol{\xi}\,dx \;-\; \langle\,\boldsymbol{f}\;,\;\boldsymbol{\xi}\,\rangle_{(\boldsymbol{H}^1_\tau(\Omega))'\times\boldsymbol{H}^1_\tau(\Omega)}.
\end{aligned}
$$

Then

$$
\|\mathbf{curl}\,\boldsymbol{\xi}\|^2_{\boldsymbol{L}^2(\Omega)} \leqslant \|\boldsymbol{\psi}_0\|_{\boldsymbol{L}^2(\Omega)}\|\mathbf{curl}\,\boldsymbol{\xi}\|_{\boldsymbol{L}^2(\Omega)} \;+\; \|\boldsymbol{f}\|_{[\boldsymbol{H}^1_\tau(\Omega)]'}\|\boldsymbol{\xi}\|_{\boldsymbol{H}^1(\Omega)}
$$

Now, as $\boldsymbol{\xi} \in X^2_T(\Omega)$, we deduce from (3.14) that

$$
\|\mathbf{curl}\,\boldsymbol{\xi}\|_{\boldsymbol{L}^2(\Omega)} \leqslant C\left(\;\|\boldsymbol{\psi}_0\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{f}\|_{[\boldsymbol{H}^1_\tau(\Omega)]'}\;\right) \leq C\|\boldsymbol{f}\|_{[\boldsymbol{H}^1_\tau(\Omega)]'}. \qquad (3.19)
$$

Consequently, by (3.17), we have

$$
\|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} \leqslant \;\|\boldsymbol{\psi}_0 - \mathbf{curl}\,\boldsymbol{\xi}\|_{\boldsymbol{L}^2(\Omega)} + \|\sum_{i=1}^{I}\langle(\boldsymbol{\psi}_0 - \mathbf{curl}\,\boldsymbol{\xi})\cdot\boldsymbol{n}\;1\,\rangle_{\Gamma_i}\nabla q_i^N\;\|_{\boldsymbol{L}^2(\Omega)}. \qquad (3.20)
$$

But note that for every $\boldsymbol{h} \in \boldsymbol{L}^2(\Omega)$, with $\operatorname{div}\boldsymbol{h} = 0$, we have

$$
\|\sum_{i=1}^{I}\langle\,\boldsymbol{h}\cdot\boldsymbol{n}\;,\;1\,\rangle_{\Gamma_i}\nabla q_i^N\;\|_{\boldsymbol{L}^2(\Omega)} \;\leqslant\; \sum_{i=1}^{I}|\langle\boldsymbol{h}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}|\,\|\nabla \boldsymbol{q}_i^N\|_{\boldsymbol{L}^2(\Omega)} \leqslant C\sum_{i=1}^{I}\|\boldsymbol{h}\cdot\boldsymbol{n}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma_i)}
$$

and

$$
\sum_{i=1}^{I}\|\boldsymbol{h}\cdot\boldsymbol{n}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma_i)} = \|\boldsymbol{h}\cdot\boldsymbol{n}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \leqslant C\|\boldsymbol{h}\|_{\boldsymbol{L}^2(\Omega)}.
$$

Applying these inequalities for $\boldsymbol{h} = \boldsymbol{\psi}_0 - \mathbf{curl}\,\boldsymbol{\xi}$ and using estimates (3.11), (3.19) and (3.20) we get the estimate (3.9). Finally, using (3.18), (3.11), (3.17) and (3.19) we deduce the estimate (3.8). $\square$

4. Application to MHD problem

In this section we will establish the existence of weak solution for the (MHD) equations. We will apply Leray-Schauder fixed theorem and the results obtained in Section 3 for weak and very weak vector potentials.

We define the following Hilbert spaces

$$\boldsymbol{V} = \left\{\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega); \quad \operatorname{div} \boldsymbol{u} = 0 \ \text{ in } \Omega\right\},$$
$$\boldsymbol{W} = \{\boldsymbol{B} \in \boldsymbol{H}^1(\Omega); \ \operatorname{div} \boldsymbol{B} = 0 \ \text{ in } \Omega,\ \boldsymbol{B}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma, \quad \int_{\Sigma_j} \boldsymbol{B}\cdot\ \boldsymbol{n} = 0,\ 1 \leqslant j \leqslant J\},$$
$$\boldsymbol{Z} = \ \boldsymbol{V} \times \boldsymbol{W},$$

and we set

$$\|\,(\boldsymbol{u},\boldsymbol{B})\,\|_{\boldsymbol{Z}} \ = \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\boldsymbol{B}\|_{\boldsymbol{H}^1(\Omega)}.$$

Theorem 4.1. *Let* $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$, $\ \boldsymbol{k} \in \boldsymbol{L}^{\frac{6}{5}}(\Omega)$ *with*

$$\operatorname{div} \boldsymbol{k} = 0 \ \textit{in } \Omega, \quad \boldsymbol{k}\cdot\boldsymbol{n} = 0 \ \textit{on } \Gamma \quad \textit{and} \quad \int_\Omega \boldsymbol{k}\cdot\boldsymbol{\varphi}\, dx = 0 \quad \forall \boldsymbol{\varphi} \in \boldsymbol{K}_T(\Omega) \tag{4.1}$$

Then Problem (MHD) has at least one weak solution

$$(\boldsymbol{u},\boldsymbol{B},\pi) \in \boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^1(\Omega) \times L^2(\Omega)$$

satisfying the following estimate

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\boldsymbol{B}\|_{\boldsymbol{H}^1(\Omega)} \leqslant C\left(\|\boldsymbol{f}\|_{\boldsymbol{H}^{-1}(\Omega)} + \|\boldsymbol{k}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)}\right) \tag{4.2}$$

Moreover

$$\boldsymbol{B} \in \boldsymbol{W}^{2,\frac{6}{5}}(\Omega).$$

Proof. **i) Necessary condition.** Let $(\boldsymbol{u},\boldsymbol{B},\pi) \in \boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^1(\Omega) \times L^2(\Omega)$ solution of Problem (MHD). We firstly observe that since div $\boldsymbol{u} =$ div $\boldsymbol{B} = 0$, then

$$\textbf{curl}\ (\boldsymbol{u} \times \boldsymbol{B}) = \boldsymbol{B}\cdot\nabla\boldsymbol{u} - \boldsymbol{u}\cdot\nabla\boldsymbol{B}. \tag{4.3}$$

Setting $\boldsymbol{\psi} = \lambda\,\textbf{curl}\,\boldsymbol{B} - \boldsymbol{u}\times\boldsymbol{B}$, we have $\boldsymbol{k} = \textbf{curl}\,\boldsymbol{\psi}$ with $\boldsymbol{\psi} \in \boldsymbol{L}^2(\Omega)$, $\textbf{curl}\,\boldsymbol{\psi} \in \boldsymbol{L}^{\frac{6}{5}}(\Omega)$ and $\boldsymbol{\psi}\times\boldsymbol{n} = \boldsymbol{0}$ in $\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$. So, div $\boldsymbol{k} = 0$ in Ω and $\boldsymbol{k}\times\boldsymbol{n} \in \boldsymbol{W}^{-\frac{5}{6},\frac{6}{5}}(\Gamma)$. We will prove that

$$\boldsymbol{k}\cdot\boldsymbol{n} = 0 \ \text{ on } \ \Gamma. \tag{4.4}$$

Indeed for any $\chi \in H^2(\Omega)$ we have

$$\int_\Omega \mathbf{curl}\, \boldsymbol{\psi} \cdot \nabla\chi \; dx = \langle \boldsymbol{k}\cdot\boldsymbol{n}, \chi\rangle_{W^{-\frac{5}{6},\frac{6}{5}}(\Gamma)\times W^{\frac{5}{6},6}(\Gamma)} \tag{4.5}$$

and

$$\int_\Omega \mathbf{curl}\, \boldsymbol{\psi} \cdot \nabla\chi \; dx = -\langle \boldsymbol{\psi}\times\boldsymbol{n}, \nabla\chi\rangle_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)\times \boldsymbol{H}^{\frac{1}{2}}(\Gamma)} = 0 \tag{4.6}$$

because when χ describes $H^2(\Omega)$, $\chi_{/\Gamma}$ describes $H^{\frac{3}{2}}(\Gamma)$.
That means that $\boldsymbol{k}\cdot\boldsymbol{n} = 0$ in $H^{-\frac{3}{2}}(\Gamma)$ and also in $W^{-\frac{5}{6},\frac{6}{5}}(\Gamma)$.

We also have by Green formula, for all $\boldsymbol{\varphi} \in \boldsymbol{K}_T(\Omega)$,

$$\int_\Omega \boldsymbol{k}\cdot\boldsymbol{\varphi}\; dx = \lambda \int_\Omega \boldsymbol{\psi}\cdot \mathbf{curl}\,\boldsymbol{\varphi}\; dx \; + \; \lambda\int_\Gamma (\boldsymbol{\psi}\times\boldsymbol{n})\cdot\boldsymbol{\varphi}\; dx = 0.$$

ii) Existence. We will use Leray-Schauder fixed point theorem to show the existence of weak solution. For proving the compactness, the idea is to apply the estimates of the very week vector potential obtained in Section 3.

Let $(\boldsymbol{u}, \boldsymbol{B}) \in \boldsymbol{Z}$ given and let us consider the following system denoted by $\widehat{MHD}$:

$$\begin{cases} -\nu\Delta\hat{\boldsymbol{u}} + \dfrac{1}{\rho}\nabla\hat{\pi} = \boldsymbol{f} - (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u} + \dfrac{1}{\rho\mu}(\boldsymbol{B}\cdot\nabla)\,\boldsymbol{B} - \dfrac{1}{2\rho\mu}\nabla\left(|\boldsymbol{B}|^2\right) & \text{in } \Omega, \\ -\lambda\Delta\hat{\boldsymbol{B}} = \boldsymbol{k} + (\boldsymbol{B}\cdot\nabla)\,\boldsymbol{u} - (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{B} & \text{in } \Omega, \\ \operatorname{div}\hat{\boldsymbol{u}} = \operatorname{div}\hat{\boldsymbol{B}} = 0 & \text{in } \Omega, \\ \hat{\boldsymbol{u}} = \boldsymbol{0}, \;\; \hat{\boldsymbol{B}}\cdot\boldsymbol{n} = 0, \;\; \mathbf{curl}\,\hat{\boldsymbol{B}}\times\boldsymbol{n} \; = \; \boldsymbol{0} & \text{on } \Gamma, \\ \displaystyle\int_{\Sigma_j} \hat{\boldsymbol{B}}\cdot\boldsymbol{n} \; = \; 0, \qquad\qquad 1 \leqslant j \leqslant J. & \end{cases}$$

Because the RHS of the first equation belongs to $\boldsymbol{H}^{-1}(\Omega)$, we know that there exists a unique solution $(\hat{\boldsymbol{u}}, \hat{\pi}) \in \boldsymbol{H}_0^1(\Omega \times L^2(\Omega)/\mathbb{R}$, with div $\hat{\boldsymbol{u}} = \boldsymbol{0}$ in Ω. The RHS of the second equation belongs to $\boldsymbol{L}^{\frac{6}{5}}(\Omega)$ and it is with divergence free thanks to (4.3).

Now we need to verify that this RHS satisfies the compatibility conditions (2.11)-(2.12). Indeed, we observe that for any $\boldsymbol{\varphi} \in \boldsymbol{K}_T(\Omega)$, we have:

$$\int_\Omega [(\boldsymbol{B}\cdot\nabla)\boldsymbol{u} - (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{B}]\cdot\boldsymbol{\varphi} = \int_\Omega (\boldsymbol{u}\otimes\boldsymbol{B} - \boldsymbol{B}\otimes\boldsymbol{u}) : \nabla\boldsymbol{\varphi} = 0$$

because

$$\int_\Omega \boldsymbol{B}\otimes\boldsymbol{u} : \nabla\boldsymbol{\varphi} = \int_\Omega B_k u_i \frac{\partial\varphi_i}{\partial x_k} = \int_\Omega B_i u_k \frac{\partial\varphi_k}{\partial x_i} = \int_\Omega B_i u_k \frac{\partial\varphi_i}{\partial x_k} = \int_\Omega \boldsymbol{u}\otimes\boldsymbol{B} : \nabla\boldsymbol{\varphi}.$$

Above, we used the implicit summation on the repeated induces and the fact that $\mathbf{curl}\,\boldsymbol{\varphi} = \mathbf{0}$, which proves the first compatibility condition (2.11). Now, as $\boldsymbol{B} \cdot \boldsymbol{n} = 0$ and $\boldsymbol{u} = \mathbf{0}$ on Γ, we have

$$(\boldsymbol{B} \cdot \nabla)\boldsymbol{u} = (\boldsymbol{B}_\tau \cdot \nabla_\tau)\boldsymbol{u} = \mathbf{0} \quad \text{on } \Gamma \tag{4.7}$$

and of course $(\boldsymbol{u} \cdot \nabla)\boldsymbol{B} = \mathbf{0}$ on Γ. Then

$$[\boldsymbol{k} + (\boldsymbol{B} \cdot \nabla)\,\boldsymbol{u} - (\boldsymbol{u} \cdot \nabla)\,\boldsymbol{B}] \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \Gamma,$$

which proves the compatibility condition (2.12).

We are now in position to use Theorem 2.4, there exists a unique solution $\hat{\boldsymbol{B}} \in \boldsymbol{H}^1(\Omega)$ satisfying the corresponding equations of $\widehat{MHD}$.

Let us consider the following operator

$$T : \boldsymbol{Z} \longrightarrow \boldsymbol{Z} \times L^2(\Omega)/\mathbb{R} \longrightarrow \boldsymbol{Z}$$
$$(\boldsymbol{u}, \boldsymbol{B}) \longmapsto (\hat{\boldsymbol{u}}, \hat{\boldsymbol{B}}, \widehat{\pi}) \longmapsto (\hat{\boldsymbol{u}}, \hat{\boldsymbol{B}})$$

where $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{B}}, \widehat{\pi})$ is the unique weak solution of $\widehat{MHD}$. We realize that a fixed point of the operator T is a weak solution of (MHD). So, we must prove that T is a compact operator on $\boldsymbol{Z}$ and

$$\begin{cases} \exists C > 0 \text{ such that } \|(\boldsymbol{u}, \boldsymbol{B})\|_{\boldsymbol{Z}} \leqslant C, \ \forall (\boldsymbol{u}, \boldsymbol{B}) \in \boldsymbol{Z} \\ \text{and } \forall\, \alpha \in [0,1] \text{ such that } (\boldsymbol{u}, \boldsymbol{B}) = \alpha T(\boldsymbol{u}, \boldsymbol{B}), \end{cases} \tag{4.8}$$

(see [9], Theorem 11.3).

1/ Let us prove that T is compact. Suppose $(\boldsymbol{u}, \boldsymbol{B}) \in \boldsymbol{Z}$, we consider the sequence $(\boldsymbol{u}_k, \boldsymbol{B}_k)_{n \in \mathbb{N}} \in \boldsymbol{Z}$ such that

$$(\boldsymbol{u}_k, \boldsymbol{B}_k) \rightharpoonup (\boldsymbol{u}, \boldsymbol{B}) \ \text{ in } \boldsymbol{Z} - \text{weak}.$$

Let us define $(\hat{\boldsymbol{u}}_k, \hat{\boldsymbol{B}}_k) = T(\boldsymbol{u}_k, \boldsymbol{B}_k)$ for all $n \in \mathbb{N}$. Then $(\hat{\boldsymbol{u}}_k - \hat{\boldsymbol{u}},\ \hat{\boldsymbol{B}}_k - \hat{\boldsymbol{B}})$ satisfies the following system denoted by $(MHD)_k$

$$\begin{cases} -\nu\Delta(\hat{\boldsymbol{u}}_k - \hat{\boldsymbol{u}}) + \dfrac{1}{\rho}\nabla(\hat{\pi}_k - \hat{\pi}) = (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u} - (\boldsymbol{u}_k\cdot\nabla)\,\boldsymbol{u}_k + & \\ +\dfrac{1}{\rho\mu}\left[(\boldsymbol{B}_k\cdot\nabla)\,\boldsymbol{B}_k - (\boldsymbol{B}\cdot\nabla)\,\boldsymbol{B}\right] + \dfrac{1}{2\rho\mu}\left[\nabla\left(|\boldsymbol{B}_k|^2 - |\boldsymbol{B}|^2\right)\right] & \text{in} \quad \Omega, \\ -\lambda\Delta(\hat{\boldsymbol{B}}_k - \hat{\boldsymbol{B}}) = (\boldsymbol{B}_k\cdot\nabla)\,\boldsymbol{u}_k - (\boldsymbol{B}\cdot\nabla)\,\boldsymbol{u} - (\boldsymbol{u}_k\cdot\nabla)\,\boldsymbol{B}_k + (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{B} & \text{in} \quad \Omega, \\ \operatorname{div}(\hat{\boldsymbol{u}}_n - \hat{\boldsymbol{u}}) \;= 0 & \text{in} \quad \Omega, \\ \operatorname{div}(\hat{\boldsymbol{B}}_k - \hat{\boldsymbol{B}}) \;= 0 & \text{in} \quad \Omega, \\ \hat{\boldsymbol{u}}_k - \hat{\boldsymbol{u}} = \boldsymbol{0} & \text{on} \quad \Gamma, \\ (\hat{\boldsymbol{B}}_k - \hat{\boldsymbol{B}})\cdot\boldsymbol{n} = 0, \quad \mathbf{curl}\,(\hat{\boldsymbol{B}}_k - \hat{\boldsymbol{B}})\times\boldsymbol{n} \;= \boldsymbol{0} & \text{on} \quad \Gamma, \\ \displaystyle\int_{\Sigma_j}(\hat{\boldsymbol{B}}_k - \hat{\boldsymbol{B}})\cdot\boldsymbol{n} = \; 0 & 1 \leqslant j \leqslant J. \end{cases}$$

We will prove that

$$\hat{\boldsymbol{u}}_k \longrightarrow \hat{\boldsymbol{u}} \quad \text{and} \quad \hat{\boldsymbol{B}}_k \longrightarrow \hat{\boldsymbol{B}} \;\text{ in }\; \boldsymbol{H}^1(\Omega), \quad \text{as } k \to \infty.$$

i) By applying usual estimates for weak solutions to the Stokes problem, we have

$$\nu\|\hat{\boldsymbol{u}}_k - \hat{\boldsymbol{u}}\|_{\boldsymbol{H}^1(\Omega)} \;\leqslant C\|\,(\boldsymbol{u}_k\cdot\nabla)\,\boldsymbol{u}_k - (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u}\|_{\boldsymbol{H}^{-1}(\Omega)} +$$
$$+\;\|\,(\boldsymbol{B}_k\cdot\nabla)\,\boldsymbol{B}_k - (\boldsymbol{B}\cdot\nabla)\,\boldsymbol{B}\|_{\boldsymbol{H}^{-1}(\Omega)} + \|\nabla\left(|\boldsymbol{B}_k|^2 - |\boldsymbol{B}|^2\right)\|_{\boldsymbol{H}^{-1}(\Omega)}.$$

Note that for any $(\boldsymbol{v},\boldsymbol{w})\in\boldsymbol{V}$, we have $(\boldsymbol{v}\cdot\nabla)\boldsymbol{w} = \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{w})$. So, we get

$$\begin{aligned} \|\,(\boldsymbol{u}_k\cdot\nabla)\,\boldsymbol{u}_k - (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u}\|_{\boldsymbol{H}^{-1}(\Omega)} &= \|\operatorname{div}\left((\boldsymbol{u}_k\otimes\boldsymbol{u}_k) - (\boldsymbol{u}\otimes\boldsymbol{u})\right)\|_{\boldsymbol{H}^{-1}(\Omega)} \\ &\leqslant C\|\boldsymbol{u}_k\otimes\boldsymbol{u}_k - \boldsymbol{u}\otimes\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \\ &\leqslant C\left(\|\boldsymbol{u}_k\|_{\boldsymbol{L}^6(\Omega)} + \|\boldsymbol{u}\|_{\boldsymbol{L}^6(\Omega)}\right)\|\boldsymbol{u}_k - \boldsymbol{u}\|_{\boldsymbol{L}^3(\Omega)} \\ &\leqslant C\|\boldsymbol{u}_k - \boldsymbol{u}\|_{\boldsymbol{L}^3(\Omega)} \longrightarrow 0, \text{ as } \; k\to\infty \end{aligned}$$

because $\boldsymbol{H}^1(\Omega)\hookrightarrow\boldsymbol{L}^6(\Omega)$ and the embedding $\boldsymbol{H}^1(\Omega)\hookrightarrow\boldsymbol{L}^3(\Omega)$ is compact. With similar arguments, we show that

$$\|\,(\boldsymbol{B}_k\cdot\nabla)\,\boldsymbol{B}_k - (\boldsymbol{B}\cdot\nabla)\,\boldsymbol{B}\|_{\boldsymbol{H}^{-1}(\Omega)} + \;\|\nabla\left(|\boldsymbol{B}_k|^2 - |\boldsymbol{B}_k|^2\right)\|_{\boldsymbol{H}^{-1}(\Omega)} \longrightarrow 0 \;\text{ as }\; k\to\infty.$$

Therefore

$$\hat{\boldsymbol{u}}_k \longrightarrow \hat{\boldsymbol{u}} \;\text{ in }\; \boldsymbol{H}^1(\Omega) \quad \text{ as } \; k\to\infty. \tag{4.9}$$

ii) Now, we will show that $\hat{\boldsymbol{B}}_k \to \hat{\boldsymbol{B}}$ in $\boldsymbol{H}^1(\Omega)$ as $k \to \infty$.

Setting now

$$\boldsymbol{F}_k = \boldsymbol{f}_k - \boldsymbol{g}_k,$$

where

$$\boldsymbol{f}_k = (\boldsymbol{B}_k \cdot \nabla)\boldsymbol{u}_k - (\boldsymbol{B} \cdot \nabla)\boldsymbol{u}, \qquad \boldsymbol{g}_k = (\boldsymbol{u}_k \cdot \nabla)\boldsymbol{B}_k - (\boldsymbol{u} \cdot \nabla)\boldsymbol{B}. \tag{4.10}$$

and

$$\boldsymbol{z}_k = \lambda\, \mathbf{curl}\, (\hat{\boldsymbol{B}}_k - \hat{\boldsymbol{B}}), \tag{4.11}$$

we have

$$\boldsymbol{z}_k \in \boldsymbol{L}^2(\Omega), \quad \text{div } \boldsymbol{z}_k = 0, \quad \mathbf{curl}\ \boldsymbol{z}_k = \boldsymbol{F}_k \ \text{ in } \Omega \ \text{ and } \boldsymbol{z}_k \times \boldsymbol{n} = \boldsymbol{0} \ \text{ on } \ \Gamma.$$

Because (4.11), we deduce from Lemma 3.5 of [2] that

$$\forall \ \ 1 \leqslant i \leqslant I, \quad \langle\, \boldsymbol{z}_k \cdot \boldsymbol{n}\ ,\ 1\, \rangle_{\Gamma_i} = 0.$$

As $\boldsymbol{F}_k \in \boldsymbol{L}^{\frac{3}{2}}(\Omega) \hookrightarrow \boldsymbol{L}^{\frac{6}{5}}(\Omega)$, from Theorem 2.1 i) we know that $\boldsymbol{z}_k \in \boldsymbol{W}^{1,\frac{6}{5}}(\Omega)$.

Since $\boldsymbol{z}_k$ satisfies the same properties that the vector potential given by Theorem 3.4, hence

$$\|\boldsymbol{z}_k\|_{\boldsymbol{L}^2(\Omega)} \leqslant C\|\boldsymbol{F}_k\|_{[\boldsymbol{H}^1_\tau(\Omega)]'}. \tag{4.12}$$

But $\boldsymbol{F}_k = \text{div}\,(\boldsymbol{B}_k \otimes \boldsymbol{u}_k - \boldsymbol{B} \otimes \boldsymbol{u} + \boldsymbol{u}_k \otimes \boldsymbol{B}_k - \boldsymbol{u} \otimes \boldsymbol{B})$, then

$$\begin{aligned}\|\boldsymbol{F}_k\|_{[\boldsymbol{H}^1_\tau(\Omega)]'} &\leqslant C\|\boldsymbol{B}_k \otimes \boldsymbol{u}_k - \boldsymbol{B} \otimes \boldsymbol{u} + \boldsymbol{u}_k \otimes \boldsymbol{B}_k - \boldsymbol{u} \otimes \boldsymbol{B}\|_{\boldsymbol{L}^2(\Omega)} \\ &\leqslant C\|\boldsymbol{B}_k \otimes \boldsymbol{u}_k - \boldsymbol{B} \otimes \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{u}_k \otimes \boldsymbol{B}_k - \boldsymbol{u} \otimes \boldsymbol{B}\|_{\boldsymbol{L}^2(\Omega)}.\end{aligned}$$

Writing

$$\boldsymbol{B}_k \otimes \boldsymbol{u}_k - \boldsymbol{B} \otimes \boldsymbol{u} = (\boldsymbol{B}_k - \boldsymbol{B}) \otimes \boldsymbol{u}_k + \boldsymbol{B} \otimes (\boldsymbol{u}_k - \boldsymbol{u}),$$

we get

$$\|\boldsymbol{B}_k \otimes \boldsymbol{u}_k - \boldsymbol{B} \otimes \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \leqslant \|\boldsymbol{B}_k - \boldsymbol{B}\|_{\boldsymbol{L}^4(\Omega)}\|\boldsymbol{u}\|_{\boldsymbol{L}^4(\Omega)} + \|\boldsymbol{B}\|_{\boldsymbol{L}^4(\Omega)}\|\boldsymbol{u}_k - \boldsymbol{u}\|_{\boldsymbol{L}^4(\Omega)}.$$

Because $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^4(\Omega)$ compactly, we deduce that

$$\boldsymbol{B}_k \otimes \boldsymbol{u}_k - \boldsymbol{B} \otimes \boldsymbol{u} \longrightarrow \boldsymbol{0} \ \text{ in } \boldsymbol{L}^2(\Omega), \ \text{ as } k \to \infty$$

With similar arguments we show that

$$\boldsymbol{u}_k \otimes \boldsymbol{B}_k \longrightarrow \boldsymbol{u} \otimes \boldsymbol{B} \quad \text{in} \quad \boldsymbol{L}^2(\Omega), \quad \text{as } k \to \infty,$$

and finally

$$\boldsymbol{z}_k \longrightarrow \boldsymbol{0} \quad \text{in} \quad \boldsymbol{L}^2(\Omega), \quad \text{as } k \to \infty,$$

which means that

$$\textbf{curl}\, \hat{\boldsymbol{B}}_k \longrightarrow \hat{\boldsymbol{B}} \;\; \text{in} \;\; \boldsymbol{L}^2(\Omega) \;\; \text{as } k \to \infty.$$

By (3.14), we deduce that

$$\hat{\boldsymbol{B}}_k \longrightarrow \hat{\boldsymbol{B}} \;\; \text{in} \;\; \boldsymbol{H}^1(\Omega), \quad \text{as } k \to \infty$$

and finally T is compact.

2) Let us show the condition (4.8). Let $(\boldsymbol{u}, \boldsymbol{B}) = \alpha T(\boldsymbol{u}, \boldsymbol{B})$ with $(\boldsymbol{u}, \boldsymbol{B}) \in \boldsymbol{Z}$ and $\alpha \in [0,1]$. As $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{B}}) = T(\boldsymbol{u}, \boldsymbol{B})$ then $(\boldsymbol{u}, \boldsymbol{B}) = \alpha\, (\hat{\boldsymbol{u}}, \hat{\boldsymbol{B}}) = (\alpha\hat{\boldsymbol{u}}, \alpha\hat{\boldsymbol{B}})$ and $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{B}}) = T(\alpha\hat{\boldsymbol{u}}, \alpha\hat{\boldsymbol{B}})$ satisfies the following system:

$$\begin{cases} -\nu\Delta\hat{\boldsymbol{u}} + \dfrac{1}{\rho}\nabla\hat{\pi} = \boldsymbol{f} - \alpha^2\,(\hat{\boldsymbol{u}}\cdot\nabla)\,\hat{\boldsymbol{u}} + \dfrac{\alpha^2}{\rho\mu}\left(\hat{\boldsymbol{B}}\cdot\nabla\right)\hat{\boldsymbol{B}} - \dfrac{\alpha^2}{2\rho\mu}\nabla\left(|\hat{\boldsymbol{B}}|^2\right) & \text{in} \quad \Omega, \\ -\lambda\Delta\hat{\boldsymbol{B}} = \boldsymbol{k} + \alpha^2\left(\hat{\boldsymbol{B}}\cdot\nabla\right)\hat{\boldsymbol{u}} - \alpha^2\,(\hat{\boldsymbol{u}}\cdot\nabla)\,\hat{\boldsymbol{B}} & \text{in} \quad \Omega, \\ \operatorname{div}\hat{\boldsymbol{u}} \;=\; 0 & \text{in} \quad \Omega, \\ \operatorname{div}\hat{\boldsymbol{B}} \;=\; 0 & \text{in} \quad \Omega, \\ \hat{\boldsymbol{u}} \;=\; \boldsymbol{0} & \text{on} \quad \Gamma, \\ \hat{\boldsymbol{B}}\cdot\boldsymbol{n} = 0, \quad \textbf{curl}\,\hat{\boldsymbol{B}}\times\boldsymbol{n} \;=\; \boldsymbol{0} & \text{on} \quad \Gamma, \\ \displaystyle\int_{\sum_j}\hat{\boldsymbol{B}}\cdot\boldsymbol{n} \;=\; 0 & 1\leqslant j\leqslant J. \end{cases}$$

Multiplying the first equation by $\hat{\boldsymbol{u}}$ and the second one by $\hat{\boldsymbol{B}}$ and integrating by parts, we get

$$\nu\int_\Omega|\nabla\hat{\boldsymbol{u}}|^2\,\mathrm{dx} = \int_\Omega\boldsymbol{f}\cdot\hat{\boldsymbol{u}}\,\mathrm{dx} + \frac{\alpha^2}{\rho\mu}\int_\Omega(\hat{\boldsymbol{B}}\cdot\nabla)\hat{\boldsymbol{B}}\cdot\hat{\boldsymbol{u}}\,\mathrm{dx} \tag{4.13}$$

and

$$\lambda\int_\Omega|\textbf{curl}\,\hat{\boldsymbol{B}}|^2\,\mathrm{dx} \;=\; -\alpha^2\int_\Omega(\hat{\boldsymbol{B}}\cdot\nabla)\hat{\boldsymbol{B}}\cdot\hat{\boldsymbol{u}}\,\mathrm{dx} + \int_\Omega\boldsymbol{k}\cdot\hat{\boldsymbol{B}}\mathrm{dx}. \tag{4.14}$$

Multiplying then (4.14) by $\dfrac{1}{\rho\mu}$ and summing, we obtain

$$\nu\|\nabla\hat{\boldsymbol{u}}\|^2_{\boldsymbol{L}^2(\Omega)}+\frac{\lambda}{\rho\mu}\|\mathbf{curl}\,\hat{\boldsymbol{B}}\|^2_{\boldsymbol{L}^2(\Omega)}\leqslant C\left[\|\boldsymbol{f}\|_{\boldsymbol{H}^{-1}(\Omega)}\|\hat{\boldsymbol{u}}\|_{\boldsymbol{H}^1(\Omega)}+\|\boldsymbol{k}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)}\|\hat{\boldsymbol{B}}\|_{\boldsymbol{L}^6(\Omega)}\right].$$

Therefore

$$\|\hat{\boldsymbol{u}}\|_{\boldsymbol{H}^1(\Omega)}+\|\hat{\boldsymbol{B}}\|_{\boldsymbol{H}^1(\Omega)}\leqslant C[\|\boldsymbol{f}\|_{\boldsymbol{H}^{-1}(\Omega)}+\|\boldsymbol{k}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)}]$$

and

$$\|(\boldsymbol{u},\boldsymbol{B})\|_{\boldsymbol{Z}}=\alpha\|(\hat{\boldsymbol{u}},\hat{\boldsymbol{B}})\|_{\boldsymbol{Z}}\leqslant C_1 \tag{4.15}$$

where $C_1=C_1\left(\Omega,\nu,\lambda,\rho,\mu,\|f\|_{\boldsymbol{H}^{-1}(\Omega)},\|\boldsymbol{k}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)}\right)$ is a positive constant independent of $(\boldsymbol{u},\boldsymbol{B})$ and α.

iii) Regularity of $\boldsymbol{B}$. Finally as $(\boldsymbol{B}\cdot\nabla)\boldsymbol{u}-(\boldsymbol{u}\cdot\nabla)\boldsymbol{B}\in\boldsymbol{L}^{\frac{3}{2}}(\Omega)$ and $\boldsymbol{k}\in\boldsymbol{L}^{\frac{6}{5}}(\Omega)$, we deduce by Theorem 2.4 that $\boldsymbol{B}\in\boldsymbol{W}^{2,\frac{6}{5}}(\Omega)$. □

Remark 5. With the same proof, we can obtain similar results if we replace the Dirichlet boundary condition by the Navier-type boundary condition:

$$\boldsymbol{u}\cdot\boldsymbol{n}\ =\ 0,\qquad \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n}=\mathbf{0}\quad\text{on }\Gamma.$$

5. Regularity of the weak solution

In this section, we will study the regularity of the weak solution of the problem (MHD). The demonstration is based on the results of regularity of solution for the Stokes and the Poisson equations and the Sobolev embedding.

Theorem 5.1. **(Regularity $\mathbf{W^{1,p}(\Omega)}$ with $\mathbf{p\geq 2}$)** *Let*

$$\boldsymbol{f}\in\boldsymbol{W}^{-1,p}(\Omega)\quad\text{and}\quad\boldsymbol{k}\in\boldsymbol{L}^{r(p)}(\Omega)\quad\text{with}\ \ r(p)=\frac{3p}{p+3}$$

and satisfying the condition (4.1) *Then the weak solution for the (MHD) system given by Theorem* (4.1) *satisfies*

$$(\boldsymbol{u},\boldsymbol{B},\pi,)\in\boldsymbol{W}^{1,p}(\Omega)\times\boldsymbol{W}^{2,r(p)}(\Omega)\times L^p(\Omega). \tag{5.1}$$

Proof. We can rewrite the problem (MHD) in the following form:

$$\begin{cases} -\nu\Delta \boldsymbol{u} + \dfrac{1}{\rho}\nabla\pi = \boldsymbol{f} + \boldsymbol{h} & \text{in } \Omega, \\ -\lambda\Delta \boldsymbol{B} = \quad \boldsymbol{g} + \boldsymbol{k} & \text{in } \Omega \\ \operatorname{div} \boldsymbol{u} = \boldsymbol{0}, \quad \operatorname{div} \boldsymbol{B} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0}, \quad \boldsymbol{B}\cdot\boldsymbol{n} = 0, \quad \mathbf{curl}\,\boldsymbol{B}\times\boldsymbol{n} = \boldsymbol{0} & \text{on } \Gamma, \\ \displaystyle\int_{\sum_j} \boldsymbol{B}\cdot\boldsymbol{n} \; = \; 0 & 1 \leqslant j \leqslant J, \end{cases}$$

where

$$\boldsymbol{h} = -\left(\boldsymbol{u}\cdot\nabla\right)\boldsymbol{u} + \frac{1}{\rho\mu}\left(\boldsymbol{B}\cdot\nabla\right)\boldsymbol{B} - \frac{1}{2\rho\mu}\nabla\left(|\boldsymbol{B}|^2\right)$$

and

$$\boldsymbol{g} = \left(\boldsymbol{B}\cdot\nabla\right)\boldsymbol{u} - \left(\boldsymbol{u}\cdot\nabla\right)\boldsymbol{B}.$$

Let $(\boldsymbol{u}, \boldsymbol{B}, \pi) \in \boldsymbol{H}^1(\Omega) \times \boldsymbol{W}^{2,\frac{6}{5}}(\Omega) \times L^2(\Omega)$ be a weak solution for the problem (MHD). According to the Sobolev embedding and the Hölder inequality, the functions $\boldsymbol{h}$ and $\boldsymbol{g}$ belong to $\boldsymbol{L}^{\frac{3}{2}}(\Omega) \hookrightarrow \boldsymbol{W}^{-1,3}(\Omega)$. We have two cases:

i) Case $\mathbf{2 \leqslant p \leqslant 3}$: Then we have $\boldsymbol{f} + \boldsymbol{h} \in \boldsymbol{W}^{-1,p}(\Omega)$. By the regularity of the Stokes equations we deduce that $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ and $\pi \in L^p(\Omega)$. Let us pass to the regularity of $\boldsymbol{B}$. We have $\boldsymbol{g} + \boldsymbol{k} \in \boldsymbol{L}^{r(p)}(\Omega)$ because $r(p) \leq \frac{3}{2}$. Thanks to Theorem 2.4, we have $\boldsymbol{B} \in \boldsymbol{W}^{2,r(p)}(\Omega)$.

ii) Case $\mathbf{p > 3}$: We know that $\boldsymbol{u} \in \boldsymbol{W}^{1,3}(\Omega)$ and $\boldsymbol{B} \in \boldsymbol{W}^{2,\frac{3}{2}}(\Omega) \hookrightarrow \boldsymbol{W}^{1,3}(\Omega)$. Now $\boldsymbol{h} \in \boldsymbol{L}^s(\Omega)$ and $\boldsymbol{g} \in \boldsymbol{L}^s(\Omega)$ for all $s < 3$. But for any $r > 1$, in particular for $r \geq p$, there is some $s < 3$ such that $\boldsymbol{L}^s(\Omega) \hookrightarrow \boldsymbol{W}^{-1,r}(\Omega)$. By the regularity of the Stokes equations with Dirichlet boundary conditions, we obtain $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$. Concerning the regularity of $\boldsymbol{B}$ we have $\boldsymbol{g} + \boldsymbol{k} \in \boldsymbol{L}^{r(p)}(\Omega)$ because $r(p) < 3$. By Theorem 2.4 we deduce that $\boldsymbol{B} \in \boldsymbol{W}^{2,r(p)}(\Omega)$. □

Theorem 5.2. (**Regularity $\mathbf{W^{2,p}(\Omega)}$ with $\mathbf{p \geq \frac{6}{5}}$**) *Let*

$$\boldsymbol{f} \in \boldsymbol{L}^p(\Omega) \quad \text{and} \quad \boldsymbol{k} \in \boldsymbol{L}^p(\Omega)$$

satisfying the condition (4.1). Then the weak solution for the (MHD) system given by Theorem 4.1 satisfies

$$(\boldsymbol{u}, \boldsymbol{B}, \pi) \in \boldsymbol{W}^{2,p}(\Omega) \times \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega). \tag{5.2}$$

Proof. Because $\boldsymbol{f}$ and $\boldsymbol{k}$ belong to $\boldsymbol{L}^{\frac{6}{5}}(\Omega)$ which is included in $\boldsymbol{H}^{-1}(\Omega)$, we know by Theorem 4.1 that there exists a weak solution $(\boldsymbol{u}, \boldsymbol{B}, \pi)$ for the problem (MHD). Then we have $\boldsymbol{h} \in \boldsymbol{L}^{\frac{3}{2}}(\Omega)$ and $\boldsymbol{g} \in \boldsymbol{L}^{\frac{3}{2}}(\Omega)$. We have two cases:

i) Case $\mathbf{\frac{6}{5} \leqslant p \leqslant \frac{3}{2}}$: We have $\boldsymbol{f} + \boldsymbol{h} \in \boldsymbol{L}^p(\Omega)$ and also $\boldsymbol{g} + \boldsymbol{k} \in \boldsymbol{L}^p(\Omega)$. By the regularity of the Stokes equation we have $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$ and $\pi \in W^{1,p}(\Omega)$ and thanks to Theorem 2.4, $\boldsymbol{B} \in \boldsymbol{W}^{2,p}(\Omega)$.

ii) Case $\mathbf{\frac{3}{2} < p < 3}$: From the above result, now we know that $(\boldsymbol{u}, \boldsymbol{B}) \in \boldsymbol{W}^{2,\frac{3}{2}}(\Omega) \times \boldsymbol{W}^{2,\frac{3}{2}}(\Omega)$. But note that $\boldsymbol{W}^{2,\frac{3}{2}}(\Omega) \hookrightarrow \boldsymbol{W}^{1,3}(\Omega) \hookrightarrow \boldsymbol{L}^r(\Omega)$, for all $1 \leqslant r < +\infty$. It follows that $(\boldsymbol{B} \cdot \nabla)\boldsymbol{u}$ and $(\boldsymbol{u} \cdot \nabla)\boldsymbol{B}$ belong to $\boldsymbol{L}^s(\Omega)$ for any $1 \leqslant s < 3$ and then $\boldsymbol{g} \in \boldsymbol{L}^p(\Omega)$. Thanks to Theorem 2.4, we deduce that $\boldsymbol{B} \in \boldsymbol{W}^{2,p}(\Omega)$. By the same arguments, we have that $\boldsymbol{h} \in \boldsymbol{L}^p(\Omega)$, and by the regularity of the Stokes equation, we get $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$ and $\pi \in W^{1,p}(\Omega)$.

iii) Case $\mathbf{3 \leqslant p < \infty}$: From the previous case, we know that $(\boldsymbol{u}, \boldsymbol{B}) \in \boldsymbol{W}^{2,q}(\Omega) \times \boldsymbol{W}^{2,q}(\Omega)$ for any $q < 3$. But $\boldsymbol{W}^{2,q}(\Omega) \hookrightarrow \boldsymbol{L}^\infty(\Omega)$ and $\boldsymbol{W}^{1,q}(\Omega) \hookrightarrow \boldsymbol{L}^t(\Omega)$ for all $1 \leqslant t < \infty$. It follows that $(\boldsymbol{B} \cdot \nabla)\boldsymbol{u}$ and $(\boldsymbol{u} \cdot \nabla)\boldsymbol{B}$ belong in particular to $\boldsymbol{L}^p(\Omega)$ and then $\boldsymbol{g} \in \boldsymbol{L}^p(\Omega)$. Thanks to Theorem 2.4, we deduce that $\boldsymbol{B} \in \boldsymbol{W}^{2,p}(\Omega)$. By the same arguments, we have that $\boldsymbol{h} \in \boldsymbol{L}^p(\Omega)$, and by the regularity of the Stokes equation, we get $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$ and $\pi \in W^{1,p}(\Omega)$. □

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