On a dual to the properties of Hurwitz polynomials I

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In this paper we develop necessary and sufficient conditions for describe the family of anti-Hurwitz polynomials, introduced by Vergara-Hermosilla et al. in [9]. Specifically, we studied a dual version of the Theorem of Routh-Hurwitz and present explicit criteria for polynomials of low order and derivatives. Another contribution of this work is establishing a dual version of the Hermite-Biehler Theorem. To this aim, we give extensions of the boundary crossing Theorems and a zero exclusion Principle for anti-Hurwitz polynomials.

1 Introduction

In this work we present the first part, of a series of three works, of a new approach about the classification of the roots of real polynomials in one variable in the right half complex plane. This new idea arises from the need to obtain simple explicit criteria for the area of the complex plane not covered by the theory of Hurwitz polynomials (also known as stable polynomials). In fact, our results are natural extensions of the classical Theorems of Routh-Hurwitz and Hermite-Biehler for the complement zone; $\mathbb{C}_+ = \{z \in \mathbb{C} \, : \, Re\,(z) > 0\}$.

In the literature we highlight as main references for the study of roots of real polynomials on the left half complex plane and its applications to system theory in a general framework the books of Gantmacher [3] and [4], and the book of Iooss and Joseph [6]. Chappellat, Mansour and Brattacharyya present classic stability criteria with elementary demonstrations in their article [1] while new and interesting ideas about the demonstration of these results have been developed by Holtz in [5]. For a generalization to real polynomials in several variables we mention the work of Fettweis in [2]. The approach introduced in this work consist of a systematic use of the linear transformation $z \mapsto -z$, on the properties that define the Hurwitz polynomials, which leads us to use and explore the notion originally introduced by Vergara-Hermosilla et al. in [9] about anti-Hurwitz polynomials. This notion can be recast as a dual result to the main necessary and sufficient conditions on stable polynomials. What is more, our Theorems

and Propositions also depend on the coefficients of the polynomial in question which makes it more manipulable for applications in science and engineering. To this end, in Section 5 we will apply our results to a family of polynomials associated with a system of PDE's that describe interactions fluid-structure, for details see [9] or Vergara-Hermosilla [7]. Now, we are in a position of stablish our first main result, which read as:

Proposition 1.1 *Let $f(X) = a_0X^n + a_1X^{n-1} + \cdots a_{n-1}X + a_n \in \mathbb{R}[X]$ of degree ≥ 3. Then $f(X)$ is an anti-Hurwitz polynomial, if and only if it satisfies the conditions:*

(1) $(-1)^i a_i > 0$, *for all* $i \in \{0, \ldots, n\}$.

(2) $(-1)^{\lfloor \frac{i+1}{2} \rfloor} \Delta_i > 0$, *for all* $i \in \{1, \ldots, n\}$.

Like the practical use of of the Routh-Hurwitz criterion is usually limited, in the context of direct computations, to polynomials of low degrees (3rd, 4th, or 5th), we develop an alternative result, which is more versatile and, as in the previous case, this extends to the dual zone the classical results on Hermite-Biehler theory. In fact, our second main result reads as:

Theorem 1.2 *A real polinomial $p(X)$ is Anti-Hurwitz, if and only if satisfies the anti-alternancy property.*

For the proof of this result is based in the dual version of the boundary crossing Theorem, a zero exclusion principle for anti-Hurwitz polynomials and the definition of a anti-alternancy property.

The paper is organized in the following manner.

- In Section 2 we state the main definitions and properties that describe the Hurwitz polynomials emphasizing the Hurwitz matrix and the Theorem of Routh-Hurwitz.
- In Section 3 we define the anti-Hurwitz polynomials, demonstrate our first main result, and establish explicit criteria for real polynomials of less than or equal order 4 and derivatives.
- In Section 4 we introduce the dual versions of establish a dual versions of Boundary Crossing Theorems and we proof our second main result.
- Finally, in Section 5 we apply our results for obtain information about the behavior of the roots of the family of *viscous polynomials* defined in [9].

2 Hurwitz polynomials

For $n \in \mathbb{N}$ we denote by $\mathcal{P}_n$ the set of all degree n polynomials with real coefficients i.e.,

$$\mathcal{P}_n = \deg^{-1}(\{n\}) \subset \mathbb{R}[X].$$

Definition 2.1 A polynomial $f(X) \in \mathbb{R}[X]$ is *Hurwitz* if the real part of all its complex roots is negative i.e., $\mathrm{Re}(u) < 0$ for any $u \in \mathbb{Z}$ satisfying $f(u) = 0$.

Let $\mathcal{H}$ denote the set of all Hurwitz polynomials, and we set $\mathcal{H}_n = \mathcal{H} \cap \mathcal{P}_n$. The set of all Hurwitz polynomials in $\mathcal{H}_n$ with positive coefficients is denoted by $\mathcal{H}_n^+$.

Theorem 2.1 (Stodola condition) *If a polynomial $f(X) \in \mathbb{R}[X]$ is Hurwitz, then all its coefficients are of the same sign.*

Proof The roots of a real polynomial are symmetric with respect to the real line. For $f(X)$, we can write

$$f(X) = a_0 \prod_k (X - s_k) \prod_j (X - \alpha_j - i\beta_j) \prod_j (X - \alpha_j + i\beta_j),$$

where each s_k are real roots, and $\alpha_j \pm i\beta_j$ are complex roots of $f(X)$ with nonzero imaginary part. Note that s_j, α_j are negative. Since the expressions $(X - s_k)$ and $X^2 - 2\alpha_j X + (\alpha_j^2 + \beta_j^2)$ have positive coefficients, their product has the same property. □

Let $f(X) = a_0 X^n + \cdots + a_{n-1} X + a_n \in \mathcal{P}_n$ be a polynomial. The *Hurwitz matrix* of a polynomial, denoted as $H(f(X))$, is the square matrix of size n defined as follows:

$$\begin{pmatrix}
a_1 & a_3 & a_5 & \cdots & 0 & 0 & 0 \\
a_0 & a_2 & a_4 & \cdots & 0 & 0 & 0 \\
0 & a_1 & a_3 & \cdots & 0 & 0 & 0 \\
0 & a_0 & a_2 & \cdots & 0 & 0 & 0 \\
0 & 0 & a_1 & \cdots & 0 & 0 & 0 \\
0 & 0 & a_0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & a_{n-2} & a_n & 0 \\
0 & 0 & 0 & \cdots & a_{n-3} & a_{n-1} & 0 \\
0 & 0 & 0 & \cdots & a_{n-4} & a_{n-2} & a_n
\end{pmatrix}$$

For every $k \in \{1, \ldots, n\}$, let $H_k(f(X))$ denote the square matrix of size k obtained from the first k rows and columns of $H(f(X))$, and we set:

$$\Delta_k = \det(H_k(f(X))).$$

Theorem 2.2 (Routh-Hurwitz) *A polynomial* $f(X) = a_0X^n + \cdots + a_{n-1}X + a_n \in \mathcal{P}_n$ *with* $a_0 > 0$ *is Hurwitz if and only if* $\Delta_k > 0$ *for all* $k \in \{1, \ldots, n\}$.

For a proof of this result see for instance [3], [6] or [5].

3 Anti-Hurwitz Polynomials

In this section we establish the definition of anti-Hurwitz polynomials and a dual criterion to the Theorem of Routh-Hurwitz. To this end, we introduce the following definition.

Definition 3.1 A polynomial $f(X) \in \mathcal{P}_n$ is said to be *anti-Hurwitz* if the real part of all its complex roots is positive, i.e., $\mathrm{Re}(u) > 0$ for all $u \in \mathbb{Z}$ satisfying $f(u) = 0$.

Lemma 3.1 *A polynomial* $f(X) \in \mathcal{P}_n$ *is anti-Hurwitz if and only if* $f(-X)$ *is Hurwitz.*

Proof Let $f(X)$ be an anti-Hurwitz polynomial and u a complex root of $f(-X)$. Then $f(-u) = 0$ and $\mathrm{Re}(-u) > 0$, i.e., $\mathrm{Re}(u) < 0$. Therefore $f(-X)$ is Hurwitz. On the other hand, if $f(-X)$ is a Hurwitz polynomial and u a complex root of $f(X)$, then $f(u) = f(-(-u)) = 0$. In this case, $\mathrm{Re}(-u) < 0$, i.e., $\mathrm{Re}(u) > 0$. Hence, $f(X)$ is anti-Hurwitz. □

Lemma 3.2 *Let* $f(X) \in \mathbb{R}[X]$ *be a polynomial of degree* n *and* Δ_i *the determinant of the Hurwitz submatrix* $H_i(f(X))$*, for* $1 \leq i \leq n$*. Then we have*

$$\Delta_i^- = (-1)^{\lfloor \frac{i+1}{2} \rfloor} \Delta_i, \tag{3–1}$$

where Δ_i^- *is the determinant of* i*-th Hurwitz submatrix* $H_i(f(-X))$.

Proof The matrix for $H_i(f(-X))$ is written as

$$\begin{pmatrix}
-a_1 & -a_3 & -a_5 & \cdots & 0 & 0 & 0 \\
a_0 & a_2 & a_4 & \cdots & 0 & 0 & 0 \\
0 & -a_1 & -a_3 & \cdots & 0 & 0 & 0 \\
0 & a_0 & a_2 & \cdots & 0 & 0 & 0 \\
0 & 0 & -a_1 & \cdots & 0 & 0 & 0 \\
0 & 0 & a_0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & a_{n-2} & a_n & 0 \\
0 & 0 & 0 & \cdots & -a_{n-3} & -a_{n-1} & 0 \\
0 & 0 & 0 & \cdots & a_{n-4} & a_{n-2} & a_n
\end{pmatrix}.$$

Comparing it with the matrix of $H_i(f(X))$, we immediately see that

$$\Delta_i^- = (-1)^{\lfloor \frac{i+1}{2} \rfloor} \Delta_i.$$

□

Proposition 3.3 *Let $f(X) = a_0X^n + a_1X^{n-1} + \cdots a_{n-1}X + a_n \in \mathbb{R}[X]$ of degree ≥ 3. Then $f(X)$ is an anti-Hurwitz polynomial, if and only if it satisfies the conditions:*

(1) $(-1)^i a_i > 0$, *for all* $i \in \{0, \ldots, n\}$.

(2) $(-1)^{\lfloor \frac{i+1}{2} \rfloor} \Delta_i > 0$, *for all* $i \in \{1, \ldots, n\}$.

Proof By lemma (3.1), we know that $f(X)$ is an anti-Hurwitz polynomial if and only if $f(-X)$ is a Hurwitz polynomial. In this case, the coefficient of X^i in $f(-X)$ is $(-1)^i a_{n-i}$. Without loss of generality, we may suppose that $a_0 > 0$. Now the Stodola Condition (2.1) and Theorem (2.2), gives us that $(-1)^i a_{n-i} > 0$, for $i \in \{0,\ 1,\ ...,\ n\}$ and $\Delta_i^- > 0$. Hence, we conclude by Lemma (3.2). □

3.1 Criteria for polynomials in low degrees and derivatives

In this Section we will establish simple criteria on the property of anti-Hurwitz, applicable to real polynomials of less than or equal order 4 and derivatives of polynomials. To this end we consider a polynomial $p(X) \in \mathbb{R}[X]$ and the necessary and sufficient conditions developed in the Proposition 3.3.

The criteria read as:

- The polynomial $p(X) = X^2 + a_1X + a_2$ is an anti-Hurwitz polynomial, if and only if
$$-a_1,\ a_2 > 0.$$
- The polynomial $p(X) = X^3 + a_1X^2 + a_2X + a_3$ is an anti-Hurwitz polynomial, if and only if
$$-a_1,\ a_2,\ -a_3 > 0 \quad \text{and} \quad a_2 - a_1a_2 > 0.$$
- The polynomial $p(X) = X^4 + a_1X^3 + a_2X^2 + a_3X + a_4$ is an anti-Hurwitz polynomial, if and only if
$$-a_1,\ a_2,\ -a_3,\ a_4 > 0 \quad \text{and} \quad a_1a_2a_3 - a_3^2 - a_1^2a_4 > 0.$$
- Let $p(X)$ be an anti-Hurwitz polynomial of degree n and let $P'(X)$ denote the first-order derivative of $p(X)$ with respect to X. Then $-p'(X)$ is again an anti-Hurwitz polynomial.

4 A dual version of the Theorem of Hermite-Biehler

In this Section we establish a dual version of the Theorem of Hermite-Biehler for anti-Hurwitz polynomials. To this end, we need to introduce dual versions of Boundary Crossing Theorems. We begin the Section with the following definition.

Definition 4.1 Let $p(X) \in \mathbb{R}[X]$ and $w \in \mathbb{R}$. The argument of $p(iw)$ is called the *phase* of $p(iw)$.

Lemma 4.1 *Let $p(X) = a_X^n + \cdots + a_1 X + a_0$ be an anti-Hurwitz polynomial of degree n. Then, $\arg p(iw)$ is a strictly decreasing function. Moreover, the net change in the phase from $-\infty$ to $+\infty$ is*

$$\lim_{w\to+\infty} p(iw) - \lim_{w\to-\infty} p(iw) = n\pi. \tag{4–1}$$

Proof By the fundamental theorem of algebra, we can write $p(X)$ as a product of its roots

$$p(X) = a_n(X - \alpha_1 - i\beta_1)(X - \alpha_2 - i\beta_2)\cdots(X - \alpha_n - i\beta_n).$$

Plugging $X = iw$, we get

$$p(iw) = a_n(-\alpha_1 + i(w - \beta_1))(-\alpha_2 + i(w - \beta_2))\cdots(-\alpha_n + i(w - \beta_n)),$$

and so, we obtain

$$\arg p(iw) = \arg(a_n) + \arctan\left(\frac{w-\beta_1}{-\alpha_1}\right) + \cdots + \arctan\left(\frac{w-\beta_n}{-\alpha_n}\right). \tag{4–2}$$

Differentiating the above expression with respect to w, we get

$$\frac{d}{dw}\arg p(iw) = \frac{1}{1+\left(\frac{w-\beta_1}{-\alpha_1}\right)^2}\left(-\frac{1}{\alpha_1}\right) + \cdots + \frac{1}{1+\left(\frac{w-\beta_1}{-\alpha_n}\right)^2}\left(-\frac{1}{\alpha_n}\right).$$

Since $p(X)$ is an anti-Hurwitz polynomial, we have that $\alpha_k > 0$ for $k \in \{1, ..., n\}$. Therefore, $\arg p(iw)$ is decreasing is a decreasing function in w. Now, from eq. (4–2), we have

$$\lim_{w\to+\infty} \arg p(iw) = \arg(\alpha_n) - \frac{n\pi}{2};$$
$$\lim_{w\to-\infty} \arg p(iw) = \arg(\alpha_n) + \frac{n\pi}{2}.$$

The claim now follows. □

In the following we enunciate two classic results on stability, whose demonstrations can be consulted in the article of Chappellat et al. [1].

Proposition 4.2 *Let* $a_n \neq 0$, $p(X) = a_nX^n + \cdots + a_1X + a_0 = a_n \prod_{j=1}^{m}(X-\omega_j)^{t_j}$, *and* $q(X) = (a_n+\varepsilon_n)X^n + \cdots + (a_1+\varepsilon_1)X + (a_0+\varepsilon_0)$. *Consider the circle* C_k *of radius* r_k *centered at* ω_k. *Let* r_k *be fixed such that* $0 \leq r_k < \min|\omega_k - \omega_j|$, *for* $j \in \{1,2,\ldots,k-1,k+1,...,m\}$. *Then, there exists an* $\varepsilon > 0$ *such that for all* $|\varepsilon_1|,|\varepsilon_2|,\ldots,|\varepsilon_n| < \varepsilon$, $q(X)$ *has precisely* t_k *zeros inside the circle* C_k.

Corollary 4.3 *Fix* m *circles* $C_1,\ldots,C_m$ *that are pairwise disjoint and centered at* $\omega_1,\omega_2,\ldots,\omega_m$ *respectively. Then, by repeatedly applying Theorem* (4.2), *it is always possible to find an* $\varepsilon > 0$ *such that for any* $|\varepsilon_1|,|\varepsilon_2|,\ldots,|\varepsilon_n| < \varepsilon$, $q(X)$ *has precisely* t_k *zeros inside each of the circles* C_k.

Remark 4.1 In the previous Corollary, we note that $q(X)$ always has $t_1+t_2+\cdots+t_m = n$ zeros and must therefore remain of degree n, so necessarily we have $\varepsilon < |a_n|$.

In the following we denote the set of anti-Hurwitz polynomials of degree n by $\mathcal{AH}_n$.

Remark 4.2 By Proposition (4.2), Corollary (4.3) and Remark (4.1), we see that if $p(X) = a_nX^n + \cdots + a_1X + a_0 \in \mathcal{AH}_n$, then there exists an $\varepsilon > 0$ such that for all $|\varepsilon_1|,|\varepsilon_2|,\ldots,|\varepsilon_n| < \varepsilon$, the polynomial $q(X) = (a_n+\varepsilon_n)X^n + \cdots + (a_1+\varepsilon_1)X + (a_0+\varepsilon_0) \in \mathcal{AH}_n$.

4.1 Boundary crossing Theorems

Let $p(\lambda,X)$ be a family of degree n polynomials with real coefficients, which is continuous with respect to $\lambda \in [a,b]$. In other words, $p(\lambda,X)$ can be written as

$$p(\lambda,X) = a_n(\lambda)X^n + \cdots + a_1(\lambda)X + a_0(\lambda),$$

where $a_0(\lambda)$, $a_1(\lambda)$, ..., $a_n(\lambda)$ are continuous functions in λ and $a_n(\lambda) \neq 0$ for all λ.

Theorem 4.4 *Suppose that* $p(a,X)$ *has all its roots in* $S \subset \mathbb{C}$, *where* $p(b,X)$ *has at leat one root in* $U = \mathbb{C} \setminus S$. *Then, there exist at least one* $\rho \in (a,b]$ *such that*

(1) $p(\rho,X)$ *has all its roots in* $S \cup \partial S$.

(2) $p(\rho,X)$ *has at least one root in* ∂S.

The proof of the Theorem above can be see in [1]. A direct consequence of the last Theorem relevant for the case of anti-Hurwitz families of polinomials is given in the following Corollary:

Corollary 4.5 *Suppose $\{f(\lambda,X)\}_{\lambda\in[a,b]}$ is a family of real polinomials in the variable X wich depends continuously on the $\lambda\in[a,b]$ and that the family is of degree constant. If $p(a,X)$ has all its roots on $\mathbb{C}^+$ and $p(b,X)$ has at least one root on $\mathbb{C}^-$, then there exist $\rho\in(a,b]$ such that*

(1) *$p(\rho,X)$ has all its roots in $\mathbb{C}^-\cup i\mathbb{R}$.*

(2) *$p(\rho,X)$ has at least one root in $i\mathbb{R}$.*

Theorem 4.6 *Let $\{f_n(X)\}_{n\in\mathbb{N}}$ be a sequence of anti-Hurwitz polynomials of degree least or equal to N such that $f_n(X)\to q(X)$. Then, the roots of $q(X)$ remain in $\mathbb{C}^+\cup i\mathbb{R}$.*

Proof We consider the polynomials $q(X)=a_0+a_1X+\cdots a_nX^N$, and $f_m(X)=a_0^m+a_1^mX+\cdots a_N^mX^N$. We suppose that $q(X)$ has a root $\overline{X}\in\mathbb{C}^-$. We know that there is a circle C with center $\overline{X}$ such that $C\subset\mathcal{C}^-$. Then, by Theorem 4.2 there is $\varepsilon>0$ such that if $|\varepsilon_i|<\varepsilon$, for all $i=0,\ 1,\ ...,\ N$, then

$$p(X)=(b_0+\varepsilon_0)+(b_1+\varepsilon_1)X+\cdots+(b_N+\varepsilon_N)X^N$$

has at least one root inside of C. How $f_n(X)\to q(X)$, then there is $f_k(X)=a_0^k+a_1^kX+\cdots a_N^kX^N$ such that $\left|b_0-a_0^k\right|,\left|b_1-a_1^k\right|,\ldots,\left|b_N-a_N^k\right|<\varepsilon$. Then, the following polynomial

$$f_k(X)=b_0+(a_0^k-b_0)+[b_1+(a_1^k-b_1)]X+\cdots+[b_N+(a_N^k-b_N)]X^N,$$

has a root in $\mathbb{C}^-$, which is a contradiction with the fact that $\{f_n(X)\}_{n\in\mathbb{N}}$ is a sequence of anti-Hurwitz polynomials. □

Theorem 4.7 (Zero exclusion principle) *Suppose $\{f(\lambda,X)\}_{\lambda\in\Omega}$ is a family of real polinomials in the variable X wich depends continuously on the $\lambda\in\Omega\subset\mathbb{R}^n$, with Ω pathwise connected. Suppose moreover that the family $f(\lambda,X)$ is of degree constant and there is at least one anti-Hurwitz polynomial. Then, the family $\{f(\lambda,X)\}_{\lambda\in\Omega}\subset\mathcal{AH}$, if and only if $p(\lambda,iw)\neq0$, for all $w\in\mathbb{R}$ and $\lambda\in\Omega$.*

Proof $\Rightarrow$): This is a direct consequence of Theorem 4.1.

$\Leftarrow$): Let $f(\lambda,X) \in \{f(\lambda,X)\}_{\lambda\in\Omega}$ an arbitrary polynomial and $f(\lambda_0,X)$ the anti-Hurwitz polynomial on $f(\lambda_0,X) \in \{f(\lambda,X)\}_{\lambda\in\Omega}$. We conside the path $\gamma : [a,b] \to \Omega$ such that $\gamma(a) = \lambda_0$ and $\gamma(b) = \lambda$ and the subfamily: $f(\gamma(X),X) = p(\lambda,X)$. We can see that $p(a,X) = f(\lambda_0,X)$ is anti-Hurwitz. Suppose that $p(b,X) = f(\lambda_0,X)$ does not an anti-Hurwitz polynomial, and hence has a root in $\mathbb{C}^- \cup i\mathbb{R}$. If $p(b,iw) = f(\lambda_0,iw) = 0$ is a contradiction. If $p(b,X)$ has a root in $\mathbb{C}^-$, then by Theorem 4.4 there is $\rho \in (a,b]$ such that

(1) $p(\rho,X)$ has all its roots in $\mathbb{C}^- \cup i\mathbb{R}$.

(2) $p(\rho,X)$ has at least one root in $i\mathbb{R}$.

By 2) there is w_0 such that $p(\rho,iw_0) = f(\gamma(\rho),iw_0) = 0$, but this is a contradiction. Therefore $p(b,X) = f(\lambda,X)$ is anti-Hurwitz for all $\lambda \in \Omega$.

□

Given a real polinomial $p(X) = a_0 + a_1X + \cdots + a_nX^n$, we note that

$$p(X) = (a_0 + a_2X^2 + a_4X^4 + \cdots) + X(a_1 + a_3X^2 + a_5X^4 + \cdots).$$

By evaluate iw, we obtain

$$p(iw) = (a_0 - a_2w^2 + a_4w^4 + \cdots) + iw(a_1 - a_3w^2 - a_5X^4 + \cdots).$$

Considering this, we consider the following notations

- $p^e(X) = a_0 - a_2X^2 + a_4X^4 + \cdots$
- $p^o(X) = a_1 - a_3X^2 - a_5X^4 + \cdots$
- $p^{even}(X) = a_0 + a_2X^2 + a_4X^4 + \cdots$
- $p^{odd}(X) = a_1 + a_3X^2 + a_5X^4 + \cdots$

Definition 4.2 A real polynomial $p(X) = a_0 + a_1X + \cdots + a_nX^n$ satisfies the anti-alternancy property if and only if

(1) The principal coeffients of $p^{even}(X)$ and $p^{odd}(X)$ has different sign.

(2) All the roots of $p^e(X)$ and $p^o(X)$ are reals and its negatives roots are interspersed, i.e.

$$0 > X_{e,1} > X_{o,1} > X_{e,2} > X_{o,2} > \cdots .$$

Theorem 4.8 (Dual version of Hermite-Biehler's Theorem) *A real polinomial $p(X)$ is Anti-Hurwitz, if and only if satisfies the anti-alternancy property.*

Proof By Theorem 4.1 that the phase of $p(iw)$ strictly decreases for $w \in \mathbb{R}$ from $\frac{n\pi}{2}$ to $-\frac{n\pi}{2}$ and the change in the phase is $2m\pi$, wich is equivalent to m turns in $w \in (-\infty, +\infty)$, or $m/2$ turns on $w \in (-\infty, 0)$. We note that for $w \in (-\infty, 0)$ the roots of $p^e(w)$ and $p^o(w)$ must be ordering in the following manner:

$$0 > X_{e,1} > X_{o,1} > X_{e,2} > X_{o,2} > \cdots > X_{o,m-1} > X_{e,m}. \tag{4–3}$$

In fact, in every turn it goes through by two roots of $p^e(w)$, and by two roots of $p^o(w)$. Then, in $m/2$ turns it goes through by m roots of of $p^e(w)$ and by m roots of $p^o(w)$. We note that, every one is real and negative, and then, we obtain part (2) of property of anti-alternancy. For the converse, assume that $p(X)$ satisfies the anti-intelacing property, and suposes without loss of generality of p is of degree $n = 2m$ and that the coefficient a_{2m} is positive. Let us consider the roots of $p^{even}(X)$ and $P^{odd}(X)$ in the form

$$0 > X^p_{R,1} > X^p_{L,1} > X^p_{R,2} > X^p_{L,2} > \cdots > X^p_{L,m-1} > X^p_{R,m}. \tag{4–4}$$

Now, let us consider a polynomial $q(X) = q_0 + \cdots q_{2m}X^{2m}$ that is known to be anti-Hurwitz, of the same degree $2m$, and with its leader coefficients positive. With this assumption on $q(X)$, we know from the first part of that $q(X)$ satisfies the anti-interlacing Theorem so that $q^{even}(X)$ has m negatives roots and q^{odd} has $m-1$ negative roots, both set of roots such that

$$0 > X^q_{R,1} > X^q_{L,1} > X^q_{R,2} > X^q_{L,2} > \cdots > X^q_{L,m-1} > X^q_{R,m}. \tag{4–5}$$

We note that for $q(iw)$, it has no imaginary roots, then for any $w \in \mathbb{R}$, $f(iw) \neq 0$. By taking $\lambda \in (0, 1)$, we have

$$0 > \lambda X^p_{e,1} + (1-\lambda)X^q_{e,1} > \lambda X^p_{o,1} + (1-\lambda)X^q_{o,1} \cdots > \lambda X^p_{e,m-1} + (1-\lambda)X^q_{e,m}.$$

Consider now the polynomial $p_\lambda(X)$ given by

$$p^e_\lambda(X) := ((1-\lambda)q_{2m} + \lambda p_{2m}) \prod_{i=1}^{m} (X^2 - ((1-\lambda)X^q_{e,i} + \lambda X^p_{e,i})^2),$$

$$p^o_\lambda(X) := ((1-\lambda)q_{2m-1} + \lambda p_{2m-1}) \prod_{i=1}^{m-1} (X^2 - ((1-\lambda)X^q_{o,i} + \lambda X^p_{o,i})^2).$$

We can see that the coefficients of p_λ are a family of polynomial functions in λ, wich are continuous on $[0, 1]$. Moreover, the coefficient of the leader degree term of $p_\lambda(X)$ remains positive as $\lambda \in [0, 1]$. Moreover, we note that for $\lambda = 0$, we have $p_0(X) = q(X)$. Then, how q is an anti-Hurwitz polynomial. This implies that the family $p_\lambda(X)$ has al least an element that is anti-Hurwitz. Hence, by the principle of exclusion of zero all the elements of the family are anti-Hurwitz polynomials, in particular $p_1(X) = p(X)$. □

5 Applications

In this section we consider a family of reals polynomials called 'viscous polynomials' introduced by Vergara-Hermosilla et al. in [9]

(5–1)

$$P_T(\lambda) = \left(1+\frac{l^3}{12}\right)\lambda^4 + l^2\sqrt{\mu}\lambda^3 + \left(l\mu - \frac{2}{\mu}\left(1+\frac{l^3}{12}\right)\right)\lambda^2 - \frac{l^2}{\sqrt{\mu}}\lambda + \frac{1}{\mu^2}\left(1+\frac{l^3}{12}\right),$$

where l and μ are free parameters in $\mathbb{R}^+$. The viscous polynomials arises naturally when considering the transfer function of a system that models the vertical movement of a solid floating in a viscous fluid, studied by Vergara-Hermosilla et al. in [8] and [9], in fact, the name of the family of polynomials is originally due to the fact that the parameters l and μ represent a measure associated with the size of the floating structure and the viscosity coefficient, respectively. Our objective in this section is to use the criteria developed in section 2 to obtain information on the location of the roots.

To this end, we can check easily that:

(1) When dividing $P_T(\lambda)$ by the coefficient of the term with exponent 4, we obtain the equivalent polynomial

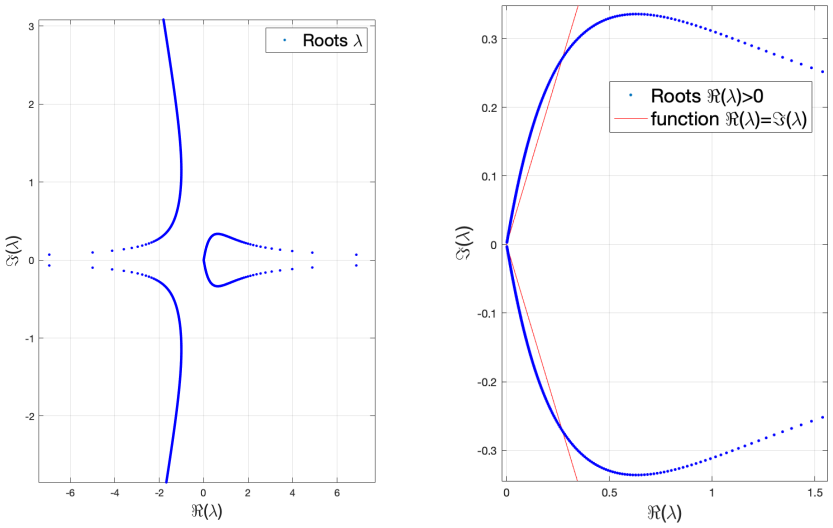
(5–2)

$$Q_T(\lambda) = \lambda^4 + \frac{l^2\sqrt{\mu}}{\left(1+\frac{l^3}{12}\right)}\lambda^3 + \frac{\left(l\mu - \frac{2}{\mu}\left(1+\frac{l^3}{12}\right)\right)}{\left(1+\frac{l^3}{12}\right)}\lambda^2 - \frac{l^2}{\sqrt{\mu}\left(1+\frac{l^3}{12}\right)}\lambda + \frac{1}{\mu^2}.$$

In this polynomial we can see that the coefficients of the terms with exponents 3 and 2 have the same sign, by considering the criteria developed in Section 3.1, we can conclude that the viscous polynomial is not anti-Hurwitz.

(2) In a similar form, we can see that the there are coefficients in $P_T(\lambda)$ with different sign, then using the Stodola condition give in Theorem 2.1, we conclude that the viscous polynomial is not Hurwitz.

In conclusion, due to the polynomial $P_T(\lambda)$ have degree 4, is not Hurwitz and is not anti-Hurwitz, we will always have two roots in the right complex half plane and two roots in the left complex half plane. In fact, in the following figure we can see numerical evidence about the behavior of the roots of the viscous polynomial with suitable parameters.



(a) (b)

Figure 5.1: Evolution of the four roots λ_i in the complex plane, as a function of μ. (a): global picture with 4 trajectories. (b): zoom in the right-half plane $\mathrm{Re}(\sigma) > 0$, 2 trajectories crossing the segment $\mathrm{Re}(\lambda) = |\Im(\lambda)|$ for a critical value μ^c of the viscosity.

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References

[1] H. CHAPPELLAT, M. MANSOUR, AND S. BHATTACHARYYA, *Elementary proofs of some classical stability criteria*, IEEE Transactions on Education, 33 (1990), pp. 232–239.

[2] A. FETTWEIS, *A new approach to hurwitz polynomials in several variables*, Circuits, Systems and Signal Processing, 5 (1986), pp. 405–417.

[3] F. R. GANTMACHER, *Matrix theory*, Chelsea, New York, 21 (1959), p. 6.

[4] F. R. GANTMACHER, *The theory of matrices, chelsea pub*, Co., New York, 2 (1959).

[5] O. HOLTZ, *Hermite{biehler, routh{hurwitz, and total positivity*, Linear Algebra and its Applications, 372 (2003), pp. 105 – 110.

[6] G. IOOSS AND D. D. JOSEPH, *Elementary stability and bifurcation theory*, Springer Science & Business Media, 2012.

[7] G. VERGARA-HERMOSILLA, *Relations between fractional calculus and interactions fluid-structure*, hal-02506981, (2020).

[8] G. VERGARA-HERMOSILLA, D. MATIGNON, AND M. TUCSNAK, *Well-posedness and input-output stability for a system modelling rigid structures floating in a viscous fluid*, hal-02475576v2, (2019).

[9] ——, *Asymptotic behaviour of a system modelling rigid structures floating in a viscous fluid*, hal-02475583, (2020).

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