



# Truncations of $W \infty$ Algebras

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► **To cite this version:**

| Mohammed Akram Fellah. Truncations of  $W \infty$  Algebras. 2020. hal-02512564

**HAL Id: hal-02512564**

**<https://hal.archives-ouvertes.fr/hal-02512564>**

Preprint submitted on 19 Mar 2020

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# Truncations of $W_\infty$ Algebras

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We introduce a new class of Vertex Operator Algebras  $Y^+$  and their duals, which generalize the standard W-algebras  $W_N$  of type  $sl(N)$ . These algebras can be defined in terms of junctions of boundary conditions and interfaces in the GL-twisted  $N = 4$  Super Yang Mills gauge theory. The aim of these technical calculations is to find the relation of these ortho-symplectic Y-algebras to truncations of even  $W_\infty$ .

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## 1 $\tilde{Y}^+$ as even $\mathcal{W}_\infty$

We expect relation of ortho-symplectic Y-algebras to truncations of even  $\mathcal{W}_\infty$ . These algebras contain roughly the same amount of generators as even  $\mathcal{W}_\infty$  but extra, finite number of generators must be added. It would be nice to explore if one of the following happens:

- There exists a  $Z_2$  projection that reduce ortho-symplectic Y-algebras to truncations of even  $\mathcal{W}_\infty$ ,
- Even  $\mathcal{W}_\infty$  in general does not contain consistent truncations unless we add one or few extra fields.
- There is no relation between these two.

I suspect the first to be the case. Similar construction appeared in [13] to construct special classes of truncations of  $\mathcal{N} = 1$   $\mathcal{W}_\infty$ . It would be nice to implement their argument here.

### 1.1 Parameters of $\tilde{Y}^+$

Recall definition of  $\tilde{Y}^+$  algebras

$$\begin{aligned}\tilde{Y}_{0,N,N}^+[\Psi] &= \frac{OSp(1|2N)_{-\Psi+2N+1} \times Sb^{OSp(1|2N)}}{Sp(1|2N)_{-\Psi+2N+2}} \\ \tilde{Y}_{0,M,N}^+[\Psi] &= \frac{W_{2N-2M}[OSp(1|2N)_{-\Psi+2N+1}]}{OSp(1|2M)_{-\Psi+2M+2}}\end{aligned}\tag{1.1}$$

together with

$$\begin{aligned}\tilde{Y}_{L,0,0}^+[\Psi] &= \frac{SO(2L+1)_{-\Psi-2L+1} \times Ff^{SO(2L+1)}}{SO(2L+1)_{-\Psi-2L+1}} \\ \tilde{Y}_{L,0,N}^+[\Psi] &= \frac{W_{2N}[OSp(2L+1|2N)_{-\Psi+2N-2L+1}]}{SO(2L+1)_{-\Psi-2L+2}}.\end{aligned}\tag{1.2}$$

We again assume that D3-branes can be removed if they recombine at the corner.

Central charge can be again written in factorized form as<sup>1</sup>

$$c = \frac{1}{2}(\lambda_1 + 1)(\lambda_2 + 1)(\lambda_3 + 1)\tag{1.3}$$

where

$$\begin{aligned}\lambda_1 &= 2L - 2N(1 - \Psi) - 2M\Psi \\ \lambda_2 &= \frac{X}{\Psi - 1} \\ \lambda_3 &= -\frac{X}{\Psi}.\end{aligned}\tag{1.4}$$

These parameters again permute under the duality transformations.

Vacuum character can be most easily calculated for  $\tilde{Y}_{L,0,N}^+$ . Characters of the other representations are conjecturally the same as their S-duals. Using formulas in the appendix one identifies the character with

$$\tilde{\chi}_{L,0,N}^+ = \oint dV_{SO(2L+1)} \chi_{DS}^{Sp(2N)} \chi_F[N, SO(2L+1)].\tag{1.5}$$

## 1.2 Parameters of $Y^+$ and their duals

$Y^+$  algebras are defined as

$$\begin{aligned}Y_{0,M,N}^+[\Psi] &= \frac{W_{2N-2M} Sp(2N)_{\frac{\Psi}{2}-N-1}}{Sp(2M)_{\frac{\Psi}{2}-M-\frac{3}{2}}} \quad N > M \\ Y_{L,0,0}^+[\Psi] &= \frac{SO(2L)_{-\frac{\Psi}{2}-L+1} \times Ff^{SO(2L)}}{SO(2L)_{-\frac{\Psi}{2}+M-L+\frac{3}{2}}} \\ Y_{L,0,N}^+[\Psi] &= \frac{W_{2N} [OSp(2L|2N)_{-\frac{\Psi}{2}+N-L+1}]}{SO(2L)_{-\frac{\Psi}{2}+N-L+\frac{3}{2}}}.\end{aligned}\tag{1.6}$$

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<sup>1</sup>Note the  $\pm 1$  difference compared to the unitary case.

One can see it contains the cosets used in [6] (see also [9, 11]). These algebras have dual construction in terms of

$$\begin{aligned}
Y_{0,M,M}^-[\Psi] &= \frac{SO(2M+1)_{-\Psi-2M+2}}{SO(2M)_{-\Psi-2M+2}} \\
Y_{0,M,M+1}^-[\Psi] &= \frac{SO(2M+2)_{-\Psi-2M}}{SO(2M+1)_{-\Psi-2M}} \\
Y_{0,M,N}^-[\Psi] &= \frac{W_{2N-2M-1}SO(2N)_{\Psi-2N+2}}{SO(2M+1)_{\Psi-2M}} \\
Y_{L,0,0}^-[\Psi] &= \frac{OSp(1|L)_{-\Psi+2L+2}}{Sp(2L)_{-\Psi+2L+2}} \\
Y_{L,0,N}^-[\Psi] &= \frac{W_{2N-1}[OSp(2N|2L)_{\Psi+2L+2}]}{OSp(1|2L)_{\Psi+2L}}.
\end{aligned} \tag{1.7}$$

Central charges of these algebras are

$$\begin{aligned}
c_{L,M,N}^-[\Psi] &= \tilde{c}_{L,M,N}^-[\Psi] \\
&= -\frac{(2(L-M)-1)(2(L-M)+1)(L-M)}{\Psi-1} \\
&\quad + \frac{2(2(L-N)+1)(L-N+1)(L-N)}{\Psi} \\
&\quad + 2\Psi(2(M-N)+1)(M-N+1)(M-N) \\
&\quad - 2L(1+6M^2+M(6-12N)-6N+6N^2) \\
&\quad + 4M^3-3M(1-2N)^2+N(5-12N+8N^2)
\end{aligned} \tag{1.8}$$

$$\begin{aligned}
c_{L,M,N}^+[\Psi] &= -\frac{2(M-L)(2(M-L)+1)(M-L+1)}{1-\Psi} \\
&\quad - \frac{2(N-L)(2(N-L)+1)(N-L+1)}{\Psi} \\
&\quad + \Psi(2(M-N)-1)(2(M-N)+1)(M-N) \\
&\quad + L(1-12(M-N)^2)-N+2(M-N)^2(3+2M+4N).
\end{aligned} \tag{1.9}$$

Calculation of vacuum characters goes in the same way as in the previous cases. Assuming the duality transformations that will be discussed below, we can restrict only to the calculation of characters for  $Y_{0,M,N}^-[\Psi]$ . Vacuum character can be again described as

$$\chi_{0,M,N}^- = \oint dV_{SO(2L+1)} \chi_{DS}^{SO(2N-2L-1)} \chi_B[N-M, SO(2M+1)]. \tag{1.10}$$

### 1.3 Duality transformations

Truncations given by algebra  $\tilde{Y}^+$  enjoy the same duality transformations as truncations of  $\mathcal{W}_\infty$ . There seem to be triality transformation on the parameter space of even  $\mathcal{W}_\infty$  at least if we restrict to minimal models of this class.

In the case of  $Y^-$  there is no triality (or duality) action induced on the vertex operator algebras. On the other hand, there is a duality action mapping above to two dual descriptions given by

$$Y_{L,M,N}^-[\Psi] = Y_{M,L,N}^-[\frac{1}{\Psi}] \quad (1.11)$$

and

$$Y_{L,M,N}^+[\Psi] = Y_{M,N,L}^-[1 - \frac{1}{\Psi}]. \quad (1.12)$$

## A Characters building blocks

Let us introduce following notation for different terms appearing in the calculation of characters of algebras in the main text

$$\begin{aligned} \chi_{DS}^{SO(2N+1)} &= \chi_{DS}^{Sp(2N)} = \prod_{n=0}^{\infty} \prod_{i=1}^N \frac{1}{1 - q^{2i+n}} \\ \chi_{DS}^{SO(2N)} &= \chi_{DS}^{Sp(2N)} = \prod_{n=0}^{\infty} \frac{1}{1 - q^{N+n}} \prod_{i=1}^N \frac{1}{1 - q^{2i+n}} \end{aligned} \quad (A.1)$$

together with

$$\begin{aligned} \chi_F(N, SO(2M+1)) &= \prod_{n=0}^{\infty} \frac{1}{1 - q^{n+N+1}} \prod_{j=1}^M \frac{1}{1 - x_j q^{n+N+1}} \frac{1}{1 - x_j^{-1} q^{n+N+1}} \\ \chi_F(N, SO(2M)) &= \chi_F(N, Sp(2M)) = \prod_{n=0}^{\infty} \prod_{j=1}^M \frac{1}{1 - x_j q^{n+N+1}} \frac{1}{1 - x_j^{-1} q^{n+N+1}} \\ \chi_B(N, SO(2M+1)) &= \prod_{n=0}^{\infty} (1 + q^{n+N+1}) \prod_{j=1}^M (1 + x_j q^{n+N+1})(1 + x_j^{-1} q^{n+N+1}) \\ \chi_B(N, SO(2M)) &= \chi_B(N, Sp(2M)) = \prod_{n=0}^{\infty} \prod_{j=1}^M (1 + x_j q^{n+N+1})(1 + x_j^{-1} q^{n+N+1}) \end{aligned} \quad (A.2)$$

and the measure given by

$$\oint dV_{SO(2L+1)} = \frac{1}{2^L L!} \oint \prod_{i=1}^L \frac{dx_i}{x_i} \prod_{i=1}^L \left[ (1-x_i)(1-x_i^{-1}) \prod_{j=i+1}^L (1-x_i x_j)(1-x_i^{-1} x_j)(1-x_i x_j^{-1})(1-x_i^{-1} x_j^{-1}) \right]$$

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