Towards a further understanding of the dynamics in NNLIF models: blow-up, global existence and coupled networks.

Pierre Roux, Delphine Salort

Towards a further understanding of the dynamics in the excitatory NNLIF neuron model: blow-up and global existence.

Pierre Roux* Delphine Salort †

July 1, 2021

Abstract

The Nonlinear Noisy Leaky Integrate and Fire (NNLIF) model is widely used to describe the dynamics of neural networks after a diffusive approximation of the mean-field limit of a stochastic differential equation. In previous works, many qualitative results were obtained: global existence in the inhibitory case, finite-time blow-up in the excitatory case, convergence towards stationary states in the weak connectivity regime. In this article, we refine some of these results in order to foster the understanding of the model. We prove with deterministic tools that blow-up is systematic in highly connected excitatory networks. Then, we show that a relatively weak control on the firing rate suffices to obtain global-in-time existence of classical solutions.

Key-words: Leaky integrate and fire models, noise, blow-up, neural networks, delay, global existence.

AMS Class. No: 35K60, 82C31, 92B20, 35Q84

*Laboratory of Computational and Quantitative Biology (LCQB), UMR 7238 CNRS, Sorbone Université, 75205 Paris Cedex 06, France and Laboratoire de Mathématiques d'Orsay (LMO), Université Paris-Sud, Paris-Saclay, Orsay, France. pierre.roux@universite-paris-saclay.fr

†Laboratory of Computational and Quantitative Biology (LCQB), UMR 7238 CNRS, Sorbone Université, 75205 Paris Cedex 06, France. delphine.salort@upcm.fr

1 Introduction

Since the biologically plausible mathematical models in neuroscience tend to be far too complex to be tackled numerically or analytically, researchers in computational neurosciences proposed simpler models aimed at encompassing the qualitative complexity of neural networks in a tractable framework. A popular approach is to study mean field differential models arising from stochastic differential equations. Over the last decade, many partial differential models were studied: Fokker-Planck equations with Poisson discharges [29]; population density models of integrate and fire neurons with jumps [18] and Fokker-Planck equations including conductance variables [33, 32], which are formal mean-field limits of integrate and fire networks assuming Poisson approximations [34, 15, 16]; time elapsed models [30, 31, 26], recently derived as mean-field limits of Hawkes processes [13, 12]; the mean-field McKean-Vlasov equations [25] related to the behaviour of Fitzhugh-Nagumo neurons [19], etc.

We focus here on the so-called Nonlinear Noisy Leaky Integrate & Fire model, NNLIF model in short, proposed in the late 90s by Brunel and Hakim in [2, 3]. In this model, neurons are described *via* their membrane potential v. If they reach a critical or *threshold value* V_F, the neurons emit an action potential and their voltage values return to the *reset value* V_R ($V_R < V_F$). Let the function $p(\cdot, t)$ represent the probability density of the electric potential of a randomly chosen neuron at time t. We consider the following PDE model (see [1, 2, 3] for its derivation):

$$\frac{\partial p}{\partial t}(v,t) + \frac{\partial}{\partial v}[(-v + bN(t))p(v,t)] - a\frac{\partial^2 p}{\partial v^2}(v,t) = N(t)\delta(v - V_R), \qquad v \leq V_F, \tag{1.1}$$

where we denote by δ the Dirac mass at point 0. For the sake of clearness, we will often write $\delta_{V_R} = \delta(v - V_R)$. The firing rate N of the network is given by

$$N(t) = -a\frac{\partial p}{\partial v}(V_F, t) \geq 0. \tag{1.2}$$

The parameter $a > 0$ is the *diffusion coefficient* and b is the *connectivity parameter*. If b is positive, the neural network is average-excitatory; if b is negative, the network is average-inhibitory.

The PDE (1.1) is completed with initial and boundary conditions

$$p(v,0) = p^0(v) \geq 0, \qquad \int_{-\infty}^{V_F} p^0(v)dv = 1 \quad \text{and} \quad p(V_F, t) = p(-\infty, t) = 0. \tag{1.3}$$

The deterministic NNLIF model arises as the probability density of the following non-linear stochastic mean-field equation:

$$X_t = X_0 + \int_0^t f(X_s)ds + \alpha\mathbb{E}[M_t] + \sigma B_t - M_t, \tag{1.4}$$

where $\sigma > 0$, $\alpha \in \mathbb{R}$ are parameters, $(B_t)_{t\geqslant 0}$ is a standard Brownian motion in $\mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function and $X_0 < 1$ almost surely. The process M_t counts the number of times X_t hits the threshold $V_F = 1$ before time t and $\mathbb{E}[M_t]$ represents the expected number of times the threshold is reached before t. This stochastic process was at first studied by Delarue, Inglis, Rubenthaler and Tanré in [15, 16]; the process X_t was meant to describe the potential of a typical neuron in an infinite mean-field network. As we further explain below this type of equation was investigated in a very similar mathematical setting in [20, 21, 28, 27] with the aim of modelling large portfolio credit problems or systems of banks with mutual exposures in mathematical finance.

When $f(x) = -x$, according to [15][Lemma 4.2, (iii)], the density $p(v,t)dv = \mathbb{P}(X_t \in dv)$ is a solution in the sense of distributions to (1.1)-(1.2)-(1.3) with $b = \alpha$, $a = \frac{\sigma^2}{2}$, $V_F = 1$, $V_R = 0$, $N(t) = \frac{d}{dt}\mathbb{E}[M_t]$, $p^0(v)dv = \mathbb{P}(X_0 \in dv)$. The connection between problems (1.1)-(1.2)-(1.3) and (1.4) has been the subject of recent studies: [24, 7].

The aim of this article is to investigate two aspects of the system (1.1)-(1.2)-(1.3).

The first one concerns finite-time blow-up. More precisely, our goal is to better understand which combinations of the initial condition p^0 and the connectivity strength b give rise to blow-up events. A first result was obtained by Cáceres, Carrillo and Perthame in [4][Theorem 2.2] with deterministic techniques. They proved that in the excitatory case $b > 0$, finite-time blow-up occurs in at least two cases: if the initial condition is concentrated enough around the firing potential V_F depending on the value of $b > 0$ or if b is large enough depending on the initial condition p^0.

We prove here that if we choose b large enough, then blow-up occurs independently of the initial condition p^0. Note that finite-time blow-up was also studied in the stochastic system (1.4). Hambly, Ledger and Søjmark proved in [21][Theorem 1.1] that for all initial condition X_0 there exists a large enough connectivity α such that the solution blows-up in finite time; their bound on α depends only on the first moment of the law of X_0.

There is another interesting question, which we are not addressing in this article: is it possible to extend the solutions after a blow-up event? Carrillo, González, Gualdani and Schonbek proved in [11][Theorem 1.1] that a classical solution blows-up in finite time if and only if the firing rate $N(t)$ diverges in finite time. Delarue, Inglis, Rubenthaler and Tanré proved in [16] that it is possible in some cases to continue solutions of the stochastic problem (1.4) after a blow-up event: they define a notion of *physical solution* which is weaker than the notion of solutions the same authors proposed before in [15]. They view blow-up as a synchronisation of some macroscopic part of the network. To sum up their idea: in case of blow-up, the particle defined by (1.4) feels a *kick* after being reset at V_R and makes a corresponding instantaneous jump towards V_F. If the jump doesn't send it back to V_F the solution is said to be physical and can continue; the jump is proportional to the size of the part of the network which synchronises during the blow-up event. When viewed through the mathematical finance viewpoint, the jumps in *physical solutions* are even more important since they may represent, for example, systematic crises in a system of banks with mutual exposures. Hence, many recent articles focused on the parameter range which yields blowing-up solutions. Nadtochiy and Shkolnikov proved in [27][Theorem 2.6] the uniqueness of *physical solutions* as long as $\|N(\cdot)\|_{L^2} < +\infty$. Hambly, Ledger and Søjmark proved in [21][Theorem 1.2] that these *physical solutions* of (1.4) have minimal jumps after blow-up and they made progresses towards a proof of unicity and blow-up rate for the *physical solutions* in the space of càdlàg functions ([21][Theorem 1.8] and the discussion therein). This last result resolved previous ambiguity about the validity of the propagation of chaos for the underlying particle system studied, *e.g.*, in [16, 27]. Recently, more methods have been developed to extend solutions of similar stochastic equations in spite of the presence of blow-up ([17], [23]). In particular, Delarue, Nadtochiy and Shkolnikov proved in [17][Theorem 1.4] the uniqueness of global-in-time *physical solutions* for the supercooled Stefan problem (which is similar to the NNLIF model after rescaling, [11]) under mild hypotheses; the authors also proved that blow-up points are at most countable and they characterised the smoothness of the firing rate after blow-up events (however, they didn't rule out the possibility of their accumulation).

Secondly, we investigate global-in-time existence of classical solutions for system (2.1) in the excitatory case ($b > 0$). As we said above, given $b > 0$ there are blowing-up solutions for initial conditions $p_b^0(\cdot)$ concentrated enough around V_F ([4][Theorem 2.2]). Hence, we cannot hope to find a criterion for global-in-time existence independently of the initial condition. In this article, we prove that given an initial condition p^0, we can find a value $b^*(p^0)$ such that if $0 < b < b^*$, then the classical solution is global-in-time (see Theorem 4.5). Such a result was obtained for the associated stochastic equation (1.4) in [15][Theorem 2.4]: for any initial condition $X_0 = x_0 < 1$, there exists $\alpha^*(x_0) \in]0, 1[$ such that for all $\alpha \in]0, \alpha^*[$, there exists a global-in-time solution of (1.4) such that $t \mapsto \mathbb{E}[M_t]$ is C^1 on $\mathbb{R}_+$ (*i.e.* the firing rate is continuous). Note that in [27, 21], the firing rate $N(t)$ is in L^2_{loc} until the first time of explosion ($\mathbb{E}[M_t]$ is in H^1). However, up until now there did not exist any deterministic method to obtain a global-in-time existence result for problem (1.1)-(1.2)-(1.3) in the excitatory case. Indeed, local-in-time classical solutions were constructed by Carrillo, Gonzales, Gualdani and Schonbeck in [11][Theorem 1.1] but global-in-time existence was only proved in the inhibitory case ($b < 0$). Let us mention that this result for the inhibitory case was obtained with another method (universal super-solutions) in [10][Corollary 4.5].

The main difficulty in the proof of Theorem 4.5 is proving that

$$T^* := \sup\{t > 0 \mid N(t) < +\infty\} = +\infty$$

in order to apply the result of [11][Theorem 1.1] which states that T^* is the maximal time of existence. To do this, we proceed in two steps. First, we prove uniform *a priori* estimates on the firing rate $N(t)$ in L^q, $q > 2$. The strategy is to construct new entropy estimates inspired from the method the authors of [10][Theorem 3.1] used to prove uniform L^2 estimates on $N(t)$. Second, we recast the problem (1.1)-(1.2)-(1.3) into a Stefan-like free boundary problem *via* a change of variables as proposed in [11], which allows us to obtain an implicit Duhamel formula on $N(t)$; we use this formula to lift our L^q estimates ($q > 2$) to L^∞ estimates. Let us mention that $q > 2$ is necessary to conduct our proof (see Remark 4.8).

To complete the state of the art on this type of equations, note that other variants of the classical NNLIF system were proposed and studied. In [6], Cáceres and Perthame considered the NNLIF system with a refractory period. In [9], Cáceres and Schneider proposed a general system with excitatory-inhibitory coupling, a refractory state and a synaptic delay (the transmission of pulses in not instantaneous any more). Numerical simulations are provided that show stable periodic solutions in the delayed case. Cáceres, Roux, Salort and Schneider ([8]) proved some results on the NNLIF equation with synaptic delay: global-in-time existence for all smooth enough initial data for both positive and negative values of b, convergence towards stationary states for small connectivities b, L^2 estimates on the firing rate. Some papers proposed suitable numerical methods for NNLIF type systems: [5], [35], [22], [36].

This article is organised as follows. In Section 2, we introduce the definitions of solution we will use throughout the article. Section 3 focuses on finite-time blow-up: we prove that for b large enough, there is no global-in-time solution to (2.1) – even in the weak sense – independently of the initial datum. Section 4 is about global-in-time existence of the solutions of (2.1); we first derive, in Subsection 4.1, uniform L^q estimates on the firing rate N for small enough connectivities; then, in Subsection 4.2, we use the previous estimates to prove global-in-time existence in the excitatory case for sufficiently small connectivity parameters $0 < b < b^*(p^0)$.

2 Notions of solution

We are interested in the classical NNLIF system which writes as follows. Let $V_R, V_F \in \mathbb{R}$ such that $V_R < V_F$. Let $b \in \mathbb{R}$, $a \in \mathbb{R}_+^*$. We study the system

$$\begin{gathered}
\frac{\partial p}{\partial t} + \frac{\partial}{\partial v}\Big[(-v + bN(t))p\Big] - a\frac{\partial^2 p}{\partial v^2} = \delta_{V_R} N(t), \\
N(t) = -a\frac{\partial p}{\partial v}(V_F, t), \quad p(V_F, t) = 0, \quad p(-\infty, t) = 0, \\
p(v, 0) = p^0(v) \geqslant 0, \quad \int_{-\infty}^{V_F} p^0(v)dv = 1.
\end{gathered} \tag{2.1}$$

We define both classical and weak solutions for this system, following [11] and [4]:

Definition 2.1 *We say that* (p,N) *is a* classical *(fast-decreasing)* solution*, of system* (2.1) *on* $[0,T^*[\,,\ T^*\in\mathbb{R}_+^*\cup\{+\infty\}$ *if*

- $p\in\mathscr{C}^0(\,]-\infty,V_F]\times[0,T^*[\,)\cap\mathscr{C}^{2,1}\Big((\,]-\infty,V_R[\cup]V_R,V_F])\times[0,T^*[\Big)\cap L^\infty\Big([0,T^*[\,,L^1_+(\,]-\infty,V_F])\Big)$ *and* $N\in\mathscr{C}^0([0,T^*[\,)$*;*
- *Functions* p *and* N *satisfy* (2.1) *in the classical sense on* $]-\infty,V_R[\cup]V_R,V_F]$ *and in the sense of distributions in* $]-\infty,V_F]$*;*
- $\forall t\in[0,T^*[\,,\forall Q\in\mathbb{R}[X]$*,* $\lim_{v\to-\infty}Q(v)p(v,t)=0$ *and* $\lim_{v\to-\infty}Q(v)\dfrac{\partial p}{\partial v}(v,t)=0$ *(fast-decreasing).*

And, for weak solutions:

Definition 2.2 *Let* $T^*\in\mathbb{R}_+^*\cup\{+\infty\}$*. Let* $p\in L^\infty([0,T^*[\,,L^1_+(\,]-\infty,V_F[\,))$ *and* $N\in L^1_{loc,+}([0,T^*[\,)$*. The pair* (p,N) *is said to be a* weak solution *of* (2.1) *if for every test function* $\phi\in\mathscr{C}^\infty(\,]-\infty,V_F])$ *such that for almost every* $t\in[0,T^*[\,$,

$$\Big(\phi(\cdot)p(\cdot,t)\ ,\ \frac{\partial^2\phi}{\partial v^2}(\cdot)p(\cdot,t)\ ,\ (|v|+1)\frac{\partial\phi}{\partial v}(\cdot,t)p(\cdot,t)\Big)\in\Big(L^1(\,]-\infty,V_F[\,)\Big)^3,\tag{2.2}$$

we have

$$\frac{d}{dt}\int_{-\infty}^{V_F}p(v,t)\phi(v)dv=\int_{-\infty}^{V_F}\Big[\big(-v+bN(t)\big)\frac{\partial\phi}{\partial v}+a\frac{\partial^2\phi}{\partial v^2}\Big]p(v,t)dv+N(t)\Big(\phi(V_R)\ -\ \phi(V_F)\Big),\tag{2.3}$$

where the time derivative is to be taken in the sense of distributions and where $\lim_{t\to0}\big\|p(\cdot,t)-p^0(\cdot)\big\|_{L^1}=0$*. A weak solution is said to be fast-decreasing if for all polynomial function* $Q\in\mathbb{R}[X]$*, for almost every* $t\in[0,T^*[\,$*, we have* $\lim_{v\to-\infty}Q(v)p(v,t)=0$*.*

Remark 2.3 *If we apply the definition of weak solution with the test function* $\phi=1$*, we obtain for* $t\in\mathbb{R}_+$*,*

$$\int_{-\infty}^{V_F}p(v,t)dv=\int_{-\infty}^{V_F}p^0(v)dv=1.$$

For the sake of brevity in our results, we will refer to the following regularity assumptions on initial data.

Assumptions 2.4 (Initial data for classical solutions) $p^0\in\mathscr{C}^0(\,]-\infty,V_F])\cap\mathscr{C}^1(\,]-\infty,V_R[\cup]V_R,V_F])\cap L^1(\,]-\infty,V_F])$ *is non-negative, fast decreasing at* $-\infty$ *and* $p^0(V_F)=0$*.* $\dfrac{dp^0}{dv}$ *admits finite left and right limits at* V_R *and is fast decreasing at* $-\infty$*.*

To sum up, the following is known ([11], [4],[10]) on the solutions of the deterministic problem (2.1):

- For any initial datum satisfying Assumptions (2.4), there exists a unique local-in-time classical solution of (2.1) and its maximal time of existence satisfies $T^*=\sup\{t>0\mid N(t)<+\infty\}$.

- The shape of stationary states is known (see (4.3) below).

- In the inhibitory or in the linear case ($b \leqslant 0$), $T^* = +\infty$, there exists a unique stationary state and the solutions are uniformly bounded; the firing rate N is also uniformly bounded. There exists a constant $C > 0$ such that if $-C < b \leqslant 0$, the solution converges exponentially fast towards the stationary state (p_∞, N_∞).

- In the excitatory case ($b > 0$), there exists p^0 such that no weak solution can be global-in-time. If b is large enough, there is no stationary state. If b is small enough, there is a unique stationary state; if b is small enough regarding p^0, the solution converges (up to it's time of existence) exponentially fast towards the unique stationary state. For intermediate values of b, two or more stationary states can coexist.

3 Blow-up for all initial conditions when b is large enough

We extend here the result of [4] on finite-time blow-up in the NNLIF system which states that for every positive b there exist blowing-up solutions from initial data $p_b^0(\cdot)$ concentrated enough around V_F. We prove that for large enough positive values of b all solutions blow-up in finite time regardless of the initial repartition of their mass.

The main idea is to use an evolution equation for the linear moment in order to control the quantity of mass that stays near V_F: we prove that if we wait for a long enough time t_0, some quantity of the initial mass ($\left\|p^0\right\|_1 = 1$ by hypothesis) will stay between V_R and V_F and this quantity is independent from the initial condition p^0. Then, we use the method of [4] and we prove, thanks to our control on the mass between V_R and V_F, that the exponential moment is unbounded, thus reaching a contradiction with mass conservation.

Theorem 3.1 *We have*

- *If* $V_F \leqslant 0$, *then if* $b \geqslant V_F - V_R$, *no fast-decreasing weak solution of* (2.1) *can be global in time.*

- *If* $V_F > 0$, *then if*
$$b > \max\left(V_F - V_R \ , \ \frac{V_F}{4} \inf_{y \in]1, \sqrt{1+\frac{a}{4}}[} (1+y)^2 \left(1 - e^{-4\frac{V_F - V_R}{V_F}\frac{1}{y^2-1}}\right) e^{\frac{2}{y-1}} \right),$$
no fast-decreasing weak solution of (2.1) *can be global in time.*

Proof. Assume (p, N) is a global in time fast-decreasing weak solution. We denote
$$M_\phi(t) = \int_{-\infty}^{V_F} (V_F - v) p(v,t) dv$$

and we use the definition of weak solution with the test function $\phi(v) = V_F - v$; it yields

$$\frac{d}{dt}M_\phi(t) = V_F - M_\phi(t) + N(t)(V_F - V_R - b).$$

If $b \geqslant V_F - V_R$, $\dfrac{d}{dt}M_\phi(t) \leqslant V_F - M_\phi(t)$, that is to say

$$\frac{d}{dt}\Big(M_\phi(t) - V_F\Big) \leqslant -\Big(M_\phi(t) - V_F\Big).$$

Using Grönwall's lemma, we deduce

$$M_\phi(t) - V_F \leqslant \Big(M_\phi(0) - V_F\Big)e^{-t}.$$

If $M_\phi(0) \leqslant V_F$, then $M_\phi(t) \leqslant V_F$ for all $t \in \mathbb{R}_+^*$ and otherwise we have the exponential convergence of $M_\phi(t)$ towards V_F. Thus, we have

$$\forall \varepsilon \in \mathbb{R}_+^*, \exists t_0 \in \mathbb{R}_+^* \text{ such that } \forall t \in [t_0, +\infty[\,,\ M_\phi(t) \leqslant V_F + \varepsilon.$$

• Assume $V_F < 0$:

There exists $\varepsilon \in \mathbb{R}_+^*$ small enough such that $V_F + \varepsilon < 0$ and then the quantity $M_\phi(t)$ becomes negative after the time t_0, which is a contradiction. Therefore, no weak solution can be global in time in this case.

• Assume $V_F \geqslant 0$:

We have, for all $\eta \in \mathbb{R}_+^*$,

$\forall t \in [t_0, +\infty]$,

$$\eta \int_{-\infty}^{V_F-\eta} p(v,t)dv \leqslant \int_{-\infty}^{V_F-\eta} (V_F - v)p(v,t)dv \leqslant M_\phi(t) \leqslant V_F + \varepsilon,$$

and by choosing $\eta = (V_F + \varepsilon)\xi$, where $\xi \in]1, +\infty[$ is a yet unchosen value, we obtain

$$\forall t \in [t_0, +\infty], \quad \int_{-\infty}^{V_F-\eta} p(v,t)dv \leqslant \frac{V_F + \varepsilon}{\eta} < \frac{1}{\xi}.$$

Since p is a probability density on $]-\infty, V_F[$ we have,

$$\int_{V_F-\eta}^{V_F} p(v,t)dv \geqslant \frac{\xi - 1}{\xi}.$$

Let's take now the test function defined by $\psi(v) = e^{\mu v}$. We denote

$$M_\mu(t) = \int_{-\infty}^{V_F} p(v,t)e^{\mu v}dv.$$

The previous computations yield,

$$\forall t \in [t_0, +\infty], \quad M_\mu(t) \geqslant \int_{V_F-\eta}^{V_F} e^{\mu v}p(v,t)dv \geqslant e^{\mu(V_F-\eta)} \int_{V_F-\eta}^{V_F} p(v,t)dv \geq \frac{\xi - 1}{\xi}e^{\mu(V_F-\eta)}.$$

We write the weak solution definition for ψ:

$$\frac{d}{dt}M_\mu(t) = \mu \int_{-\infty}^{V_F} (-v + bN(t))e^{\mu v}p(v,t)dv + a\mu^2 M_\mu(t) + N(t)(e^{\mu V_R} - e^{\mu V_F}),$$

which gives the following bound, for $t > t_0$:

$$\frac{d}{dt}M_\mu(t) \geqslant -V_F\mu M_\mu(t) + \mu b N(t) M_\mu(t) + a\mu^2 M_\mu(t) + N(t)(e^{\mu V_R} - e^{\mu V_F}).$$

The later writes

$$\frac{d}{dt}M_\mu(t) \geqslant N(t)\Big(b\mu M_\mu(t) + e^{\mu V_R} - e^{\mu V_F}\Big) + \mu(a\mu - V_F)M_\mu(t).$$

Thus, we have,

$$\frac{d}{dt}M_\mu(t) \geqslant N(t)\Big(\mu b\frac{\xi-1}{\xi}e^{\mu(V_F-\eta)} + e^{\mu V_R} - e^{\mu V_F}\Big) + \mu(a\mu - V_F)M_\mu(t).$$

If b satisfies

$$b > \frac{\xi}{\xi-1}\frac{e^{\mu V_F} - e^{\mu V_R}}{\mu e^{\mu(V_F-\eta)}} = \frac{\xi}{\xi-1}\frac{e^{\mu V_F} - e^{\mu V_R}}{\mu e^{\mu V_F}} e^{\mu V_F \xi} e^{\mu\varepsilon\xi}, \tag{3.1}$$

we then have

$$\forall t \in [t_0, +\infty[\,, \quad \frac{d}{dt}M_\mu(t) \geqslant \mu(a\mu - V_F)M_\mu(t).$$

If we also assume that $\mu > \dfrac{V_F}{a}$, by Grönwall's lemma and since $M_\mu(t_0) > 0$, we obtain

$$\lim_{t\to+\infty} M_\mu(t) = +\infty,$$

which is a contradiction with the inequality

$$\forall t \in \mathbb{R}_+^*, \quad M_\mu(t) < e^{\mu V_F}\int_{-\infty}^{V_F} p(v,t)dv = e^{\mu V_F}.$$

Therefore, if b satisfies condition (3.1) for some $\varepsilon > 0$, the solution cannot be global in time. As $\varepsilon \in \mathbb{R}_+^*$ can be arbitrarily small, if

$$b > \frac{\xi}{\xi-1}\frac{e^{\mu V_F} - e^{\mu V_R}}{\mu e^{\mu V_F}} e^{\mu V_F \xi}, \tag{3.2}$$

then there is no global-in-time fast-decreasing weak solution.

Taking the infimum on $\xi \in]1, +\infty[$ and $\mu \in]\frac{V_F}{a}, +\infty[$ in the right-hand side of (3.2) and applying Lemma A.1 of the appendix, we get the final result. $\square$

Remark 3.2 *When $V_F \leq 0$ the bound is optimal; indeed, for $b < V_F - V_R$ there exists at least one stationary state, which constitutes a global in time fast-decreasing solution. In the case $V_F > 0$, the optimal bound can be strictly above $V_F - V_R$ because for some parameters there exist stationary states in the case $b > V_F - V_R$ (see [4][Theorem 3.1]). Yet, we probably have a sub-optimal lower bound for b because of the limitations of our method. Finding the optimal bound and linking it to the existence of stationary states is an interesting open question.*

Remark 3.3 *In the article [4], the authors study the number of stationary states depending on the parameters.*

They find that there is no stationary state under the condition

$$b > \max\left(2(V_F - V_R)\,,\, 2V_F \int_0^{+\infty} \left(e^{\frac{sV_F}{\sqrt{a}}} - e^{\frac{sV_R}{\sqrt{a}}} \right) \frac{e^{-s^2}}{s} ds \right). \tag{3.3}$$

We did a numerical quantitative comparison (see Figure 1) between (3.3) *and the lower bound for* b *we provide in Theorem 3.1. Indeed, if every solution blows-up in finite time, there can be no stationary state. The numerical simulations indicate that our bound is better when* a *or* $V_F - V_R$ *are large.*

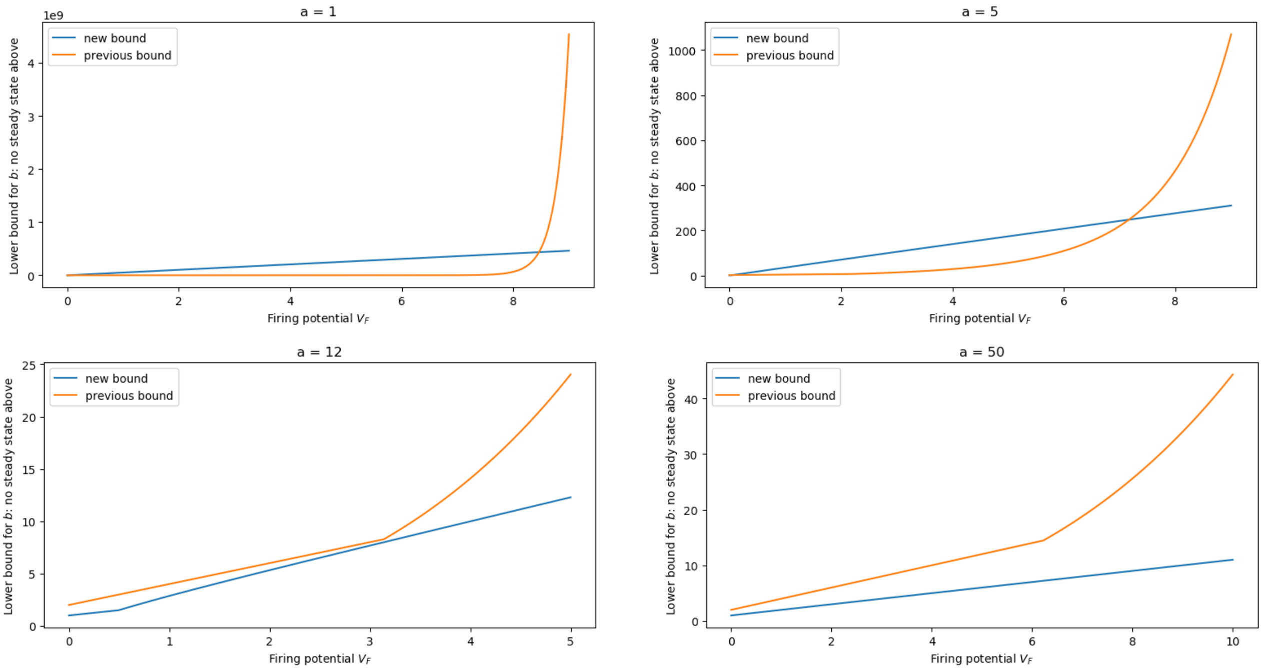

Figure 1: *Comparison between the lower bound* (3.3) *in [4] and the lower bound in Theorem 3.1 for non-existence of stationary states, for different values of* a*. The new bound is always better when* $V_F - V_R$ *is large enough. We set* $V_R = -1$ *for convenience but it does not impact the results.*

4 Global-in-time existence in excitatory networks

In this section, we prove global-in-time existence in an excitatory network for a wide class of initial data. To do so, we first improve the L^2 estimates on the firing rate in [10][Theorem 3.1] into L^q estimates. Then, we take advantage of the L^q estimates (we choose $q = 3$ for clarity), $q > 2$, and we obtain global-in-time existence for appropriate parameters and initial data by following ideas of [11].

4.1 *A priori* L^q estimates for the firing rate

For the sake of clarity, we assume in all this section that

$$0 < V_R < V_F. \tag{4.1}$$

This can be done without loss of generality by using the rescaling $\bar{p}(v,t) = \alpha p(\beta v + \gamma t, t)$, $\bar{N}(t) = \frac{\alpha}{\beta} N(t)$, with $\alpha, \beta > 0,\ \gamma \in \mathbb{R}$.

The authors of [10][Theorem 3.1] proved uniform L^2 estimates on the firing rate: for all $b \in]-\infty, \eta]$,

$$\int_0^T N(t)^2 dt \leqslant C(1+T).$$

In the excitatory case $b > 0$, $\eta = \eta(p^0) > 0$ depends on the initial datum p^0. Making more smallness hypotheses on b, they used these L^2 estimates in order to prove convergence to the unique (unique because b is then small enough) stationary state *via* an entropy method and a Poincaré-like inequality. As explained in Remark 4.8 bellow, these L^2 estimates are not sufficient to prove global-in-time existence; the convergence to stationary state result in [10] is only valid up to an unknown time of existence. Thus we extend the result to L^q estimates under similar hypotheses. More precisely, we prove:

Theorem 4.1 *Let $b_1 \in \mathbb{R}_+^*$ be such that there exists a stationary state (p_∞^1, N_∞^1). Let $V_M \in]V_R, V_F[$; denote*

$$S_q(b_1, V_M) = \int_{V_M}^{V_F} \frac{p^0(v)^q}{p_\infty^1(v)^{q-1}} dv.$$

For every exponent $q \in \mathbb{N},\ q \geqslant 2$, for every initial condition p^0 such that $S_q(b_1, V_M) < +\infty$, for every fast-decreasing classical solution of (2.1) *from p^0,*

- *There exists a constant $C_q \in \mathbb{R}_+^*$ depending only on V_F, q, b_1 and a (thus independent from $S_q(b_1, V_M)$) and a constant $T \in \mathbb{R}_+$ which depends only on V_M and $S_q(b_1, V_M)$ such that for every interval $I \subset]T, +\infty[$ and for all $b \in]-\infty, 0]$,*

$$\int_I N(t)^q dt \leqslant C_q(1 + |I|).$$

- *Let $b \in \mathbb{R}_+^*$. If b is small enough regarding $S_q(b_1, V_M)$ and V_M, there exists a constant $C_q \in \mathbb{R}_+^*$ depending only on V_F, q, a, b_1, V_M and $S_q(b_1, V_M)$ such that for every interval $I \subset \mathbb{R}_+$,*

$$\int_I N(t)^q dt \leqslant C_q(1 + |I|).$$

Let us first make some comments on the term

$$S_q(b_1, V_M) = \int_{V_M}^{V_F} \frac{p^0(v)^q}{p_\infty^1(v)^{q-1}} dv$$

and the technical hypothesis $S_q(b_1, V_M) < +\infty$.

- The main idea is that these uniform estimates on the firing rate $N(t)$ depend only upon the initial repartition of mass in a neighbourhood of V_F: if we choose any parameter V_M such that $V_R < V_M < V_F$, the proof of Theorem 4.1 only uses the equations on $[V_M, V_F]$ and the shape of the initial condition p^0 in $]-\infty, V_M[$ doesn't have any impact on the results. Hence, if $b < 0$ or if $b > 0$ is small enough regarding $S_q(b_1, V_M)$ and V_M, we can control uniformly the L^q norm of the firing rate.

- The technical hypothesis $S_q(b_1, V_M) < +\infty$ is necessary to apply our method because we study at some point an ODE whose initial condition is close to $S_q(b_1, V_M)$. Thus, we need this initial quantity to be finite because otherwise the computations don't make sense. It is not a restrictive hypothesis as a sufficient condition to have it is simply (see computation (4.27) below): $p^0 \in L^\infty(]V_R, V_F[)$ and
$$\limsup_{v \to V_F} \frac{p^0(v)}{V_F - v} < +\infty.$$
Note that the rate of decay at the boundary is an important matter for this type of equation (see remark 4.10 below).

Then, let us recall from [4][Section 3.] that the stationary states (p_∞, N_∞) satisfy
$$\begin{aligned} \frac{\partial}{\partial v}\Big[(-v + bN_\infty)p_\infty\Big] - a\frac{\partial^2 p_\infty}{\partial v^2} &= \delta_{V_R} N_\infty, \\ N_\infty = -a\frac{\partial p_\infty}{\partial v}(V_F), \quad p_\infty(V_F) &= p_\infty(-\infty) = 0, \end{aligned} \tag{4.2}$$
and are of the form
$$p_\infty(v) = \frac{N_\infty}{a} \mathbf{e}^{-\frac{(v-bN_\infty)^2}{2a}} \int_{\max(v,V_R)}^{V_F} \mathbf{e}^{\frac{(w-bN_\infty)^2}{2a}} dw. \tag{4.3}$$
where N_∞ is a solution of a fixed-point equation defined and studied in [4].

Let (p^1_∞, N^1_∞) be a stationary state associated to a parameter $b_1 > 0$. Let (p, N) be the unique classical solution associated with the parameter b and the suitable initial condition p^0. We denote
$$H(v,t) = \frac{p(v,t)}{p^1_\infty(v)} \qquad \text{and} \qquad W(v,t) = p^1_\infty(v) H(v,t)^q = \frac{p(v,t)^q}{p^1_\infty(v)^{q-1}}.$$
We will also use the following cutoff function
$$\forall v \in]-\infty, V_F], \quad \gamma(v) = e^{\frac{-1}{\beta - (V_F - v)^2}} \text{ if } v > \alpha, \qquad \gamma(v) = 0 \text{ otherwise}, \tag{4.4}$$
where $\beta = (V_F - \alpha)^2$ and $\alpha \in]V_R, V_F[$. Function γ has some handy properties one can find in [10][Lemma 3.4]:

Lemma 4.2 (Carrillo, Perthame, Salort, Smets) *Let $0 < V_R < V_F$ and let $\alpha \in]V_R, V_F[$. Then γ defined in (4.4) is a positive increasing function on $]\alpha, V_F[$ and the following properties hold:*

1. $\lim_{v \to V_F} \frac{\gamma'(v)}{\gamma(v)} = 0$;

2. *There exists a constant $C > 0$ such that $\gamma'^2 + \gamma''^2 + \gamma'''^2 \leqslant C\gamma$;*

3. There exists a constant $\eta \in]\alpha, V_F[$ such that $\gamma"(\eta) = 0$ and for all $v \in]\eta, V_F[$, $\gamma''(v) \leqslant 0$.

The following technical lemma is proved in the appendix. The proof is purely computational.

Lemma 4.3 *Let (p, N) a fast-decreasing classical solution of (2.1) for a parameter $b \in \mathbb{R}$ and let $b_1 \in \mathbb{R}_+^*$ such that there exists at least one stationary state (p_∞^1, N_∞^1) for equation (2.1). Then, with the notations above,*

$$\begin{aligned}\frac{d}{dt}\int_{-\infty}^{V_F} W(v,t)\gamma(v)dv \leqslant \int_{-\infty}^{V_F} \big(-v+bN(t)\big)W(v,t)\gamma'(v)dv + a\frac{\partial W}{\partial v}(V_F,t)\gamma(V_F) - aW(V_F,t)\gamma'(V_F)\\ - aq(q-1)\int_{-\infty}^{V_F} p_\infty^1(v)H(v,t)^{q-2}\left(\frac{\partial H}{\partial v}(v,t)\right)^2\gamma(v)dv + a\int_{-\infty}^{V_F} W(v,t)\gamma''(v)dv\\ - (q-1)(bN(t)-b_1N_\infty^1)\int_{-\infty}^{V_F}\frac{\partial p_\infty^1}{\partial v}(v)H(v,t)^q\gamma(v)dv. \quad (4.5)\end{aligned}$$

We now prove another technical lemma, this proof is inspired from the proof of Theorem 3.1 in [10].

Lemma 4.4 *Let $\alpha \in]V_R, V_F[$. If α is close enough to V_F,*

- *If $b \leqslant 0$, there exist constants $C_1, C_2 \in \mathbb{R}_+^*$ depending only on α, V_F, q, b_1 and a such that*

$$\begin{aligned}\frac{d}{dt}\int_{-\infty}^{V_F} W(v,t)\gamma(v)dv \leqslant -\frac{\gamma(V_F)}{{N_\infty^1}^{q-1}}N(t)^q\\ + (q-1)(bN(t)-C_1)\int_{-\infty}^{V_F} W(v,t)\gamma(v)dv + C_2\int_\alpha^{V_F} H(v,t)^{q-2}p_\infty^1(v)dv. \quad (4.6)\end{aligned}$$

- *If $b > 0$, there exist constants $C_3, C_4, C_5, C_6, C_7 \in \mathbb{R}_+^*$ depending only on α, V_F, q, b_1 and a such that*

$$\begin{aligned}\frac{d}{dt}\int_{-\infty}^{V_F} W(v,t)\gamma(v)dv \leqslant -\frac{\gamma(V_F)}{{N_\infty^1}^{q-1}}N(t)^q\\ + b^qN(t)^q\Big(C_3 + C_4\int_{-\infty}^{V_F} W(v,t)\gamma(v)dv + C_5\int_\alpha^{V_F} H(v,t)^{q-2}p_\infty^1(v)dv\Big)\\ + C_6\int_\alpha^{V_F} H(v,t)^{q-2}p_\infty^1(v)dv - C_7\int_{-\infty}^{V_F} W(v,t)\gamma(v)dv. \quad (4.7)\end{aligned}$$

Proof.

• Inhibitory case:

Looking at the expression of p_∞^1 in formula (4.3), we can see that if α is close enough to V_F, p_∞^1 is of class $\mathscr{C}^\infty$ on $[\alpha, V_F]$ and we have

$$\forall v \in [\alpha, V_F], \quad \frac{\partial p_\infty^1}{\partial v}(v) \leqslant -p_\infty^1(v).$$

Since $b \leqslant 0$, we have

$$-(bN(t)-b_1N_\infty^1)\int_{-\infty}^{V_F}\frac{\partial p_\infty^1}{\partial v}(v)H(v,t)^q\gamma(v)dv \leqslant (bN(t)-b_1N_\infty^1)\int_{-\infty}^{V_F} W(v,t)\gamma(v)dv. \quad (4.8)$$

Moreover, since γ is increasing, b is non-positive and α is positive, we have

$$\int_{-\infty}^{V_F} \big(-v + bN(t)\big) W(v,t)\gamma'(v)dv \leqslant 0. \tag{4.9}$$

Using boundary conditions and the facts that $\partial_v p(V_F,t) = \lim_{v\to V_F} \frac{-p(v,t)}{V_F - v}$ and $\partial_v p^1_\infty(V_F) = \lim_{v\to V_F} \frac{-p^1_\infty(v)}{V_F-v} > 0$, we compute

$$W(V_F,t) = \lim_{v\to V_F} \left(\frac{p(v,t)^{q-1}}{p^1_\infty(v)^{q-1}} p(v,t)\right) = \lim_{v\to V_F} \left[\left(\frac{\frac{-p(v,t)}{V_F-v}}{\frac{-p^1_\infty(v)}{V_F-v}}\right)^{q-1} p(v,t)\right] = \left(\frac{\frac{\partial p}{\partial v}(V_F,t)}{\frac{\partial p^1_\infty}{\partial v}(V_F)}\right)^{q-1} p(V_F,t) = 0. \tag{4.10}$$

and

$$\begin{aligned}
-a\frac{\partial W}{\partial v}(V_F,t) &= -a \lim_{v\to V_F} \frac{q\frac{\partial p}{\partial v}(v,t)p(v,t)^{q-1}p^1_\infty(v)^{q-1} - (q-1)\frac{\partial p^1_\infty}{\partial v}(v)p^1_\infty(v)^{q-2}p(v,t)^q}{p^1_\infty(v)^{2(q-1)}}\\
&= \lim_{v\to V_F} \frac{q\left(-a\frac{\partial p}{\partial v}(v,t)\right)\left(a\frac{p(v,t)}{V_F-v}\right)^{q-1}\left(a\frac{p^1_\infty(v)}{V_F-v}\right)^{q-1} - (q-1)\left(-a\frac{\partial p^1_\infty}{\partial v}(v)\right)\left(a\frac{p^1_\infty(v)}{V_F-v}\right)^{q-2}\left(a\frac{p(v,t)}{V_F-v}\right)^{q}}{\left(a\frac{p^1_\infty(v)}{V_F-v}\right)^{2(q-1)}}\\
&= \frac{qN(t)^q {N^1_\infty}^{q-1} - (q-1)N(t)^q {N^1_\infty}^{q-1}}{{N^1_\infty}^{2(q-1)}} = \frac{N(t)^q}{{N^1_\infty}^{q-1}}.
\end{aligned} \tag{4.11}$$

The function γ being non-negative and its second order derivative being non-positive on an interval $[\eta, V_F]$ for some $\eta \in]\alpha, V_F[$ (see Lemma 4.2), we deduce from (4.5), (4.8), (4.9), (4.10) and (4.11) the inequality

$$\begin{aligned}
\frac{d}{dt}\int_{-\infty}^{V_F} W(v,t)\gamma(v)dv \;&\leqslant\; -\frac{N(t)^q}{{N^1_\infty}^{q-1}}\gamma(V_F) - aq(q-1)\int_{-\infty}^{V_F} H(v,t)^{q-2}p^1_\infty(v)\left(\frac{\partial H}{\partial v}(v,t)\right)^2\gamma(v)dv\\
&\quad + a\int_{-\infty}^{\eta} W(v,t)\gamma''(v)dv + (q-1)(bN(t) - b_1N^1_\infty)\int_{-\infty}^{V_F} W(v,t)\gamma(v)dv.
\end{aligned} \tag{4.12}$$

Now, let's control the term containing γ''. We compute, using integration by parts and $\gamma(\eta) = 0$ (Lemma 4.2),

$$\begin{aligned}
\int_{-\infty}^{\eta} W(v,t)\gamma''(v)dv &= \int_{-\infty}^{\eta} H^{q-1}\frac{\partial}{\partial v}\left(\int_{-\infty}^{v} p(w,t)dw\right)\gamma'' dv\\
&= -\int_{-\infty}^{\eta}\frac{\partial H^{q-1}}{\partial v}\left(\int_{-\infty}^{v} p(w,t)dw\right)\gamma'' dv - \int_{-\infty}^{\eta} H^{q-1}\left(\int_{-\infty}^{v} p(w,t)dw\right)\gamma''' dv\\
&= -(q-1)\int_{-\infty}^{\eta}\frac{\partial H}{\partial v}H^{q-2}\left(\int_{-\infty}^{v} p(w,t)dw\right)\gamma'' dv - \int_{-\infty}^{\eta} H^{q-1}\left(\int_{-\infty}^{v} p(w,t)dw\right)\gamma''' dv.
\end{aligned}$$

Moreover, since p is a probability density,

$$\left|\int_{-\infty}^{\eta} W(v,t)\gamma''(v)dv\right| \leqslant (q-1)\int_{-\infty}^{\eta}\left|\frac{\partial H}{\partial v}\right| H^{q-2}\left|\gamma''(v)\right| dv + \int_{-\infty}^{\eta} H^{q-1}\left|\gamma'''(v)\right| dv.$$

If α is close enough to V_F, there exists $\tilde{C} \in \mathbb{R}^*_+$ such that $p^1_\infty \geqslant \tilde{C}$ on $[\alpha, \eta]$, and with Peter and Paul inequality,

$$\begin{aligned}
\left|\int_{-\infty}^{\eta} W(v,t)\gamma''(v)dv\right| \;&\leqslant\; \varepsilon\tilde{C}(q-1)\int_{-\infty}^{\eta} H^{q-2}\left(\frac{\partial H}{\partial v}\right)^2 p^1_\infty\left|\gamma''\right|^2 dv + \frac{\tilde{C}(q-1)}{\varepsilon}\int_{\alpha}^{\eta} H^{q-2}p^1_\infty dv\\
&\quad + \varepsilon\tilde{C}\int_{-\infty}^{\eta} H^q p^1_\infty\left|\gamma'''\right|^2 dv + \frac{\tilde{C}(q-1)}{\varepsilon}\int_{\alpha}^{\eta} H^{q-2}p^1_\infty dv.
\end{aligned}$$

There exists $C \in \mathbb{R}_+^*$ such that $\gamma''^2 \leqslant C\gamma$ and $\gamma'''^2 \leqslant C\gamma$ (Lemma 4.2); thus there exists $\bar{C} \in \mathbb{R}_+^*$ such that

$$\begin{aligned}\left|\int_{-\infty}^{\eta} W(v,t)\gamma''(v)dv\right| \quad \leqslant \quad & \varepsilon\bar{C}(q-1)\int_{-\infty}^{\eta} H^{q-2}(v,t)p_\infty^1(v)\left(\frac{\partial H}{\partial v}(v,t)\right)^2\gamma(v)dv \\ & + \varepsilon\bar{C}\int_{-\infty}^{\eta} W(v,t)\gamma(v)dv + \frac{2\bar{C}(q-1)}{\varepsilon}\int_{\alpha}^{\eta} H^{q-2}(v,t)p_\infty^1(v)dv.\end{aligned}$$

We have eventually

$$\begin{aligned}\left|\int_{-\infty}^{\eta} W(v,t)\gamma''(v)dv\right| \quad \leqslant \quad & \varepsilon\bar{C}(q-1)\int_{-\infty}^{V_F} H^{q-2}(v,t)p_\infty^1(v)\left(\frac{\partial H}{\partial v}(v,t)\right)^2\gamma(v)dv \\ & + \varepsilon\bar{C}\int_{-\infty}^{V_F} W(v,t)\gamma(v)dv + \frac{2\bar{C}(q-1)}{\varepsilon}\int_{\alpha}^{V_F} H^{q-2}(v,t)p_\infty^1(v)dv.\end{aligned} \tag{4.13}$$

Combining (4.12) and (4.13), and choosing ε small enough, there exist constants $C_1, C_2 \in \mathbb{R}_+^*$ depending only on γ, q, a and b_1 such that

$$\begin{aligned}\frac{d}{dt}\int_{-\infty}^{V_F} W(v,t)\gamma(v)dv \leqslant -\frac{\gamma(V_F)}{{N_\infty^1}^{q-1}}N(t)^q + (q-1)(bN(t)-C_1)\int_{-\infty}^{V_F} W(v,t)\gamma(v)dv \\ + C_2\int_{\alpha}^{V_F} H(v,t)^{q-2}p_\infty^1(v)dv.\end{aligned} \tag{4.14}$$

- **Excitatory case:**

Like in the previous case, we have

$$\int_{-\infty}^{V_F} -vW(v,t)\gamma'(v)dv \leqslant 0$$

and, for α close enough to V_F,

$$b_1 N_\infty^1 \int_{-\infty}^{V_F} \frac{\partial p_\infty^1}{\partial v}(v)H(v,t)^q\gamma(v)dv \leqslant -b_1 N_\infty^1\int_{-\infty}^{V_F} W(v,t)\gamma(v)dv.$$

The term containing γ'' can be handled the same way as before. There remain the terms

$$J_1(t) = bN(t)\int_{-\infty}^{V_F} W(v,t)\gamma'(v)dv \qquad \text{and} \qquad J_2(t) = -bN(t)\int_{-\infty}^{V_F}\frac{\partial p_\infty^1}{\partial v}(v)H(v,t)^q\gamma(v)dv.$$

In order to obtain a bound for J_1, we use a partition of unity: by properties of γ (Lemma 4.2), there exists $\omega \in]\alpha, V_F[$ such that $\gamma' \leqslant \gamma$ on $]\omega, V_F]$; we choose two functions $\gamma_1, \gamma_2 \in \mathscr{C}^\infty(]-\infty, V_F])$ and a value $\varepsilon_1 \in \mathbb{R}_+^*$ small enough such that $\gamma_1 + \gamma_2 = 1$ and

- γ_1 is non-increasing on $]-\infty, V_F]$ and $\gamma_1 \equiv 0$ on $[\omega + \varepsilon_1, V_F]$;
- γ_2 is non-decreasing on $]-\infty, V_F]$ and $\gamma_2 \equiv 0$ on $]-\infty, \omega]$.

Then,

$$\int_{-\infty}^{V_F} W(v,t)\gamma'(v)dv = \int_{-\infty}^{V_F} W(v,t)\gamma'(v)\gamma_1(v)dv + \int_{-\infty}^{V_F} W(v,t)\gamma'(v)\gamma_2(v)dv.$$

Since $\gamma_2 = 0$ on $]-\infty, \omega]$, $\gamma_2(v) \leqslant 1$ for all $v \in]-\infty, V_F]$ and $\gamma' \leqslant \gamma$ on $]\omega, V_F]$,

$$\int_{-\infty}^{V_F} W(v,t)\gamma'(v)\gamma_2(v)dv = \int_{\omega}^{V_F} W(v,t)\gamma'(v)\gamma_2(v)dv \leqslant \int_{\omega}^{V_F} W(v,t)\gamma(v)dv \leqslant \int_{-\infty}^{V_F} W(v,t)\gamma(v)dv.$$

Using the inequality $jk \leqslant \frac{\varepsilon}{q} j^q + \frac{q-1}{q\varepsilon^{\frac{q}{q-1}}} k^{\frac{q}{q-1}}$, we obtain for all $\varepsilon \in \mathbb{R}_+^*$,

$$bN(t) \int_{-\infty}^{V_F} W(v,t)\gamma'(v)\gamma_2(v)dv \leqslant \Big(\frac{(q-1)\varepsilon^{\frac{1}{q-1}}}{q} + \frac{b^q N(t)^q}{q\varepsilon} \Big) \int_{-\infty}^{V_F} W(v,t)\gamma(v)dv.$$

We do for the term γ_1 what we did previously unto the term γ'', which yields

$$\begin{aligned}
\int_{-\infty}^{V_F} W(v,t)\gamma'(v)\gamma_1(v)dv &= \int_{-\infty}^{V_F} H^{q-1} \frac{\partial}{\partial v} \left(\int_{-\infty}^{v} p(w,t)dw \right) \gamma'(v)\gamma_1(v)dv \\
&= -\int_{-\infty}^{V_F} \left(\int_{-\infty}^{v} p(w,t)dw \right) \left(\frac{\partial H^{q-1}}{\partial v} \gamma'\gamma_1 + H^{q-1} \left(\gamma'\gamma_1\right)' \right) dv \\
&= -\int_{-\infty}^{V_F} \left(\int_{-\infty}^{v} p(w,t)dw \right) \left((q-1) \frac{\partial H}{\partial v} H^{q-2} \gamma'\gamma_1 + H^{q-1} \left(\gamma'\gamma_1\right)' \right) dv.
\end{aligned}$$

Moreover, since p is a probability density,

$$\left| \int_{-\infty}^{V_F} W(v,t)\gamma'(v)\gamma_1(v)dv \right| \leqslant (q-1) \int_{-\infty}^{V_F} \left| \frac{\partial H}{\partial v} \right| H^{q-2} \left| \gamma'(v)\gamma_1(v) \right| dv + \int_{-\infty}^{V_F} H^{q-1} \left| \left(\gamma'\gamma_1\right)'(v) \right| dv.$$

If α is close enough to V_F, there exists like before C_ω such that $p^1_\infty > C_\omega$ on $[\alpha, \omega + \varepsilon_1]$, and $\gamma'(v)^2 \leqslant C\gamma(v)$. Then, there exists $\bar{C}_2 \in \mathbb{R}_+^*$ such that, by Peter and Paul inequality,

$$bN(t) \int_{-\infty}^{V_F} \left| \frac{\partial H}{\partial v} \right| H^{q-2} \left| \gamma'\gamma_1 \right| dv \leqslant \bar{C}_2 \varepsilon \int_{-\infty}^{V_F} H^{q-2} p^1_\infty \left(\frac{\partial H}{\partial v} \right)^2 \gamma dv + \frac{\bar{C}_2}{\varepsilon} b^2 N(t)^2 \int_{\alpha}^{\omega+\varepsilon_1} H^{q-2} p^1_\infty dv. \tag{4.15}$$

Hence, using again $jk \leqslant \frac{\varepsilon}{q} j^q + \frac{q-1}{q\varepsilon^{\frac{q}{q-1}}} k^{\frac{q}{q-1}}$, there exists $\bar{C}_3, s \in \mathbb{R}_+^*$ such that

$$\begin{aligned}
bN(t) \int_{-\infty}^{V_F} \left| \frac{\partial H}{\partial v} \right| H^{q-2} \left| \gamma'\gamma_1 \right| dv \leqslant \bar{C}_2 \varepsilon \int_{-\infty}^{V_F} H^{q-2} p^1_\infty \left(\frac{\partial H}{\partial v} \right)^2 \gamma dv \\
+ \frac{\bar{C}_2}{\varepsilon} (\bar{C}_3 \varepsilon^s + \frac{2}{q} b^q N(t)^q) \int_{\alpha}^{V_F} H^{q-2} p^1_\infty dv.
\end{aligned} \tag{4.16}$$

Besides, using the properties of γ and γ_1, we have

$$\begin{aligned}
|(\gamma'\gamma_1)'(v)|^2 &\leqslant \left(|\gamma''(v)\gamma_1(v)| + |\gamma'(v)\gamma_1'(v)| \right)^2 \\
&\leqslant \gamma''(v)^2\gamma_1^2 + 2|\gamma''(v)|\gamma_1(v)\gamma'(v)|\gamma_1'(v)| + \gamma'(v)^2\gamma_1'(v)^2 \\
&\leqslant \gamma''(v)^2\gamma_1^2 + \left(\gamma''(v)^2 + \gamma'(v)^2\right) \left\|\gamma_1'\right\|_\infty + \gamma'(v)^2 \left\|\gamma_1'^2\right\|_\infty \\
&\leqslant C(1 + 2\left\|\gamma_1'\right\|_\infty + \left\|\gamma_1'^2\right\|_\infty)\gamma(v).
\end{aligned} \tag{4.17}$$

The method used to prove (4.16) combined with (4.17) gives that there exist $\bar{C}_4, \bar{C}_5, \bar{C}_6, s_3, s_4 \in \mathbb{R}_+^*$ such that

$$bN(t) \int_{-\infty}^{V_F} H^{q-1} \left| (\gamma'\gamma_1)'(v) \right| dv \leqslant \bar{C}_4 \varepsilon \int_{-\infty}^{V_F} W\gamma dv + \left(\bar{C}_5 \varepsilon^{s_3} + \frac{\bar{C}_6}{\varepsilon^{s_4}} b^q N(t)^q \right) \int_{\alpha}^{\omega+\varepsilon_1} H^{q-2} p^1_\infty dv. \tag{4.18}$$

Collecting the previous results, we get the following bound for J_1: there exist $\bar{C}_7, \bar{C}_8, \bar{C}_9, s_5, s_6 \in \mathbb{R}_+^*$ such that

$$\begin{aligned}
J_1(t) \leqslant \varepsilon \bar{C}_7 \left(\int_{-\infty}^{V_F} H^{q-2} p^1_\infty \left(\frac{\partial H}{\partial v} \right)^2 \gamma dv + \int_{-\infty}^{V_F} W\gamma dv \right) + \frac{\bar{C}_8 b^q N(t)^q}{\varepsilon^{s_5}} \int_{-\infty}^{V_F} W\gamma dv \\
+ \frac{\bar{C}_9 (1 + b^q N(t)^q)}{\varepsilon^{s_6}} \int_{\alpha}^{V_F} H^{q-2} p^1_\infty dv.
\end{aligned} \tag{4.19}$$

Let's now bound J_2. Integration by parts yields

$$\int_{-\infty}^{V_F} \frac{\partial p_\infty^1}{\partial v}(v)H(v,t)^q\gamma(v)dv = -\int_{-\infty}^{V_F} W(v,t)\gamma'(v)dv - q\int_{-\infty}^{V_F} \frac{\partial H}{\partial v} H^{q-1}p_\infty^1\gamma dv.$$

The term

$$bN(t)\int_{-\infty}^{V_F} W(v,t)\gamma'(v)dv$$

can be controlled by inequality (4.19). The term

$$bN(t)q\int_{-\infty}^{V_F} \frac{\partial H}{\partial v} H^{q-1}p_\infty^1\gamma dv$$

can be bounded, thanks to inequality Peter and Paul, which gives

$$bN(t)q\int_{-\infty}^{V_F} \left|\frac{\partial H}{\partial v}\right| H^{q-1}p_\infty^1\gamma dv \leqslant \varepsilon\bar{C}_{10}\int_{-\infty}^{V_F} H^{q-1}\left(\frac{\partial H}{\partial v}\right)^2 p_\infty^1\gamma dv + \frac{\bar{C}_{11}b^2N(t)^2}{\varepsilon}\int_{-\infty}^{V_F} W\gamma dv,$$

where $C_{10}, C_{11} \in \mathbb{R}_+^*$. The previous method yields the following bound for J_2: there exist $\bar{C}_{12}, \bar{C}_{13}, \bar{C}_{14}, s_7, s_8 \in \mathbb{R}_+^*$ such that

$$J_1(t) \leqslant \varepsilon\bar{C}_{12}\left(\int_{-\infty}^{V_F} H^{q-2}p_\infty^1\left(\frac{\partial H}{\partial v}\right)^2\gamma dv + \int_{-\infty}^{V_F} W\gamma dv\right) + \frac{\bar{C}_{13}b^qN(t)^q}{\varepsilon^{s_7}}\int_{-\infty}^{V_F} W\gamma dv$$
$$+ \frac{\bar{C}_{14}(1+b^qN(t)^q)}{\varepsilon^{s_8}}\int_{\alpha}^{V_F} H^{q-2}p_\infty^1 dv. \quad (4.20)$$

Combining (4.5), (4.19) and (4.20), and choosing ε small enough, we obtain that there exist $C_3, C_4, C_5, C_6, C_7 \in \mathbb{R}_+^*$ depending only on γ, q, b_1, a and N_∞^1 such that

$$\frac{d}{dt}\int_{-\infty}^{V_F} W(v,t)\gamma(v)dv \leqslant -\frac{\gamma(V_F)}{{N_\infty^1}^{q-1}}N(t)^q + b^qN(t)^q\Big(C_3 + C_4\int_{-\infty}^{V_F} W(v,t)\gamma(v)dv + C_5\int_{\alpha}^{V_F} H(v,t)^{q-2}p_\infty^1(v)dv\Big)$$
$$+ C_6\int_{\alpha}^{V_F} H(v,t)^{q-2}p_\infty^1(v)dv - C_7\int_{-\infty}^{V_F} W(v,t)\gamma(v)dv. \quad (4.21)$$

□

We are now able to prove the main theorem of this subsection. We proceed by a kind of bootstrap method.

Proof of Theorem 4.1. We are going to use Lemma (4.4). Let's denote for clarity

$$H(v,t) = \frac{p(v,t)}{p_\infty^1(v)}, \qquad W(q;v,t) = p_\infty^1(v)H(v,t)^q = \frac{p(v,t)^q}{p_\infty^1(v)^{q-1}},$$

$$\gamma(\alpha;v) = 1_{[\alpha,V_F]}(v)e^{\frac{-1}{\beta-(V_F-v)^2}} \qquad \text{and} \qquad I_{q,\alpha}(t) = \int_{-\infty}^{V_F} W(q;v,t)\gamma(\alpha;v)dv.$$

Assume that for a natural number $q \geqslant 2$ and a real number $\alpha_{q-2} \in]V_R, V_F[$ close enough to V_F to apply Lemma 4.4 we have a constant K_{q-2} such that

$$\int_{\alpha_{q-2}}^{V_F} H(v,t)^{q-2}p_\infty^1(v)dv \leqslant K_{q-2}. \quad (4.22)$$

Then, Lemma (4.4) reduces in the inhibitory case to

$$\frac{d}{dt}I_{q,\alpha_{q-2}}(t) \;\leqslant\; -\frac{\gamma(\alpha_{q-2};V_F)}{{N_\infty^1}^{q-1}}N(t)^q \;+(q-1)(bN(t)-C_1)I_{q,\alpha_{q-2}}(t)+C_2K_{q-2}. \tag{4.23}$$

Let us proceed as in the proof of [10][Theorem 3.1]. By assumptions we have $I_{q,\alpha_{q-2}}(0)<+\infty$. Since $b\leqslant 0$, (4.23) implies $I'_{q,\alpha_{q-2}}(t)\leqslant C_2K_{q-2}-(q-1)C_1I'_{q,\alpha_{q-2}}(t)$. Thus, assuming t is large enough depending on $I_{q,\alpha_{q-2}}(0)$, we have

$$I_{q,\alpha_{q-2}}(t)\leqslant 2\frac{C_2K_{q-2}}{(q-1)C_1}.$$

Hence, integrating (4.23) in time, we get

$$\int_J N(t)^q dt\leqslant C_q(1+|J|),$$

with $C_q\in\mathbb{R}_+^*$ a constant and for all interval $J\subset[T_0,+\infty[$ with T_0 depending upon $I_{q,\alpha_{q-2}}(0)$.

In the excitatory case, Lemma (4.4) and (4.22) yield

$$\frac{d}{dt}I_{q,\alpha_{q-2}}(t) \;\leqslant\; N(t)^q\Big(b^q\left(C_3+C_5K_{q-2}\right)+b^qC_4I_{q,\alpha_{q-2}}(t)-\frac{\gamma(\alpha_{q-2};V_F)}{{N_\infty^1}^{q-1}}\Big)+C_6K_{q-2}-C_7I_{q,\alpha_{q-2}}(t). \tag{4.24}$$

Define

$$L=\max\left(I_{q,\alpha_{q-2}}(0),\frac{C_6K_{q-2}}{C_7}\right).$$

If b is small enough to have

$$b^q\left(C_3+C_5K_{q-2}\right)+b^qC_4L-\frac{\gamma(\alpha_{q-2};V_F)}{{N_\infty^1}^{q-1}}<0,$$

then for all $t>0$, $I_{q,\alpha_{q-2}}(t)\leqslant L$ and integrating (4.24) between 0 and t we get

$$\int_0^t N(t)^q dt\leqslant C(1+t),$$

with $C_q\in\mathbb{R}_+^*$ a constant. As a by-product, we have uniform boundedness of $I_{q,\alpha_{q-2}}(t)$ in both inhibitory and excitatory cases and hence there exists a constant $\bar K_q>0$ such that

$$\int_{\alpha_{q-2}}^{V_F} W(q;v,t)\gamma(\alpha_{q-2};v)dv=\int_{\alpha_{q-2}}^{V_F} H(v,t)^q p_\infty^1(v)\gamma(\alpha_{q-2};v)dv\leqslant \bar K_q. \tag{4.25}$$

By choosing a new value $\alpha_q\in]V_R,V_F[$ such that $\alpha_{q-2}<\alpha_q$, we have the property (4.22) for this new α_q because there exists a constant C such that

$$\int_{\alpha_q}^{V_F} H(v,t)^q p_\infty^1(v)dv\leqslant C\int_{\alpha_{q-2}}^{V_F} H(v,t)^q p_\infty^1(v)\gamma(\alpha_{q-2};v)dv\leqslant C\bar K_q=:K_q.$$

Therefore, we can easily prove (4.22) by induction: for $q=2$, the result is obvious because

$$\int_{\alpha_0}^{V_F} H^{q-2}(v,t)p_\infty^1(v)dv=\int_{\alpha_0}^{V_F} p_\infty^1(v)dv\leqslant\int_{-\infty}^{V_F} p_\infty^1(v)dv=1;$$

for $q=3$, the result holds because of mass conservation of p:

$$\int_{\alpha_1}^{V_F} H^{q-2}(v,t)p^1_\infty(v)dv = \int_{\alpha_1}^{V_F} \frac{p(v,t)}{p^1_\infty(v)}p^1_\infty(v)dv = \int_{\alpha_1}^{V_F} p(v,t)dv \leqslant \int_{-\infty}^{V_F} p(v,t)dv = 1.$$

Then, induction works if we choose an increasing sequence of numbers $\alpha_q < V_F$, $q \in \mathbb{N}$. $\square$

4.2 Global-in-time existence

We now use uniform L^3 estimates in order to prove global-in-time existence for small enough excitatory connectivities. More precisely, we prove:

Theorem 4.5 *Let $b_1 \in \mathbb{R}^*_+$ be such that there exists a stationary state (p^1_∞, N^1_∞). Let $V_M \in]V_R, V_F[$; denote*

$$S_3(b_1, V_M) = \int_{V_M}^{V_F} \frac{p^0(v)^3}{p^1_\infty(v)^2}dv.$$

If p^0 satisfies Assumptions 2.4, there exists $b^* \in \mathbb{R}^*_+$ depending only upon $S_3(b_1, V_M)$ and V_M such that for all $b \in [0, b^*[$, there exists a unique global-in-time classical solution (p, N) of (2.1).

In [11][Proposition 4.3], the authors use an L^∞ bound on N to obtain global-in-time existence of classical solutions in the inhibitory case $b < 0$. We are going to adapt their method in the excitatory case $b > 0$.

When constructing local-in-time solutions to (2.1), the authors of [11] introduce the change of variables

$$y = e^t v, \quad \tau = \frac{1}{2}(e^{2t} - 1) \qquad \text{or equivalently} \qquad v = \frac{y}{\sqrt{2\tau + 1}} \quad t = \frac{1}{2}\log(2\tau + 1).$$

It allows us to write (2.1) as an equivalent free-boundary Stefan-like problem on the time-dependent domain $]-\infty, s(\tau)]$ where the free boundary $s(\tau)$ evolves depending on the solution up to time τ (see it's expression in (4.26)). As in [11], we assume without loss of generality (using the same rescaling as in hypothesis (4.1)) that $V_F = 0$, $a = 1$ and $-v + bN(t)$ is replaced by $-v + b_0 + bN(t)$, $b_0 \in \mathbb{R}$. We denote

$$w(y,\tau) = \alpha(\tau)p\Big(y\alpha(\tau), -ln(\alpha(\tau))\Big), \qquad \alpha(\tau) = \frac{1}{\sqrt{2\tau + 1}}, \qquad M(\tau) = \alpha(\tau)^2 N(t) = -\frac{dw}{dy}(0,\tau),$$

or equivalently

$$p(v,t) = e^t w\left(e^t v, \frac{1}{2}(e^{2t} - 1)\right).$$

We then make the change of variable

$$x = y - b_0(\sqrt{1 + 2\tau} - 1) - b\int_0^\tau \frac{M(s)}{\alpha(s)}ds.$$

with the associated new unknown $u(x,\tau) = w(y,\tau)$. The pair (u, M) is then a solution to

$$\begin{cases} \dfrac{\partial u}{\partial \tau}(x,\tau) = \dfrac{\partial^2 u}{\partial y^2}(x,\tau) + M(\tau)\delta_{s(\tau)+\frac{V_R}{\alpha(\tau)}}(x) & x \in]-\infty, s(\tau)], \tau \in \mathbb{R}_+ \\ M(\tau) = -\dfrac{\partial u}{\partial x}(s(\tau),\tau) & \tau \in \mathbb{R}_+ \\ s(\tau) = s(0) - b_0(\sqrt{1+2\tau}-1) - b\displaystyle\int_0^\tau \frac{M(s)}{\alpha(s)}ds & \tau \in \mathbb{R}_+ \\ u(-\infty,\tau) = u(s(\tau),\tau) = 0 & \tau \in \mathbb{R}_+ \\ u(x,0) = u^0(x) & x \in]-\infty, s(0)]. \end{cases} \tag{4.26}$$

For this system, we recall the notion of classical solution introduced in [11].

Definition 4.6 (Classical solutions for the Stefan-like problem) *Assume u_0 satisfies Assumptions 2.4. We say that u is a classical solution of (4.26) with initial datum u^0 on the interval $J = [0,T[$ or $J = [0,T]$, for a given $T > 0$ if*

1. *M is a continuous function on J, u is continuous in the region $\{(x,\tau) : -\infty < x \le s(\tau), \tau \in J\}$ and for all $\tau \in J$, $u(\cdot,\tau) \in L^1(]-\infty, s(\tau)[)$.*
2. *If we denote $s_1(\tau) := s(\tau) + \frac{V_R}{\alpha(\tau)}$, then $\partial_{xx}u$ and $\partial_\tau u$ are continuous in the region $\{(x,\tau) : x \in]-\infty, s(\tau)[\backslash\{s_1(\tau)\}, \tau \in J\backslash\{0\}\}$.*
3. *$\partial_x u(s_1(\tau)^-,\tau)$, $\partial_x u(s_1(\tau)^+,\tau)$, $\partial_x u(s(\tau)^-,t)$ are well defined and $\partial_x u$ vanishes at $-\infty$.*
4. *Equations (4.26) are satisfied in the sense of distributions in the region $\{(x,\tau) : -\infty < x \le s(\tau), \tau \in J\}$ and pointwise (in the classical sense of a function's derivative) in $\{(x,\tau) : x \in]-\infty, s(\tau)[\backslash\{s_1(\tau)\}, \tau \in J\backslash\{0\}\}$.*

The article [11][Theorem 4.2] provides a criterium for global-in-time existence:

Theorem 4.7 (Carrillo, Gonzales, Gualdani, Schonbeck) *Assume u_0 satisfies Assumptions 2.4. There exists a unique maximal classical solution of (4.26) in the sense of Definition 4.6 and it's maximal time of existence T^* satisfies*

$$T^* = \sup\{\tau \in \mathbb{R}_+ \mid M(\tau) < +\infty\}.$$

We are now able to prove our main theorem:

Proof of Theorem 4.5. First, note that by Assumptions 2.4, p^0 has a derivative on the left of V_F, and thus

$$\lim_{v\to V_F^-} \frac{p^0(v)^3}{p_\infty^1(v)^2} = \lim_{v\to V_F^-} \left(\frac{\frac{p^0(v)}{V_F-v}}{\frac{p_\infty^1(v)}{V_F-v}}\right)^2 p^0(v) = \left(\frac{\frac{dp^0}{dv}(V_F^-)}{N_\infty^1}\right)^2 p^0(V_F) = 0. \tag{4.27}$$

Hence, since p^0 is C^1 on $]V_R, V_F[$ and $p^1_\infty > 0$ on $]V_R, V_F[$, we have $S_3(b_1, V_M) < +\infty$, which is a technical hypothesis of Theorem 4.1.

We follow the ideas of the proof of global-in-time existence in the inhibitory case in the article [11][Proposition 4.3 and Theorem 4.4]. We derive the equivalent free boundary Stefan-like problem as we recall above and we assume the local-in-time solution of (4.26) from initial datum u^0 exists on the maximal time interval $[0, \tau_0[$. There exists a value $\varepsilon \in \mathbb{R}_+^*$, chosen small enough for what follows, such that,

$$U_{\tau_0-\varepsilon} = \sup_{x \in]-\infty, s(\tau_0-\varepsilon)]} \left| \frac{du}{dx}(x, \tau_0 - \varepsilon) \right| < +\infty.$$

We are going to prove that there exists $K_\infty \in \mathbb{R}_+^*$ such that

$$\sup_{\tau \in [\tau_0-\varepsilon, \tau_0]} M(\tau) < K_\infty.$$

Then, Theorem 4.7 gives the existence of the solution on $[0, \tau_0 + \varepsilon]$. If ε does not depend on $U_{\tau_0-\varepsilon}$, we can repeat the argument and we have global-in-time existence for the solution.

For the sake of clarity, we assume without loss of generality that $\tau_0 - \varepsilon = 0$. We are thus going to find a bound for M on $[0, \tau_0[$ with respect to τ_0, for τ_0 small enough. We recall first the expression of the heat kernel:

$$G(x, \tau, \xi, \eta) = \frac{1}{\sqrt{4\pi(\tau - \eta)}} e^{-\frac{(x-\xi)^2}{4(\tau-\eta)}}.$$

It is proved in [11], sections 2 and 3 that, recalling that $t = \frac{1}{2}\log(2\tau + 1)$, the continuous function M satisfies

$$M(\tau) = \alpha(\tau)^2 N(t) = e^{-2t} N(t)$$

and for all $\tau \in [0, \tau_0[$,

$$\begin{aligned} M(\tau) \;&=\; -2\int_{-\infty}^{s(0)} G(s(\tau), \tau, \xi, 0)\frac{du^0}{dx}(\xi)d\xi \;+\; 2\int_0^\tau M(\eta)\frac{\partial}{\partial x}G(s(\tau), \tau, s(\eta), \eta)d\eta \\ &\quad -\; 2\int_0^\tau M(\eta)\frac{\partial}{\partial x}G(s(\tau), \tau, s(\eta) + \frac{V_R}{\alpha(\eta)}, \eta)d\eta. \end{aligned} \tag{4.28}$$

We denote,

$$\Phi(\tau) = \sup_{s \in [0,\tau]} M(s).$$

• Firstly, by properties of G,

$$\left| 2\int_{-\infty}^{s(0)} G(s(\tau), \tau, \xi, 0)\frac{du^0}{dx}(\xi)d\xi \right| \leqslant 2 \left\| \frac{du^0}{dx} \right\|_\infty . \tag{4.29}$$

Applying Hölder inequality, we obtain

$$2\int_0^\tau M(\eta)\frac{\partial}{\partial x}G(s(\tau), \tau, s(\eta), \eta)d\eta \leqslant 2\left(\int_0^\tau M(\eta)^3 d\eta\right)^{\frac{1}{3}} \left(\int_0^\tau \left|\frac{\partial}{\partial x}G(s(\tau), \tau, s(\eta), \eta)\right|^{\frac{3}{2}} d\eta\right)^{\frac{2}{3}}, \tag{4.30}$$

and

$$2\int_0^\tau M(\eta)\frac{\partial}{\partial x}G(s(\tau),\tau,s(\eta)+\frac{V_R}{\alpha(\eta)},\eta)d\eta$$
$$\leqslant 2\left(\int_0^\tau M(\eta)^3 d\eta\right)^{\frac{1}{3}}\left(\int_0^\tau\left|\frac{\partial}{\partial x}G(s(\tau),\tau,s(\eta)+\frac{V_R}{\alpha(\eta)},\eta)\right|^{\frac{3}{2}}d\eta\right)^{\frac{2}{3}}. \quad (4.31)$$

• Then, by Theorem 4.1, there exists a constant $C\in\mathbb{R}_+^*$ such that

$$\begin{aligned}\int_0^\tau M(\eta)^3 d\eta &= \int_0^{\frac{1}{2}(e^{2t}-1)} e^{-6s}N(s)^3 ds\\ &\leqslant C(1+\frac{1}{2}(e^{2t}-1)) = C(1+\tau),\end{aligned} \quad (4.32)$$

which yields

$$\left(\int_0^\tau M(\eta)^3 d\eta\right)^{\frac{1}{3}}\leqslant C^{\frac{1}{3}}(1+\tau_0)^{\frac{1}{3}}. \quad (4.33)$$

• We come back to the bound (4.30). We have

$$s(\tau)=s(0)-b_0(\sqrt{2\tau+1}-1)-b\int_0^\tau\frac{M(s)}{\alpha(s)}ds.$$

Since α^{-1} is 1-Lipschitz,

$$\begin{aligned}|s(\tau)-s(\eta)| &\leqslant |b_0|(\sqrt{2\tau+1}-\sqrt{2\eta+1})+b\int_\eta^\tau M(s)\sqrt{2s+1}ds\\ &\leqslant |b_0||\tau-\eta|+b\sqrt{2\tau_0+1}\Phi(\tau)|\tau-\eta|\\ &\leqslant \left(|b_0|+b\sqrt{2\tau_0+1}\Phi(\tau)\right)|\tau-\eta|.\end{aligned}$$

Moreover, we compute

$$\frac{\partial}{\partial x}G(x,\tau,\xi,\eta)=-\frac{1}{2\sqrt{4\pi}}\frac{(x-\xi)}{(\tau-\eta)^{\frac{3}{2}}}e^{-\frac{(x-\xi)^2}{4(\tau-\eta)}},$$

and with the previous bound, we have

$$\begin{aligned}\int_0^\tau\left|\frac{\partial}{\partial x}G(s(\tau),\tau,s(\eta),\eta)\right|^{\frac{3}{2}}d\eta &= \frac{1}{8\pi^{\frac{3}{4}}}\int_0^\tau\frac{(s(\tau)-s(\eta))^{\frac{3}{2}}}{(\tau-\eta)^{\frac{9}{4}}}e^{-\frac{3(s(\tau)-s(\eta))^2}{8(\tau-\eta)}}d\eta\\ &\leqslant \frac{|b_0|+b\sqrt{2\tau_0+1}\Phi(\tau)}{8\pi^{\frac{3}{4}}}\int_0^\tau\frac{1}{|\tau-\eta|^{\frac{3}{4}}}d\eta\\ &= \frac{|b_0|+b\sqrt{2\tau_0+1}\Phi(\tau)}{6\pi^{\frac{3}{4}}}\tau^{\frac{1}{4}}.\end{aligned} \quad (4.34)$$

Hence,

$$\left|2\int_0^\tau M(\eta)\frac{\partial}{\partial x}G(s(\tau),\tau,s(\eta),\eta)d\eta\right|\leqslant 2C^{\frac{1}{3}}(1+\tau_0)^{\frac{1}{3}}\frac{\left(|b_0|+b\sqrt{2\tau_0+1}\Phi(\tau)\right)^{\frac{2}{3}}}{6^{\frac{2}{3}}\pi^{\frac{1}{2}}}\tau_0^{\frac{1}{6}}.$$

and applying the inequality $(a_1+a_2)^{\frac{2}{3}}\leqslant 2^{\frac{1}{3}}|a_1|^{\frac{2}{3}}+2^{\frac{1}{3}}|a_2|^{\frac{2}{3}}$, we come to

$$\left|2\int_0^\tau M(\eta)\frac{\partial}{\partial x}G(s(\tau),\tau,s(\eta),\eta)d\eta\right|\leqslant\frac{2^{\frac{2}{3}}|b_0|^{\frac{2}{3}}C^{\frac{1}{3}}(1+\tau_0)^{\frac{1}{3}}}{3^{\frac{2}{3}}\pi^{\frac{1}{2}}}\tau_0^{\frac{1}{6}}+\frac{2^{\frac{2}{3}}b^{\frac{2}{3}}(2\tau_0+1)^{\frac{1}{3}}C^{\frac{1}{3}}(1+\tau_0)^{\frac{1}{3}}}{3^{\frac{2}{3}}\pi^{\frac{1}{2}}}\tau_0^{\frac{1}{6}}\Phi(\tau)^{\frac{2}{3}}. \quad (4.35)$$

• We now come back to the bound (4.31). Using (4.33) and Hölder's inequality, we have

$$\begin{aligned}
|s(\tau)-s(\eta)-\tfrac{V_R}{\alpha(\tau)}| &\geqslant |\tfrac{|V_R|}{\alpha(\tau)}-|s(\tau)-s(\eta)|| \\
&\geqslant |V_R|-b_0\tau_0-b\textstyle\int_\eta^\tau \frac{M(s)}{\alpha(s)}ds \\
&\geqslant |V_R|-b_0\tau_0-bC^{\frac13}(1+\tau_0)^{\frac13}(2\tau_0+1)^{\frac13}\tau_0^{\frac23}.
\end{aligned} \tag{4.36}$$

If τ_0 is small enough regarding b_0 and C, we have $|V_R|-b_0\tau_0-bC^{\frac13}(1+\tau_0)^{\frac13}(2\tau_0+1)^{\frac13}\tau_0^{\frac23}>0$. We denote $K(\tau_0)=|V_R|-b_0\tau_0-bC^{\frac13}(1+\tau_0)^{\frac13}(2\tau_0+1)^{\frac13}\tau_0^{\frac23}$. We use inequality $ye^{-y^2}\leqslant e^{-\frac{y^2}{2}}$ in order to obtain

$$\left|\frac{\partial}{\partial x}G(x,\tau,\xi,\eta)\right|\leqslant\frac{1}{\sqrt{4\pi}(\tau-\eta)}e^{-\frac{|x-\xi|^2}{8(\tau-\eta)}}. \tag{4.37}$$

Using the change of variable $z=\frac{\sqrt3 K(\varepsilon)}{4\sqrt{\tau-\eta}}$, we compute:

$$\begin{aligned}
\int_0^\tau\left|\frac{\partial}{\partial x}G(s(\tau),\tau,s(\eta)+\frac{V_R}{\alpha(\eta)},\eta)\right|^{\frac32}d\eta &\leqslant \frac{1}{(4\pi)^{\frac34}}\int_0^\tau\frac{1}{(\tau-\eta)^{\frac32}}e^{-\frac{3|s(\tau)-s(\eta)-\frac{V_R}{\alpha(\tau)}|^2}{16(\tau-\eta)}}d\eta \\
&\leqslant \frac{1}{(4\pi)^{\frac34}}\int_0^\tau\frac{1}{(\tau-\eta)^{\frac32}}e^{-\frac{3K(\tau_0)^2}{16(\tau-\eta)}}d\eta \\
&= \frac{\sqrt2}{\sqrt3(\pi)^{\frac34}K(\tau_0)}\int_{\frac{\sqrt3K(\tau_0)}{4\sqrt\tau}}^{+\infty}e^{-z^2}dz.
\end{aligned}$$

Hence,

$$2\int_0^\tau M(\eta)\frac{\partial}{\partial x}G(s(\tau),\tau,s(\eta)+\frac{V_R}{\alpha(\eta)},\eta)d\eta\leqslant C^{\frac13}(1+\tau_0)^{\frac13}\frac{2^{\frac13}}{3^{\frac13}\sqrt\pi K(\tau_0)^{\frac23}}\left(\int_{\frac{\sqrt3K(\tau_0)}{4\sqrt\tau}}^{+\infty}e^{-z^2}dz\right)^{\frac23}. \tag{4.38}$$

• Collecting inequalities (4.29), (4.35) and (4.38), we derive the following bound for M on $[0,\tau_0[$,

$$\begin{aligned}
M(\tau) \leqslant\ & 2\left\|\frac{du^0}{dx}\right\|_\infty+\frac{2^{\frac23}|b_0|^{\frac23}C^{\frac13}(1+\tau_0)^{\frac13}}{3^{\frac23}\pi^{\frac12}}\tau_0^{\frac16}+\frac{2^{\frac23}b^{\frac23}(2\tau_0+1)^{\frac13}C^{\frac13}(1+\tau_0)^{\frac13}}{3^{\frac23}\pi^{\frac12}}\tau_0^{\frac16}\Phi(\tau)^{\frac23} \\
&+\frac{2^{\frac13}C^{\frac13}(1+\tau_0)^{\frac13}}{3^{\frac13}\sqrt\pi K(\tau_0)^{\frac23}}\left(\int_{\frac{\sqrt3K(\tau_0)}{4\sqrt\tau}}^{+\infty}e^{-z^2}dz\right)^{\frac23}.
\end{aligned} \tag{4.39}$$

As a consequence, there exists $\Lambda\in\mathbb{R}_+^*$ depending only on b_0, b, $\frac{d}{dx}u^0$ and V_R such that, if τ_0 is chosen small enough with respect to b,

$$M(\tau)\leqslant\Lambda+\frac12\Phi(\tau)^{\frac23}.$$

Denote $\Psi(\tau)=\max(\Phi(\tau),1)$; we have $\Phi(\tau)^{\frac23}\leqslant\Psi(\tau)^{\frac23}\leqslant\Psi(\tau)$. Taking the supremum on both sides of the inequality and the maximum with 1, we obtain $\Psi(\tau)\leqslant 2\Lambda$, which in turn implies

$$M(\tau)\leqslant 2\Lambda. \tag{4.40}$$

□

Remark 4.8 *In order to prove this result, we needed a bound on N in L^3; the L^2 estimate proved in article [10] was not enough. Indeed, in bounds (4.30) and (4.31), we have L^3 norms of M in the right-hand side. Applying*

Hölder inegality with exponent 2 instead of 3 would lead to the integral

$$\int_0^\tau \frac{(s(\tau)-s(\eta))^2}{(\tau-\eta)^3} e^{-\frac{(s(\tau)-s(\eta))^2}{4(\tau-\eta)}} d\eta,$$

in computation (4.34)*, and then the following diverging integral would appear:*

$$\int_0^\tau \frac{1}{|\tau-\eta|} d\eta.$$

However, with the L^3 estimate of M, we have instead in (4.34) *the converging integral*

$$\int_0^\tau \frac{1}{|\tau-\eta|^{\frac{3}{4}}} d\eta.$$

Remark 4.9 *The convergence to stationary state proved in [10][Theorem 3.5] with the help of L^2 bounds on N is a priori and thus only valid up to an unknown time of existence; global-in-time existence is not granted. Moreover, this result demands severe restrictions on the size of b: the non-linearity has to be weak enough in order to apply the Entropy method and the Poincaré-like inequality. In our result, we only demand b to be small with respect to $S_3(b_1, V_M)$ and V_M. Although we have no proof of it, we may have proved global-in-time existence for parameters where two or more stationary states coexist, which is impossible in a priori convergence results of [10][Theorem 2.1 and Theorem 3.5].*

Remark 4.10 *The assumptions on the initial datum p^0 are similar to what can be found in the early probabilistic literature. Namely, in the articles [15, 16] and in the note [14][Lemma 2.1], the authors use the working assumption that law ν of X_0 is differentiable at V_F and satisfies*

$$\forall x \in]V_F-\varepsilon, V_F],\ \nu(dx) \leqslant \beta(V_F-x)dx, \tag{4.41}$$

for some $\beta, \varepsilon > 0$. Assumptions (2.4) *could be weakened merely to* (4.41) *with technical work, since more regularity is granted after any non-zero period of time (see [21][Proposition 2.1]).*

However, note that the question of the decay of p^0 near V_F is a subtle and important one, since the short-time regularity of the solutions depends upon the boundary decay of the initial condition. In [24], the authors assume the initial density to be 0 in a vicinity of V_F; in [27], the result is proved under the hypothesis that the initial density decays like $o(x^{\frac{1}{2}})$ near the origin ($V_F = 0$ in this work). The general results of [21] which allowed to fully prove propagation of chaos for the underlying particle system rely on the assumption that the initial density decays like $\mathcal{O}(x^\beta)$ for some $0 < \beta < 1$. It can be seen in [21][Theorem 1.7] that the exponent β has an effect upon the behaviour of solutions in short time.

5 Conclusion

In this article, we covered two aspects of the NNLIF model in the excitatory case: finite-time blow-up and global-in-time existence.

First, we proved systematic blow-up in the high connectivity case, hence upgrading previous results about blow-up and stationary states non-existence ([4]). Our bound for b doesn't seem to be optimal though, and the obtention of an optimal bound is an open problem. It is reasonable to think that as soon as there is no stationary state, every solution blows-up in finite time. In the case $V_F \leqslant 0$ our bound is optimal. Note that our bound in the case $V_F > 0$ converges to the bound for $V_F = 0$ when V_F goes to 0.

Then, we upgraded previously obtained uniform L^2 estimates on the firing rate into uniform L^q estimates at a fairly reasonable price in terms of hypotheses. Using L^3 estimates, we proved global-in-time existence when $b > 0$ is small enough regarding p^0. Previous deterministic results were *a priori* convergence to stationary states in cases when the stationary state is unique ($b > 0$ very small). Although we do make smallness hypotheses on b, we don't demand existence of a unique stationary state in our global existence theorem. The result had been obtained for the stochastic counterpart (1.4) in [15] and other works ([20, 21, 28, 27]) focused on global solvability for problem (1.4) in the broader setting of *physical solutions* were solutions can continue after a blow-up event.

We didn't address two difficult questions: non-existence of periodic solutions for the non-delayed NNLIF model and convergence to stationary states for large and negative values of b. These two questions are linked, since there is numerical evidence for periodic solutions in the NNLIF model with synaptic delay when $b < 0$ and $|b|$ is large. All attempts to find periodic solutions in the non-delayed NNLIF model failed, which may be an indicator of unconditional convergence towards stationary states in the inhibitory case. Yet, there is no efficient tool to prove such a conjecture, because general entropy methods break in the high connectivity regime. More broadly, there is a need for the development of new methods in order to tackle the remaining questions around NNLIF-type models, as they still contain complexity we can scarcely fathom with numerical exploration (see [22] for the most recent numerical insights).

A Technical results

We first prove the following technical result which is used at the end of the proof of Theorem 3.1.

Lemma A.1 *Let $V_F, a \geqslant 0$ and $V_R < V_F$. Then,*

$$\inf_{\xi,\mu \in]1,+\infty[\times]\frac{V_F}{a},+\infty[} \frac{\xi}{\xi-1}\frac{e^{\mu V_F}-e^{\mu V_R}}{\mu e^{\mu V_F}} e^{\mu V_F \xi} = \frac{V_F}{4} \inf_{y\in]1,\sqrt{1+\frac{a}{4}}[} (1+y)^2 \left(1-e^{-4\frac{V_F-V_R}{V_F}\frac{1}{y^2-1}}\right) e^{\frac{2}{y-1}}. \tag{A.1}$$

Proof. We study the auxiliary function $f:]1,+\infty[\to \mathbb{R}_+^*$ defined by

$$f(\xi) = \frac{\xi}{\xi-1}\frac{e^{\mu V_F}-e^{\mu V_R}}{\mu e^{\mu V_F}} e^{\mu V_F \xi}.$$

If $V_F = 0$ and $V_R = 0$, its minimum is $\frac{1}{\mu}(1-e^{\mu V_R})$, and this quantity goes to 0 when μ goes to infinity and the right-hand side of (A.1) is also 0; hence the result. Otherwise, we compute

$$f'(\xi) = \frac{e^{\mu V_F \xi}}{\xi-1}\Big[V_F\xi - \frac{1}{\mu(\xi-1)}\Big]\frac{e^{\mu V_F}-e^{\mu V_R}}{e^{\mu V_F}}.$$

This derivative is negative on $]1,\bar{\xi}[$ and positive on $]\bar{\xi},+\infty[$, where

$$\bar{\xi} = \frac{1}{2}\left(1+\sqrt{1+\frac{4}{\mu V_F}}\right),$$

and f reaches its global minimum for $\xi = \bar{\xi}$, and that minimum is

$$\begin{aligned}
J(\mu) &= \frac{\frac{1}{2}\left(1+\sqrt{1+\frac{4}{\mu V_F}}\right)}{\frac{1}{2}\left(1+\sqrt{1+\frac{4}{\mu V_F}}\right)-1}\frac{e^{\mu V_F}-e^{\mu V_R}}{\mu e^{\mu V_F}} e^{\mu V_F \frac{1}{2}\left(1+\sqrt{1+\frac{4}{\mu V_F}}\right)} \\
&= \frac{\mu V_F}{4}\left(1+\sqrt{1+\frac{4}{\mu V_F}}\right)^2 \frac{1}{\mu}\left(1-e^{\mu(V_R-V_F)}\right) e^{\frac{\mu V_F}{2}\left(1+\sqrt{1+\frac{4}{\mu V_F}}\right)} \\
&= \frac{V_F}{4}\left(1+\sqrt{1+\frac{4}{\mu V_F}}\right)^2 \left(1-e^{V_F\mu\frac{V_R-V_F}{V_F}}\right) e^{\frac{\mu V_F}{2}\left(1+\sqrt{1+\frac{4}{\mu V_F}}\right)}.
\end{aligned}$$

With the change of variable $x = \dfrac{V_F\mu}{4}$, it's equivalent to finding, for x in $[\dfrac{4}{a},+\infty[$, the minimum of

$$K(x) = \frac{V_F}{4}\left(1+\sqrt{1+\frac{1}{x}}\right)^2\left(1-e^{-4\frac{V_F-V_R}{V_F}x}\right)e^{2x\left(1+\sqrt{1+\frac{1}{x}}\right)}.$$

With the change of variable $y = \sqrt{1+\frac{1}{x}}$, it's equivalent to finding, for y in $[1,\sqrt{1+\dfrac{a}{4}}]$, the minimum of

$$L(y) = \frac{V_F}{4}(1+y)^2\left(1-e^{-4\frac{V_F-V_R}{V_F}\frac{1}{y^2-1}}\right)e^{\frac{2}{y-1}}.$$

$\square$

Then we prove the technical lemma 4.3 stated above.

Proof of Lemma 4.3. We first compute the partial derivatives of W:

$$\frac{\partial W}{\partial t} = qH^{q-1}\frac{\partial p}{\partial t} = -qH^{q-1}\frac{\partial}{\partial v}[(-v+bN(t))p] + aqH^{q-1}\frac{\partial^2 p}{\partial v^2} + qH^{q-1}N(t)\delta_{V_R}, \tag{A.2}$$

and

$$\begin{aligned}
\frac{\partial H}{\partial v} &= \frac{1}{p_\infty^1}\big(\frac{\partial p}{\partial v} - H\frac{\partial p_\infty^1}{\partial v}\big),\\
\frac{\partial W}{\partial v} &= qH^{q-1}\frac{\partial p}{\partial v} - (q-1)H^q\frac{\partial p_\infty^1}{\partial v},\\
\frac{\partial^2 W}{\partial v^2} &= q(q-1)\frac{\partial H}{\partial v}H^{q-2}\frac{\partial p}{\partial v} + qH^{q-1}\frac{\partial^2 p}{\partial v^2} - q(q-1)H^{q-1}\frac{\partial H}{\partial v}\frac{\partial p_\infty^1}{\partial v} - (q-1)H^q\frac{\partial^2 p_\infty^1}{\partial v^2}.
\end{aligned}$$

Thus, using the equation (4.2) for p_∞^1,

$$aqH^{q-1}\frac{\partial^2 p}{\partial v^2} = a\frac{\partial^2 W}{\partial v^2} - aq(q-1)p_\infty^1 H^{q-2}\left(\frac{\partial H}{\partial v}\right)^2 + (q-1)H^q\Big((-v+b_1N_\infty^1)\frac{\partial p_\infty^1}{\partial v} - p_\infty^1 - N_\infty^1\delta_{V_R}\Big). \quad \text{(A.3)}$$

Moreover,

$$\begin{aligned}
\frac{\partial}{\partial v}\Big[\big(-v+bN(t)\big)W\Big] &= \frac{\partial}{\partial v}\Big[\big(-v+bN\big)pH^{q-1}\Big] = H^{q-1}\frac{\partial}{\partial v}\Big[\big(-v+bN(t)\big)p\Big] + \big(-v+bN(t)\big)pH^{q-2}\frac{\partial H}{\partial v}\\
&= H^{q-1}\frac{\partial}{\partial v}\Big[\big(-v+bN(t)\big)p\Big] + (q-1)H^{q-1}\big(-v+bN(t)\big)\frac{\partial p}{\partial v}\\
&\qquad -(q-1)H^q\big(-v+bN(t)\big)\frac{\partial p_\infty^1}{\partial v}.
\end{aligned}$$

Hence,

$$-qH^{q-1}\frac{\partial}{\partial v}\Big[\big(-v+bN(t)\big)p\Big] = -\frac{\partial}{\partial v}\Big[\big(-v+bN(t)\big)W\Big] - (q-1)H^q\Big(\big(-v+bN(t)\big)\frac{\partial p_\infty^1}{\partial v} - p_\infty^1\Big). \quad \text{(A.4)}$$

Putting (A.3) and (A.4) into (A.2), we obtain

$$\begin{aligned}
&\frac{\partial W}{\partial t} + \frac{\partial}{\partial v}\Big[\big(-v+bN(t)\big)W\Big] - a\frac{\partial^2 W}{\partial v^2} + aq(q-1)p_\infty^1 H^{q-2}\left(\frac{\partial H}{\partial v}\right)^2\\
&\qquad + (q-1)(bN(t)-b_1N_\infty^1)H^q\frac{\partial p_\infty^1}{\partial v} = N_\infty^1\left(q\frac{N(t)}{N_\infty^1} - (q-1)H\right)H^{q-1}\delta_{V_R}. \quad \text{(A.5)}
\end{aligned}$$

We then multiply this identity by γ and we integrate; since the support of γ is included in $[\alpha, V_F]$, with $\alpha > V_R > 0$ by hypothesis (4.1) made without loss of generality, we have the result by integration by parts. $\square$

Acknowledgements: Delphine Salort was supported by the grant ANR ChaMaNe, ANR-19-CE40-0024. We want to thank Étienne Tanré for useful discussions about the probabilistic point of view over the NNLIF model. We are grateful for the careful reading and numerous suggestions from two anonymous referees.

References

[1] R. Brette and W. Gerstner, *Adaptive exponential integrate-and-fire model as an effective description of neural activity*, Journal of neurophysiology, 94 (2005), pp. 3637–3642.

[2] N. Brunel, *Dynamics of sparsely connected networks of excitatory and inhibitory spiking networks*, J. Comp. Neurosci., 8 (2000), pp. 183–208.

[3] N. Brunel and V. Hakim, *Fast global oscillations in networks of integrate-and-fire neurons with long firing rates*, Neural Computation, 11 (1999), pp. 1621–1671.

[4] M. J. Cáceres, J. A. Carrillo, and B. Perthame, *Analysis of nonlinear noisy integrate & fire neuron models: blow-up and steady states*, Journal of Mathematical Neuroscience, 1-7 (2011).

[5] M. J. Cáceres, J. A. Carrillo, and L. Tao, *A numerical solver for a nonlinear Fokker-Planck equation representation of neuronal network dynamics*, J. Comp. Phys., 230 (2011), pp. 1084–1099.

[6] M. J. Cáceres and B. Perthame, *Beyond blow-up in excitatory integrate and fire neuronal networks: refractory period and spontaneous activity*, Journal of theoretical Biology, 350 (2014), pp. 81–89.

[7] M. J. Cáceres and A. Ramos-Lora, *An understanding of the physical solutions and the blow-up phenomenon for nonlinear noisy leaky integrate and fire neuronal models*, arXiv preprint arXiv:2011.05860, (2020).

[8] M. J. Cáceres, P. Roux, D. Salort, and R. Schneider, *Global-in-time solutions and qualitative properties for the NNLIF neuron model with synaptic delay*, Communications in Partial Differential Equations, 44 (2019), pp. 1358–1386.

[9] M. J. Cáceres and R. Schneider, *Analysis and numerical solver for excitatory-inhibitory networks with delay and refractory periods*, ESAIM: Mathematical Modelling and Numerical Analysis, 52 (2018), pp. 1733–1761.

[10] J. Carrillo, B. Perthame, D. Salort, and D. Smets, *Qualitative properties of solutions for the Noisy Integrate & Fire model in computational neuroscience*, Nonlinearity, 25 (2015), pp. 3365–3388.

[11] J. A. Carrillo, M. d. M. González, M. P. Gualdani, and M. E. Schonbek, *Classical solutions for a nonlinear Fokker-Planck equation arising in computational neuroscience*, Comm. in Partial Differential Equations, 38 (2013), pp. 385–409.

[12] J. Chevallier, *Mean-field limit of generalized Hawkes processes*, Stochastic Processes and their Applications, 127 (2017), pp. 3870–3912.

[13] J. Chevallier, M. J. Cáceres, M. Doumic, and P. Reynaud-Bouret, *Microscopic approach of a time elapsed neural model*, Mathematical Models and Methods in Applied Sciences, 25 (2015), pp. 2669–2719.

[14] F. Delarue, J. Inglis, S. Rubenthaler, and E. Tanré, *First hitting times for general non-homogeneous 1d diffusion processes: density estimates in small time*, (2013).

[15] ——, *Global solvability of a networked integrate-and-fire model of McKean–Vlasov type*, The Annals of Applied Probability, 25 (2015), pp. 2096–2133.

[16] ——, *Particle systems with a singular mean-field self-excitation. Application to neuronal networks*, Stochastic Processes and their Applications, 125 (2015), pp. 2451–2492.

[17] F. Delarue, S. Nadtochiy, and M. Shkolnikov, *Global solutions to the supercooled Stefan problem with blow-ups: regularity and uniqueness*, arXiv preprint arXiv:1902.05174, (2019).

[18] G. Dumont and J. Henry, *Synchronization of an excitatory integrate-and-fire neural network*, Bull. Math. Biol., 75 (2013), pp. 629–648.

[19] R. Fitzhugh, *Impulses and physiological states in theoretical models of nerve membrane*, Biophysical journal, 1 (1961), pp. 445–466.

[20] B. Hambly and S. Ledger, *A stochastic Mckean–Vlasov equation for absorbing diffusions on the half-line*, The Annals of Applied Probability, 27 (2017), pp. 2698–2752.

[21] B. Hambly, S. Ledger, and A. Søjmark, *A McKean–Vlasov equation with positive feedback and blow-ups*, The Annals of Applied Probability, 29 (2019), pp. 2338–2373.

[22] J. Hu, J.-G. Liu, Y. Xie, and Z. Zhou, *A structure preserving numerical scheme for Fokker-Planck equations of neuron networks: numerical analysis and exploration*, arXiv preprint arXiv:1911.07619, (2019).

[23] S. Ledger and A. Søjmark, *Uniqueness for contagious McKean–Vlasov systems in the weak feedback regime*, Bulletin of the London Mathematical Society, 52 (2020), pp. 448–463.

[24] J.-g. Liu, Z. Wang, Y. Zhang, and Z. Zhou, *Rigorous justification of the Fokker-Planck equations of neural networks based on an iteration perspective*, arXiv preprint arXiv:2005.08285, (2020).

[25] S. Mischler, C. Quininao, and J. Touboul, *On a kinetic Fitzhugh–Nagumo model of neuronal network*, Communications in Mathematical Physics, 342 (2016), pp. 1001–1042.

[26] S. Mischler and Q. Weng, *Relaxation in time elapsed neuron network models in the weak connectivity regime*, ACAP, (2018), pp. 1–30.

[27] S. Nadtochiy and M. Shkolnikov, *Particle systems with singular interaction through hitting times: application in systemic risk modeling*, Annals of Applied Probability, 29 (2019), pp. 89–129.

[28] ———, *Mean field systems on networks, with singular interaction through hitting times*, Annals of Probability, 48 (2020), pp. 1520–1556.

[29] K. Newhall, G. Kovačič, P. Kramer, D. Zhou, A. V. Rangan, and D. Cai, *Dynamics of current-based, Poisson driven, Integrate-and-Fire neuronal networks*, Comm. in Math. Sci., 8 (2010), pp. 541–600.

[30] K. Pakdaman, B. Perthame, and D. Salort, *Dynamics of a structured neuron population*, Nonlinearity, 23 (2010), pp. 55–75.

[31] K. Pakdaman, B. Perthame, and D. Salort, *Relaxation and self-sustained oscillations in the time elapsed neuron network model*, SIAM Journal on Applied Mathematics, 73 (2013), pp. 1260–1279.

[32] B. Perthame and D. Salort, *On a voltage-conductance kinetic system for integrate & fire neural networks*, Kinet. Relat. Models, 6 (2013), pp. 841–864.

[33] A. V. Rangan, G. Kovačič, and D. Cai, *Kinetic theory for neuronal networks with fast and slow excitatory conductances driven by the same spike train*, Physical Review E, 77 (2008), pp. 1–13.

[34] A. Renart, N. Brunel, and X.-J. Wang, *Mean-field theory of irregularly spiking neuronal populations and working memory in recurrent cortical networks*, in Computational Neuroscience: A comprehensive approach, J. Feng, ed., Chapman & Hall/CRC Mathematical Biology and Medicine Series, 2004.

[35] D. Sharma and P. Singh, *Discontinuous Galerkin approximation for excitatory-inhibitory networks with delay and refractory periods*, International Journal of Modern Physics C (IJMPC), 31 (2020), pp. 1–25.

[36] D. Sharma, P. Singh, R. P. Agarwal, and M. E. Koksal, *Numerical approximation for Nonlinear Noisy Leaky Integrate-and-Fire neuronal model*, Mathematics, 7 (2019), p. 363.