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Abstract. We consider power distribution networks containing source nodes and electricity consumer nodes. These nodes are interconnected by a switched network. Configuring such networks consists in deciding which switches are activated and the orientation of the links between these switches, to obtain a directed acyclic graph (DAG) from the producer nodes to the consumer nodes. This DAG is valid if the electric induced flow satisfies the demand of each consumer without exceeding the production capacity of each source and the flow capacity of each switch. Besides, unlike conventional flows, the distribution of the flow on the DAG cannot be arbitrarily chosen and in some cases, network operators prefer that consumers should only be supplied by a single source. We focus on planar graphs and show that the problem of deciding if such a valid DAG exists is NP-complete in general case and weakly NP-complete in case there is only one source.

Keywords: Flows · Power Grids · Complexity Theory · Algorithms

1 Introduction

Reliability and resilience are two key characteristics of a modern power grid. The configuration of a network consists in deciding which components (lines, sources, switches) should be activated or not. The reliability of such a configuration can be defined as the capacity of the electrical system to supply the electricity in quantity and with the quality demanded by the users. To be resilient, such a configuration must avoid the snowball effect during reconfiguration after a breakdown or malfunction.

In this article, to guarantee reliability and resilience, the objective is to find the configuration satisfying the consumer demands and being the most balanced in terms of power load [14, 6, 17]. To obtain such a guarantee, we consider two types of properties of a configuration. First, balance the demand rate on all sources as much as possible, to make them all capable of responding to an increase in production in the same proportion [3]. Second, each consumer, and therefore each activated switch, must be powered by a single source, to ensure that the failure of an element has an impact on only one source [12]; it is said that such a configuration is without double adduction (WDA).

In terms of optimization, the problem of finding a reliable configuration has been considered with the objective to minimize the cost of investment of electric lines and switches [4, 8, 15]. The resilience of configurations can be taken into account by considering various graph theory metrics [2]. Note that in this context, balanced configuration, in terms of the power of loads [16], can also be provided by configuring the network in balanced pairwise disjoint subnetworks [17, 9, 11], in particular subtrees, which corresponds to avoid double adductions (WDA). Besides, since the grid topologies are generally geographically planar, we focus on the planar graphs modeling them.

Configuration problems with resilience objectives are often considered through a graph theory point of view [15, 2]. Some solutions consider graph partitioning [11] and others consider graph covering by spanning subtrees or sub-DAGS [12]. In particular, several approaches propose solving Steiner tree problems with a goal of load balancing between the sources on the one hand and / or the switches on the other hand [10, 19]; in [3] we have proposed an algorithmic approach for the search for a configuration balancing the proportion of use of sources.

Indeed in our context, the electric flow in a network is a direct consequence of the chosen configuration, the consumers and the sources capacity [16]. Thus the objective is not to compute an electric flow in a graph (such as in [5]) but rather to determine the best set of disjoint spanning sub-DAGs of the whole network covering all consumers and optimizing the balance of proportional use of sources capacities, with or without considering the WDA property. Moreover, since the distribution networks are often planar [1, 18], we focus here on optimizing configurations in planar graphs.

The rest of the paper is organized as follows. First, we define our model of the power grid and the related computational problems then we present and prove complexity results for the most useful cases of the problem.

2 Model

2.1 Distribution network topology

We consider a connected graph $G = (V = S \cup W \cup P, E)$ in which vertices in S represent *electrical sources*, W *switches* and P *consumers*, with E a set of edges. Each vertex $x \in S$ is characterized by a maximum production capacity denoted $Prod(x) \geq 0$. Each vertex $y \in P$ is characterized by a called power, denoted by $Pow(y)$. Each $w \in W$ is characterized by a flow capacity $Cap(w)$.

2.2 Activation and orientation of the network

An **activation** of G is a function $\alpha : W \to \{0, 1\}$. We denote by W^1_α the subset of vertices $x \in W$ such that $\alpha(x) = 1$. We define $G_\alpha = (V_\alpha, E_\alpha)$ the subgraph of G induced by $V_\alpha = S \cup W^1_\alpha \cup P$.

An **orientation** $\mathcal{O}_\alpha$ of G_α is a function associating each edge $[x, y] \in E_\alpha$ with a couple (x, y) or (y, x) corresponding to an orientation of this edge. We denote by $G^{\mathcal{O}_\alpha}$ the digraph obtained by applying such an orientation to G_α.

Let α be an activation of G and $\mathcal{O}_\alpha$ be an orientation of G_α such that $G^{\mathcal{O}_\alpha}$ is a Directed Acyclic Graph (DAG) whose leaves are vertices in P. Such a DAG is said to be **coherent** iff each vertex $v \in W^1_\alpha \cup P$ is included in at least one path from a vertex of S to a vertex of P and each vertex $s \in S$ has no input arc. Let us underline that to be coherent, the directed graph $G^{\mathcal{O}_\alpha}$ must be acyclic. It is not possible to have cycles of electricity. We now define the main properties provided by orientation and activation, then we consider the network reliability.

Definition 1. *A valid DAG $G^{\mathcal{O}_\alpha}_\alpha$ is said to be* **without double adduction (WDA)** *iff for each consumer $p \in P$ there exists one and only one source $s \in S$ such that there exists a path from s to p in $G^{\mathcal{O}_\alpha}_\alpha$.*

This definition means that in a configuration modeled by a valid DAG, each consumer is filled by exactly one source. Such a property is often required for some fault tolerance behaviors [17, 9, 11]. Note also that this WDA property deeply changes the nature of the problem.

2.3 Flow in a oriented and activated DAG

Given an activation α and an orientation $\mathcal{O}_\alpha$ of G such that $G^{\mathcal{O}_\alpha}$ is coherent, we compute a flow F in $G^{\mathcal{O}_\alpha}$ as follows. For each vertex y of $G^{\mathcal{O}_\alpha}$ (except the sources), let $\Gamma^+(y)$ (with cardinal $d^+(y)$) be its set of successors in $G^{\mathcal{O}_\alpha}$ ($\Gamma^-(y)$ and $d^-(y)$ for the predecessors). The flow on each arc (x, y) of $G^{\mathcal{O}_\alpha}$ is

$$F(x,y) = \frac{\sum\limits_{z \in \Gamma^+(y)} F(y,z)}{d^-(y)} + \begin{cases} \frac{Pow(y)}{d^-(y)} & \text{if } y \in P, \\ 0 & \text{otherwise} \end{cases}$$

Given that G^{O_α} is coherent, $d^-(y) > 0$ for all $y \in W^1_\alpha \cup P$.

The flow coming out of y (plus the possible power called in y) is distributed equitably over all the arcs entering y. Note that, consequently, for a given activation and orientation, the flow is calculated by going up from the vertices of P and is unique. Considering such a flow F, for each source or switch $x \in S \cup W^1_\alpha$, we note

$$Load(x) = \sum_{z \in \Gamma^+(x)} F(x,z).$$

Definition 2. *$G^{\mathcal{O}_\alpha}$ is* **valid** *for $Load$, $Prod$ and Cap iff it is coherent and*

- *for each $w \in W^1_\alpha$ we have $Load(w) \le Cap(w)$,*
- *for each $s \in S$, we have $Load(s) \le Prod(s)$.*

If $G^{\mathcal{O}_\alpha}$ is valid and without double adduction then it is **WDA-valid**.

Definition 3. *Given a planar graph $G = (V = S \cup W \cup P, E)$, three functions Prod, Cap and Pow, the problem VALID (resp. WDA-VALID) is determining if it exists an activation α and an orientation $\mathcal{O}_\alpha$ of G_α such that $G^{\mathcal{O}_\alpha}$ is a DAG and is valid (resp. WDA-valid) for Load, Prod and Cap.*

Note that we are considering only planar graphs and that the instances of **valid** and **WDA-valid** limited with a unique source ($|S| = 1$) are equivalent.

3 Complexity

As we will see, the VALID and WDA-VALID problems are NP-complete. Clearly they belong to NP.

Lemma 1. *VALID and WDA-VALID problems belong to NP.*

Proof. Given an activation α and an orientation $\mathcal{O}_\alpha$, we can, in polynomial time, determine if $G^{\mathcal{O}_\alpha}$ is a DAG; check that, for every node $v \in P \cup W^1_\alpha$, at least one source fills v; check the WDA constraint (for the WDA-VALID problem only); compute the flow from the leaves to the sources and then check the capacity and production constraints. Thus, it is possible to check if a DAG is valid or WDA-valid in polynomial time. □

The nature of the problems being quite different, it is necessary to use two different polynomial reductions to prove the hardness. However, in both cases, we use the monotone planar 3-SAT problem.

Let $\phi(X, C)$ be a 3-SAT formula with a given set of variables X and a given set of clauses C. Let $G_{\phi(X,C)}$ be a graph whose vertices are $V_\phi = X \cup C$ and the set of edges E_ϕ is such that for each variable $x \in X$ appearing in a clause $c \in C$ (negated or not) there exists an edge (x, c). The graph $G_{\phi(X,C)}$ is called the incidence graph of $\phi(X, C)$. The planar 3-SAT problem is, given a formula $\phi(X, C)$ and an incidence graph $G_{\phi(X,C)}$, to determine whether $\phi(X, C)$ is satisfiable. This problem is NP-complete. It is also NP-complete if the incidence graph $G_{\phi(X,C)}$ is planar after adding a cycle whose vertices are exactly the variables X and this cycle is part of the inputs [13].

The monotone planar 3-SAT problem is a variant where the clauses are monotone (i.e. each clause contains only unnegated variables or only negated ones) and $G_{\phi(X,C)}$ is a planar embedding in which all the unnegated clauses are on the same side of the variable cycle, and all the negated are on the other side. The monotone planar 3-SAT problem is also NP-complete [7].

An example of monotone planar 3-SAT is a formula $\phi(X, C) = (\overline{x_1} \vee \overline{x_2} \vee \overline{x_5}) \wedge (x_1 \vee x_4 \vee x_5) \wedge (\overline{x_2} \vee \overline{x_3} \vee \overline{x_4}) \wedge (x_1 \vee x_2 \vee x_3)$ and an incidence graph $G_{\phi(X,C)}$ (Figure 1).

Lemma 2. *Given a planar graph G and its embedding, replacing an edge (u, v) by a path u, w, v (with w a new vertex) or connecting a new vertex v to a single vertex u keeps the graph planar. Splitting a vertex $x_i \in X$ of the variable cycle into two connected vertices w_i and $\overline{w_i}$ respectively inside and outside the variable*

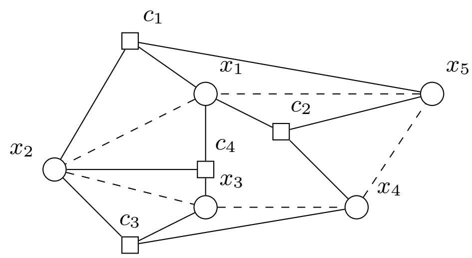

Fig. 1. The incidence graph $G_{\phi(X,C)}$ of the formula $\phi(X, C)$. Squares are used for clauses and circles for variables. Dashed lines represent the variable cycle. The negated clauses c_1 and c_3 are outside the cycle while the unnegated clauses c_2 and c_4 are inside.

cycle also keeps the graph planar as long as the incident edges $[u, x_i]$ are replaced by edges $[u, w_i]$ if u is inside the variable cycle and $[u, \overline{w_i}]$ otherwise.

Sketch of proof of lemma 2. Neither of these transformations requires more than a local modification of the embedding without violating the planarity. □

We can recursively use this property to transform an edge into an arbitrarily long path or to grow a tree from a vertex.

3.1 WDA-VALID Problem

Theorem 1. *Given an instance of Problem WDA-VALID, i.e. a planar graph $G = (V = S \cup W \cup P, E)$ and three functions Pow, Prod and Cap, deciding if there exists a pair $\alpha, \mathcal{O}_\alpha$ such that the DAG $G_\alpha^{\mathcal{O}_\alpha}$ is WDA-valid for Pow, Prod and Cap is NP-complete, even if Pow an arbitrary function, $Prod(u) = \infty, \forall u \in S$ and $Cap(v) = \infty, \forall v \in W$.*

As seen the WDA-VALID problem is in NP thus to prove it completeness we present the following reduction from an monotone planar 3-SAT instance $(\phi(X, C), G_{\phi(X,C)} = (X \cup C, E_\phi))$ into a WDA-VALID instance *i.e.* a graph $G = (V = S \cup W \cup P, E)$.

1. Create $G = (V = S \cup W \cup P, E)$ with $S = \emptyset$, $P = C$, $W = X$, $E = E_\phi$.
2. Delete from E the variable cycle, i.e. all edges $[x_i, x_j]$ with x_i , $x_j \in W$.
3. Each vertex x_i is split in two vertices w_i, $\overline{w_i}$ connected by an edge $[w_i, \overline{w_i}]$. The incident edges connecting c_j to x_i are now connecting c_j to w_i if x_i is unnegated in c_j and c_j to $\overline{w_i}$ otherwise.
4. For each w_i and $\overline{w_i} \in W$ add 2 vertices s_i, and $\overline{s_i} \in S \in W$ and the two edges $[s_i, w_i]$ and $[\overline{s_i}, \overline{w_i}]$.
5. Replace each edge $[c_j, w_i]$ and $[c_j, \overline{w_i}]$ by a path c_j, w_{ij}, w_i and $c_j, \overline{w_{ij}}, \overline{w_i}$.

For example, for the formula of Figure 1 we represent the instance of WDA-VALID obtained by the above reduction (Figure 2).

The graph G is obtained from a the planar graph $G_{\phi(X,C)}$ by only deleting edges (step 2) or using the transformation of lemma 2 thus G is planar.

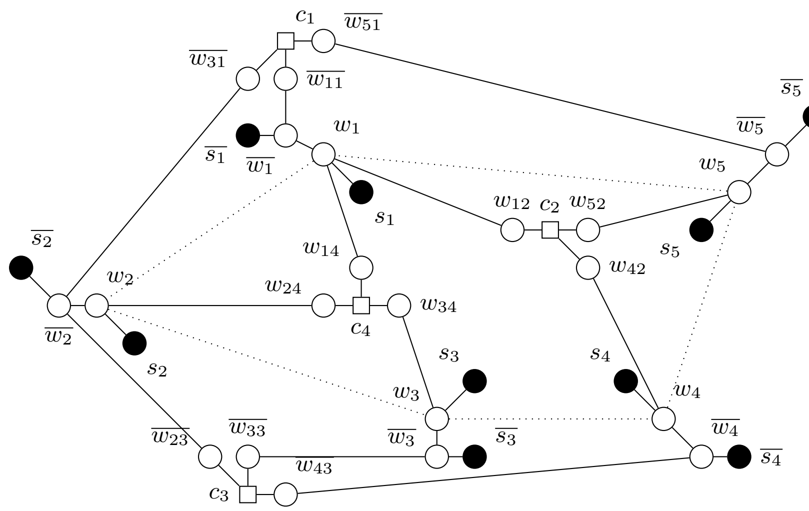

Fig. 2. Reduction from the instance of Figure 1 to an instance of the WDA-VALID problem. Switches, consumers and sources are respectively circles, rectangles and filled circles. The dotted lines represent the variable cycle which is deleted at step 2.

Proof of Theorem 1. Suppose that it exists a thruth assignement for the formula $\phi(X, C)$. We can use this assignment to build a pair of activation α and orientation $\mathcal{O}_\alpha$ such that $G^{\mathcal{O}_\alpha}$ is a valid DAG. For each clause c_j, we arbitrary select one of the literals satisfying it and we activate the switches along the following path: if this literal is the variable x_i we activate the switches of along the path $s_i, w_i, w_{i,j}, c_j$, and if it is $\overline{x_i}$ along the path $\overline{s_i}, \overline{w_i}, \overline{w_{i,j}}, c_j$. All other switches are off. The orientation is defined from the sources to the consumers.

As we build this solution from a solution of the 3-SAT instance, we never activate both w_i and $\overline{w_i}$ and each consumer is connected to a source using only one path. Then the DAG we obtain is without double adduction, it is a valid solution for the instance of WDA-VALID.

Suppose now it exists a valid DAG $G^{\mathcal{O}_\alpha}$ for the instance of WDA-VALID, without losing generality, we can consider that all active vertices are contained in directed paths from sources to consumers. If not, such vertices can be deactivated. For example if the vertices s_i, w_i, $w_{i,j}$ and c_j are all active and orientations of the edges are $[w_i, w_{i,j}]$ and $[c_j, w_{i,j}]$ then we can simplify the DAG and deactivate $w_{i,j}$. As $G^{\mathcal{O}_\alpha}$ is valid, then each consumer c_j appears in only one path from a source and it is its extremity. If not, two sources are in the same directed path and the DAG is not WDA-valid.

Finally, w_i and $\overline{w_i}$ can't be active together. By contradiction, this means it exists two directed paths $s_i, w_i, w_{i,j}, c_j$ and $\overline{s_i}, \overline{w_i}, \overline{w_{i,j'}}, c_j'$. As w_i and $\overline{w_i}$ are both activated, we have to choose an orientation for the edge $[w_i, \overline{w_i}]$ but in both cases either c_j or c'_j will be connected to two sources. It is impossible in a WDA-valid instance.

So, it exists exactly one path from a source s_i or $\overline{s_i}$ to each consumer c_j. We can use these paths to create a satisfying assignment. As it exists a polynomial time reduction from the monotone planar 3-SAT problem to the WDA-VALID problem and WDA-VALID belongs to NP (Lemma 1) then it is NP-complete. □

3.2 VALID Problem

Theorem 2. *Given an instance of VALID, i.e. a planar graph $G = (V = S \cup W \cup P, E)$ and three functions Pow, $Prod$ and Cap, deciding if there exists a pair $\alpha, \mathcal{O}_\alpha$ such that the DAG $G_\alpha^{\mathcal{O}_\alpha}$ is valid for Pow, $Prod$ and Cap is NP-complete.*

As seen the VALID problem is in NP thus to prove its completeness we give a polynomial time reduction from an instance $(\phi(X,C), G_{\phi(X,C)})$ of monotone planar 3-SAT to a VALID instance not constrainted by the switches capacity, *i.e.* a graph $G = (V = S \cup W \cup P, E)$ and three functions Cap, Pow and $Prod$, such that for any switch $w \in W$, $Cap(w) = \sum_{p \in P} Pow(p)$.

We define two useful parameters to simplify the calculations: $\beta = 10m + 1$ and $\gamma = n \cdot (6\beta + 6m) - 4m - 3$.

We build G from the graph $G_{\phi(X,C)}$. Each variable $x_i \in X$ $(1 \le i \le n)$ is replaced by a tree containing 13 nodes in G (this tree is drawn in Figure 3): five switches $\{w^2_{x_i}, w^1_{x_i}, w_{x_i}, w^1_{\bar{x}_i}, w^2_{\bar{x}_i}\} \subset W$; five sources $\{s^2_{x_i}, s^1_{x_i}, s_{x_i}, s^1_{\bar{x}_i}, s^2_{\bar{x}_i}\} \subset S$ with the following respective production capacities $(4\beta, \beta + m, \gamma + \beta + m, \beta + m, 4\beta)$; and three consumers $\{p^2_{x_i}, p_{x_i}, p^2_{\bar{x}_i}\} \subset P$ with the following called powers $\{4\beta, \gamma, 4\beta\}$. We link those nodes with 12 edges of E as drawn in the central part of Figure 3.

Each clause $c_j = (x_1 \vee x_2 \vee x_3)$ is replaced by two switches $\{w_{C_j}, w^0_{C_j}\} \subset P$; a source $s^0_{C_j}$ with production capacity $Prod(s^0_{C_j}) = \gamma + 2$; two consumers $\{p_{C_j}, p^0_{C_j}\} \subset P$ with the following respective called powers $(4, \gamma)$; and four edges $[s^0_{C_j}, w^0_{C_j}]$, $[w_{C_j}, p_{C_j}]$, $[w^0_{C_j}, p^0_{C_j}]$ and $[w^0_{C_j}, w_{C_j}]$. The edges initially between x_i and c_j (resp. $\bar{x}_i$) are now linking w_{x_i} (resp. $w_{\bar{x}_i}$) to w_{C_j}.

This graph is planar as it built from a planar graph $G_{\phi(X,C)}$ using transformations of Lemma 2. Let α and $\mathcal{O}_\alpha$ be valid activation and orientation of G, if such a solution exists. Let us show some useful properties of that solution. Figure 3 illustrates such a gadget with all the consequences of those properties. Some obvious orientations of edges (from sources to switches or from switches to consumers) for the solution to be coherent are directly given in this figure.

The following property is straightforward as a feasible solution is coherent.

Property 1. For all $i \in [\![1;n]\!]$, $\alpha(w^2_{x_i}) = \alpha(w_{x_i}) = \alpha(w^2_{\bar{x}_i}) = 1$. For all $j \in [\![1;m]\!]$, $w_{C_j} = wC^0_j = 1$. Every edge linked to a consumer is directed toward that node.

Property 2. For all $j \in [\![1;m]\!]$, $\mathcal{O}([w_{C_j}, w^0_{C_j}]) = (w^0_{C_j}, w_{C_j})$.

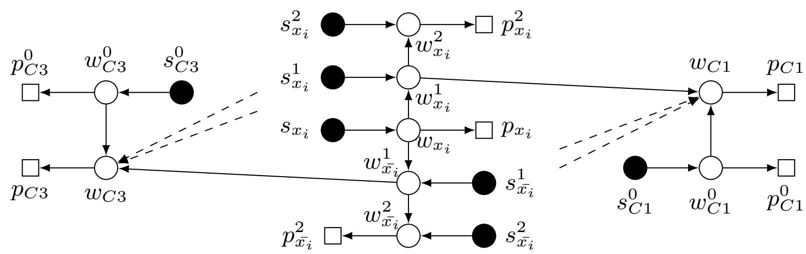

Fig. 3. Gadget associated with a variable x_i and two clauses, C_1 containing the literal x_1 and C_3 containing $\bar{x_1}$. On the gadget are also drawn the necessary activations and orientations of that instance in case it is valid. An activated switch is drawn thick. $w^1_{x_i}$ and $w^1_{\bar{x}_i}$ may or may not be activated.

Proof. Both $w^0_{C_j}$ and w_{C_j} are active (Prop. 1). Let us suppose that the edge is orientated $(w_{C_j}, w^0_{C_j})$. In this case, $s^0_{C_j}$ produces a flow of $\frac{\gamma}{2}$ for the consumer $p^0_{C_j}$. The total production of all the other sources is $n\cdot(\gamma+11\beta+3m)+(m-1)\cdot(\gamma+2)$. The total called power not filled by $s^0_{C_j}$ is at least $n\cdot(\gamma+8\beta)+m\cdot(4+\gamma)-\frac{\gamma}{2}$. By the conservation of the flow and as all the consumers must be filled, we must satisfy the following constraint:

$$\begin{aligned} n\cdot(\gamma+11\beta+3m)+(m-1)\cdot(\gamma+2) &\geq n\cdot(\gamma+8\beta)+m\cdot(4+\gamma)-\frac{\gamma}{2} \\ n\cdot(3\beta+3m)+(m-1)\cdot 2 &\geq \gamma+4m-\frac{\gamma}{2} \\ n\cdot(6\beta+6m)-4m-4 &\geq \gamma \end{aligned}$$

However $\gamma = n\cdot(6\beta+6m)-4m-3$. By this contradiction, the property is proved. □

Property 3. For all $i \in [\![1;n]\!]$, if $\alpha(w^1_{x_i}) = 1$ and $Load(w^1_{x_i}) > 0$ then, the orientation $\mathcal{O}([w^2_{x_i}, w^1_{x_i}]) = (w^1_{x_i}, w^2_{x_i})$ and, for all j such that C_j contains the litteral x_i, $\mathcal{O}([w^1_{x_i}, w_{C_j}]) = (w^1_{x_i}, w_{C_j})$. The same property occurs for $\bar{x}_i$.

Proof. If $\mathcal{O}([w^2_{x_i}, w^1_{x_i}]) = (w^2_{x_i}, w^1_{x_i})$ and $Load(w^1_{x_i}) > 0$ then $Load(w^2_{x_i}) > 4\beta$ and all that flow must come from $s^2_{x_i}$. This is not possible as $Prod(s^2_{x_i}) = 4\beta$. Consequently, $[w^1_{x_i}, w^2_{x_i}]$ is directed toward $w^2_{x_i}$ and $Load(w^1_{x_i}) \geq 2\beta$.

If we now assume that, for some C_j containing the litteral x_i, $\mathcal{O}([w^1_{x_i}, w_{C_j}]) = (w_{C_j}, w^1_{x_i})$, then $Load(w_{C_j}) \geq 4 + \frac{2\beta}{d^-(w^1_{x_i})} \geq 4 + \frac{2\beta}{m+2}$. By Property 2, a part of that flow comes from $s^0_{C_j}$ through $w^0_{C_j}$. Consequently, $Load(s^0_{C_j}) = Load(w^0_{C_j}) = \gamma + \frac{Load(w_{C_j})}{d^-(w_{C_j})} \geq \gamma + \frac{Load(w_{C_j})}{4} \geq \gamma + \frac{\beta}{2(m+2)}$. However, as $\beta > 10m$ and $Prod(s^0_{C_j}) = \gamma + 2$, the source $s^0_{C_j}$ cannot satisfy its production capacity constraint. □

Property 4. For all $i \in [\![1;n]\!]$, if $\alpha(w^1_{x_i}) = 1$ then $\mathcal{O}([w_{x_i}, w^1_{x_i}]) = (w_{x_i}, w^1_{x_i})$. The same property occurs for $\bar{x}_i$.

Proof. If we assume the contrary, then by Properties 2 and 3, all the flow going through the arc $(w^1_{x_i}, w_{x_i})$ comes from the source $s^1_{x_i}$. Thus $Load(s^1_{x_i}) \geq 2\beta + \frac{\gamma}{3}$. However, as $\beta > m$ and $Prod(s^1_{x_i}) = \beta + m$, the source $s^1_{x_i}$ cannot satisfy its production capacity constraint.

□

The only decision that should be made is to activate or not the nodes $w^1_{x_i}$ and $w^1_{\bar{x}_i}$.

Property 5. It is not possible to activate both $w^1_{x_i}$ and $w^1_{\bar{x}_i}$.

Proof. If $w^1_{x_i}$ and $w^1_{\bar{x}_i}$ were both activated, $Load(w^1_{x_i}) \geq 2\beta$ and $Load(w^1_{\bar{x}_i}) \geq 2\beta$ due to the flows going to $w^2_{x_i}$ and $w^2_{\bar{x}_i}$. As the switch w_{x_i} is activated, half of these values is sent by w_{x_i} (the other half is produced by the sources $s^1_{x_i}$ and $s^1_{\bar{x}_i}$). However w_{x_i} also sends γ units of flow to p_{x_i}. Consequently $Load(s_{x_i}) = Load(w_{x_i}) \geq 2\beta + \gamma > \beta + m + \gamma$: the capacity constraint is not satisfied for s_{x_i}. Thus it is not possible to activate the two switches $w^1_{x_i}$ and $w^1_{\bar{x}_i}$. □

Lemma 3. *Given a boolean formula φ and an instance $\mathcal{J}$ of VALID built from φ using the former reduction, if φ is satisfiable then $\mathcal{J}$ is valid.*

Proof. If the formula φ can be satisfied then there exists a truth affectation of the variables. We activate $w^1_{x_i}$ if x_i if true and $w^1_{\bar{x}_i}$ otherwise.

For each clause $C_j = (l_1 \vee l_2 \vee l_3)$, at least one of the three literals is true. Let l_i be that literal, then $s^1_{l_i}$ fills p_{C_j}. Each other consumer node p_{x_i}, $p^2_{x_i}$ and $p^2_{\bar{x}_i}$ is filled at least by the corresponding source in the gadget of x_i.

The capacity constraint is satisfied for all the nodes.

- $Load(s^0_{C_j}) \leq \gamma + \frac{Load(w_{C_j})}{2} \leq \gamma + 2 = Prod(s^0_{C_j})$;
- $Load(s^2_{x_i})$ and $Load(s^2_{\bar{x}_i})$ are either 2β or 4β and $Prod(s^2_{x_i}) = Prod(s^2_{\bar{x}_i}) = 4\beta$.
- If $\alpha(w^1_{x_i}) = 1$, then $Load(s^1_{x_i}) \leq \frac{2\beta+2m}{2} = Prod(s^1_{x_i})$. Similarly for $w^1_{\bar{x}_i}$.
- Finally $Load(s_{x_i}) \leq \gamma + \beta + m = Prod(s_{x_i})$ as $w^1_{x_i}$ and $w^1_{\bar{x}_i}$ cannot be both activated.

□

Lemma 4. *If $\mathcal{J}$ is valid then φ is satisfiable.*

Proof. If there exists a valid activation α and a valid orientation $\mathcal{O}_\alpha$ of G, then we define the following truth affectation: x_i is true if and only if $w^1_{x_i}$ is activated.

Each consumer node is filled by at least one source. The consumer p_{C_j} cannot be completely filled by $s^0_{C_j}$, otherwise $Load(s^0_{C_j})$ would be $\gamma + 4$, greater than $Prod(s^0_{C_j}) = \gamma + 2$.

Thus, for each clause $C_j = (l_1 \vee l_2 \vee l_3)$, there exists a path in $G^{\mathcal{O}_\alpha}_\alpha$ from some source to p_{C_j} and that path necessarily goes through one of the nodes $w^1_{l_i}$. Consequently, that literal is true and the clause is also true: the formula is satisfied by this assignment. □

Theorem 3. *The Problem VALID is NP-complete, even if the maximum capacity of the switches is arbitrarily large.*

Proof. The proof is a direct consequence of lemmas 1, 3 and 4 □

Corollary 1. *The Problem VALID is NP-complete, even if the maximum capacity of the sources is arbitrarily large.*

Proof. By Theorem 3, the problem VALID is NP-Complete. We can easily transform an instance of VALID to an instance where all the sources have an arbitrarily large production capacity by adding, for each source s, a new switch w_s with capacity $Cap(w_s) = Prod(s)$. We then delete the edges incident to s and link w_s to s and each previous neighbour of s. Finally, we can now arbitrarily increase the value of $Prod(s)$ without changing the feasible flows. □

But relaxing the constraints on both the capacity and production makes the Problem VALID easy to solve. We can produce a solution to such instances by choosing an orientation of the edges from the sources to the producers. With only one of these constraints, the problem VALID becomes NP-Complete.

3.3 Single source cases

In this part, we show that, even when there is a unique source, the problems remains at least weakly NP-Complete. We recall that the two problems VALID and WDA-VALID are equivalent when $|S| = 1$ as there cannot be any consumer powered by two sources. We then only focus on VALID for the proof.

We give a polynomial time reduction from the PARTITION problem, which consists, given a set $X = \{x_1, x_2, \ldots, x_n\}$ of integers, in the search for a subset $I \subset [\![1; n]\!]$ such that $\sum_{i \in I} x_i = \sum_{i \notin I} x_i$. Let X be an instance of the PARTITION problem. We may assume that $\sum_{i=1}^{n} x_i$ is even otherwise there is trivially no solution. Let $A = (n+1)\max(X)$ and $B = \sum_{i=1}^{n} x_i/2$, we build an instance $(G = (V = S \cup W \cup P, E), Pow, Cap, Prod)$ of VALID as follows. S, the unique source, has an infinite production. Two switches W two switches w and $\bar{w}$ with capacity $A + B$ and two consumers p and $\bar{p}$ with called power A are created. We connect w to s and p, and $\bar{w}$ to s and $\bar{p}$. For each integer $x_i \in X$, we add two switches w_i and $\bar{w}_i$ with capacity $\frac{3x_i}{4}$ and one consumer p_i with call powers x_i. We add the path $(w, w_i, p_i, \bar{w}_i, \bar{w})$ and the edge $[w_i, \bar{w}_i]$. As shown in Figure 4, the graph is planar.

Theorem 4. *VALID is (weakly) NP-Complete even if there is only one source.*

Proof. First, note that, for every $I \subset [\![1; n]\!]$, $\sum\limits_{i \in I} x_i = \sum\limits_{i \notin I} x_i = B$ if and only if $\frac{3}{4}\sum\limits_{i \in I} x_i + \frac{1}{4}\sum\limits_{i \notin I} x_i = \frac{1}{4}\sum\limits_{i \in I} x_i + \frac{3}{4}\sum\limits_{i \notin I} x_i = B$.

Let $(\alpha, \mathcal{O})$ be a feasible solution, then we can prove that every switch must be activated. Let $i \in [\![1; n]\!]$, if $\alpha(\bar{w}_i) = 0$, then $\alpha(w_i) = 1$ and $\mathcal{O}([w_i, p_i]) = (w_i, p_i)$, otherwise, p_i is not powered and $G^{\mathcal{O}_\alpha}$ is not coherent. Consequently,

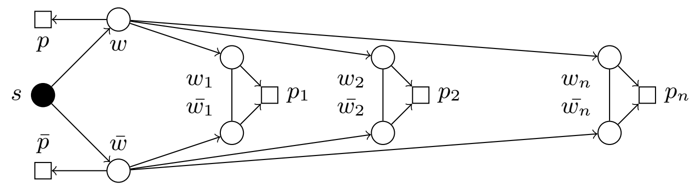

Fig. 4. Exemple of graph obtained with the reduction from PARTITION to VALID. On this graph are drawn the necessary orientations proved with Theorem 4.

$Load(w_i) \geq Pow(p_i) = x_i > Cap(w_i) = \frac{3}{4}x_i$. Similarly, $\alpha(w_i) \neq 0$. For the same reason, $\mathcal{O}([w_i, p_i]) = (w_i, p_i)$ and $\mathcal{O}([\bar{w_i}, p_i]) = (\bar{w_i}, p_i)$. In addition, $\alpha(w) \neq 0$ and $\alpha(\bar{w}) \neq 0$, otherwise, p or $\bar{p}$ are not powered. Finally $\mathcal{O}([w, w_i]) = (w, w_i)$ and $\mathcal{O}([\bar{w}, \bar{w_i}]) = (\bar{w}, \bar{w_i})$. Indeed, if, for instance, we assume that $[w, w_i]$ is directed from w_i, then the called power of p partially goes through that arc and $Load(w_i) \geq F(w_i, w) \geq \frac{F(w,p)}{d^-(w)} \geq \frac{A}{n+1} \geq \max(X) \geq x_i > Cap(w_i)$. Figure 4 shows all those orientations on the graph.

As a consequence, the only choice left in order to build a feasible solution is the direction of the edge $[w_i, \bar{w_i}]$. Let I be the set of indexes $i \in [\![1; n]\!]$ for which that edge is directed from w_i. In that case, if $i \in I$, $F(w, w_i) = F(w_i, \bar{w_i}) + F(w_i, p_i) = \frac{1}{2}F(\bar{w_i}, p_i) + F(w_i, p_i) = \frac{3}{4}x_i$ and $F(w, \bar{w_i}) = \frac{1}{2}F(\bar{w_i}, p_i) = \frac{1}{4}x_i$. Similarly, if $i \notin I$, $F(w, w_i) = \frac{1}{4}x_i$ and $F(w, \bar{w_i}) = \frac{3}{4}x_i$. Consequently, $Load(w) = A + \frac{3}{4}\sum_{i \in I} x_i + \frac{1}{4}\sum_{i \notin I} x_i$ and $Load(\bar{w}) = A + \frac{1}{4}\sum_{i \in I} x_i + \frac{3}{4}\sum_{i \notin I} x_i$. As $Cap(w) = Cap(\bar{w}) = A + B$, there is a feasible solution if and only if there exists $I \subset [\![1; n]\!]$ such that $\frac{3}{4}\sum_{i \in I} x_i + \frac{1}{4}\sum_{i \notin I} x_i = \frac{1}{4}\sum_{i \in I} x_i + \frac{3}{4}\sum_{i \notin I} x_i = B$ if and only if the PARTITION instance is positive.

As the described transformation is polynomial, there exists a polynomial Karp reduction from PARTITION to VALID. □

4 Conclusion

In this paper, we have presented the problem of configuring electrical grids taking into account the actual specificities. We have demonstrated that these specificities make the configuration problem NP-complete (at least weakly), even in the most restricted cases (planar graph, only one source, and only a few conditions on the productions, capacities, and consumptions). Besides, this difficulty is only related to questions about the existence of solutions.

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