GLOBAL EXISTENCE FOR NAVIER-STOKES SYSTEM WITH STRONG DISSIPATION IN ONE DIRECTION

Marius Paicu, Ping Zhang

GLOBAL EXISTENCE FOR NAVIER-STOKES SYSTEM WITH STRONG DISSIPATION IN ONE DIRECTION

MARIUS PAICU AND PING ZHANG

Abstract. We consider the 3D incompressible Navier-Stokes equations with different viscous coefficients in each variables. In particular, when one of this viscous coefficient is large enough compared to the initial data, we prove the global well-posedness of this problem. More generally, we obtain the existence of a global strong solution when the initial data verify an anisotropic smallness condition which take into account the different role of the horizontal and vertical viscosity.

1. Introduction

We consider the anisotropic incompressible Navier-Stokes system with different vertical and horizontal viscosities. Our goal is to study the role that plays the large viscous coefficient in one direction in the global existence of the strong solution for the Navier-Stokes system. Let us recall the anisotropic Navier-Stokes equations which describes the evolution of a fluid with "turbulent viscosity" in $\mathbb{R}^3$

$$(1.1)\qquad (NS_v)\begin{cases}\partial_t u + u\nabla u - \nu_{\mathrm{h}}\Delta_{\mathrm{h}} u - \nu_{\mathrm{v}}\partial_3^2 u = -\nabla p & \text{in} \quad [0,T]\times\mathbb{R}^3\\ \operatorname{div} u = 0\\ u|_{t=0} = u_0,\end{cases}$$

where $\Delta_h = \partial_{x_1}^2 + \partial_{x_2}^2$ is the horizontal Laplacian, $\nu_{\mathrm{h}} > 0$ and $\nu_{\mathrm{v}} > 0$ are respectively the horizontal and the vertical viscous coefficients, $u = (u_1, u_2, u_3)$ denotes velocity field of the fluid, and p the scalar pressure, the Lagrange multiplier which assure the divergence free condition on the velocity field.

When $\nu_h = \nu_v$ we obtain the classical Navier-Stokes system, whereas when $\nu_v = 0$ (NS_v) is the anisotropic Navier-Stokes system arising from geophysical fluid mechanics (see [7]). The Navier-Stokes system with large vertical viscosity is an usual model to study the evolution of the fluid in a thin domain in the vertical direction (see [30]). Our main motivation for us to study Navier-Stokes system with large vertical viscous coefficient comes from the study of Navier-Stokes system on thin domains.

The first important result about the classical Navier-Stokes system, was obtained by J. Leray in the seminar paper [22] in 1933. He proved that given a every finite energy initial data, (1.1) has a global in time weak solution which verify the energy estimate. This solution is unique in $\mathbb{R}^2$ but unfortunately the solution is not known to be unique in dimension three. The Fujita-Kato theorem [15] give a partial answer to the construction of global unique solution to (1.1). Indeed, the theorem of Fujita-Kato [15] allows to construct local in time unique solution to (1.1) in the homogeneous Sobolev spaces $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, or in the Lebsegue space $L^3(\mathbb{R}^3)$ (see [18]). Moreover, if the initial data is small compared the the viscosity, that is, $\|u_0\|_{\dot{H}^{\frac{1}{2}}} \leq c\nu$, the the strong solution exists globally in time. This result was generalized by M. Cannone, Y. Meyer et F. Planchon [5] for initial data in Besov spaces with negative index. The end-point result in this direction is given by D. Tataru and H. Koch [19]. They proved that given

Date: December 11, 2018.

initial data in the derivatives of BMO space with the norm sufficiently small compared to the viscosity, then (1.1) has a unique global solution.

We remark that all the norms in $\dot{H}^{\frac{1}{3}}(\mathbb{R}^3)$, $L^3(\mathbb{R}^3)$, $B_{p,\infty}^{-1+\frac{3}{p}}(\mathbb{R}^3)$ and ∇BMO, are sclaing-invariant under the following scaling transform:

$$u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x) \quad \text{and} \quad u_{0,\lambda}(x) = \lambda u_0(\lambda x). \tag{1.2}$$

We notice that any solution u of (1.1) on $[0,T]$, u_λ is also a solution of (1.1) on $[0,T/\lambda^2]$. We remark that the largest space, which belongs to $\mathcal{S}'(\mathbb{R}^3)$ and the norm of which is scaling invariant under (1.2), is $\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$. So, a large initial data means a large data compared to the viscosity coefficient in the space $\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$.

We recall also some examples of large initial data generates unique global solution to (1.1). G. Raugel and G. Sell [30] obtained the global well-posedness of (1.1) in a thin domain. This result was generalized by G. Raguel, G. Sell and D. Iftimie in [17] where loosely speaking, they proved that (1.1) has a unique global periodic solution provided that if the initial data u_0 can be splitting as $u_0 = v_0 + w_0$, with v_0 being a bidimensional solenoidal vector field in $L^2(\mathbb{T}^2_{\mathrm{h}})$ and $w_0 \in H^{\frac{1}{2}}(\mathbb{T}^3)$, such that

$$\|w_0\|_{H^{\frac{1}{2}}(\mathbb{T}^3)} \exp\left(\frac{\|u_0\|^2_{L^2(\mathbb{T}^2_{\mathrm{h}})}}{\nu^2}\right) \leq c\nu.$$

J.-Y. Chemin and I. Gallagher construct in [10] the first example of initial data which is effectively large in $B^{-1}_{\infty,\infty}$ and gives rise to a global strong solution. The initial data is highly oscillating in the vertical variable and is give by $u_0^N = (Nu_{\mathrm{h}}(x_h)\cos(Nx_3), -\operatorname{div}_{\mathrm{h}} u_h(x_h) sin(Nx_3))$, where $\|u_{\mathrm{h}}\|_{L^2(\mathbb{T}^2_h)} \leq C(\ln N)^{\frac{1}{9}}$. They also consider the case of large quasi-2D initial data and more generally the case of slowly varying in one direction data in the well-prepared situation in [12]. More recently, the case of slowly varying in one direction data in the ill-prepared situation was studied in [13] by performing estimates in analytic functions spaces and by obtaining a global Cauchy-Kowalewkaya type theorem.

Our goal in this paper, is to study the role of a large vertical viscosity in order to obtain the global well-posendess of the (NS_v) system.

As a consequence of ours results is that, we obtain the global well-posedness of (NS_v) system for any large initial data, if the vertical viscosity is large enough. Moreover, we obtain the result of the global existence and uniqueness of the solution even in the case of a small horizontal viscosity converging to zero and a large enough vertical viscosity.

The first main result of this paper states as follows:

Theorem 1.1. *Let $\bar{u}_0^{\mathrm{h}}$ be in $H^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}_{\mathrm{v}}; \dot{H}^{-\delta}(\mathbb{R}_{\mathrm{h}})) \cap L^\infty(\mathbb{R}_{\mathrm{v}}; H^1(\mathbb{R}_{\mathrm{h}}))$ for some $\delta \in]0,1[$. Let us assume also that $\bar{u}_0^{\mathrm{h}}$, $\partial_z \bar{u}_0^{\mathrm{h}}$ and $\partial_z^2 \bar{u}_0^{\mathrm{h}}$ belongs to $L^2(\mathbb{R}_{\mathrm{v}}; \dot{H}^{-\frac{1}{2}} \cap \dot{H}^{\frac{1}{2}}(\mathbb{R}_{\mathrm{h}}))$. Let $v_0 \in \mathcal{B}^{\frac{1}{2},0}(\mathbb{R}^3) \cap B^{0,\frac{1}{2}}(\mathbb{R}^3)$ be a solenoidal vector field. We assume that $\nu_{\mathrm{v}} \geq \nu_{\mathrm{h}} > 0$. Then for some $\theta \in [0,1/2[$, there exists a positive constant $\mathfrak{E}_{\delta,\nu_{\mathrm{h}}}(\bar{u}_0^{\mathrm{h}})$, which depends on the norms of $\bar{u}_0^{\mathrm{h}}$ above, such that if ν_{v} is so large that*

$$\left(\mathcal{A}_0^{\frac{1}{2}} \nu_{\mathrm{h}}^{-\frac{5}{8}} \nu_{\mathrm{v}}^{-\frac{1}{8}} + \varepsilon^{1-\theta} \nu_{\mathrm{v}}^{-\frac{\theta}{2}}\right) \mathfrak{E}_{\delta,\nu_h}(\bar{u}_0^{\mathrm{h}}) \leq c_0 \nu_{\mathrm{h}} \quad \textit{with} \quad \mathcal{A}_0 \stackrel{def}{=} \|v_0\|_{\mathcal{B}^{\frac{1}{2},0}} \big(\|v_0\|_{B^{0,\frac{1}{2}}} + \|v_0\|_{\mathcal{B}^{\frac{1}{2},0}}\big) \tag{1.3}$$

for some c_0 sufficiently small, the initial data

$$u_{0,\varepsilon}(x_{\mathrm{h}}, x_3) = \big(\bar{u}_0^{\mathrm{h}}(x_{\mathrm{h}}, \varepsilon x_3), 0\big) + v_0(x) \tag{1.4}$$

generates a unique global solution u to (1.1) in the space $C_b(\mathbb{R}^+; \dot{B}^{0,\frac{1}{2}}(\mathbb{R}^3))$ with $\nabla u \in L^2(\mathbb{R}^+; \dot{B}^{0,\frac{1}{2}}(\mathbb{R}^3))$.

The definitions of the Besov norms will be presented in Section 3. The exact form of the constant $\mathfrak{E}_{\delta,\nu_h}(\bar{u}_0^{\rm h})$ will be given by (5.8).

We remark that the norms of v_0 in $\mathcal{B}^{\frac{1}{2},0}(\mathbb{R}^3)$ and $B^{0,\frac{1}{2}}(\mathbb{R}^3)$ are scaling invariant under the scaling transformation (1.2).

In the case when $\nu_{\rm h}=0$ in (1.1), that is

$$
\begin{cases}
\partial_t u + u\cdot\nabla u - \nu_{\rm v}\partial_3^2 u = -\nabla p \quad \text{for } (t,x)\in\mathbb{R}^+\times\Omega,\\
\operatorname{div} u = 0,\\
u|_{t=0}=u_0,\\
u|_{x_3=0}=u|_{x_3=1}=0,
\end{cases}
\tag{1.5}
$$

where $\Omega=\mathbb{R}^2\times]0,1[$, we have the following global well-posedness result for (1.5):

Theorem 1.2. *Given solenoidal vector field $u_0\in L^2(\mathbb{R}^3)\cap\mathcal{B}^2_h(\mathbb{R}^3)$, then there exists a small enough positive constant c so that if*

$$
\|u_0\|_{L^2}+\|u_0\|_{\mathcal{B}^2_{\rm h}}\le c\nu_{\rm v},
\tag{1.6}
$$

(1.5) *has a unique solution $u\in C_b([0,\infty[;L^2\cap\mathcal{B}^2_h(\mathbb{R}^3))$ with $\partial_3 u\in L^2(\mathbb{R}^+;L^2\cap\mathcal{B}^2_h(\mathbb{R}^3))$.*

Let us complete this section by the notations of the paper:

Let A,B be two operators, we denote $[A;B]=AB-BA$, the commutator between A and B. For $a\lesssim b$, we mean that there is a uniform constant C, which may be different on different lines, such that $a\le Cb$. We denote by $(a|b)$ the $L^2(\mathbb{R}^3)$ inner product of a and b, $(d_\ell)_{\ell\in\mathbb{Z}}$ will be a generic element of $\ell^1(\mathbb{Z})$ so that $\sum_{j\in\mathbb{Z}}d_j=1$.

For X a Banach space and I an interval of $\mathbb{R}$, we denote by $\mathcal{C}(I;\,X)$ the set of continuous functions on I with values in X, and by $\mathcal{C}_b(I;\,X)$ the subset of bounded functions of $\mathcal{C}(I;\,X)$. For $q\in[1,+\infty]$, the notation $L^q(I;\,X)$ stands for the set of measurable functions on I with values in X, such that $t\longmapsto\|f(t)\|_X$ belongs to $L^q(I)$.

2. Ideas of the proof and structure of the paper

Let us construct v_L via

$$
\begin{cases}
\partial_t v_L - \nu_{\rm h}\Delta_{\rm h}v_L - \nu_{\rm v}\partial_z^2 v_L = 0 \quad \text{for } (t,x_{\rm h},z)\in\mathbb{R}^+\times\mathbb{R}^3,\\
v_L|_{t=0}=v_0.
\end{cases}
\tag{2.1}
$$

We construct $(\bar{u}^{\rm h},p^{\rm h})$ through

$$
\begin{cases}
\partial_t\bar{u}^{\rm h}+\bar{u}^{\rm h}\cdot\nabla_{\rm h}\bar{u}^{\rm h}-\nu_{\rm h}\Delta_{\rm h}\bar{u}^{\rm h}-\nu_{\rm v}\varepsilon^2\partial_z^2\bar{u}^{\rm h}=-\nabla_{\rm h}p^{\rm h} \quad \text{for } (t,x_{\rm h},z)\in\mathbb{R}^+\times\mathbb{R}^3,\\
\operatorname{div}_{\rm h}\bar{u}^{\rm h}=0,\\
\bar{u}^{\rm h}|_{t=0}=\bar{u}^{\rm h}_0.
\end{cases}
\tag{2.2}
$$

We write

$$
u=v_L+[\bar{u}^{\rm h}]_\varepsilon+R \quad \text{and} \quad \pi=p-p^{\rm h} \quad \text{with} \quad [f]_\varepsilon(t,x_{\rm h},x_3)=f(t,x_{\rm h},\varepsilon x_3).
\tag{2.3}
$$

Then it follows form (1.1) and (2.1), (2.2) that $(R,\nabla\pi)$ verifies

$$
\begin{cases}
\partial_t R+u\cdot\nabla R+R\cdot\nabla(v_L+[\bar{u}^{\rm h}]_\varepsilon)-\nu_{\rm h}\Delta_{\rm h}R-\nu_{\rm v}\partial_3^2R+\nabla\pi=F,\\
\quad \text{with} \quad F=(F^{\rm h},F^{\rm v}) \quad \text{and}\\
F^{\rm h}=-v_L\cdot\nabla(v_L^{\rm h}+[\bar{u}^{\rm h}]_\varepsilon)-[\bar{u}^{\rm h}]_\varepsilon\cdot\nabla_{\rm h}v_L^{\rm h} \quad \text{and}\\
F^{\rm v}=-v_L\cdot\nabla v_L^3-[\bar{u}^{\rm h}]_\varepsilon\cdot\nabla_h v_L^3-\partial_3[p^{\rm h}]_\varepsilon,\\
\operatorname{div} R=0,\\
R|_{t=0}=0,
\end{cases}
\tag{2.4}
$$

where p^{h} is determined by

$$p^{\mathrm{h}} = \sum_{i,j=1}^{2} (-\Delta_{\mathrm{h}})^{-1}\partial_i\partial_j(\bar{u}^i\bar{u}^j), \tag{2.5}$$

which is obtained by taking $\mathrm{div}_{\mathrm{h}}$ to the momentum equation of (2.2).

In order to explain the main idea to the proof of Theorem 1.1. Let us first assume $\bar{u}_0^{\mathrm{h}} = 0$ in (1.2). Instead of assume v_0 in the critical spaces, we assume $v_0 \in H^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$. Then we have the following simplified version of Theorem 1.1.

Theorem 2.1. *Let $v_0 \in H^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ be a solenoidal vector field, we denote*

$$E(v_0) \stackrel{\text{def}}{=} \|v_0\|_{L^2}^3 \|\nabla_{\mathrm{h}} v_0\|_{L^2} + \|v_0\|_{L^\infty}^2 \|v_0\|_{L^2}^2. \tag{2.6}$$

We assume that $\nu_{\mathrm{v}} \geq \nu_{\mathrm{h}} > 0$. Then there exist two positive constants c, C so that if

$$\frac{E(v_0)}{\sqrt{\nu_{\mathrm{h}}\nu_{\mathrm{v}}}} \exp\Big(\frac{\|v_0\|_{H^{0,1}}^4}{C\nu_{\mathrm{h}}^4}\Big) \leq c\nu_{\mathrm{h}}^3, \tag{2.7}$$

then (1.1) *has a unique global solution u with*

$$u \in C([0,\infty[; H^{0,1}(\mathbb{R}^3)) \quad \text{and} \quad \nabla u \in L^2(\mathbb{R}_+; H^{0,1}(\mathbb{R}^3)).$$

We begin the proof of the above theorem by the following useful lemmas.

Lemma 2.1. *Let a be a free divergence vector field in $H^{0,1}$ with $\nabla_{\mathrm{h}} a$ belonging to $H^{0,1}$. Let $b \in H^{0,1}$ with $\nabla_{\mathrm{h}} b \in H^{0,1}$. Then one has*

$$\big(a\nabla b | b\big)_{H^{0,1}} \leq C\|a\|_{H^{0,1}}^{\frac12} \|\nabla_{\mathrm{h}} a\|_{H^{0,1}}^{\frac12} \|b\|_{H^{0,1}}^{\frac12} \|\nabla_{\mathrm{h}} b\|_{H^{0,1}}^{\frac32}.$$

Proof. We recall that $\|a\|_{H^{0,1}} = \big(\|a\|_{L^2}^2 + \|\partial_3 a\|_{L^2}^2\big)^{\frac12}$. Then It is easy to observe that due to $\operatorname{div} a = 0$, $\big(a\nabla b|b\big)_{L^2} = 0$, so that there holds

$$\begin{aligned}
\big(a\nabla b|b\big)_{H^{0,1}} =& \big(a\nabla b|b\big)_{L^2} + \big(\partial_3(a\nabla b)|\partial_3 b\big)_{L^2}\\
=& \big(a\nabla\partial_3 b|\partial_3 b\big)_{L^2} + \big(\partial_3 a\nabla b|\partial_3 b\big)_{L^2}\\
=& \big(\partial_3 a_{\mathrm{h}}\nabla_{\mathrm{h}} b|\partial_3 b\big)_{L^2} + \big(\partial_3 a_3\partial_3 b|\partial_3 b\big)_{L^2}.
\end{aligned}$$

Notice that

$$\|f\|_{L^\infty_{\mathrm{v}}(L^2_{\mathrm{h}})} \lesssim \|f\|_{L^2}^{\frac12} \|\partial_3 f\|_{L^2}^{\frac12},$$

and $\partial_3 a_3 = -\operatorname{div}_{\mathrm{h}} a_h$, we deduce from the law of product in Sobolev spaces that

$$\begin{aligned}
|\big(\operatorname{div}_{\mathrm{h}} a_{\mathrm{h}}\partial_3 b|\partial_3 b\big)_{L^2}| \leq& \|\operatorname{div}_h a_h\|_{L^\infty_{\mathrm{v}}(\dot{H}_{\mathrm{h}}^{-\frac12})} \|(\partial_3 b)^2\|_{L^1_{\mathrm{v}}(\dot{H}_{\mathrm{h}}^{\frac12})}\\
\leq& \|\nabla_{\mathrm{h}} a\|_{L^\infty_{\mathrm{v}}(\dot{H}_{\mathrm{h}}^{-\frac12})} \|\partial_3 b\|^2_{L^2_{\mathrm{v}}(\dot{H}_{\mathrm{h}}^{\frac34})}\\
\lesssim& \|a\|_{L^\infty_{\mathrm{v}}(\dot{H}_{\mathrm{h}}^{\frac12})} \|\partial_3 b\|_{L^2}^{\frac12} \|\nabla_{\mathrm{h}}\partial_3 b\|_{L^2}^{\frac32}.
\end{aligned}$$

On the other hand, we write

$$\frac{d}{dx_3}\int_{\mathbb{R}^2} |\xi_{\mathrm{h}}| |\hat{f}(\xi_{\mathrm{h}}, x_3)|^2 \, d\xi_{\mathrm{h}} = 2\int_{\mathbb{R}^2} |\xi_{\mathrm{h}}| \hat{f}(\xi_{\mathrm{h}}, x_3)\partial_3 \hat{f}(\xi_{\mathrm{h}}, x_3) \, d\xi_{\mathrm{h}}.$$

Integrating the above inequality over $]-\infty, x_3]$ and using Hölder's inequality, we achieve

$$\|f\|_{L^\infty_{\mathrm{v}}(\dot{H}_{\mathrm{h}}^{\frac12})} \lesssim \|f\|_{L^2}^{\frac12} \|\nabla_{\mathrm{h}}\partial_3 f\|_{L^2}^{\frac12}.$$

As a result, it comes out

$$
| \big(\mathrm{div_h}\, a_{\mathrm{h}}\partial_3 b|\partial_3 b\big)_{L^2} | \lesssim \|a\|_{L^2}^{\frac12}\|\nabla_{\mathrm{h}}\partial_3 a\|_{L^2}^{\frac12}\|\partial_3 b\|_{L^2}^{\frac12}\|\nabla_{\mathrm{h}}\partial_3 b\|_{L^2}^{\frac32}. \tag{2.8}
$$

Whereas observe that

$$
\begin{aligned}
| \big(\partial_3 a_{\mathrm{h}}\nabla_{\mathrm{h}} b|\partial_3 b\big)_{L^2} | \leq & \|\partial_3 a_{\mathrm{h}}\|_{L^2_{\mathrm{v}}(L^4_{\mathrm{h}})}\|\nabla_{\mathrm{h}} b\|_{L^\infty_{\mathrm{v}}(L^2_{\mathrm{h}})}\|\partial_3 b\|_{L^2_{\mathrm{v}}(L^4_{\mathrm{h}})}\\
\lesssim & \|\partial_3 a_{\mathrm{h}}\|_{L^2}^{\frac12}\|\nabla_{\mathrm{h}}\partial_3 a_{\mathrm{h}}\|_{L^2}^{\frac12}\|\nabla_{\mathrm{h}} b\|_{L^2}^{\frac12}\|\nabla_{\mathrm{h}}\partial_3 b\|_{L^2}^{\frac12}\|\partial_3 b\|_{L^2}^{\frac12}\|\nabla_{\mathrm{h}}\partial_3 b\|_{L^2}^{\frac12},
\end{aligned}
$$

which together with (2.8) ensures the lemma. □

The next lemma is concerned with the linear equation (2.1), which tells us the small quantities that will be used in what follows

Lemma 2.2. *Let $v_0 \in L^2(\mathbb{R}^3)$ with $\nabla_{\mathrm{h}} v_0 \in L^2(\mathbb{R}^3)$. Let v_L be the corresponding solution of* (2.1). *Then we have*

$$
\|\partial_3 v_L\|_{L^2_t(L^2)}^2 \leq \frac{\|v_0\|_{L^2}^2}{\nu_{\mathrm{v}}} \quad \text{and} \quad \|\partial_3\nabla_{\mathrm{h}} v_L\|_{L^2_t(L^2)}^2 \leq \frac{\|\nabla_{\mathrm{h}} v_0\|_{L^2}^2}{\nu_{\mathrm{v}}}
$$

Proof. Indeed by applying standard energy method to (2.1), we get

$$
\begin{aligned}
&\|v_L(t)\|_{L^2}^2 + 2\nu_{\mathrm{h}}\int_0^t \|\nabla_{\mathrm{h}} v_L(t')\|_{L^2}^2\,dt' + 2\nu_{\mathrm{v}}\int_0^t \|\partial_3 v_L(t')\|_{L^2}^2\,dt' = \|v_0\|_{L^2}^2,\\
&\|\nabla_{\mathrm{h}} v_L(t)\|_{L^2}^2 + 2\nu_{\mathrm{h}}\int_0^t \|\nabla_{\mathrm{h}}^2 v_L(t')\|_{L^2}^2\,dt' + 2\nu_{\mathrm{v}}\int_0^t \|\partial_3\nabla_{\mathrm{h}} v_L(t')\|_{L^2}^2 = \|\nabla_{\mathrm{h}} v_0\|_{L^2}^2,
\end{aligned}
$$

which implies the lemma. □

Lemma 2.3. *Let $E(v_0)$ be given by* (2.6). *Then under the assumptions of Lemma 2.2, one has*

$$
\int_0^\infty \|v_L\otimes v_L(t)\|_{H^{0,1}}^2\,dt \leq \frac{E(u_0)}{\sqrt{\nu_{\mathrm{h}}\nu_{\mathrm{v}}}}.
$$

Proof. We begin by writing that

$$
\begin{aligned}
\int_0^\infty \|v_L\otimes v_L(t)\|_{H^{0,1}}^2\,dt =& \int_0^\infty \big(\|v_L\otimes v_L(t)\|_{L^2}^2 + \|\partial_3(v_L\otimes v_L)(t)\|_{L^2}^2\\
\leq& C\int_0^\infty \big(\|v_L(t)\|_{L^4}^4 + \|v_L\partial_3 v_L(t)\|_{L^2}^2\big)\,dt.
\end{aligned} \tag{2.9}
$$

Applying Lemma 2.2 gives

$$
\begin{aligned}
\int_0^\infty \|v_L\partial_3 v_L(t)\|_{L^2}^2\,dt \leq& \int_0^\infty \|v_3^L(t)\|_{L^\infty}^2\|\partial_3 v_L(t)\|_{L^2}^2\,dt\\
\leq& \frac{\|v_0\|_{L^\infty}^2\|v_0\|_{L^2}^2}{\nu_{\mathrm{v}}}.
\end{aligned} \tag{2.10}
$$

To handle the other term in (2.9), by applying the Sobolev embedding of $\dot H^{\frac14}(\mathbb{R}_{\mathrm{v}}) \hookrightarrow L^4(\mathbb{R}_{\mathrm{v}})$, we obtain

$$
\|v_L(t,x_{\mathrm{h}},\cdot)\|_{L^4(\mathbb{R}_{\mathrm{v}})} \leq C\|v_L(t,x_{\mathrm{h}},\cdot)\|_{L^2_{\mathrm{v}}}^{\frac34}\|\partial_3 v_L(t,x_{\mathrm{h}},\cdot)\|_{L^2_{\mathrm{v}}}^{\frac14},
$$

which together with Lemma 2.2 implies that

$$
\begin{aligned}
\int_0^\infty \|v_L(t)\|_{L^4}^4\,dt \lesssim& \int_0^\infty\int_{\mathbb{R}^2_{\mathrm{h}}} \|v_L(t,x_{\mathrm{h}},\cdot)\|_{L^2_{\mathrm{v}}}^3\|\partial_3 v_L(t,x_{\mathrm{h}},\cdot)\|_{L^2_v}\,dx_{\mathrm{h}}\,dt\\
\lesssim& \|v_L\|_{L^6(\mathbb{R}^+;L^6_{\mathrm{h}}(L^2_{\mathrm{v}}))}^3\|\partial_3 v_L\|_{L^2_t(L^2)}^2\\
\lesssim& \nu_{\mathrm{v}}^{-\frac12}\|v_L\|_{L^6(\mathbb{R}^+;L^6_{\mathrm{h}}(L^2_{\mathrm{v}}))}^3\|v_0\|_{L^2}.
\end{aligned}
$$

Whereas notice that Sobolev imbedding Theorem implies $\dot{H}^{\frac{2}{3}}(\mathbb{R}^2_h) \hookrightarrow L^6(\mathbb{R}^2_h)$, we write

$$\|v_L(t,\cdot,x_3)\|_{L^6_{\rm h}} \leq C\|v_L(t,\cdot,x_3)\|^{\frac{1}{3}}_{L^2_{\rm h}}\|\nabla_{\rm h}v_L(t,\cdot,x_3)\|^{\frac{2}{3}}_{L^2_{\rm h}}.$$

Taking the $L^2_{\rm v}$ norm leads to

$$\|v_L(t)\|_{L^6_{\rm h}(L^2_{\rm v})} \leq \|v_L(t)\|_{L^2_{\rm v}(L^6_{\rm h})} \leq C\|v_L(t)\|^{\frac{1}{3}}_{L^2(\mathbb{R}^3)}\|\nabla_{\rm h}v_L(t)\|^{\frac{2}{3}}_{L^2(\mathbb{R}^3)},$$

which together with Lemma 2.2 ensures that

$$\begin{aligned}\|v_L\|^3_{L^6(\mathbb{R}^+;L^2_{\rm v}(L^6_{\rm h}))} \lesssim&\|v_L\|_{L^\infty(\mathbb{R}^+;L^2)}\|\nabla_{\rm h}v_L\|_{L^\infty(\mathbb{R}^+;L^2)}\|\nabla_{\rm h}v_L\|_{L^2(\mathbb{R}^+;L^2)}\\ \lesssim&\nu_{\rm h}^{-\frac{1}{2}}\|v_0\|^2_{L^2}\|\nabla_{\rm h}v_0\|^2_{L^2}.\end{aligned}$$

This gives rise to

$$\int_0^\infty \|v_L(t)\|^4_{L^4}\,dt \leq \frac{\|v_0\|^3_{L^2}\|\nabla_h v_0\|_{L^2}}{\sqrt{\nu_{\rm h}\nu_{\rm v}}}.$$

Along with (2.9) and (2.10), we complete the proof of the lemma. □

Proof of the theorem 1. Let v_L be determined by (2.1). We write

$$u = v_L + R.$$

Inserting the above substitution into (1.1) yields

$$\begin{cases}\partial_t R + R\cdot\nabla R - \nu_{\rm h}\Delta_{\rm h}R - \nu_{\rm v}\partial_3^2 R + \nabla p = -R\cdot\nabla v_L - v_L\cdot\nabla R - v_L\cdot\nabla v_L,\\ \operatorname{div} R = 0,\\ R(0) = 0.\end{cases}\tag{2.11}$$

By taking $H^{0,1}$ scalar product of R equation of (2.11) with R, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|R\|^2_{H^{0,1}}+&\nu_{\rm h}\|\nabla_{\rm h}R\|^2_{H^{0,1}} + \nu_{\rm v}\|\partial_3 R\|^2_{H^{0,1}} = -\big(R\cdot\nabla R|R\big)_{H^{0,1}}\\ &-\big(v_L\cdot\nabla R|R\big)_{H^{0,1}} - \big(R\cdot\nabla v_L|R\big)_{H^{0,1}} - \big(v_L\cdot\nabla v_L|R\big)_{H^{0,1}}.\end{aligned}\tag{2.12}$$

Applying Lemma 2.1 gives

$$|\ \big(R\cdot\nabla R|R\big)_{H^{0,1}}\ |\leq \|R\|_{H^{0,1}}\|\nabla_{\rm h}R\|^2_{H^{0,1}},$$

and

$$|\ \big(v_L\cdot\nabla R|R\big)_{H^{0,1}}\ |\leq C\|v_L\|^{1/2}_{H^{0,1}}\|\nabla_{\rm h}v_L\|^{1/2}_{H^{0,1}}\|R\|^{1/2}_{H^{0,1}}\|\nabla_h R\|^{3/2}_{H^{0,1}}.$$

Then by applying Young's inequality, $xy \leq \frac{3}{4}x^{\frac{4}{3}} + \frac{1}{4}y^4$, we achieve

$$|\ \big(v_L\cdot\nabla R|R\big)_{H^{0,1}}\ |\leq C\nu_{\rm h}^{-3}\|v_L\|^2_{H^{0,1}}\|\nabla_{\rm h}v_L\|^2_{H^{0,1}}\|R\|^2_{H^{0,1}} + \frac{\nu_{\rm h}}{100}\|\nabla_h R\|^2_{H^{0,1}}.$$

Similarly, notice that $H^{\frac{1}{2}}(\mathbb{R}^2_{\rm h}) \subset L^4(\mathbb{R}^2_{\rm h})$, we get

$$\begin{aligned}|\ \big(R\cdot\nabla v_L|R\big)_{H^{0,1}}\ |\leq&\|R\|^2_{L^4_{\rm h}(H^1_{\rm v})}\|\nabla v_L\|_{H^{0,1}}\\ \leq&\|R\|_{H^{0,1}}\|\nabla_{\rm h}R\|_{H^{0,1}}\|\nabla v_L\|_{H^{0,1}}\\ \leq&C{\nu_{\rm h}}^{-1}\|\nabla v_L\|^2_{H^{0,1}}\|R\|^2_{H^{0,1}} + \frac{\nu_{\rm h}}{100}\|\nabla_{\rm h}R\|^2_{H^{0,1}}.\end{aligned}$$

For the last term in (2.11), we first get, by using integrating by parts, that

$$\big(v_L\cdot\nabla v_L|R\big)_{H^{0,1}} = -\big(v_L\otimes v_L|\nabla R\big)_{H^{0,1}},$$

which implies

$$
\begin{aligned}
|\left(v_L\cdot\nabla v_L|R\right)_{H^{0,1}}|\leq&\|v_L\otimes v_L\|_{H^{0,1}}\|\nabla R\|_{H^{0,1}}\\
\leq& C\nu_{\rm h}^{-1}\|v_L\otimes v_L\|_{H^{0,1}}^2+\frac{\nu_{\rm h}}{100}\|\nabla R\|_{H^{0,1}}.
\end{aligned}
$$

Let us denote

$$
T^\star\stackrel{\rm def}{=}\left\{\ T<t^*,\ \|R\|_{L^\infty_T(H^{0,1})}\leq\frac{\nu_{\rm h}}{4}\ \right\}. \tag{2.13}
$$

Then by substituting the above estimates into (2.12), for $t\leq T^\star$, we arrive at

$$
\begin{aligned}
\frac{d}{dt}\|R\|_{H^{0,1}}^2+\nu_{\rm h}\|\nabla_{\rm h}R\|_{H^{0,1}}^2+\nu_{\rm v}\|\partial_3R\|_{H^{0,1}}^2\leq& C\nu_{\rm h}^{-1}\|v_L\otimes v_L\|_{H^{0,1}}^2\\
&+C\Big({\nu_{\rm h}}^{-1}\|\nabla v_L\|_{H^{0,1}}^2+\nu_{\rm h}^{-3}\|v_L\|_{H^{0,1}}^2\|\nabla_{\rm h}v_L\|_{H^{0,1}}^2\Big)\|R\|_{H^{0,1}}^2.
\end{aligned} \tag{2.14}
$$

Applying Gronwall's inequality gives rise to (2.14)

$$
\begin{aligned}
\|R\|_{L^\infty_t(H^{0,1})}^2\leq& C\nu_{\rm h}^{-1}\|v_L\otimes v_L\|_{L^2_t(H^{0,1})}^2\\
&\times\exp\Big(\nu_{\rm h}^{-1}\|v_L\|_{L^\infty_t(H^{0,1})}^2\|\nabla_{\rm h}v_L\|_{L^2_t(H^{0,1})}^2+\nu_{\rm h}^{-1}\|\nabla v_L\|_{L^2_t(H^{0,1})}^2\Big),
\end{aligned}
$$

from which, Lemmas 2.2 and 2.3, for $t\leq T^\star$, we infer

$$
\|R\|_{L^\infty_t(H^{0,1})}^2\leq\nu_{\rm h}^{-\frac32}\nu_{\rm v}^{-\frac12}E(v_0)\exp\left(C\nu_{\rm h}^{-4}\|v_0\|_{H^{0,1}}^4+C\nu_{\rm h}^{-2}\|v_0\|_{H^{0,1}}^2\right).
$$

Then under the smallness condition (2.7), we have

$$
\|R\|_{L^\infty_t(H^{0,1})}\leq\frac{\nu_{\rm h}}{8}\quad\text{for}\quad t\leq T^\star, \tag{2.15}
$$

which contradicts with (2.13). This in turn shows that $T^\star=T^*=\infty$. Furthermore inserting the estimate (2.15) into (2.14) shows that $\nabla R\in L^2(\mathbb{R}^+;H^{0,1})$. this completes the proof of Theorem 2.1. $\square$

The organization of this paper is as follows:

In the third section, we shall recall some basic facts on Littlewood-Paley theory.

In the fourth section, we present the *priori* estimates for smooth enough solutions of (2.1) and (2.2).

In the fifth section, we prove Theorem 1.1.

In the sixth section, we present the proof of Theorem 1.2.

3. Basics on Littlewood-Paley theory

Before we present the function spaces we are going to work with in this context, let us briefly recall some basic facts on Littlewood-Paley theory (see e.g. [6]). Let φ and χ be smooth functions supported in $\mathcal{C}\stackrel{\rm def}{=}\{\tau\in\mathbb{R}^+,\ \frac34\leq\tau\leq\frac83\}$ and $\mathfrak{B}\stackrel{\rm def}{=}\{\tau\in\mathbb{R}^+,\ \tau\leq\frac43\}$ respectively such that

$$
\sum_{j\in\mathbb{Z}}\varphi(2^{-j}\tau)=1\quad\text{for}\quad\tau>0\quad\text{and}\quad\chi(\tau)+\sum_{j\geq0}\varphi(2^{-j}\tau)=1\quad\text{for}\quad\tau\geq0.
$$

For $a\in\mathcal{S}'(\mathbb{R}^3)$, we set

$$
\begin{aligned}
&\Delta_k^{\rm h}a\stackrel{\rm def}{=}\mathcal{F}^{-1}(\varphi(2^{-k}|\xi_{\rm h}|)\widehat{a}),\qquad S_k^{\rm h}a\stackrel{\rm def}{=}\mathcal{F}^{-1}(\chi(2^{-k}|\xi_{\rm h}|)\widehat{a}),\\
&\Delta_\ell^{\rm v}a\stackrel{\rm def}{=}\mathcal{F}^{-1}(\varphi(2^{-\ell}|\xi_3|)\widehat{a}),\qquad S_\ell^{\rm v}a\stackrel{\rm def}{=}\mathcal{F}^{-1}(\chi(2^{-\ell}|\xi_3|)\widehat{a}),\quad\text{and}\\
&\Delta_ja\stackrel{\rm def}{=}\mathcal{F}^{-1}(\varphi(2^{-j}|\xi|)\widehat{a}),\qquad S_ja\stackrel{\rm def}{=}\mathcal{F}^{-1}(\chi(2^{-j}|\xi|)\widehat{a}),
\end{aligned} \tag{3.1}
$$

where $\xi_{\rm h}=(\xi_1,\xi_2)$, $\xi=(\xi_{\rm h},\xi_3)$, $\mathcal{F}a$ and $\widehat{a}$ denote the Fourier transform of the distribution a. The dyadic operators satisfy the property of almost orthogonality:

$$\Delta_k\Delta_j a\equiv 0 \quad \text{if} \quad |k-j|\geq 2 \quad \text{and} \quad \Delta_k(S_{j-1}a\Delta_j b)\equiv 0 \quad \text{if} \quad |k-j|\geq 5. \tag{3.2}$$

Similar properties hold for $\Delta_k^{\rm h}$ and $\Delta_\ell^{\rm v}$.

Due to the anisotropic spectral properties of the linear equation, (2.1), we need also the following anisotropic type Besov norm:

Definition 3.1. *Let $s_1,s_2\in\mathbb{R}$ and $a\in\mathcal{S}_h'(\mathbb{R}^3)$, we define the norm*

$$\|a\|_{\mathcal{B}^{s_1,s_2}}\stackrel{\rm def}{=}\Big(2^{\ell s_2}\big(2^{ks_1}\|\Delta_k^{\rm h}\Delta_\ell^{\rm v}a\|_{L^2}\big)_{\ell^2}\Big)_{\ell^1}.$$

In particular, when $s_1=0$ we denote $\mathcal{B}^{0,s_2}$ by B^{0,s_2} with

$$\|a\|_{B^{0,s_2}}=\sum_{\ell\in\mathbb{Z}}2^{\ell s_2}\|\Delta_\ell^{\rm v}a\|_{L^2}.$$

We recall the classical homogeneous anisotropic Sobolev norm as follows

$$\|a\|_{\dot H^{s_1,s_2}}\stackrel{\rm def}{=}\Big(\sum_{k,\ell\in\mathbb{Z}}2^{2ks_1}2^{2\ell s_2}\|\Delta_k^{\rm h}\Delta_\ell^{\rm v}a\|_{L^2}^2\Big)^{\frac12}.$$

In order to obtain a better description of the regularizing effect for the transport-diffusion equation, we will use Chemin-Lerner type spaces:

Definition 3.2. *Let $p\in[1,+\infty]$ and $T\in]0,+\infty]$. We define the norms of $\widetilde{L}^p_T(\mathcal{B}^{s_1,s_2}(\mathbb{R}^3))$ and $\widetilde{L}^p_T(B^{0,s_2})$ by*

$$\|a\|_{\widetilde{L}^p_T(\mathcal{B}^{s_1,s_2})}\stackrel{\rm def}{=}\Big(2^{\ell s_2}\big(2^{ks_1}\|\Delta_k^{\rm h}\Delta_\ell^{\rm v}a\|_{L^p_T(L^2)}\big)_{\ell^2}\Big)_{\ell^1},$$

and

$$\|a\|_{\widetilde{L}^p_T(B^{0,s_2})}\stackrel{\rm def}{=}\sum_{\ell\in\mathbb{Z}}2^{\ell s_2}\|\Delta_\ell^{\rm v}a\|_{L^p_T(L^2)}$$

respectively.

In particular, when $p=2$, we have

$$\begin{aligned}\|a\|_{\widetilde{L}^2_T(\mathcal{B}^{0,s_2})}&=\sum_{\ell\in\mathbb{Z}}2^{\ell s_2}\big(\sum_{k\in\mathbb{Z}}\|\Delta_k^{\rm h}\Delta_\ell^{\rm v}a\|_{L^2_T(L^2)}^2\big)^{\frac12}\\&=\sum_{\ell\in\mathbb{Z}}2^{\ell s_2}\|\Delta_\ell^{\rm v}a\|_{L^2_T(L^2)}=\|a\|_{\widetilde{L}^2_T(B^{0,s_2})}.\end{aligned}\tag{3.3}$$

In order to study a fluid evolving between two parallel plans, namely to prove Theorem 1.2, we also need the following norms:

Definition 3.3. *Let $s\in\mathbb{R}$ and $p\in[1,\infty]$, we define*

$$\|a\|_{\mathcal{B}^s_{\rm h}}\stackrel{\rm def}{=}\sum_{k\in\mathbb{Z}}2^{ks}\|\Delta_k^{\rm h}a\|_{L^2}\quad\textit{and}\quad\|a\|_{\widetilde{L}^p_T(\mathcal{B}^s_{\rm h})}\stackrel{\rm def}{=}\sum_{k\in\mathbb{Z}}2^{ks}\|\Delta_k^{\rm h}a\|_{L^p_T(L^2)}.\tag{3.4}$$

To over come the difficulty that one can not use Gronwall's argument for the Chemin-Lerner type norms, we also need the time-weighted Chemin-Lerner norm introduced by the authors in [28]

Definition 3.4. *Let $f(t)\in L^1_{loc}(\mathbb{R}_+)$, $f(t)\geq 0$. We define*

$$\|u\|_{\widetilde{L^2_{T,f}}(\mathcal{B}^{0,s_2})}=\sum_{\ell\in\mathbb{Z}}2^{\ell s_2}\Big(\int_0^T f(t)\|\Delta_\ell^{\rm v}u(t)\|_{L^2}^2dt\Big)^{\frac12}.$$

We also recall the following anisotropic Bernstein type lemma from [14, 27].

Lemma 3.1. *Consider $\mathcal{B}_{\mathrm{h}}$ a ball of $\mathbb{R}^2_{\mathrm{h}}$ and $\mathcal{C}_{\mathrm{h}}$ a ring of $\mathbb{R}^2_{\mathrm{h}}$; fix $1 \leq p_2 \leq p_1 \leq \infty$. Then the following properties hold:*
- If the support of $\widehat{a}$ is included in $2^k\mathcal{B}_{\mathrm{h}}$, then

$$\|\partial^\alpha_{x_{\mathrm{h}}} a\|_{L^{p_1}_{\mathrm{h}}} \lesssim 2^{k(|\alpha|+2(1/p_2-1/p_1))}\|a\|_{L^{p_2}_{\mathrm{h}}}.$$

- If the support of $\widehat{a}$ is included in $2^k\mathcal{C}_{\mathrm{h}}$, then

$$\|a\|_{L^{p_1}_{\mathrm{h}}} \lesssim 2^{-k}\|\nabla_{\mathrm{h}} a\|_{L^{p_1}_{\mathrm{h}}}.$$

To study product laws between distributions in the anisotropic Besov spaces, we need to modify the isotropic para-differential decomposition of Bony [4] to the setting of anisotropic version. We first recall the isotropic para-differential decomposition from [4]: let a and b be in $\mathcal{S}'(\mathbb{R}^3)$,

$$\begin{aligned}
&ab = T_a b + T_b a + R(a,b) \quad \text{or} \quad ab = T_a b + \bar{R}(a,b) \quad \text{where}\\
&T_a b = \sum_{j\in\mathbb{Z}} S_{j-1}a\Delta_j b, \quad \bar{R}(a,b) = \sum_{j\in\mathbb{Z}} \Delta_j a S_{j+2} b \quad \text{and}\\
&R(a,b) = \sum_{j\in\mathbb{Z}} \Delta_j a\tilde{\Delta}_j b, \quad \text{with} \quad \tilde{\Delta}_j b = \sum_{\ell=j-1}^{j+1} \Delta_\ell a.
\end{aligned} \tag{3.5}$$

In what follows, we shall also use the anisotropic version of Bony's decomposition for both horizontal and vertical variables.

As an application of the above basic facts on Littlewood-Paley theory, we present the following product laws in the anisotropic Besov spaces.

Lemma 3.2. *Let $\tau_1, \tau_2 \in]-1/2, 1/2[$ with $\tau_1 + \tau_2 > 0$. Let $a \in B^{0,\tau_1}$ with $\nabla_{\mathrm{h}} a \in B^{0,\tau_1}$, let $b \in B^{0,\tau_2}$ with $\nabla_{\mathrm{h}} b \in B^{0,\tau_2}$. Then one has*

$$\|ab\|_{B^{0,\tau_1+\tau_2-\frac12}} \lesssim \|a\|^{\frac12}_{B^{0,\tau_1}}\|\nabla_{\mathrm{h}} a\|^{\frac12}_{B^{0,\tau_1}}\|b\|^{\frac12}_{B^{0,\tau_2}}\|\nabla_{\mathrm{h}} b\|^{\frac12}_{B^{0,\tau_2}}. \tag{3.6}$$

Proof. We first get, by applying Bony's decomposition in vertical variables, that

$$ab = T^{\mathrm{v}}_a b + T^{\mathrm{v}}_b a + R^{\mathrm{v}}(a,b). \tag{3.7}$$

Next, we just handle term by term above. Indeed due to $\tau_1 \leq \frac12$, we have

$$\begin{aligned}
\|S^{\mathrm{v}}_\ell a\|_{L^\infty_{\mathrm{v}}(L^4_{\mathrm{h}})} &\lesssim \sum_{\ell'\leq\ell-1} 2^{\frac{\ell'}{2}}\|\Delta^{\mathrm{v}}_{\ell'} a\|_{L^2_{\mathrm{v}}(L^4_{\mathrm{h}})}\\
&\lesssim \sum_{\ell'\leq\ell-1} 2^{\frac{\ell'}{2}}\|\Delta^{\mathrm{v}}_{\ell'} a\|^{\frac12}_{L^2}\|\Delta^{\mathrm{v}}_{\ell'}\nabla_{\mathrm{h}} a\|^{\frac12}_{L^2}\\
&\lesssim 2^{\ell\left(\frac12-\tau_1\right)}\|a\|^{\frac12}_{B^{\tau_1}}\|\nabla_{\mathrm{h}} a\|^{\frac12}_{B^{\tau_1}}.
\end{aligned}$$

from which, and the support properties to the Fourier transform of the terms in $T^{\mathrm{v}}_a b$, we infer

$$\begin{aligned}
\|\Delta^{\mathrm{v}}_\ell T^{\mathrm{v}}_a b\|_{L^2} &\lesssim \sum_{|\ell'-\ell|\leq4} \|S^{\mathrm{v}}_{\ell'-1} a\|_{L^\infty_{\mathrm{v}}(L^4_{\mathrm{h}})}\|\Delta^{\mathrm{v}}_{\ell'} b\|_{L^2_{\mathrm{v}}(L^4_{\mathrm{h}})}\\
&\lesssim \sum_{|\ell'-\ell|\leq4} \|S^{\mathrm{v}}_{\ell'-1} a\|_{L^\infty_{\mathrm{v}}(L^4_{\mathrm{h}})}\|\Delta^{\mathrm{v}}_{\ell'} b\|^{\frac12}_{L^2}\|\Delta^{\mathrm{v}}_{\ell'}\nabla_{\mathrm{h}} b\|^{\frac12}_{L^2}\\
&\lesssim d_{ell} 2^{-\ell(\tau_1+\tau_2-\frac12)}\|a\|^{\frac12}_{B^{0,\tau_1}}\|\nabla_{\mathrm{h}} a\|^{\frac12}_{B^{0,\tau_1}}\|b\|^{\frac12}_{B^{0,\tau_2}}\|\nabla_{\mathrm{h}} b\|^{\frac12}_{B^{0,\tau_2}}.
\end{aligned}$$

The same estimate holds for $T^{\mathrm{v}}_b a$.

Whereas by applying Lemma 3.1, one has

$$\begin{aligned}\|\Delta_\ell^{\rm v} R^{\rm v}(a,b)\|_{L^2} \lesssim & 2^{\frac{\ell}{2}} \sum_{\ell' \geq \ell-3} \|\Delta_{\ell'}^{\rm v} a\|_{L^2_{\rm v}(L^4_{\rm h})} \|\widetilde{\Delta}_{\ell'}^{\rm v} b\|_{L^2_{\rm v}(L^4_{\rm h})} \\ \lesssim & 2^{\frac{\ell}{2}} \sum_{\ell' \geq \ell-3} \|\Delta_{\ell'}^{\rm v} a\|_{L^2}^{\frac12} \|\Delta_{\ell'}^{\rm v} \nabla_{\rm h} a\|_{L^2}^{\frac12} \|\widetilde{\Delta}_{\ell'}^{\rm v} b\|_{L^2}^{\frac12} \|\widetilde{\Delta}_{\ell'}^{\rm v} \nabla_{\rm h} b\|_{L^2}^{\frac12} \\ \lesssim & d_\ell 2^{-\ell(\tau_1+\tau_2-\frac12)} \|a\|_{B^{0,\tau_1}}^{\frac12} \|\nabla_{\rm h} a\|_{B^{0,\tau_1}}^{\frac12} \|b\|_{B^{0,\tau_2}}^{\frac12} \|\nabla_{\rm h} b\|_{B^{0,\tau_2}}^{\frac12},\end{aligned}$$

where in the last step, we used the fact that $\tau_1+\tau_2>0$, so that

$$\sum_{\ell' \geq \ell-3} d_{\ell'} 2^{-\ell'(\tau_1+\tau_2)} \lesssim d_\ell 2^{-\ell(\tau_1+\tau_2)}.$$

This completes the proof of the lemma. □

Remark 3.1. *We remark that the law of product* (3.6) *works for Chemin-Lerner norms as well.*

4. The estimate of v_L and $\bar{u}^{\rm h}$

The goal of this section is to present the estimates of v_L and $\bar{u}^{\rm h}$.

Proposition 4.1. *Let $v_0 \in \mathcal{B}^{\frac12,0} \cap \mathcal{B}^{0,\frac12}$ and v_L be the corresponding solution of* (2.1). *Then we have*

$$\begin{aligned}& \|\nabla_{\rm h} v_L\|_{\widetilde{L}^2_t(B^{0,\frac12})} \lesssim (\nu_{\rm h}\nu_{\rm v})^{-\frac14} \|v_0\|_{\mathcal{B}^{\frac12,0}}, \\ & \|v_L\|_{\widetilde{L}^\infty_t(B^{0,\frac12})} + \nu_{\rm h}^{\frac12} \|\nabla_{\rm h} v_L\|_{\widetilde{L}^2_t(B^{0,\frac12})} + \nu_{\rm v}^{\frac12} \|\partial_z v_L\|_{\widetilde{L}^2_t(B^{0,\frac12})} \lesssim \|v_0\|_{B^{0,\frac12}}.\end{aligned} \tag{4.1}$$

Proof. We get, by first applying the operator $\Delta_k^{\rm h}\Delta_\ell^{\rm v}$ to the system (2.1) and then taking L^2 inner product of the resulting equation with $\Delta_k^{\rm h}\Delta_\ell^{\rm v} v_L$, that

$$\frac12 \frac{d}{dt} \|\Delta_k^{\rm h}\Delta_\ell^{\rm v} v_L(t)\|_{L^2}^2 + \nu_{\rm h} \|\Delta_k^{\rm h}\Delta_\ell^{\rm v} \nabla_{\rm h} v_L\|_{L^2}^2 + \nu_{\rm v} \|\Delta_k^{\rm h}\Delta_\ell^{\rm v} \partial_z v_L\|_{L^2}^2 = 0.$$

Integrating the above equality over $[0,t]$, and taking square root of the resulting equality, we write

$$\|\Delta_k^{\rm h}\Delta_\ell^{\rm v} v_L\|_{L^\infty_t(L^2)} + \sqrt{\nu_{\rm h}} \|\Delta_k^{\rm h}\Delta_\ell^{\rm v} \nabla_{\rm h} v_L\|_{L^2_t(L^2)} + \sqrt{\nu_{\rm v}} \|\Delta_k^{\rm h}\Delta_\ell^{\rm v} \partial_z v_L(t)\|_{L^2_t(L^2)} \leq \|\Delta_k^{\rm h}\Delta_\ell^{\rm v} v_0\|_{L^2}.$$

Taking ℓ^2 norm with respect to $k \in \mathbb{Z}$ and then taking ℓ^1 norm with respect to $\ell \in \mathbb{Z}$, we achieve

$$\|v_L\|_{\widetilde{L}^\infty_t(\mathcal{B}^{\frac12,0})} + \sqrt{\nu_{\rm h}} \|\nabla_{\rm h} v_L\|_{\widetilde{L}^2_t(\mathcal{B}^{\frac12,0})} + \sqrt{\nu_{\rm v}} \|\partial_z v_L\|_{\widetilde{L}^2_t(\mathcal{B}^{\frac12,0})} \leq \|v_0\|_{\mathcal{B}^{\frac12,0}}. \tag{4.2}$$

Whereas it follows from Fourier-Plancherel equality that

$$\begin{aligned}2^\ell \|\Delta_\ell^{\rm v} \nabla_{\rm h} v_L\|_{L^2_t(L^2)}^2 = & 2^\ell \sum_{k\in\mathbb{Z}} \|\Delta_k^{\rm h}\Delta_\ell^{\rm v} \nabla_{\rm h} v_L\|_{L^2_t(L^2)}^2 \\ \lesssim & 2^\ell \sum_{k\in\mathbb{Z}} 2^{2k} \|\Delta_k^{\rm h}\Delta_\ell^{\rm v} v_L\|_{L^2_t(L^2)}^2 \\ \lesssim & \Big(\sum_{k\in\mathbb{Z}} 2^{3k} \|\Delta_k^{\rm h}\Delta_\ell^{\rm v} v_L\|_{L^2_t(L^2)}^2\Big)^{\frac12} \Big(\sum_{k\in\mathbb{Z}} 2^k 2^{2\ell} \|\Delta_k^{\rm h}\Delta_\ell^{\rm v} v_L\|_{L^2_t(L^2)}^2\Big)^{\frac12},\end{aligned}$$

from which and (4.2), we infer

$$
\begin{aligned}
\|\nabla_{\rm h} v_L\|_{\widetilde{L}^2_t(B^{0,\frac12})} \lesssim & \sum_{\ell\in\mathbb{Z}} 2^{\frac{\ell}{2}} \|\Delta_\ell^{\rm v}\nabla_{\rm h} v_L\|_{L^2_t(L^2)} \\
\lesssim & \Bigl(\sum_{\ell\in\mathbb{Z}}\Bigl(\sum_{k\in\mathbb{Z}} 2^k\|\Delta_k^{\rm h}\Delta_\ell^{\rm v}\nabla_{\rm h} v_L\|_{L^2_t(L^2)}^2\Bigr)^{\frac12}\Bigr)^{\frac12} \\
& \times \Bigl(\sum_{\ell\in\mathbb{Z}}\Bigl(\sum_{k\in\mathbb{Z}} 2^k\|\Delta_k^{\rm h}\Delta_\ell^{\rm v}\partial_z v_L\|_{L^2_t(L^2)}^2\Bigr)^{\frac12}\Bigr)^{\frac12} \\
\lesssim & (\nu_{\rm h}\nu_{\rm v})^{-\frac14}\bigl(\sqrt{\nu_{\rm h}}\|\nabla_{\rm h} v_L\|_{\widetilde{L}^2_t(\mathcal{B}^{\frac12,0})}\bigr)^{\frac12}\bigl(\sqrt{\nu_{\rm v}}\|\partial_3 v_L\|_{\widetilde{L}^2_t(\mathcal{B}^{\frac12,0})}\bigr)^{\frac12} \\
\lesssim & (\nu_{\rm h}\nu_{\rm v})^{-\frac14}\|v_0\|_{\mathcal{B}^{\frac12,0}}.
\end{aligned}
$$

This leads to the first inequality of (4.1).

On the other hand, we get, by first applying the operator $\Delta_\ell^{\rm v}$ to the system (2.1) and then taking L^2 inner product of the resulting equation with $\Delta_\ell^{\rm v} v_L$, that

$$
\frac12\frac{d}{dt}\|\Delta_\ell^{\rm v} v_L(t)\|_{L^2}^2+\nu_{\rm h}\|\nabla_{\rm h}\Delta_\ell^{\rm v} v_L\|_{L^2}^2+\nu_{\rm v}\|\partial_z\Delta_\ell^{\rm v} v_L\|_{L^2}^2=0.
$$

Integrating the above equality over $[0,t]$, and taking square root of the resulting equality, and then taking ℓ^1 norm with respect to $\ell\in\mathbb{Z}$, we obtain the second inequality of (4.1). This completes the proof of the proposition. □

Lemma 4.1. *Let $\bar u_0^{\rm h}$ and $\nabla_{\rm h}\bar u_0^{\rm h}$ be in $L^2(\mathbb{R}^3)\cap L^\infty(\mathbb{R}_{\rm v};L^2(\mathbb{R}^2_{\rm h}))$. Then (2.2) has a unique global solution so that*

$$
\|\bar u^{\rm h}\|_{L^\infty_t(L^\infty_{\rm v}(L^2_{\rm h}))}\leq \|\bar u_0^{\rm h}\|_{L^\infty_{\rm v}(L^2_{\rm h})}. \tag{4.3}
$$

If moreover, $\bar u_0^{\rm h}\in L^\infty(\mathbb{R}_{\rm v};\dot H^{-\delta}(\mathbb{R}^2_{\rm h}))$ for some $\delta\in]0,1[$, then we have

$$
\begin{aligned}
\int_0^\infty \|\nabla_{\rm h}\bar u^{\rm h}(t)\|_{L^\infty_{\rm v}(L^2_{\rm h})}^2\,dt \leq & A_{\nu_{\rm h},\delta}(\bar u_0^{\rm h}) \quad \textit{with} \\
A_{\nu_{\rm h},\delta}(\bar u_0^{\rm h}) = & C_\delta\exp\bigl(C_\delta\nu_{\rm h}^{-2}\|\bar u_0^{\rm h}\|_{L^\infty_{\rm v}(L^2_{\rm h})}^2(1+\nu_{\rm h}^{-2}\|\bar u_0^{\rm h}\|_{L^\infty_{\rm v}(L^2_{\rm h})}^2)\bigr) \\
& \times\biggl(\frac{\|\nabla_{\rm h}\bar u_0^{\rm h}\|_{L^\infty_{\rm v}(L^2_{\rm h})}^2\|\bar u_0^{\rm h}\|_{L^\infty_{\rm v}((\dot B^{-\delta}_{2,\infty})_{\rm h})}^{\frac2\delta}}{\|\bar u_0^{\rm h}\|_{L^\infty_{\rm v}(L^2_{\rm h})}^{\frac2\delta}}+\|\bar u_0^{\rm h}\|_{L^\infty_{\rm v}(L^2_{\rm h})}^2\biggr).
\end{aligned}
\tag{4.4}
$$

Proof. Theorem 1.2 of [14] ensures the global existence of solutions to (2.2). Moreover, (2.4) of [14] gives (4.3). To prove the estimate (4.4), we introduce

$$
w^{\rm h}(t,x_{\rm h},z)\stackrel{\rm def}{=}\nu_{\rm h}^{-\frac12}\bar u^{\rm h}(t,\nu_{\rm h}^{\frac12}x_{\rm h},z)\quad\text{and}\quad q^{\rm h}(t,x_{\rm h},z)\stackrel{\rm def}{=}\nu_{\rm h}^{-1}p^{\rm h}(t,\nu_{\rm h}^{\frac12}x_{\rm h},z). \tag{4.5}
$$

Then in view of (2.2), we write

$$
\begin{cases}
\partial_t w^{\rm h}+w^{\rm h}\cdot\nabla_{\rm h} w^{\rm h}-\Delta_{\rm h} w^{\rm h}-\frac{\nu_{\rm v}}{\nu_{\rm h}}\varepsilon^2\partial_z^2 w^{\rm h}=-\nabla_{\rm h} q^{\rm h}\quad\text{for } (t,x)\in\mathbb{R}^+\times\mathbb{R}^3,\\
\operatorname{div}_{\rm h} w^{\rm h}=0,\\
w^{\rm h}|_{t=0}=w_0^{\rm h}=\nu_{\rm h}^{-\frac12}\bar u_0^{\rm h}(\nu_{\rm h}^{\frac12}x).
\end{cases}
\tag{4.6}
$$

It follows from Theorem 1.2 of [14] that

$$\begin{aligned}\int_0^\infty \|\nabla_{\rm h} w^{\rm h}(t)\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}\,dt \leq & C_\delta \exp\bigl(C_\delta \|w_0^{\rm h}\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}(1+\|w_0^{\rm h}\|^2_{L^\infty_{\rm v}(L^2_{\rm h})})\bigr)\\ & \times\left(\frac{\|\nabla_{\rm h} w_0^{\rm h}\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}\|w_0^{\rm h}\|^{\frac{2}{\delta}}_{L^\infty_{\rm v}((\dot B^{-\delta}_{2,\infty})_{\rm h})}}{\|w_0^{\rm h}\|^{\frac{2}{\delta}}_{L^\infty_{\rm v}(L^2_{\rm h})}}+\|w_0^{\rm h}\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}\right).\end{aligned}\tag{4.7}$$

Yet by virtue of (4.5), we have

$$\begin{aligned}&\|w_0^{\rm h}\|_{L^\infty_v(L^2_{\rm h})}=\nu_{\rm h}^{-1}\|\bar u^{\rm h}\|_{L^\infty_v(L^2_{\rm h})},\quad \|\nabla_{\rm h} w_0^{\rm h}\|_{L^\infty_v(L^2_{\rm h})}=\nu_{\rm h}^{-\frac12}\|\nabla_{\rm h}\bar u^{\rm h}\|_{L^\infty_v(L^2_{\rm h})}\\ &\|w_0^{\rm h}\|_{L^\infty_{\rm v}((\dot B^{-\delta}_{2,\infty})_{\rm h})}=\nu_{\rm h}^{-1-\frac{\delta}{2}}\|u_0^{\rm h}\|_{L^\infty_{\rm v}((\dot B^{-\delta}_{2,\infty})_{\rm h})},\quad\text{and}\\ &\int_0^\infty \|\nabla_{\rm h} w^{\rm h}(t)\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}\,dt=\nu_{\rm h}^{-2}\int_0^\infty \|\nabla_{\rm h}\bar u^{\rm h}(t)\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}\,dt,\end{aligned}$$

from which and (4.7), we deduce (4.4). □

Proposition 4.2. *Under the assumptions of Lemma 4.1, for any $t>0$, we have*

$$\begin{aligned}&\|\bar u^{\rm h}\|_{\widetilde L^\infty_t(B^{0,\frac12})}+\sqrt{\nu_{\rm h}}\|\nabla_{\rm h}\bar u^{\rm h}\|_{\widetilde L^2_t(B^{0,\frac12})}+\sqrt{\nu_{\rm v}}\varepsilon\|\partial_z \bar u^{\rm h}\|_{\widetilde L^2_t(B^{0,\frac12})}\lesssim B_{\delta,\nu_{\rm h}}(\bar u_0^{\rm h})\quad\textit{with}\\ &\qquad B_{\delta,\nu_{\rm h}}(\bar u_0^{\rm h})\stackrel{\rm def}{=}\|\bar u_0^{\rm h}\|_{B^{0,\frac12}}\exp\Bigl(C\nu_{\rm h}^{-3}\|\bar u_0^{\rm h}\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}A_{\nu_{\rm h},\delta}(\bar u_0^{\rm h})\Bigr),\end{aligned}\tag{4.8}$$

where $A_{\nu_{\rm h},\delta}(\bar u_0^{\rm h})$ is given by (4.4).

Proof. Let us denote

$$g(t)\stackrel{\rm def}{=}\|\bar u^{\rm h}(t)\|^4_{L^\infty_{\rm v}(L^4_{\rm h})}\quad\text{and}\quad f_\lambda(t)\stackrel{\rm def}{=}f(t)\exp\left(-\lambda\int_0^t g(t')\,dt'\right).\tag{4.9}$$

Then by virtue of (2.2), we write

$$\partial_t\bar u^{\rm h}_\lambda+\lambda g(t)\bar u^{\rm h}_\lambda+\bar u^{\rm h}\cdot\nabla_{\rm h}\bar u^{\rm h}_\lambda-\nu_{\rm h}\Delta_{\rm h}\bar u^{\rm h}_\lambda-\nu_{\rm v}\varepsilon^2\partial_3^2\bar u^{\rm h}_\lambda=-\nabla_{\rm h}p^{\rm h}_\lambda.$$

Applying $\Delta_\ell^{\rm v}$ to the above equation and then taking L^2 inner product of the resulting equation with $\Delta_\ell^{\rm v}\bar u^{\rm h}$, we obtain

$$\begin{aligned}\frac12\frac{d}{dt}\|\Delta_\ell^{\rm v}\bar u^{\rm h}(t)\|^2_{L^2}&+\lambda g(t)\|\Delta_\ell^{\rm v}\bar u^{\rm h}(t)\|^2_{L^2}\\ &+\nu_{\rm h}\|\Delta_\ell^{\rm v}\nabla_{\rm h}\bar u^{\rm h}\|^2_{L^2}+\varepsilon^2\nu_{\rm v}\|\Delta_\ell^{\rm v}\partial_3\bar u^{\rm h}\|^2_{L^2}=-\bigl(\bar u^{\rm h}\otimes\bar u^{\rm h}_\lambda|\Delta_\ell^v\nabla_{\rm h}\bar u^{\rm h}\bigr)_{L^2}.\end{aligned}\tag{4.10}$$

By applying Bony's decomposition (3.5) for the vertical variable, one has

$$\bar u^{\rm h}\otimes\bar u^{\rm h}_\lambda=2T^{\rm v}(\bar u^{\rm h},\bar u^{\rm h}_\lambda)+R^{\rm v}(\bar u^{\rm h},\bar u^{\rm h}_\lambda).$$

Due to the support properties to the Fourier of the terms in $T^{\rm v}(\bar u^{\rm h},\bar u^{\rm h}_\lambda)$, we have

$$\begin{aligned}&\int_0^t\bigl|\bigl(\Delta_\ell^{\rm v}T^{\rm v}(\bar u^{\rm h},\bar u^{\rm h}_\lambda)|\Delta_\ell^v\nabla_{\rm h}\bar u^{\rm h}\bigr)_{L^2}\bigr|\,dt'\\ &\lesssim\sum_{|\ell'-\ell|\leq4}\int_0^t\|S^{\rm v}_{\ell'-1}\bar u^{\rm h}\|_{L^\infty_{\rm v}(L^4_{\rm h})}\|\Delta^{\rm v}_{\ell'}\bar u^{\rm h}\|_{L^2_{\rm v}(L^4_{\rm h})}\|\Delta_\ell^{\rm v}\nabla_{\rm h}\bar u^{\rm h}\|_{L^2}\,dt'\\ &\lesssim\sum_{|\ell'-\ell|\leq4}\int_0^t\|\bar u^{\rm h}\|_{L^\infty_{\rm v}(L^4_{\rm h})}\|\Delta^{\rm v}_{\ell'}\bar u^{\rm h}\|^{\frac12}_{L^2}\|\Delta^{\rm v}_{\ell'}\nabla_{\rm h}\bar u^{\rm h}\|^{\frac12}_{L^2}\|\Delta_\ell^{\rm v}\nabla_{\rm h}\bar u^{\rm h}\|_{L^2}\,dt'.\end{aligned}$$

Applying Hölder's inequality and using Definition 3.4, we get

$$
\begin{aligned}
&\int_0^t \big|\big(\Delta_\ell^{\rm v} T^{\rm v}(\bar u^{\rm h},\bar u_\lambda^{\rm h}) | \Delta_\ell^{v}\nabla_{\rm h}\bar u^{\rm h}\big)_{L^2}\big|\,dt'\\
&\lesssim \sum_{|\ell'-\ell|\le 4}\Big(\int_0^t \|\bar u^{\rm h}\|_{L^\infty_{\rm v}(L^4_{\rm h})}^4 \|\Delta_{\ell'}^{\rm v}\bar u_\lambda^{\rm h}\|_{L^2}^2\,dt\Big)^{\frac14}\|\Delta_{\ell'}^{\rm v}\nabla_{\rm h}\bar u_\lambda^{\rm h}\|_{L^2_t(L^2)}^{\frac12}\|\Delta_\ell^{\rm v}\nabla_{\rm h}\bar u_\lambda^{\rm h}\|_{L^2_t(L^2)}\\
&\lesssim d_\ell^2 2^{-\ell}\|\bar u_\lambda^{\rm h}\|_{\widetilde L^2_{t,g}(B^{0,\frac12})}^{\frac12}\|\nabla_{\rm h}\bar u_\lambda^{\rm h}\|_{\widetilde L^2_t(B^{0,\frac12})}^{\frac32}.
\end{aligned}
$$

Along the same line, we have

$$
\begin{aligned}
&\int_0^t \big|\big(\Delta_\ell^{\rm v}(R^{\rm v}(\bar u^{\rm h},\bar u_\lambda^{\rm h}) | \Delta_\ell^{\rm v}\nabla_{\rm h}\bar u_\lambda^{\rm h}\big)_{L^2}\big|\,dt'\\
&\lesssim \sum_{\ell'\ge \ell-3}\int_0^t \|\widetilde\Delta_{\ell'}^{\rm v}\bar u^{\rm h}\|_{L^\infty_{\rm v}(L^4_{\rm h})}\|\Delta_{\ell'}^{\rm v}\bar u_\lambda^{\rm h}\|_{L^2_{\rm v}(L^4_h)}\|\Delta_\ell^{\rm v}\nabla_{\rm h}\bar u_\lambda^{\rm h}\|_{L^2}\,dt'\\
&\lesssim \sum_{\ell'\ge \ell-3}\Big(\int_0^t \|\bar u^{\rm h}\|_{L^\infty_{\rm v}(L^4_{\rm h})}^4 \|\Delta_{\ell'}^{\rm v}\bar u_\lambda^{\rm h}\|_{L^2}^2\,dt'\Big)^{\frac14}\|\Delta_{\ell'}^{\rm v}\nabla_{\rm h}\bar u_\lambda^{\rm h}\|_{L^2_t(L^2)}^{\frac12}\|\Delta_\ell^{\rm v}\nabla_{\rm h}\bar u_\lambda^{\rm h}\|_{L^2_t(L^2)}\\
&\lesssim d_\ell 2^{-\ell}\|\bar u_\lambda^{\rm h}\|_{\widetilde L^2_{t,g}(B^{0,\frac12})}^{\frac12}\|\nabla_{\rm h}\bar u^{\rm h}\|_{\widetilde L^2_t(B^{0,\frac12})}^{\frac32}\sum_{\ell'\ge\ell-3} d_{\ell'}2^{-\frac{\ell-\ell'}{2}},
\end{aligned}
$$

which implies

$$
\int_0^t \big|\big(\Delta_\ell^{\rm v}(R^{\rm v}(\bar u^{\rm h},\bar u_\lambda^{\rm h}) | \Delta_\ell^{\rm v}\nabla_{\rm h}\bar u_\lambda^{\rm h}\big)_{L^2}\big|\,dt' \lesssim d_\ell^2 2^{-\ell}\|\bar u_\lambda^{\rm h}\|_{\widetilde L^2_{t,g}(B^{0,\frac12})}^{\frac12}\|\nabla_{\rm h}\bar u_\lambda^{\rm h}\|_{\widetilde L^2_t(B^{0,\frac12})}^{\frac32}.
$$

As a result, it comes out

$$
\int_0^t \big|\big(\Delta_\ell^{\rm v}(\bar u^{\rm h}\otimes\bar u_\lambda^{\rm h}) | \Delta_\ell^{\rm v}\nabla_{\rm h}\bar u_\lambda^{\rm h}\big)_{L^2}\big|\,dt \lesssim d_\ell^2 2^{-\ell}\|\bar u_\lambda^{\rm h}\|_{\widetilde L^2_{t,g}(B^{0,\frac12})}^{\frac12}\|\nabla_{\rm h}\bar u_\lambda^{\rm h}\|_{\widetilde L^2_t(B^{0,\frac12})}^{\frac32}. \tag{4.11}
$$

Integrating (4.10) over $[0,t]$ and then inserting (4.11) into the resulting inequality, we obtain

$$
\begin{aligned}
\|\Delta_\ell^{\rm v}\bar u^{\rm h}\|_{L^\infty_t(L^2)}^2 &+\lambda\int_0^t g(t)\|\Delta_\ell^{\rm v}\bar u_\lambda^{\rm h}(t)\|_{L^2}^2\,dt+\nu_{\rm h}\|\Delta_\ell^{\rm v}\nabla_{\rm h}\bar u_\lambda^{\rm h}\|_{L^2_T(L^2)}^2\\
&+\varepsilon^2\nu_{\rm v}\|\Delta_\ell^{\rm v}\partial_z\bar u_\lambda^{\rm h}\|_{L^2_T(L^2)}^2 \lesssim d_\ell^2 2^{-\ell}\|\bar u_\lambda^{\rm h}\|_{\widetilde L^2_{t,g}(\mathcal B^{0,\frac12})}^{\frac12}\|\nabla_{\rm h}\bar u_\lambda^{\rm h}\|_{\widetilde L^2_t(\mathcal B^{0,\frac12})}^{\frac32}.
\end{aligned}
$$

Multiplying the above inequality by 2^ℓ and taking square root of the above inequality and then summing up the resulting inequality over $\mathbb Z$, we achieve

$$
\begin{aligned}
&\|\bar u_\lambda^{\rm h}\|_{\widetilde L^\infty_t(B^{0,\frac12})}+\sqrt\lambda\|\bar u_\lambda^{\rm h}\|_{\widetilde L^2_{t,g}(B^{0,\frac12})}+\sqrt{\nu_{\rm h}}\|\nabla_{\rm h}\bar u^{\rm h}\|_{\widetilde t^2_T(B^{0,\frac12})}+\sqrt{\nu_{\rm v}}\varepsilon\|\partial_z\bar u^{\rm h}\|_{\widetilde L^2_t(B^{0,\frac12})}\\
&\le \|\bar u_0^{\rm h}\|_{\mathcal B^{0,\frac12}}+C\|\bar u_\lambda^{\rm h}\|_{\widetilde L^2_{t,g}(B^{0,\frac12})}^{\frac14}\|\nabla_{\rm h}\bar u_\lambda^{\rm h}\|_{\widetilde L^2_t(B^{0,\frac12})}^{\frac34}\\
&\le \|\bar u_0^{\rm h}\|_{\mathcal B^{0,\frac12}}+\frac{\sqrt{\nu_{\rm h}}}{2}\|\nabla_{\rm h}\bar u^{\rm h}\|_{\widetilde L^2_T(B^{0,\frac12})}+C\nu_{\rm h}^{-\frac32}\|\bar u_\lambda^{\rm h}\|_{\widetilde L^2_{t,g}(B^{0,\frac12})}.
\end{aligned}
$$

Taking $\lambda=C^2\nu_{\rm h}^{-3}$ in the above inequality leads to

$$
\|\bar u_\lambda^{\rm h}\|_{\widetilde L^\infty_t(B^{0,\frac12})}+\sqrt{\nu_{\rm h}}\|\nabla_{\rm h}\bar u_\lambda^{\rm h}\|_{\widetilde L^2_t(B^{0,\frac12})}+\sqrt{\nu_{\rm v}}\varepsilon\|\partial_3\bar u_\lambda^{\rm h}\|_{\widetilde L^2_t(B^{0,\frac12})}\le 2\|\bar u_0^{\rm h}\|_{B^{0,\frac12}},
$$

from which, and (4.9), we infer

$$\Big(\|\bar{u}^{\rm h}\|_{\widetilde{L}^\infty_t(B^{0,\frac12})}+\sqrt{\nu_{\rm h}}\|\nabla_{\rm h}\bar{u}^{\rm h}\|_{\widetilde{L}^2_t(B^{0,\frac12})}+\sqrt{\nu_{\rm v}}\varepsilon\|\partial_z\bar{u}^{\rm h}\|_{\widetilde{L}^2_t(B^{0,\frac12})}\Big)\exp\left(-\lambda\int_0^t g(t')\,dt'\right)$$
$$\leq \|\bar{u}^{\rm h}_\lambda\|_{\widetilde{L}^\infty_T(B^{0,\frac12})}+\sqrt{\nu_{\rm h}}\|\nabla_{\rm h}\bar{u}^{\rm h}_\lambda\|_{\widetilde{L}^2_T(B^{0,\frac12})}+\sqrt{\nu_{\rm v}}\varepsilon\|\partial_z\bar{u}^{\rm h}_\lambda\|_{\widetilde{L}^2_T(B^{0,\frac12})}$$
$$\leq 2\|\bar{u}^{\rm h}_0\|_{B^{0,\frac12}},$$

Note that

$$\|\bar{u}^{\rm h}(t)\|_{L^\infty_{\rm v}(L^4_{\rm h})}\leq C\|\bar{u}^{\rm h}(t)\|^{\frac12}_{L^\infty_{\rm v}(L^2_{\rm h})}\|\nabla_{\rm h}\bar{u}^{\rm h}(t)\|^{\frac12}_{L^\infty_{\rm v}(L^2_{\rm h})},$$

we deduce that

$$\Big(\|\bar{u}^{\rm h}\|_{\widetilde{L}^\infty_t(\mathcal{B}^{0,\frac12})}+\sqrt{\nu_{\rm h}}\|\nabla_{\rm h}\bar{u}^{\rm h}\|_{\widetilde{L}^2_t(\mathcal{B}^{0,\frac12})}+\sqrt{\nu_{\rm v}}\varepsilon\|\partial_z\bar{u}^{\rm h}\|_{\widetilde{L}^2_t(\mathcal{B}^{0,\frac12})}\Big)$$
$$\leq 2\|\bar{u}^{\rm h}_0\|_{\mathcal{B}^{0,\frac12}}\exp\Big(\frac{C}{\nu_{\rm h}^3}\|\bar{u}^{\rm h}\|^2_{L^\infty_t(L^\infty_{\rm v}(L^2_{\rm h}))}\int_0^t\|\nabla_{\rm h}\bar{u}^{\rm h}(t')\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}\,dt'\Big).$$

This together with (4.3) and (4.4) ensures (4.8). □

Proposition 4.3. *Let $\bar{u}^{\rm h}$ be a smooth enough solution of* (2.2)*. Then for any $s\in]-1,0[$, we have*

$$\|\bar{u}^{\rm h}\|^2_{L^\infty_t(\dot{H}^{s,0})}+\nu_{\rm h}\|\nabla_{\rm h}\bar{u}^{\rm h}\|^2_{L^2_t(\dot{H}^{s,0})}\leq\|\bar{u}^{\rm h}_0\|^2_{\dot{H}^{s,0}}\exp\left(\frac{C}{\nu_{\rm h}}A_{\nu_{\rm h},\delta}(\bar{u}^{\rm h}_0)\right)\stackrel{def}{=}\mathfrak{C}^1_{\nu_{\rm h},\delta}(\bar{u}^{\rm h}_0,s);\tag{4.12}$$

$$\|\partial_z\bar{u}^{\rm h}\|^2_{L^\infty_t(\dot{H}^{s,0})}+\nu_{\rm h}\|\nabla_{\rm h}\partial_z\bar{u}^{\rm h}\|^2_{L^2_t(\dot{H}^{s,0})}\leq\|\partial_z\bar{u}^{\rm h}_0\|^2_{\dot{H}^{s,0}}\exp\left(\frac{C}{\nu_{\rm h}}A_{\nu_{\rm h},\delta}(\bar{u}^{\rm h}_0)\right)\stackrel{def}{=}\mathfrak{C}^2_{\nu_{\rm h},\delta}(\bar{u}^{\rm h}_0,s);\tag{4.13}$$

$$\begin{aligned}\|\partial_z^2\bar{u}^{\rm h}\|^2_{L^\infty_t(H^{s,0})}+\nu_{\rm h}\|\nabla_{\rm h}\partial_z^2\bar{u}^{\rm h}\|^2_{L^2_t(\dot{H}^{s,0})}&\leq\big(\|\partial_z^2\bar{u}^{\rm h}_0\|^2_{\dot{H}^{s,0}}+\mathfrak{C}^2_{\nu_{\rm h},\delta}(\bar{u}^{\rm h}_0,s)\big)\\&\times\exp\left(\frac{C}{\nu_{\rm h}}A_{\nu_{\rm h},\delta}(\bar{u}^{\rm h}_0)+\frac{C}{\nu_{\rm h}^2}\mathfrak{C}^2_{\nu_{\rm h},\delta}(\bar{u}^{\rm h}_0,0)\right)\stackrel{def}{=}\mathfrak{C}^3_{\nu_{\rm h},\delta}(\bar{u}^{\rm h}_0,s).\end{aligned}\tag{4.14}$$

Proof. In fact, (4.12) and (4.13) hold for $s\in]-1,1[$, which is a direct consequence of Lemma 4.2 of [14] and Lemma 4.1. To prove (4.14), we get, by first applying ∂_z^2 to (2.2) and then taking $\dot{H}^s_{\rm h}$ inner product of the resulting equation with $\partial_z^2\bar{u}^{\rm h}$, that

$$\begin{aligned}&\frac12\frac{d}{dt}\|\partial_z^2\bar{u}^{\rm h}(t,\cdot,z)\|^2_{\dot{H}^s_{\rm h}}+\nu_{\rm h}\|\nabla_{\rm h}\partial_z^2\bar{u}^{\rm h}(t,\cdot,z)\|^2_{\dot{H}^s_{\rm h}}+\nu_{\rm v}\varepsilon^2\|\partial_z^2\bar{u}^{\rm h}(t,\cdot,z)\|^2_{\dot{H}^s_{\rm h}}\\&-\frac{\nu_{\rm v}\varepsilon^2}{2}\partial_z^2\|\partial_z^2\bar{u}^{\rm h}(t,\cdot,z)\|^2_{\dot{H}^s_{\rm h}}=-\big(\bar{u}^{\rm h}(t,\cdot,z)\nabla_{\rm h}\partial_z^2\bar{u}^{\rm h}(t,\cdot,z)|\partial_z^2\bar{u}^{\rm h}\big)_{\dot{H}^s_{\rm h}}\\&-2\big(\partial_z\bar{u}^{\rm h}(t,\cdot,z)\nabla_{\rm h}\partial_z\bar{u}^{\rm h}(t,\cdot,z)|\partial_z^2\bar{u}^{\rm h}\big)_{\dot{H}^s_{\rm h}}-\big(\partial_z^2\bar{u}^{\rm h}(t,\cdot,z)\nabla_{\rm h}\bar{u}^{\rm h}(t,\cdot,z)|\partial_z^2\bar{u}^{\rm h}\big)_{\dot{H}^s_{\rm h}}.\end{aligned}\tag{4.15}$$

Applying Lemma 1.1 of [9] yields

$$\big|\big(\bar{u}^{\rm h}(t,\cdot,z)\nabla_{\rm h}\partial_z^2\bar{u}^{\rm h}(t,\cdot,z)|\partial_z^2\bar{u}^{\rm h}\big)_{\dot{H}^s_{\rm h}}\big|\lesssim\|\nabla_{\rm h}\bar{u}^{\rm h}(t,\cdot,z)\|_{L^2_{\rm h}}\|\nabla_{\rm h}\partial_z^2\bar{u}^{\rm h}(t,\cdot,z)\|_{\dot{H}^s_{\rm h}}\|\partial_z^2\bar{u}^{\rm h}(t,\cdot,z)\|_{\dot{H}^s_{\rm h}}.$$

Due to $s\in]-1,0[$, the law of product in Sobolev spaces implies that

$$\big|\big(\partial_z^2\bar{u}^{\rm h}(t,\cdot,z)\nabla_{\rm h}\bar{u}^{\rm h}(t,\cdot,z)|\partial_z^2\bar{u}^{\rm h}\big)_{\dot{H}^s_{\rm h}}\big|\lesssim\|\nabla_{\rm h}\partial_z^2\bar{u}^{\rm h}(t,\cdot,z)\|_{\dot{H}^s_{\rm h}}\|\nabla_{\rm h}\bar{u}^{\rm h}(t,\cdot,z)\|_{L^2_{\rm h}}\|\partial_z^2\bar{u}^{\rm h}(t,\cdot,z)\|_{\dot{H}^s_{\rm h}}.$$

Similarly, one has

$$\begin{aligned}&\big|\big(\partial_z\bar{u}^{\rm h}(t,\cdot,z)\nabla_{\rm h}\partial_z\bar{u}^{\rm h}(t,\cdot,z)|\partial_z^2\bar{u}^{\rm h}\big)_{\dot{H}^s_{\rm h}}\big|\\&\qquad\lesssim\|\nabla_{\rm h}\partial_z\bar{u}^{\rm h}(t,\cdot,z)\|_{\dot{H}^s_{\rm h}}\|\nabla_{\rm h}\partial_z\bar{u}^{\rm h}(t,\cdot,z)\|_{L^2_{\rm h}}\|\partial_z^2\bar{u}^{\rm h}(t,\cdot,z)\|_{\dot{H}^s_{\rm h}}.\end{aligned}$$

Inserting the above estimates into (4.15) and integrating the resulting equality with respect to z gives rise to

$$\frac{1}{2}\frac{d}{dt}\|\partial_z^2\bar u^{\rm h}(t)\|_{\dot H^{s,0}}^2+\nu_{\rm h}\|\nabla_{\rm h}\partial_z^2\bar u^{\rm h}(t)\|_{\dot H^{s,0}}^2\leq C\Big(\|\nabla_{\rm h}\bar u^{\rm h}\|_{L^\infty_{\rm v}(L^2_{\rm h})}\|\nabla_{\rm h}\partial_z^2\bar u^{\rm h}\|_{\dot H^{s,0}}\|\partial_z^2\bar u^{\rm h}\|_{\dot H^{s,0}}$$
$$+\|\nabla_{\rm h}\partial_z\bar u^{\rm h}\|_{L^\infty_{\rm v}(\dot H^s_{\rm h})}\|\nabla_{\rm h}\partial_z\bar u^{\rm h}\|_{L^2}\|\partial_z^2\bar u^{\rm h}\|_{\dot H^{s,0}}\Big).$$

Due to

$$\|\nabla_{\rm h}\partial_z\bar u^{\rm h}\|_{L^\infty_{\rm v}(\dot H^s_{\rm h})}\lesssim\|\nabla_{\rm h}\partial_z\bar u^{\rm h}\|_{\dot H^{s,0}}^{\frac12}\|\nabla_{\rm h}\partial_z^2\bar u^{\rm h}\|_{\dot H^{s,0}}^{\frac12},$$

we get, by applying Young's inequality, that

$$\frac{1}{2}\frac{d}{dt}\|\partial_z^2\bar u^{\rm h}(t)\|_{\dot H^{s,0}}^2+\nu_{\rm h}\|\nabla_{\rm h}\partial_z^2\bar u^{\rm h}(t)\|_{\dot H^{s,0}}^2\leq\frac{\nu_{\rm h}}{2}\|\nabla_{\rm h}\partial_z^2\bar u^{\rm h}(t)\|_{\dot H^{s,0}}^2$$
$$+\nu_{\rm h}\|\nabla_{\rm h}\partial_z\bar u^{\rm h}(t)\|_{\dot H^{s,0}}^2+\frac{C}{\nu_{\rm h}}\big(\|\nabla_{\rm h}\bar u^{\rm h}\|_{L^\infty_{\rm v}(L^2_{\rm h})}^2+\|\nabla_{\rm h}\partial_z\bar u^{\rm h}\|_{L^2}^2\big)\|\partial_z^2\bar u^{\rm h}\|_{\dot H^{s,0}}.$$

Applying Gronwall's inequality and using (4.13) leads to (4.14). □

Corollary 4.1. *Under the assumption of Proposition 4.3, for any $\theta\in]0,1/2[$, we have*

$$\|\bar u^{\rm h}\|_{L^\infty_t(\mathcal B^{\frac12,\frac12-\theta})}+\nu_{\rm h}^{\frac12}\big(\|\bar u^{\rm h}\|_{L^2_t(\mathcal B^{\frac12,\frac12-\theta})}+\|\partial_z\bar u^{\rm h}\|_{L^2_t(\mathcal B^{\frac12,\frac12-\theta})}\big)\leq\mathfrak D_{\nu_{\rm h},\delta}(\bar u_0^{\rm h})\quad\textit{with}$$
$$\mathfrak D_{\nu_{\rm h},\delta}(\bar u_0^{\rm h})\stackrel{\rm def}{=}C\Big(\sum_{i=1}^2\mathfrak C^i_{\nu_{\rm h},\delta}(\bar u_0^{\rm h},1/2)^{\frac12}+\sum_{i=1}^3\mathfrak C^i_{\nu_{\rm h},\delta}(\bar u_0^{\rm h},-1/2)^{\frac12}\Big).\tag{4.16}$$

Proof. Indeed it follows from interpolation inequality in Besov spaces that

$$\|\bar u^{\rm h}\|_{L^\infty_t(\mathcal B^{\frac12,\frac12-\theta})}+\nu_{\rm h}^{\frac12}\|\bar u^{\rm h}\|_{L^2_t(\mathcal B^{\frac12,\frac12-\theta})}$$
$$\lesssim\|\bar u^{\rm h}\|_{L^\infty_t(H^{\frac12,0})}^{\frac12+\theta}\|\|\partial_z\bar u^{\rm h}\|_{L^\infty_t(H^{\frac12,0})}^{\frac12-\theta}+\nu_{\rm h}^{\frac12}\|\bar u^{\rm h}\|_{L^2_t(H^{\frac12,0})}^{\frac12+\theta}\|\partial_z\bar u^{\rm h}\|_{L^2_t(H^{\frac12,0})}^{\frac12-\theta},$$

and

$$\|\partial_z\bar u^{\rm h}\|_{L^2_t(\mathcal B^{\frac12,\frac12-\theta})}\lesssim\|\partial_z\bar u^{\rm h}\|_{L^2_t(H^{\frac12,0})}^{\frac12+\theta}\|\partial_z^2\bar u^{\rm h}\|_{L^2_t(H^{\frac12,0})}^{\frac12-\theta},$$

which together with Proposition 4.3 ensures (4.16). □

5. The proof of Theorem 1.1

The goal of this section is to present the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $\theta\in[0,1/2[$, we denote

$$f_1(t)\stackrel{\rm def}{=}\|\bar u^{\rm h}(t)\|_{B^{0,\frac12}}^2\|\nabla_{\rm h}\bar u^{\rm h}(t)\|_{B^{0,\frac12}}^2,\quad f_2(t)\stackrel{\rm def}{=}\|\nabla_\varepsilon\bar u^{\rm h}(t)\|_{B^{0,\frac12}}^2,$$
$$f_3(t)\stackrel{\rm def}{=}\|\bar u^{\rm h}(t)\|_{\mathcal B^{\frac12,\frac12-\theta}}^{\frac{2}{1-\theta}}+\|\partial_z\bar u^{\rm h}(t)\|_{\mathcal B^{\frac12,\frac12}}^2\quad\text{and}$$
$$R_\lambda(t)\stackrel{\rm def}{=}R(t)\exp\Big(-\sum_{i=1}^3\lambda_i\int_0^t f_i(t')\,dt'\Big).\tag{5.1}$$

And similar notations for π_λ and F_λ. Then it follows from (2.4) that

$$\partial_tR_\lambda+\big(\sum_{i=1}^3\lambda_if_i(t)\big)R_\lambda+u\cdot\nabla R_\lambda+R_\lambda\cdot\nabla(v_L+[\bar u^{\rm h}]_\varepsilon)$$
$$-\nu_{\rm h}\Delta_{\rm h}R_\lambda-\nu_{\rm v}\partial_3^2R_\lambda+\nabla\pi_\lambda=F_\lambda,\tag{5.2}$$

where $F_\lambda = (F_\lambda^{\mathrm{h}}, F_\lambda^3)$ with (F^{h}, F^3) being given by (2.4).

Applying the operator Δ_ℓ^{v} to (5.2) and taking L^2 inner product of the resulting equation with $\Delta_\ell^{\mathrm{v}} R_\lambda$ yields

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\Delta_\ell^{\mathrm{v}} R_\lambda(t)\|_{L^2}^2 + \sum_{i=1}^3 \lambda_i f_i(t)\|\Delta_\ell^{\mathrm{v}} R_\lambda(t)\|_{L^2}^2 + \nu_{\mathrm{h}}\|\nabla_{\mathrm{h}}\Delta_\ell^{\mathrm{v}} R_\lambda\|_{L^2}^2 + \nu_{\mathrm{v}}\|\partial_3\Delta_\ell^{\mathrm{v}} R_\lambda\|_{L^2}^2 \\
= -\big(\Delta_\ell^{\mathrm{v}}(u\cdot\nabla R_\lambda + R_\lambda\cdot\nabla(v_L + [\bar u^{\mathrm{h}}]_\varepsilon))|\Delta_\ell^{\mathrm{v}} R_\lambda\big)_{L^2} + \big(\Delta_\ell^{\mathrm{v}} F_\lambda|\Delta_\ell^{\mathrm{v}} R_\lambda\big)_{L^2}.
\end{aligned}
\tag{5.3}
$$

The estimate of the above terms relies on the following lemmas:

Lemma 5.1. *There holds*

$$
\int_0^t \big|\big(\Delta_\ell^{\mathrm{v}}(a_\otimes[\bar u^{\mathrm{h}}]_\varepsilon)|\Delta_\ell^{\mathrm{v}} b\big)_{L^2}\big|\,dt \lesssim d_\ell^2 2^{-\ell}\|a\|_{\widetilde L^2_{t,f_1}(B^{0,\frac12})}^{\frac12}\|\nabla_{\mathrm{h}} a\|_{\widetilde L^2_t(B^{0,\frac12})}^{\frac12}\|b\|_{\widetilde L^2_t(B^{0,\frac12})}.
$$

Lemma 5.2. *There holds*

$$
\int_0^t \big|\big(\Delta_\ell^{\mathrm{v}}([\bar u^{\mathrm{h}}]_\varepsilon\cdot\nabla_{\mathrm{h}} v_L)|\Delta_\ell^{\mathrm{v}} a\big)_{L^2}\big|\,dt' \lesssim d_\ell^2 2^{-\ell}\|\nabla_{\mathrm{h}} v_L\|_{\widetilde L^2_t(B^{0,\frac12})}\|a\|_{\widetilde L^2_{t,f_1}(B^{0,\frac12})}^{\frac12}\|\nabla_{\mathrm{h}} a\|_{\widetilde L^2_t(B^{0,\frac12})}^{\frac12}.
$$

Lemma 5.3. *There holds*

$$
\begin{aligned}
\int_0^t e^{-\lambda_2\int_0^{t'} f_2(\tau)\,d\tau}\big|\big(\Delta_\ell^{\mathrm{v}}(v_L\cdot[\nabla_\varepsilon \bar u^{\mathrm{h}}]_\varepsilon)|\Delta_\ell^{\mathrm{v}} a\big)_{L^2}\big|\,dt' &\lesssim d_\ell^2 2^{-\ell}\lambda_2^{-\frac14}\|v_L\|_{\widetilde L^\infty_t(\mathcal{B}^{0,\frac12})}^{\frac12} \\
&\quad\times \|\nabla_{\mathrm{h}} v_L\|_{\widetilde L^2_t(B^{0,\frac12})}^{\frac12}\|a\|_{\widetilde L^2_{t,f_2}(B^{0,\frac12})}^{\frac12}\|\nabla_{\mathrm{h}} a\|_{\widetilde L^2_t(B^{0,\frac12})}^{\frac12}.
\end{aligned}
$$

Lemma 5.4. *Let p^{h} be given by (2.5). Then for $\theta\in[0,1/2[$, we have*

$$
\begin{aligned}
&\int_0^t \exp\Big(-\lambda_3\int_0^{t'} f_3(\tau)\,d\tau\Big)\big|\big(\Delta_\ell^{\mathrm{v}}\partial_3[p^{\mathrm{h}}]_\varepsilon|\Delta_\ell^{\mathrm{v}} a\big)_{L^2}\big|\,dt' \\
&\qquad\lesssim d_\ell^2 2^{-\ell}\varepsilon^{1-\theta}\lambda_3^{-\frac12}\|a\|_{\widetilde L^2_{t,f_3}(\mathcal{B}^{0,\frac12})}^{1-\theta}\|\partial_3 a\|_{\widetilde L^2_t(\mathcal{B}^{0,\frac12})}^{\theta}.
\end{aligned}
$$

Let us admit the above Lemmas for the time being and continue to handle the terms in the second line of (5.3). We first get, by using integration by parts, that

$$
\begin{aligned}
\int_0^t \big(\Delta_\ell^{\mathrm{v}}(R\cdot\nabla R_\lambda)|\Delta_\ell^{\mathrm{v}} R_\lambda\big)_{L^2}\,dt' &= -\int_0^t \big(\Delta_\ell^{\mathrm{v}}(R\otimes R_\lambda)|\Delta_\ell^{\mathrm{v}}\nabla R_\lambda\big)_{L^2}\,dt', \\
\int_0^t \big(\Delta_\ell^{\mathrm{v}}(v_L\cdot\nabla v_L)_\lambda|\Delta_\ell^{\mathrm{v}} R_\lambda\big)_{L^2}\,dt' &= \int_0^t \big(\Delta_\ell^{\mathrm{v}}(v_L\otimes v_L)_\lambda|\Delta_\ell^{\mathrm{v}}\nabla R_\lambda\big)_{L^2}\,dt'.
\end{aligned}
$$

Then applying the law of product, Lemma 3.2, gives

$$
\begin{aligned}
\int_0^t \big|\big(\Delta_\ell^{\mathrm{v}}(R\cdot\nabla R_\lambda)|\Delta_\ell^{\mathrm{v}} R_\lambda\big)_{L^2}\big|\,dt' \leq& \|\Delta_\ell^{\mathrm{v}}(R\otimes R_\lambda)\|_{L^2_T(L^2)}\|\Delta_\ell^{\mathrm{v}}\nabla R_\lambda\|_{L^2_t(L^2)} \\
\lesssim& d_\ell^2 2^{-\ell}\|R\otimes R_\lambda\|_{\widetilde L^2_t(B^{0,\frac12})}\|\nabla R_\lambda\|_{\widetilde L^2_t(B^{0,\frac12})} \\
\lesssim& d_\ell^2 2^{-\ell}\|R\|_{\widetilde L^\infty_t(B^{0,\frac12})}\|\nabla_{\mathrm{h}} R_\lambda\|_{\widetilde L^2_t(B^{0,\frac12})}\|\nabla R_\lambda\|_{\widetilde L^2_t(B^{0,\frac12})},
\end{aligned}
$$

and

$$
\int_0^t \big|\big(\Delta_\ell^{\mathrm{v}}(v_L\cdot\nabla v_L)|\Delta_\ell^{\mathrm{v}} R_\lambda\big)_{L^2}\big|\,dt' \lesssim d_\ell^2 2^{-\ell}\|v_L\|_{\widetilde L^\infty_t(B^{0,\frac12})}\|\nabla_{\mathrm{h}} v_L\|_{\widetilde L^2_t(B^{0,\frac12})}\|\nabla R_\lambda\|_{\widetilde L^2_t(B^{0,\frac12})}.
$$

Along the same line, by using integrating by parts, one has

$$\int_0^t \big(\Delta_\ell^{\mathrm{v}}(v_L\cdot\nabla R_\lambda+R_\lambda\cdot\nabla v_L)|\Delta_\ell^{\mathrm{v}}R_\lambda\big)_{L^2}\,dt' \\ =-\int_0^t \big(\Delta_\ell^{\mathrm{v}}(v_L\otimes R_\lambda+R_\lambda\otimes v_L)|\Delta_\ell^{\mathrm{v}}\nabla R_\lambda\big)_{L^2}\,dt'.$$

It follows from the law of product in anisotropic Besov space, Lemma 3.2, that

$$\|v_L\otimes R_\lambda\|_{\widetilde{L}^2_t(B^{0,\frac12})}\lesssim\|v_L\|^{\frac12}_{\widetilde{L}^\infty_t(B^{0,\frac12})}\|\nabla_{\mathrm{h}}v_L\|^{\frac12}_{\widetilde{L}^\infty_t(B^{0,\frac12})}\|R_\lambda\|^{\frac12}_{\widetilde{L}^\infty_t(B^{0,\frac12})}\|\nabla_{\mathrm{h}}R_\lambda\|^{\frac12}_{\widetilde{L}^2_t(B^{0,\frac12})},$$

which implies

$$\int_0^t \big|\big(\Delta_\ell^{\mathrm{v}}(v_L\cdot\nabla R_\lambda+R_\lambda\cdot\nabla v_L)|\Delta_\ell^{\mathrm{v}}R_\lambda\big)_{L^2}\big|\,dt' \\ \lesssim d_\ell^2 2^{-\ell}\|v_L\|^{\frac12}_{\widetilde{L}^\infty_t(B^{0,\frac12})}\|\nabla_{\mathrm{h}}v_L\|^{\frac12}_{\widetilde{L}^2_t(B^{0,\frac12})}\|R_\lambda\|^{\frac12}_{\widetilde{L}^\infty_t(B^{0,\frac12})}\|\nabla_{\mathrm{h}}R_\lambda\|^{\frac12}_{\widetilde{L}^2_t(B^{0,\frac12})}\|\nabla R_\lambda\|_{\widetilde{L}^2_t(B^{0,\frac12})}.$$

Whereas we get, by using integrating by parts, that

$$\int_0^t \big(\Delta_\ell^{\mathrm{v}}([\bar u^{\mathrm{h}}]_\varepsilon\cdot\nabla_{\mathrm{h}} R_\lambda+R_\lambda\cdot\nabla [\bar u^{\mathrm{h}}]_\varepsilon)|\Delta_\ell^{\mathrm{v}}R_\lambda\big)_{L^2}\,dt' \\ =-\int_0^t \big(\Delta_\ell^{\mathrm{v}}([\bar u^{\mathrm{h}}]_\varepsilon\otimes R_\lambda)|\Delta_\ell^{\mathrm{v}}\nabla_{\mathrm{h}} R_\lambda\big)_{L^2}\,dt'-\int_0^t \big(\Delta_\ell^{\mathrm{v}}(R_\lambda\otimes[\bar u^{\mathrm{h}}]_\varepsilon)|\Delta_\ell^{\mathrm{v}}\nabla R_\lambda\big)_{L^2}\,dt'.$$

Then applying Lemma 5.1 yields

$$\int_0^t \big|\big(\Delta_\ell^{\mathrm{v}}([\bar u^{\mathrm{h}}]_\varepsilon\cdot\nabla_{\mathrm{h}} R_\lambda+R_\lambda\cdot\nabla [\bar u^{\mathrm{h}}]_\varepsilon)|\Delta_\ell^{\mathrm{v}}R_\lambda\big)_{L^2}\big|\,dt' \\ \lesssim d_\ell^2 2^{-\ell}\|R\|^{\frac12}_{\widetilde{L}^2_{t,f_1}(B^{0,\frac12})}\|\nabla_{\mathrm{h}}R_\lambda\|^{\frac12}_{\widetilde{L}^2_t(B^{0,\frac12})}\|\nabla R_\lambda\|_{\widetilde{L}^2_t(B^{0,\frac12})}.$$

Applying Lemma 5.2 gives

$$\int_0^t \big|\big(\Delta_\ell^{\mathrm{v}}([\bar u^{\mathrm{h}}]_\varepsilon\cdot\nabla_{\mathrm{h}} v_L)|\Delta_\ell^{\mathrm{v}}R_\lambda\big)_{L^2}\big|\,dt'\lesssim d_\ell^2 2^{-\ell}\|\nabla_{\mathrm{h}}v_L\|_{\widetilde{L}^2_T(B^{0,\frac12})}\|R_\lambda\|^{\frac12}_{\widetilde{L}^2_{t,f_1}(B^{0,\frac12})}\|\nabla_{\mathrm{h}}R_\lambda\|^{\frac12}_{\widetilde{L}^2_t(B^{0,\frac12})}.$$

Applying Lemma 5.3 yields

$$\int_0^t \big|\big(\Delta_\ell^{\mathrm{v}}(v_L\cdot[\nabla_\varepsilon\bar u^{\mathrm{h}}]_\varepsilon)_\lambda|\Delta_\ell^{\mathrm{v}}R_\lambda\big)_{L^2}\big|\,dt'\lesssim d_\ell^2 2^{-\ell}\lambda_2^{-\frac14}\|v_L\|^{\frac12}_{\widetilde{L}^\infty_t(B^{0,\frac12})} \\ \times\|\nabla_{\mathrm{h}}v_L\|^{\frac12}_{\widetilde{L}^2_t(B^{0,\frac12})}\|R_\lambda\|^{\frac12}_{\widetilde{L}^2_{t,f_2}(B^{0,\frac12})}\|\nabla_{\mathrm{h}}R_\lambda\|^{\frac12}_{\widetilde{L}^2_t(B^{0,\frac12})}.$$

Finally applying Lemma 5.4 gives rise to

$$\int_0^t \big|\big(\Delta_\ell^{\mathrm{v}}(\partial_3[p^{\mathrm{h}}]_\varepsilon)_\lambda|\Delta_\ell^{\mathrm{v}}R^3\big)_{L^2}\big|\,dt'\lesssim d_\ell^2 2^{-\ell}\varepsilon^{1-\theta}\lambda_3^{-\frac12}\|\partial_3R^3\|^{1-\theta}_{\widetilde{L}^2_{t,f_3}(B^{0,\frac12})}\|\partial_3R^3\|^{\theta}_{\widetilde{L}^2_t(B^{0,\frac12})}.$$

Let us denote

$$\mathcal{A}_t\stackrel{\text{def}}{=}\|v_L\|_{\widetilde{L}^\infty_t(B^{0,\frac12})}\|\nabla_{\mathrm{h}}v_L\|_{\widetilde{L}^2_t(B^{0,\frac12})}.$$

Then it follows from Proposition 4.1 that

$$\mathcal{A}_t\lesssim(\nu_{\mathrm{h}}\nu_{\mathrm{v}})^{-\frac14}\mathcal{A}_0\quad\text{with}\quad\mathcal{A}_0\stackrel{\text{def}}{=}\|v_0\|_{\mathcal{B}^{\frac12,0}}\big(\|v_0\|_{B^{0,\frac12}}+\|v_0\|_{\mathcal{B}^{\frac12,0}}\big).\tag{5.4}$$

Integrating (5.3) over $[0,t]$ and inserting the above estimates into the resulting inequality, then we take the square root of resulting inequality to achieve

$$
\begin{aligned}
\|\Delta_\ell^{\mathrm{v}} R_\lambda\|_{L_t^\infty(L^2)} &+ \sum_{i=1}^{3} \lambda_i^{\frac12}\Big(\int_0^t f_i(t')\|\Delta_\ell^{\mathrm{v}} R_\lambda(t')\|_{L^2}^2\,dt'\Big)^{\frac12} \\
&\qquad\qquad + \nu_{\mathrm{h}}^{\frac12}\|\nabla_{\mathrm{h}}\Delta_\ell^{\mathrm{v}} R_\lambda\|_{L_t^2(L^2)} + \nu_{\mathrm{v}}^{\frac12}\|\partial_3\Delta_\ell^{\mathrm{v}} R_\lambda\|_{L_t^2(L^2)} \\
\lesssim d_\ell 2^{-\frac{\ell}{2}}\Big(&\|\nabla R_\lambda\|_{\widetilde{L}_t^2(B^{0,\frac12})}^{\frac12}\big[\mathcal{A}_t^{\frac12} + \|R\|_{\widetilde{L}_t^\infty(B^{0,\frac12})}^{\frac12}\|\nabla_{\mathrm{h}} R_\lambda\|_{\widetilde{L}_t^2(B^{0,\frac12})}^{\frac12} \\
&+ \big(\|R_\lambda\|_{\widetilde{L}_{t,f_1}^2(B^{0,\frac12})}^{\frac14} + \mathcal{A}_t^{\frac14}\|R\|_{\widetilde{L}_t^\infty(B^{0,\frac12})}^{\frac14}\big)\|\nabla_{\mathrm{h}} R_\lambda\|_{\widetilde{L}_t^2(B^{0,\frac12})}^{\frac14}\big] \\
&+ \big[\lambda_2^{-\frac18}\mathcal{A}_t^{\frac14}\|R_\lambda\|_{\widetilde{L}_{t,f_2}^2(B^{0,\frac12})}^{\frac14} + \|\nabla_{\mathrm{h}} v_L\|_{\widetilde{L}_T^2(B^{0,\frac12})}^{\frac12}\|R_\lambda\|_{\widetilde{L}_{t,f_1}^2(B^{0,\frac12})}^{\frac14}\big] \\
&\times \|\nabla_{\mathrm{h}} R_\lambda\|_{\widetilde{L}_t^2(B^{0,\frac12})}^{\frac14} + \varepsilon^{\frac{1-\theta}{2}}\lambda_3^{-\frac14}\|R^3\|_{\widetilde{L}_{t,f_3}^2(B^{0,\frac12})}^{\frac{1-\theta}{2}}\|\partial_3 R^3\|_{\widetilde{L}_t^2(B^{0,\frac12})}^{\frac{\theta}{2}}\Big).
\end{aligned}
$$

Multiplying $2^{\frac{\ell}{2}}$ to the above inequality and summing the resulting inequality for $\ell \in \mathbb{Z}$ leads to

$$
\begin{aligned}
\|R_\lambda\|_{\widetilde{L}_t^\infty(B^{0,\frac12})} &+ \sum_{i=1}^{3} \lambda_i^{\frac12}\|R_\lambda\|_{\widetilde{L}_{t,f_i}^2(B^{0,\frac12})} + \nu_{\mathrm{h}}^{\frac12}\|\nabla_{\mathrm{h}} R_\lambda\|_{\widetilde{L}_t^2(B^{0,\frac12})} + \nu_{\mathrm{v}}^{\frac12}\|\partial_3 R_\lambda\|_{\widetilde{L}_t^2(B^{0,\frac12})} \\
\leq C\Big(&\|\nabla R_\lambda\|_{\widetilde{L}_t^2(B^{0,\frac12})}^{\frac12}\big[\mathcal{A}_t^{\frac12} + \|R\|_{\widetilde{L}_t^\infty(B^{0,\frac12})}^{\frac12}\|\nabla_{\mathrm{h}} R_\lambda\|_{\widetilde{L}_t^2(B^{0,\frac12})}^{\frac12} \\
&+ \big(\mathcal{A}_t^{\frac14}\|R\|_{\widetilde{L}_t^\infty(B^{0,\frac12})}^{\frac14} + \|R_\lambda\|_{\widetilde{L}_{t,f_1}^2(B^{0,\frac12})}^{\frac14}\big)\|\nabla_{\mathrm{h}} R_\lambda\|_{\widetilde{L}_t^2(B^{0,\frac12})}^{\frac14}\big] \\
&+ \big[\lambda_2^{-\frac18}\mathcal{A}_t^{\frac14}\|R_\lambda\|_{\widetilde{L}_t^\infty(B^{0,\frac12})}^{\frac14} + \|\nabla_{\mathrm{h}} v_L\|_{\widetilde{L}_T^2(B^{0,\frac12})}^{\frac12}\|R_\lambda\|_{\widetilde{L}_{t,f_1}^2(B^{0,\frac12})}^{\frac14}\big] \\
&\times \|\nabla_{\mathrm{h}} R_\lambda\|_{\widetilde{L}_t^2(B^{0,\frac12})}^{\frac14} + \varepsilon^{\frac{1-\theta}{2}}\lambda_3^{-\frac14}\|R^3\|_{\widetilde{L}_{t,f_3}^2(B^{0,\frac12})}^{\frac{1-\theta}{2}}\|\partial_3 R^3\|_{\widetilde{L}_t^2(B^{0,\frac12})}^{\frac{\theta}{2}}\Big).
\end{aligned}
\tag{5.5}
$$

Let us assume that $\nu_{\mathrm{v}} \geq \nu_{\mathrm{h}}$. Then by applying Young's inequality, we obtain

$$\mathcal{A}_t^{\frac12}\|\nabla R_\lambda\|_{\widetilde{L}_t^2(B^{0,\frac12})}^{\frac12} \leq C\nu_{\mathrm{h}}^{-\frac12}\mathcal{A}_t^{\frac12} + \frac{\nu_{\mathrm{h}}^{\frac12}}{100}\|\nabla R_\lambda\|_{\widetilde{L}_t^2(B^{0,\frac12})};$$

$$\mathcal{A}^{\frac14}\|R\|_{\widetilde{L}_t^\infty(B^{0,\frac12})}^{\frac14}\|\nabla_{\mathrm{h}} R_\lambda\|_{\widetilde{L}_t^2(B^{0,\frac12})}^{\frac14}\|\nabla R_\lambda\|_{\widetilde{L}_t^2(B^{0,\frac12})}^{\frac12} \leq C\mathcal{A}_t\nu_{\mathrm{h}}^{-\frac32}\|R\|_{\widetilde{L}_t^\infty(B^{0,\frac12})} + \frac{\nu_{\mathrm{h}}^{\frac12}}{100}\|\nabla R_\lambda\|_{\widetilde{L}_t^2(B^{0,\frac12})};$$

$$C\|R\|_{\widetilde{L}_{t,f_1}^2(B^{0,\frac12})}^{\frac14}\|\nabla_{\mathrm{h}} R_\lambda\|_{\widetilde{L}_t^2(B^{0,\frac12})}^{\frac14}\|\nabla R_\lambda\|_{\widetilde{L}_t^2(B^{0,\frac12})}^{\frac12} \leq C\nu_{\mathrm{h}}^{-\frac32}\|R\|_{\widetilde{L}_{t,f_1}^2(B^{0,\frac12})} + \frac{\nu_{\mathrm{h}}^{\frac12}}{100}\|\nabla R_\lambda\|_{\widetilde{L}_t^2(B^{0,\frac12})};$$

and

$$
\begin{aligned}
C\lambda_2^{-\frac14}\mathcal{A}_t^{\frac18}\|R_\lambda\|_{\widetilde{L}_{t,f_2}^2(B^{0,\frac12})}^{\frac14}\|\nabla_{\mathrm{h}} R_\lambda\|_{\widetilde{L}_t^2(B^{0,\frac12})}^{\frac14} \leq& C\mathcal{A}_t^{\frac12}\lambda_2^{-\frac14}\nu_{\mathrm{h}}^{\frac12} \\
&+ C\nu_{\mathrm{h}}^{-\frac32}\|R_\lambda\|_{\widetilde{L}_{t,f_2}^2(B^{0,\frac12})} + \frac{\nu_{\mathrm{h}}^{\frac12}}{100}\|\nabla R_\lambda\|_{\widetilde{L}_t^2(B^{0,\frac12})};
\end{aligned}
$$

$$\|\nabla_{\rm h} v_L\|_{\widetilde{L}^2_T(B^{0,\frac12})}^{\frac12}\|R_\lambda\|_{\widetilde{L}^2_{t,f_1}(B^{0,\frac12})}^{\frac14}\|\nabla_{\rm h} R_\lambda\|_{\widetilde{L}^2_t(B^{0,\frac12})}^{\frac14}$$
$$\leq C\nu_{\rm h}^{-\frac12}\|\nabla_{\rm h} v_L\|_{\widetilde{L}^2_T(B^{0,\frac12})}+\frac{C}{2}\nu_{\rm h}^{-\frac32}\|R_\lambda\|_{\widetilde{L}^2_{t,f_1}(B^{0,\frac12})}+\frac{\nu_{\rm h}^{\frac12}}{100}\|\nabla R_\lambda\|_{\widetilde{L}^2_t(B^{0,\frac12})};$$

$$C\varepsilon^{\frac{1-\theta}{2}}\lambda_3^{-\frac14}\|R^3\|_{\widetilde{L}^2_{t,f_3}(\mathcal{B}^{0,\frac12})}^{\frac{1-\theta}{2}}\|\partial_3 R^3\|_{\widetilde{L}^2_t(\mathcal{B}^{0,\frac12})}^{\frac{\theta}{2}}\leq C\varepsilon^{1-\theta}\nu_{\rm v}^{-\frac{\theta}{2}}\lambda_3^{-\frac12}+\|R^3\|_{\widetilde{L}^2_{t,f_3}(\mathcal{B}^{0,\frac12})}+\frac{\nu_{\rm v}^{\frac12}}{100}\|\partial_3 R_\lambda\|_{\widetilde{L}^2_t(B^{0,\frac12})}.$$

Inserting the above estimates into (5.5) leads to

$$\begin{aligned}
&\|R_\lambda\|_{\widetilde{L}^\infty_t(B^{0,\frac12})}+\sum_{i=1}^3\lambda_i^{\frac12}\|R\|_{\widetilde{L}^2_{t,f_i}(B^{0,\frac12})}+\nu_{\rm h}^{\frac12}\|\nabla_{\rm h} R_\lambda\|_{\widetilde{L}^2_t(B^{0,\frac12})}+\nu_{\rm v}^{\frac12}\|\partial_3 R_\lambda\|_{\widetilde{L}^2_t(B^{0,\frac12})}\\
&\leq C\Big(\mathcal{A}_t^{\frac12}\nu_{\rm h}^{-\frac12}+\mathcal{A}_t^{\frac12}\lambda_2^{-\frac14}\nu_{\rm h}^{\frac12}+\nu_{\rm h}^{-\frac12}\|\nabla_{\rm h} v_L\|_{\widetilde{L}^2_T(B^{0,\frac12})}+\varepsilon^{1-\theta}\nu_{\rm v}^{-\frac{\theta}{2}}\Big)\\
&\quad+C\mathcal{A}_t\nu_{\rm h}^{-\frac32}\|R\|_{\widetilde{L}^\infty_t(B^{0,\frac12})}+\big(\frac{\nu_h^{\frac12}}{4}+C\|R\|_{\widetilde{L}^\infty_t(B^{0,\frac12})}^{\frac12}\big)\|\nabla R_\lambda\|_{\widetilde{L}^2_t(B^{0,\frac12})}\\
&\quad+C\nu^{-\frac32}\big(\|R\|_{\widetilde{L}^2_{t,f_1}(B^{0,\frac12})}+\|R\|_{\widetilde{L}^2_{t,f_2}(B^{0,\frac12})}\big)+\|R^3\|_{\widetilde{L}^2_{t,f_3}(B^{0,\frac12})}
\end{aligned}$$

Let us denote

$$T^\star\stackrel{\rm def}{=}\Big\{\ T<T^*:\ \|R\|_{\widetilde{L}^\infty_t(B^{0,\frac12})}\leq\frac{\nu_{\rm h}}{16C^2}\quad\Big\},\tag{5.6}$$

and we take

$$\lambda_1=\lambda_2=C^2\nu_{\rm h}^{-3}\quad\text{and}\quad\lambda_3=1.\tag{5.7}$$

Then for $t\leq T^\star$, we deduce from (5.4) that

$$\|R_\lambda\|_{\widetilde{L}^\infty_t(B^{0,\frac12})}+\nu_{\rm h}^{\frac12}\|\nabla_{\rm h} R_\lambda\|_{\widetilde{L}^2_t(B^{0,\frac12})}+\nu_{\rm v}^{\frac12}\|\partial_3 R_\lambda\|_{\widetilde{L}^2_t(B^{0,\frac12})}\leq C\Big(\mathcal{A}_0^{\frac12}\nu_{\rm h}^{-\frac58}\nu_{\rm v}^{-\frac18}+\varepsilon^{1-\theta}\nu_{\rm v}^{-\frac{\theta}{2}}\Big),$$

which implies that

$$\begin{aligned}
&\|R\|_{\widetilde{L}^\infty_t(B^{0,\frac12})}+\nu_{\rm h}^{\frac12}\|\nabla_{\rm h} R\|_{\widetilde{L}^2_t(B^{0,\frac12})}+\nu_{\rm v}^{\frac12}\|\partial_3 R\|_{\widetilde{L}^2_t(B^{0,\frac12})}\leq C\Big(\mathcal{A}_0^{\frac12}\nu_{\rm h}^{-\frac58}\nu_{\rm v}^{-\frac18}+\varepsilon^{1-\theta}\nu_{\rm v}^{-\frac{\theta}{2}}\Big)\\
&\times\exp\Big(C\nu_{\rm h}^{-3}\int_0^t\big(1+\|\bar u^{\rm h}(t')\|_{B^{0,\frac12}}^2\big)\|\nabla_\varepsilon\bar u^{\rm h}(t')\|_{B^{0,\frac12}}^2\,dt'\\
&\qquad+C\int_0^t\big(\|\bar u^{\rm h}(t')\|_{\mathcal{B}^{\frac12,\frac12-\theta}}^{\frac{2}{1-\theta}}+\|\partial_z\bar u^{\rm h}(t)\|_{\mathcal{B}^{\frac12,\frac12}}^2\big)\,dt'\Big),
\end{aligned}$$

from which and Proposition 4.2 and Corollary 4.1, we deduce that for $t\leq T^\star$,

$$\begin{aligned}
&\|R\|_{\widetilde{L}^\infty_t(B^{0,\frac12})}+\nu_{\rm h}^{\frac12}\|\nabla_{\rm h} R\|_{\widetilde{L}^2_t(B^{0,\frac12})}+\nu_{\rm v}^{\frac12}\|\partial_3 R\|_{\widetilde{L}^2_t(B^{0,\frac12})}\\
&\leq C\Big(\mathcal{A}_0^{\frac12}\nu_{\rm h}^{-\frac58}\nu_{\rm v}^{-\frac18}+\varepsilon^{1-\theta}\nu_{\rm v}^{-\frac{\theta}{2}}\Big)\mathfrak{E}_{\delta,\nu_h}(\bar u_0^{\rm h})\quad\text{with}\\
&\mathfrak{E}_{\delta,\nu_h}(\bar u_0^{\rm h})\stackrel{\rm def}{=}\exp\Big(C\nu_{\rm h}^{-4}\big(1+\mathfrak{B}^4_{\delta,\nu_h}(\bar u_0^{\rm h})+C\nu_{\rm h}^{-1}\big(\mathfrak{D}_{\delta,\nu_{\rm h}}(\bar u_0^{\rm h})^{\frac{2}{1-\theta}}+\mathfrak{D}_{\delta,\nu_{\rm h}}(\bar u_0^{\rm h})^2\Big).
\end{aligned}\tag{5.8}$$

□

Now let us present the proof of Lemmas 5.1 to 5.4.

Proof of Lemma 5.1. Applying Bony's decomposition yields

$$a\otimes[\bar u^{\rm h}]_\varepsilon=T^{\rm v}_a[\bar u^{\rm h}]_\varepsilon+\bar R^{\rm v}(a,[\bar u^{\rm h}]_\varepsilon).$$

We first observe that

$$
\begin{aligned}
&\int_0^t \big|\big(\Delta_\ell^{\rm v} T^{\rm v}_a[\bar u^{\rm h}]_\varepsilon|\Delta_\ell^{\rm v} b\big)_{L^2}\big|\,dt'\\
&\leq \sum_{|\ell'-\ell|\leq 4}\int_0^t \|S^{\rm v}_{\ell'-1}a\|_{L^\infty_{\rm v}(L^4_{\rm h})}\|\Delta^{\rm v}_{\ell'}[\bar u^{\rm h}]_\varepsilon\|_{L^2_{\rm v}(L^4_{\rm h})}\|\Delta_\ell^{\rm v} b\|_{L^2}\,dt'\\
&\lesssim \sum_{|\ell'-\ell|\leq 4} 2^{-\frac{\ell'}{2}}\int_0^t d_{\ell'}(t')\|a\|_{L^\infty_{\rm v}(L^4_{\rm h})}\|[\bar u^{\rm h}]_\varepsilon\|^{\frac12}_{\mathcal{B}^{0,\frac12}}\|[\nabla_{\rm h}\bar u^{\rm h}]_\varepsilon\|^{\frac12}_{\mathcal{B}^{0,\frac12}}\|\Delta_\ell^{\rm v} b\|_{L^2}\,dt'\\
&\lesssim \sum_{|\ell'-\ell|\leq 4} d_{\ell'}2^{-\frac{\ell'}{2}}\int_0^t \|a\|_{L^\infty_{\rm v}(L^4_{\rm h})}\|\bar u^{\rm h}\|^{\frac12}_{\mathcal{B}^{0,\frac12}}\|\nabla_{\rm h}\bar u^{\rm h}\|^{\frac12}_{\mathcal{B}^{0,\frac12}}\|\Delta_\ell^{\rm v} b\|_{L^2}\,dt'.
\end{aligned}
$$

On the other hand, it follows from Lemma 3.1 that

$$
\begin{aligned}
\|\Delta_\ell^{\rm v} a\|_{L^\infty_{\rm v}(L^4_{\rm h})}\lesssim & 2^{\frac{\ell}{2}}\|\Delta_\ell^{\rm v} a\|_{L^2_{\rm v}(L^4_{\rm h})}\\
\lesssim & 2^{\frac{\ell}{2}}\|\Delta_\ell^{\rm v} a\|^{\frac12}_{L^2}\|\Delta_\ell^{\rm v}\nabla_{\rm h} a\|^{\frac12}_{L^2},
\end{aligned}
$$

from which and Definition 3.4, we infer

$$
\begin{aligned}
&\Big(\int_0^t \|\bar u^{\rm h}\|_{\mathcal{B}^{0,\frac12}}\|\nabla_{\rm h}\bar u^{\rm h}\|_{\mathcal{B}^{0,\frac12}}\|a\|^2_{L^\infty_{\rm v}(L^4_{\rm h})}\,dt'\Big)^{\frac12}\\
&\quad\leq\sum_{\ell\in\mathbb{Z}} 2^{\frac{\ell}{2}}\Big(\int_0^t \|\bar u^{\rm h}\|_{\mathcal{B}^{0,\frac12}}\|\nabla_{\rm h}\bar u^{\rm h}\|_{\mathcal{B}^{0,\frac12}}\|\Delta_\ell^{\rm v} a\|_{L^2}\|\Delta_\ell^{\rm v}\nabla_{\rm h} a\|_{L^2}\,dt'\Big)^{\frac12}\\
&\quad\lesssim\sum_{\ell\in\mathbb{Z}} 2^{\frac{\ell}{2}}\Big(\int_0^t f_1(t')\|\Delta_\ell^{\rm v} a\|^2_{L^2}\,dt'\Big)^{\frac12}\|\Delta_\ell^{\rm v}\nabla_{\rm h} a\|^{\frac12}_{L^2_t(L^2)}\\
&\quad\lesssim\|a\|^{\frac12}_{\widetilde{L}^2_{t,f_1}(B^{0,\frac12})}\|\nabla_{\rm h} a\|^{\frac12}_{\widetilde{L}^2_t(B^{0,\frac12})}.
\end{aligned}
$$

As a result, it comes out

$$
\int_0^t \big|\big(\Delta_\ell^{\rm v} T^{\rm v}_a[\bar u^{\rm h}]_\varepsilon|\Delta_\ell^{\rm v} b\big)_{L^2}\big|\,dt\lesssim d_\ell^2 2^{-\ell}\|a\|^{\frac12}_{\widetilde{L}^2_{t,f_1}(\mathcal{B}^{0,\frac12})}\|\nabla_{\rm h} a\|^{\frac12}_{\widetilde{L}^2_t(\mathcal{B}^{0,\frac12})}\|b\|_{\widetilde{L}^2_t(\mathcal{B}^{0,\frac12})}. \tag{5.9}
$$

Exactly along the same line, we have

$$
\begin{aligned}
&\int_0^t \big|\big(\Delta_\ell^{\rm v}\bar R^{\rm v}([\bar u^{\rm h}]_\varepsilon,a)|\Delta_\ell^{\rm v} b\big)_{L^2}\big|\,dt'\\
&\lesssim\sum_{\ell'\geq\ell-N_0}\int_0^t \|\Delta^{\rm v}_{\ell'}a\|_{L^2_{\rm v}(L^4_{\rm h})}\|S^{\rm v}_{\ell'+2}([\bar u^{\rm h}]_\varepsilon)\|_{L^\infty_{\rm v}(L^4_{\rm h})}\|\Delta_\ell^{\rm v} b\|_{L^2}\,dt'\\
&\lesssim\sum_{\ell'\geq\ell-N_0}\int_0^t \|\bar u^{\rm h}\|_{L^\infty_{\rm v}(L^4_{\rm h})}\|\Delta^{\rm v}_{\ell'}a\|^{\frac12}_{L^2}\|\Delta^{\rm v}_{\ell'}\nabla_{\rm h} a\|^{\frac12}_{L^2}\|\Delta_\ell^{\rm v} b\|_{L^2}\,dt'\\
&\lesssim\sum_{\ell'\geq\ell-N_0}\Big(\int_0^t f_1(t')\|\Delta^{\rm v}_{\ell'}a\|^2_{L^2}\,dt'\Big)^{\frac14}\|\Delta^{\rm v}_{\ell'}\nabla_{\rm h} a\|^{\frac12}_{L^2_t(L^2)}\|\Delta_\ell^{\rm v} b\|_{L^2_t(L^2)}\\
&\lesssim d_\ell 2^{-\ell}\|a\|^{\frac12}_{\widetilde{L}^2_{t,f_1}(\mathcal{B}^{0,\frac12})}\|\nabla_{\rm h} a\|^{\frac12}_{\widetilde{L}^2_t(\mathcal{B}^{0,\frac12})}\|b\|_{\widetilde{L}^2_t(\mathcal{B}^{0,\frac12})}\sum_{\ell'\geq\ell-N_0} d_{\ell'}2^{-\frac{\ell'-\ell}{2}},
\end{aligned}
$$

which implies

$$
\int_0^t \big|\big(\Delta_\ell^{\rm v}(\bar R^{\rm v}(a,[\bar u^{\rm h}]_\varepsilon)|\Delta_\ell^{\rm v} b\big)_{L^2}\big|\,dt'\lesssim d_\ell^2 2^{-\ell}\|a\|^{\frac12}_{\widetilde{L}^2_{t,f_1}(\mathcal{B}^{0,\frac12})}\|\nabla_{\rm h} a\|^{\frac12}_{\widetilde{L}^2_t(\mathcal{B}^{0,\frac12})}\|b\|_{\widetilde{L}^2_t(\mathcal{B}^{0,\frac12})}.
$$

Along with (5.9), we complete the proof of Lemma 5.1. $\square$

Proof of Lemma 5.2. Applying Bony's decomposition in the vertical variable gives

$$[\bar{u}^{\rm h}]_\varepsilon\cdot\nabla_{\rm h}v_L=T_{[\bar{u}^{\rm h}]_\varepsilon}\nabla_{\rm h}v_L+\bar{R}^{\rm v}([\bar{u}^{\rm h}]_\varepsilon,\nabla_{\rm h}v_L).$$

We first observe that

$$\begin{aligned}&\int_0^t\big|\big(\Delta_\ell^{\rm v}T^{\rm v}_{[\bar{u}^{\rm h}]_\varepsilon}\nabla_{\rm h}v_L|\Delta_\ell^{\rm v}a\big)_{L^2}\big|\,dt'\\
&\lesssim\sum_{|\ell'-\ell|\leq4}\int_0^t\|S^{\rm v}_{\ell'-1}([\bar{u}^{\rm h}]_\varepsilon)\|_{L^\infty_{\rm v}(L^4_{\rm h})}\|\Delta_{\ell'}^{\rm v}\nabla_{\rm h}v_L\|_{L^2}\|\Delta_\ell^{\rm v}a\|_{L^2_{\rm v}(L^4_{\rm h})}\,dt'\\
&\lesssim\sum_{|\ell'-\ell|\leq4}\int_0^t\|\bar{u}^{\rm h}\|_{L^\infty_{\rm v}(L^4_{\rm h})}\|\Delta_{\ell'}^{\rm v}\nabla_{\rm h}v_L\|_{L^2}\|\Delta_\ell^{\rm v}a\|_{L^2}^{\frac12}\|\Delta_\ell^{\rm v}\nabla_{\rm h}a\|_{L^2}^{\frac12}\,dt'\\
&\lesssim\sum_{|\ell'-\ell|\leq4}\|\Delta_{\ell'}^{\rm v}\nabla_{\rm h}v_L\|_{L^2_t(L^2)}\Big(\int_0^t f_1(t)\|\Delta_\ell^{\rm v}a\|_{L^2}^2\,dt\Big)^{\frac14}\|\Delta_\ell^{\rm v}\nabla_{\rm h}a\|_{L^2_t(L^2)}^{\frac12}\\
&\lesssim d_\ell^2 2^{-\ell}\|\nabla_{\rm h}v_L\|_{\widetilde{L}^2_t(B^{0,\frac12})}\|a\|^{\frac12}_{\widetilde{L}^2_{t,f_1}(B^{0,\frac12})}\|\nabla_{\rm h}a\|^{\frac12}_{\widetilde{L}^2_t(\mathcal{B}^{0,\frac12})}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}&\int_0^t\big|\big(\Delta_\ell^{\rm v}(\bar{R}^{\rm v}([\bar{u}^{\rm h}]_\varepsilon,\nabla_{\rm h}v_L)|\Delta_\ell^{\rm v}a\big)_{L^2}\big|\,dt'\\
&\lesssim\sum_{\ell'\geq\ell-N_0}\int_0^t\|\Delta_{\ell'}^{\rm v}([\bar{u}^{\rm h}]_\varepsilon)\|_{L^2_{\rm v}(L^4_{\rm h})}\|S^{\rm v}_{\ell'+2}\nabla_{\rm h}v_L\|_{L^\infty_{\rm v}(L^2_{\rm h})}\|\Delta_\ell^{\rm v}a\|_{L^2_{\rm v}(L^4_{\rm h})}\,dt'\\
&\lesssim\sum_{\ell'\geq\ell-N_0}2^{-\frac{\ell'}{2}}\int_0^T d_{\ell'}(t)\|\nabla_{\rm h}v_L\|_{B^{0,\frac12}}\|\bar{u}^{\rm h}\|^{\frac12}_{B^{0,\frac12}}\|\nabla_{\rm h}\bar{u}^{\rm h}\|^{\frac12}_{B^{0,\frac12}}\|\Delta_\ell^{\rm v}a\|_{L^2}^{\frac12}\|\Delta_\ell^{\rm v}\nabla_{\rm h}a\|_{L^2}^{\frac12}\,dt',\end{aligned}$$

then applying Hölder's inequality and using Definition 3.4, we obtain

$$\begin{aligned}&\int_0^t\big|\big(\Delta_\ell^{\rm v}(\bar{R}^{\rm v}([\bar{u}^{\rm h}]_\varepsilon,\nabla_{\rm h}v_L)|\Delta_\ell^{\rm v}a\big)_{L^2}\big|\,dt'\\
&\lesssim\sum_{\ell'\geq\ell-N_0}d_{\ell'}2^{-\frac{\ell'}{2}}\|\nabla_{\rm h}v_L\|_{\widetilde{L}^2_t(B^{0,\frac12})}\Big(\int_0^t f_1(t')\|\Delta_\ell^{\rm v}a\|_{L^2}^2\,dt'\Big)^{\frac14}\|\Delta_\ell^{\rm v}\nabla_{\rm h}a\|_{L^2_t(L^2)}^{\frac12}\\
&\lesssim d_\ell^2 2^{-\ell}\|\nabla_{\rm h}v_L\|_{\widetilde{L}^2_t(B^{0,\frac12})}\|a\|^{\frac12}_{\widetilde{L}^2_{t,f_1}(B^{0,\frac12})}\|\nabla_{\rm h}a\|^{\frac12}_{\widetilde{L}^2_t(B^{0,\frac12})}.\end{aligned}$$

This completes the proof of the lemma. $\square$

Proof of Lemma 5.3. By applying Bony's decomposition in the vertical variable, we get

$$v_L\cdot[\nabla_\varepsilon\bar{u}^{\rm h}]_\varepsilon=T^{\rm v}_{v_L}[\nabla_\varepsilon\bar{u}^{\rm h}]_\varepsilon+\bar{R}^{\rm v}(v_L,[\nabla_\varepsilon\bar{u}^{\rm h}]_\varepsilon).$$

Note that

$$\begin{aligned}&\int_0^t e^{-\lambda_2\int_0^{t'}f_2(\tau)\,d\tau}\big|\big(\Delta_\ell^{\rm v}T^{\rm v}_{v_L}[\nabla_\varepsilon\bar{u}^{\rm h}]_\varepsilon|\Delta_\ell^{\rm v}a\big)_{L^2}\big|\,dt'\\
&\lesssim\sum_{|\ell'-\ell|\leq4}\int_0^t e^{-\lambda_2\int_0^{t'}f_2(\tau)\,d\tau}\|S^{\rm v}_{\ell'-1}v_L\|_{L^\infty_{\rm v}(L^4_{\rm h})}\|\Delta_{\ell'}^{\rm v}[\nabla_\varepsilon\bar{u}^{\rm h}]_\varepsilon\|_{L^2}\|\Delta_\ell^{\rm v}a\|_{L^2_{\rm v}(L^4_{\rm h})}\,dt'\\
&\lesssim\sum_{|\ell'-\ell|\leq4}2^{-\frac{\ell'}{2}}\int_0^t d_{\ell'}(t')e^{-\lambda_2\int_0^{t'}f_2(\tau)\,d\tau}\|v_L\|_{L^\infty_{\rm v}(L^4_{\rm h})}\|\nabla_\varepsilon\bar{u}^{\rm h}\|_{\mathcal{B}^{0,\frac12}}\|\Delta_\ell^{\rm v}a\|_{L^2}^{\frac12}\|\Delta_\ell^{\rm v}\nabla_{\rm h}a\|_{L^2}^{\frac12}\,dt'.\end{aligned}$$

Applying Hölder's inequality gives

$$
\begin{aligned}
&\int_0^t e^{-\lambda_2\int_0^{t'} f_2(\tau)\,d\tau}\big|\big(\Delta_\ell^{\rm v}T^{\rm v}_{v_L}[\nabla_\varepsilon\bar u^{\rm h}]_\varepsilon|\Delta_\ell^{\rm v}a\big)_{L^2}\big|\,dt'\\
&\lesssim\sum_{|\ell'-\ell|\le4}d_{\ell'}2^{-\frac{\ell'}{2}}\|v_L\|^{\frac12}_{\widetilde L^\infty_t(B^{0,\frac12})}\|\nabla_{\rm h}v_L\|^{\frac12}_{\widetilde L^2_t(B^{0,\frac12})}\|\nabla_{\rm h}\Delta_\ell^{\rm v}a\|^{\frac12}_{L^2_t(\mathcal B^{0,\frac12})}\\
&\qquad\qquad\times\Big(\int_0^t f_2(t')e^{-4\lambda_2\int_0^{t'} f_2(\tau)\,d\tau}\,dt'\Big)^{\frac14}\Big(\int_0^t f_2(t')\|\Delta_\ell^{\rm v}a\|_{L^2}^2\,dt'\Big)^{\frac14}\\
&\lesssim d_\ell^2 2^{-\ell}\lambda_2^{-\frac14}\|v_L\|^{\frac12}_{\widetilde L^\infty_t(\mathcal B^{0,\frac12})}\|\nabla_{\rm h}v_L\|^{\frac12}_{\widetilde L^2_t(\mathcal B^{0,\frac12})}\|a\|^{\frac12}_{\widetilde L^2_{t,f_2}(B^{0,\frac12})}\|\nabla_{\rm h}a\|^{\frac12}_{\widetilde L^2_t(B^{0,\frac12})}.
\end{aligned}\tag{5.10}
$$

On the other hand, we observe that

$$
\begin{aligned}
&\int_0^t e^{-\lambda_2\int_0^{t'} f_2(\tau)\,d\tau}\big|\big(\Delta_\ell^{\rm v}\bar R^{\rm v}(v_L,[\nabla_\varepsilon\bar u^{\rm h}]_\varepsilon))|\Delta_\ell^{\rm v}a\big)_{L^2}\big|\,dt'\\
&\lesssim\sum_{\ell'\ge\ell-N_0}\int_0^t e^{-\lambda_2\int_0^{t'} f_2(\tau)\,d\tau}\|\Delta_{\ell'}^{\rm v}v_L\|_{L^2_{\rm v}(L^4_{\rm h})}\|S^{\rm v}_{\ell'+2}[\nabla_\varepsilon\bar u^{\rm h}]_\varepsilon\|_{L^\infty_{\rm v}(L^2_{\rm h})}\|\Delta_\ell^{\rm v}a\|_{L^2_{\rm v}(L^4_{\rm h})}\,dt'\\
&\lesssim\sum_{\ell'\ge\ell-N_0}\int_0^t e^{-\lambda_2\int_0^{t'} f_2(\tau)\,d\tau}\|\Delta_{\ell'}^{v}v_L\|^{\frac12}_{L^2}\|\Delta_{\ell'}^{v}\nabla_{\rm h}v_L\|^{\frac12}_{L^2}\|\nabla_\varepsilon\bar u^{\rm h}\|_{\mathcal B^{0,\frac12}}\|\Delta_\ell^{\rm v}a\|^{\frac12}_{L^2}\|\Delta_\ell^{\rm v}\nabla_{\rm h}a\|^{\frac12}_{L^2}\,dt'.
\end{aligned}
$$

Then along the same line to proof of (5.10), we arrive at

$$
\begin{aligned}
&\int_0^t e^{-\lambda_2\int_0^{t'} f_2(\tau)\,d\tau}\big|\big(\Delta_\ell^{\rm v}\bar R^{\rm v}(v_L,[\nabla_\varepsilon\bar u^{\rm h}]_\varepsilon))|\Delta_\ell^{\rm v}a\big)_{L^2}\big|\,dt'\\
&\lesssim d_\ell^2 2^{-\ell}\lambda_2^{-\frac14}\|v_L\|^{\frac12}_{\widetilde L^\infty_t(B^{0,\frac12})}\|\nabla_{\rm h}v_L\|^{\frac12}_{\widetilde L^2_t(\mathcal B^{0,\frac12})}\|a\|^{\frac12}_{\widetilde L^2_{t,f_2}(B^{0,\frac12})}\|\nabla_{\rm h}a\|^{\frac12}_{\widetilde L^2_t(\mathcal B^{0,\frac12})}.
\end{aligned}
$$

This together with (5.10) completes the proof of the lemma. □

Proof of Lemma 5.4. In view of (2.5), we get, by applying Bony's decomposition in the vertical variable that

$$
\begin{aligned}
\partial_3[p^{\rm h}]_\varepsilon=&\varepsilon2\sum_{i,j=1}^2(-\Delta_{\rm h})^{-1}\partial_i\partial_j\big([\partial_z\bar u^i]_\varepsilon[\bar u^j]_\varepsilon\big)\\
=&2\varepsilon\sum_{i,j=1}^2(-\Delta_{\rm h})^{-1}\partial_i\partial_j\big(T^{\rm v}_{[\partial_z\bar u^i]_\varepsilon}[\bar u^j]_\varepsilon+\bar R^{\rm v}([\partial_z\bar u^i]_\varepsilon,[\bar u^j]_\varepsilon)\big).
\end{aligned}\tag{5.11}
$$

We first observe from Bernstein inequality that for any $\theta\in[0,1/2[$

$$
\begin{aligned}
&\int_0^t\exp\Big(-\lambda_3\int_0^{t'} f_3(\tau)\,d\tau\Big)\big|\big((-\Delta_{\rm h})^{-1}\partial_i\partial_j\Delta_\ell^{\rm v}T^{\rm v}_{[\partial_z\bar u^i]_\varepsilon}[\bar u^j]_\varepsilon|\Delta_\ell^{\rm v}a\big)_{L^2}\big|\,dt'\\
&\lesssim\sum_{|\ell'-\ell|\le4}\int_0^t\exp\Big(-\lambda_3\int_0^{t'} f_3(\tau)\,d\tau\Big)\|S^{\rm v}_{\ell'-1}([\partial_z\bar u^i]_\varepsilon)\|_{L^\infty_{\rm v}(L^4_{\rm h})}\|\Delta^{\rm v}_{\ell'}([\bar u^j]_\varepsilon)\|_{L^2_{\rm v}(L^4_{\rm h})}\|\Delta_\ell^{\rm v}a\|_{L^2}\,dt'\\
&\lesssim 2^{-\ell\theta}\sum_{|\ell'-\ell|\le4}\int_0^t\exp\Big(-\lambda_3\int_0^{t'} f_3(\tau)\,d\tau\Big)\|\partial_3\bar u^{\rm h}\|_{L^\infty_{\rm v}(\dot H^{\frac12}_{\rm h})}\|\Delta^{\rm v}_{\ell'}([|D_{\rm h}|^{\frac12}\bar u^j]_\varepsilon)\|_{L^2}\\
&\qquad\qquad\qquad\qquad\times\|\Delta_\ell^{\rm v}\partial_3a\|^{\theta}_{L^2}\|\Delta_\ell^{\rm v}a\|^{1-\theta}_{L^2}\,dt',
\end{aligned}
$$

where $|D_{\rm h}|^{\frac{1}{2}}$ denote Fourier multiplier with symbol $|\xi_{\rm h}|^{\frac{1}{2}}$. Then applying Hölder's inequality yields

$$
\begin{aligned}
&\int_0^t \exp\Big(-\lambda_3\int_0^{t'} f_3(\tau)\,d\tau\Big)\big|\big((-\Delta_{\rm h})^{-1}\partial_i\partial_j\Delta_\ell^{\rm v}T^{\rm v}_{[\partial_3\bar{u}^i]_\varepsilon}[\bar{u}^j]_\varepsilon|\Delta_\ell^{\rm v}a\big)_{L^2}\big|\,dt'\\
&\lesssim\varepsilon^{-\theta}2^{-\ell\theta}\sum_{|\ell'-\ell|\leq 4}2^{-\ell'(\frac{1}{2}-\theta)}\int_0^t d_{\ell'}(t')\exp\Big(-\lambda_3\int_0^{t'} f_3(\tau)\,d\tau\Big)\||\partial_z\bar{u}^{\rm h}\|_{\mathcal{B}^{\frac{1}{2},\frac{1}{2}}}\\
&\qquad\qquad\qquad\qquad\times\|\bar{u}^{\rm h}\|_{\mathcal{B}^{\frac{1}{2},\frac{1}{2}-\theta}}\|\Delta_\ell^{\rm v}\partial_3 a\|_{L^2}^{\theta}\|\Delta_\ell^{\rm v}a\|_{L^2}^{1-\theta}\,dt'\\
&\lesssim\varepsilon^{-\theta}2^{-\ell\theta}\sum_{|\ell'-\ell|\leq 4}d_{\ell'}2^{-\ell'(\frac{1}{2}-\theta)}\Big(\int_0^t f_3(t')\exp\Big(-2\lambda_3\int_0^{t'} f_3(\tau)\,d\tau\Big)\,dt'\Big)^{\frac{1}{2}}\\
&\qquad\times\Big(\int_0^t\|\bar{u}^{\rm h}\|^{\frac{2}{1-\theta}}_{\mathcal{B}^{\frac{1}{2},\frac{1}{2}-\theta}}\|\Delta_\ell^{\rm v}a\|_{L^2}^2\,dt'\Big)^{\frac{1-\theta}{2}}\|\Delta_\ell^{\rm v}\partial_3 a\|^{\theta}_{L^2_t(L^2)}\\
&\lesssim\varepsilon^{-\theta}d_\ell^2 2^{-\ell}\lambda_3^{-\frac{1}{2}}\|a\|^{1-\theta}_{\widetilde{L}^2_{t,f_3}(B^{0,\frac{1}{2}})}\|\partial_3 a\|^{\theta}_{\widetilde{L}^2_t(B^{0,\frac{1}{2}})}.
\end{aligned}
$$

Along the same line, we have

$$
\begin{aligned}
&\int_0^t \exp\Big(-\lambda_3\int_0^{t'} f_3(\tau)\,d\tau\Big)\big|\big((-\Delta_{\rm h})^{-1}\partial_i\partial_j\Delta_\ell^{\rm v}(\bar{R}^{\rm v}([\partial_3\bar{u}^i]_\varepsilon,[\bar{u}^j]_\varepsilon)|\Delta_\ell^{\rm v}a\big)_{L^2}\big|\,dt'\\
&\lesssim\sum_{\ell'\geq\ell-N_0}\int_0^t \exp\Big(-\lambda_3\int_0^{t'} f_3(\tau)\,d\tau\Big)\|\Delta_{\ell'}^{\rm v}([\partial_z\bar{u}^i]_\varepsilon)\|_{L^2_{\rm v}(L^4_{\rm h})}\|S^{\rm v}_{\ell'+2}([\bar{u}^j]_\varepsilon)\|_{L^\infty_{\rm v}(L^4_{\rm h})}\|\Delta_\ell^{\rm v}a\|_{L^2}\,dt'\\
&\lesssim\varepsilon^{-\theta}2^{-\ell\theta}\sum_{\ell'\geq\ell-N_0}d_{\ell'}2^{-\ell'(\frac{1}{2}-\theta)}\int_0^t\exp\Big(-\lambda_3\int_0^{t'} f_3(\tau)\,d\tau\Big)\|\partial_3\bar{u}^{\rm h}\|_{\mathcal{B}^{\frac{1}{2},\frac{1}{2}}}\\
&\qquad\qquad\qquad\qquad\times\|\bar{u}^{\rm h}\|_{\mathcal{B}^{\frac{1}{2},\frac{1}{2}-\theta}}\|\Delta_\ell^{\rm v}\partial_3 R^3\|_{L^2}^{\theta}\|\Delta_\ell^{\rm v}R^3\|_{L^2}^{1-\theta}\,dt',
\end{aligned}
$$

which together with the fact that $\theta\in[0,1/2[$ implies that

$$
\begin{aligned}
&\int_0^t \exp\Big(-\lambda_3\int_0^{t'} f_3(\tau)\,d\tau\Big)\big|\big((-\Delta_{\rm h})^{-1}\partial_i\partial_j\Delta_\ell^{\rm v}(\bar{R}^{\rm v}([\partial_3\bar{u}^i]_\varepsilon,[\bar{u}^j]_\varepsilon)|\Delta_\ell^{\rm v}a\big)_{L^2}\big|\,dt'\\
&\lesssim d_\ell\varepsilon^{-\theta}2^{-\ell(\theta+\frac{1}{2})}\|a\|^{1-\theta}_{\widetilde{L}^2_{t,f_3}(B^{0,\frac{1}{2}})}\|\partial_3 a\|^{\theta}_{\widetilde{L}^2_t(B^{0,\frac{1}{2}})}\sum_{\ell'\geq\ell-N_0}d_{\ell'}2^{-\ell'(\frac{1}{2}-\theta)}\\
&\lesssim\varepsilon^{-\theta}d_\ell^2 2^{-\ell}\lambda_3^{-\frac{1}{2}}\|a\|^{1-\theta}_{\widetilde{L}^2_{t,f_3}(B^{0,\frac{1}{2}})}\|\partial_3 a\|^{\theta}_{\widetilde{L}^2_t(B^{0,\frac{1}{2}})}.
\end{aligned}
$$

$\square$

6. The case of fluid evolving between two parallel plans with Dirichlet boundary condition

The goal of this section is to study the global well-posedness result to the anisotropic Navier-Stokes system with only vertical viscosity, (1.5). To do it, let us first present the following lemmas:

Lemma 6.1. *Let f satisfy $f|_\Omega=0$ and $\partial_3 f\in\mathcal{B}^1_h(\mathbb{R}^3)$ (see Definition 3.3). Then one has*

$$\|f\|_{L^\infty}\leq C\|\partial_3 f\|_{\mathcal{B}^1_{\rm h}}.$$

Proof. We first get, by applying Lemma 3.1, that

$$\begin{aligned}\|f\|_{L^\infty} &\leq \sum_{k\in\mathbb{Z}} \|\Delta_k^{\rm h} f\|_{L^\infty} \lesssim \sum_{k\in\mathbb{Z}} 2^k \|\Delta_k^{\rm h} f\|_{L^\infty_{\rm v}(L^2_{\rm h})}\\ &\lesssim \sum_{k\in\mathbb{Z}} 2^k \|\Delta_k^{\rm h} f\|_{L^2}^{\frac12} \|\Delta_k^{\rm h}\partial_3 f\|_{L^2}^{\frac12}.\end{aligned}$$

Recalling that the Poincar'e inequality holds in the strip Ω with Dirichlet boundary condition

$$\|f\|_{L^2(\Omega)} \leq C\|\partial_z f\|_{L^2(\Omega)}. \tag{6.1}$$

So that we obtain

$$\|f\|_{L^\infty} \lesssim \sum_{k\in\mathbb{Z}} 2^k \|\Delta_k^{\rm h}\partial_3 f\|_{L^2} \lesssim \|\partial_3 f\|_{\mathcal{B}^1_{\rm h}}.$$

This completes the proof of the lemma. □

Lemma 6.2. *Let $u = (u^{\rm h}, u^3)$ be a smooth solenoidal vector field. Then one has*

$$\int_0^t \mid \big(\Delta_k^{\rm h}(u\cdot\nabla u)|\Delta_k^{\rm h} u\big)_{L^2} \mid\, dt' \lesssim d_k^2 2^{-4k} \|u\|_{\widetilde{L}^\infty_t(\mathcal{B}^2_{\rm h})}\big(\|\partial_3 u\|^2_{L^2_t(L^2)} + \|\partial_3 u\|^2_{\widetilde{L}^2_t(\mathcal{B}^2)}\big).$$

Proof. To estimate the trilinear term $\big(\Delta_k^{\rm h}(u\cdot\nabla u)|\Delta_k^{\rm h} u\big)_{L^2}$, we have to distinguish the terms containing the horizontal derivatives with the term containing the vertical derivative. We write

$$\begin{gathered}\big(\Delta_k^{\rm h}(u\cdot\nabla u)|\Delta_k^{\rm h} u\big)_{L^2} = I_{\rm h} + I_{\rm v} \quad \text{with}\\ I_{\rm h} \stackrel{\rm def}{=} \big(\Delta_k^{\rm h}(u^{\rm h}\cdot\nabla_{\rm h} u)|\Delta_k u\big)_{L^2} \quad \text{and} \quad I_{\rm v} \stackrel{\rm def}{=} \big(\Delta_k^{h}(u^3\partial_3 u)|\Delta_k^{\rm h} u\big).\end{gathered} \tag{6.2}$$

We start with the estimate of $I_{\rm h}$. Applying Bony's decomposition (3.5) in the horizontal variables gives

$$u^{\rm h}\cdot\nabla_{\rm h} u = T^{\rm h}_{u^{\rm h}}\nabla_{\rm h} u + \bar{R}(u^{\rm h}, \nabla_{\rm h} u).$$

Then through a commutator's argument, we write

$$\begin{aligned}\int_0^t \big(\Delta_k^{\rm h} T^{\rm h}_{u^{\rm h}}\nabla_{\rm h} u|\Delta_k^{\rm h} u\big)_{L^2}\, dt' = \sum_{|k'-k|\leq 4} \Big(&\int_0^t \big([\Delta_k^{\rm h}, S^{\rm h}_{k'-1}u^{\rm h}]\cdot\Delta^{\rm h}_{k'} u|\Delta_k^{\rm h} u\big)_{L^2}\, dt'\\ &+ \int_0^t \big((S^{\rm h}_{k'-1}u^{\rm h} - S^{\rm h}_{k-1}u^{\rm h})\cdot\Delta^{\rm h}_{k'}\Delta^{\rm h}_k u|\Delta_k^{\rm h} u\big)_{L^2}\, dt'\Big)\\ &+ \int_0^t \big(S^{\rm h}_{k-1}u^{\rm h}\cdot\nabla_{\rm h}\Delta_k^{\rm h} u|\Delta_k^{\rm h} u\big)_{L^2}\, dt'\\ &\stackrel{\rm def}{=} I^1_{\rm h}(t) + I^2_{\rm h}(t) + I^3_{\rm h}(t).\end{aligned}$$

it follows from standard commutator's estimate (see [6]) and Lemma 3.1 that

$$\begin{aligned}\|[\Delta_k^h, S^{\rm h}_{k'-1}u^{\rm h}]\nabla_{\rm h}\Delta^{\rm h}_{k'} u^{\rm h}\|_{L^2} \leq& C2^{-k}\|\nabla_{\rm h} S^{\rm h}_{k'-1}u^{\rm h}\|_{L^\infty}\|\Delta^{\rm h}_{k'}\nabla_{\rm h} u^h\|_{L^2}\\ \leq& C\|\nabla_{\rm h} u\|_{L^\infty}\|\Delta^{\rm h}_{k'} u\|_{L^2},\end{aligned}$$

so that we deduce from (6.1) and Lemma 6.1 that

$$\begin{aligned}\mid I^1_{\rm h}(t)\mid \lesssim& \sum_{|k'-k|\leq 4} \|\nabla_{\rm h} u\|_{L^2_t(L^\infty)}\|\Delta^{\rm h}_{k'} u\|_{L^\infty_t(L^2)}\|\Delta^{\rm h}_{k'}\partial_3 u\|_{L^2_t(L^2)}\\ \lesssim& d_k^2 2^{-4k}\|u\|_{\widetilde{L}^\infty_t(\mathcal{B}^2_{\rm h})}\|\partial_3 u\|^2_{\widetilde{L}^2_t(\mathcal{B}^2)}.\end{aligned}$$

The same estimate holds for $I^2_{\rm h}(t)$.

To handle $I_{\mathrm{h}}^3(t)$, we first get, by using integrating by parts and using $\operatorname{div} u=0$, that

$$\begin{aligned}I_{\mathrm{h}}^3(t)=&-\frac{1}{2}\int_0^t\big(S_{k-1}\operatorname{div}_{\mathrm{h}}u^{\mathrm{h}}\Delta_k^{\mathrm{h}}u|\Delta_k^{\mathrm{h}}u\big)_{L^2}\,dt'\\=&\frac{1}{2}\int_0^t\big(S_{k-1}\partial_3u^3\Delta_k^{\mathrm{h}}u|\Delta_k^{\mathrm{h}}u\big)_{L^2}\,dt',\end{aligned}$$

from which, we infer

$$\begin{aligned}|\ I_{\mathrm{h}}^3(t)\ |\lesssim&\int_0^t\|S_{k-1}^{\mathrm{h}}\partial_3u^3\|_{L^2_{\mathrm{v}}(L^\infty_{\mathrm{h}})}\|\Delta_k^{\mathrm{h}}u\|_{L^4_{\mathrm{v}}(L^2_{\mathrm{h}})}^2\,dt'\\\lesssim&\int_0^t\|\partial_3u^3\|_{\mathcal{B}^1_{\mathrm{h}}}\|\Delta_k^{\mathrm{h}}u\|_{L^2}^{\frac{3}{2}}\|\Delta_k^{\mathrm{h}}\partial_3u\|_{L^2}^{\frac{1}{2}}\,dt'.\end{aligned}$$

Applying (6.1) gives

$$\begin{aligned}|\ I_{\mathrm{h}}^3(t)\ |\lesssim&\int_0^t\|\partial_3u\|_{L^2}^{\frac{1}{2}}\|\partial_3u^3\|_{\mathcal{B}^2_{\mathrm{h}}}^{\frac{1}{2}}\|\Delta_k^{\mathrm{h}}u\|_{L^2}\|\Delta_k^{\mathrm{h}}\partial_3u\|_{L^2}\,dt'\\\lesssim&\|\partial_3u\|_{L^2_t(L^2)}^{\frac{1}{2}}\|\partial_3u^3\|_{L^2_t(\mathcal{B}^2_{\mathrm{h}})}^{\frac{1}{2}}\|\Delta_k^{\mathrm{h}}u\|_{L^\infty_t(L^2)}\|\Delta_k^{\mathrm{h}}\partial_3u\|_{L^2_t(L^2)}\\\lesssim&d_k^22^{-4k}\|u\|_{\widetilde{L}^\infty_t(\mathcal{B}^2_{\mathrm{h}})}\|\partial_3u\|_{L^2_t(L^2)}^{\frac{1}{2}}\|\partial_3u\|_{\widetilde{L}^2_t(\mathcal{B}^2)}^{\frac{3}{2}}.\end{aligned}$$

As a result, it comes out

$$|\int_0^t\big(\Delta_k^{\mathrm{h}}T^{\mathrm{h}}_{u^{\mathrm{h}}}\nabla_{\mathrm{h}}u|\Delta_k^{\mathrm{h}}u\big)_{L^2}\,dt'\ |\lesssim d_k^22^{-4k}\|u\|_{\widetilde{L}^\infty_t(\mathcal{B}^2_{\mathrm{h}})}\big(\|\partial_3u\|_{L^2_t(L^2)}^2+\|\partial_3u\|_{\widetilde{L}^2_t(\mathcal{B}^2)}^2\big).\tag{6.3}$$

Whereas observing that

$$\begin{aligned}|\int_0^t\big(\Delta_k^{\mathrm{h}}\bar{R}^{\mathrm{h}}(u^{\mathrm{h}},\nabla_{\mathrm{h}}u)|\Delta_k^{\mathrm{h}}u\big)_{L^2}\,dt'\ |\lesssim&\sum_{k'\geq k-N_0}\int_0^t\|\Delta_{k'}^{\mathrm{h}}u^{\mathrm{h}}\|_{L^2}\|S_{k'-1}^{\mathrm{h}}\nabla_{\mathrm{h}}u\|_{L^\infty}\|\Delta_k^{\mathrm{h}}u\|_{L^2}\,dt'\\\lesssim&\sum_{k'\geq k-N_0}\|\nabla_{\mathrm{h}}u\|_{L^2_t(L^\infty)}\|\Delta_{k'}^{\mathrm{h}}u^{\mathrm{h}}\|_{L^\infty_t(L^2)}\|\Delta_k^{\mathrm{h}}u\|_{L^2_t(L^2)},\end{aligned}$$

which together with (6.1) and Lemma 6.1 ensures that Whereas observing that

$$\begin{aligned}|\int_0^t\big(\Delta_k^{\mathrm{h}}\bar{R}^{\mathrm{h}}(u^{\mathrm{h}},\nabla_{\mathrm{h}}u)|\Delta_k^{\mathrm{h}}u\big)_{L^2}\,dt'\ |\lesssim&d_k2^{-2k}\sum_{k'\geq k-N_0}d_{k'}2^{-2k'}\|u\|_{\widetilde{L}^\infty_t(\mathcal{B}^2_{\mathrm{h}})}\|\partial_3u\|_{\widetilde{L}^2_t(\mathcal{B}^2_{\mathrm{h}})}^2\\\lesssim&d_k^22^{-4k}\|u\|_{\widetilde{L}^\infty_t(\mathcal{B}^2_{\mathrm{h}})}\|\partial_3u\|_{\widetilde{L}^2_t(\mathcal{B}^2)}^2.\end{aligned}$$

Along with (6.3), we infer

$$|\ I_{\mathrm{h}}\ |\lesssim d_k^22^{-4k}\|u\|_{\widetilde{L}^\infty_t(\mathcal{B}^2_{\mathrm{h}})}\big(\|\partial_3u\|_{L^2_t(L^2)}^2+\|\partial_3u\|_{\widetilde{L}^2_t(\mathcal{B}^2)}^2\big).\tag{6.4}$$

To handle $I_{\mathrm{v}}(t)$, we first get, by applying Bony's decomposition (3.5), that

$$u^3\partial_3u=T^{\mathrm{h}}_{u^3}\partial_3u+\bar{R}^{\mathrm{h}}(u^3,\partial_3u).$$

We first observe from Lemma 6.1 that

$$\begin{aligned}\int_0^t|\ \big(\Delta_k^{\mathrm{h}}T^{\mathrm{h}}_{u^3}\partial_3u|\Delta_k^{\mathrm{h}}u\big)_{L^2}\ |\ dt'\lesssim&\sum_{|k'-k|\leq4}\int_0^t\|S_{k'-1}^{\mathrm{h}}u^3\|_{L^\infty}\|\Delta_{k'}^{\mathrm{h}}\partial_3u\|_{L^2}\|\Delta_k^{\mathrm{h}}u\|_{L^2}\,dt'\\\lesssim&\sum_{|k'-k|\leq4}\|\partial_3u^3\|_{L^2_t(\mathcal{B}^1)}\|\Delta_{k'}^{\mathrm{h}}\partial_3u\|_{L^2_t(L^2)}\|\Delta_k^{\mathrm{h}}u\|_{L^\infty_t(L^2)}\\\lesssim&d_k^22^{-4k}\|u\|_{\widetilde{L}^\infty_t(\mathcal{B}^2_{\mathrm{h}})}|\partial_3u\|_{L^2_t(L^2)}^{\frac{1}{2}}\|\partial_3u\|_{\widetilde{L}^2_t(\mathcal{B}^2)}^{\frac{3}{2}}.\end{aligned}$$

Similarly, we get, by applying (6.1), that

$$
\begin{aligned}
&\int_0^t |\big(\Delta_k^{\rm h} R^{\rm h}(u^3,\partial_3 u)|\Delta_k^{\rm h} u\big)_{L^2}|\, dt'\\
&\lesssim \sum_{k'\geq k-N_0}\int_0^t \|\Delta_{k'}^{\rm h} u^3\|_{L^\infty_{\rm v}(L^2)}\|S^{\rm h}_{k'+2}\partial_3 u\|_{L^2_{\rm v}(L^\infty_{\rm h})}\|\Delta_k^{\rm h} u\|_{L^2}\, dt'\\
&\lesssim \sum_{k'\geq k-N_0}\int_0^t \|\partial_3 u\|_{\mathcal{B}^1_{\rm h}}\|\Delta_{k'}^{\rm h} u^3\|_{L^2}^{\frac12}\|\Delta_{k'}^{\rm h}\partial_3 u^3\|_{L^2}^{\frac12}\|\Delta_k^{\rm h} u\|_{L^2}\, dt'\\
&\lesssim \sum_{k'\geq k-N_0}\|\partial_3 u^3\|_{L^2_t(\mathcal{B}^1)}\|\Delta_{k'}^{\rm h}\partial_3 u\|_{L^2_t(L^2)}\|\Delta_k^{\rm h} u\|_{L^\infty_t(L^2)}\\
&\lesssim d_k^2 2^{-4k}\|u\|_{\widetilde{L}^\infty_t(\mathcal{B}^2_{\rm h})}|\partial_3 u\|_{L^2_t(L^2)}^{\frac12}\|\partial_3 u\|_{\widetilde{L}^2_t(\mathcal{B}^2)}^{\frac32}.
\end{aligned}
$$

This leads to

$$
|\, I_{\rm v}\,| \lesssim d_k^2 2^{-4k}\|u\|_{\widetilde{L}^\infty_t(\mathcal{B}^2_{\rm h})}|\partial_3 u\|_{L^2_t(L^2)}^{\frac12}\|\partial_3 u\|_{\widetilde{L}^2_t(\mathcal{B}^2)}^{\frac32}.
$$

Together with (6.4), we complete the proof of the lemma. □

Let now present the proof of Theorem 1.2.

Proof of Theorem 1.2. By taking L^2 inner product of the momentum equation of (1.5), we get

$$
\frac12\frac{d}{dt}\|u(t)\|_{L^2}^2+\nu_{\rm v}\|\partial_3 u\|_{L^2}^2=0.
$$

Integrating the above inequality over $[0,t]$ leads to

$$
\|u\|_{L^\infty(L^2)}+\nu^{\frac12}\|\partial_3 u\|_{L^2_t(L^2)}\leq 2\|u_0\|_{L^2}. \tag{6.5}
$$

On the other hand, by applying the operator Δ_k^h to the momentum equation of (1.5) and performing L^2 inner product with the resulting equation, we obtain

$$
\frac12\frac{d}{dt}\|\Delta_k^{\rm h} u\|_{L^2}^2+\nu_{\rm v}\|\partial_3\Delta_k^h u\|_{L^2}^2\leq |\, \big(\Delta_k^{\rm h}(u\cdot\nabla u)|\Delta_k^{\rm h} u\big)_{L^2}\,|\,.
$$

Integrating the above inequality over $[0,t]$ and applying Lemma 6.2 yields

$$
\begin{aligned}
\|\Delta_k^{\rm h} u\|_{L^\infty_t(L^2)}^2+&\nu_{\rm v}\|\partial_3\Delta_k^h u\|_{L^2_t(L^2)}^2\\
&\leq \|\Delta_k^{\rm h} u_0\|_{L^2}^2+Cd_k^2 2^{-4k}\|u\|_{\widetilde{L}^\infty_t(\mathcal{B}^2_{\rm h})}\big(\|\partial_3 u\|_{L^2_t(L^2)}^2+\|\partial_3 u\|_{\widetilde{L}^2_t(\mathcal{B}^2)}^2\big).
\end{aligned}
$$

Taking square root of the above inequality and multiplying the resulting inequality by 2^{2k}, and then summing up the resulting inequality for $k\in\mathbb{Z}$, we achieve

$$
\|u\|_{\widetilde{L}^\infty_t(\mathcal{B}^2_{\rm h})}+\nu_{\rm v}^{\frac12}\|\partial_3 u\|_{\widetilde{L}^2_t(\mathcal{B}^2)}\leq C\Big(\|u_0\|_{\mathcal{B}^2_{\rm h}}+\|u\|_{\widetilde{L}^\infty_t(\mathcal{B}^2_{\rm h})}^{\frac12}\big(\|\partial_3 u\|_{L^2_t(L^2)}+\|\partial_3 u\|_{\widetilde{L}^2_t(\mathcal{B}^2)}\big)\Big). \tag{6.6}
$$

Summing up (6.5) with (6.6) gives rise to

$$
\begin{aligned}
\|u\|_{L^\infty(L^2)}+&\|u\|_{\widetilde{L}^\infty_t(\mathcal{B}^2_{\rm h})}+\nu^{\frac12}\big(\|\partial_z u\|_{L^2_t(L^2)}+\|\partial_3 u\|_{\widetilde{L}^2_t(\mathcal{B}^2)}\big)\\
&\leq C\Big(\|u_0\|_{L^2}+\|u_0\|_{\mathcal{B}^2_{\rm h}}+\|u\|_{\widetilde{L}^\infty_t(\mathcal{B}^2_{\rm h})}^{\frac12}\big(\|\partial_3 u\|_{L^2_t(L^2)}+\|\partial_3 u\|_{\widetilde{L}^2_t(\mathcal{B}^2)}\big)\Big).
\end{aligned} \tag{6.7}
$$

Let us denote

$$
T^\star \stackrel{\rm def}{=} \big\{T<T^*,\ 4C^2\|u\|_{\widetilde{L}^\infty_t(\mathcal{B}^2_{\rm h})}\leq \nu_{\rm v}\ \big\}. \tag{6.8}
$$

Then for $t \leq T^\star$, we deduce from (6.7) that

$$\|u\|_{L^\infty(L^2)} + \|u\|_{\widetilde{L}^\infty_t(\mathcal{B}^2_{\rm h})} + \frac{\nu^{\frac{1}{2}}}{2}\big(\|\partial_z u\|_{L^2_t(L^2)} + \|\partial_3 u\|_{\widetilde{L}^2_t(\mathcal{B}^2)}\big) \leq C\big(\|u_0\|_{L^2} + \|u_0\|_{\mathcal{B}^2_{\rm h}}\big). \tag{6.9}$$

In particular, under the assumption of (1.6), for $t \leq T^\star$, we have

$$\|u\|_{L^\infty(L^2)} + \|u\|_{\widetilde{L}^\infty_t(\mathcal{B}^2_{\rm h})} + \frac{\nu^{\frac{1}{2}}}{2}\big(\|\partial_z u\|_{L^2_t(L^2)} + \|\partial_3 u\|_{\widetilde{L}^2_t(\mathcal{B}^2)}\big) \leq \frac{\nu_{\rm v}}{8C^2}.$$

This contradicts with the definition of $T^\star$ given by (6.8). This in turn shows that $T^\star = T^* = \infty$. We complete the proof of Theorem 1.2. $\square$

References

[1] J. D. Avrin, Large-eigevalue global existence and regularity results for the Navier-Stokes equations, J. Differential Equations, 127, no. 2, 365-390, 1996.

[2] A.Babin, A. Mahalov, B.Nicolaenko, Regularity and integrability of 3D Euler and Navier- Stokes equations for rotating fluids, Asymptotic Analysis, 15, 103-150, 1997.

[3] A.Babin, A. Mahalov, B.Nicolaenko, Global Regularity of 3D Rotating Navier-Stokes Equations for Resonant Domains, Indiana Univ. Math. Journal, Vol. 48, no. 3, 1133-1176, 1999.

[4] J. M. Bony, Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires, *Annales Scientifiques de l' École Normale Supérieure*, **14**, 1981, pages 209–246.

[5] M. Cannone, Y. Meyer and F. Planchon, Solutions autosimilaires des équations de Navier-Stokes, Séminaire "Équations aux Dérivées Partielles" de l'École polytechnique, Exposé VIII, 1993–1994.

[6] H. Bahouri, J. Y. Chemin and R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations,* Grundlehren der Mathematischen Wissenschaften, Springer 2011.

[7] J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier, Mathematical geophysics. An introduction to rotating fluids and the Navier-Stokes equations. Oxford Lecture Series in Mathematics and its Applications, 32. The Clarendon Press, Oxford University Press, Oxford, 2006.

[8] J.-Y. Chemin, I. Gallagher and M. Paicu, Global regularity for some classes of large solutions to the Navier-Stokes equations, preprint, 2008.

[9] J.-Y. Chemin, Remarques sur l'existence globale pour le systéme de Navier-Stokes incompressible, *SIAM J. Math. Anal.*, **23** (1992), 20-28.

[10] J.-Y. Chemin and I. Gallagher, On the global wellposedness of the 3-D Navier-Stokes equations with large initial data, *Annales de l'École Normale Supérieure* (**39**), 2006, pages 679–698.

[11] J.-Y. Chemin and I. Gallagher, Wellposedness and stability results for the Navier-Stokes equations in $\mathbf{R}^3$ *to appear in Annales de l'Institut Henri Poincaré, Analyse Non Linéaire*

[12] J.-Y. Chemin and I. Gallagher, Large, global solutions to the Navier-Stokes equations, slowly varying in one direction, *to appear in Transactions of the Americal Mathematical Society*

[13] J.-Y. Chemin, I. Gallagher and M. Paicu, Global regularity for some classes of large solutions to the Navier-Stokes equations, Ann. of Math. (2), 173 (2011), 983-1012.

[14] J.-Y. Chemin and P. Zhang, Remarks on the global solutions of 3-D Navier-Stokes system with one slow variable *Comm. Partial Differential Equations*, **40** (2015), 878-896.

[15] H. Fujita and T. Kato, On the Navier-Stokes initial value problem I, *Archive for Rational Mechanics and Analysis*, **16** (1964), pages 269–315.

[16] Y. Giga and T. Miyakawa, Solutions in L^r of the Navier-Stokes initial value problem, *Arch. Rational Mech. Anal.* **89** (1985), no. 3, pages 267–281.

[17] D. Iftimie, G. Raugel and G. R. Sell, Navier-Stokes equations in thin 3D domains with Navier boundary conditions, *Indiana University Mathematical Journal*, **56** (2007), pages 1083–1156.

[18] T. Kato, *Strong L^p-solutions of the Navier-Stokes equation in $\mathbb{R}^m$, with applications to weak solutions.* Mathematische Zeitschrift, **187** (1984), pages 471-480 .

[19] H. Koch and D. Tataru, Well–posedness for the Navier–Stokes equations, *Advances in Mathematics*, **157** (2001), pages 22–35.

[20] S. Leibovich, A. Mahalov, E. Titi, Invariant helical subspaces for the Navier-Stokes equations. *Arch. Rational Mech. Anal.* **112** (1990), no. 3, pages 193–222.

[21] P.-G. Lemarié-Rieusset, Recent developments in the Navier-Stokes problem. Chapman and Hall/CRC Research Notes in Mathematics, **43**, 2002.

[22] J. Leray, Essai sur le mouvement d'un liquide visqueux emplissant l'espace, *Acta Matematica*, **63** (1933), pages 193–248.

[23] J. Leray. Etude de diverses équations intégrales non linéaires et de quelques problèmes que pose l'hydrodynamique. *J. Math. Pures. Appl.* **12** (1933), pages 1–82.
[24] Yanlin Liu and P. Zhang, Global well-posedness of 3-D anisotropic Navier-Stokes system with large vertical viscous coefficient, arXiv:1708.04731.
[25] A. Mahalov and B. Nicolaenko, Global solvability of three-dimensional Navier-Stokes equations with uniformly high initial vorticity, (Russian. Russian summary) *Uspekhi Mat. Nauk* **58**, 2003, pages 79–110; translation in *Russian Math. Surveys* **58** (2003), pages 287–318.
[26] Y. Meyer. *Wavelets, Paraproducts and Navier–Stokes.* Current Developments in Mathematics, International Press, Cambridge, Massachussets, 1996.
[27] M. Paicu, 'Equation anisotrope de Navier-Stokes dans des espaces critiques, *Rev. Mat. Iberoamericana*, **21** (2005), 179-235.
[28] M. Paicu and P. Zhang, Global solutions to the 3-D incompressible anisotropic Navier-Stokes system in the critical spaces, *Comm. Math. Phys.* **307** (2011), 713-759.
[29] G. Ponce, R. Racke, T. Sideris, E. Titi, Global stability of large solutions to the 3D Navier-Stokes equations. *Comm. Math. Phys.* **159** (1994), no. 2, pages 329–341.
[30] G. Raugel and G. R. Sell, Navier-Stokes equations on thin 3D domains. I. Global attractors and global regularity of solutions, *Journal of the American Mathematical Society*, **6** (1993), pages 503–568.
[31] M. Ukhovskii, V. Iudovich, Axially symmetric flows of ideal and viscous fluids filling the whole space. *Prikl. Mat. Meh.* 32 59–69 (Russian); translated as *J. Appl. Math. Mech.* **32** (1968) pages 52–61.
[32] F. Weissler, The Navier-Stokes Initial Value Problem in L^p, *Archiv for Rational Mechanics and Analysis*, **74** (1980), pages 219-230.

(M. PAICU) Université Bordeaux, Institut de Mathématiques de Bordeaux, F-33405 Talence Cedex, France
E-mail address: marius.paicu@math.u-bordeaux.fr

(P. ZHANG) Academy of Mathematics & Systems Science and Hua Loo-Keng Key Laboratory of Mathematics, The Chinese Academy of Sciences, Beijing 100190, CHINA, and School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, CHINA.
E-mail address: zp@amss.ac.cn