ON THE HYDROSTATIC APPROXIMATION OF THE NAVIER-STOKES EQUATIONS IN A THIN STRIP

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Abstract. In this paper, we first prove the global well-posedness of a scaled anisotropic Navier-Stokes system and the hydrostatic Navier-Stokes system in a 2-D striped domain with small analytic data in the tangential variable. Then we justify the limit from the anisotropic Navier-Stokes system to the hydrostatic Navier-Stokes system with analytic data.

Keywords: Incompressible Navier-Stokes Equations, Hydrostatic approximation, Radius of analyticity.

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1. Introduction

This paper is concerned with the study of the Navier-Stokes system in a thin-striped domain and the hydrostatic approximation of these equations when the depth of the domain and the viscosity converge to zero simultaneously in a related way. This is a classical model in geophysical fluid dynamics where the vertical dimension of the domain is very small compared with the horizontal dimension of the domain. In this case, the rescaled viscosity is not isotropic and we have to use the anisotropic Navier-Stokes system with a "turbulent" viscosity. The formal limit thus obtained is the hydrostatic Navier-Stokes equations which are currently used as a standard model to describes the atmospheric flows and also oceanic flows in oceanography (see [23, 24]).

The other motivation of this paper comes from the boundary layer theory obtained by vanishing viscosity limit of Navier-Stokes system with Dirichlet boundary condition. The governing equation to describe the motion of the fluid in this thin boundary layer was derived by Prandtl [25] in 1904 in order to explain the disparity between the boundary conditions verified by ideal fluid and viscous fluid with vanishing viscosity. Heuristically, these boundary layers are of amplitude $O(1)$ and of thickness $O(\sqrt{\nu})$ where $\nu = \varepsilon^2$ is the viscosity of the fluid. In order to focus only on the boundary layer, we shall consider here the Navier-Stokes equations in a thin strip, $\{(x,y) \in \mathbb{R}^2 : \ 0 < y < \varepsilon \}$, which is consistent with the physical parameters in geophysical flows.

When we consider Dirichlet boundary conditions on the top and the bottom of a 2-D striped domain, we are able to prove the global well-posedness of both the anisotropic Navier-Stokes system and the hydrostatic/Prandtl approximate equations when the initial data is small and analytic in the tangential variable. This should be regarded as a global Cauchy-Kowalevskaya theorem for small analytic data, which originates from [5]. The proof of this type of results requires the control of the loss of the radius of the analyticity of the solution and the estimate of the solution itself simultaneously. Taking the advantage of the Poincaré inequality in the strip, we are able to control the analyticity of the solution globally in time. We also rigorously prove the convergence of the anisotropic Navier-Stokes system to the hydrostatic/Prandtl

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equations in the natural framework of the analytic data in the tangential variable. We now present a precise description of the problem that we shall investigate.

We consider two-dimensional incompressible Navier-Stokes equations in a thin strip: $\mathcal{S}^\varepsilon \stackrel{\text{def}}{=} \big\{(x,y)\in\mathbb{R}^2:\ 0<y<\varepsilon\ \big\}$,

$$\left\{\begin{array}{l}\partial_t U + U\cdot\nabla U - \varepsilon^2\Delta U + \nabla P = 0 \quad \text{in } \ \mathcal{S}^\varepsilon\times]0,\infty[,\\ \operatorname{div} U = 0,\end{array}\right. \tag{1.1}$$

where $U(t,x,y)$ denotes the velocity of the fluid and $P(t,x,y)$ denotes the scalar pressure function which guarantees the divergence free condition of the velocity field U. We complement the system (1.1) with the non-slip boundary condition

$$U|_{y=0} = U|_{y=\varepsilon} = 0,$$

and the initial condition

$$U|_{t=0} = \Big(u_0\big(x,\frac{y}{\varepsilon}\big), \varepsilon v_0\big(x,\frac{y}{\varepsilon}\big)\Big) = U_0^\varepsilon \quad \text{in } \ \mathcal{S}^\varepsilon.$$

As in [2, 16], we write

$$U(t,x,y) = \Big(u^\varepsilon\big(t,x,\frac{y}{\varepsilon}\big), \varepsilon v^\varepsilon\big(t,x,\frac{y}{\varepsilon}\big)\Big) \quad \text{and} \quad P(t,x,y) = p^\varepsilon\big(t,x,\frac{y}{\varepsilon}\big). \tag{1.2}$$

Let $\mathcal{S} \stackrel{\text{def}}{=} \big\{(x,y)\in\mathbb{R}^2:\ 0<y<1\big\}$. Then the system (1.1) becomes the following scaled anisotropic Navier-Stokes system:

$$\left\{\begin{array}{l}\partial_t u^\varepsilon + u^\varepsilon\partial_x u^\varepsilon + v^\varepsilon\partial_y u^\varepsilon - \varepsilon^2\partial_x^2 u^\varepsilon - \partial_y^2 u^\varepsilon + \partial_x p^\varepsilon = 0 \ \text{ in } \mathcal{S}\times]0,\infty[,\\ \varepsilon^2\left(\partial_t v^\varepsilon + u^\varepsilon\partial_x v^\varepsilon + v^\varepsilon\partial_y v^\varepsilon - \varepsilon^2\partial_x^2 v^\varepsilon - \partial_y^2 v^\varepsilon\right) + \partial_y p^\varepsilon = 0,\\ \partial_x u^\varepsilon + \partial_y v^\varepsilon = 0,\\ \left(u^\varepsilon, v^\varepsilon\right)|_{t=0} = \left(u_0, v_0\right),\end{array}\right. \tag{1.3}$$

together with the boundary condition

$$\left(u^\varepsilon, v^\varepsilon\right)|_{y=0} = \left(u^\varepsilon, v^\varepsilon\right)|_{y=1} = 0. \tag{1.4}$$

Formally taking $\varepsilon\to 0$ in the system (1.3), we obtain the hydrostatic Navier-Stokes/Prandtl equations:

$$\left\{\begin{array}{l}\partial_t u + u\partial_x u + v\partial_y u - \partial_y^2 u + \partial_x p = 0 \ \text{ in } \mathcal{S}\times]0,\infty[,\\ \partial_y p = 0\\ \partial_x u + \partial_y v = 0,\\ u|_{t=0} = u_0,\end{array}\right. \tag{1.5}$$

together with the boundary condition

$$\left(u, v\right)|_{y=0} = \left(u, v\right)|_{y=1} = 0. \tag{1.6}$$

The goal of this paper is to justify the limit from the system (1.3) to the system (1.5). The first step is to establish the well-posedness of the two systems. Similar to the Prandtl equation, the nonlinear term $v\partial_y u$ in (1.5) will lead to one derivative loss in the x variable in the process of energy estimates. Thus, it is natural to work with analytic data in order to overcome this difficulty if we don't impose extra structural assumptions on the initial data [10, 26]. Indeed, for the data which is analytic in x, y variables, Sammartino and Caflisch [27] established the local well-posedness result of (1.5) in the upper half space. Later, the analyticity in y variable was removed by Lombardo, Cannone and Sammartino in [17]. The main argument used in [27, 17] is to apply the abstract Cauchy-Kowalewskaya (CK) theorem. We also mention a well-posedness result of Prandtl system for a class of data with Gevrey regularity [11]. Lately, for a class of convex data, Gérard-Varet, Masmoudi and Vicol [12] proved the well-posedness of the system (1.5) in the Gevrey class.

Our result complete the result of [17] in the sense that we obtain the global well-posedness for the hydrostatic Navier-Stokes equations in bi-dimensional strip with small analytic data. We also prove the convergence of the rescaled 2D Navier-Stokes equations (1.3) to the hydrostatic equations (1.5) in the analytic spaces. We remark that Kukavica et al [15] proved the local existence of analytic solutions for the hydrostatic Euler equations, and in [14], the authors derived the hydrostatic Euler equations as the zero viscosity limit for analytic solutions of the primitive equations.

Now let us state our main results.

The first result is the global well-posedness of the system (1.3) with small analytic data in x variable. The global well-posedness and the global analyticity of the solutionis to the classical 2-D Navier-Stokes system are well-known (see [9] for instance). However, the main interesting point here is that the smallness of data is independent of ε and there holds the global uniform estimate (1.8) with respect to the parameter ε. Moreover, all our results are valid in the multi-dimensional case, by changing the space $\mathcal{B}^{\frac{1}{2}}$ (see (2.1)) into the corresponding space $\mathcal{B}^{\frac{d-1}{2}}$ where d is the dimension of the space, with the obvious change in the definition and in the proofs.

Theorem 1.1. *Let $a > 0$. We assume that the initial data satisfies*

$$\left\|e^{a|D_x|}(u_0, \varepsilon v_0)\right\|_{\mathcal{B}^{\frac{1}{2}}} \leq c_0 a \tag{1.7}$$

for some c_0 sufficiently small. Then the system (1.3) *has a unique global solution* (u, v) *so that*

$$\begin{aligned} \|e^{\mathfrak{K}t}(u^\varepsilon_\Psi, \varepsilon v^\varepsilon_\Psi)\|_{\widetilde{L}^\infty(\mathbb{R}^+;\mathcal{B}^{\frac{1}{2}})} + &\|e^{\mathfrak{K}t}\partial_y(u^\varepsilon_\Psi, \varepsilon v^\varepsilon_\Psi)\|_{\widetilde{L}^2(\mathbb{R}^+;\mathcal{B}^{\frac{1}{2}})} \\ &+\varepsilon\left\|e^{\mathfrak{K}t}(u^\varepsilon_\Psi, \varepsilon v^\varepsilon_\Psi)\right\|_{\widetilde{L}^2(\mathbb{R}^+;\mathcal{B}^{\frac{3}{2}})} \leq C\left\|e^{a|D_x|}(u_0, \varepsilon v_0)\right\|_{\mathcal{B}^{\frac{1}{2}}}, \end{aligned} \tag{1.8}$$

where $(u^\varepsilon_\Psi, v^\varepsilon_\Psi)$ will be given by (3.1) *and the constant $\mathfrak{K}$ is determined by Poincaré inequality on the strip $\mathcal{S}$ (see* (3.6)*), and the functional spaces will be presented in Section 2.*

The second result is the global well-posedness of the hydrostatic Navier-Stokes system (1.5) with small analytic data in x variable. We remark that similar global result seems open for the Prandtl equation, where only a lower bound of the lifespan to the solution was obtained (see [13, 28]).

Theorem 1.2. *Let $a > 0$. We assume that the initial data satisfies*

$$\left\|e^{a|D_x|}u_0\right\|_{\mathcal{B}^{\frac{1}{2}}} \leq c_1 a \tag{1.9}$$

for some c_1 sufficiently small and there holds the compatibility condition $\int_0^1 u_0 dy = 0$. Then the system (1.5) has a unique global solution u so that

$$\|e^{\mathfrak{K}t}u_\Phi\|_{\widetilde{L}^\infty(\mathbb{R}^+;\mathcal{B}^{\frac{1}{2}})} + \|e^{\mathfrak{K}t'}\partial_y u_\Phi\|_{\widetilde{L}^2(\mathbb{R}^+;\mathcal{B}^{\frac{1}{2}})} \leq C\left\|e^{a|D_x|}u_0\right\|_{\mathcal{B}^{\frac{1}{2}}}, \tag{1.10}$$

where u_Φ will be determined by (4.3)*. Furthermore, if $e^{a|D_x|}u_0 \in \mathcal{B}^{\frac{5}{2}}, e^{a|D_x|}\partial_y u_0 \in \mathcal{B}^{\frac{3}{2}}$ and*

$$\left\|e^{a|D_x|}u_0\right\|_{\mathcal{B}^{\frac{1}{2}}} \leq \frac{c_2 a}{1 + \left\|e^{a|D_x|}u_0\right\|_{\mathcal{B}^{\frac{3}{2}}}} \tag{1.11}$$

for some c_2 sufficiently small, then exists a positive constant C so that for $\lambda = C^2\big(1 + \big\|e^{a|D_x|}u_0\big\|_{\mathcal{B}^{\frac{3}{2}}}\big)$ and $1 \leq s \leq \frac{5}{2}$, one has

$$\begin{aligned} &\|e^{\mathfrak{K}t}u_\Phi\|_{\widetilde{L}^\infty(\mathbb{R}^+;\mathcal{B}^s)} + \|e^{\mathfrak{K}t}\partial_y u_\Phi\|_{\widetilde{L}^2(\mathbb{R}^+;\mathcal{B}^s)} \leq C\big\|e^{a|D_x|}u_0\big\|_{\mathcal{B}^s},\\ &\|e^{\mathfrak{K}t}(\partial_t u)_\Phi\|_{\widetilde{L}^2(\mathbb{R}^+;\mathcal{B}^{\frac{3}{2}})} + \|e^{\mathfrak{K}t}\partial_y^2 u_\Phi\|_{\widetilde{L}^\infty(\mathbb{R}^+;\mathcal{B}^{\frac{3}{2}})} \leq C\big(\big\|e^{a|D_x|}\partial_y u_0\big\|_{\mathcal{B}^{\frac{3}{2}}} + \big\|e^{a|D_x|}u_0\big\|_{\mathcal{B}^{\frac{5}{2}}}\big). \end{aligned} \tag{1.12}$$

The main idea to prove the above two theorems is to control the new unknown u_Ψ defined by (3.1), where u is the horizontal velocity and u_Ψ is a weighted function of u in the dual Fourier variable with an exponential function of $(a - \lambda\eta(t))|\xi|$, which is equivalent to the analyticity of the solution in the horizontal variable. By the classical Cauchy-Kowalewskaya theorem, one expects the radius of the analyticity of the solutions decay in time and so the exponent, which corresponds to the width of the analyticity strip, is allowed to vary with time. Using energy estimates on the equation satisfied by u_Ψ and the control of the quantity which describes "the loss of the analyticity radius", we shall show that the analyticity strip persists globally in time. Consequently, our result is a global Cauchy-Kowalewskaya type theorem.

The third result is concerning the convergence from the scaled anisotropic Navier-Stokes system (1.3) to the hydrostatic Navier-Stokes system (1.5), which corresponds to [27] for the vanishing viscosity of the analytical solutions of Navier-Stokes system in the half space. Compared with [27], here we proved the convergence globally in time.

Theorem 1.3. *Let $a > 0$ and $(u_0^\varepsilon, v_0^\varepsilon)$ satisfy (1.7). Let u_0 satisfy $e^{a|D_x|}u_0 \in \mathcal{B}^{\frac{1}{2}} \cap \mathcal{B}^{\frac{5}{2}}, e^{a|D_x|}\partial_y u_0 \in \mathcal{B}^{\frac{3}{2}}$, and there holds (1.11) for some c_2 sufficiently small and the compatibility condition $\int_0^1 u_0 dy = 0$. Then we have*

$$\begin{aligned} &\|(w^1_\Theta, \varepsilon w^2_\Theta)\|_{\widetilde{L}^\infty_t(\mathcal{B}^{\frac{1}{2}})} + \|\partial_y(w^1_\Theta, \varepsilon w^2_\Theta)\|_{\widetilde{L}^2_t(\mathcal{B}^{\frac{1}{2}})} + \varepsilon\|(w^1_\Theta, \varepsilon w^2_\Theta)\|_{\widetilde{L}^2_t(\mathcal{B}^{\frac{3}{2}})}\\ &\qquad\qquad \leq C\Big(\big\|e^{a|D_x|}(u_0^\varepsilon - u_0, \varepsilon(v_0^\varepsilon - v_0))\big\|_{\mathcal{B}^{\frac{1}{2}}} + M\varepsilon\Big). \end{aligned} \tag{1.13}$$

Here $w^1 \stackrel{\text{def}}{=} u^\varepsilon - u$, $w^2 \stackrel{\text{def}}{=} v^\varepsilon - v$ and v_0 is determined from u_0 via $\partial_x u_0 + \partial_y v_0 = 0$ and $v_0|_{y=0} = v_0|_{y=1} = 0$, and $(w^1_\Theta, \varepsilon w^2_\Theta)$ will be given by (5.3).

We remark that without the smallness conditions (1.7) and (1.11), we can prove the convergence of the system (1.3) to the system (1.5) on a fixed time interval $[0, T]$ for $T < T^*$, where T^* is the lifetime of the solution of the hydrostatic Navier-Stokes equation with the large initial data u_0.

We end this introduction by the notations that will be used in all that follows. For $a \lesssim b$, we mean that there is a uniform constant C, which may be different on different lines, such that $a \leq Cb$. We denote by $(a|b)_{L^2}$ the $L^2(\mathcal{S})$ inner product of a and b. We designate by $L^p_T(L^q_{\rm h}(L^r_{\rm v}))$ the space $L^p(]0, T[; L^q(\mathbb{R}_x; L^r(\mathbb{R}_y)))$. Finally, we denote by $(d_k)_{k\in\mathbb{Z}}$ (resp. $(d_k(t))_{k\in\mathbb{Z}}$) to be a generic element of $\ell^1(\mathbb{Z})$ so that $\sum_{k\in\mathbb{Z}} d_k = 1$ (resp. $\sum_{k\in\mathbb{Z}} d_k(t) = 1$).

2. Littlewood-Paley theory and functional framework

In this work, since the solution is analytic in x and Sobolev in y, we have to establish the product laws in anisotropic regularity spaces. On the other hand, in order to prove a version of global Cauchy-Kowalewskaya type theorem, we need to control simultaneously the analytic radius and the estimate of the solution itself. For these purposes, it seems more convenient to introduce the Littlewood-Paley decomposition in the horizontal variable x. Let us recall from [1] that

$$\Delta_k^{\mathrm{h}} a = \mathcal{F}^{-1}(\varphi(2^{-k}|\xi|)\widehat{a}), \qquad S_k^{\mathrm{h}} a = \mathcal{F}^{-1}(\chi(2^{-k}|\xi|)\widehat{a}), \tag{2.1}$$

where $\mathcal{F}a$ and $\widehat{a}$ denote the partial Fourier transform of the distribution a with respect to x variable, that is, $\widehat{a}(\xi, y) = \mathcal{F}_{x\to\xi}(a)(\xi, y)$, and $\chi(\tau)$, $\varphi(\tau)$ are smooth functions such that

$$\mathrm{Supp}\ \varphi \subset \Big\{\tau \in \mathbb{R} \,/\ \ \frac{3}{4} \leq |\tau| \leq \frac{8}{3}\Big\} \quad \text{and} \quad \forall \tau > 0\,,\ \sum_{j\in\mathbb{Z}} \varphi(2^{-j}\tau) = 1,$$

$$\mathrm{Supp}\ \chi \subset \Big\{\tau \in \mathbb{R} \,/\ \ |\tau| \leq \frac{4}{3}\Big\} \qquad \text{and} \qquad \chi(\tau) + \sum_{j\geq 0} \varphi(2^{-j}\tau) = 1.$$

Let us also recall the functional spaces we are going to use.

Definition 2.1. *Let s in $\mathbb{R}$. For u in $S'_h(\mathcal{S})$, which means that u belongs to $S'(\mathcal{S})$ and satisfies $\lim_{k\to-\infty} \|S_k^{\mathrm{h}} u\|_{L^\infty} = 0$, we set*

$$\|u\|_{\mathcal{B}^s} \stackrel{\mathrm{def}}{=} \big\|\big(2^{ks}\|\Delta_k^{\mathrm{h}} u\|_{L^2}\big)_{k\in\mathbb{Z}}\big\|_{\ell^1(\mathbb{Z})}.$$

- *For $s \leq \frac{1}{2}$, we define $\mathcal{B}^s(\mathcal{S}) \stackrel{\mathrm{def}}{=} \big\{u \in S'_h(\mathcal{S}) \bigm| \|u\|_{\mathcal{B}^s} < \infty\big\}$.*
- *If k is a positive integer and if $\frac{1}{2} + k < s \leq \frac{3}{2} + k$, then we define $\mathcal{B}^s(\mathcal{S})$ as the subset of distributions u in $S'_h(\mathcal{S})$ such that $\partial_x^k u$ belongs to $\mathcal{B}^{s-k}(\mathcal{S})$.*

In order to obtain a better description of the regularizing effect of the diffusion equation, we need to use Chemin-Lerner type spaces $\widetilde{L}^\lambda_T(\mathcal{B}^s(\mathcal{S}))$.

Definition 2.2. *Let $p \in [1, +\infty]$ and $T \in]0, +\infty]$. We define $\widetilde{L}^p_T(\mathcal{B}^s(\mathcal{S}))$ as the completion of $C([0,T];\ S(\mathcal{S}))$ by the norm*

$$\|a\|_{\widetilde{L}^p_T(\mathcal{B}^s)} \stackrel{\mathrm{def}}{=} \sum_{k\in\mathbb{Z}} 2^{ks}\Big(\int_0^T \|\Delta_k^{\mathrm{h}}\, a(t)\|_{L^2}^p\, dt\Big)^{\frac{1}{p}}$$

with the usual change if $p = \infty$.

In order to overcome the difficulty that one can not use Gronwall type argument in the framework of Chemin-Lerner space, we need to use the time-weighted Chemin-Lerner norm, which was introduced by the first two authors in [20].

Definition 2.3. *Let $f(t) \in L^1_{loc}(\mathbb{R}_+)$ be a nonnegative function. We define*

$$\|a\|_{\widetilde{L}^p_{t,f}(\mathcal{B}^s)} \stackrel{\mathrm{def}}{=} \sum_{k\in\mathbb{Z}} 2^{ks}\Big(\int_0^t f(t')\|\Delta_k^{\mathrm{h}} a(t')\|_{L^2}^p\, dt'\Big)^{\frac{1}{p}}. \tag{2.2}$$

For the convenience of the readers, we recall the following anisotropic Bernstein type lemma from [7, 19].

Lemma 2.1. *Let $\mathcal{B}_{\mathrm{h}}$ be a ball of $\mathbb{R}_{\mathrm{h}}$, and $\mathcal{C}_{\mathrm{h}}$ a ring of $\mathbb{R}_{\mathrm{h}}$; let $1 \leq p_2 \leq p_1 \leq \infty$ and $1 \leq q \leq \infty$. Then there holds:*
If the support of $\widehat{a}$ is included in $2^k\mathcal{B}_{\mathrm{h}}$, then

$$\|\partial_x^\alpha a\|_{L^{p_1}_{\mathrm{h}}(L^q_{\mathrm{v}})} \lesssim 2^{k\left(|\alpha|+\left(\frac{1}{p_2}-\frac{1}{p_1}\right)\right)}\|a\|_{L^{p_2}_{\mathrm{h}}(L^q_{\mathrm{v}})}.$$

If the support of $\widehat{a}$ is included in $2^k\mathcal{C}_{\mathrm{h}}$, then

$$\|a\|_{L^{p_1}_{\mathrm{h}}(L^q_{\mathrm{v}})} \lesssim 2^{-kN}\|\partial_x^N a\|_{L^{p_1}_{\mathrm{h}}(L^q_{\mathrm{v}})}.$$

In the following context, we shall constantly use Bony's decomposition (see [4]) for the horizontal variable:

$$fg = T^{\rm h}_f g + T^{\rm h}_g f + R^{\rm h}(f,g), \tag{2.3}$$

where

$$T^{\rm h}_f g \stackrel{\rm def}{=} \sum_k S^{\rm h}_{k-1} f \Delta^{\rm h}_k g, \quad \text{and} \quad R^{\rm h}(f,g) \stackrel{\rm def}{=} \sum_k \Delta^{\rm h}_k f \widetilde{\Delta}^{\rm h}_k g$$

with $\widetilde{\Delta}^{\rm h}_k g \stackrel{\rm def}{=} \sum_{|k-k'|\leq 1} \Delta^{\rm h}_{k'} g.$

3. Global well-posedness of the system (1.3)

In this section, we establish the global well-posedness of the scaled anisotropic Navier-Stokes system (1.3) with small analytic data.

Proof of Theorem 1.1. As in [5, 6, 8, 21, 22, 28], for any locally bounded function Ψ on $\mathbb{R}^+ \times \mathbb{R}$, we define

$$u^\varepsilon_\Psi(t,x,y) \stackrel{\rm def}{=} \mathcal{F}^{-1}_{\xi\to x}\big(e^{\Psi(t,\xi)}\widehat{u}^\varepsilon(t,\xi,y)\big). \tag{3.1}$$

We introduce a key quantity $\eta(t)$ to describe the evolution of the analytic band of u^ε :

$$\begin{cases} \dot\eta(t) = \varepsilon\|\partial_x u^\varepsilon_\Psi(t)\|_{\mathcal{B}^{\frac12}} + \|\partial_y u^\varepsilon_\Psi(t)\|_{\mathcal{B}^{\frac12}}, \\ \eta|_{t=0} = 0. \end{cases} \tag{3.2}$$

Here the phase function Ψ is defined by

$$\Psi(t,\xi) \stackrel{\rm def}{=} (a-\lambda\eta(t))|\xi|. \tag{3.3}$$

In the rest of this section, we shall prove that under the assumption of (1.7), there holds the *a priori* estimate (1.8) for smooth enough solutions of (1.3), and neglect the regularization procedure. For simplicity, we shall neglect the script ε. Then in view of (1.3) and (3.1), we observe that (u_Ψ, v_Ψ) verifies

$$\begin{cases} \partial_t u_\Psi + \lambda\dot\eta(t)|D_x|u_\Psi + (u\partial_x u)_\Psi + (v\partial_y u)_\Psi - \varepsilon^2\partial_x^2 u_\Psi - \partial_y^2 u_\Psi + \partial_x p_\Psi = 0, \\ \varepsilon^2\left(\partial_t v_\Psi + \lambda\dot\eta(t)|D_x|v_\Psi + (u\partial_x v)_\Psi + (v\partial_y v)_\Psi - \varepsilon^2\partial_x^2 v_\Psi - \partial_y^2 v_\Psi\right) + \partial_y p_\Psi = 0, \\ \partial_x u_\Psi + \partial_y v_\Psi = 0 \quad \text{for} \quad (t,x,y)\in\mathbb{R}_+\times\mathcal{S}, \\ (u_\Psi, v_\Psi)\,|_{y=0} = (u_\Psi, v_\Psi)\,|_{y=1} = 0, \end{cases} \tag{3.4}$$

where $|D_x|$ denotes the Fourier multiplier with symbol $|\xi|$.

By applying the dyadic operator $\Delta^{\rm h}_k$ to (3.4) and then taking the L^2 inner product of the resulting equation with $\big(\Delta^{\rm h}_k u_\Psi, \Delta^{\rm h}_k v_\Psi\big)$, we find

$$\begin{aligned} &\frac12\frac{d}{dt}\big\|\Delta^{\rm h}_k(u_\Psi,\varepsilon v_\Psi)(t)\big\|^2_{L^2} + \lambda\dot\eta\big(|D_x|\Delta^{\rm h}_k(u_\Psi,\varepsilon v_\Psi)\ |\ \Delta^{\rm h}_k(u_\Psi,\varepsilon v_\Psi)\big)_{L^2} \\ &\quad + \varepsilon^2\big\|\partial_x\Delta^{\rm h}_k(u_\Psi,\varepsilon v_\Psi)\big\|^2_{L^2} + \big\|\partial_y\Delta^{\rm h}_k(u_\Psi,\varepsilon v_\Psi)\big\|^2_{L^2} \\ &\quad = -\left(\Delta^{\rm h}_k\,(u\partial_x u)_\Psi\,|\Delta^{\rm h}_k u_\Psi\right)_{L^2} - \left(\Delta^{\rm h}_k\,(v\partial_y u)_\Psi\,|\Delta^{\rm h}_k u_\Psi\right)_{L^2} \\ &\qquad - \varepsilon^2\big(\Delta^{\rm h}_k\,(u\partial_x v)_\Psi\,|\Delta^{\rm h}_k v_\Psi\big)_{L^2} - \varepsilon^2\big(\Delta^{\rm h}_k\,(v\partial_y v)_\Psi\,|\Delta^{\rm h}_k v_\Psi\big)_{L^2}, \end{aligned} \tag{3.5}$$

where we used the fact that $\partial_x u_\Psi + \partial_y v_\Psi = 0$, so that

$$\big(\nabla\Delta^{\rm h}_k p_\Psi\ |\ \Delta^{\rm h}_k(u_\Psi, v_\Psi)\big)_{L^2} = 0.$$

While due to $(u_\Psi, v_\Psi)\,|_{y=0} = (u_\Psi, v_\Psi)\,|_{y=1} = 0$, by applying Poincaré inequality, we have

$$\mathfrak{K}\|\Delta_k(u_\Psi, \varepsilon v_\Psi)\|_{L^2}^2 \leq \frac{1}{2}\big\|\partial_y\Delta_k(u_\Psi, \varepsilon v_\Psi)\big\|_{L^2}^2. \tag{3.6}$$

Then by using Lemma 2.1 and by multiplying (3.5) by $e^{2\mathfrak{K}t}$ and then integrating the resulting inequality over $[0,t]$, we achieve

$$\begin{aligned}
&\frac{1}{2}\|e^{\mathfrak{K}t'}\Delta_k^{\rm h}(u_\Psi, \varepsilon v_\Psi)\|_{L_t^\infty(L^2)}^2 + \lambda 2^k\int_0^t \dot{\eta}(t')\|e^{\mathfrak{K}t'}\Delta_k^{\rm h}(u_\Psi, \varepsilon v_\Psi)(t')\|_{L^2}^2\,dt'\\
&+\frac{1}{2}\int_0^t e^{2\mathfrak{K}t'}\Big(\|\Delta_k^{\rm h}\partial_y u_\Psi\|_{L^2}^2 + c\varepsilon^2\big(2^{2k}\big(\|\Delta_k^{\rm h}u_\Psi\|_{L^2}^2 + \varepsilon^2\|\Delta_k^{\rm h}v_\Psi\|_{L^2}^2\big) + \|\Delta_k^{\rm h}\partial_y v_\Psi\|_{L^2}^2\big)\Big)\,dt'\\
&\leq \big\|e^{a|D_x|}\Delta_k^{\rm h}(u_0, \varepsilon v_0)\big\|_{L^2}^2 + \int_0^t\big|\big(e^{\mathfrak{K}t'}\Delta_k^{\rm h}\,(u\partial_x u)_\Psi\,|e^{\mathfrak{K}t'}\Delta_k^{\rm h}u_\Psi\big)_{L^2}\big|\,dt'\\
&+\int_0^t\big|\big(e^{\mathfrak{K}t'}\Delta_k^{\rm h}\,(v\partial_y u)_\Psi\,|e^{\mathfrak{K}t'}\Delta_k^{\rm h}u_\Psi\big)_{L^2}\big|\,dt' + \varepsilon^2\int_0^t\big|\big(e^{\mathfrak{K}t'}\Delta_k^{\rm h}\,(u\partial_x v)_\Psi\,|e^{\mathfrak{K}t'}\Delta_k^{\rm h}v_\Psi\big)_{L^2}\big|\,dt'\\
&+\varepsilon^2\int_0^t\big|\big(e^{\mathfrak{K}t'}\Delta_k^{\rm h}\,(v\partial_y v)_\Psi\,|e^{\mathfrak{K}t'}\Delta_k^{\rm h}v_\Psi\big)_{L^2}\big|\,dt'.
\end{aligned}\tag{3.7}$$

In what follows, we shall always assume that $t < T^*$ with T^* being determined by

$$T^* \stackrel{\text{def}}{=} \sup\big\{\ t>0,\quad \eta(t) < a/\lambda\big\}. \tag{3.8}$$

So that by virtue of (3.3), for any $t < T^*$, there holds the following convex inequality

$$\Psi(t,\xi) \leq \Psi(t,\xi-\eta) + \Psi(t,\eta) \quad \text{for} \quad \forall\ \xi,\eta\in\mathbb{R}. \tag{3.9}$$

The estimate of (3.7) relies on the following lemmas. The first two lemmas concern the estimate of the horizontal component of the velocity. The first one is the energy estimate in analytic framework for the convection term $u\partial_x u$.

Lemma 3.1. *For any $s\in]0,1]$ and $t\leq T^*$, there holds*

$$\int_0^t\big|\big(e^{\mathfrak{K}t'}\Delta_k^{\rm h}(u\partial_x w)_\Psi \mid e^{\mathfrak{K}t'}\Delta_k^{\rm h}w_\Psi\big)_{L^2}\big|\,dt' \lesssim d_k^2 2^{-2ks}\|e^{\mathfrak{K}t'}w_\Psi\|_{\widetilde{L}^2_{t,\dot{\eta}(t)}(\mathcal{B}^{s+\frac{1}{2}})}^2. \tag{3.10}$$

The next lemma is concerned with the analytical energy estimate for the term $v\partial_y u$, which is the term responsible for the loss of one derivative in the tangential direction. It is the estimate of this term that the analyticity in the tangential variable is crucially used.

Lemma 3.2. *For any $s\in]0,1]$ and $t\leq T^*$, there holds*

$$\int_0^t\big|\big(e^{\mathfrak{K}t'}\Delta_k^{\rm h}(v\partial_y u)_\Psi \mid e^{\mathfrak{K}t'}\Delta_k^{\rm h}u_\Psi\big)_{L^2}\big|\,dt' \lesssim d_k^2 2^{-2ks}\|e^{\mathfrak{K}t'}u_\Psi\|_{\widetilde{L}^2_{t,\dot{\eta}(t)}(\mathcal{B}^{s+\frac{1}{2}})}^2. \tag{3.11}$$

The last lemma will be useful to control the vertical component of the velocity field in the analytic spaces.

Lemma 3.3. *For $t\leq T^*$, there holds*

$$\varepsilon^2\int_0^t\big|\big(e^{\mathfrak{K}t'}\Delta_k^{\rm h}(v\partial_y v)_\Psi \mid e^{\mathfrak{K}t'}\Delta_k^{\rm h}v_\Psi\big)_{L^2}\big|\,dt' \lesssim d_k^2 2^{-k}\big\|e^{\mathfrak{K}t'}(u_\Psi, \varepsilon v_\Psi)\big\|_{\widetilde{L}^2_{t,\dot{\eta}(t)}(\mathcal{B}^1)}^2. \tag{3.12}$$

Let us admit the above lemmas for the time being and continue our proof. Indeed, thanks to Lemmas 3.1-3.3, we deduce from (3.7) that

$$\begin{aligned}&\frac{1}{2}\|e^{\mathfrak{R}t'}\Delta_k^{\rm h}(u_\Psi,\varepsilon v_\Psi)\|_{L^\infty_t(L^2)}^2+\lambda 2^k\int_0^t\dot\eta(t')\|e^{\mathfrak{R}t'}\Delta_k^{\rm h}(u_\Psi,\varepsilon v_\Psi)(t')\|_{L^2}^2\,dt'\\&\quad+\frac{c}{2}\int_0^t e^{2\mathfrak{R}t'}\Big(\|\Delta_k^{\rm h}\partial_y(u_\Psi,\varepsilon v_\Psi)\|_{L^2}^2+\varepsilon^2 2^{2k}\|\Delta_k^{\rm h}(u_\Psi,\varepsilon v_\Psi)\|_{L^2}^2\Big)\,dt'\\&\leq\big\|e^{a|D_x|}\Delta_k^{\rm h}(u_0,\varepsilon v_0)\big\|_{L^2}^2+Cd_k^2 2^{-k}\big\|e^{\mathfrak{R}t'}(u_\Psi,\varepsilon v_\Psi)\big\|^2_{\widetilde{L}^2_{t,\dot\eta(t)}(\mathcal{B}^1)}.\end{aligned}$$

By multiplying the above inequality by 2^k and then taking square root of the resulting inequality, and finally by summing up the resulting ones over $\mathbb{Z}$, we find that for $t\leq T^*$

$$\begin{aligned}&\|e^{\mathfrak{R}t'}(u_\Psi,\varepsilon v_\Psi)\|_{\widetilde{L}^\infty_t(\mathcal{B}^{\frac12})}+\sqrt{\lambda}\|e^{\mathfrak{R}t'}(u_\Psi,\varepsilon v_\Psi)\|_{\widetilde{L}^2_{t,\dot\eta(t)}(\mathcal{B}^1)}+c\|e^{\mathfrak{R}t'}\partial_y(u_\Psi,\varepsilon v_\Psi)\|_{\widetilde{L}^2_t(\mathcal{B}^{\frac12})}\\&\quad+c\varepsilon\big\|e^{\mathfrak{R}t'}(u_\Psi,\varepsilon v_\Psi)\|_{\widetilde{L}^2_t(\mathcal{B}^{\frac32})}\leq\big\|e^{a|D_x|}(u_0,\varepsilon v_0)\big\|_{\mathcal{B}^{\frac12}}+C\|e^{\mathfrak{R}t'}(u_\Psi,\varepsilon v_\Psi)\|_{\widetilde{L}^2_{t,\dot\eta(t)}(\mathcal{B}^1)}.\end{aligned}$$

Taking $\lambda=C^2$ in the above inequality leads to

$$\begin{aligned}&\|e^{\mathfrak{R}t'}(u_\Psi,\varepsilon v_\Psi)\|_{\widetilde{L}^\infty_t(\mathcal{B}^{\frac12})}+c\|e^{\mathfrak{R}t'}\partial_y(u_\Psi,\varepsilon v_\Psi)\|_{\widetilde{L}^2_t(\mathcal{B}^{\frac12})}\\&\qquad+c\varepsilon\big\|e^{\mathfrak{R}t'}(u_\Psi,\varepsilon v_\Psi)\|_{\widetilde{L}^2_t(\mathcal{B}^{\frac32})}\leq\big\|e^{a|D_x|}(u_0,\varepsilon v_0)\big\|_{\mathcal{B}^{\frac12}}\quad\text{for } t\leq T^*.\end{aligned}\tag{3.13}$$

Then for $t\leq T^*$, we deduce from (3.2) that

$$\begin{aligned}\eta(t)=&\int_0^t\big(\varepsilon\|\partial_x u^\varepsilon_\Psi(t')\|_{\mathcal{B}^{\frac12}}+\|\partial_y u^\varepsilon_\Psi(t')\|_{\mathcal{B}^{\frac12}}\big)\,dt'\\\leq&\Big(\int_0^t e^{-2\mathfrak{R}t'}\,dt'\Big)^{\frac12}\Big(\int_0^t\big(\varepsilon\|e^{\mathfrak{R}t'}\partial_x u^\varepsilon_\Psi(t')\|_{\mathcal{B}^{\frac12}}+\|e^{\mathfrak{R}t'}\partial_y u^\varepsilon_\Psi(t')\|_{\mathcal{B}^{\frac12}}\big)^2\,dt'\Big)^{\frac12}\\\leq&C\big\|e^{\mathfrak{R}t'}(\varepsilon\partial_x u^\varepsilon_\Psi,\partial_y u^\varepsilon_\Psi)\big\|_{\widetilde{L}^2_t(\mathcal{B}^{\frac12})}\\\leq&C\big\|e^{a|D_x|}(u_0,\varepsilon v_0)\big\|_{\mathcal{B}^{\frac12}}.\end{aligned}$$

In particular, if we take c_0 in (1.7) to be so small that

$$C\big\|e^{a|D_x|}(u_0,\varepsilon v_0)\big\|_{\mathcal{B}^{\frac12}}\leq\frac{a}{2\lambda},\tag{3.14}$$

we deduce by a continuous argument that T^* determined by (3.8) equals $+\infty$ and (1.8) holds. This completes the proof of Theorem 1.1. □

Now let us present the proof of Lemmas 3.1 to 3.3. Indeed, we observe that it amounts to prove these lemmas for $\mathfrak{R}=0$. Without loss of generality, we may assume that $\widehat{u}\geq 0$ and $\widehat{v}\geq 0$ (and similar assumption for the proof of the product law in the rest of this paper, one may check [6] for detail).

Proof of Lemma 3.1. We first get, by applying Bony's decomposition (2.3) for the horizontal variable to $u\partial_x w$, that

$$u\partial_x w=T^{\rm h}_u\partial_x w+T^{\rm h}_{\partial_x w}u+R^h(u,\partial_x w).$$

Accordingly, we shall handle the following three terms:

- <u>Estimate of $\int_0^t\big(\Delta_k^{\rm h}(T^{\rm h}_u\partial_x w)_\Psi\mid\Delta_k^{\rm h}w_\Psi\big)_{L^2}\,dt'$</u>

Considering the support properties to the Fourier transform of the terms in $T^{\mathrm{h}}_u\partial_x w$, we infer

$$\int_0^t \big|\big(\Delta_k^{\mathrm{h}}(T^{\mathrm{h}}_u\partial_x w)_\Psi \mid \Delta_k^{\mathrm{h}} w_\Psi\big)_{L^2}\big|\,dt' \lesssim \sum_{|k'-k|\leq 4}\int_0^t \|S^{\mathrm{h}}_{k'-1}u_\Psi(t')\|_{L^\infty}\|\Delta_{k'}^{\mathrm{h}}\partial_x w_\Psi(t')\|_{L^2}\|\Delta_k^{\mathrm{h}} w_\Psi(t')\|_{L^2}\,dt'.$$

However, it follows from Lemma 2.1 and Poincaré inequality that

$$\begin{aligned}\|\Delta_k^{\mathrm{h}}u_\Psi(t)\|_{L^\infty} &\lesssim 2^{\frac{k}{2}}\|\Delta_k^{\mathrm{h}}u_\Psi(t)\|_{L^2_{\mathrm{h}}(L^\infty_{\mathrm{v}})}\\ &\lesssim 2^{\frac{k}{2}}\|\Delta_k^{\mathrm{h}}u_\Psi(t)\|_{L^2}^{\frac12}\|\Delta_k^{\mathrm{h}}\partial_y u_\Psi(t)\|_{L^2}^{\frac12}\\ &\lesssim 2^{\frac{k}{2}}\|\Delta_k^{\mathrm{h}}\partial_y u_\Psi(t)\|_{L^2}\lesssim d_j(t)\|\partial_y u_\Psi(t)\|_{\mathcal{B}^{\frac12}},\end{aligned} \tag{3.15}$$

so that

$$\|S^{\mathrm{h}}_{k'-1}u_\Psi(t)\|_{L^\infty}\lesssim \|\partial_y u_\Psi(t)\|_{\mathcal{B}^{\frac12}},$$

which implies that

$$\int_0^t \big|\big(\Delta_k^{\mathrm{h}}(T^{\mathrm{h}}_u\partial_x w)_\Psi \mid \Delta_k^{\mathrm{h}} w_\Psi\big)_{L^2}\big|\,dt' \lesssim \sum_{|k'-k|\leq 4}2^{k'}\int_0^t \|\partial_y u_\Psi(t)\|_{\mathcal{B}^{\frac12}}\|\Delta_{k'}^{\mathrm{h}} w_\Psi(t)\|_{L^2}\|\Delta_k^{\mathrm{h}} w_\Psi(t')\|_{L^2}\,dt'.$$

Applying Hölder inequality and using Definition 2.3 gives

$$\begin{aligned}\int_0^t \big|\big(\Delta_k^{\mathrm{h}}(T^{\mathrm{h}}_u\partial_x w)_\Psi \mid \Delta_k^{\mathrm{h}} w_\Psi\big)_{L^2}\big|\,dt' \lesssim& \sum_{|k'-k|\leq 4}2^{k'}\Big(\int_0^t \|\partial_y u_\Psi(t')\|_{\mathcal{B}^{\frac12}}\|\Delta_{k'}^{\mathrm{h}} w_\Psi(t')\|_{L^2}^2\,dt'\Big)^{\frac12}\\ &\times\Big(\int_0^t \|\partial_y u_\Psi(t')\|_{\mathcal{B}^{\frac12}}\|\Delta_{k}^{\mathrm{h}} w_\Psi(t')\|_{L^2}^2\,dt'\Big)^{\frac12}\\ \lesssim& d_k 2^{-2ks}\|w_\Psi\|^2_{\widetilde{L}^2_{t,\dot\eta(t)}(\mathcal{B}^{s+\frac12})}\Big(\sum_{|k'-k|\leq 4}d_{k'}2^{(k-k')(s-\frac12)}\Big)\\ \lesssim& d_k^2 2^{-2ks}\|w_\Psi\|^2_{\widetilde{L}^2_{t,\dot\eta(t)}(\mathcal{B}^{s+\frac12})}.\end{aligned}$$

• <u>Estimate of $\int_0^t\big(\Delta_k^{\mathrm{h}}(T^{\mathrm{h}}_{\partial_x w}u)_\Psi \mid \Delta_k^{\mathrm{h}} w_\Psi\big)_{L^2}\,dt'$</u>

Again considering the support properties to the Fourier transform of the terms in $T^{\mathrm{h}}_{\partial_x w}u$ and thanks to (3.15), we have

$$
\begin{aligned}
&\int_0^t \big|\big(\Delta_k^{\mathrm{h}}(T^{\mathrm{h}}_{\partial_x w}u)_\Psi \mid \Delta_k^{\mathrm{h}} w_\Psi\big)_{L^2}\big|\,dt' \\
&\lesssim \sum_{|k'-k|\leq 4}\int_0^t \|S^{\mathrm{h}}_{k'-1}\partial_x w_\Psi(t')\|_{L^\infty_{\mathrm{h}}(L^2_{\mathrm{v}})}\|\Delta^{\mathrm{h}}_{k'}u_\Psi(t')\|_{L^2_{\mathrm{h}}(L^\infty_{\mathrm{v}})}\|\Delta_k^{\mathrm{h}}w_\Psi(t')\|_{L^2}\,dt' \\
&\lesssim \sum_{|k'-k|\leq 4} 2^{-\frac{k'}{2}}\int_0^t d_{k'}(t)\|S^{\mathrm{h}}_{k'-1}\partial_x w_\Psi(t')\|_{L^\infty_{\mathrm{h}}(L^2_{\mathrm{v}})}\|\partial_y u_\Psi(t')\|_{\mathcal{B}^{\frac12}}\|\Delta_k^{\mathrm{h}}w_\Psi(t')\|_{L^2}\,dt' \\
&\lesssim \sum_{|k'-k|\leq 4} d_{k'}2^{-\frac{k'}{2}}\Big(\int_0^t \|S^{\mathrm{h}}_{k'-1}\partial_x w_\Psi(t')\|^2_{L^\infty_{\mathrm{h}}(L^2_{\mathrm{v}})}\|\partial_y u_\Psi(t')\|_{\mathcal{B}^{\frac12}}\,dt'\Big)^{\frac12} \\
&\qquad\qquad\qquad\times\Big(\int_0^t \|\Delta_k^{\mathrm{h}}w_\Psi(t')\|^2_{L^2}\|\partial_y u_\Psi(t')\|_{\mathcal{B}^{\frac12}}\,dt'\Big)^{\frac12}.
\end{aligned}
$$

Yet we observe from Definition 2.3 and $s\leq 1$ that

$$
\begin{aligned}
&\Big(\int_0^t \|S^{\mathrm{h}}_{k'-1}\partial_x w_\Psi(t')\|^2_{L^\infty_{\mathrm{h}}(L^2_{\mathrm{v}})}\|\partial_y u_\Psi(t')\|_{\mathcal{B}^{\frac12}}\,dt'\Big)^{\frac12} \\
&\lesssim \sum_{\ell\leq k'-2} 2^{\frac{3\ell}{2}}\Big(\int_0^t \|\Delta^{\mathrm{h}}_\ell w_\Psi(t')\|^2_{L^2}\|\partial_y u_\Psi(t')\|_{\mathcal{B}^{\frac12}}\,dt'\Big)^{\frac12} \\
&\lesssim \sum_{\ell\leq k'-2} d_\ell 2^{\ell(1-s)}\|w_\Psi\|_{\widetilde{L}^2_{t,\dot\eta(t)}(\mathcal{B}^{s+\frac12})} \\
&\lesssim 2^{k'(1-s)}\|w_\Psi\|_{\widetilde{L}^2_{t,\dot\eta(t)}(\mathcal{B}^{s+\frac12})}.
\end{aligned}
$$

So that it comes out

$$
\int_0^t \big|\big(\Delta_k^{\mathrm{h}}(T^{\mathrm{h}}_{\partial_x w}u)_\Psi \mid \Delta_k^{\mathrm{h}} w_\Psi\big)_{L^2}\big|\,dt' \lesssim d_k^2 2^{-2ks}\|w_\Psi\|^2_{\widetilde{L}^2_{t,\dot\eta(t)}(\mathcal{B}^{s+\frac12})}.
$$

• <u>Estimate of $\int_0^t \big(\Delta_k^{\mathrm{h}}(R^{\mathrm{h}}(u,\partial_x w))_\Psi \mid \Delta_k^{\mathrm{h}} w_\Psi\big)_{L^2}\,dt'$</u>

Again considering the support properties to the Fourier transform of the terms in $R^{\mathrm{h}}(u,\partial_x w)$, we get, by applying lemma 2.1 and (3.15), that

$$
\begin{aligned}
&\int_0^t \big|\big(\Delta_k^{\mathrm{h}}(R^{\mathrm{h}}(u,\partial_x w))_\Psi \mid \Delta_k^{\mathrm{h}} w_\Psi\big)_{L^2}\big|\,dt' \\
&\lesssim 2^{\frac{k}{2}}\sum_{k'\geq k-3}\int_0^t \|\widetilde{\Delta}^{\mathrm{h}}_{k'}u_\Psi(t')\|_{L^2_{\mathrm{h}}(L^\infty_{\mathrm{v}})}\|\Delta^{\mathrm{h}}_{k'}\partial_x w_\Psi(t')\|_{L^2}\|\Delta_k^{\mathrm{h}}w_\Psi(t')\|_{L^2}\,dt' \\
&\lesssim 2^{\frac{k}{2}}\sum_{k'\geq k-3}2^{\frac{k'}{2}}\int_0^t \|\partial_y u_\Psi(t')\|_{\mathcal{B}^{\frac12}}\|\Delta^{\mathrm{h}}_{k'}w_\Psi(t')\|_{L^2}\|\Delta_k^{\mathrm{h}}w_\Psi(t')\|_{L^2}\,dt'.
\end{aligned}
$$

Applying Hölder inequality and using Definition 2.3 yields

$$
\begin{aligned}
&\int_0^t \big|\big(\Delta_k^{\rm h}(R^{\rm h}(u,\partial_x w))_\Psi \mid \Delta_k^{\rm h} w_\Psi\big)_{L^2}\big|\,dt'\\
\lesssim & 2^{\frac{k}{2}}\sum_{k'\geq k-3} 2^{\frac{k'}{2}}\Big(\int_0^t \|\Delta_{k'}^{\rm h} w_\Psi(t')\|_{L^2}^2\|\partial_y u_\Psi(t')\|_{\mathcal{B}^{\frac12}}\,dt'\Big)^{\frac12}\\
&\qquad\qquad\qquad\times\Big(\int_0^t \|\Delta_k^{\rm h} w_\Psi(t')\|_{L^2}^2\|\partial_y u_\Psi(t')\|_{\mathcal{B}^{\frac12}}\,dt'\Big)^{\frac12}\\
\lesssim & d_k 2^{-2ks}\|w_\Psi\|^2_{\widetilde{L}^2_{t,\dot\eta(t)}(\mathcal{B}^{s+\frac12})}\Big(\sum_{k'\geq k-3} d_{k'}2^{(k-k')s}\Big)\\
\lesssim & d_k^2 2^{-2ks}\|w_\Psi\|^2_{\widetilde{L}^2_{t,\dot\eta(t)}(\mathcal{B}^{s+\frac12})},
\end{aligned}
$$

where we used the fact that $s>0$ in the last step.

By summing up the above estimates, we conclude the proof of (3.10). □

Remark 3.1. *In the particular case when* $w=u$ *in* (3.10), (3.10) *holds for any* $s>0$, *that is*

$$
\int_0^t \big|\big(e^{\mathfrak{R}t'}\Delta_k^{\rm h}(u\partial_x u)_\Psi \mid e^{\mathfrak{R}t'}\Delta_k^{\rm h} u_\Psi\big)_{L^2}\big|\,dt' \lesssim d_k^2 2^{-2ks}\|e^{\mathfrak{R}t'}u_\Psi\|^2_{\widetilde{L}^2_{t,\dot\eta(t)}(\mathcal{B}^{s+\frac12})}. \tag{3.16}
$$

It follows from the proof of Lemma 3.1 that we only need to prove

$$
\int_0^t \big|\big(\Delta_k^{\rm h}(T^{\rm h}_{\partial_x u}u)_\Psi \mid \Delta_k^{\rm h} u_\Psi\big)_{L^2}\big|\,dt' \lesssim d_k^2 2^{-2ks}\|u_\Psi\|^2_{\widetilde{L}^2_{t,\dot\eta(t)}(\mathcal{B}^{s+\frac12})} \quad \textit{for any } s>0. \tag{3.17}
$$

Indeed in view of (3.15), *we infer*

$$
\begin{aligned}
&\int_0^t \big|\big(\Delta_k^{\rm h}(T^{\rm h}_{\partial_x u}u)_\Psi \mid \Delta_k^{\rm h} u_\Psi\big)_{L^2}\big|\,dt'\\
&\lesssim \sum_{|k'-k|\leq 4}\int_0^t \|S^{\rm h}_{k'-1}\partial_x u_\Psi(t')\|_{L^\infty}\|\Delta_{k'}^{\rm h}u_\Psi(t')\|_{L^2}\|\Delta_k^{\rm h}u_\Psi(t')\|_{L^2}\,dt'\\
&\lesssim \sum_{|k'-k|\leq 4}2^{k'}\int_0^t \|\partial_y u_\Psi(t')\|_{\mathcal{B}^{\frac12}}\|\Delta_{k'}^{\rm h}u_\Psi(t')\|_{L^2}\|\Delta_k^{\rm h}u_\Psi(t')\|_{L^2}\,dt'\\
&\lesssim \sum_{|k'-k|\leq 4}2^{k'}\Big(\int_0^t \|\Delta_{k'}^{\rm h} u_\Psi(t')\|_{L^2}^2\|\partial_y u_\Psi(t')\|_{\mathcal{B}^{\frac12}}\,dt'\Big)^{\frac12}\\
&\qquad\qquad\qquad\times\Big(\int_0^t \|\Delta_k^{\rm h} u_\Psi(t')\|_{L^2}^2\|\partial_y u_\Psi(t')\|_{\mathcal{B}^{\frac12}}\,dt'\Big)^{\frac12},
\end{aligned}
$$

which leads to (3.17).

Proof of Lemma 3.2. We first get, by applying Bony's decomposition (2.3) for the horizontal variable to $v\partial_y u$, that

$$
v\partial_y u = T^{\rm h}_v\partial_y u + T^{\rm h}_{\partial_y u}v + R^h(v,\partial_y u).
$$

Accordingly, we shall handle the following three terms:

- Estimate of $\int_0^t \big(\Delta_k^{\rm h}(T^{\rm h}_v\partial_y u)_\Psi \mid \Delta_k^{\rm h}u_\Psi\big)_{L^2}\,dt'$

We first observe that

$$
\begin{aligned}
\int_0^t \big|\big(\Delta_k^{\rm h}(T^{\rm h}_v\partial_y u)_\Psi \mid \Delta_k^{\rm h} u_\Psi\big)_{L^2}\big|\,dt'&\\
&\lesssim \sum_{|k'-k|\leq 4}\int_0^t \|S^{\rm h}_{k'-1}v_\Psi(t')\|_{L^\infty}\|\Delta_{k'}^{\rm h}\partial_y u_\Psi(t)\|_{L^2}\|\Delta_k^{\rm h}u_\Psi(t')\|_{L^2}\,dt'\\
&\lesssim \sum_{|k'-k|\leq 4} d_{k'}2^{-\frac{k'}{2}}\int_0^t \|S^{\rm h}_{k'-1}v_\Psi(t')\|_{L^\infty}\|\partial_y u_\Psi(t')\|_{\mathcal{B}^{\frac12}}\|\Delta_k^{\rm h}u_\Psi(t')\|_{L^2}\,dt'.
\end{aligned}
$$

Due to $\partial_x u+\partial_y v=0$ and (1.4), we write $v(t,x,y)=-\int_0^y \partial_x u(t,x,y')\,dy'$. Then we deduce from Lemma 2.1 that

$$
\begin{aligned}
\|\Delta_k^{\rm h} v_\Psi(t)\|_{L^\infty}&\leq \int_0^1 \|\Delta_k^{\rm h}\partial_x u_\Psi(t,\cdot,y')\|_{L^\infty_{\rm h}}\,dy'\\
&\lesssim 2^{\frac{3k}{2}}\int_0^1 \|\Delta_k^{\rm h}u_\Psi(t,\cdot,y')\|_{L^2_h}\,dy' \lesssim 2^{\frac{3k}{2}}\|\Delta_k^{\rm h}u_\Psi(t)\|_{L^2},
\end{aligned}\tag{3.18}
$$

from which and $s\leq 1$, we infer

$$
\begin{aligned}
&\Big(\int_0^t \|S^{\rm h}_{k'-1}v_\Psi(t')\|_{L^\infty}^2\|\partial_y u_\Psi(t')\|_{\mathcal{B}^{\frac12}}\,dt'\Big)^{\frac12}\\
&\leq \sum_{\ell\leq k'-2} 2^{\frac{3\ell}{2}}\Big(\int_0^t \|\Delta_\ell^{\rm h}u_\Psi(t)\|_{L^2}^2\|\partial_y u_\Psi(t')\|_{\mathcal{B}^{\frac12}}\,dt'\Big)^{\frac12}\\
&\lesssim \sum_{\ell\leq k'-2} d_\ell 2^{\ell(1-s)}\|u_\Psi\|_{\widetilde{L}^2_{t,\dot\eta(t)}(\mathcal{B}^{s+\frac12})}\\
&\lesssim 2^{k'(1-s)}\|u_\Psi\|_{\widetilde{L}^2_{t,\dot\eta(t)}(\mathcal{B}^{s+\frac12})}.
\end{aligned}\tag{3.19}
$$

Consequently, by virtue of Definition 2.3, we obtain

$$
\begin{aligned}
\int_0^t \big|\big(\Delta_k^{\rm h}(T^{\rm h}_v\partial_y u)_\Psi \mid \Delta_k^{\rm h} u_\Psi\big)_{L^2}\big|\,dt'&\\
&\lesssim \sum_{|k'-k|\leq 4} d_{k'}2^{-\frac{k'}{2}}\Big(\int_0^t \|S^{\rm h}_{k'-1}v_\Psi(t')\|_{L^\infty}^2\|\partial_y u_\Psi(t')\|_{\mathcal{B}^{\frac12}}\,dt'\Big)^{\frac12}\\
&\qquad\times\Big(\int_0^t \|\Delta_k^{\rm h}u_\Psi(t')\|_{L^2}^2\|\partial_y u_\Psi(t')\|_{\mathcal{B}^{\frac12}}\,dt'\Big)^{\frac12}\\
&\lesssim d_k^2 2^{-2ks}\|u_\Psi\|^2_{\widetilde{L}^2_{t,\dot\eta(t)}(\mathcal{B}^{s+\frac12})}.
\end{aligned}
$$

• <u>Estimate of $\int_0^t \big(\Delta_k^{\rm h}(T^{\rm h}_{\partial_y u}v)_\Psi \mid \Delta_k^{\rm h}u_\Psi\big)_{L^2}\,dt'$</u>

Notice that

$$
\begin{aligned}
&\int_0^t \big|\big(\Delta_k^{\rm h}(T^{\rm h}_{\partial_y u}v)_\Psi \mid \Delta_k^{\rm h} u_\Psi\big)_{L^2}\big|\,dt'\\
&\lesssim \sum_{|k'-k|\leq 4}\int_0^t \|S^{\rm h}_{k'-1}\partial_y u_\Psi(t')\|_{L^\infty_{\rm h}(L^2_{\rm v})}\|\Delta_{k'}^{\rm h}v_\Psi(t)\|_{L^2_{\rm h}(L^\infty_{\rm v})}\|\Delta_k^{\rm h}u_\Psi(t')\|_{L^2}\,dt',
\end{aligned}
$$

which together with (3.18) ensures that

$$\begin{aligned}
&\int_0^t \big|\big(\Delta_k^{\rm h}(T^{\rm h}_{\partial_y u}v)_\Psi \mid \Delta_k^{\rm h}u_\Psi\big)_{L^2}\big|\,dt'\\
&\lesssim \sum_{|k'-k|\leq 4} 2^{k'}\int_0^t \|\partial_y u_\Psi(t')\|_{\mathcal{B}^{\frac12}}\|\Delta_{k'}^{\rm h}u_\Psi(t')\|_{L^2}\|\Delta_k^{\rm h}u_\Psi(t')\|_{L^2}\,dt'\\
&\lesssim \sum_{|k'-k|\leq 4} 2^{k'}\Big(\int_0^t \|\Delta_{k'}^{\rm h}u_\Psi(t')\|_{L^2}^2\|\partial_y u_\Psi(t')\|_{\mathcal{B}^{\frac12}}\,dt'\Big)^{\frac12}\\
&\qquad\qquad\times\Big(\int_0^t \|\Delta_{k}^{\rm h}u_\Psi(t')\|_{L^2}^2\|\partial_y u_\Psi(t')\|_{\mathcal{B}^{\frac12}}\,dt'\Big)^{\frac12}.
\end{aligned}$$

Then thanks to Definition 2.3, we arrive at

$$\int_0^t \big|\big(\Delta_k^{\rm h}(T^{\rm h}_{\partial_y u}v)_\Psi \mid \Delta_k^{\rm h}u_\Psi\big)_{L^2}\big|\,dt' \lesssim d_k^2 2^{-2ks}\|u_\Psi\|^2_{\widetilde{L}^2_{t,\dot\eta(t)}(\mathcal{B}^{s+\frac12})}.$$

• <u>Estimate of $\int_0^t\big(\Delta_k^{\rm h}(R^{\rm h}(v,\partial_y u))_\Psi \mid \Delta_k^{\rm h}u_\Psi\big)_{L^2}\,dt'$</u>

We get, by applying lemma 2.1 and (3.18), that

$$\begin{aligned}
&\int_0^t \big|\big(\Delta_k^{\rm h}(R^{\rm h}(v,\partial_y u))_\Psi \mid \Delta_k^{\rm h}u_\Psi\big)_{L^2}\big|\,dt'\\
&\lesssim 2^{\frac{k}{2}}\sum_{k'\geq k-3}\int_0^t \|\Delta_{k'}^{\rm h}v_\Psi(t')\|_{L^2_{\rm h}(L^\infty_{\rm v})}\|\widetilde{\Delta}_{k'}^{\rm h}\partial_y u_\Psi(t')\|_{L^2}\|\Delta_k^{\rm h}u_\Psi(t')\|_{L^2}\,dt'\\
&\lesssim 2^{\frac{k}{2}}\sum_{k'\geq k-3}2^{\frac{k'}{2}}\int_0^t \|\Delta_{k'}^{\rm h}u_\Psi(t')\|_{L^2}\|\partial_y u_\Psi(t')\|_{\mathcal{B}^{\frac12}}\|\Delta_k^{\rm h}u_\Psi(t')\|_{L^2}\,dt'\\
&\lesssim 2^{\frac{k}{2}}\sum_{k'\geq k-3}2^{\frac{k'}{2}}\Big(\int_0^t \|\Delta_{k'}^{\rm h}u_\Psi(t')\|_{L^2}^2\|\partial_y u_\Psi(t')\|_{\mathcal{B}^{\frac12}}\,dt'\Big)^{\frac12}\\
&\qquad\qquad\times\Big(\int_0^t \|\Delta_{k}^{\rm h}u_\Psi(t')\|_{L^2}^2\|\partial_y u_\Psi(t')\|_{\mathcal{B}^{\frac12}}\,dt'\Big)^{\frac12},
\end{aligned}$$

which together with Definition 2.3 and $s>0$ ensures that

$$\begin{aligned}
\int_0^t \big|\big(\Delta_k^{\rm h}(R^{\rm h}(v,\partial_y u)_\Psi \mid \Delta_k^{\rm h}u_\Psi\big)_{L^2}\big|\,dt' &\lesssim d_k 2^{-2ks}\|u_\Psi\|^2_{\widetilde{L}^2_{t,\dot\eta(t)}(\mathcal{B}^{s+\frac12})}\Big(\sum_{k'\geq k-3} d_{k'}2^{(k-k')s}\Big)\\
&\lesssim d_k^2 2^{-2ks}\|u_\Psi\|^2_{\widetilde{L}^2_{t,\dot\eta(t)}(\mathcal{B}^{s+\frac12})}.
\end{aligned}$$

By summing up the above estimates, we achieve (3.11). □

Proof of Lemma 3.3. We first get, by applying Bony's decomposition (2.3) for the horizontal variable to $v\partial_y v$, that

$$v\partial_y v = T_v^{\rm h}\partial_y v + T^{\rm h}_{\partial_y v}v + R^h(v,\partial_y v).$$

Let us handle the following three terms:

• <u>Estimate of $\int_0^t\big(\Delta_k^{\rm h}(T_v^{\rm h}\partial_y v)_\Psi \mid \Delta_k^{\rm h}v_\Psi\big)_{L^2}\,dt'$</u>

Due to $\partial_y v = -\partial_x u$, one has

$$
\begin{aligned}
&\varepsilon^2 \int_0^t \big|\big(\Delta_k^{\rm h}(T_v^{\rm h}\partial_y v)_\Psi \mid \Delta_k^{\rm h} v_\Psi\big)_{L^2}\big|\,dt'\\
&\qquad \lesssim \varepsilon^2 \sum_{|k'-k|\leq 4} \int_0^t \|S_{k'-1}^{\rm h} v_\Psi(t')\|_{L^\infty} \|\Delta_{k'}^{\rm h}\partial_y v_\Psi(t)\|_{L^2} \|\Delta_k^{\rm h} v_\Psi(t')\|_{L^2}\,dt'\\
&\qquad \lesssim \varepsilon \sum_{|k'-k|\leq 4} 2^{-\frac{k'}{2}} \int_0^t \|S_{k'-1}^{\rm h} v_\Psi(t')\|_{L^\infty} \varepsilon \|\partial_x u_\Psi(t')\|_{\mathcal{B}^{\frac12}} \|\Delta_k^{\rm h} v_\Psi(t')\|_{L^2}\,dt'\\
&\qquad \lesssim \varepsilon \sum_{|k'-k|\leq 4} 2^{-\frac{k'}{2}} \Big(\int_0^t \|S_{k'-1}^{\rm h} v_\Psi(t')\|_{L^\infty}^2 \varepsilon \|\partial_x u_\Psi(t')\|_{\mathcal{B}^{\frac12}}\,dt'\Big)^{\frac12}\\
&\qquad\qquad\qquad \times \Big(\int_0^t \|\Delta_k^{\rm h} v_\Psi(t')\|_{L^2}^2 \varepsilon \|\partial_x u_\Psi(t')\|_{\mathcal{B}^{\frac12}}\,dt'\Big)^{\frac12}.
\end{aligned}
$$

Yet we get, by a similar derivation of (3.19), that

$$
\Big(\int_0^t \|S_{k'-1}^{\rm h} v_\Psi(t')\|_{L^\infty}^2 \varepsilon \|\partial_x u_\Psi(t')\|_{\mathcal{B}^{\frac12}}\,dt'\Big)^{\frac12} \lesssim d_{k'} 2^{\frac{k'}{2}} \|u_\Psi\|_{\widetilde{L}^2_{t,\dot\eta(t)}(\mathcal{B}^1)}.
$$

Hence we deduce from Definition 2.3 that

$$
\varepsilon^2 \int_0^t \big|\big(\Delta_k^{\rm h}(T_v^{\rm h}\partial_y v)_\Psi \mid \Delta_k^{\rm h} v_\Psi\big)_{L^2}\big|\,dt' \leq d_k^2 2^{-k} \|u_\Psi\|_{\widetilde{L}^2_{t,\dot\eta(t)}(\mathcal{B}^1)} \varepsilon \|v_\Psi\|_{\widetilde{L}^2_{t,\dot\eta(t)}(\mathcal{B}^1)}.
$$

- Estimate of $\int_0^t \big(\Delta_k^{\rm h}(T_{\partial_y v}^{\rm h} v)_\Psi \mid \Delta_k^{\rm h} v_\Psi\big)_{L^2}\,dt'$

  Notice that

$$
\begin{aligned}
&\int_0^t \big|\big(\Delta_k^{\rm h}(T_{\partial_y v}^{\rm h} v)_\Psi \mid \Delta_k^{\rm h} v_\Psi\big)_{L^2}\big|\,dt'\\
&\qquad \lesssim \sum_{|k'-k|\leq 4} \int_0^t \|S_{k'-1}^{\rm h} \partial_x u_\Psi(t')\|_{L^\infty} \|\Delta_{k'}^{\rm h} v_\Psi(t)\|_{L^2} \|\Delta_k^{\rm h} v_\Psi(t')\|_{L^2}\,dt',
\end{aligned}
$$

which together with (3.15) ensures that

$$
\begin{aligned}
&\int_0^t \big|\big(\Delta_k^{\rm h}(T_{\partial_y v}^{\rm h} v)_\Psi \mid \Delta_k^{\rm h} v_\Psi\big)_{L^2}\big|\,dt'\\
&\qquad \lesssim \sum_{|k'-k|\leq 4} 2^{k'} \int_0^t \|\partial_y u_\Psi(t')\|_{\mathcal{B}^{\frac12}} \|\Delta_{k'}^{\rm h} v_\Psi(t)\|_{L^2} \|\Delta_k^{\rm h} v_\Psi(t')\|_{L^2}\,dt'\\
&\qquad \lesssim \sum_{|k'-k|\leq 4} 2^{k'} \Big(\int_0^t \|\Delta_{k'}^{\rm h} v_\Psi(t)\|_{L^2}^2 \|\partial_y u_\Psi(t')\|_{\mathcal{B}^{\frac12}}\,dt'\Big)^{\frac12}\\
&\qquad\qquad\qquad \times \Big(\int_0^t \|\Delta_k^{\rm h} v_\Psi(t')\|_{L^2}^2 \|\partial_y u_\Psi(t')\|_{\mathcal{B}^{\frac12}}\,dt'\Big)^{\frac12}
\end{aligned}
$$

Then thanks to Definition 2.3, we arrive at

$$
\int_0^t \big|\big(\Delta_k^{\rm h}(T_{\partial_y v}^{\rm h} v)_\Psi \mid \Delta_k^{\rm h} v_\Psi\big)_{L^2}\big|\,dt' \lesssim d_k^2 2^{-k} \|v_\Psi\|_{\widetilde{L}^2_{t,\dot\eta(t)}(\mathcal{B}^1)}^2.
$$

- Estimate of $\int_0^t \big(\Delta_k^{\rm h}(R^{\rm h}(v,\partial_y v))_\Psi \mid \Delta_k^{\rm h} v_\Psi\big)_{L^2}\,dt'$

Due to $\partial_x u + \partial_y v = 0$, we get, by applying lemma 2.1 and (3.18), that

$$\begin{aligned}
&\int_0^t \big|\big(\Delta_k^{\rm h}(R^{\rm h}(v,\partial_y v))_\Psi \mid \Delta_k^{\rm h} v_\Psi\big)_{L^2}\big|\,dt' \\
&\lesssim 2^{\frac{k}{2}} \sum_{k'\geq k-3} \int_0^t \|\Delta_{k'}^{\rm h} v_\Psi(t')\|_{L^2} \|\widetilde{\Delta}_{k'}^{\rm h}\partial_x u_\Psi(t')\|_{L^2_{\rm h}(L^\infty_{\rm v})} \|\Delta_k^{\rm h} v_\Psi(t')\|_{L^2}\,dt' \\
&\lesssim 2^{\frac{k}{2}} \sum_{k'\geq k-3} 2^{\frac{k'}{2}} \int_0^t \|\Delta_{k'}^{\rm h} v_\Psi(t')\|_{L^2} \|\partial_y u_\Psi(t')\|_{\mathcal{B}^{\frac12}} \|\Delta_k^{\rm h} v_\Psi(t')\|_{L^2}\,dt' \\
&\lesssim 2^{\frac{k}{2}} \sum_{k'\geq k-3} 2^{\frac{k'}{2}} \Big(\int_0^t \|\Delta_{k'}^{\rm h} v_\Psi(t')\|_{L^2}^2 \|\partial_y u_\Psi(t')\|_{\mathcal{B}^{\frac12}}\,dt'\Big)^{\frac12} \\
&\qquad\qquad \times \Big(\int_0^t \|\Delta_k^{\rm h} v_\Psi(t')\|_{L^2}^2 \|\partial_y u_\Psi(t')\|_{\mathcal{B}^{\frac12}}\,dt'\Big)^{\frac12},
\end{aligned}$$

which together with Definition 2.3 and $s>0$ ensures that

$$\int_0^t \big|\big(\Delta_k^{\rm h}(R^{\rm h}(v,\partial_y u))_\Psi \mid \Delta_k^{\rm h} u_\Psi\big)_{L^2}\big|\,dt' \lesssim d_k^2 2^{-k} \|v_\Psi\|_{\widetilde{L}^2_{t,\dot\eta(t)}(\mathcal{B}^1)}^2.$$

By summing up the above estimates, we obtain (3.12). This concludes the proof of Lemma 3.3. $\square$

4. Global well-posedness of the system (1.5)

In this section, we study the global well-posedness of the hydrostatic approximate equations (1.5) with small analytic data.

Due to the compatibility condition $\partial_x \int_0^1 u_0 dy = 0$, we deduce from $\partial_x u + \partial_y v = 0$ that

$$\partial_x \int_0^1 u(t,x,y)dy = 0,$$

which together with the fact: $u(t,x,y) \to 0$ as $|x| \to +\infty$, ensures that

$$\int_0^1 u(t,x,y)dy = 0. \tag{4.1}$$

Then by integrating the equation $\partial_t u + u\partial_x u + v\partial_y u - \partial_y^2 u + \partial_x p = 0$ for $y \in [0,1]$ and using the fact that $\partial_y p = 0$, we obtain

$$\partial_x p = \partial_y u(t,x,1) - \partial_y u(t,x,0) - \frac12 \partial_x \int_0^1 u^2(t,x,y)\,dy. \tag{4.2}$$

We now define

$$u_\Phi(t,x,y) \stackrel{\text{def}}{=} \mathcal{F}^{-1}_{\xi\to x}\big(e^{\Phi(t,\xi)}\widehat{u}(t,\xi,y)\big) \quad \text{with} \quad \Phi(t,\xi) \stackrel{\text{def}}{=} (a-\lambda\theta(t))|\xi|, \tag{4.3}$$

where the quantity $\theta(t)$ describes the evolution of the analytic band of u, which is determined by

$$\dot\theta(t) = \|\partial_y u_\Phi(t)\|_{\mathcal{B}^{\frac12}} \quad \text{with} \quad \theta|_{t=0} = 0. \tag{4.4}$$

We remark that as in the previous section, the damping obtained by the Poincaré inequality in the strip helps to obtain this global control of $\theta(t)$ which describes the "loss of the analyticity".

Proof of Theorem 1.2. In view of (1.5) and (4.3), we observe that u_Φ verifies

$$\partial_t u_\Phi + \lambda\dot\theta(t)|D_x|u_\Phi + (u\partial_x u)_\Phi + (v\partial_y u)_\Phi - \partial_y^2 u_\Phi + \partial_x p_\Phi = 0, \tag{4.5}$$

where $|D_x|$ denotes the Fourier multiplier with symbol $|\xi|$.

By applying $\Delta_k^{\rm h}$ to (4.5) and taking L^2 inner product of the resulting equation with $\Delta_k^{\rm h} u_\Phi$, we find

$$\begin{aligned}&\frac12\frac{d}{dt}\|\Delta_k^{\rm h}u_\Phi(t)\|_{L^2}^2 + \lambda\dot\theta\big(|D_x|\Delta_k^{\rm h}u_\Phi \mid \Delta_k^{\rm h}u_\Phi\big)_{L^2} + \|\Delta_k^{\rm h}\partial_y u_\Psi\|_{L^2}^2\\ &\quad = -\big(\Delta_k^{\rm h}(u\partial_x u)_\Phi|\Delta_k^{\rm h}u_\Phi\big)_{L^2} - \big(\Delta_k^{\rm h}(v\partial_y u)_\Phi|\Delta_k^{\rm h}u_\Phi\big)_{L^2} - \big(\Delta_k^{\rm h}\partial_x p_\Phi|\Delta_k^{\rm h}u_\Phi\big)_{L^2}.\end{aligned} \tag{4.6}$$

Thanks to (1.6) and $\partial_x u + \partial_y v = 0$, we get, by using integration by parts, that

$$\begin{aligned}\big(\Delta_k^{\rm h}\partial_x p_\Phi|\Delta_k^{\rm h}u_\Phi\big)_{L^2} &= -\big(\Delta_k^{\rm h}p_\Phi|\Delta_k^{\rm h}\partial_x u_\Phi\big)_{L^2}\\ &=\big(\Delta_k^{\rm h}p_\Phi|\Delta_k^{\rm h}\partial_y v_\Phi\big)_{L^2} = -\big(\Delta_k^{\rm h}\partial_y p_\Phi|\Delta_k^{\rm h}v_\Phi\big)_{L^2} = 0.\end{aligned}$$

Then by using Lemma 2.1, (3.6) and by multiplying (4.6) by $e^{2\mathfrak{R}t}$ and then integrating the resulting inequality over $[0,t]$, we achieve

$$\begin{aligned}&\frac12\|e^{\mathfrak{R}t'}\Delta_k^{\rm h}u_\Phi\|_{L^\infty_t(L^2)}^2 + \lambda 2^k\int_0^t\dot\theta(t')\|e^{\mathfrak{R}t'}\Delta_k^{\rm h}u_\Phi(t')\|_{L^2}^2\,dt' + \frac12\|e^{\mathfrak{R}t'}\Delta_k^{\rm h}\partial_y u_\Psi\|_{L^2_t(L^2)}^2\\ &\leq\big\|e^{a|D_x|}\Delta_k^{\rm h}u_0\big\|_{L^2}^2 + \int_0^t\big|\big(e^{\mathfrak{R}t'}\Delta_k^{\rm h}(u\partial_x u)_\Phi\,|e^{\mathfrak{R}t'}\Delta_k^{\rm h}u_\Phi\big)_{L^2}\big|\,dt'\\ &\qquad + \int_0^t\big|\big(e^{\mathfrak{R}t'}\Delta_k^{\rm h}(v\partial_y u)_\Phi\,|e^{\mathfrak{R}t'}\Delta_k^{\rm h}u_\Phi\big)_{L^2}\big|\,dt'.\end{aligned} \tag{4.7}$$

In what follows, we shall always assume that $t < T^\star$ with $T^\star$ being determined by

$$T^\star \overset{\rm def}{=} \sup\big\{\ t>0,\quad \theta(t) < a/\lambda\big\}. \tag{4.8}$$

So that by virtue of (4.3), for any $t\leq T^\star$, there holds the following convex inequality

$$\Phi(t,\xi)\leq\Phi(t,\xi-\eta)+\Phi(t,\eta)\quad\text{for}\quad\forall\ \xi,\eta\in\mathbb{R}. \tag{4.9}$$

Then we deduce from Lemma 3.1 that for any $s\in]0,1]$ and $t\leq T^\star$

$$\int_0^t\big|\big(e^{\mathfrak{R}t'}\Delta_k^{\rm h}(u\partial_x u)_\Phi \mid e^{\mathfrak{R}t'}\Delta_k^{\rm h}u_\Phi\big)_{L^2}\big|\,dt' \lesssim d_k^2 2^{-2ks}\|e^{\mathfrak{R}t'}u_\Phi\|^2_{\widetilde{L}^2_{t,\dot\theta(t)}(\mathcal{B}^{s+\frac12})}.$$

Whereas it follows from Lemma 3.2 that for any $s\in]0,1]$ and $t\leq T^\star$

$$\int_0^t\big|\big(e^{\mathfrak{R}t'}\Delta_k^{\rm h}(v\partial_y u)_\Phi \mid e^{\mathfrak{R}t'}\Delta_k^{\rm h}u_\Phi\big)_{L^2}\big|\,dt' \lesssim d_k^2 2^{-2ks}\|e^{\mathfrak{R}t'}u_\Phi\|^2_{\widetilde{L}^2_{t,\dot\theta(t)}(\mathcal{B}^{s+\frac12})}.$$

Inserting the above estimates into (4.7) gives rise to

$$\begin{aligned}&\frac12\|e^{\mathfrak{R}t'}\Delta_k^{\rm h}u_\Phi\|_{L^\infty_t(L^2)}^2 + \lambda 2^k\int_0^t\dot\theta(t')\|e^{\mathfrak{R}t'}\Delta_k^{\rm h}u_\Phi(t')\|_{L^2}^2\,dt' + \frac12\|e^{\mathfrak{R}t'}\Delta_k^{\rm h}\partial_y u_\Psi\|_{L^2_t(L^2)}^2\\ &\qquad\qquad\leq \big\|e^{a|D_x|}\Delta_k^{\rm h}u_0\big\|_{L^2}^2 + Cd_k^2 2^{-2ks}\|e^{\mathfrak{R}t'}u_\Phi\|^2_{\widetilde{L}^2_{t,\dot\theta(t)}(\mathcal{B}^{s+\frac12})}.\end{aligned}$$

Then for any $s\in]0,1]$, by multiplying the above inequality by 2^{2ks} and then taking square root of the resulting inequality, and finally by summing up the resulting ones over $\mathbb{Z}$, we obtain

$$\|e^{\mathfrak{K}t'}u_\Phi\|_{\widetilde{L}^\infty_t(\mathcal{B}^s)}+\sqrt{\lambda}\|e^{\mathfrak{K}t'}u_\Phi\|_{\widetilde{L}^2_{t,\dot{\theta}(t)}(\mathcal{B}^{s+\frac12})}+\|e^{\mathfrak{K}t'}\partial_y u_\Phi\|_{\widetilde{L}^2_t(\mathcal{B}^s)}$$
$$\leq\big\|e^{a|D_x|}u_0\big\|_{\mathcal{B}^s}+C\|e^{\mathfrak{K}t'}u_\Phi\|_{\widetilde{L}^2_{t,\dot{\theta}(t)}(\mathcal{B}^{s+\frac12})}.$$

Taking $\lambda=C^2$ in the above inequality leads to

$$\|e^{\mathfrak{K}t'}u_\Phi\|_{\widetilde{L}^\infty_t(\mathcal{B}^s)}+\|e^{\mathfrak{K}t'}\partial_y u_\Phi\|_{\widetilde{L}^2_t(\mathcal{B}^s)}\leq\big\|e^{a|D_x|}u_0\big\|_{\mathcal{B}^s}\quad\text{for}\quad s\in]0,1]\quad\text{and}\quad t\leq T^\star.\tag{4.10}$$

In particular, we deduce from (4.10) for $s=\frac12$ and (4.4) that

$$\begin{aligned}\theta(t)=&\int_0^t\|\partial_y u_\Phi(t')\|_{\mathcal{B}^{\frac12}}\,dt'\\ \leq&\Big(\int_0^t e^{-2\mathfrak{K}t'}\,dt'\Big)^{\frac12}\Big(\int_0^t\|e^{\mathfrak{K}t'}\partial_y u_\Phi(t')\|^2_{\mathcal{B}^{\frac12}}\,dt'\Big)^{\frac12}\\ \leq&C\|e^{\mathfrak{K}t'}\partial_y u_\Phi\|_{\widetilde{L}^2_t(\mathcal{B}^{\frac12})}\leq C\big\|e^{a|D_x|}u_0\big\|_{\mathcal{B}^{\frac12}}.\end{aligned}$$

Then if we take c_1 in (1.9) to be so small that

$$C\big\|e^{a|D_x|}u_0\big\|_{\mathcal{B}^{\frac12}}\leq\frac{a}{2\lambda},\tag{4.11}$$

we deduce by a continuous argument that $T^\star$ determined by (4.8) equals $+\infty$ and (1.10) holds. Then Theorem 1.2 is proved provided that we present the proof of (1.12), which relies on the the following propositions. The first proposition states the propagation of any $\mathcal{B}^s$ regularity on the solution of the hydrostatic Navier-Stokes equations.

Proposition 4.1. *Under the assumption of* (1.11), *for any* $s>0$, *there exists a positive constant* C *so that for* $\lambda=C^2\big(1+\big\|e^{a|D_x|}u_0\big\|_{\mathcal{B}^{\frac32}}\big)$, *there holds*

$$\|e^{\mathfrak{K}t}u_\Phi\|_{\widetilde{L}^\infty(\mathbb{R}^+;\mathcal{B}^s)}+\|e^{\mathfrak{K}t}\partial_y u_\Phi\|_{\widetilde{L}^2(\mathbb{R}^+;\mathcal{B}^s)}\leq C\big\|e^{a|D_x|}u_0\big\|_{\mathcal{B}^s}\tag{4.12}$$

The second proposition allow to control two derivatives in the normal direction $\partial_y^2 u$ in any $\mathcal{B}^s$, despite de difficulties raised by the boundary conditions. This will be useful in the section last section when we prove the global convergence in the Theorem 1.3.

Proposition 4.2. *Under the assumption of* (1.11), *for any* $s>0$, *there exists a positive constant* C *so that for* $\lambda=C^2\big(1+\big\|e^{a|D_x|}u_0\big\|_{\mathcal{B}^{\frac32}}\big)$, *there holds*

$$\begin{aligned}&\|e^{\mathfrak{K}t'}\partial_y u_\Phi\|_{\widetilde{L}^\infty_t(\mathcal{B}^s)}+\|e^{\mathfrak{K}t'}\partial_y^2 u_\Phi\|_{\widetilde{L}^2_t(\mathcal{B}^s)}\\ &\qquad\leq C\Big(\big\|e^{a|D_x|}\partial_y u_0\big\|_{\mathcal{B}^s}+\big\|e^{a|D_x|}u_0\big\|_{\mathcal{B}^s}+\big\|e^{a|D_x|}u_0\big\|_{\mathcal{B}^{s+1}}\Big).\end{aligned}\tag{4.13}$$

We admit the above propositions for the time being and continue our proof of Theorem 1.2.

As a matter of fact, it remains to present the estimate of $\|e^{\mathfrak{K}t}(\partial_t u)_\Phi\|_{\widetilde{L}^2(\mathbb{R}^+;\mathcal{B}^{\frac32})}$. Indeed, by applying $\Delta_k^{\rm h}$ to (1.5) and then taking L^2 inner product of resulting equation with $e^{2\mathfrak{K}t}\Delta_k^{\rm h}(\partial_t u)_\Phi$, we obtain

$$\begin{aligned}\|e^{\mathfrak{K}t}\Delta_k^{\rm h}(\partial_t u)_\Phi\|^2_{L^2}=&e^{2\mathfrak{K}t}\big(\Delta_k^{\rm h}\partial_y^2 u_\Phi|\Delta_k^{\rm h}(\partial_t u)_\Phi\big)_{L^2}\\ &-e^{2\mathfrak{K}t}\big(\Delta_k^{\rm h}(u\partial_x u)_\Phi|\Delta_k^{\rm h}(\partial_t u)_\Phi\big)_{L^2}-e^{2\mathfrak{K}t}\big(\Delta_k^{\rm h}(v\partial_y u)_\Phi|\Delta_k^{\rm h}(\partial_t u)_\Phi\big)_{L^2},\end{aligned}$$

Using the fact that $(\partial_t u)_\Phi = \partial_t u_\Phi + \dot\theta(t)|D_x|u_\Phi$, we remark that

$$e^{2\mathfrak{K}t}\big(\Delta_k^{\rm h}\partial_y^2 u_\Phi|\Delta_k^{\rm h}(\partial_t u)_\Phi\big)_{L^2} = -(\frac{1}{2}\frac{d}{dt}\|e^{\mathfrak{K}t}\Delta_k^h\partial_y u_\phi\|_{L^2}^2 + \dot\theta(t)2^k\|e^{\mathfrak{K}t}\Delta_k^h\partial_y u_\Phi\|_{L^2}^2)$$

from which, we deduce that

$$\begin{aligned}\|e^{\mathfrak{K}t'}\Delta_k^{\rm h}\,(\partial_t u)_\Phi\,\|_{L^2_t(L^2)} + \|e^{\mathfrak{K}t'}\Delta_k^{\rm h}\partial_y u_\Phi\|_{L^\infty_t(L^2)} \leq C\Big(&\|\Delta_k^{\rm h}\partial_y e^{a|D_x|}u_0\|_{L^2}\\ &+\big\|e^{\mathfrak{K}t'}\Delta_k^h(u\partial_x u)_\Phi\big\|_{L^2_t(L^2)} + \big\|e^{\mathfrak{K}t'}\Delta_k^h(v\partial_y u)_\Phi\|_{L^2_t(L^2)}\Big).\end{aligned}$$

This gives rise to

$$\begin{aligned}\big\|e^{\mathfrak{K}t'}(\partial_t u)_\Phi\big\|_{\widetilde L^2_t(\mathcal{B}^{\frac32})} + \|e^{\mathfrak{K}t'}\partial_y u_\Phi\|_{\widetilde L^\infty_t(\mathcal{B}^{\frac32})} \leq C\Big(&\big\|e^{a|D_x|}\partial_y u_0\big\|_{\mathcal{B}^{\frac32}}\\ &+\big\|e^{\mathfrak{K}t'}(u\partial_x u)_\Phi\big\|_{\widetilde L^2_t(\mathcal{B}^{\frac32})} + \big\|e^{\mathfrak{K}t'}(v\partial_y u)_\Phi\big\|_{\widetilde L^2_t(\mathcal{B}^{\frac32})}\Big).\end{aligned} \tag{4.14}$$

Yet it follows from the law of product in anisotropic Besov space and Poincaré inequality that

$$\begin{aligned}&\big\|e^{\mathfrak{K}t'}(u\partial_x u)_\Phi\big\|_{\widetilde L^2_t(\mathcal{B}^{\frac32})} \lesssim \|u_\Phi\|_{\widetilde L^\infty_t(\mathcal{B}^{\frac12})}\big\|e^{\mathfrak{K}t'}\partial_y u_\Phi\big\|_{\widetilde L^2_t(\mathcal{B}^{\frac52})};\\ &\big\|e^{\mathfrak{K}t'}(v\partial_y u)_\Phi\big\|_{\widetilde L^2_t(\mathcal{B}^{\frac32})} \lesssim \|u_\Phi\|_{\widetilde L^\infty_t(\mathcal{B}^{\frac12})}\big\|e^{\mathfrak{K}t'}\partial_y u_\Phi\big\|_{\widetilde L^2_t(\mathcal{B}^{\frac52})} + \|u_\Phi\|_{\widetilde L^\infty_t(\mathcal{B}^{\frac52})}\big\|e^{\mathfrak{K}t'}\partial_y u_\Phi\big\|_{\widetilde L^2_t(\mathcal{B}^{\frac12})}.\end{aligned}$$

Inserting the above estimates into (4.14) and then using (1.9), (1.10) and Proposition 4.1, we achieve

$$\big\|e^{\mathfrak{K}t'}(\partial_t u)_\Phi\big\|_{\widetilde L^2_t(\mathcal{B}^{\frac32})} + \|e^{\mathfrak{K}t'}\partial_y u_\Phi\|_{\widetilde L^\infty_t(\mathcal{B}^{\frac32})} \lesssim \big\|e^{a|D_x|}\partial_y u_0\big\|_{\mathcal{B}^{\frac32}} + \big\|e^{a|D_x|}u_0\big\|_{\mathcal{B}^{\frac52}}.$$

This completes the proof of Theorem 1.2. □

Now let us present the proof of the above two propositions.

Proof of Proposition 4.1. We first deduce from Remark 3.1 that for any $s>0$

$$\int_0^t\big|\big(\Delta_k^{\rm h}\,(u\partial_x u)_\Phi\,|\Delta_k^{\rm h}u_\Phi\big)_{L^2}\big|\,dt' \lesssim d_k^2 2^{-2ks}\|u_\Phi\|^2_{\widetilde L^2_{t,\dot\theta(t)}(\mathcal{B}^{s+\frac12})}. \tag{4.15}$$

While it follows from the proof of Lemma 3.2 that

$$\int_0^t\big|\big(\Delta_k^{\rm h}(T^{\rm h}_{\partial_y u}v + R^{\rm h}(v,\partial_y u))_\Phi|\Delta_k^{\rm h}u_\Phi\big)_{L^2}\big|\,dt' \lesssim d_k^2 2^{-2ks}\|u_\Phi\|^2_{\widetilde L^2_{t,\dot\theta(t)}(\mathcal{B}^{s+\frac12})}.$$

In view of (3.18), we have

$$\|\Delta_k^{\rm h}v_\Phi(t)\|_{L^\infty} \lesssim d_k(t)2^{\frac k2}\|u_\Phi(t)\|^{\frac12}_{\mathcal{B}^{\frac32}}\|\partial_y u_\Phi(t)\|^{\frac12}_{\mathcal{B}^{\frac12}},$$

so that there holds

$$
\begin{aligned}
&\int_0^t \big|\big(\Delta_k^{\rm h}(T_v^{\rm h}\partial_y u)_\Phi \mid \Delta_k^{\rm h} u_\Phi\big)_{L^2}\big|\,dt' \\
&\lesssim \sum_{|k'-k|\leq 4}\int_0^t \|S_{k'-1}^{\rm h} v_\Phi(t')\|_{L^\infty}\|\Delta_{k'}^{\rm h}\partial_y u_\Phi(t)\|_{L^2}\|\Delta_k^{\rm h} u_\Phi(t')\|_{L^2}\,dt' \\
&\lesssim \sum_{|k'-k|\leq 4} 2^{\frac{k'}{2}}\|u_\Phi\|_{L^\infty_t(\mathcal{B}^{\frac32})}^{\frac12}\|\Delta_{k'}^{\rm h}\partial_y u_\Phi\|_{L^2_t(L^2)}\Big(\int_0^t \|\partial_y u_\Phi(t')\|_{\mathcal{B}^{\frac12}}\|\Delta_k^{\rm h} u_\Phi(t')\|_{L^2}^2\,dt'\Big)^{\frac12} \\
&\lesssim d_k^2 2^{-2ks}\|u_\Phi\|_{L^\infty_t(\mathcal{B}^{\frac32})}^{\frac12}\|\partial_y u_\Phi\|_{\widetilde{L}^2_t(\mathcal{B}^s)}\|u_\Phi\|_{\widetilde{L}^2_{t,\dot\theta(t)}(\mathcal{B}^{s+\frac12})}.
\end{aligned}
\tag{4.16}
$$

As a result, it comes out

$$
\begin{aligned}
\int_0^t \big|\big(\Delta_k^{\rm h}(v\partial_y u)_\Phi | \Delta_k^{\rm h} u_\Phi\big)_{L^2}\big|\,dt' \lesssim d_k^2 2^{-2ks}\|u_\Phi\|_{\widetilde{L}^2_{t,\dot\theta(t)}(\mathcal{B}^{s+\frac12})} \\
\times\Big(\|u_\Phi\|_{\widetilde{L}^2_{t,\dot\theta(t)}(\mathcal{B}^{s+\frac12})} + \|u_\Phi\|_{\widetilde{L}^\infty_t(\mathcal{B}^{\frac32})}^{\frac12}\|\partial_y u_\Phi\|_{\widetilde{L}^2_t(\mathcal{B}^s)}\Big).
\end{aligned}
\tag{4.17}
$$

By virtue of (4.15) and (4.17), we deduce from (4.7) that

$$
\begin{aligned}
&\frac12\|e^{\mathfrak{K}t'}\Delta_k^{\rm h} u_\Phi\|_{L^\infty_t(L^2)}^2 + \lambda 2^k\int_0^t \dot\theta(t')\|e^{\mathfrak{K}t'}\Delta_k^{\rm h} u_\Phi(t')\|_{L^2}^2\,dt' + \frac12\|e^{\mathfrak{K}t'}\Delta_k^{\rm h}\partial_y u_\Phi\|_{L^2_t(L^2)}^2 \\
&\leq \frac12\big\|e^{a|D_x|}\Delta_k^{\rm h} u_0\big\|_{L^2}^2 + C d_k^2 2^{-2ks}\|e^{\mathfrak{K}t'}u_\Phi\|_{\widetilde{L}^2_{t,\dot\theta(t)}(\mathcal{B}^{s+\frac12})} \\
&\qquad\qquad \times\Big(\|e^{\mathfrak{K}t'}u_\Phi\|_{\widetilde{L}^2_{t,\dot\theta(t)}(\mathcal{B}^{s+\frac12})} + \|u_\Phi\|_{\widetilde{L}^\infty_t(\mathcal{B}^{\frac32})}^{\frac12}\|e^{\mathfrak{K}t'}\partial_y u_\Phi\|_{\widetilde{L}^2_t(\mathcal{B}^s)}\Big),
\end{aligned}
$$

from which, we infer

$$
\begin{aligned}
\|e^{\mathfrak{K}t'}u_\Phi\|_{\widetilde{L}^\infty_t(\mathcal{B}^s)} + \sqrt{\lambda}\|e^{\mathfrak{K}t'}u_\Phi\|_{\widetilde{L}^2_{t,\dot\theta(t)}(\mathcal{B}^{s+\frac12})} + \|e^{\mathfrak{K}t'}\partial_y u_\Phi\|_{\widetilde{L}^2_t(\mathcal{B}^s)} \leq C\Big(\big\|e^{a|D_x|}u_0\big\|_{\mathcal{B}^s} \\
+ \|e^{\mathfrak{K}t'}u_\Phi\|_{\widetilde{L}^2_{t,\dot\theta(t)}(\mathcal{B}^{s+\frac12})} + \|u_\Phi\|_{\widetilde{L}^\infty_t(\mathcal{B}^{\frac32})}^{\frac14}\|e^{\mathfrak{K}t'}\partial_y u_\Phi\|_{\widetilde{L}^2_t(\mathcal{B}^s)}^{\frac12}\|e^{\mathfrak{K}t'}u_\Phi\|_{\widetilde{L}^2_{t,\dot\theta(t)}(\mathcal{B}^{s+\frac12})}^{\frac12}\Big).
\end{aligned}
$$

Applying Young's inequality yields

$$
\begin{aligned}
C\|u_\Phi\|_{\widetilde{L}^\infty_t(\mathcal{B}^{\frac32})}^{\frac14}\|e^{\mathfrak{K}t'}\partial_y u_\Phi\|_{\widetilde{L}^2_t(\mathcal{B}^s)}^{\frac12}\|e^{\mathfrak{K}t'}u_\Phi\|_{\widetilde{L}^2_{t,\dot\theta(t)}(\mathcal{B}^{s+\frac12})}^{\frac12} \\
\leq C\|u_\Phi\|_{\widetilde{L}^\infty_t(\mathcal{B}^{\frac32})}^{\frac12}\|e^{\mathfrak{K}t'}u_\Phi\|_{\widetilde{L}^2_{t,\dot\theta(t)}(\mathcal{B}^{s+\frac12})} + \frac12\|e^{\mathfrak{K}t'}\partial_y u_\Phi\|_{\widetilde{L}^2_t(\mathcal{B}^s)}.
\end{aligned}
$$

Therefore if we take

$$
\lambda \geq C^2\big(1+\|u_\Phi\|_{\widetilde{L}^\infty_t(\mathcal{B}^{\frac32})}\big),
\tag{4.18}
$$

we obtain

$$
\|e^{\mathfrak{K}t'}u_\Phi\|_{\widetilde{L}^\infty_t(\mathcal{B}^s)} + \|e^{\mathfrak{K}t'}\partial_y u_\Phi\|_{\widetilde{L}^2_t(\mathcal{B}^s)} \leq C\big\|e^{a|D_x|}u_0\big\|_{\mathcal{B}^s}.
\tag{4.19}
$$

which in particular implies that under the condition (4.18), there holds

$$
\|u_\Phi\|_{\widetilde{L}^\infty_t(\mathcal{B}^{\frac32})} \leq C\big\|e^{a|D_x|}u_0\big\|_{\mathcal{B}^{\frac32}}.
$$

Then by taking $\lambda = C^2\big(1+\|e^{a|D_x|}u_0\|_{\mathcal{B}^{\frac{3}{2}}}\big)$, (4.18) holds. Therefore under the condition (1.11), both (4.11) and (4.18) hold, and thus (4.19) holds for any $t>0$, which leads to (4.12). This completes the proof of the proposition. □

Proof of Proposition 4.2. Due to $\partial_x u + \partial_y v = 0$, we get, by applying ∂_y to (1.5), that

$$\partial_t\partial_y u + u\partial_x\partial_y u + v\partial_y^2 u - \partial_y^3 u + \partial_x\partial_y p = 0,$$

from which, using that $-\partial_y^2 u + \partial_x p$ is vanishing on the boundary, we get, by using a similar derivation of (4.7), that

$$\begin{aligned}
&\frac{1}{2}\|e^{\mathfrak{R}t'}\Delta_k^{\rm h}\partial_y u_\Phi\|_{L^\infty_t(L^2)}^2 + \lambda 2^k\int_0^t \dot\theta(t')\|e^{\mathfrak{R}t'}\Delta_k^{\rm h}\partial_y u_\Phi(t')\|_{L^2}^2\,dt' + \frac{1}{2}\|e^{\mathfrak{R}t'}\Delta_k^{\rm h}\partial_y^2 u_\Phi\|_{L^2_t(L^2)}^2\\
\leq&\frac{1}{2}\big\|e^{a|D_x|}\Delta_k^{\rm h}\partial_y u_0\big\|_{L^2}^2 + \int_0^t\big|\big(e^{\mathfrak{R}t'}\Delta_k^{\rm h}\left(u\partial_x\partial_y u\right)_\Phi\,|e^{\mathfrak{R}t'}\Delta_k^{\rm h}\partial_y u_\Phi\big)_{L^2}\big|\,dt'\\
&+\int_0^t\big|\big(e^{\mathfrak{R}t'}\Delta_k^{\rm h}\left(v\partial_y^2 u\right)_\Phi\,|e^{\mathfrak{R}t'}\Delta_k^{\rm h}\partial_y u_\Phi\big)_{L^2}\big|\,dt' + \int_0^t\big|\big(e^{\mathfrak{R}t'}\Delta_k^{\rm h}\partial_x p_\Phi|e^{\mathfrak{R}t'}\Delta_k^{\rm h}\partial_y^2 u_\Phi\big)_{L^2}\big|\,dt'.
\end{aligned}\tag{4.20}$$

It follows from the proof of Lemma 3.1 that for any $s>0$

$$\int_0^t\big|\big(\Delta_k^{\rm h}\big(T^{\rm h}_u\partial_x\partial_y u + R^{\rm h}(u,\partial_x\partial_y u)\big)_\Phi|\Delta_k^{\rm h}\partial_y u_\Phi\big)_{L^2}\big|\,dt' \lesssim d_k^2 2^{-2ks}\|\partial_y u_\Phi\|_{\widetilde L^2_{t,\dot\theta(t)}(\mathcal{B}^{s+\frac{1}{2}})}.$$

While we deduce from Lemma 2.1 and Definition 2.3 that

$$\begin{aligned}
&\int_0^t\big|\big(\Delta_k^{\rm h}(T^{\rm h}_{\partial_x\partial_y u}u)_\Phi\ |\ \Delta_k^{\rm h}\partial_y u_\Phi\big)_{L^2}\big|\,dt'\\
\lesssim&\sum_{|k'-k|\leq 4}\int_0^t\|S^{\rm h}_{k'-1}\partial_x\partial_y u_\Phi(t')\|_{L^\infty_{\rm h}(L^2_{\rm v})}\|\Delta^{\rm h}_{k'}u_\Phi(t')\|_{L^2_{\rm h}(L^\infty_{\rm v})}\|\Delta_k^{\rm h}\partial_y u_\Phi(t')\|_{L^2}\,dt'\\
\lesssim&\sum_{|k'-k|\leq 4}2^{k'}\int_0^t\|\partial_y u_\Phi(t')\|_{\mathcal{B}^{\frac{1}{2}}}\|\Delta^{\rm h}_{k'}\partial_y u_\Phi(t')\|_{L^2}\|\Delta_k^{\rm h}\partial_y u_\Phi(t')\|_{L^2}\,dt'\\
\lesssim&\sum_{|k'-k|\leq 4}2^{k'}\Big(\int_0^t\|\partial_y u_\Phi(t')\|_{\mathcal{B}^{\frac{1}{2}}}\|\Delta^{\rm h}_{k'}\partial_y u_\Phi(t')\|_{L^2}^2\,dt'\Big)^{\frac{1}{2}}\\
&\qquad\qquad\times\Big(\int_0^t\|\partial_y u_\Phi(t')\|_{\mathcal{B}^{\frac{1}{2}}}\|\Delta_k^{\rm h}\partial_y u_\Phi(t')\|_{L^2}^2\,dt'\Big)^{\frac{1}{2}}\\
\lesssim& d_k^2 2^{-2ks}\|\partial_y u_\Phi\|_{\widetilde L^2_{t,\dot\theta(t)}(\mathcal{B}^{s+\frac{1}{2}})}^2.
\end{aligned}\tag{4.21}$$

As a result, it comes out that for any $s>0$,

$$\int_0^t\big|\big(\Delta_k^{\rm h}(u\partial_x\partial_y u)_\Phi|\Delta_k^{\rm h}\partial_y u_\Phi\big)_{L^2}\big|\,dt' \lesssim d_k^2 2^{-2ks}\|\partial_y u_\Phi\|_{\widetilde L^2_{t,\dot\theta(t)}(\mathcal{B}^{s+\frac{1}{2}})}^2.\tag{4.22}$$

On the other hand, we deduce from Lemma 2.1 and (3.18) that for any $s > 0$

$$\begin{aligned}
&\int_0^t \bigl|\bigl(\Delta_k^{\mathrm{h}}\bigl(R^{\mathrm{h}}(v,\partial_y^2 u)\bigr)_\Phi|\Delta_k^{\mathrm{h}}\partial_y u_\Phi\bigr)_{L^2}\bigr|\,dt'\\
\lesssim& 2^{\frac{k}{2}}\sum_{k'\geq k-3}\int_0^t \|\Delta_{k'}^{\mathrm{h}} v_\Phi(t')\|_{L^2_{\mathrm{h}}(L^\infty_{\mathrm{v}})}\|\widetilde{\Delta}_{k'}^{\mathrm{h}}\partial_y^2 u_\Phi(t')\|_{L^2}\|\Delta_{k'}^{\mathrm{h}}\partial_y u_\Phi(t')\|_{L^2}\,dt'\\
\lesssim& 2^{\frac{k}{2}}\sum_{k'\geq k-3}\int_0^t \|u_\Phi(t')\|_{\mathcal{B}^{\frac{3}{2}}}^{\frac{1}{2}}\|\partial_y u_\Phi(t')\|_{\mathcal{B}^{\frac{1}{2}}}^{\frac{1}{2}}\|\widetilde{\Delta}_{k'}^{\mathrm{h}}\partial_y^2 u_\Phi(t')\|_{L^2}\|\Delta_{k'}^{\mathrm{h}}\partial_y u_\Phi(t')\|_{L^2}\,dt'\\
\lesssim& 2^{\frac{k}{2}}\sum_{k'\geq k-3}\|u_\Phi\|_{L^\infty_t(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}}\|\widetilde{\Delta}_{k'}^{\mathrm{h}}\partial_y^2 u_\Phi\|_{L^2_t(L^2)}\Bigl(\int_0^t \|\partial_y u_\Phi(t')\|_{\mathcal{B}^{\frac{1}{2}}}\|\Delta_{k'}^{\mathrm{h}}\partial_y u_\Phi(t')\|_{L^2}^2\,dt'\Bigr)^{\frac{1}{2}}\\
\lesssim& d_k^2 2^{-2ks}\|u_\Phi\|_{L^\infty_t(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}}\|\partial_y^2 u_\Phi\|_{\widetilde{L}^2_t(\mathcal{B}^s)}\|\partial_y u_\Phi\|_{\widetilde{L}^2_{t,\dot{\theta}(t)}(\mathcal{B}^{s+\frac{1}{2}})}.
\end{aligned}$$

And the proof of (4.16) ensures that

$$\begin{aligned}
&\int_0^t \bigl|\bigl(\Delta_k^{\mathrm{h}}(T_v^{\mathrm{h}}\partial_y^2 u)_\Phi \mid \Delta_k^{\mathrm{h}}\partial_y u_\Phi\bigr)_{L^2}\bigr|\,dt'\\
&\qquad\qquad\lesssim d_k^2 2^{-2ks}\|u_\Phi\|_{L^\infty_t(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}}\|\partial_y^2 u_\Phi\|_{\widetilde{L}^2_t(\mathcal{B}^s)}\|\partial_y u_\Phi\|_{\widetilde{L}^2_{t,\dot{\theta}(t)}(\mathcal{B}^{s+\frac{1}{2}})}.
\end{aligned}$$

Finally, by using integration by parts, we have

$$\begin{aligned}
\int_0^t \bigl|\bigl(\Delta_k^{\mathrm{h}}\bigl(T^{\mathrm{h}}_{\partial_y^2 u}v\bigr)_\Phi|\Delta_k^{\mathrm{h}}\partial_y u_\Phi\bigr)_{L^2}\bigr|\,dt' \leq& \int_0^t \bigl|\bigl(\Delta_k^{\mathrm{h}}\bigl(T^{\mathrm{h}}_{\partial_y u}\partial_y v\bigr)_\Phi|\Delta_k^{\mathrm{h}}\partial_y u_\Phi\bigr)_{L^2}\bigr|\,dt\\
&+\int_0^t \bigl|\bigl(\Delta_k^{\mathrm{h}}\bigl(T^{\mathrm{h}}_{\partial_y u}v\bigr)_\Phi|\Delta_k^{\mathrm{h}}\partial_y^2 u_\Phi\bigr)_{L^2}\bigr|\,dt.
\end{aligned}$$

Due to $\partial_x u + \partial_y v = 0$, we deduce from a similar derivation of (4.21) that

$$\begin{aligned}
&\int_0^t \bigl|\bigl(\Delta_k^{\mathrm{h}}\bigl(T^{\mathrm{h}}_{\partial_y u}\partial_y v\bigr)_\Phi|\Delta_k^{\mathrm{h}}\partial_y u_\Phi\bigr)_{L^2}\bigr|\,dt\\
\lesssim&\sum_{|k'-k|\leq 4}\int_0^t \|S_{k'-1}^{\mathrm{h}}\partial_y u_\Phi(t')\|_{L^\infty_{\mathrm{h}}(L^2_{\mathrm{v}})}\|\Delta_{k'}^{\mathrm{h}}\partial_x u_\Phi(t')\|_{L^2_{\mathrm{h}}(L^\infty_{\mathrm{v}})}\|\Delta_k^{\mathrm{h}}\partial_y u_\Phi(t')\|_{L^2}\,dt'\\
\lesssim&\sum_{|k'-k|\leq 4}2^{k'}\int_0^t \|\partial_y u_\Phi(t')\|_{\mathcal{B}^{\frac{1}{2}}}\|\Delta_{k'}^{\mathrm{h}}\partial_y u_\Phi(t')\|_{L^2}\|\Delta_k^{\mathrm{h}}\partial_y u_\Phi(t')\|_{L^2}\,dt'\\
\lesssim& d_k^2 2^{-2ks}\|\partial_y u_\Phi\|_{\widetilde{L}^2_{t,\dot{\theta}(t)}(\mathcal{B}^{s+\frac{1}{2}})}^2.
\end{aligned}$$

While we observe that

$$\begin{aligned}
&\int_0^t \big|\big(\Delta_k^{\rm h}\big(T^{\rm h}_{\partial_y u}v\big)_\Phi|\Delta_k^{\rm h}\partial_y^2 u_\Phi\big)_{L^2}\big|\,dt\\
&\lesssim \sum_{|k'-k|\leq 4}\int_0^t \|S^{\rm h}_{k'-1}\partial_y u_\Phi(t')\|_{L^\infty_{\rm h}(L^2_{\rm v})}\|\Delta^{\rm h}_{k'}v_\Phi(t')\|_{L^2_{\rm h}(L^\infty_{\rm v})}\|\Delta_k^{\rm h}\partial_y^2u_\Phi(t')\|_{L^2}\,dt'\\
&\lesssim \sum_{|k'-k|\leq 4}2^{k'}\int_0^t \|\partial_y u_\Phi(t')\|_{\mathcal{B}^{\frac12}}\|\Delta^{\rm h}_{k'}u_\Phi(t')\|_{L^2}\|\Delta_k^{\rm h}\partial_y^2u_\Phi(t')\|_{L^2}\,dt'\\
&\lesssim d_k^2 2^{-2ks}\|\partial_y u_\Phi\|_{\widetilde{L}^2_t(\mathcal{B}^{\frac12})}\|u_\Phi\|_{L^\infty_t(\mathcal{B}^{s+1})}\|\partial_y^2u_\Phi\|_{\widetilde{L}^2_t(\mathcal{B}^s)}.
\end{aligned}$$

This gives rise to

$$\begin{aligned}
\int_0^t \big|\big(\Delta_k^{\rm h}\big(T^{\rm h}_{\partial_y^2 u}v\big)_\Phi|\Delta_k^{\rm h}\partial_y u_\Phi\big)_{L^2}\big|\,dt' \lesssim & d_k^2 2^{-2ks}\Big(\|\partial_y u_\Phi\|^2_{\widetilde{L}^2_{t,\dot\theta(t)}(\mathcal{B}^{s+\frac12})}\\
&+\|\partial_y u_\Phi\|_{\widetilde{L}^2_t(\mathcal{B}^{\frac12})}\|u_\Phi\|_{L^\infty_t(\mathcal{B}^{s+1})}\|\partial_y^2u_\Phi\|_{\widetilde{L}^2_t(\mathcal{B}^s)}\Big).
\end{aligned}$$

By summarizing the above estimates, we obtain

$$\begin{aligned}
&\int_0^t \big|\big(\Delta_k^{\rm h}(v\partial_y^2 u)_\Phi|\Delta_k^{\rm h}\partial_y u_\Phi\big)_{L^2}\big|\,dt'\\
&\lesssim d_k^2 2^{-2ks}\Big(\|u_\Phi\|^{\frac12}_{L^\infty_t(\mathcal{B}^{\frac32})}\|\partial_y u_\Phi\|_{\widetilde{L}_{t,\dot\theta(t)}(\mathcal{B}^{s+\frac12})}\|\partial_y^2u_\Phi\|_{\widetilde{L}^2_t(\mathcal{B}^s)}\\
&\quad+\|\partial_y u_\Phi\|^2_{\widetilde{L}_{t,\dot\theta(t)}(\mathcal{B}^{s+\frac12})}+\|\partial_y u_\Phi\|_{\widetilde{L}^2_t(\mathcal{B}^{\frac12})}\|u_\Phi\|_{L^\infty_t(\mathcal{B}^{s+1})}\|\partial_y^2u_\Phi\|_{\widetilde{L}^2_t(\mathcal{B}^s)}\Big).
\end{aligned}\tag{4.23}$$

To handle the last term in (4.20), we denote

$$I_k(t)\stackrel{\rm def}{=}\big(e^{\mathfrak{K}t}\Delta_k^h\partial_x p_\Phi(t)|e^{\mathfrak{K}t}\Delta_k^h\partial_y^2u_\Phi(t)\big)_{L^2}.$$

Then in view of (4.2), we write

$$\begin{aligned}
I_k(t)=&\int_{\mathbb{R}} e^{\mathfrak{K}t}\Delta_k^h\partial_x p_\Phi\cdot e^{\mathfrak{K}t}\big(\Delta_k^{\rm h}\partial_y u(t,x,1)-\Delta_k^{\rm h}\partial_y u(t,x,0)\big)\,dx\\
=&\int_{\mathbb{R}}\big(e^{\mathfrak{K}t}\Delta_k^{\rm h}(\partial_y u_\Phi(t,x,1)-\partial_y u_\Phi(t,x,0))\big)^2\,dx\\
&-\frac12\int_{\mathbb{R}}\Big(e^{\mathfrak{K}t}\int_0^1\Delta_k^h\partial_x(u^2)_\Phi\,dy\Big)\cdot e^{\mathfrak{K}t}\Delta_k^{\rm h}(\partial_y u_\Phi(t,x,1)-\partial_y u_\Phi(t,x,0))\,dx,
\end{aligned}$$

from which, we infer

$$|I_k(t)|\leq C\big(\|e^{\mathfrak{K}t}\Delta_k^{\rm h}\partial_y u_\Phi(t)\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}+\|e^{\mathfrak{K}t}\Delta_k^{\rm h}\partial_x(u^2)_\Phi(t)\|^2_{L^1_{\rm v}(L^2_{\rm h})}\big).$$

Applying the Bony decomposition yields for any $s>0$,

$$\begin{aligned}
\|e^{\mathfrak{K}t'}\Delta_k^{\rm h}\partial_x(u^2)_\Phi\|_{L^2_t(L^1_{\rm v}(L^2_{\rm h}))}\leq&\|e^{\mathfrak{K}t'}\Delta_k^{\rm h}\partial_x(u^2)_\Phi\|_{L^2_t(L^2)}\\
\lesssim& d_k2^{-ks}\|u_\Phi\|_{L^2_t(L^\infty)}\|e^{\mathfrak{K}t'}u_\Phi\|_{\widetilde{L}^\infty_t(\mathcal{B}^{s+1})}\\
\lesssim& d_k2^{-ks}\|\partial_y u_\Phi\|_{L^2_t(\mathcal{B}^{\frac12})}\|e^{\mathfrak{K}t'}u_\Phi\|_{\widetilde{L}^\infty_t(\mathcal{B}^{s+1})}.
\end{aligned}$$

While notice that

$$\int_0^1 \Delta_k^{\mathrm{h}}\partial_y u(t,x,y)\,dy=0,$$

for any fixed (t,x), there exists $Y_0^k(t,x)$ so that $\Delta_k^{\mathrm{h}}\partial_y u_\Phi(t,x,Y_0^k(t,x))=0$. Then we have

$$\begin{aligned}\left(\Delta_k^{\mathrm{h}}\partial_y u_\Phi(t,x,y)\right)^2=&\int_{Y_0^k(t,x)}^y \partial_y\left(\Delta_k^{\mathrm{h}}\partial_y u_\Phi(t,x,y')\right)^2\,dy'\\ \leq&2\|\Delta_k^{\mathrm{h}}\partial_y u_\Phi(t,x,\cdot)\|_{L^2_{\mathrm{v}}}\|\Delta_k^{\mathrm{h}}\partial_y^2 u_\Phi(t,x,\cdot)\|_{L^2_{\mathrm{v}}},\end{aligned}$$

which implies

$$\|e^{\mathfrak{R}t}\Delta_k^{\mathrm{h}}\partial_y u_\Phi\|^2_{L^\infty_{\mathrm{v}}(L^2_{\mathrm{h}})}\leq 2\|e^{\mathfrak{R}t}\Delta_k^{\mathrm{h}}\partial_y u_\Phi(t)\|_{L^2}\|e^{\mathfrak{R}t}\Delta_k^{\mathrm{h}}\partial_y^2 u_\Phi(t)\|_{L^2}.$$

As a result, it comes out

$$\begin{aligned}\int_0^t |I_k(t')|\,dt'\leq&\frac14\|e^{\mathfrak{R}t'}\Delta_k^{\mathrm{h}}\partial_y^2u_\Phi\|^2_{L^2_t(L^2)}+C\|e^{\mathfrak{R}t'}\Delta_k^{\mathrm{h}}\partial_y u_\Phi\|^2_{L^2_t(L^2)}\\ &+Cd_k^2 2^{-2ks}\|\partial_y u_\Phi\|^2_{L^2_t(\mathcal{B}^{\frac12})}\|e^{\mathfrak{R}t'}u_\Phi\|^2_{\widetilde{L}^\infty_t(\mathcal{B}^{s+1})}\\ \leq&Cd_k^2 2^{-2ks}\big(\|e^{\mathfrak{R}t'}\partial_y u_\Phi\|^2_{\widetilde{L}^2_t(\mathcal{B}^s)}+\|\partial_y u_\Phi\|^2_{L^2_t(\mathcal{B}^{\frac12})}\|e^{\mathfrak{R}t'}u_\Phi\|^2_{\widetilde{L}^\infty_t(\mathcal{B}^{s+1})}\big)\\ &+\frac14\|e^{\mathfrak{R}t'}\Delta_k^{\mathrm{h}}\partial_y^2u_\Phi\|^2_{L^2_t(L^2)}.\end{aligned}\tag{4.24}$$

By inserting (4.22), (4.23) and (4.24) into (4.20) and then repeating the last step of the proof of Proposition 4.1, we obtain

$$\begin{aligned}&\|e^{\mathfrak{R}t'}\partial_y u_\Phi\|_{\widetilde{L}^\infty_t(\mathcal{B}^s)}+\sqrt{\lambda}\|e^{\mathfrak{R}t'}\partial_y u_\Phi\|_{\widetilde{L}^2_{t,\dot\theta(t)}(\mathcal{B}^{s+\frac12})}+\|e^{\mathfrak{R}t'}\partial_y^2 u_\Phi\|_{\widetilde{L}^2_t(\mathcal{B}^s)}\\ &\leq\big\|e^{a|D_x|}\partial_y u_0\big\|_{\mathcal{B}^s}+C\Big(\|e^{\mathfrak{R}t'}\partial_y u_\Phi\|_{\widetilde{L}^2_{t,\dot\theta(t)}(\mathcal{B}^{s+\frac12})}+\|e^{\mathfrak{R}t'}\partial_y u_\Phi\|_{\widetilde{L}^2_t(\mathcal{B}^s)}\\ &\quad+\|\partial_y u_\Phi\|_{L^2_t(\mathcal{B}^{\frac12})}\|e^{\mathfrak{R}t'}u_\Phi\|_{\widetilde{L}^\infty_t(\mathcal{B}^{s+1})}+\|e^{\mathfrak{R}t'}\partial_y^2u_\Phi\|^{\frac12}_{\widetilde{L}^2_t(\mathcal{B}^s)}\\ &\quad\times\big(\|u_\Phi\|^{\frac14}_{L^\infty_t(\mathcal{B}^{\frac32})}\|e^{\mathfrak{R}t'}\partial_y u_\Phi\|^{\frac12}_{\widetilde{L}_{t,\dot\theta(t)}(\mathcal{B}^{s+\frac12})}+\|\partial_y u_\Phi\|^{\frac12}_{\widetilde{L}^2_t(\mathcal{B}^{\frac12})}\|e^{\mathfrak{R}t'}u_\Phi\|^{\frac12}_{L^\infty_t(\mathcal{B}^{s+1})}\big)\Big).\end{aligned}$$

Applying Young's inequality yields

$$\begin{aligned}&\|e^{\mathfrak{R}t'}\partial_y u_\Phi\|_{\widetilde{L}^\infty_t(\mathcal{B}^s)}+\sqrt{\lambda}\|e^{\mathfrak{R}t'}\partial_y u_\Phi\|_{\widetilde{L}^2_{t,\dot\theta(t)}(\mathcal{B}^{s+\frac12})}+\|e^{\mathfrak{R}t'}\partial_y^2 u_\Phi\|_{\widetilde{L}^2_t(\mathcal{B}^s)}\\ &\leq\big\|e^{a|D_x|}\partial_y u_0\big\|_{\mathcal{B}^s}+C\Big(\big(1+\|u_\Phi\|^{\frac12}_{L^\infty_t(\mathcal{B}^{\frac32})}\big)\|e^{\mathfrak{R}t'}\partial_y u_\Phi\|_{\widetilde{L}^2_{t,\dot\theta(t)}(\mathcal{B}^{s+\frac12})}\\ &\quad+\|e^{\mathfrak{R}t'}\partial_y u_\Phi\|_{\widetilde{L}^2_t(\mathcal{B}^s)}+\|\partial_y u_\Phi\|_{\widetilde{L}^2_t(\mathcal{B}^{\frac12})}\|e^{\mathfrak{R}t'}u_\Phi\|_{L^\infty_t(\mathcal{B}^{s+1})}\Big)+\frac12\|e^{\mathfrak{R}t'}\partial_y^2 u_\Phi\|_{\widetilde{L}^2_t(\mathcal{B}^s)},\end{aligned}$$

from which, (1.9), (1.10) and Proposition 4.1, we infer

$$\begin{aligned}&\|e^{\mathfrak{R}t'}\partial_y u_\Phi\|_{\widetilde{L}^\infty_t(\mathcal{B}^s)}+\sqrt{\lambda}\|e^{\mathfrak{R}t'}\partial_y u_\Phi\|_{\widetilde{L}^2_{t,\dot\theta(t)}(\mathcal{B}^{s+\frac12})}+\|e^{\mathfrak{R}t'}\partial_y^2 u_\Phi\|_{\widetilde{L}^2_t(\mathcal{B}^s)}\\ &\leq\big\|e^{a|D_x|}\partial_y u_0\big\|_{\mathcal{B}^s}+C\Big(\big(1+\big\|e^{a|D_x|}u_0\big\|^{\frac12}_{\mathcal{B}^{\frac32}}\big)\|e^{\mathfrak{R}t'}\partial_y u_\Phi\|_{\widetilde{L}^2_{t,\dot\theta(t)}(\mathcal{B}^{s+\frac12})}\\ &\qquad+\big\|e^{a|D_x|}u_0\big\|_{\mathcal{B}^s}+\big\|e^{a|D_x|}u_0\big\|_{\mathcal{B}^{s+1}}\Big).\end{aligned}$$

Taking $\lambda = C^2\big(1 + \|e^{a|D_x|}u_0\|_{\mathcal{B}^{\frac{3}{2}}}\big)$ in the above inequality leads to (4.13). This completes the proof of Proposition 4.2. $\square$

5. The Convergence to the hydrostatic Navier-Stokes system

In this section, we justify the limit from the scaled anisotropic Navier-Stokes system to the hydrostatic Navier-Stokes system in a 2-D striped domain. As in the first sections, the main idea will be to obtain a control of the difference between the two solutions in analytic spaces, by using energy estimates with exponential weights in the Fourier variable. As previously, the exponent of the exponential weight is depending on time but shall take into account now the "loss of the analyticity" for both solutions, of the rescaled Navier-Stokes system and respectively of the hydrostatic Navier-Stokes equations.

To this end, we introduce

$$w_\varepsilon^1 \stackrel{\text{def}}{=} u^\varepsilon - u, \quad w_\varepsilon^2 \stackrel{\text{def}}{=} v^\varepsilon - v, \quad q_\varepsilon \stackrel{\text{def}}{=} p^\varepsilon - p.$$

Then $(w_\varepsilon^1, w_\varepsilon^2, q_\varepsilon)$ verifies

$$\left\{\begin{array}{l} \partial_t w_\varepsilon^1 - \varepsilon^2\partial_x^2 w_\varepsilon^1 - \partial_y^2 w_\varepsilon^1 + \partial_x q_\varepsilon = R_\varepsilon^1 \ \text{ in } \mathcal{S}\times]0,\infty[,\\ \varepsilon^2\left(\partial_t w_\varepsilon^2 - \varepsilon^2\partial_x^2 w_\varepsilon^2 - \partial_y^2 w_\varepsilon^2\right) + \partial_y q_\varepsilon = R_\varepsilon^2,\\ \partial_x w_\varepsilon^1 + \partial_y w_\varepsilon^2 = 0,\\ \left(w_\varepsilon^1, w_\varepsilon^2\right)|_{y=0} = \left(w_\varepsilon^1, w_\varepsilon^2\right)|_{y=1} = 0,\\ \left(w_\varepsilon^1, w_\varepsilon^2\right)|_{t=0} = \left(u_0^\varepsilon - u_0, v_0^\varepsilon - v_0\right), \end{array}\right. \tag{5.1}$$

where v_0 is determined from u_0 via $\partial_x u_0 + \partial_y v_0 = 0$ and $v_0|_{y=0} = v_0|_{y=1} = 0$, and

$$\begin{aligned} R_\varepsilon^1 &= \varepsilon^2\partial_x^2 u - \big(u^\varepsilon\partial_x u^\varepsilon - u\partial_x u\big) - \big(v^\varepsilon\partial_y u^\varepsilon - v\partial_y u\big),\\ R_\varepsilon^2 &= -\varepsilon^2\big(\partial_t v - \varepsilon^2\partial_x^2 v - \partial_y^2 v + u^\varepsilon\partial_x v^\varepsilon + v^\varepsilon\partial_y v^\varepsilon\big). \end{aligned} \tag{5.2}$$

Let us define

$$u_\Theta(t,x,y) \stackrel{\text{def}}{=} \mathcal{F}^{-1}_{\xi\to x}\big(e^{\Theta(t,\xi)}\widehat{u}(t,\xi,y)\big) \quad \text{and} \quad \Theta(t,\xi) \stackrel{\text{def}}{=} \big(a - \mu\zeta(t)\big)|\xi|, \tag{5.3}$$

where $\mu \geq \lambda$ will be determined later, and $\zeta(t)$ is given by

$$\zeta(t) = \int_0^t \big(\|(\partial_y u_\Psi^\varepsilon, \varepsilon\partial_x u_\Psi^\varepsilon)(t')\|_{\mathcal{B}^{\frac{1}{2}}} + \|\partial_y u_\Phi(t')\|_{\mathcal{B}^{\frac{1}{2}}}\big)\,dt'.$$

Similar notation for $(w_\varepsilon^1)_\Theta$ and so on.

It is easy to observe that if we take c_0 in (1.7) and c_1 in (1.9) small enough, then $\Theta(t) \geq 0$ and

$$\Theta(t,\xi) \leq \min\left(\Psi(t,\xi), \Phi(t,\xi)\right).$$

Thanks to Theorem 1.2, we deduce that

$$\|u_\Psi^\varepsilon\|_{\widetilde{L}^\infty(\mathbb{R}^+;\mathcal{B}^{\frac{1}{2}})} + \|u_\Phi\|_{\widetilde{L}^\infty(\mathbb{R}^+;\mathcal{B}^{\frac{1}{2}}\cap\mathcal{B}^{\frac{5}{2}})} + \|\partial_y u_\Phi\|_{\widetilde{L}^2(\mathbb{R}^+;\mathcal{B}^{\frac{1}{2}}\cap\mathcal{B}^{\frac{5}{2}})} + \|(\partial_t u)_\Phi\|_{\widetilde{L}^2(\mathbb{R}^+;\mathcal{B}^{\frac{3}{2}})} \leq M, \tag{5.4}$$

where u_Ψ^ε and u_Φ are determined respectively by (3.1) and (4.3) and $M \geq 1$ is a constant independent of ε.

In what follows, we shall neglect the subscript ε in $(w_\varepsilon^1, w_\varepsilon^2)$.

Proof of Theorem 1.3. In view of (5.1), we get, by using a similar derivation of (3.7), that

$$\begin{aligned}
&\|\Delta_k^{\rm h}(w_\Theta^1,\varepsilon w_\Theta^2)\|_{L_t^\infty(L^2)}^2+\mu 2^k\int_0^t\dot\zeta(t')\|\Delta_k^{\rm h}(w_\Theta^1,\varepsilon w_\Theta^2)(t')\|_{L^2}^2\,dt'\\
&\quad+\int_0^t\bigl(\|\Delta_k^{\rm h}\partial_y(w_\Theta^1,\varepsilon w_\Theta^2)(t')\|_{L^2}^2+\varepsilon^2 2^{2k}\|\Delta_k^{\rm h}(w_\Theta^1,\varepsilon w_\Theta^2)(t')\|_{L^2}^2\bigr)\,dt'\\
&\leq\bigl\|e^{a|D_x|}\Delta_k^{\rm h}(u_0^\varepsilon-u_0,\varepsilon(v_0^\varepsilon-v_0))\bigr\|_{L^2}^2\\
&\quad+\int_0^t\bigl|\bigl(\Delta_k^{\rm h}R_\Theta^1|\Delta_k^{\rm h}w_\Theta^1\bigr)_{L^2}\bigr|\,dt'+\int_0^t\bigl|\bigl(\Delta_k^{\rm h}R_\Theta^2|\Delta_k^{\rm h}w_\Theta^2\bigr)_{L^2}\bigr|\,dt'.
\end{aligned}\tag{5.5}$$

We now claim that

$$\begin{aligned}
\int_0^t\bigl|\bigl(\Delta_k^{\rm h}R_\Theta^1|\Delta_k^{\rm h}w_\Theta^1\bigr)_{L^2}\bigr|\,dt'\lesssim d_k^2 2^{-k}\Bigl(&\varepsilon\|\partial_y u_\Theta\|_{\widetilde L_t^2(\mathcal B^{\frac32})}\|\varepsilon w_\Theta^1\|_{\widetilde L_t^2(\mathcal B^{\frac32})}\\
&+\|u_\Theta\|_{L_t^\infty(\mathcal B^{\frac32})}^{\frac12}\|\partial_y w_\Theta^1\|_{\widetilde L_t^2(\mathcal B^{\frac12})}\|w_\Theta^1\|_{\widetilde L_{t,\dot\zeta(t)}^2(\mathcal B^1)}+\|w_\Theta^1\|_{\widetilde L_{t,\dot\zeta(t)}^2(\mathcal B^1)}^2\Bigr),
\end{aligned}\tag{5.6}$$

and

$$\begin{aligned}
\int_0^t\bigl|\bigl(\Delta_k^{\rm h}R_\Theta^2|\Delta_k^{\rm h}w_\Theta^2\bigr)_{L^2}\bigr|\,dt'\lesssim d_k^2 2^{-k}\Bigl\{&\bigl\|(w_\Theta^1,\varepsilon w_\Theta^2)\bigr\|_{\widetilde L_{t,\dot\zeta(t)}^2(\mathcal B^1)}^2+\varepsilon^2\bigl\|(\partial_y w_\Theta^2,\varepsilon\partial_x w_\Theta^2)\bigr\|_{\widetilde L_t^2(\mathcal B^{\frac12})}\\
&\times\Bigl(\|(\partial_t u)_\Theta\|_{\widetilde L_t^2(\mathcal B^{\frac32})}+\|\partial_y u_\Theta\|_{\widetilde L_t^2(\mathcal B^{\frac32})}+\varepsilon\|\partial_y u_\Theta\|_{\widetilde L_t^2(\mathcal B^{\frac52})}\Bigr)\\
&+\varepsilon^2\|w_\Theta^2\|_{\widetilde L_{t,\dot\zeta(t)}^2(\mathcal B^1)}\Bigl(\|w_\Theta^2\|_{\widetilde L_{t,\dot\zeta(t)}^2(\mathcal B^1)}+\|u_\Theta^\varepsilon\|_{L_t^\infty(\mathcal B^{\frac12})}^{\frac12}\|\partial_y u_\Theta\|_{\widetilde L_t^2(\mathcal B^2)}\\
&\qquad+\|u_\Theta\|_{L_t^\infty(\mathcal B^{\frac32})}^{\frac12}\bigl(\|\partial_y w_\Theta^2\|_{\widetilde L_t^2(\mathcal B^{\frac12})}+\|\partial_y u_\Theta\|_{\widetilde L_t^2(\mathcal B^{\frac32})}\bigr)\Bigr)\Bigr\}.
\end{aligned}\tag{5.7}$$

By virtue of (5.4), (5.6) and (5.7), we infer

$$\begin{aligned}
\sum_{i=1}^2\int_0^t\bigl|\bigl(\Delta_k^{\rm h}R_\Theta^i|\Delta_k^{\rm h}w_\Theta^i\bigr)_{L^2}\bigr|\,dt'\lesssim&\, d_k^2 2^{-k}\Bigl(M\varepsilon\bigl\|\bigl(\varepsilon\partial_x(w_\Theta^1,\varepsilon w_\Theta^2),\varepsilon\partial_y w_\Theta^2\bigr)\bigr\|_{\widetilde L_t^2(\mathcal B^{\frac12})}\\
&+M^{\frac12}\bigl\|\partial_y(w_\Theta^1,\varepsilon w_\Theta^2)\bigr\|_{\widetilde L_t^2(\mathcal B^{\frac12})}\bigl\|(w_\Theta^1,\varepsilon w_\Theta^2)\bigr\|_{\widetilde L_{t,\dot\zeta(t)}^2(\mathcal B^1)}\\
&+M^{\frac32}\varepsilon\|\varepsilon w_\Theta^2\|_{\widetilde L_{t,\dot\zeta(t)}^2(\mathcal B^1)}+\bigl\|(w_\Theta^1,\varepsilon w_\Theta^2)\bigr\|_{\widetilde L_{t,\dot\zeta(t)}^2(\mathcal B^1)}^2\Bigr),
\end{aligned}$$

from which and (5.5), we deduce that

$$\begin{aligned}
&\|(w_\Theta^1,\varepsilon w_\Theta^2)\|_{\widetilde L_t^\infty(\mathcal B^{\frac12})}+\mu^{\frac12}\|(w_\Theta^1,\varepsilon w_\Theta^2)\|_{\widetilde L_{t,\dot\zeta_1(t)}^2(\mathcal B^1)}+\|\partial_y(w_\Theta^1,\varepsilon w_\Theta^2)\|_{\widetilde L_t^2(\mathcal B^{\frac12})}\\
&+\varepsilon\|(w_\Theta^1,\varepsilon w_\Theta^2)\|_{\widetilde L_t^2(\mathcal B^{\frac32})}\leq C\bigl\|e^{a|D_x|}\left(u_0^\varepsilon-u_0,\varepsilon(v_0^\varepsilon-v_0)\right)\bigr\|_{\mathcal B^{\frac12}}\\
&\qquad+C\Bigl(\sqrt{M\varepsilon}\bigl\|\bigl(\varepsilon\partial_x(w_\Theta^1,\varepsilon w_\Theta^2),\varepsilon\partial_y w_\Theta^2\bigr)\bigr\|_{\widetilde L_t^2(\mathcal B^{\frac12})}^{\frac12}\\
&\qquad+M^{\frac14}\bigl\|\partial_y(w_\Theta^1,\varepsilon w_\Theta^2)\bigr\|_{\widetilde L_t^2(\mathcal B^{\frac12})}^{\frac12}\bigl\|(w_\Theta^1,\varepsilon w_\Theta^2)\bigr\|_{\widetilde L_{t,\dot\zeta(t)}^2(\mathcal B^1)}^{\frac12}\\
&\qquad+M^{\frac34}\varepsilon^{\frac12}\|\varepsilon w_\Theta^2\|_{\widetilde L_{t,\dot\zeta(t)}^2(\mathcal B^1)}^{\frac12}+\bigl\|(w_\Theta^1,\varepsilon w_\Theta^2)\bigr\|_{\widetilde L_{t,\dot\zeta(t)}^2(\mathcal B^1)}\Bigr).
\end{aligned}\tag{5.8}$$

Applying Young's inequality gives rise to

$$\begin{aligned}&\|(w^1_\Theta,\varepsilon w^2_\Theta)\|_{\widetilde{L}^\infty_t(\mathcal{B}^{\frac12})}+\mu^{\frac12}\|(w^1_\Theta,\varepsilon w^2_\Theta)\|_{\widetilde{L}^2_{t,\dot\zeta(t)}(\mathcal{B}^1)}+\|\partial_y(w^1_\Theta,\varepsilon w^2_\Theta)\|_{\widetilde{L}^2_t(\mathcal{B}^{\frac12})}\\&\quad+\varepsilon^2\|(w^1_\Theta,\varepsilon w^2_\Theta)\|_{\widetilde{L}^2_t(\mathcal{B}^{\frac32})}\\&\leq C\Big(\big\|e^{a|D_x|}\,(u^\varepsilon_0-u_0,\varepsilon(v^\varepsilon_0-v_0))\big\|_{\mathcal{B}^{\frac12}}+M\big(\varepsilon+\big\|(w^1_\Theta,\varepsilon w^2_\Theta)\big\|_{\widetilde{L}^2_{t,\dot\zeta(t)}(\mathcal{B}^1)}\big)\Big).\end{aligned}$$

Taking $\mu=C^2M^2$ leads to (1.11). This completes the proof of the theorem. $\square$

Now let us present the proof of (5.6) and (5.7).

Proof of (5.6). According (5.2), we write

$$R^1_\varepsilon=\varepsilon^2\partial_x^2u-\big(u^\varepsilon\partial_xw^1+w^1\partial_xu\big)-\big(v^\varepsilon\partial_yw^1+w^2\partial_yu\big).$$

We first observe that

$$\varepsilon^2\int_0^t\big|\big(\Delta^{\rm h}_k\partial_x^2u_\Theta|\Delta^{\rm h}_kw^1_\Theta\big)_{L^2}\big|\,dt'\leq Cd_k^22^{-k}\varepsilon\|\partial_yu_\Theta\|_{\widetilde{L}^2_t(\mathcal{B}^{\frac32})}\|\varepsilon w^1_\Theta\|_{\widetilde{L}^2_t(\mathcal{B}^{\frac32})}. \tag{5.9}$$

• The estimate of $\int_0^t\big|\big(\Delta^{\rm h}_k(u^\varepsilon\partial_xw^1+w^1\partial_xu)_\Theta|\Delta^{\rm h}_kw^1_\Theta\big)_{L^2}\big|\,dt'$.

It follows from Lemma 3.1 that

$$\int_0^t\big|\big(\Delta^{\rm h}_k(u^\varepsilon\partial_xw^1)_\Theta|\Delta^{\rm h}_kw^1_\Theta\big)_{L^2}\big|\,dt'\lesssim d_k^22^{-k}\|w^1_\Theta\|^2_{\widetilde{L}^2_{t,\dot\zeta(t)}(\mathcal{B}^1)}. \tag{5.10}$$

By applying Bony's decomposition (2.3) for the horizontal variable to $w^1\partial_xu$, we obtain

$$w^1\partial_xu=T^{\rm h}_{w^1}\partial_xu+T^{\rm h}_{\partial_xu}w^1+R^h(w^1,\partial_xu).$$

Notice that

$$\|\Delta^{\rm h}_{k'}\partial_xu_\Theta(t')\|_{L^2_{\rm h}(L^\infty_{\rm v})}\lesssim d_{k'}(t)\|u_\Theta(t')\|^{\frac12}_{\mathcal{B}^{\frac32}}\|\partial_yu_\Theta(t')\|^{\frac12}_{\mathcal{B}^{\frac12}},$$

we infer

$$\begin{aligned}&\int_0^t\big|\big(\Delta^{\rm h}_k(T^{\rm h}_{w^1}\partial_xu)_\Theta\mid\Delta^{\rm h}_kw^1_\Theta\big)_{L^2}\big|\,dt'\\&\lesssim\sum_{|k'-k|\leq4}\int_0^t\|S^{\rm h}_{k'-1}w^1_\Theta(t')\|_{L^\infty_{\rm h}(L^2_{\rm v})}\|\Delta^{\rm h}_{k'}\partial_xu_\Theta(t')\|_{L^2_{\rm h}(L^\infty_{\rm v})}\|\Delta^{\rm h}_kw^1_\Theta(t')\|_{L^2}\,dt'\\&\lesssim\sum_{|k'-k|\leq4}d_{k'}\|u_\Theta\|^{\frac12}_{L^\infty_t(\mathcal{B}^{\frac32})}\|S^{\rm h}_{k'-1}w^1_\Theta\|_{L^2_t(L^\infty_{\rm h}(L^2_{\rm v}))}\Big(\int_0^t\|\partial_yu_\Theta(t')\|_{\mathcal{B}^{\frac12}}\|\Delta^{\rm h}_kw^1_\Theta(t')\|^2_{L^2}\,dt'\Big)^{\frac12}\\&\lesssim d_k^22^{-k}\|u_\Theta\|^{\frac12}_{L^\infty_t(\mathcal{B}^{\frac32})}\|\partial_yw^1_\Theta\|_{\widetilde{L}^2_t(\mathcal{B}^{\frac12})}\|w^1_\Theta\|_{\widetilde{L}^2_{t,\dot\zeta(t)}(\mathcal{B}^1)}.\end{aligned}$$

While observing that

$$\begin{aligned}\|S^{\rm h}_{k'-1}\partial_xu_\Theta(t')\|_{L^\infty}&\lesssim\sum_{\ell\leq k'-2}2^{\frac{3\ell}{2}}\|\Delta^{\rm h}_\ell u_\Theta(t')\|^{\frac12}_{L^2}\|\Delta^{\rm h}_\ell\partial_yu_\Theta(t')\|^{\frac12}_{L^2}\\&\lesssim d_{k'}(t)2^{k'}\|\partial_yu_\Theta(t')\|_{\mathcal{B}^{\frac12}},\end{aligned}$$

we deduce

$$
\begin{aligned}
&\int_0^t \big|\big(\Delta_k^{\rm h}(T^{\rm h}_{\partial_x u} w^1)_\Theta \mid \Delta_k^{\rm h} w^1_\Theta\big)_{L^2}\big|\, dt'\\
&\lesssim \sum_{|k'-k|\le 4} \int_0^t \|S^{\rm h}_{k'-1}\partial_x u_\Theta(t')\|_{L^\infty} \|\Delta^{\rm h}_{k'} w^1_\Theta(t')\|_{L^2} \|\Delta^{\rm h}_k w^1_\Theta(t')\|_{L^2}\, dt'\\
&\lesssim \sum_{|k'-k|\le 4} 2^{k'} \Big(\int_0^t \|\Delta^{\rm h}_{k'} w^1_\Theta(t')\|^2_{L^2} \|\partial_y u_\Theta(t')\|_{\mathcal{B}^{\frac12}}\, dt'\Big)^{\frac12}\\
&\qquad\qquad\qquad\qquad \times \Big(\int_0^t \|\Delta^{\rm h}_{k} w^1_\Theta(t')\|^2_{L^2} \|\partial_y u_\Theta(t')\|_{\mathcal{B}^{\frac12}}\, dt'\Big)^{\frac12}\\
&\lesssim d_k^2 2^{-k} \|w^1_\Theta\|^2_{\widetilde{L}^2_{t,\dot\zeta(t)}(\mathcal{B}^1)}.
\end{aligned}
$$

Along the same line, we have

$$
\begin{aligned}
&\int_0^t \big|\big(\Delta_k^{\rm h}(R^{\rm h}(w^1, \partial_x u)_\Theta \mid \Delta_k^{\rm h} w^1_\Theta\big)_{L^2}\big|\, dt'\\
&\lesssim 2^{\frac k2} \sum_{k'\ge k-3} \int_0^t \|\Delta^{\rm h}_{k'} w^1_\Theta(t')\|_{L^2} \|\widetilde{\Delta}^{\rm h}_{k'}\partial_x u_\Theta(t')\|_{L^2_{\rm h}(L^\infty_{\rm v})} \|\Delta^{\rm h}_k w^1_\Theta(t')\|_{L^2}\, dt'\\
&\lesssim 2^{\frac k2} \sum_{k'\ge k-3} 2^{\frac{k'}{2}} \Big(\int_0^t \|\Delta^{\rm h}_{k'} w^1_\Theta(t')\|^2_{L^2} \|\partial_y u_\Theta(t')\|_{\mathcal{B}^{\frac12}}\, dt'\Big)^{\frac12}\\
&\qquad\qquad\qquad\qquad \times \Big(\int_0^t \|\Delta^{\rm h}_{k} w^1_\Theta(t')\|^2_{L^2} \|\partial_y u_\Theta(t')\|_{\mathcal{B}^{\frac12}}\, dt'\Big)^{\frac12}\\
&\lesssim d_k^2 2^{-k} \|w^1_\Theta\|^2_{\widetilde{L}^2_{t,\dot\zeta(t)}(\mathcal{B}^1)}.
\end{aligned}
$$

As a result, it comes out

$$
\begin{aligned}
&\int_0^t \big|\big(\Delta_k^{\rm h}(w^1\partial_x u)_\Theta \mid \Delta_k^{\rm h} w^1_\Theta\big)_{L^2}\big|\, dt'\\
&\lesssim d_k^2 2^{-k} \|w^1_\Theta\|_{\widetilde{L}^2_{t,\dot\zeta(t)}(\mathcal{B}^1)} \Big(\|w^1_\Theta\|_{\widetilde{L}^2_{t,\dot\zeta(t)}(\mathcal{B}^1)} + \|u_\Theta\|^{\frac12}_{L^\infty_t(\mathcal{B}^{\frac32})} \|\partial_y w^1_\Theta\|_{\widetilde{L}^2_t(\mathcal{B}^{\frac12})}\Big).
\end{aligned}
\tag{5.11}
$$

• <u>The estimate of $\int_0^t \big|\big(\Delta_k^{\rm h}(v^\varepsilon\partial_y w^1)_\Theta | \Delta_k^{\rm h} w^1_\Theta\big)_{L^2}\big|\, dt'$.</u>

We write

$$
v^\varepsilon \partial_y w^1 = w^2\partial_y w^1 + v\partial_y w^1.
$$

We first deduce from Lemma 3.2 that

$$
\int_0^t \big|\big(\Delta_k^{\rm h}(w^2\partial_y w^1)_\Theta | \Delta_k^{\rm h} w^1_\Theta\big)_{L^2}\big|\, dt' \lesssim d_k^2 2^{-k} \|w^1_\Theta\|^2_{\widetilde{L}^2_{t,\dot\zeta(t)}(\mathcal{B}^1)}.
\tag{5.12}
$$

Whereas by applying Bony's decomposition (2.3) for the horizontal variable to $v\partial_y w^1$, we find

$$
v\partial_y w^1 = T^{\rm h}_v \partial_y w^1 + T^{\rm h}_{\partial_y w^1} v + R^h(v, \partial_y w^1).
$$

It follows from (3.15) that

$$\begin{aligned}\|S^{\rm h}_{k'-1}v_\Theta(t')\|_{L^\infty}\lesssim&\sum_{\ell\leq k'-2}2^{\frac{3\ell}{2}}\|\Delta_\ell^{\rm h}u_\Theta(t')\|_{L^2}^{\frac12}\|\Delta_\ell^{\rm h}\partial_y u_\Theta(t')\|_{L^2}^{\frac12}\\ \lesssim& d_{k'}(t)2^{\frac{k'}{2}}\|u_\Theta(t')\|_{\mathcal{B}^{\frac32}}^{\frac12}\|\partial_y u_\Theta(t')\|_{\mathcal{B}^{\frac12}}^{\frac12},\end{aligned}$$

from which, we infer

$$\begin{aligned}&\int_0^t\big|\big(\Delta_k^{\rm h}(T^{\rm h}_v\partial_y w^1)_\Theta\ |\ \Delta_k^{\rm h}w^1_\Theta\big)_{L^2}\big|\,dt'\\ &\lesssim\sum_{|k'-k|\leq4}\int_0^t\|S^{\rm h}_{k'-1}v_\Theta(t')\|_{L^\infty}\|\Delta_{k'}^{\rm h}\partial_y w^1_\Theta(t')\|_{L^2}\|\Delta_k^{\rm h}w^1_\Theta(t')\|_{L^2}\,dt'\\ &\lesssim\sum_{|k'-k|\leq4}2^{\frac{k'}{2}}\|u_\Theta\|^{\frac12}_{L^\infty_t(\mathcal{B}^{\frac32})}\|\Delta_{k'}^{\rm h}\partial_y w^1_\Theta(t')\|_{L^2_t(L^2)}\Big(\int_0^t\|\partial_y u_\Theta(t')\|_{\mathcal{B}^{\frac12}}\|\Delta_k^{\rm h}w^1_\Theta(t')\|^2_{L^2}\,dt'\Big)^{\frac12}\\ &\lesssim d_k^2 2^{-k}\|u_\Theta\|^{\frac12}_{L^\infty_t(\mathcal{B}^{\frac32})}\|\partial_y w^1_\Theta\|_{\widetilde{L}^2_t(\mathcal{B}^{\frac12})}\|w^1_\Theta\|_{\widetilde{L}^2_{t,\dot\zeta(t)}(\mathcal{B}^1)}.\end{aligned}$$

Whereas thanks to (3.18), we get

$$\begin{aligned}&\int_0^t\big|\big(\Delta_k^{\rm h}(T^{\rm h}_{\partial_y w^1}v)_\Theta\ |\ \Delta_k^{\rm h}w^1_\Theta\big)_{L^2}\big|\,dt'\\ &\lesssim\sum_{|k'-k|\leq4}\int_0^t\|S^{\rm h}_{k'-1}\partial_y w^1_\Theta(t')\|_{L^\infty_{\rm h}(L^2_{\rm v})}\|\Delta_{k'}^{\rm h}v_\Theta(t')\|_{L^2_{\rm h}(L^\infty_{\rm v})}\|\Delta_k^{\rm h}w^1_\Theta(t')\|_{L^2}\,dt'\\ &\lesssim\sum_{|k'-k|\leq4}d_{k'}\|S^{\rm h}_{k'-1}\partial_y w^1_\Theta\|_{L^2_t(L^\infty_{\rm h}(L^2_{\rm v}))}\|u_\Theta\|^{\frac12}_{L^\infty_t(\mathcal{B}^{\frac32})}\Big(\int_0^t\|\Delta_k^{\rm h}w^1_\Theta(t')\|^2_{L^2}\|\partial_y u_\Theta(t')\|_{\mathcal{B}^{\frac12}}\,dt'\Big)^{\frac12}\\ &\lesssim d_k^2 2^{-k}\|u_\Theta\|^{\frac12}_{L^\infty_t(\mathcal{B}^{\frac32})}\|\partial_y w^1_\Theta\|_{\widetilde{L}^2_t(\mathcal{B}^{\frac12})}\|w^1_\Theta\|_{\widetilde{L}^2_{t,\dot\zeta(t)}(\mathcal{B}^1)}.\end{aligned}$$

Along the same line, we obtain

$$\begin{aligned}&\int_0^t\big|\big(\Delta_k^{\rm h}(R^{\rm h}(v,\partial_y w^1))_\Theta\ |\ \Delta_k^{\rm h}w^1_\Theta\big)_{L^2}\big|\,dt'\\ &\lesssim 2^{\frac{k}{2}}\sum_{k'\geq k-3}\int_0^t\|\Delta_{k'}^{\rm h}v_\Theta(t')\|_{L^\infty_{\rm h}(L^2_{\rm v})}\|\widetilde{\Delta}_{k'}^{\rm h}\partial_y w^1_\Theta(t')\|_{L^2}\|\Delta_k^{\rm h}w^1_\Theta(t')\|_{L^2}\,dt'\\ &\lesssim 2^{\frac{k}{2}}\sum_{k'\geq k-3}\|u_\Theta\|^{\frac12}_{L^\infty_t(\mathcal{B}^{\frac32})}\|\widetilde{\Delta}_{k'}^{\rm h}\partial_y w^1_\Theta\|_{L^2_t(L^2)}\Big(\int_0^t\|\Delta_k^{\rm h}w^1_\Theta(t')\|^2_{L^2}\|\partial_y u_\Theta(t')\|_{\mathcal{B}^{\frac12}}\,dt'\Big)^{\frac12}\\ &\lesssim d_k^2 2^{-k}\|u_\Theta\|^{\frac12}_{L^\infty_t(\mathcal{B}^{\frac32})}\|\partial_y w^1_\Theta\|_{\widetilde{L}^2_t(\mathcal{B}^{\frac12})}\|w^1_\Theta\|_{\widetilde{L}^2_{t,\dot\zeta(t)}(\mathcal{B}^1)}.\end{aligned}$$

As a consequence, we arrive at

$$\int_0^t\big|\big(\Delta_k^{\rm h}(v\partial_y w^1)_\Theta\ |\ \Delta_k^{\rm h}w^1_\Theta\big)_{L^2}\big|\,dt'\lesssim d_k^2 2^{-k}\|u_\Theta\|^{\frac12}_{L^\infty_t(\mathcal{B}^{\frac32})}\|\partial_y w^1_\Theta\|_{\widetilde{L}^2_t(\mathcal{B}^{\frac12})}\|w^1_\Theta\|_{\widetilde{L}^2_{t,\dot\zeta(t)}(\mathcal{B}^1)}.\tag{5.13}$$

• <u>The estimate of $\int_0^t\big|\big(\Delta_k^{\rm h}(w^2\partial_y u)_\Theta|\Delta_k^{\rm h}w^1_\Theta\big)_{L^2}\big|\,dt'$.</u>

By applying Bony's decomposition (2.3) for the horizontal variable to $w^2\partial_y u$, we write

$$w^2\partial_y u = T^{\rm h}_{w^2}\partial_y u + T^{\rm h}_{\partial_y u}w^2 + R^h(w^2,\partial_y u).$$

In view of (3.19), we have

$$\Bigl(\int_0^t \|S^{\rm h}_{k'-1}w^2_\Theta(t')\|^2_{L^\infty}\|\partial_y u_\Theta(t')\|_{\mathcal{B}^{\frac12}}\,dt'\Bigr)^{\frac12}\lesssim d_{k'}2^{\frac{k'}{2}}\|w^1_\Theta\|_{\widetilde{L}^2_{t,\dot{\zeta}(t)}(\mathcal{B}^1)},$$

so that we get, by applying Hölder's inequality, that

$$\begin{aligned}
&\int_0^t \bigl|\bigl(\Delta_k^{\rm h}(T^{\rm h}_{w^2}\partial_y u)_\Theta \mid \Delta_k^{\rm h}w^1_\Theta\bigr)_{L^2}\bigr|\,dt'\\
&\lesssim \sum_{|k'-k|\leq 4} 2^{-\frac{k'}{2}}\int_0^t \|S^{\rm h}_{k'-1}w^2_\Theta(t')\|_{L^\infty}\|\partial_y u_\Theta(t')\|_{\mathcal{B}^{\frac12}}\|\Delta_k^{\rm h}w^1_\Theta(t')\|_{L^2}\,dt'\\
&\lesssim \sum_{|k'-k|\leq 4} 2^{-\frac{k'}{2}}\Bigl(\int_0^t \|S^{\rm h}_{k'-1}w^2_\Theta(t')\|^2_{L^\infty}\|\partial_y u_\Theta(t')\|_{\mathcal{B}^{\frac12}}\,dt'\Bigr)^{\frac12}\\
&\qquad\qquad\times\Bigl(\int_0^t \|\Delta_k^{\rm h}w^1_\Theta(t')\|^2_{L^2}\|\partial_y u_\Theta(t')\|_{\mathcal{B}^{\frac12}}\,dt'\Bigr)^{\frac12}\\
&\lesssim d_k^2 2^{-k}\|w^1_\Theta\|^2_{\widetilde{L}^2_{t,\dot{\zeta}(t)}(\mathcal{B}^1)}.
\end{aligned}$$

While thanks to (3.18), we find

$$\begin{aligned}
&\int_0^t \bigl|\bigl(\Delta_k^{\rm h}(T^{\rm h}_{\partial_y u}w^2)_\Theta \mid \Delta_k^{\rm h}w^1_\Theta\bigr)_{L^2}\bigr|\,dt'\\
&\lesssim \sum_{|k'-k|\leq 4}\int_0^t \|S^{\rm h}_{k'-1}\partial_y u_\Theta(t')\|_{L^\infty_{\rm h}(L^2_{\rm v})}\|\Delta_{k'}^{\rm h}w^2_\Theta(t')\|_{L^2_{\rm h}(L^\infty_{\rm v})}\|\Delta_k^{\rm h}w^1_\Theta(t')\|_{L^2}\,dt'\\
&\lesssim \sum_{|k'-k|\leq 4} 2^{k'}\int_0^t \|\partial_y u_\Theta(t')\|_{\mathcal{B}^{\frac12}}\|\Delta_{k'}^{\rm h}w^1_\Theta(t')\|_{L^2}\|\Delta_k^{\rm h}w^1_\Theta(t')\|_{L^2}\,dt'\\
&\lesssim d_k^2 2^{-k}\|w^1_\Theta\|^2_{\widetilde{L}^2_{t,\dot{\zeta}(t)}(\mathcal{B}^1)}.
\end{aligned}$$

Along the same line, we obtain

$$\begin{aligned}
&\int_0^t \bigl|\bigl(\Delta_k^{\rm h}(R^{\rm h}(w^2,\partial_y u))_\Theta \mid \Delta_k^{\rm h}w^1_\Theta\bigr)_{L^2}\bigr|\,dt'\\
&\lesssim 2^{\frac{k}{2}}\sum_{k'\geq k-3}\int_0^t \|\Delta_{k'}^{\rm h}w^2_\Theta(t')\|_{L^2_{\rm h}(L^\infty_{\rm v})}\|\widetilde{\Delta}_{k'}^{\rm h}\partial_y u_\Theta(t')\|_{L^2}\|\Delta_k^{\rm h}w^1_\Theta(t')\|_{L^2}\,dt'\\
&\lesssim 2^{\frac{k}{2}}\sum_{k'\geq k-3}2^{\frac{k'}{2}}\int_0^t \|\Delta_{k'}^{\rm h}w^1_\Theta(t')\|_{L^2}\|\partial_y u_\Theta(t')\|_{\mathcal{B}^{\frac12}}\|\Delta_k^{\rm h}w^1_\Theta(t')\|_{L^2}\,dt'\\
&\lesssim d_k^2 2^{-k}\|w^1_\Theta\|^2_{\widetilde{L}^2_{t,\dot{\zeta}(t)}(\mathcal{B}^1)}.
\end{aligned}$$

This gives rise to

$$\int_0^t \bigl|\bigl(\Delta_k^{\rm h}(w^2\partial_y u)_\Theta \mid \Delta_k^{\rm h}w^1_\Theta\bigr)_{L^2}\bigr|\,dt'\lesssim d_k^2 2^{-k}\|w^1_\Theta\|^2_{\widetilde{L}^2_{t,\dot{\zeta}(t)}(\mathcal{B}^1)}.\tag{5.14}$$

By summing up (5.9-5.14), we conclude the proof of (5.6). $\square$

Proof of (5.7). We first observe from $\partial_x u+\partial_y v=0$ and Poincaré inequality that

$$\begin{aligned}
&\varepsilon^2\int_0^t\big|\big(\Delta_k^{\rm h}(\partial_t v)_\Theta|\Delta_k^{\rm h}w^2_\Theta\big)_{L^2}\big|dt'\lesssim\varepsilon^2 d_k^2 2^{-k}\|(\partial_t u)_\Theta\|_{\widetilde{L}^2_t(\mathcal{B}^{\frac32})}\|\partial_y w^2_\Theta\|_{\widetilde{L}^2_t(\mathcal{B}^{\frac12})},\\
&\varepsilon^2\int_0^t\big|\big(\Delta_k^{\rm h}(\partial_y^2 v)_\Theta|\Delta_k^{\rm h}w^2_\Theta\big)_{L^2}\big|dt'\lesssim\varepsilon^2 d_k^2 2^{-k}\|\partial_y u_\Theta\|_{\widetilde{L}^2_t(\mathcal{B}^{\frac32})}\|\partial_y w^2_\Theta\|_{\widetilde{L}^2_t(\mathcal{B}^{\frac12})},\\
&\varepsilon^4\int_0^t\big|\big(\Delta_k^{\rm h}(\partial_x^2 v)_\Theta|\Delta_k^{\rm h}w^2_\Theta\big)_{L^2}\big|dt'\lesssim\varepsilon^4 d_k^2 2^{-k}\|\partial_y u_\Theta\|_{\widetilde{L}^2_t(\mathcal{B}^{\frac52})}\|w^2_\Theta\|_{\widetilde{L}^2_t(\mathcal{B}^{\frac32})}.
\end{aligned}\tag{5.15}$$

• The estimate of $\int_0^t\big|\big(\Delta_k^{\rm h}(u^\varepsilon\partial_x v^\varepsilon)_\Theta|\Delta_k^{\rm h}w^1_\Theta\big)_{L^2}\big|\,dt'$.

We write

$$u^\varepsilon\partial_x v^\varepsilon=u^\varepsilon\partial_x w^2+u^\varepsilon\partial_x v.$$

It follows from Lemma 3.1 that

$$\int_0^t\big|\big(\Delta_k^{\rm h}(u^\varepsilon\partial_x w^2)_\Theta|\Delta_k^{\rm h}w^2_\Theta\big)_{L^2}\big|\,dt'\lesssim d_k^2 2^{-k}\|w^2_\Theta\|^2_{\widetilde{L}^2_{t,\dot\zeta(t)}(\mathcal{B}^1)}.\tag{5.16}$$

By applying Bony's decomposition for the horizontal variable to $u^\varepsilon\partial_x v$ gives

$$u^\varepsilon\partial_x v=T^{\rm h}_{u^\varepsilon}\partial_x v+T^{\rm h}_{\partial_x v}u^\varepsilon+R^{\rm h}(u^\varepsilon,\partial_x v).$$

Due to

$$\|S^{\rm h}_{k'-1}u^\varepsilon_\Theta(t')\|_{L^\infty}\lesssim\|u^\varepsilon_\Theta(t')\|^{\frac12}_{\mathcal{B}^{\frac12}}\|\partial_y u^\varepsilon_\Theta(t')\|^{\frac12}_{\mathcal{B}^{\frac12}},$$

and (3.18), we have

$$\begin{aligned}
&\int_0^t\big|\big(\Delta_k^{\rm h}(T^{\rm h}_{u^\varepsilon}\partial_x v)_\Theta\mid\Delta_k^{\rm h}w^2_\Theta\big)_{L^2}\big|\,dt'\\
&\lesssim\sum_{|k'-k|\leq4}\int_0^t\|S^{\rm h}_{k'-1}u^\varepsilon_\Theta(t')\|_{L^\infty}\|\Delta^{\rm h}_{k'}\partial_x v_\Theta(t')\|_{L^2}\|\Delta_k^{\rm h}w^2_\Theta(t')\|_{L^2}\,dt'\\
&\lesssim\sum_{|k'-k|\leq4}2^{2k'}\|u^\varepsilon_\Theta\|^{\frac12}_{L^\infty_t(\mathcal{B}^{\frac12})}\|\Delta_k^{\rm h}u_\Theta\|_{L^2_t(L^2)}\Big(\int_0^t\|\partial_y u^\varepsilon_\Theta(t')\|_{\mathcal{B}^{\frac12}}\|\Delta_k^{\rm h}w^2_\Theta(t')\|^2_{L^2}\,dt'\Big)^{\frac12}\\
&\lesssim d_k^2 2^{-k}\|u^\varepsilon_\Theta\|^{\frac12}_{L^\infty_t(\mathcal{B}^{\frac12})}\|\partial_y u_\Theta\|_{\widetilde{L}^2_t(\mathcal{B}^2)}\|w^1_\Theta\|^2_{\widetilde{L}^2_{t,\dot\zeta(t)}(\mathcal{B}^1)}.
\end{aligned}$$

While again thanks to (3.18), we find

$$\|S^{\rm h}_{k'-1}\partial_x v_\Theta(t')\|_{L^\infty}\lesssim 2^{\frac{k'}{2}}\|\partial_y u_\Theta(t')\|_{\mathcal{B}^2},$$

which leads to

$$\begin{aligned}
&\int_0^t\big|\big(\Delta_k^{\rm h}(T^{\rm h}_{\partial_x v}u^\varepsilon)_\Theta\mid\Delta_k^{\rm h}w^2_\Theta\big)_{L^2}\big|\,dt'\\
&\lesssim\sum_{|k'-k|\leq4}\int_0^t\|S^{\rm h}_{k'-1}\partial_x v_\Theta(t')\|_{L^\infty}\|\Delta^{\rm h}_{k'}u^\varepsilon_\Theta(t')\|_{L^2}\|\Delta_k^{\rm h}w^2_\Theta(t')\|_{L^2}\,dt'\\
&\lesssim\sum_{|k'-k|\leq4}d_{k'}\|\partial_y u_\Theta\|_{\widetilde{L}^2_t(\mathcal{B}^2)}\|u^\varepsilon\|^{\frac12}_{L^\infty_t(\mathcal{B}^{\frac12})}\Big(\int_0^t\|\partial_y u^\varepsilon_\Theta(t')\|_{\mathcal{B}^{\frac12}}\|\Delta_k^{\rm h}w^2_\Theta(t')\|^2_{L^2}\,dt'\Big)^{\frac12}\\
&\lesssim d_k^2 2^{-k}\|u^\varepsilon_\Theta\|^{\frac12}_{L^\infty_t(\mathcal{B}^{\frac12})}\|\partial_y u_\Theta\|_{\widetilde{L}^2_t(\mathcal{B}^2)}\|w^1_\Theta\|^2_{\widetilde{L}^2_{t,\dot\zeta(t)}(\mathcal{B}^1)}.
\end{aligned}$$

Along the same line, we obtain

$$\begin{aligned}
&\int_0^t \big|\big(\Delta_k^{\rm h}(R^{\rm h}(u^\varepsilon,\partial_x v))_\Theta \mid \Delta_k^{\rm h} w^2_\Theta\big)_{L^2}\big|\,dt'\\
\lesssim & 2^{\frac{k}{2}}\sum_{k'\geq k-3}\int_0^t \|\Delta_{k'}^{\rm h} u^\varepsilon_\Theta(t')\|_{L^2_{\rm h}(L^\infty_{\rm v})}\|\widetilde{\Delta}_{k'}^{\rm h}\partial_x v_\Theta(t')\|_{L^2}\|\Delta_k^{\rm h} w^1_\Theta(t')\|_{L^2}\,dt'\\
\lesssim & 2^{\frac{k}{2}}\sum_{k'\geq k-3} 2^{\frac{3k'}{2}}\|u^\varepsilon_\Theta\|^{\frac12}_{L^\infty_t(\mathcal{B}^{\frac12})}\|\Delta_k^{\rm h} u_\Theta\|_{L^2_t(L^2)}\Big(\int_0^t \|\partial_y u^\varepsilon_\Theta(t')\|_{\mathcal{B}^{\frac12}}\|\Delta_k^{\rm h} w^2_\Theta(t')\|^2_{L^2}\,dt'\Big)^{\frac12}\\
\lesssim & d_k^2 2^{-k}\|u^\varepsilon_\Theta\|^{\frac12}_{L^\infty_t(\mathcal{B}^{\frac12})}\|\partial_y u_\Theta\|_{\widetilde{L}^2_t(\mathcal{B}^2)}\|w^2_\Theta\|_{\widetilde{L}^2_{t,\dot\zeta(t)}(\mathcal{B}^1)}.
\end{aligned}$$

This gives rise to

$$\int_0^t \big|\big(\Delta_k^{\rm h}(u^\varepsilon\partial_x v)_\Theta \mid \Delta_k^{\rm h} w^2_\Theta\big)_{L^2}\big|\,dt' \lesssim d_k^2 2^{-k}\|u^\varepsilon_\Theta\|^{\frac12}_{L^\infty_t(\mathcal{B}^{\frac12})}\|\partial_y u_\Theta\|_{\widetilde{L}^2_t(\mathcal{B}^2)}\|w^2_\Theta\|_{\widetilde{L}^2_{t,\dot\zeta(t)}(\mathcal{B}^1)}. \tag{5.17}$$

• <u>The estimate of $\int_0^t \big|\big(\Delta_k^{\rm h}(v^\varepsilon\partial_y v^\varepsilon)_\Theta | \Delta_k^{\rm h} w^1_\Theta\big)_{L^2}\big|\,dt'$.</u>

We first note that

$$v^\varepsilon\partial_y v^\varepsilon = v\partial_y w^2 + w^2\partial_y w^2 + v\partial_y v + w^2\partial_y v.$$

We first deduce Lemma 3.3 that

$$\varepsilon^2\int_0^t \big|\big(\Delta_k^{\rm h}(w^2\partial_y w^2)_\Theta \mid \Delta_k^{\rm h} w^2_\Theta\big)_{L^2}\big|\,dt' \lesssim d_k^2 2^{-k}\big\|(w^1_\Theta,\varepsilon w^2_\Theta)\big\|^2_{\widetilde{L}^2_{t,\dot\zeta(t)}(\mathcal{B}^1)}.$$

It follows from (5.13) that

$$\int_0^t \big|\big(\Delta_k^{\rm h}(v\partial_y w^2)_\Theta \mid \Delta_k^{\rm h} w^2_\Theta\big)_{L^2}\big|\,dt' \lesssim d_k^2 2^{-k}\|u_\Theta\|^{\frac12}_{L^\infty_t(\mathcal{B}^{\frac32})}\|\partial_y w^2_\Theta\|_{\widetilde{L}^2_t(\mathcal{B}^{\frac12})}\|w^2_\Theta\|_{\widetilde{L}^2_{t,\dot\zeta(t)}(\mathcal{B}^1)}.$$

And (5.11) ensures that

$$\begin{aligned}
&\int_0^t \big|\big(\Delta_k^{\rm h}(w^2\partial_x u)_\Theta \mid \Delta_k^{\rm h} w^2_\Theta\big)_{L^2}\big|\,dt'\\
&\qquad\lesssim d_k^2 2^{-k}\|w^2_\Theta\|_{\widetilde{L}^2_{t,\dot\zeta(t)}(\mathcal{B}^1)}\Big(\|w^2_\Theta\|_{\widetilde{L}^2_{t,\dot\zeta(t)}(\mathcal{B}^1)} + \|u_\Theta\|^{\frac12}_{L^\infty_t(\mathcal{B}^{\frac32})}\|\partial_y w^2_\Theta\|_{\widetilde{L}^2_t(\mathcal{B}^{\frac12})}\Big).
\end{aligned}$$

We deduce from the proof of (5.13) that

$$\begin{aligned}
\int_0^t \big|\big(\Delta_k^{\rm h}(v\partial_y v)_\Theta \mid \Delta_k^{\rm h} w^2_\Theta\big)_{L^2}\big|\,dt' \lesssim & d_k^2 2^{-k}\|u_\Theta\|^{\frac12}_{L^\infty_t(\mathcal{B}^{\frac32})}\|\partial_y v_\Theta\|_{\widetilde{L}^2_t(\mathcal{B}^{\frac12})}\|w^2_\Theta\|_{\widetilde{L}^2_{t,\dot\zeta(t)}(\mathcal{B}^1)}\\
\lesssim & d_k^2 2^{-k}\|u_\Theta\|^{\frac12}_{L^\infty_t(\mathcal{B}^{\frac32})}\|\partial_y u_\Theta\|_{\widetilde{L}^2_t(\mathcal{B}^{\frac32})}\|w^2_\Theta\|_{\widetilde{L}^2_{t,\dot\zeta(t)}(\mathcal{B}^1)}.
\end{aligned}$$

As a result, it comes out

$$\begin{aligned}
\varepsilon^2\int_0^t \big|\big(\Delta_k^{\rm h}(v^\varepsilon\partial_y v^\varepsilon)_\Theta | \Delta_k^{\rm h} w^1_\Theta\big)_{L^2}\big|\,dt' \lesssim & d_k^2 2^{-k}\Big(\big\|(w^1_\Theta,\varepsilon w^2_\Theta)\big\|^2_{\widetilde{L}^2_{t,\dot\zeta(t)}(\mathcal{B}^1)}\\
&+\varepsilon^2\|u_\Theta\|^{\frac12}_{L^\infty_t(\mathcal{B}^{\frac32})}\big(\|\partial_y w^2_\Theta\|_{\widetilde{L}^2_t(\mathcal{B}^{\frac12})} + \|\partial_y u_\Theta\|_{\widetilde{L}^2_t(\mathcal{B}^{\frac32})}\big)\|w^2_\Theta\|_{\widetilde{L}^2_{t,\dot\zeta(t)}(\mathcal{B}^1)}\Big).
\end{aligned} \tag{5.18}$$

Summing up (5.15-5.18) gives rise to (5.7). □

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