A proof-based framework for several types of grounding

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Abstract

By adopting a proof-theoretic perspective on grounding, we provide a general framework where several notions of grounding, such as complete and mediate, partial and immediate and partial mediate, are defined and compared with the more widespread notions of full and partial grounding. As a result, we get a cartography of the recent work on grounding as well as the observation of important and interesting features of this notion.

1 Introduction

Theres exists a famous distinction, which dates back at least as far as (Aristotle, 1994, I.13), has been analysed in the Middle Ages and was then adopted by (Bolzano, 2014, §525), between two types of proofs, namely *proofs-that* and *proofs-why*. Both proofs-that and proofs-why are objects that start with some premisses and end with a conclusion, however, while in proofs-that from the premisses it simply follows *that* the conclusion is true, in proofs-why the premisses represent the *reasons why* the conclusion is true. Hence, proofs-that are composed of inferential steps: in each of these steps the truth of the premisses allows one to infer that the conclusion is true as well. Proofs-why, on the other hand, are explanatory proofs: they establish not just the truth of the conclusion but reveal the premisses to be the grounds of the truth of the conclusion.

At the beginning of the last century, the study of proofs-that has proven to be fertile: the well-notion of *derivability* can be looked at as the best formalization of the logical aspects of the concept of proof-that. The formalization of proofs-why has not encountered the same success: it has been ignored or forgotten by the great developments of logic of the last century. Nowadays, we however assist to a change of this trend. There is indeed a growing, thriving and impressive interest for the concept of *grounding*, that is taken to be an explanatory relation which is non-causal in nature (e.g. see Betti (2010); Fine (2012)). Grounding is studied by different perspectives: most research belongs to metaphysics (e.g. Audi (2012); Clark and Liggins (2012)), but there also exist some attempts which aim at getting a formalization of the notion of grounding (e.g. Correia (2014); Schnieder (2011)). In particular, in Poggiolesi (2016b, 2018, 2021) this is done by relying on the link between grounding and proofs-why, two notions that are intimately connected: a relation of grounding between some truths M and a truth A can be seen as emanating from a proof-why which ends in A and starts with the reasons M behind it. In other words, the work of Poggiolesi represents the attempt of formalizing grounding by adopting a proof(-why)-theoretic perspective.[1]

When seen from this perspective, it is natural to ask about the relationship between the notion of grounding (and related proofs-why) and the notion of (non-explanatory) proof-that. Poggiolesi (2016b) has already moved some steps towards this direction since she has proposed a definition of the notion of logical grounding as a special type of derivability. However, as there exists several types of grounding, namely complete and immediate, complete and mediate, partial and immediate and partial and mediate,[2] Poggiolesi's approach is limited to only one of them, namely complete and immediate grounding, and thus leaves open the question of the relationships between other types of grounding and the notion of derivability. The aim of this paper is to answer this question and thus to provide derivability-based definitions of an array of other grounding notions.

A central contribution of such an array of definitions is that it provides a single and harmonious framework where several types of grounding are defined in terms of derivability. Moreover, as we will show, Fine's notions of full and partial grounding[3]—which are doubtless the most popular in the current literature—are amenable to treatment in this framework, and we situate them in the wide cartography of grounding and derivability. Finally, analyzing the notion of grounding from the point of view of proofs allows us to re-connect to features of grounding that have been emphasized in the past but are currently neglected; a notable example is the property of *analyticity*. A proof is said to be analytic when it is self-contained: every element which occurs in the proof, it will also occurs in the conclusion; this way in analytic proofs one can assist to a complexity-reduction from the conclusion to the premisses: we indeed move from more complex concepts to simpler ones. There exists a long and illustrious philosophical tradition (e.g. see Descartes (1997); Arnauld (2011); Bolzano (2014)) that links explanatory proofs with analytic proofs (see also Rumberg (2013)). In particular, if grounding is seen as emanating from an explanatory proof and explanatory proofs are special types of proofs-that, then, according to this tradition, they are special types of analytic proofs-that. However, in the current literature on grounding, where the links to proofs are

[1]As a result, Poggiolesi's work can also be read as the attempt of formalizing proofs-why.

[2]The multiset of all, and only, those truths each of which contributes to ground a truth A is a *complete* ground of A. On the other hand, each of the truths that compose the complete ground of A, as well as each strict sub-multiset of them, is said to be a *partial* ground of A. As for the distinction immediate and mediate, if we could describe it in proof-theoretical terms, we would say that *immediate* grounding corresponds to a single (irreflexive) grounding(proof)-step, while *mediate* grounding corresponds to a sequence of several steps of immediate grounding. In other words, while immediate grounding is a relation that does not seem to be reducible further, mediate grounding is definable as the transitive closure of immediate grounding.

[3]As explained in detail in Poggiolesi (2016b,a), the distinctions complete-partial and full-partial, although similar, are not the same. According to Fine (2012) A is a *partial* ground of C if A on its own or together with some other truths is a ground of C. Thus, given that A and B are the full ground of $A \wedge B$, each of A and B will be a partial ground of $A \wedge B$. On the other hand, A is a *full* ground of C if the truth of A is sufficient to guarantee the truth of C.

disregarded, the question of analyticity is largely ignored. One of the main results of this paper is to show that, in order to define mediate grounding, we can restrict ourselves to analytic derivations. This result, which emerges naturally under a proof-theoretical view on grounding, seems to vindicate the aforementioned philosophical tradition, to reconnect such tradition to current research on grounding and to assess rigorously an important feature of this notion.

The paper is organized as follows. In Section 2, we will briefly remind the reader the definition of complete and immediate grounding presented in Poggiolesi (2016b). In Section 3 we will introduce a definition of partial and immediate grounding, whilst in Section 4 we will introduce a definition of complete grounding. Section 5 will serve to deal with the notion of partial and mediate grounding. In Section 6 we will compare our results to Fine's notions of full and partial (both immediate and mediate) grounding. Finally in Section 7 we will draw some general conclusions.

2 A definition of the notion of complete and immediate logical grounding

We use this section to briefly recall the definition proposed in Poggiolesi (2016b) of the notion of complete and immediate logical grounding, which will play an important role in the sequel. The definition is motivated by two very simple insights. The first consists in taking seriously the idea that grounding can be seen as a proof-why and that proofs-why are a special type of proofs-that; if this is the case, and derivations are the formal contemporary counterparts of proofs-that, then a necessary condition for having a grounding relation between a (multi)-set of formulas M and a formula A is that A is derivable from M. This condition is called *positive derivability.* The second idea consists in identifying what makes grounding a *special* type of derivation. Which formal constraints should be added to positive derivability to get an explanatory relation? Poggiolesi identified two constraints. The first is complexity: a grounding relation is a special type of derivation where the premisses are always less complex than their conclusion; a precise definition of complexity is for this goal formulated. The second constraint is what Poggiolesi calls *negative* derivability and corresponds to the request that not only A should be derivable from M but also $\neg A$ should be derivable from the negation of each element in M. Negative derivability is supposed to capture the idea of variation: in a grounding relation grounds and conclusion are so strictly related that if the grounds are modified (we take its negation instead of the ground itself), then the modification affects the conclusion. Hence, according to Poggiolesi's account, a grounding relation is characterized by positive and negative derivability, plus a complexity increase from the ground(s) to the conclusion.

Note that the account put forward in Poggiolesi (2016b) involves a distinction between grounds and robust conditions that can be described briefly on the example of a disjunction like $A \vee B$, in a situation where the formula A is true.

Figure 1: Axioms and Rules of the Classical Sequent Calculus.

$$A, M \Rightarrow N, A \qquad \frac{M \Rightarrow N, A}{\neg A, M \Rightarrow N}\neg L \qquad \frac{A, M \Rightarrow N}{M \Rightarrow N, \neg A}\neg R$$

$$\frac{A, B, M \Rightarrow N}{A \wedge B, M \Rightarrow N}\wedge L \qquad \frac{M \Rightarrow N, A \quad P \Rightarrow Q, B}{M, P \Rightarrow N, Q, A \wedge B}\wedge R$$

$$\frac{A, M \Rightarrow N \quad B, P \Rightarrow Q}{A \vee B, M, P \Rightarrow N, Q}\vee L \qquad \frac{M \Rightarrow N, A, B}{M \Rightarrow N, A \vee B}\vee R$$

In this case, A is certainly a ground for $A \vee B$; but in order for A to be the *complete* ground for $A \vee B$, it is necessary to specify that B is false (i.e. that B is not also a ground for $A \vee B$). In other words, it is the falsity of B that ensures that, or is a *(robust) condition* for A to be the complete ground of $A \vee B$. Thus, A is the complete and immediate formal ground for $A \vee B$ under the robust condition that B is false. The reader is referred to Poggiolesi (2016b) for a detailed explanation and discussion of the idea of robust conditions in a grounding framework. Robust conditions are denoted by square brackets and will be introduced in Definition 2.11.

We now present the formalism inspired by these ideas. Once more, the reader is referred to Poggiolesi (2016b) for a detailed clarification of the various notions. We slightly change the presentation of some of them to make it easier to adapt them to other notions of grounding.

Definition 2.1. The classical language $\mathcal{L}^c$ is composed of a denumerable stock of propositional atoms (p, q, r, ...), the logical operators $\neg$, $\wedge$ and $\vee$ and the parentheses (,). The connectives $\rightarrow$ and $\leftrightarrow$ are defined as usual; the symbol $\bot$ is defined as $A \wedge \neg A$.

Once the classical language $\mathcal{L}^c$ is given, we can define the notion of classical derivability in the standard way,[4] by means of the classical sequent calculus **C** (e.g. see Troelstra and Schwichtenberg (1996)). We will write $\vdash_{\mathbf{C}} M \Rightarrow A$ to denote the fact that the sequent composed by the multiset M[5] and the formula A is derivable in classical sequent calculus **C**.

We now introduce the key notion of g-complexity, which is a way of assigning a number to each formula of the language $\mathcal{L}^c$. The way that number is calculated reflects deep grounding-relevant features. As we will see, g-complexity straightforwardly leads to the identification of the relation of *being completely*

[4] Note that Poggiolesi works with derivability in the classical Hilbert system, we chose to work with the sequent calculus for its usefulness in the definition of mediate grounding, see Section 4.

[5] We work with multisets of formulas rather than with sets of formulas because we need to take into account the number of occurrences of each formula of M.

and immediately less g-complex, which will play a central role in the definition of the notion of complete and immediate grounding.

Definition 2.2. As is standard, we call atoms as well as negation of atoms *literals*. $l, l', ...$ denote literals.

Definition 2.3. The g-complexity of a formula $A \in \mathcal{L}^c$, $gcm(A)$, is defined in the following way:

- $gcm(l) = 0$,
- $gcm(\neg\neg A) = gcm(A) + 1$,
- $gcm(A \circ B) = gcm\neg(A \circ B) = gcm(A) + gcm(B) + 1$.

where the symbol $\circ$ stands for either conjunction or disjunction.

To understand the notion of g-complexity, it must be kept in mind that grounding is concerned entirely with truths. Accordingly, the appropriate notion of complexity should track relationships among the truths expressed by the formulas if they were true. If A and B express truths, then the truth expressed by $A \wedge B$ or $A \vee B$ is obtained from the previous truths using a single operation, just as the formulas $A \wedge B$ and $A \vee B$ are constructed from the formulas A and B using a single connective. Counting the connective in this case is faithful to the relationship of interest among truths and indeed $gcm(A \circ B) = gcm(A) + gcm(B)) +1$.

The negation is different, because there is no sense in which the negation of a formula is a truth constructed from the formula itself. Consider for instance the formulas p and $\neg p$ (namely the literals). p is atomic and so has g-complexity 0, but does that mean that $\neg p$ should count as having g-complexity 1? That would be justified if the truth $\neg p$ (when it is a truth) was constructed from the truth p; but this is not the case in general, not least because when one of the formulas is a truth, the other is not. From the point of view of grounding, which deals solely in truths, there is no truth from which $\neg p$ can be formally constructed, so, like p, it is atomic. Similar points hold for formulas of the form A, $\neg A$, where A is either a conjunction or a disjunction: the complexity of the latter cannot be counted as one more than the complexity of the former, since it is not reducible to it. Therefore in the formula $\neg A$ (where A does not itself start with a negation), the only g-complexity to count is that of A. This is precisely what Definition 2.3 does, by setting the complexity of $A \circ B$ and $\neg(A \circ B)$ on the same level.

The case of the double negation, however, is different. A formula like $\neg\neg A$, if true, can be reduced to another, simpler truth, namely A. Moreover, this reduction is direct: there is no "intermediate" truth that one passes through to obtain the former from the latter. Thus, it makes sense to count the g-complexity of $\neg\neg A$ as equal to that of A plus one.

Let us now move to the introduction of some notions which are central to define the relation of *being completely and immediately less g-complex*.

Figure 2: Admissible rules of the classical sequent calculus.

$$\frac{M \Rightarrow N}{P, M \Rightarrow N, Q}\, W \qquad \frac{M \Rightarrow N, A \quad A, P \Rightarrow Q}{M, P \Rightarrow N, Q}\, cut$$

$$\frac{A_i, M \Rightarrow N}{A_1 \wedge A_2, M \Rightarrow N}\, \wedge L' \qquad \frac{M \Rightarrow N, A \quad M \Rightarrow N, B}{M \Rightarrow N, A \wedge B}\, \wedge R'$$

$$\frac{A, M \Rightarrow N \quad B, M \Rightarrow N}{A \vee B, M \Rightarrow N}\, \vee L' \qquad \frac{M \Rightarrow N, A_i}{M \Rightarrow N, A_1 \vee A_2}\, \vee R'$$

where $i = \{1, 2\}$

Definition 2.4. Let D be a formula. The *converse* of D, written D^*, is defined in the following way

$$D^* = \begin{cases} \neg^{n-1}E, & if\ D = \neg^n E\ and\ \text{n}\ is\ odd \\ \neg^{n+1}E, & if\ D = \neg^n E\ and\ \text{n}\ is\ even \end{cases}$$

where the principal connective of E is not a negation, $n \geqslant 0$ and 0 is taken to be an even number.[6]

Let us provide some examples that help to clarify the notion of converse of a formula. If $D = \neg\neg\neg\neg p$, then its converse, D^*, is $\neg\neg\neg\neg\neg p$. If $D = \neg(A \wedge B)$, then its converse, D^*, is $(A \wedge B)$; finally, if $D = (A \vee B)$, then its converse, D^*, is $\neg(A \vee B)$. From now on we will use capital letters to refer to objects of $\mathbb{PF}$ and their converse.

Definition 2.5. Consider a formula A. We will say that A is *a-c equiv* (for associatively and commutatively equivalent) to B, if, and only if, A can be obtained from B by applications of associativity and commutativity of conjunction and disjunction.

Let us provide some exemples of formulas A and B such that A is *a-c equiv* to B. $A \wedge (B \wedge C)$ is *a-c equiv* to $C \wedge (A \wedge B)$. $\neg((E \vee F) \wedge (G \wedge (H \vee D)))$ is *a-c equiv* to $\neg((F \vee E) \wedge (G \wedge (D \vee H)))$, but also to $\neg((G \wedge (D \vee H)) \wedge (F \vee E))$. $A \wedge ((B \vee C) \vee (D \vee E))$ is *a-c equiv* to $A \wedge ((D \vee B) \vee (E \vee C))$.

Definition 2.6. For any two formulas A, B, $A \cong B$ if, and only if:

A is *a-c equiv* to B or A is *a-c equiv* to B^*

As extensively discussed in Poggiolesi (2016b), two formulas A and B stand in the relation denoted by $\cong$ when they *are about*, or *pertain to*, or *concern* the same issues. The relation $\cong$ is thus analogous (though not equivalent) to the notion of factual equivalence discussed in Correia (2014, 2016). This relation can be easily extended to multisets.

[6]Note that $\neg^0 E$ is just E. Also we keep the term *converse* for continuity with Poggiolesi's work. However, one should not confuse $*$ with an idempotent operator.

Definition 2.7. For any two multisets M, N, $M \cong N$ if, and only if, there exists a bijection between M and N sending each $A \in M$ to a $B \in N$ such that $A \cong B$.[7]

The relation $\cong$ allows us to define the notion of g-subformula, which is the analogue of the notion of subformula in the grounding framework.

Definition 2.8. A is a *g-subformula* of B if, and only if, one of the following holds:

- $A \cong B$
- $B \cong \neg\neg C$ and A is a g-subformula of C,
- $B \cong (C \circ D)$ and A is a g-subformula of C or a g-subformula of D.

Consider the formula $\neg\neg(p \wedge (q \wedge r))$. Some of its g-subformulas are: $\neg\neg((p \wedge q) \wedge r)$, $\neg\neg\neg(p \wedge (q \wedge r))$, $p \wedge (q \wedge r)$, $\neg(q \wedge (p \wedge r))$, $p \wedge q$, $q \wedge p$, $\neg(q \wedge r)$, r, $\neg r$, p, $\neg p$, q, $\neg q$.

The notion of *immediate* g-subformula can be defined similarly.

Definition 2.9. A is an *immediate g-subformula*[8] of B if, and only if, one of the following holds:

- $B \cong \neg\neg C$ and $A \cong C$,
- $B \cong (C \circ D)$ and $A \cong C$ or $A \cong D$.

We have now all the elements needed to establish when a multiset M is completely and immediately less g-complex than a formula C. The insight is that M is completely and immediately less g-complex than a formula C when it contains *all* (immediate) g-subformulas of C which are such that the sum of their g-complexity is *one less than* that of C.

Definition 2.10. Given a multiset of formulas M and a formula C of the classical language $\mathcal{L}^c$, we say that M is *completely and immediately less g-complex* than C, if, and only if:

- $C \cong \neg\neg B$ and $M \cong \{B\}$, or
- $C \cong (B \circ D)$ and $M \cong \{B, D\}$.

Definition 2.11. For any consistent multiset of formulas $C \cup M$ such that C and the formulas of M are in $\mathcal{L}^c$, we say that, under the robust condition C (that may be empty), M *completely and immediately logically grounds* A, in symbols $[C]\ M \mid\!\sim$ A, if and only if:

[7] Although the definition of the relation $\cong$ between multisets is not required for Poggiolesi's definition of the notion of complete and immediate grounding, we introduce it here because it is related to the other notions of this section and it will become useful in the next section.

[8] Although the notion of immediate g-subformula is not required for Poggiolesi's definition of the notion of complete and immediate grounding, we introduce it here because it is related to the other notions of this section and it will become useful in the next section.

- $\vdash_{\mathbf{C}} M \Rightarrow A$ (*positive derivability*),
- $\vdash_{\mathbf{C}} C, \neg(M) \Rightarrow \neg A$ (*negative derivability*),
- $C \cup M$ is completely and immediately less g-complex than A in the sense of Definition 2.10.

where $\neg(M) := \{\neg B | B \in M\}$.

Under the robust condition C, the multiset M completely and immediately formally grounds A if, and only if, (i) A is derivable from M – positive derivability; (ii) $\neg A$ is derivable from $\neg(M)$ plus C – negative derivability; (iii) $C \cup M$ is completely and immediately less g-complex than A.

3 Partial and immediate logical grounding

We now extend Poggiolesi's definition beyond the case of complete and immediate grounding. We begin by introducing a definition of the notion of partial and immediate grounding which draws on the intuitions concerning g-complexity and derivability used in the previous Section.

Definition 3.1. For any formula B of the classical language $\mathcal{L}^c$, we say that $\{B\}$ *partially and immediately logically grounds* A, in symbols $\{B\}$ $||\!\sim$ A, if and only if:

- either $\vdash_{\mathbf{C}} A \Rightarrow B$
- or (exclusive)[9] $\vdash_{\mathbf{C}} \neg A \Rightarrow \neg B$
- B is an immediate g-subformula of A, according to Definition 2.9.

Concerning this definition, note first of all that since the notion at stake is immediate grounding we can limit ourselves to considering one formula at a time. In fact, since complete immediate grounds are a multiset composed by at most two formulas, partial immediate ground being a (proper) subset will be a singleton. Secondly, we need to ensure that the definition is adequate: that is, that it takes into account all cases of partial ground. As stated above, a partial ground is one that can be modified so to become a case of complete ground, notably by adding the missing grounds or robust conditions. The previous definition will be adequate if the formulas that it picks out as partial grounds are precisely those with this property. The following result shows that this is so.

Theorem 3.2. *Given two formulas A, B of the classical language $\mathcal{L}^c$, $\{B\}$ is a partial and immediate ground of A according to Definition 3.1 if, and only if, there exists a formula C such that $\{C, B\}$ is a complete and immediate ground of*

[9]We remind the reader that the exclusive *or* stands for either one disjunct or the other but not both.

A, or $\{B\}$ is a complete and immediate ground of A under the robust condition C^* (where for the relation of complete and immediate grounding, we refer to Definition 2.11).

Proof. Let us distinguish cases according to the form of A.

[a] A is of the form $\neg\neg D$. According to Definition 2.10, the multisets which are completely and immediately less g-complex than $\neg\neg D$ are $\{D\}$ and $\{\neg D\}$ (as well as all formulas that are ac-equivalent to D and $\neg D$; the points made here hold for them as well). The latter does not enjoy neither positive nor negative derivability with $\neg\neg D$, the former enjoys them both. Hence none of them, nor any other formula, can be a partial ground for $\neg\neg D$ according to Definition 3.1. On the other hand, since according to Definition 2.11 $\{D\}$ is the only complete and immediate ground of $\neg\neg D$ and $\{D\}$ has no non-empty proper subset, no partial ground of the formula $\neg\neg D$ can be obtained from the relation of complete and immediate grounding.

[b] A is of the form $D_1 \wedge D_2$. According to Definition 2.10, the multisets which are completely and immediately less g-complex than $D_1 \wedge D_2$ are $\{D_1, D_2\}$, $\{D_1^*, D_2\}$, $\{D_1, D_2^*\}$, $\{D_1^*, D_2^*\}$.[10] The proper subsets of these mutlisets are $\{D_1\}$, $\{D_2\}$, $\{D_1^*\}$, $\{D_2^*\}$. Amongst them only the former two can enter in a partial grounding relation with $D_1 \wedge D_2$. Each of them enjoy negative derivability, but not positive derivability, with $D_1 \wedge D_2$ (whilst neither positive nor negative derivability are enjoyed by $\{D_1^*\}$ or $\{D_2^*\}$ and $D_1 \wedge D_2$). So both $\{D_1\}$ and $\{D_2\}$ are the only partial and immediate ground of $D_1 \wedge D_2$ according to Definition 3.1. On the other hand, since $\{D_1, D_2\}$ is the only complete and immediate ground of $D_1 \wedge D_2$ according to Definition 2.11, $\{D_1\}$ and $\{D_2\}$ are the only partial and immediate ground of $D_1 \wedge D_2$ since they are the only proper non empty subests of $\{D_1, D_2\}$.

The case where A is of the form $D_1 \vee D_2$, $\neg(D_1 \wedge D_2)$, $\neg(D_1 \vee D_2)$ can be treated analogously to [b].

□

According to Definition 3.1 some examples of partial and immediate grounding are the following. p is a partial and immediate ground of $p \wedge q$, but also of $p \vee q$ and also $\neg(\neg p \wedge \neg q)$. On the other hand $p \wedge q$ is a partial and immediate ground of $p \wedge (q \wedge r)$, but also $(p \wedge q) \wedge r$, and also $(p \wedge r) \wedge q$.

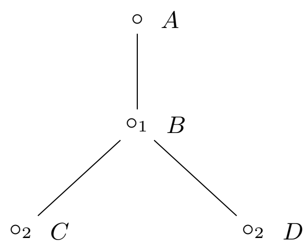
In the contemporary literature it is often said that in a grounding relation the consequent is strictly connected to its grounds or that there is an authentic dependence between the two (e.g. see Correia and Schnieder (2012); Fine (2012)). As Poggiolesi (2016b) has argued, positive and negative derivability are the formal counterpart of this idea of connection or authentic dependence. But if one compares Definition 2.11 with Definition 3.1, one straightforwardly sees that positive and negative derivability only hold in the case of complete grounding and not for partial grounding. This amounts to the fact that only

[10] We ignore ac-equivalence not to burden the paper. But of course such cases can be treated analogously.

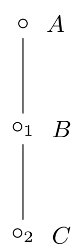
the *complete* grounds - and not partial grounds - enjoy an authentic dependence with their conclusion. In other words, whilst in a complete grounding relation the grounds and the conclusion are dependent on each other (or as Poggiolesi (2016b) puts it, they vary together or one *tracks* the truth of the other), in partial grounding relations the connection between grounds and conclusion is not as strict. This is an important feature of partial grounding that is rigorously captured by Definition 3.1.

4 Complete and mediate logical grounding

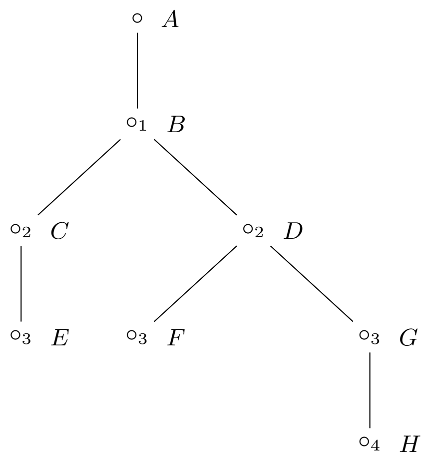
We use this section to deal with the notion of complete and mediate grounding that we aim to define in terms of derivability and g-complexity. To accomplish this task, which is far from trivial, let us then start by considering the relation of *being completely and immediately less g-complex* which ensures the grounds to be immediate. This relation evidently needs to be adapted to the mediate case. We will do this by employing a well-known method called *tableau* (e.g. see Fitting and Mendelsohn (1998)) for constructing trees of formulas, namely objects that might have this form:



or this form:



or this form:

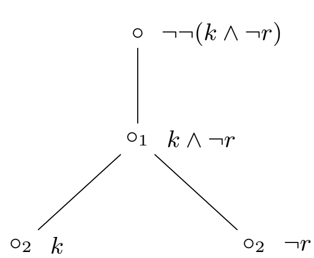


We focus on this last tree to introduce the relevant terminology. This last tree has 8 nodes, each of them associated with a formula $(A, B, C, ...H)$ and belonging to a different level of the tree (1, ... 4). The top node of a tree is usually called *root*; other nodes are called *child-nodes*; the nodes at the bottom are called *leaves*. Each sequence of nodes going from the root to one of the leaves is called a *branch* (in the tree above there are three different branches: the branch A, B, C, E, the branch A, B, D, F and the branch A, B, D, G, H). A sequence of consecutive nodes starting from the root is called a *path* (in the tree above paths include: the path A, B and the path A, B, D).

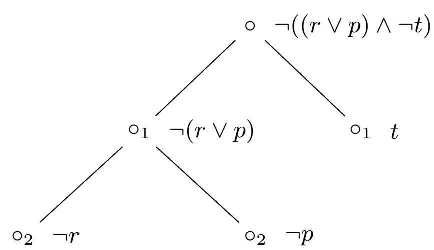
We use the tableau method to construct *trees of g-subformulas* of a given formula. The root of the tree will be associated with the formula we want to extract the g-subformulas from; child-nodes will be associated with g-subformulas of the formula at the top node. In order to extract g-subformulas from a given formula and thus construct a tree, we will use one of the following three rules:

$$\frac{\neg\neg A}{A} \qquad \frac{C \circ E}{C \mid E} \qquad \frac{\neg(C \circ E)}{C^* \mid E^*}$$

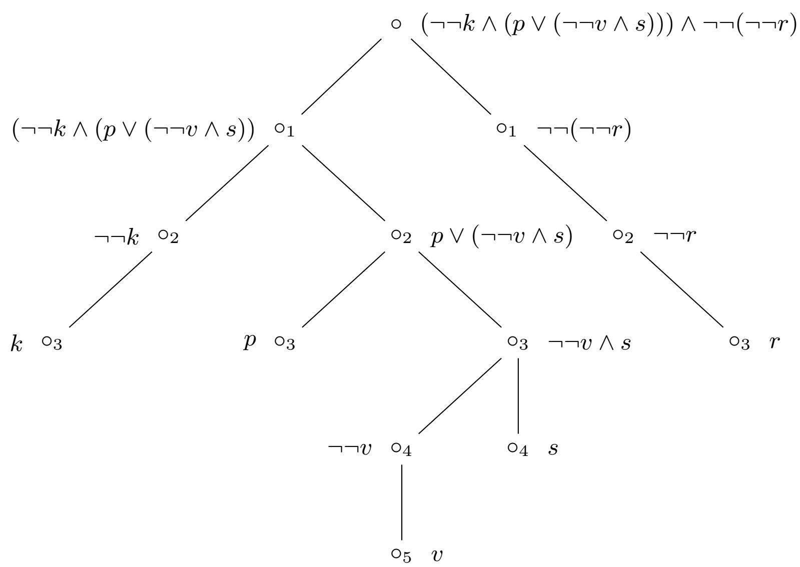
where the symbol $\circ$ stands for either conjunction or disjunction and the vertical line stands for a bifurcation. In the trees we will work with, the passage from one node to another is obtained by means of one of the rules above. Aside from the specificity of these rules, the definition of a tree is the standard one, see Fitting and Mendelsohn (1998). A g-subformula tree closes when no rule can be any longer applied, i.e. when the leafs are occupied by literals. So an example of a tree constructed with our rules is the following:



or the following:



or the following:



In each of these trees each formula associated with a node is a g-subformula of the formula at the root and is indeed obtained from this latter by means of our rules. Moreover each g-subformula occupies a different level in the tree, depending on the number of rules needed to obtain it from the main formula. Given a formula A, we will henceforth denote the tree of g-subformulas of A generated by our rules by $\mathcal{T}_A$.

Definition 4.1. Given a formula A of the language $\mathcal{L}^c$, we say that the multiset M is maximal for A if there exists a set of child-nodes $\mathcal{N}$ of the tree $\mathcal{T}_A$ such that each branch of $\mathcal{T}_A$ contains exactly one child-node in $\mathcal{N}$, and M is the multiset of formulas labelling the child-nodes in $\mathcal{N}$. Finally, let $G_{sub}(A)$ be the set of maximal multisets for A.

Suppose that A is the formula $(\neg\neg k \wedge (p \vee (\neg\neg v \wedge s))) \wedge \neg\neg(\neg\neg r)$ and that T_A is the tree of g-subformulas of A shown above. Then the multiset $M = \{k, p, v, s, r\}$ is maximal for A and belongs to $G_{sub}(A)$; but also the multiset $N = \{k, p \vee (\neg\neg v \wedge s), r\}$ is maximal for A and belongs to $G_{sub}(A)$; and also the multiset $P = \{\neg\neg k \wedge (p \vee (\neg\neg v \wedge s), \neg\neg(\neg\neg r)\}$ is maximal for A and belongs to $G_{sub}(A)$. The fact that in each multiset

$M \in G_{sub}(B)$, each branch of $\mathcal{T}_B$ is taken into account ensures that each multiset $M \in G_{sub}(B)$ is (maximal and thus) a *complete* multiset of g-subformulas of B. The fact that in each multiset $M \in G_{sub}(B)$ each branch of $\mathcal{T}_B$ is taken into account by only one formula ensures that in each multiset $M \in G_{sub}(B)$ there is no superfluous repetition. Finally, the fact that the g-subformulas of any $M \in G_{sub}(B)$ belong to *any* level of the tree $\mathcal{T}_B$ is the key-ingredient that will lead us to the definition of the notion of *mediate* grounding.

Definition 4.2. Given a formula $A \in \mathcal{L}^c$, we denote with $G^*_{sub}(A)$ the set of all multisets M such that for any $N \in G_{sub}(A)$, $M \cong N$.

Suppose that A is the formula $(\neg\neg k \wedge (p \vee (\neg\neg v \wedge s))) \wedge \neg\neg(\neg\neg r)$, so $\mathcal{T}_A$ is the tree of g-subformulas of A drawn above. Then the multiset $\{k, p, v, s, r\}$ belongs to $G_{sub}(A)$. Hence $G^*_{sub}(A)$ contains the following multisets: $\{k, p, v, s, r\}$, $\{\neg k, p, v, s, r\}$ $\{k, \neg p, v, s, r\}$, $\{k, p, \neg v, s, r\}$, $\{k, p, v, \neg s, r\}$, $\{k, p, v, s, \neg r\}$, $\{\neg k, \neg p, v, s, r\}$, $\{\neg k, p, \neg v, s, r\}$, $\{\neg k, p, v, \neg s, r\}$, $\{\neg k, p, v, s, \neg r\}$, $\{k, \neg p, \neg v, s, r\}$, and so on.

Recall that, according to Definition 2.11, for any consistent multiset of formulas $C \cup M$ such that C and M belong to the language $\mathcal{L}^c$, if, under the robust condition C (that may be empty), M completely and immediately logically grounds A, then $C \cup M$ is completely and immediately less g-complex than A according to Definition 2.10. The relation of *being completely and immediately less g-complex* forces the grounds to be the immediate g-subformulas of their conclusion and at the same time ensures that they are the maximal multiset of such g-subformulas. When it comes to complete and *mediate* grounds, by contrast, we want a relation of g-complexity that still yields a maximal multiset of g-subformulas of a given formula but can pick *any* g-subformula and not just immediate ones. Thanks to the notion of $G^*_{sub}(...)$, we can get such relation, namely the relation of *being completely and mediately less g-complex*.

Definition 4.3. Given a multiset M and a formula A both belonging to the classical language $\mathcal{L}^c$, we say that M is *completely and mediately less g-complex* than A if, and only if, M belongs to $G^*_{sub}(A)$.

Let us now turn to the notions of positive and negative derivability. At the first glance it would be tempting to straightforwardly use positive and negative derivability plus the relation of being completely and mediately less g-complex in order to get the definition of complete and mediate formal grounding. In other words it would be tempting to formulate the following definition.

Definition 4.4. $-$**Tentative**. For any consistent multiset of formulas $N \cup M$ and formula A of the language $\mathcal{L}^c$, we say that, under the robust conditions N (that may be empty), M *completely and mediately logically grounds* A, if and only if:

- $\vdash_{\mathbf{C}} M \Rightarrow A$ (*positive derivability*),
- $\vdash_{\mathbf{C}} N, \neg(M) \Rightarrow \neg A$ (*negative derivability*),
- $N \cup M$ is completely and mediately less g-complex than A according to Definition 4.3.

However this definition is problematic. As a counterexample, consider again the formula $(\neg\neg k \wedge (p \vee (\neg\neg v \wedge s))) \wedge \neg\neg(\neg\neg r)$, which we call A. It is easy to check that the multisets $\{k, p, v, s, r\}$ and $\{k, p, \neg v, \neg s, r\}$ belong to $G^*_{sub}(A)$. Moreover both multisets enjoy positive and negative derivability with respect to A. Indeed since A can be derived from the multiset $\{k, p, r\}$, and from $\{\neg p, \neg k, \neg r\}$ $\neg A$ can be derived, by weakening we can add both the literals v and s, but also the literals $\neg v$ and $\neg s$, without changing these relations. So, according to the tentative Definition, both the multiset $\{k, p, v, s, r\}$ and the multiset $\{k, p, \neg v, \neg s, r\}$ are complete and mediate grounds of the formula A. However, while we might agree that the multiset $\{k, p, v, s, r\}$ is a complete and mediate ground for A since each of its atoms seem to contribute to the truth of the formula A, nobody would ever accept $\{k, p, \neg v, \neg s, r\}$ as a complete and mediate ground of A. The tentative Definition 4.4 is thus inadequate.

The problem seems to stem from the breadth of the relation of derivability in classical logic. It works fine in cases of immediate grounding because in such cases the analysis is limited to the main connective of formulas, but with mediate grounding, where the analysis goes deeper into the formula, and possibly down to its atomic parts, the notion of derivability may loosen the connections. In order to avoid such losses and retain the spirit of positive and negative derivability as the formal counterpart of the dependence between grounds and conclusion, we need to focus on a special subset of classical derivations, namely derivations (in the classical sequent calculus) where the rule of weakening (W) is not used and where we start from axioms of the form $M \Rightarrow M$. By prohibiting weakening, we avoid the introduction of irrelevant formulas amongst the grounds and by choosing to start each derivation with axioms of the form $M \Rightarrow M$, we keep track of the connections between grounds and conclusion.

Note that although the prohibition of weakening is motivated by cases like the example above, it also benefits from a strong defense along conceptual lines. Indeed, Bolzano (e.g. see Rumberg (2013)) but also more recently Fine (2012) have suggested that the relation of grounding is non-monotone: and, as a mere logical point, weakening is typically related to monotonicity.

Definition 4.5. Given a multiset M and a formula $A \in \mathcal{L}^c$, we write $\vdash^{\star}_{\mathbf{C}} M \Rightarrow A$ to denote that there exists, in the classical sequent calculus, a derivation of the sequent $M \Rightarrow A$ where there is no use of the weakening rule and the axioms have the form $N \Rightarrow N$ (i.e. we have a derivation of the sequent $M \Rightarrow A$ that starts from $N \Rightarrow N$ and is such that the only rules used are $\neg L$, $\neg R$, $\wedge L/L'$, $\wedge R/R'$, $\vee L/L'$, $\vee R/R'$, see Figures 1 and 2).

We now have all the elements needed to introduce our definition of the notion of complete and mediate grounding.

Definition 4.6. For any consistent multiset of formulas $N \cup M$ and formula A of the language $\mathcal{L}^c$, we say that, under the robust conditions N (that may be empty), M *completely and mediately logically grounds* A, in symbols $[N]\ M \mid\!\sim^m$ A, if and only if:

- $\vdash^{\star}_{\mathbf{C}} M \Rightarrow A$ (*positive derivability*),
- $\vdash^{\star}_{\mathbf{C}} N, \neg(M) \Rightarrow \neg A$ (*negative derivability*),
- $N \cup M$ is completely and mediately less g-complex than A according to Definition 4.3.

where $\neg(M) := \{\neg B | B \in M\}$.

This is a definition of complete and mediate grounding which reveals the properties that characterize this grounding relation, namely positive and negative derivability under $\vdash^{\star}_{\mathbf{C}}$, but also the relation of being completely and mediately less g-complex. We first consider some examples of complete and mediate grounding according to our definition, then we show that the definition is adequate.

Consider again the formula $(\neg\neg k \wedge (p \vee (\neg\neg v \wedge s))) \wedge \neg\neg(\neg\neg r)$ and the three multisets $\{k, p, v, s, r\}$, $\{k, p, \neg v, \neg s, r\}$ and $\{k, p, r\}$. The latter multiset does not belong to $G^*_{sub}(A)$ so it cannot be a complete and mediate ground of A. The multiset $\{k, p, \neg v, \neg s, r\}$ does belong to $G^*_{sub}(A)$ but it enjoys neither positive nor negative derivability with A. Let us see the case of positive derivability under $\vdash^\star_{\mathbf{C}}$:[11]

$$
\begin{prooftree}
\AxiomC{$k \Rightarrow k$}
\RightLabel{$\neg^+$}
\UnaryInfC{$k \Rightarrow \neg\neg k$}
\AxiomC{$p \Rightarrow p, s, s$}
\RightLabel{$\neg L$}
\UnaryInfC{$p, \neg s \Rightarrow p, s$}
\AxiomC{$\Rightarrow v, v$}
\RightLabel{$\neg^+$}
\UnaryInfC{$\neg v \Rightarrow \neg\neg v$}
\RightLabel{$\wedge R'$}
\BinaryInfC{$p, \neg v, \neg s \Rightarrow p, \neg\neg v \wedge s$}
\RightLabel{$\vee R$}
\UnaryInfC{$p, \neg v, \neg s \Rightarrow p \vee (\neg\neg v \wedge s)$}
\RightLabel{$\wedge R'$}
\BinaryInfC{$k, p, \neg v, \neg s \Rightarrow \neg\neg k \wedge (p \vee (\neg\neg v \wedge s))$}
\AxiomC{$r \Rightarrow r$}
\RightLabel{$\neg^+$}
\UnaryInfC{$r \Rightarrow \neg\neg(\neg\neg r)$}
\RightLabel{$\wedge R'$}
\BinaryInfC{$k, p, \neg v, \neg s, r \Rightarrow (\neg\neg k \wedge (p \vee (\neg\neg v \wedge s))) \wedge \neg\neg(\neg\neg r)$}
\end{prooftree}
$$

Although we have applied all the rules that could be possibly applied (we know that since we have reached some atoms), in one case we do not have an axiom, i.e. the case of $\Rightarrow v, v$, and in another case we do not have an axiom of the desired form, namely $p \Rightarrow p, s, s$. Thus the multiset $\{k, p, \neg v, \neg s, r\}$ is not a complete and mediate ground of the formula $(\neg\neg k \wedge (p \vee (\neg\neg v \wedge s))) \wedge \neg\neg(\neg\neg r)$.

By contrast, the multiset $\{k, p, v, s, r\}$ not only belongs to $G^*_{sub}(A)$, but it also enjoys positive and negative derivability, under $\vdash^\star_{\mathbf{C}}$, with A, as we can see in the following derivations:

$$
\begin{prooftree}
\AxiomC{$k \Rightarrow k$}
\RightLabel{$\neg^+$}
\UnaryInfC{$k \Rightarrow \neg\neg k$}
\AxiomC{$p, s \Rightarrow p, s$}
\AxiomC{$v \Rightarrow v$}
\RightLabel{$\neg^+$}
\UnaryInfC{$\neg v \Rightarrow \neg\neg v$}
\RightLabel{$\wedge R'$}
\BinaryInfC{$p, v, s \Rightarrow p, \neg\neg v \wedge s$}
\RightLabel{$\vee R$}
\UnaryInfC{$p, v, s \Rightarrow p \vee (\neg\neg v \wedge s)$}
\RightLabel{$\wedge R'$}
\BinaryInfC{$k, p, v, s \Rightarrow \neg\neg k \wedge (p \vee (\neg\neg v \wedge s))$}
\AxiomC{$r \Rightarrow r$}
\RightLabel{$\neg^+$}
\UnaryInfC{$r \Rightarrow \neg\neg(\neg\neg r)$}
\RightLabel{$\wedge R'$}
\BinaryInfC{$k, p, v, s, r \Rightarrow (\neg\neg k \wedge (p \vee (\neg\neg v \wedge s))) \wedge \neg\neg(\neg\neg r)$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{$k, p, r \Rightarrow p, k, r$}
\AxiomC{$v, s \Rightarrow v, s$}
\RightLabel{$\neg^+$}
\UnaryInfC{$\neg\neg v, s \Rightarrow v, s$}
\RightLabel{$\wedge L$}
\UnaryInfC{$\neg\neg v \wedge s \Rightarrow v, s$}
\RightLabel{$\vee L'$}
\BinaryInfC{$k, p \vee (\neg\neg v \wedge s), r \Rightarrow k, p, v, s, r$}
\RightLabel{$\neg^+$}
\UnaryInfC{$\neg\neg k, p \vee (\neg\neg v \wedge s), \neg\neg(\neg\neg r) \Rightarrow k, p, v, s, r$}
\RightLabel{$\wedge L$}
\UnaryInfC{$\neg\neg k \wedge (p \vee (\neg\neg v \wedge s)), \neg\neg(\neg\neg r) \Rightarrow k, p, v, s, r$}
\RightLabel{$\wedge L$}
\UnaryInfC{$(\neg\neg k \wedge (p \vee (\neg\neg v \wedge s))) \wedge \neg\neg(\neg\neg r) \Rightarrow k, p, v, s, r$}
\RightLabel{$\neg^+$}
\UnaryInfC{$\neg k, \neg p, \neg v, \neg s, \neg r \Rightarrow \neg(\neg\neg k \wedge (p \vee (\neg\neg v \wedge s))) \wedge \neg\neg(\neg\neg r)$}
\end{prooftree}
$$

Hence the multiset $\{k, p, v, s, r\}$ is a complete and mediate ground of the formula A. Other complete and mediate grounds of the formula A include:

- $[\neg p]\{k, v, s, r\}$
- $[\neg p]\{k, \neg\neg v, s, r\}$
- $[\neg s, \neg v]\{k, p, r\}$

[11] In what follows, to shorten the derivations, we might use several applications of the same pair (R/L) of rules in one shot. We denote this by writing the connective of the pair of rules followed by a $+$.

- $[\neg(\neg\neg s \wedge v)]\{k, p, r\}$
- $[\neg(v \wedge \neg\neg s)]\{k, p, r\}$
- $\{\neg\neg k, (p \vee (\neg\neg v \wedge s)), \neg\neg r\}$
- $\{\neg\neg k, ((s \wedge \neg\neg v) \vee p), \neg\neg r\}$
- $\{\neg\neg k, (p \vee (\neg\neg v \wedge s)), \neg\neg(\neg\neg r)\}$
- $\{\neg\neg k \wedge (p \vee (\neg\neg v \wedge s)), \neg\neg(\neg\neg r)\}$

It is tedious but straightforward to verify that each of these combinations enjoy positive and negative derivability with A and they all are completely and mediately less g-complex than A, according to Definition 4.3. Note also that according to Definition 4.6 in a complete and mediate grounding relation there can be more than one robust condition and even robust conditions can be decomposed in their simpler elements.

We now need to prove that Definition 4.6 is adequate, namely that it captures all cases of complete and mediate grounding. As stated in Section 2, complete and mediate grounding should be thought of as the transitive closure of the relation of complete and immediate grounding and as such it can be defined inductively in the following way.

Definition 4.7. For any consistent multiset of formulas $N \cup M$ and formula A of the language $\mathcal{L}^c$, $[N]\ M \mid\!\sim^m A$ if, and only if,

- $[N]\ M \mid\!\sim A$, or
- if $[C]\ M' \mid\!\sim B$ and $[N']\ B, M'' \mid\!\sim^m A$, then $[N]\ M \mid\!\sim^m A$, where $N = N' \cup C$ and $M = M' \cup M''$.

Note that the second item of the definition is said to be a *cut*[12] between the relation of complete and immediate grounding and its transitive closure.

In what follows we will show that Definition 4.6 is equivalent to Definition 4.7.

Theorem 4.8. *For any consistent multiset of formulas $N \cup M$ and formula A of the language $\mathcal{L}^c$, $[N]\ M \mid\!\sim^m A$, as defined in Definition 4.6 if, and only if, $[N]\ M \mid\!\sim^m A$ as defined in Definition 4.7.*

Proof. The proof of this theorem is complicated and long and thus we divide it into two parts. By Lemma 4.9, we show the right to left direction, while by Lemma 4.11, we prove the direction from left to right. □

Lemma 4.9. *For any consistent multiset of formulas $N \cup M$ and formula A of the language $\mathcal{L}^c$, if $[N]\ M \mid\!\sim^m A$, as defined in Definition 4.6, then $[N]\ M \mid\!\sim^m A$ as defined in Definition 4.7.*

Proof. The proof is by induction on the number n of cuts used to obtain the relation $[N]\ M \mid\!\sim^m A$.

- if n is 0, then the relation of complete and mediate grounding is actually a relation of complete and immediate grounding. We then need to show that the restrictions of classical derivability to the special $\vdash^\star_{\mathbf{C}}$ does not affect immediate grounding, and that the relation of being completely and mediately less g-complex cover all the appropriate

[12]This is the technical term standardly used, for further details see Troelstra and Schwichtenberg (1996).

cases. We will prove this by distinguishing cases following the form of the formula A. We first however show that derivability under $\vdash^{\star}_{\mathbf{C}}$ takes into account ac-equivalences:

$$\dfrac{\dfrac{A \Rightarrow A \quad B \Rightarrow B}{A, B \Rightarrow B \wedge A}\,{\scriptstyle \wedge R}}{A \wedge B \Rightarrow B \wedge A}\,{\scriptstyle \wedge L} \qquad\qquad \dfrac{\dfrac{A \Rightarrow A \quad B \Rightarrow B}{A \vee B \Rightarrow B, A}\,{\scriptstyle \vee L}}{A \vee B \Rightarrow B \vee A}\,{\scriptstyle \vee R}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A \quad B \Rightarrow B}{A, B \Rightarrow A \wedge B}\,{\scriptstyle \wedge R} \quad C \Rightarrow C}{A, B, C \Rightarrow (A \wedge B) \wedge C}\,{\scriptstyle \wedge R}}{A, B \wedge C \Rightarrow (A \wedge B) \wedge C}\,{\scriptstyle \wedge L}}{A \wedge (B \wedge C) \Rightarrow (A \wedge B) \wedge C}\,{\scriptstyle \wedge L} \qquad \dfrac{\dfrac{\dfrac{A \Rightarrow A \quad \dfrac{A \Rightarrow A \quad B \Rightarrow B}{A \vee B \Rightarrow A, B}\,{\scriptstyle \vee L}}{A \vee (B \vee C) \Rightarrow A, B, C}\,{\scriptstyle \vee L}}{A \vee (B \vee C) \Rightarrow (A \vee B), C}\,{\scriptstyle \vee R}}{A \vee (B \vee C) \Rightarrow (A \vee B) \vee C}\,{\scriptstyle \vee R}$$

Now we can distinguish cases following the form of the formula A.

If A is $\neg\neg B$, then either $\{B\}$ or any multiset composed by formulas ac-equivalent to B is a complete and immediate ground of A according to Definition 2.11. As the following derivations show, $\{B\}$ enjoys positive and negative derivability under $\vdash^{\star}_{\mathbf{C}}$ with $\neg\neg B$:

$$\dfrac{\dfrac{B \Rightarrow B}{B, \neg B \Rightarrow}\,{\scriptstyle \neg L}}{B \Rightarrow \neg\neg B}\,{\scriptstyle \neg R} \qquad \dfrac{\dfrac{B \Rightarrow B}{\neg\neg B \Rightarrow B}\,{\scriptstyle \neg^{+}}}{\neg B \Rightarrow \neg\neg\neg B}\,{\scriptstyle \neg^{+}}$$

The same holds for any formula ac-equivalent to B by constructing a derivation that starts as one of those (or a combination of one of those) that takes into account ac-equivalence and then continuing with negation rules. Note that $\{B\}$ as well as any multiset composed by a formula ac-equivalent to it belongs to $G^{*}_{sub}(\neg\neg B)$ and thus it is completely and mediately less g-complex than $\neg\neg B$.

If A is $B \wedge C$, then either $\{B, C\}$ or any multiset composed by formulas ac-equivalent to B or to C is a complete and immediate ground of A according to Definition 2.11. As the following derivations show, $\{B, C\}$ enjoys positive and negative derivability under $\vdash^{\star}_{\mathbf{C}}$ with $B \wedge C$:

$$\dfrac{B \Rightarrow B \quad C \Rightarrow C}{B, C \Rightarrow B \wedge C}\,{\scriptstyle \wedge R} \qquad \dfrac{\dfrac{B, C \Rightarrow B, C}{B \wedge C \rightarrow B, C}\,{\scriptstyle \wedge L}}{\neg B, \neg C \rightarrow \neg(B \wedge C)}\,{\scriptstyle \neg^{+}}$$

The same holds for any formula ac-equivalent to B by constructing a derivation that starts as one of those (or a combination of one of those) that takes into account ac-equivalence and then continuing with the rule $\wedge R$ or $\wedge L$ and the negation rules. Note that $\{B, C\}$ as well as any multiset composed by formulas ac-equivalent to B and to C belongs to $G^{*}_{sub}(B \wedge C)$ and thus it is completely and mediately less g-complex than $B \wedge C$.

If A is $B \vee C$, then either $\{B, C\}$ or any multiset composed by formulas ac-equivalent to B or to C is a complete and immediate ground of A according to Definition 2.11. But also $\{B\}$ under the robust condition C^{*}, as well as $\{C\}$ under the robust condition B^*, are complete and immediate of A (as well as formulas that are ac-equivalent to them) according to Definition 2.11. We first show that $\{B, C\}$ enjoys positive and negative derivability under $\vdash^\star_{\mathbf{C}}$ with $B \vee C$:

$$\frac{B, C \Rightarrow B, C}{B, C \Rightarrow B \vee C}\ \vee R \qquad \frac{\dfrac{B \Rightarrow B \quad C \Rightarrow C}{B \vee C \rightarrow B, C}\ \vee L}{\neg B, \neg C \rightarrow \neg(B \vee C)}\ \neg^+$$

The same holds for any multisets containing formulas that are ac-equivalent to B or to C by constructing a derivation that starts as one of those (or a combination of one of those) that takes into account ac-equivalence and then continuing with the rule $\vee R$ or $\vee L$ and the negation rules. Note that $\{B, C\}$ as well as any multiset composed by formulas ac-equivalent to B and to C belongs to $G^*_{sub}(B \vee C)$ and thus it is completely and mediately less g-complex than $B \vee C$.

Let us move to the case where $\{B\}$ under the robust condition C^* is a complete and immediate ground of $B \vee C$ (the case where $\{C\}$ under the robust condition B^* is a complete and mediate ground of $B \vee C$ can be treated analogously). We first show that $\{B\}$ enjoys positive derivability under $\vdash^\star_{\mathbf{C}}$ with $B \vee C$ (negative derivability is the same as before):

$$\frac{B \Rightarrow B}{B \Rightarrow B \vee C}\ \vee R'$$

An analogous derivation can be constructed for formulas that are ac-equivalent to B. Finally, note that $\{B, C^*\}$, as well as any multiset composed by formulas ac-equivalent to B or to C^* belongs to $G^*_{sub}(B \vee C)$ and thus it is completely and mediately less g-complex than $B \vee C$.

If A is $\neg(B \wedge C)$ or $\neg(B \vee C)$, then these cases can be treated analogously to the previous ones.

- n is > 0. Suppose that the relation $[N]\ M \mid\!\sim^m A$ has been obtained by $[C]\ M' \mid\!\sim B$ and $[N']\ B, M'' \mid\!\sim^m A$ where $N = N' \cup C$ and $M = M' \cup M''$. By the inductive hypothesis, we have that $B, M'' \vdash^\star_{\mathbf{C}} A$. Since B is not atomic (we know that since we have $[C]\ M' \mid\!\sim B$) we can continue this derivation (upwards) applying the rules deriving B. Whatever form B might have, we know that such derivation exists and ends with axioms of the appropriate form because of what we have just proved with n= 0. This yields a derivation of $M \vdash^\star_{\mathbf{C}} A$. Similar reasoning applies to negative derivability. Finally note that from $[N']\ B, M'' \mid\!\sim^m A$, by the inductive hypothesis we have that $\{N', B, M''\}$ is completely and mediately less complex than A. But then this also holds for $\{N, M\}$, since this multiset is obtained from $\{N', B, M''\}$ by substituting B for its complete and immediate g-subformulas. Therefore $\{N, M\}$ still belongs to $G^*_{sub}(A)$. Hence all conditions are satisfied to claim that M is a complete and mediate ground of A under the robust conditions N (that may be empty), according to Definition 4.6.

□

Let us make two important remarks. The first remark is that via Lemma 4.9 we have actually shown that any complete and immediate ground according to Definition 2.11 is also a complete and mediate ground in the sense of Definition 4.6; this involves that even if in Definition 2.11 Poggiolesi uses positive and negative derivability in classical logic, she could have used positive and negative derivability restricted to our notion $\vdash^{\star}_{\mathbf{C}}$ and the resulting grounds would have been the same. Hence Definition 4.6 is a proper extension of Definition 2.11.

The second remark concerns Lemma 4.9 which tells us that any mediate grounding established via the use of n cuts can also be established without using any cut. If from a grounding point of view, this formulation of the Lemma might nor ring any bell, from a proof-theoretical point of view it surely does, since Lemma 4.9 is clearly proving a sort of cut-elimination theorem (see Indrzejczak (2010); Poggiolesi (2010)) for mediate grounding. In other words, Lemma 4.9 is telling us that the underlying proof-structure of grounding chains is purely analytic: nothing enters in the derivation that is not required to draw the conclusion. As explained in the Introduction, the importance of analyticity property for explanatory proofs has been underlined by a long and illustrious philosophical tradition. Hence Lemma 4.9 does not do anything else than confirming this tradition.

Definition 4.10. Let M be a consistent multiset of formulas of the language $\mathcal{L}^c$ which is completely and mediately less g-complex than a formula A of the language $\mathcal{L}^c$. Let $\mathcal{T}_A$ be the tree of g-subformulas of A such that each formula of M either is associated to a node of $\mathcal{T}_A$ or is in the relation $\cong$ with a formula associated to a node of $\mathcal{T}_A$. By Definition 4.2, each formula B of M is associated to a (different) node in $\mathcal{T}_A$. We associate to each formula B of M, the *tree-distance* $td(B)$, defined as the length of the path ending with the node associated with B (or to a formula in the $\cong$ relation with B) in $\mathcal{T}_A$, minus one. Let $td(M)$ be the sum of all $td(B)$ such that $B \in M$.

Lemma 4.11. *For any consistent multiset of formulas $N \cup M$ and formula A of the language $\mathcal{L}^c$, if $[N]\ M\ |\!\sim^m A$, as defined in Definition 4.7, then $[N]\ M\ |\!\sim^m A$ as defined in Definition 4.6.*

Proof. We reason by induction on $td(N \cup M)$.

- If $td(N \cup M)$ is 0, then we have a case of complete and immediate grounding. First of all, inspection of cases show that the relation of completely and mediately less g-complex between $M \cup N$ and A corresponds to the relation of completely and immediately less g-complex between $M \cup N$ and A. Moreover $M \cup N$ enjoys positive and negative derivability under $\vdash^{\star}_{\mathbf{C}}$ with A. Hence, by the way $\vdash^{\star}_{\mathbf{C}}$ is defined, $M \cup N$ also enjoys classical positive and negative derivability with A. Therefore M is a complete and mediate ground of A under the robust conditions N (that may be empty) according to Definition 4.7.

- If $td(N \cup M) > 0$, then M and the eventual robust conditions N do not contain only immediate g-subformulas of A, but also g-subformulas of a deeper tree-distance from A. In particular, there exists C in M such that the path between node associated to C and the root node in $\mathcal{T}_A$ contains intermediate nodes. Let B be the formula associated to the parent node of C on this path, let $M'' \cup \{D\}$, with M'' a subset of M and D in N, be the set of formulae in $M \cup N$ associated to the child nodes of this node, and let M' be $M \setminus M''$ and and N' be $N \setminus \{D\}$. Since (the nodes associated to) M'' and D are immediate children of (the node associated to) B in the tree, they are completely and immediately less g-complex. It is straightforward to see that whenever child nodes are connected to a parent node via a single tree-rule, they satisfy positive and negative derivability; hence this holds for M'' and D with respect to B. So, by Definition 2.11, M'' is a complete and immediate ground of B, under the robust condition D. On the other hand, by construction, it is clear that $\{B\} \cup M' \cup N'$ form a completely and mediately less g-complex multiset of g-subformulas of A. By the way positive and negative derivability under $\vdash^{\star}_{\mathbf{C}}$ have been defined, it is straightforward to see that M', N' and B enter in this relation with A. Hence, $M' \cup \{B\}$ is a complete and mediate ground of A, under the robust condition N, according to Definition 4.6. Moreover, $td(M' \cup \{B\} \cup N) < td(M)$ since $M' \cup \{B\} \cup N'$ has been obtained from $M \cup N$ by replacing formulas with a tree-distance from A higher than that of B. We can thus apply the inductive hypothesis and obtain that M', B are the complete and mediate grounds of A under the robust conditions N' according to Definition 4.6. Hence M is a complete and mediate ground of A under the robust condition N according to Definition 4.6, as required. $\square$

5 Partial and mediate logical grounding

Let us now consider the notion of partial and mediate grounding. As for the case of complete and mediate grounding, the notion of partial and mediate grounding should be seen as the transitive closure of the relation of partial and immediate grounding and as such it can be rigorously defined in the following way.

Definition 5.1. For any consistent multiset of formulas M and formula A of the language $\mathcal{L}^c$, M is a partial and mediate ground of A, in symbols $M \mid\mid\!\sim^m$ A if, and only if:

- $M \mid\mid\!\sim A$, or
- if $M' \mid\mid\!\sim B$ and $B, M'' \mid\mid\!\sim^m A$, then $M \mid\mid\!\sim^m A$, where $M = M' \cup M''$.

Unfortunately, careful reflexion shows that the notion of partial and mediate grounding cannot be defined in terms of derivability and complexity as we have done for the notions of partial and immediate and complete and mediate grounding. Partial and mediate grounding is a sequence of partial and immediate grounding steps and Definition 3.1 tells us that each step can either enjoy positive or (exclusive) negative derivability, but of course there seems to be no way to describe the order in which each step satisfies either one or the other. Consider the pairs of formulas p and $((p \vee q) \wedge (r \vee s)) \wedge f$ on the one hand, and p and $((p \wedge q) \vee (r \wedge s)) \vee f$ on the other: there seems to be a relation of partial and immediate grounding within each pair, and indeed it even seems possible to identify steps of either positive or negative derivability. However an unique and general procedure that covers them both does not seem to be describable since they differ in each step. So there seems to be no way to generalize Definition 3.1.

6 Complete and partial versus full and partial

In this section, we explore the links between the complete-partial distinction studied above, and the distinction between full and partial grounds, as introduced by Fine (2012). First of all note that the comparison of the two approaches cannot but be on the basis of their extensions: in case of the full-partial distinction, full grounding is taken as a primitive notion and thus there is no definition to rely on.

We use the symbol $<$ to denote full and immediate grounding, and the symbol $\prec$ to denote partial and immediate ground in Fine's sense. The grounding principles holding for these notions, according to Fine (2012), are given in Figure 3 and Figure 4, respectively.

Figure 3: Full and immediate grounding principles.

$A < \neg\neg A$			
$A, B < A \wedge B$	$A < A \vee B$	$B < A \vee B$	$A, B < A \vee B$
$\neg A < \neg(A \wedge B)$	$\neg B < \neg(A \wedge B)$	$\neg A, \neg B < \neg(A \wedge B)$	$\neg A, \neg B < \neg(A \vee B)$

Figure 4: Partial and immediate grounding principles.

$A \prec A \wedge B$	$B \prec A \wedge B$	$A \prec A \vee B$	$B \prec A \vee B$
$\neg A < \neg(A \wedge B)$	$\neg B < \neg(A \wedge B)$	$\neg A < \neg(A \vee B)$	$\neg B < \neg(A \vee B)$

To compare the two approaches, let us firstly note a difference between Fine's and Poggiolesi's accounts which, although transversal to the notions of full and complete, is relevant here. It concerns the treatments of negation in formulas like $\neg(A \wedge B)$ and $\neg(A \vee B)$. Whilst in Fine's account $\neg A$ and $\neg B$ are as grounds, in Poggiolesi's account the notion of converse is used. Hence, to give an example, in case of the formula $\neg(p \vee q)$, both Poggiolesi and Fine's accounts agree that the grounds are $\neg p, \neg q$, whilst in case of the formula $\neg(\neg p \vee \neg q)$, under Fine's account the grounds are $\neg\neg p$ and $\neg\neg q$, whilst under Poggiolesi's the grounds are p, q. A detailed discussion of the advantages of using the notion of converse is provided in Poggiolesi (2016a) so we do not dwell on it. Let us introduce the definition of the function τ that allows us to account for this difference.

Definition 6.1. Let M be a multiset of formulas and A a formula of the language $\mathcal{L}^c$ such that $M < A$. Then either M only contains g-subformulas of A or it contains a formula of the form $\neg\neg B$ such that B is a g-subformula of A. Let τ be the function assigning to the multiset M, the multiset M itself in case it contains only g-subformulas of A, otherwise the multiset M' which will be obtained from M by replacing each non-g-subformula of A of the form $\neg\neg B$ with the corresponding formula of the form B.

Proposition 6.2. *For any consistent multiset M and formula A of the language $\mathcal{L}^c$, we have that if $M < A$, then there exists a formula C (that may be empty) and a function τ defined as above such that $[C]\ (M)^\tau \mid\!\sim A$.*

Proof. By a simple inspection of cases. □

The notion of full ground is thus definable in terms of the notion of complete ground via the function τ. This result may help clarifying the notion of full ground itself. Indeed, although full ground is never explicitly defined, Fine seems to assume that full grounds are *sufficient* to obtain their conclusion. In cases where A is a full ground of $A \vee B$ or A, B are the full ground of $A \wedge B$, the sufficiency statement seems to work. On the other hand, the grounding principle stating that A, B are the full ground of $A \vee B$ fits less well with the idea that sufficiency characterises full grounds, for A, B are more than sufficient to obtain $A \vee B$. Someone might wonder, if one allows A, B to be full grounds in such cases, why one cannot also take A, C or B, D as full grounds of $A \vee B$. Proposition 6.2 provides a reply to such worries, insofar as it characterises the full grounds of a truth are those that either are complete grounds or may become with the help of some robust conditions.

We have not been able to find a definition of the notion of complete grounds in terms of the notion of full ground. Indeed, we conjecture that there is no way of defining complete from full ground, for two reasons. Firstly, the former involves the notion of robust condition which is absent from the latter; secondly, complete and immediate grounds are closed under ac-equivalence, while no ground-theoretic equivalence emerges from Fine's account.

Let us now move to the two notions of partial ground, one expressed by the symbol $||\!\sim$, and the other by the symbol $\prec$. If we ignore the contrast between the negation of a formula versus its converse, and the closure under ac-equivalence of the relation $||\!\sim$, the two notions coincide.

Definition 6.3. Let A, B be two formulas of the language $\mathcal{L}^c$ such that $B \prec A$. Then either B is a g-subformula of A or it might differ from a g-subformula of A because of the form $\neg\neg C$ and C is a g-subformula of A. Let ϕ be the function assigning to the formula B, the formula B itself in case it already is a g-subformula of A, otherwise the formula B' which is obtained by B by appropriately erasing the double negation.

Proposition 6.4. *For any formulas A and B of the language $\mathcal{L}^c$, if $B \prec A$, then there exists a function ϕ such that $(B)^\phi\ ||\!\sim A$.*

Proof. By a simple inspection of cases. □

Let us now move to mediate grounding. In order to obtain the notion of full and mediate grounding, we can take the transitive closure of the corresponding notion of immediate grounding.[13] So we have:

Definition 6.5. Given a multiset of formulas M and a formula A of the language $\mathcal{L}^c$, M is a full and mediate ground of A, in symbols $M <_m A$ if, and only if:

- $M < A$, or
- if $M' < B$ and $B, P' <_m A$, then $M <_m A$, where $M = M' \cup P'$.

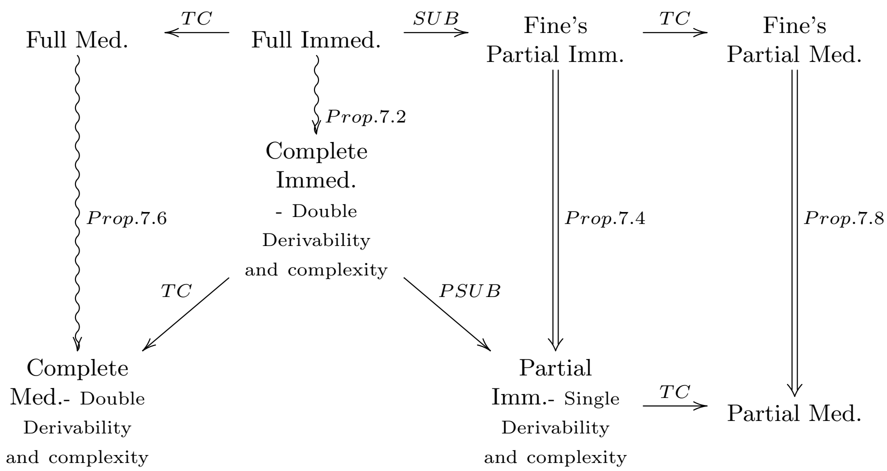
It is easy to see that the relationship between complete and full grounding at the mediate level is analogous to that between complete and full at the immediate level.

Proposition 6.6. *For any consistent multiset M and formula A of the language $\mathcal{L}^c$, we have that if $M <_m A$, then there exists a multiset N (that may be empty) and a function τ defined as above such that $[N]\ (M)^\tau \mid\!\sim A$.*

Proof. By considering Definition 4.7, Proposition 6.2 and Theorem 4.11, it is straightforward. □

[13]In Definition 6.5 we slightly deviate from Fine's original notation and formulation. However it is easy to see that the one adopted here is equivalent to that used by Fine and is more useful for our comparison.

Figure 5: Cartography of grounding



In the table TC stands for transitive closure, while SUB stands for subset and $PSUB$ for proper subset.

The same holds for partial grounding. Let us indeed define partial and mediate grounding in the Finean sense in the following way:

Definition 6.7. Given a multiset of formulas M and a formula A of the language $\mathcal{L}^c$, M is a partial and mediate ground of A, in symbols $M \prec_m A$ if, and only if:

- $M \prec A$, or
- if $M' \prec B$ and $B, P' \prec_m A$, then $M \prec_m A$, where $M = M' \cup P'$.

Therefore we have:

Proposition 6.8. *For any consistent multiset M and formula A of the language $\mathcal{L}^c$, we have that if $M \prec_m A$, then there exists a function τ such that $(M)^\tau \mid\!\sim A$.*

Proof. The proof is straightforward. □

7 Conclusions

In the recent literature on grounding two dominant distinctions are that between full and partial grounding, and that between immediate and mediate grounding. An older distinction, which is analogous to yet different from the former one, is that between complete and partial grounding. In this paper, drawing on previous work on complete and immediate grounding, we have developed a single framework where the three different notions of complete and immediate, complete and mediate and partial and immediate have been rigorously defined via the notions of derivability and complexity. These definitions not only revive an old tradition relating grounding to proof, but they illustrate the insight it can provide into grounding concepts. Moreover they have emphasized important and interesting features of grounding: in case of partial grounding the lack of a proper dependence between grounds and conclusion, and in case of complete and mediate grounding the absence of weakening as well as the presence of a purely analytic base. Finally, drawing on this rich analysis, we have also incorporated Fine's notion of full grounding by translating it in terms of complete grounding. We have thus obtained a single proof-based framework mapping out the relationships among all of the grounding notions and with other non-grounding notions.

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