New singularity invariants: the sheaf $\beta \bullet X$

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New singularity invariants : the sheaf $\beta_X^\bullet$.

Daniel Barlet.*

March 4, 2020

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ABSTRACT. The graded coherent sheaf $\alpha_X^\bullet$ constructed in [B.18] for any reduced pure dimensional complex space X is stable by exterior product but not by the de Rham differential. We construct here a new graded coherent sheaf $\beta_X^\bullet$ containing $\alpha_X^\bullet$ and stable both by exterior product and by the de Rham differential. We show that it has again the “pull-back property” for holomorphic maps $f : X \rightarrow Y$ between irreducible complex spaces such that $f(X)$ is not contained in the singular set of Y . Moreover, this graded coherent sheaf $\beta_X^\bullet$ comes with a natural coherent exhaustive filtration and this filtration is also compatible with the pull-back by such holomorphic maps. These sheaves define new invariants on singular complex spaces. We show on some simple examples that these invariants are new.

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1 Erratum for “the sheaf $\alpha_X^\bullet$ ”

The aim of this section is to correct several mistakes in [B.18]. The main mistake is in theorem 4.1.1 which is wrong in the very general setting in which it is stated.

So we begin with a much more modest version of the “pull-back” theorem for these sheaves which has a rather simple proof.

Theorem 1.0.1 *Let $f : X \rightarrow Y$ be a holomorphic map between irreducible complex spaces and assume that $f(X)$ is not contained in the singular set $S(Y)$ of Y . Then there exists a natural “pull-back map”*

$$\hat{f}^* : f^*(\alpha_Y^\bullet) \rightarrow \alpha_X^\bullet$$

which extends the usual pull-back of the sheaf $f^ : f^*(\Omega_Y^\bullet / \text{torsion}) \rightarrow \Omega_X^\bullet / \text{torsion}$. For any holomorphic maps $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ between irreducible complex spaces such that $f(X) \not\subset S(Y)$ and $g(f(X)) \not\subset S(Z)$ we have*

$$\hat{f}^*(g^*(\sigma)) = \widehat{f \circ g}^*(\sigma) \quad \forall \sigma \in \alpha_Z^\bullet.$$

PROOF. The problem is local. Let σ be a section of the sheaf $\alpha_Y^\bullet$ on an open set V in Y . Let V' be the set of regular points in V and let U'' the set of regular points in the open set $U' := f^{-1}(V')$. This is a Zariski dense open set in $U := f^{-1}(V)$ and, as σ is a holomorphic form on V' , $f^*(\sigma)$ is a well defined holomorphic form on U'' which is Zariski open and dense in U . Take a point x in U ; by definition (see proposition 2.2.4 in [3]) there exists a open neighborhood W of $y := f(x)$ in V and a monic polynomial

$$P(z) = z^k + \sum_{h=1}^k S_h \cdot z^{k-h}$$

such that S_h is a section on W of the symmetric algebra of degree h , $S_h(\Omega_Y^\bullet / \text{torsion})$, of the sheaf $\Omega_Y^\bullet / \text{torsion}$, which satisfies $P(\sigma) = 0$ in $\Gamma(W, S_k(\Omega_Y^\bullet / \text{torsion}))$. Then the pull-back $f^*(P)$ of P by f is well defined on $f^{-1}(W)$ and is a monic polynomial whose coefficients are sections on $f^{-1}(W)$ of the symmetric algebra of $\Omega_X^\bullet / \text{torsion}$. On the open set $U'' \cap f^{-1}(W)$ the holomorphic form $f^*(\sigma)$ is a root of $f^*(P)$ and so the meromorphic¹ form $f^*(\sigma)$ on $U \cap f^{-1}(W)$ is integrally dependent on the sheaf $\Omega_X^\bullet / \text{torsion}$. So it defines a unique section on U of the sheaf $\alpha_X^\bullet$.

The second assertion of the theorem is a simple consequence of the fact that the sheaf $\alpha_X^\bullet$ has no torsion. ■

The second mistake (which is consequence of the previous one) is that in definition 5.1.5 of [3] it is necessary to ask that the p -dimensional irreducible analytic subset Y is not contained in the singular set of X to define the integral on Y of a form of

¹Remember that σ is a meromorphic form on V with poles in $S(Y) \cap V$.

the type $\rho.\alpha \wedge \bar{\beta}$, where α, β are sections of the sheaf α_X^p in X .

To be clear we give here the correct statements for definition 5.1.5, lemma 5.1.6 and for the theorem 5.1.7. The proof given in [3] of this theorem is correct but makes sense only assuming that the pull-back for the sheaf $\alpha^\bullet$ are defined. This is consequence of the following hypotheses which allow to apply the corrected version of the theorem 4.1.1 above.

Definition 1.0.2 *Let X be an irreducible complex space and let $Y \subset X$ be a closed irreducible p -dimensional analytic subset in X ; assume that Y is not contained in the singular set $S(X)$ of X . We shall note $j : Y \rightarrow X$ the inclusion map. Let ρ be a continuous function with compact support in X and let α, β be sections on X of the sheaf α_X^p . We define the number $\int_Y \rho.\alpha \wedge \bar{\beta}$ as the integral*

$$\int_Y j^*(\rho).\hat{j}^*(\alpha) \wedge \overline{\hat{j}^*(\beta)}$$

which is well defined by corollary 5.1.2 of [3] using the inclusion $\alpha_Y^p \subset L_Y^p$.

Lemma 1.0.3 *Let $f : X \rightarrow Y$ be a holomorphic map between irreducible complex spaces. Let Z be a closed p -dimensional irreducible analytic subset in X such that Z is not contained in the singular set $S(X)$ of X , the restriction of f to Z is proper and $f(Z)$ is not contained in the singular set of Y . Let α, β be sections on Y of the sheaf α_Y^p and let ρ be a continuous function with compact support in Y . Then we have the equality*

$$\int_Z f^*(\rho).\hat{f}^*(\alpha) \wedge \overline{\hat{f}^*(\beta)} = \int_{f_*(Z)} \rho.\alpha \wedge \bar{\beta}.$$

If $f(Z)$ is contained in $S(Y)$ the singular set of Y and has dimension at most $p-1$ we have $\int_Z f^(\rho).\hat{f}^*(\alpha) \wedge \overline{\hat{f}^*(\beta)} = 0$.*

PROOF. The first assertion is an easy consequence of the same result when α, β are holomorphic forms, by considering a modification of Z where it is the case, using for instance, a desingularization of Z (see [4] normalizing the sheaf Ω_Z^p (see [3])). When $f(Z) \subset S(Y)$ and $f_*(Z) = 0$ the restriction of f to Z has generic rank at most $p-1$, so the pull-back of any holomorphic p -form on Y to Z is 0. Then the monic polynomial giving an integral dependence relation of α (or of β) reduces to $z^k = 0$ on $f(Z)$ and so α (and β) vanishes on Z . ■

Theorem 1.0.4 *Let X be an irreducible complex space and $(Y_t)_{t \in T}$ be an analytic family of p -cycles in X parametrized by a reduced complex space T . Assume that for t in a dense open subset T' in T no component of the cycle Y_t is contained in $S(X)$,*

the singular set of X . Let ρ be a continuous function with support in the compact set K in X and let α, β be two sections of the sheaf α_X^p . Define the function

$$\varphi : T' \rightarrow \mathbb{C} \quad \text{by} \quad \varphi(t) := \int_{Y_t} \rho \cdot \alpha \wedge \bar{\beta}.$$

Then φ is continuous on T' and locally bounded near each point in T . More precisely, for any continuous hermitian metric h on X and any compact set L in T , there exists a constant $C > 0$ (depending on K, α, β, h, L but not of the choice of ρ with support in K) such that for each $t \in T'$ we have:

$$|\varphi(t)| \leq C \cdot \int_{Y_t} |\rho| \cdot h^{\wedge p} \leq C \cdot \|\rho\| \cdot \int_{Y_t \cap K} h^{\wedge p}.$$

Remark that the subset of points $t \in T$ where the cycle Y_t has no irreducible component meeting a given compact set and contained in $S(X)$ is a closed analytic subset in T by a general result on analytic families of cycles (see for instance the proposition IV 3.5.7 in [2] in the case of compact cycles). So, assuming that T is irreducible, if there exists a point t such that Y_t has no irreducible component meeting K and contained in $S(X)$, there exists a Zariski open and dense subset T' of T which satisfies the hypothesis in the previous theorem.

The last mistake is the lemma 6.2.2 which is wrong for $k \geq 4$. The correct computation of $\alpha_{S_k}^2$ is given in the paragraph 3.3.

2 Definition of $\beta_X^\bullet$ and the pull-back property

The next sections of this paper are complements to [3]. So the notations are the same than in *loc. cit.* In particular, for a reduced pure dimensional complex space X we note $L_X^\bullet$ the graded sheaf of meromorphic forms on X which are holomorphic on any desingularization of X and $\omega_X^\bullet$ the sheaf of $\bar{\partial}$ -closed currents on X of type $(\bullet, 0)$ modulo its torsion sub-sheaf. Recall that the graded sheaf $\alpha_X^\bullet$ constructed in *loc. cit.* is a graded coherent sub-sheaf of $L_X^\bullet$ which is again a graded coherent sub-sheaf of $\omega_X^\bullet$. Of course, all these sheaves contain the sheaf $\Omega_X^\bullet / \text{torsion}$ and coincide with it on the non singular part of X .

2.1 Construction of the sheaf $\beta_M^\bullet$

Let X be a pure dimensional reduced complex space and note $\alpha_X^\bullet$ the graded sheaf on X introduced in [3].

Lemma 2.1.1 *The sheaf α_X is stable by exterior product.*

PROOF. Recall that, by definition, the sheaf $\alpha_X^\bullet$ is a sub-sheaf of the sheaf $\omega_X^\bullet$ and a section σ on the open set $U \subset X$ of the sheaf $\omega_X^\bullet$ is a section on U of $\alpha_X^\bullet$ if it may be written locally on U as $\sigma = \sum_{j \in J} \rho_j \cdot \omega_j$ where ω_j are holomorphic forms on U and ρ_j are $\mathcal{C}^\infty$ functions on the complement of the singular set S in U which are bounded near S . Is it clear that the exterior product of two such sections on U of $\alpha_X^\bullet$ can be written in the same way locally on U and then define a current on U which is $\bar{\partial}$ -closed on $U \setminus S$. So to conclude the lemma, it is enough to prove that this current admits a $\bar{\partial}$ -closed extension to U . In fact, as the sheaf $\alpha_X^\bullet$ is a sub-sheaf of the sheaf $L_X^\bullet$ obtained by the direct image of the sheaf $\Omega_{\tilde{X}}^\bullet$ where $\tau : \tilde{X} \rightarrow X$ is any desingularisation of X and as this sheaf $L_X^\bullet$ is stable by exterior product, the conclusion follows from the inclusion $L_X^\bullet \subset \omega_X^\bullet$. $\blacksquare$

Remark that the sheaf $\alpha_X^\bullet$ is a graded $\Omega_X^\bullet$ -module but is not stable in general by the de Rham differential. For instance in $X := \{(x, y, z) \in \mathbb{C}^3 \mid x \cdot y = z^2\}$ the differential form $dx \wedge dy/z = -d(z \cdot dx/x - z \cdot dy/y)$ is not in $\alpha_X^\bullet$ but the form $z \cdot dx/x - z \cdot dy/y$ is a section of α_X^1 (see [3] or paragraph 3.3).

A CONSTRUCTION. Define $\alpha_X^\bullet[0] := \alpha_X^\bullet$ and for any integer $p \geq 0$ and any integer $q \geq 0$ define

$$\alpha_X^q[p+1] := \sum_{r=0}^q \left(\alpha_X^r[p] \wedge \alpha_X^{q-r}[p] \right) + \sum_{r=0}^{q-1} \left(\alpha_X^r[p] \wedge d(\alpha_X^{q-r-1}[p]) \right) \subset L_X^q \quad (1)$$

Recall that the sheaf $L_X^\bullet$ is stable by exterior products and by the de Rham differential.

Proposition 2.1.2 *Then we have the following properties:*

1. *For each integer p the sheaf $\alpha_X^\bullet[p]$ is stable by exterior product with $\Omega_X^\bullet/\text{torsion}$. Moreover for each integers p, q we have $\alpha_X^0 \cdot \alpha_X^q[p] \subset \alpha_M^q[p]$.*
2. *For each integers p, q the sheaf $\alpha_X^q[p]$ is $\mathcal{O}_X$ -coherent sub-sheaf of L_X^q .*
3. *For each integers p, q the sub-sheaf $\alpha_X^q[p]$ is contained in $\alpha_X^q[p+1]$.*
4. *For each integers p, q and q' we have $\alpha_X^q[p] \wedge \alpha_X^{q'}[p] \subset \alpha_X^{q+q'}[p+1]$.*
5. *For each integers p, q and r we have $\alpha_X^r[p] \wedge d(\alpha_X^q[p]) \subset \alpha_X^{q+r+1}[p+1]$. In particular $d(\alpha_X^q[p]) \subset \alpha_X^{q+1}[p+1]$.*

PROOF. The property 1. is an obvious consequence of the definition of these sheaves by an induction on p .

As L_X^q is a coherent sheaf on X , to prove 2. it is enough to prove that $\alpha_X^q[p+1]$ is a finite type $\mathcal{O}_X$ -module. We shall prove this by an induction on $p \geq 0$.

So assumed that for each q the coherence of the sheaf $\alpha_X^q[p]$. Then we want to prove that $\alpha_X^q[p+1]$ is finitely generated. Let $g_{j,r}$ be a generator of the sheaf $\alpha_X^r[p]$. Then we shall show that the elements $g_{i,r} \wedge g_{j,q-r}$ and $g_{i,r} \wedge dg_{j,q-r-1}$ for all choices of i and j , generates $\alpha_X^q[p+1]$. The only point which is not obvious is the fact that for any sections $u \in \alpha_X^r[p]$ and $v \in \alpha_X^{q-r-1}[p]$ the wedge product $u \wedge dv$ is in the sheaf generated by our “candidates” generators. But then write

$$u = \sum_i a_i \cdot g_{i,r} \quad \text{and} \\ v = \sum_j b_j \cdot g_{j,q-r-1}$$

where a_i and b_j are holomorphic functions. Then

$$dv = \sum_j db_j \wedge g_{j,q-r-1} + \sum_j b_j \cdot dg_{j,q-r-1}.$$

So in the wedge products $u \wedge dv$ the terms are linear combinations of our candidates generators excepted those like $a_i \cdot g_{i,r} \wedge db_j \wedge g_{j,q-r-1}$. This point is solved by the condition 1. which is already proved. The points 3. 4. and 5. are obvious. This complete the proof of our induction. $\blacksquare$

Now remark that the sequence of coherent sub-sheaves $\alpha_X^\bullet[p]$ of the coherent sheaf $L_X^\bullet$ is increasing. So it is locally stationary on X .

Definition 2.1.3 Define the coherent sub-sheaf $\beta_X^\bullet$ as the union of the increasing sequence of coherent sub-sheaves $\alpha_X^\bullet[p], p \geq 0$ of the coherent sheaf $L_X^\bullet$.

Corollary 2.1.4 The graded sub-sheaf $\beta_X^\bullet$ of the graded coherent differential sheaf $L_X^\bullet$ is coherent, stable by exterior product and by the de Rham differential.

PROOF. The assertion is local, so we may assume that $\beta_X^\bullet = \alpha_X^\bullet[p] \quad \forall p \geq p_0$. Then the proposition is consequence of the properties 3. and 4. above. $\blacksquare$

Theorem 2.1.5 For any holomorphic map $f : X \rightarrow Y$ between irreducible complex spaces such that $f(X)$ is not contained in the singular set $S(Y)$ of Y , there exists an unique pull-back

$$\hat{f}^* : f^*(\beta_Y^\bullet) \rightarrow \beta_X^\bullet$$

which is compatible with the pull-back of the $\alpha^\bullet$ -sheaves (see section 1) and which is graded of degree 0 and compatible with the exterior product and the de Rham differential. For any holomorphic maps $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ between irreducible complex spaces, such that $f(X) \not\subset S(Y)$ and $g(f(X)) \not\subset S(Z)$ we have

$$\widehat{f \circ g}^*(\sigma) = \hat{f}^*(\hat{g}^*(\sigma)) \quad \forall \sigma \in \beta_Z^\bullet.$$

Moreover, for each integer $p \geq 0$ the pull-back $\hat{f}^*$ induces a pull-back

$$\hat{f}^*[p] : f^*(\alpha_Y^\bullet[p]) \rightarrow \alpha_X^\bullet[p]$$

and in the previous situation $\widehat{f \circ g}^*[p] = \hat{g}^*[p] \circ \hat{f}^*[p]$ for each $p \geq 0$.

So we shall construct in fact a (graded) naturally filtered sheaf $(\beta_X^\bullet, (\alpha_X^\bullet[p])_{p \in \mathbb{N}})$ such that the pull-back constructed in the previous theorem is compatible with these filtrations and with the composition of suitable holomorphic maps.

In the following we make the convention that $\alpha_X^\bullet[-1] := \alpha_X^\bullet[0] := \alpha_X^\bullet$.

PROOF. Assume that X is an irreducible complex space and that $f(X)$ is not contained in the singular locus $S(Y)$ of Y . Assume also that for some integer $p \geq 0$ we have constructed for any $q \leq p$ a pull-back morphism

$$\hat{f}_q^* : f^*(\alpha_Y^\bullet[q]) \rightarrow \alpha_X^\bullet[q]$$

with the following properties

1_p It induces the usual pull-back of the sheaves of holomorphic forms when it is restricted to the smooth parts² of X and Y . Note that this implies that the restriction of $\hat{f}_p^*$ to $f^*(\alpha_Y^\bullet[q])$ is equal to $\hat{f}_q^*$ because, by definition the sections of the sheaves under consideration are determined by their restrictions to an open dense subset.

2_p For s, t in $\alpha_Y^\bullet[p-1]$ we have $\hat{f}_p^*(s \wedge t) = \hat{f}_{p-1}^*(s) \wedge \hat{f}_{p-1}^*(t)$.

3_p For any u in $\alpha_Y^\bullet[p-1]$ such that du is in $\alpha_Y^{\bullet+1}[p]$ ³ we have $d(\hat{f}_{p-1}^*(u)) = \hat{f}_p^*(du)$.

Then we want to construct $\hat{f}_{p+1}^* : f^*(\alpha_Y^\bullet[p+1]) \rightarrow \alpha_X^\bullet[p+1]$ satisfying again the properties above for $p+1$.

It is clear that that our inductive hypothesis given by the conditions 1_p, 2_p and 3_p is true for $p=0$ (but 2₀ is obtain by looking at points in X'' and using the absence of torsion).

Now we shall show that if it is satisfied for some $p \geq 0$ then it is also satisfied for $p+1$.

²Our hypothesis implies that there exists a dense Zariski open set X'' in $X \setminus S(X)$ such that the restriction of f to X'' takes values in $Y \setminus S(Y)$.

³For $p \geq 1$ $u \in \alpha_X^\bullet[p-1]$ implies $du \in \alpha_X^{\bullet+1}[p]$ is automatic; but not for $p=0$ with our convention.

CONSTRUCTION OF $\hat{f}_{p+1}^*$. Let ξ be a section in $\alpha_Y^\bullet[p+1]$. We may write

$$\xi = \sum_{j=0}^J \beta_j \wedge \gamma_j + \sum_{j=0}^J u_j \wedge dv_j$$

where $\beta_j, \gamma_j, u_j, v_j$ are sections of the sheaf $\alpha_Y^\bullet[p]$. It is clear that our conditions $1_{p+1}, 2_{p+1}, 3_{p+1}$ implies that we must put

$$\hat{f}_{p+1}^*(\xi) = \sum_{j=0}^J \hat{f}_p^*(\beta_j) \wedge \hat{f}_p^*(\gamma_j) + \sum_{j=0}^J \hat{f}_p^*(u_j) \wedge d(\hat{f}_p^*(v_j)).$$

Now the main point is to prove that if we change the choice of writing ξ in such a way, the value of $\hat{f}_{p+1}^*(\xi)$ stays the same. In other words, we have to prove that if $\xi = 0$ is written as above then we find $\hat{f}_{p+1}^*(\xi) = 0$.

To prove this is quite simple because it is enough to look on X'' . On this open dense subset we have simply taken the usual pull-back of the holomorphic form ξ restricted to the smooth part of Y by the holomorphic map $f' : X'' \rightarrow Y \setminus S(Y)$ induced by f . As this pull-back commutes with exterior product and de Rham differential, its result is independent on the way with have written ξ above. This implies our claim because the sheaf $\alpha_X^\bullet[p+1]$ has no torsion.

To verify the properties $1_{p+1}, 2_{p+1}$ and 3_{p+1} is then obvious because it is enough to check them on X'' .

This completes the proof of the existence of pull-back morphisms $\hat{f}^*[p]$ for each $p \geq 0$ and then for the sheaves $\beta^\bullet$. And it also gives the compatibility of these pull-back with the exterior product and the de Rham differential.

The only point which we have to precise to complete the proof of the theorem 2.1.5 is the “functorial” aspect of these pull-back. But this is again an easy consequence of the non existence of torsion for the sheaves we consider. $\blacksquare$

2.2 Filtration

Proposition 2.2.1 *Let X be an irreducible complex space. Then for each $q \geq 0$ we have $\beta_X^q = \alpha_X^q[q]$. If X is normal, for $q \geq 1$ we have $\beta_X^q = \alpha_X^q[q-1]$.*

PROOF. First remark that, by definition α_X^0 is the sheaf of locally bounded meromorphic functions on X (so it is equal to $\mathcal{O}_X$ if and only if X is normal), and that $\beta_M^0 = \alpha_M^0$ by definition.

Fix an integer $q_0 \geq 1$ and assume that for any integer $q < q_0$ we have $\alpha_X^q[p-1] = \beta_X^q$ for some integer $p \geq 1$. This means that $\alpha_X^q[p-1] = \alpha_X^q[p]$ for these q . By definition we have

$$\alpha_X^{q_0}[p+1] = \sum_{h=0}^{q_0} \alpha_X^h[p] \wedge \alpha_X^{q_0-h}[p] + \sum_{h=0}^{q_0-1} d(\alpha_X^h[p]) \wedge \alpha_X^{q_0-h-1}[p].$$

But our assumption allows to replace p by $p-1$ in the right hand-side of the equality above, so we find that $\alpha_X^{q_0}[p+1] = \alpha_X^{q_0}[p]$.

Now remark that, for each $q \geq 0$, α_X^q is stable by multiplication by elements in α_X^0 , so we have

$$\alpha_X^1[1] = \alpha_X^1[0] + \sum_{i,j=1}^I \mathcal{O}_X \cdot g_j \cdot dg_i$$

where $g_1, \dots, g_I$ generate the coherent $\mathcal{O}_X$ -module α_X^0 . This implies that $\alpha_X^1[1]$ is stable by multiplication by α_X^0 and this implies the equality

$$\alpha_X^1[2] = \alpha_X^1[1] + \sum_{i,j=1}^I \mathcal{O}_X \cdot g_j \cdot dg_i = \alpha_X^1[1].$$

So we have $\alpha_X^1[1] = \beta_X^1$. This allows to begin our induction on q_0 for $q_0 = 1$ with $p = 2$.

Then by induction on $q_0 \geq 1$ we conclude that for each $q \geq 1$ we have $\beta_X^q = \alpha_X^q[q]$. In the case where X is normal, we may take $I = \{1\}$ and $g_1 = 1$ and this shows that $\alpha_X^1[0] = \beta_X^1$ and the induction gives now, if we begin with $q_0 = 1$ and $p = 1$, the equality $\alpha_X^q[q-1] = \beta_X^q$ for each $q \geq 1$. ■

REMARK. This shows that for a normal complex space we always have the equality $\beta_X^1 = \alpha_X^1$, so the sheaf $\beta_X^\bullet$ is “new” only in degrees at least equal to 2 when X is normal.

2.3 Integration and Stokes’ formula

We begin by defining the integration on p -cycles.

Lemma 2.3.1 *Let X be an irreducible complex space and u, v be sections of the sheaf β_X^p . Let Y be an irreducible p -cycle in X which is not contained in $S(X)$ and let ρ be a continuous function with compact support on X . Then the improper integral*

$$\int_Y \rho \cdot \hat{j}(u) \wedge \overline{\hat{j}(v)}$$

is absolutely convergent, where $j : Y \rightarrow X$ is the inclusion map.

PROOF. This is an easy consequence of the fact that, using the pull-back theorem, on a suitable modification of Y the p -forms u and v becomes holomorphic. ■

Remark that for any holomorphic function f on Y which does not vanish on a non empty open set in Y this integral is the limit as $\varepsilon > 0$ does to zero of the integral

$$\int_{Y \cap \{|f| > \varepsilon\}} \rho \cdot \hat{j}(u) \wedge \overline{\hat{j}(v)}.$$

Definition 2.3.2 *In the situation of the previous lemma, the number*

$$\int_X \rho \cdot \hat{j}(u) \wedge \overline{\hat{j}(v)}$$

will be called the integral of $\rho \cdot \hat{j}(u) \wedge \overline{\hat{j}(v)}$ on Y . This definition extends by linearity to general p -cycles in X which have no irreducible component in $S(X)$.

In the sequel we shall write simply $\int_Y \rho \cdot u \wedge \bar{v}$ this integral, omitting the pull-back by j . But it is necessary to keep in mind that this abuse of notation is acceptable because of the compatibility of the pull-back maps for the sheaf β with the composition of suitable maps.

Lemma 2.3.3 *Let X be an irreducible complex space and $f : X \rightarrow Y$ an holomorphic map. Let Z be an irreducible p -cycle in X such that its direct image $f_*(Z)$ is defined (as a p -cycle⁴) in Z . Assume that Z is not contained in $S(X)$ and that $f(Z)$ is not contained in $S(Y)$. Let u, v be sections of the sheaf β_Y^p and ρ a continuous function on Y with compact support. Then we have*

$$\int_{f_*(Z)} \rho \cdot u \wedge \bar{v} = \int_Z f^*(\rho) \cdot \hat{f}(u) \wedge \overline{\hat{f}(v)}.$$

PROOF. Thanks to the lemma 2.3.1 and the remark following it, this reduces to the usual change of variable in the case where $f(Z)$ is not contained in $S(Y)$. ■

Again the functoriality of the pull-back implies here the fact that we may either take the pull-back by f and restrict to Z or directly take the pull-back by the restriction of f to Z .

Proposition 2.3.4 *Let $(Y_t)_{t \in T}$ be an analytic family of p -cycles in X parametrized by a reduced complex space T and let $\rho : X \rightarrow \mathbb{C}$ be a continuous function with compact support in X . Assume that for each $t \in T$ the cycle Y_t has no irreducible component contained in $S(X)$. Let u, v be sections of the sheaf β_X^p . Then the function $\varphi : T \rightarrow \mathbb{C}$ defined by*

$$\varphi(t) = \int_{Y_t} \rho \cdot u \wedge \bar{v} \quad \forall t \in T$$

is continuous on a dense open set T' of T and is locally bounded near each point in T .

PROOF. Consider a modification $\tau : \tilde{X} \rightarrow X$ of X with center in $S(X)$ such that u and v becomes holomorphic on it. Let $\nu : \tilde{T} \rightarrow T$ be the normalization of T .

⁴Remember that this means that the restriction of f to Z is proper.

CLAIM. There exists a modification $\sigma : \Theta \rightarrow \tilde{T}$ with Θ a normal complex space and an analytic family $(\tilde{Y}_\theta)_{\theta \in \Theta}$ of p -cycles in $\tilde{X}$ such that for θ generic, $\tilde{Y}_\theta$ is the strict transform of $Y_{\nu(\sigma(\theta))}$ by τ . Moreover, for each $\theta \in \Theta$ we will have $\tau_*(\tilde{Y}_\theta) = Y_{\nu(\sigma(\theta))}$.

PROOF OF THE CLAIM. As $\tilde{T}$ is normal, we may decompose in an open neighborhood of the support of ρ the family $(Y_i)_{i \in \tilde{T}}$ as a finite sum of analytic families of p -cycles parametrized by $\tilde{T}$ such that their generic cycles are irreducible. Then, by additivity of the integral, it is enough to treat the case of such a family. This means that, without loss of generality, we may assume that the family $(Y_i)_{i \in \tilde{T}}$ has an irreducible generic cycle.

So let $G \subset \tilde{T} \times X$ the graph of the family; it is irreducible. Let Γ the strict transform of G by the modification $\text{id}_{\tilde{T}} \times \tau : \tilde{T} \times \tilde{X} \rightarrow \tilde{T} \times X$.

Now we may find a modification $\sigma : \Theta \rightarrow \tilde{T}$ such that the strict transform $\tilde{\pi} : \tilde{\Gamma} \rightarrow \Theta$ by the modification σ of the projection $\pi : \Gamma \rightarrow \tilde{T}$ becomes equidimensionnal. So, as Θ is normal, the fibres of $\tilde{\pi}$ give an analytic family of p -cycles in $\tilde{X}$ parametrized by Θ , such that for θ generic, $\tilde{Y}_\theta$ is the strict transform of $Y_{\nu(\sigma(\theta))}$. Moreover, for each $\theta \in \Theta$ we will have $\tau_*(\tilde{Y}_\theta) = Y_{\nu(\sigma(\theta))}$ because this is true for θ generic and both are analytic families of cycles in X , thanks to the direct image theorem (see [2] theorem IV 3.5.1). This proves the claim.

Then the result follows using the previous lemma and the continuity of the integration of a continuous form with compact support on a continuous family of cycles (see [2] proposition IV 2.3.1). $\blacksquare$

REMARK. The only point where we use the fact that u, v are sections of the sheaf β_X^p is when we define the integral using the definition 2.3.1 and the second part of the theorem 2.1.5 which gives that the pull-back is compatible with composition of suitable holomorphic maps. In the rest of the proof, we only use the fact that u, v are sections of L_X^p to know that on a suitable modification of X they become holomorphic forms.

Lemma 2.3.5 *Let X be a reduced complex space and let u and v be respectively sections of the sheaves β_X^{p-1} and β_X^p . Let ρ be a $\mathcal{C}^1$ -function on X with compact support. Then for any p -cycle Z in X which has no irreducible component in $S(X)$ we have*

$$\int_X d(\rho \cdot u \wedge \bar{v}) = 0.$$

PROOF. It is enough to prove this formula when Z is irreducible. As Z is not contained in the singular locus of X the result follows from the fact that on a suitable modification of Z the forms u and v becomes holomorphic so we may apply the

standard Stokes's theorem. ■

Note that the commutation of the pull-back maps with the de Rham differential is crucial here.

3 Examples

3.1 The sheaf α for a product

Proposition 3.1.1 *Let X and Y be irreducible complex spaces. Then if $p_1 : X \times Y \rightarrow X$ and $p_2 : X \times Y \rightarrow Y$ are the projections, we have for each $p \geq 0$ a natural isomorphism*

$$\theta : \bigoplus_{q=0}^p p_1^*(\alpha_X^q) \otimes_{\mathcal{O}_{X \times Y}} p_2^*(\alpha_Y^{p-q}) \rightarrow \alpha_{X \times Y}^p$$

given by exterior product.

PROOF. This is an easy exercise using two desingularizations $\sigma : \tilde{X} \rightarrow X$ and $\tau : \tilde{Y} \rightarrow Y$ which are normalizing respectively for the sheaves $\Omega_X^\bullet / \text{torsion}$ and $\Omega_Y^\bullet / \text{torsion}$, as the product map $\sigma \times \tau : \tilde{X} \times \tilde{Y} \rightarrow X \times Y$ is a desingularization of $X \times Y$ which normalizes the sheaf $\Omega_{X \times Y}^\bullet$. ■

REMARK. It is easy to extend this proposition to the sheaves $\alpha_{X \times Y}^\bullet[p]$ for each $p \geq 0$ and so to the sheaf $\beta_{X \times Y}^\bullet$.

The following trivial corollary will be used in an example below.

Corollary 3.1.2 *Let X be an irreducible complex spaces. Consider on $X \times D$ a L^{p+1} form $\omega \wedge f(z).dz$ where D is a disc in $\mathbb{C}$ with coordinate z , $f : D \rightarrow \mathbb{C}$ a holomorphic function on D which is not identically zero, and where ω is a L^p -form on X . Then ω is a section of α_X^p if and only if $\omega \wedge f(z).dz$ is a section of $\alpha_{X \times D}^{p+1}$.*

3.2 The curve $X := \{x^3 = y^5\} \subset \mathbb{C}^2$

Lemma 3.2.1 *On the curve $X := \{x^3 = y^5\} \subset \mathbb{C}^2$ we have*

$$\begin{aligned} \alpha_X^0 &= L_X^0 = \mathcal{O}_X \oplus \mathbb{C}.y^2/x \oplus \mathbb{C}.y^4/x^2 \oplus \mathbb{C}.y^3/x \oplus \mathbb{C}.y^4/x \\ \omega_X^0 &= L_X^0 + \mathcal{O}_X.y/x^2 \\ \alpha_X^1 &= \Omega_X^1 / \text{torsion} + \mathcal{O}_X.y^2.dy/x \\ \beta_X^1 &= L_X^1 = \alpha_X^1[1] = \mathcal{O}_X.y^2.dx/x^2 \\ \omega_X^1 &= \Omega_X^1 / \text{torsion} + \mathcal{O}_X.dy/x^2 \end{aligned}$$

The proof is left as an exercise to the reader. For the computation of the sheaf ω_X^0 the reader may use the fact that

$$(y/x^2).(3x^2 - 5y^4.dy)/(x^3 - y^5) = (3y.dx - 5x.dy)/(x^3 - y^5)$$

in $H_X^1(\Omega_{\mathbb{C}^2}^1)$ and the characterization given in [1] of the sheaf $\omega_X^\bullet$ in terms of the fundamental class of X .

3.3 The surfaces S_k

Consider the surfaces $S_k := \{(x, y, z) \in \mathbb{C}^3 \mid x.y = z^k\}$ for k an integer at least equal to 2.

In the following lemma, we determine the sheaves $\alpha_{S_k}^\bullet$ and $\beta_{S_k}^\bullet$. We also correct the lemma 6.2.2 of [B.18] which is wrong for $k \geq 4$.

Lemma 3.3.1 *Let $m := [k/2]$ be the integral part of $k/2$. Then we have*

$$\begin{aligned}\beta_{S_k}^1 &= \alpha_{S_k}^1 = \Omega_{S_k}^1 / \text{torsion} + \mathcal{O}_{S_k}.x.dy/z^m \\ \alpha_{S_k}^2 &= \Omega_{S_k}^2 / \text{torsion} + \mathcal{O}_{S_k}.\frac{dx \wedge dy}{z^{m-1}} \\ \beta_{S_k}^2 &= \alpha_{S_k}^2[1] = \Omega_{S_k}^2 / \text{torsion} + \mathcal{O}_{S_k}.\frac{dx \wedge dy}{z^m}.\end{aligned}$$

PROOF. The first assertion is consequence of the equality $\alpha_M^1 = \beta_M^1$ for any normal complex space which is proved in proposition 2.2.1. The computation of $\alpha_{S_k}^1$ is an obvious consequence of lemma 6.2.3 in [3]. Note that the equalities

$$x.dy/z^m + y.dx/z^m = k.z^{k-m}.dz \quad \text{and} \quad (x.dy/z^m).(y.dx/z^m) = z^{k-2m}.(dx).(dy)$$

gives the integral dependance relation of $x.dy/z^m$ on $S^\bullet(\Omega_{S_k}^1 / \text{torsion})$.

Let now prove the second assertion.

Remark first that we have on S_k the relations

$$x.dx \wedge dy = k.z^{k-1}.dx \wedge dz \quad y.dx \wedge dy = k.z^{k-1}.dz \wedge dy$$

and using the equality $x.y = z^k$ this implies

$$dx \wedge dy = k.y.dx \wedge dz/z = -k.x.dy \wedge dz/z.$$

Dividing by z^{m-1} this gives

$$\left(\frac{dx \wedge dy}{z^{m-1}}\right)^2 = -k^2.z^{k-2m}.(dx \wedge dz).(dy \wedge dz) \quad \text{in } S^2(\Omega_{S_k}^2 / \text{torsion}).$$

This prove that $dx \wedge dy/z^{m-1}$ is a section of the sheaf $\alpha_{S_k}^2$.

We want to prove now that the meromorphic form

$$\frac{dx \wedge dy}{z^m} = k.y.\frac{dx \wedge dz}{z^{m+1}} = -k.x.\frac{dy \wedge dz}{z^{m+1}}$$

which corresponds to $k^2.(a.b)^{k-m}.da \wedge db$ via the quotient map

$$q_k : \mathbb{C}^2 \rightarrow S_k \quad (a, b) \mapsto (x = a^k, y = b^k, z = a.b)$$

is not in $\alpha_{S_k}^2 = \Omega_{S_k}^2 / \text{torsion}$.

As the fiber F_0 of the sheaf $F := q_k^{-1}(\Omega_{S_k}^2 / \text{torsion})$ at 0 is the $A := \mathbb{C}\{a^k, b^k, a.b\}$ -submodule of $A.da \wedge db$ generated by $a^k.da \wedge db, b^k.da \wedge db, (a.b)^{k-1}.da \wedge db$, we have to show that $(a.b)^{k-m-1}.da \wedge db$ is not integral on F_0 . This is an easy consequence of the fact that for $q < k/2$ there is no positive constant C such that for $a > 0$ and $b > 0$ small enough we have the inequality $(a.b)^q \leq C.(a^k + b^k)$.

To prove the last assertion remark first that the form $d(x.dy/z^m)$ is in $\alpha_{S_k}^2[1] = \beta_{S_k}^2$ (this last equality is also proved in proposition 2.2.1). But we have on S_k , using the equality $[(k-1)/2] + [k/2] = k-1$

$$\begin{aligned} y.dx + x.dy &= k.z^{k-1}.dz & \text{so} \\ x.dy \wedge dx &= k.z^{k-1}.dz \wedge dx & \text{and then} \\ \frac{dy \wedge dx}{z^m} &= k.y.\frac{dz \wedge dx}{z^{m+1}} \end{aligned}$$

This gives $d(x.dy/z^m) = (1 - m/k).dx \wedge dy/z^m$.

So the inclusion of $\Omega_{S_k}^2 / \text{torsion} + \mathcal{O}_{S_k}.dx \wedge dy/z^m$ in $\beta_{S_k}^2$ is proved. The equality $\alpha_{S_k}^2[1] = \beta_{S_k}^2$ easily implies the equality in the previous inclusion, as we have the inclusion $\alpha_{S_k}^1 \wedge \alpha_{S_k}^1 \subset \Omega_{S_k}^2 / \text{torsion}$. ■

Note that the sheaf $L_{S_k}^2$ is equal to $\Omega_{S_k}^2 / \text{torsion} + \mathcal{O}_{S_k}.dx \wedge dy/z^{k-1}$. So for $k \geq 4$ we have strict inclusions between $\Omega_{S_k}^2 / \text{torsion}, \alpha_{S_k}^2, \beta_{S_k}^2$ and $L_{S_k}^2 = \omega_{S_k}^2$.

3.4 $M_k := \{x.y = u^k.v\}$

Let $m := [k/2]$ be the integral part of the integer $k \geq 1$.

Lemma 3.4.1 *The meromorphic 1-form $\omega_m := x.dy/u^m$ belongs to $\alpha_{M_k}^1$ but for $k \geq 2$ the differential $d\omega_m$ is not in $\alpha_{M_k}^2$.*

PROOF. We have

$$\begin{aligned} x.dy/u^m + y.dx/u^m &= d(xy)/u^m = d(u^k v)/u^m = k.u^{k-1-m}v.du + u^{k-m}.dv & \text{and} \\ (x.dy/u^m).(y.dx/u^m) &= x.y.(dx).(dy)/u^{2m} = u^{k-2m}.v.(dx).(dy) \end{aligned}$$

so ω_m satisfies the following integral dependance relation on $\Omega_M^1/torsion$

$$\omega_m^2 - (k.u^{k-m-1}.v.du + u^{k-m}.dv).\omega_m + u^{k-2m}.v.(dx).(dy) = 0 \quad (2)$$

Now we have

$$d\omega_m = \frac{dx \wedge dy}{u^m} - m. \frac{x.du \wedge dy}{u^{m+1}}.$$

But now we restrict this 2-form to the surface $S_k := \{v = 1\}$ which cuts the singular set of M_k only at the point $x = y = u = 0, v = 1$, and we find, as we have on this surface $x.dy + y.dx = k.u^{k-1}.du$ which implies $y.dx \wedge dy = k.u^{k-1}.du \wedge dy$ and then $u.dx \wedge dy = k.x.du \wedge dy$. So

$$(d\omega_m)|_{\{v=1\}} = (1 - m/k).dx \wedge dy/u^m$$

which is not in $\alpha_{S_k}^2$ for $k \geq 2$ (see lemma 3.3.1). ■

Lemma 3.4.2 *The 2-form $w := \omega_m \wedge dv$ belongs to $\alpha_{M_k}^2$ but dw is not in $\alpha_{M_k}^3$ for $k \geq 2$.*

PROOF. The first assertion is obvious as $\alpha_{M_k}^\bullet$ is stable by wedge products and contains $\Omega_{M_k}^\bullet/torsion$.

To prove the second assertion consider the following holomorphic map

$$\pi : S_k \times \mathbb{C} \rightarrow M_k, \quad ((x, y, u), v) \mapsto (x.v, y, u, v).$$

Then $\pi^*(dw) = dx \wedge dy \wedge dv/u^m - m.x.du \wedge dy \wedge dv/u^{m+1}$. Using the corollary 3.1.2 of the proposition 3.1.1 and the fact that we have on $S_k \times \mathbb{C}$

$$\pi^*(dw) = v.dv \wedge ((k - m).x.du \wedge dy/u^{m+1})$$

we conclude that $\pi^*(dw)$ is not a section of $\alpha_{S_k \times \mathbb{C}}^3$, concluding the proof. ■

Corollary 3.4.3 *For $k \geq 4$ we have on M_k*

$$\begin{aligned} \Omega_{M_k}^1 &\subset \alpha_{M_k}^1 = \beta_{M_k}^1 \subset L_{M_k}^1 \\ \Omega_{M_k}^2 &\subset \alpha_{M_k}^2 \subset \beta_{M_k}^2 \subset L_{M_k}^2 \\ \Omega_{M_k}^3 &\subset \alpha_{M_k}^3 \subset \beta_{M_k}^3 \subset L_{M_k}^3 \end{aligned}$$

where all inclusions are strict.

We leave to the reader the easy proof using the previous computations. ■

3.5 Fermat surfaces

We shall look now to the surfaces $F_n := \{(a, b, z) \in \mathbb{C}^3 \mid a^n - b^n = z^n\}$ for $n \geq 3$. As these surfaces are normal the only interesting sheaf is $\alpha_{F_n}^2$ because $\beta^1 = \alpha^1 = \Omega^1$ and $\alpha^2 = \beta^2$.

Lemma 3.5.1 *For $n = 2p \geq 4$ the form $(a.b)^p.da \wedge db/z^{2p-1}$ is a section of $\alpha_{F_{2p}}^2$ and the form $(a.b)^{p-1}.da \wedge db/z^{p-1}$ is also a section of $\alpha_{F_{2p}}^2$. For $n = 2p + 1 \geq 3$ the form $(a.b)^p.da \wedge db/z^{2p}$ is a section of $\alpha_{F_{2p+1}}^2$. Moreover, all these forms are not holomorphic forms.*

PROOF. On F_n we have the equalities

$$\begin{aligned} a^{n-1}.da \wedge db &= z^{n-1}.dz \wedge db \quad \text{and} \\ b^{n-1}.da \wedge db &= z^{n-1}.dz \wedge da \end{aligned}$$

so we have

$$(dz \wedge da).(dz \wedge db) = \frac{(a.b)^{n-1}.(da \wedge db)^2}{z^{2n-2}}$$

which implies

$$\left(\frac{(a.b)^p.da \wedge db}{z^{2p-1}} \right)^2 = a.b.(dz \wedge da).(dz \wedge db) \quad \text{for } n = 2p$$

and

$$\left(\frac{(a.b)^p.da \wedge db}{z^{2p}} \right)^2 = a.b.(dz \wedge da).(dz \wedge db) \quad \text{for } n = 2p + 1.$$

To prove that $(a.b)^{p-1}.da \wedge db/z^{p-1}$ is a section of $\alpha_{F_{2p}}^2$ consider the map $f : F_{2p} \rightarrow S_{2p}$ given by $f(a, b, z) = (a^p - b^p, a^p + b^p, z)$ and compute the pull back of the form $dx \wedge dy/z^{p-1}$ which is a section of $\alpha_{S_{2p}}^2$ (see above). The result follows.

To see that these forms are not holomorphic is a simple exercise using the homogeneity on F_n ; we live it to the reader. ■

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