Fourier transform of a class of radial and semi-radial functions

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Abstract

We give here the key to reduce the problem of computing a three dimensional Fourier transform - for a certain class of functions - to a one dimensional Fourier transform computation, allowing in some way to "reverse" the main theorem of [1], in this particular case. This class of functions being of the form

$$t \mapsto \frac{R(\cos(\|t\|, \sin(\|t\|)}{\|t\|^m} P(t) \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^3),$$

where $R \in \mathbb{C}[X, Y]$ and $P \in \mathbb{C}[X, Y, Z]$

Key words : Fourier Transform, radial and semi-radial functions

1 Introduction

1.1 "Universal" notation for the Fourier Transform

There is a lot of conventions for the Fourier transform, so we give here a convenient notation to take care of all the cases once for all (with $f, g : \mathbb{R}^n \to \mathbb{R}$) :

$$\mathcal{F}_n^{a,b}(f)(k) := \left(\frac{|b|}{(2\pi)^{1-a}}\right)^{n/2} \int_{\mathbb{R}^n} f(t) e^{ib\langle k,t\rangle} \, \mathrm{d}t \tag{1.1}$$

$$\left(\mathcal{F}_n^{a,b}\right)^{-1}(g)(t) := \left(\frac{|b|}{(2\pi)^{1+a}}\right)^{n/2} \int_{\mathbb{R}^n} g(k) e^{-ib\langle t,k\rangle} \, \mathrm{d}k \tag{1.2}$$

To jump from a convention to another, we have the simple following corresponding formula :

$$\mathcal{F}^{a,b}(f)(k) = \left(\frac{|b/b'|}{(2\pi)^{a'-a}}\right)^{n/2} \mathcal{F}^{a',b'}(f)\left(\tfrac{b}{b'}k\right). \tag{1.3}$$

Fourier Transform of a tempered distribution

Let us recall that if $T \in \mathcal{S}'(\mathbb{R}^n)$ is a tempered distribution, we define de Fourier transform of $\mathcal{F}_n^{a,b} T$ by

$$\langle \mathcal{F}_n^{a,b} T, \varphi \rangle := \langle T, \mathcal{F}_n^{a,b} \varphi \rangle \tag{1.4}$$

In order to give the formula to jump from a convention to another, we have to define the following operator : for $a \in \mathbb{R}$, we set $m_a : f \mapsto (x \mapsto f(ax), x \in \mathbb{R}^n)$.

This operator can be extended to the distributions : for all $a \neq 0$,

$$\langle m_a T, \varphi \rangle := \frac{1}{|a|^n} \langle T, m_{\frac{1}{a}} \varphi \rangle \tag{1.5}$$

From (1.3) and (1.5) we have

$$\mathcal{F}_n^{a,b} T = \left(\frac{|b'/b|}{(2\pi)^{a'-a}} \right)^{n/2} \mathcal{F}_n^{a',b'} \left(m_{\frac{b'}{b}} T \right). \tag{1.6}$$

(please notice that $|b/b'|$ in (1.3) "becoming" $|b'/b|$ in (1.6) is not a typo!)

1.2 Radial distributions

We recall here the definitions and de properties given and demonstrated in [1].

We set (where $\mathcal{S}(\mathbb{R}^n)$ stands for the space of Schwartz functions on $\mathbb{R}^n$)

$$\mathcal{S}_{\text{rad}}(\mathbb{R}^n) = \{\varphi \in \mathcal{S}(\mathbb{R}^n) \ : \ \varphi = \varphi \circ A, \ \forall A \in \mathcal{O}(n)\} \tag{1.7}$$

$$\mathcal{S}_{\text{rad}}(\mathbb{R}) = \mathcal{S}_{\text{even}}(\mathbb{R}) = \{\varphi \in \mathcal{S}(\mathbb{R}) \ : \ \varphi(x) = \varphi(-x)\} \tag{1.8}$$

where $\mathcal{O}(n)$ is the set of the orthogonal transformations of $\mathbb{R}^n$.

We define then the following functions :

$$\left\{ \begin{array}{rcl} \mathcal{S}(\mathbb{R}^n) & \to & \mathcal{S}_{\text{rad}}(\mathbb{R}) \\ \varphi & \mapsto & \left(r \mapsto \varphi^o(r) := \frac{1}{\omega_{n-1}} \int_{\mathbb{S}^{n-1}} \varphi(r\theta) \, \mathrm{d}\theta \right) \end{array} \right. \tag{1.9}$$

$$\left\{ \begin{array}{rcl} \mathcal{S}_{\text{rad}}(\mathbb{R}) & \to & \mathcal{S}_{\text{rad}}(\mathbb{R}^n) \\ \varphi & \mapsto & \left(x \mapsto \varphi^O(x) := \varphi(|x|) \right) \end{array} \right. \tag{1.10}$$

(where $\mathbb{S}^{n-1}$ is the unit sphere on$\mathbb{R}^n$ and ω_{n-1} its surface area; with the convention $\omega_0 = 2$ and $\varphi^o(x) = \frac{1}{2}(\varphi(x) + \varphi(-x))$, for $\varphi \in \mathcal{S}(\mathbb{R})$).

Definition 1.1 *A distribution $u \in \mathcal{S}'(\mathbb{R}^n)$ (with $\mathcal{S}'(\mathbb{R}^n)$ the space of tempered distributions on $\mathbb{R}^n$) is called radial if for all $A \in \mathcal{O}(n)$,*

$$u = u \circ A,$$

that is,

$$\langle u, \varphi \rangle = \langle u, \varphi \circ A \rangle$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$. The set of all radial tempered distributions is denoted by $\mathcal{S}'_{\text{rad}}(\mathbb{R}^n)$.

Proposition 1.2 *For $u \in \mathcal{S}'_{\text{rad}}(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$,*

$$\langle u, \varphi \rangle = \langle u, \varphi^{\text{rad}} \rangle \tag{1.11}$$

where $\varphi^{\text{rad}} := (\varphi^o)^O$ (i.e. $\varphi^{\text{rad}}(x) = \varphi^o(|x|)$).

Let us define the space

$$\mathcal{R}_n := r^{n-1}\mathcal{S}_{\mathrm{rad}}(\mathbb{R}) = \left\{\left(r \mapsto \psi(r)r^{n-1}\right),\ \psi \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R})\right\}. \tag{1.12}$$

Remark 1.3 *$\mathcal{R}_n$ is a subspace $\mathcal{S}(\mathbb{R})$ on which we can use the same topology ; we denote its dual (set of the linear continuous functions defined over $\mathcal{R}_n$) by $\mathcal{R}'_n$.*

We switch from $\mathcal{R}'_n$ to radials distributions of $\mathcal{S}'(\mathbb{R}^n)$ as follows :

- if u is radial distribution, we define $u_\diamond \in \mathcal{R}'_n$ by

$$\langle u_\diamond, \psi(r)r^{n-1}\rangle := \frac{2}{\omega_{n-1}}\langle u, \psi^O\rangle, \qquad \psi \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R}) \tag{1.13}$$

- if $u_\diamond \in \mathcal{R}'_n$, we define a radial distribution u by

$$\langle u, \varphi\rangle := \frac{\omega_{n-1}}{2}\langle u_\diamond, \varphi^o(r)r^{n-1}\rangle, \qquad \varphi \in \mathcal{S}(\mathbb{R}^n) \tag{1.14}$$

1.3 Grafakos-Teschl's Theorem

Theorem 1.4 (Grafakos-Teschl (2013)) *Given v_1 in $\mathcal{S}'(\mathbb{R})$, we define a radial distribution v_k on $\mathbb{R}^k$ ($k \in \mathbb{N}^*$) by*

$$\langle v_k, \varphi\rangle := \frac{\omega_{k-1}}{2}\langle v_1, \varphi^o(r)r^{k-1}\rangle, \qquad \varphi \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^k) \tag{1.15}$$

(if $\varphi \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^n)$, then $\varphi(x) = \varphi^o(|x|)$).
Let $u^k = \mathcal{F}_k^{a,b}(v_k)$. We have then

$$-\frac{(2\pi)^a}{|b|r}\frac{\mathrm{d}}{\mathrm{d}r}u^n_\diamond = u^{n+2}_\diamond \tag{1.16}$$

Remark 1.5
- *Dans l'article de base, on partait d'une fonction $v_0 \in \mathcal{S}'(\mathbb{R})$, et on définissait $\langle v_k, \varphi\rangle := \frac{\omega_{k-1}}{2}\langle v_0, \varphi^o(r)r^{k-1}\rangle$, mais alors, on vérifie très rapidement que $v_0 = v_1$; cela n'a pas d'intérêt de commencer par v_0.*
- *On a $u^1_\diamond = u^1$: en effet*

$$\langle u^1_\diamond, \psi(r)r^{1-1}\rangle := \frac{2}{\omega_0}\langle u, \psi^O\rangle = \langle u, \psi\rangle$$

  soit $\langle u^1_\diamond, \psi\rangle = \langle u, \psi\rangle$ pour tout $\psi \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R})$.

Corollary 1.6 *Given v_1 in $\mathcal{S}'(\mathbb{R})$, we define the radial distribution v_3 on $\mathbb{R}^3$ by*

$$\langle v_3, \varphi\rangle := \frac{\omega_2}{2}\langle v_1, \varphi^o(r)r^2\rangle, \qquad \varphi \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^3) \tag{1.17}$$

(if $\varphi \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^n)$, then $\varphi(x) = \varphi^o(|x|)$). We have then, for all $\varphi \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^3)$

$$\langle \mathcal{F}_3^{a,b}(v_3), \varphi\rangle = -\frac{(2\pi)^{a+1}}{|b|}\left\langle r\frac{\mathrm{d}}{\mathrm{d}r}(\mathcal{F}_1^{a,b}(v_1)), \varphi^o(r)\right\rangle \tag{1.18}$$

Proof: With $u^3 := \mathcal{F}_3^{a,b}(v_3)$ and $u^1 := \mathcal{F}_1^{a,b}(v_3)$ we have

$$\begin{aligned}\langle u^3, \varphi\rangle &= \frac{\omega_2}{2}\langle u^3_\diamond, \varphi^o(r) r^2\rangle && (cf.\ (1.14))\\ &= \frac{\omega_2}{2}\Big\langle -\frac{(2\pi)^a}{|b| r}\frac{\mathrm{d}}{\mathrm{d}r}u^1, \varphi^o(r) r^2\Big\rangle && (cf.\ (1.16) \text{ and Remark 1.5})\\ &= -\frac{(2\pi)^{a+1}}{|b|}\Big\langle r\frac{\mathrm{d}}{\mathrm{d}r}(u^1), \varphi^o(r)\Big\rangle\end{aligned}$$

since $\omega_2 = 4\pi$. •

2 Fourier transform of $t \mapsto \frac{R(\cos(\|t\|,\sin(\|t\|)}{\|t\|^m} \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^3)$

Lemma 2.1 *Let f be a smooth function such that*

$$t \mapsto \frac{f(\|t\|)}{\|t\|^m} \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^3). \tag{2.19}$$

If we set $T_1 = f_e T_{\frac{1}{r^m}}$ and $T_2 = f_o T_{\frac{\mathrm{sgn}(r)}{r^m}}$, (with $f_e = \frac{f(x)+f(-x)}{2}$ and $f_o = \frac{f(x)-f(-x)}{2}$) if m is even, and $T_1 = f_o T_{\frac{1}{r^m}}$ and $T_2 = f_e T_{\frac{\mathrm{sgn}(r)}{r^m}}$ otherwise, the radial distribution T defined by

$$\langle T, \varphi\rangle := \frac{\omega_2}{2}\langle T_1 + T_2, r^2\varphi^o\rangle, \qquad \varphi \in \mathcal{S}(\mathbb{R}^3)$$

verifies

$$\frac{f(\|t\|)}{\|t\|^m} = T.$$

Proof:

We set

$$j := \inf\Big((\inf_{u\in\mathbb{N}}\{u \,|\, f^{(u)}(0) \neq 0\}, m-2\Big)$$

and $n := m - j - 2$. Let us notice that, from condition (2.19) and since either $f(r) \sim C r^j$ (Taylor's Theorem) or $j = m-2$, it becomes that

$$0 \geq n. \tag{2.20}$$

1) First, we will show the result for $T := T_2 = f(r) T_{\frac{\mathrm{sgn}(r)}{r^m}}$ (f even or odd, according to m).

From (5.33), we have (with $g(r) := \frac{f(r)}{r^j}$) and $s \in \mathbb{N}$ such that $m + s = j + 2$),

$$\begin{aligned}\frac{\omega_2}{2}\langle f(r) T_{\frac{\mathrm{sgn}(r)}{r^m}}, r^2\varphi^o\rangle &= \frac{\omega_2}{2}\langle g(r) r^s \mathrm{sgn}(r), \varphi^o\rangle\\ &= \frac{\omega_2}{2}\Big(-\int_{-\infty}^0 g(u)u^s\varphi^o(u)\,\mathrm{d}u + \int_0^{+\infty} g(u)u^s\varphi^o(u)\,\mathrm{d}u\Big)\\ &= \frac{\omega_2}{2}\Big(-\int_{-\infty}^0 \frac{f(u)}{u^m}\varphi^o(u)u^2\,\mathrm{d}u + \int_0^{+\infty}\frac{f(u)}{u^m}\varphi^o(u)u^2\,\mathrm{d}u\Big)\\ &= \omega_2\int_0^{+\infty}\frac{f(u)}{u^m}\varphi^o(u)u^2\,\mathrm{d}u\\ &= \int_{\mathbb{R}^3}\frac{f(\|t\|)}{\|t\|^m}\varphi^o(\|t\|)\,\mathrm{d}t = \int_{\mathbb{R}^3}\frac{f(\|t\|)}{\|t\|^m}\varphi(t)\,\mathrm{d}t.\end{aligned}$$

2) Similarly, for $T := T_1 = f(r)T_{\frac{1}{r^m}}$ (f even or odd, according to m), From (5.32) , we have (with $g(r) := \frac{f(r)}{r^j}$) and $s \in \mathbb{N}$ such that $m + s = j + 2$),

$$\begin{aligned}
\frac{\omega_2}{2}\langle f(r)T_{\frac{1}{r^m}}, r^2\varphi^o\rangle &= \frac{\omega_2}{2}\langle g(r)r^s, \varphi^o\rangle \\
&= \frac{\omega_2}{2}\Big(\int_{-\infty}^{0} g(u)u^s\varphi^o(u)\,\mathrm{d}u + \int_{0}^{+\infty} g(u)u^s\varphi^o(u)\,\mathrm{d}u\Big) \\
&= \frac{\omega_2}{2}\Big(\int_{-\infty}^{0} \frac{f(u)}{u^m}\varphi^o(u)u^2\,\mathrm{d}u + \int_{0}^{+\infty} \frac{f(u)}{u^m}\varphi^o(u)u^2\,\mathrm{d}u\Big) \\
&= \omega_2\int_{0}^{+\infty} \frac{f(u)}{u^m}\varphi^o(u)u^2\,\mathrm{d}u \\
&= \int_{\mathbb{R}^3} \frac{f(\|t\|)}{\|t\|^m}\varphi^o(\|t\|)\,\mathrm{d}t = \int_{\mathbb{R}^3} \frac{f(\|t\|)}{\|t\|^m}\varphi(t)\,\mathrm{d}t.
\end{aligned}$$

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3 Fourier transform of $t \mapsto P(t)\frac{R(\cos(\|t\|,\sin(\|t\|)}{\|t\|^m} \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^3)$

Lemma 3.1 *Let f be a smooth function and $P \in \mathbb{C}[X, Y, Z]$, such that*

$$t \mapsto P(t)\frac{f(\|t\|)}{\|t\|^m} \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^3). \tag{3.21}$$

If we set $T_1 = f_eT_{\frac{1}{r^m}}$ and $T_2 = f_oT_{\frac{\mathrm{sgn}(r)}{r^m}}$, (with $f_e = \frac{f(x)+f(-x)}{2}$ and $f_o = \frac{f(x)-f(-x)}{2}$) if m is even, and $T_1 = f_oT_{\frac{1}{r^m}}$ and $T_2 = f_eT_{\frac{\mathrm{sgn}(r)}{r^m}}$ otherwise, the radial distribution T defined by

$$\langle T, \varphi\rangle := \frac{\omega_2}{2}\langle T_1 + T_2, r^2\varphi^o\rangle, \qquad \varphi \in \mathcal{S}(\mathbb{R}^3)$$

verifies

$$P(t)\frac{f(\|t\|)}{\|t\|^m} = P(t)T.$$

Proof: If $P(0) \neq 0$, it is a straightforward consequence of Lemma 2.1 ; hence, we will consider that $P(0) = 0$ in the sequel of this proof and, using the spherical coordinates, we then define the quantity $d \in \mathbb{N}^*$ by

$$P(rX(\varphi, \theta)) = r^d h(X(\varphi, \theta)). \tag{3.22}$$

We set

$$j = \inf\Big((\inf_{u\in\mathbb{N}}\{u \,|\, f^{(u)}(0) \neq 0\}, m - 2\Big)$$

and $n := m - j - 2$. Let us notice that, from condition (3.21) and since either $f(r) \sim Cr^j$ (Taylor's Theorem) or $j = m - 2$, it becomes that

$$d \geq n. \tag{3.23}$$

If $n \leq 0$, we will have $t \mapsto \frac{f(\|t\|)}{\|t\|^m} \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^3)$, so that Lemma 2.1 can be, again, directly applied.

Hence, in the sequel, we will assume always that $n \geq 1$.

1) First, we will show the result for $T := T_2 = f(r)T_{\frac{\mathrm{sgn}(r)}{r^m}}$ (f even or odd, according to m).

From (5.35), we have (with $g(r) := \frac{f(r)}{r^j}$)

$$\frac{\omega_2}{2}\langle f(r)T_{\frac{\operatorname{sgn}(r)}{r^m}}, r^2\varphi^o\rangle = \frac{\omega_2}{2}\langle g(r)T_{\frac{\operatorname{sgn}(r)}{r^n}}, \varphi^o\rangle + (-1)^{n-1}\frac{\omega_2}{2}C(j+2,n)\langle\delta_0^{(n-1)}, \varphi^o\rangle \tag{3.24}$$

and (with (5.37)),

$$\begin{aligned}&\left\langle T_{\frac{\operatorname{sgn}(r)}{r^n}}, g(r)\varphi^o\right\rangle\\ &= \lim_{\varepsilon\to 0^+}\int_{r>\varepsilon}\frac{g(r)\varphi^o(r)+(-1)^{n+1}g(-r)\varphi^o(-r)}{r^n}\,\mathrm{d}r\\ &\quad+\frac{1}{(n-1)!}\left(\ln\varepsilon\Big((g\varphi^o)^{(n-1)}(\varepsilon)+(g\varphi^o)^{(n-1)}(-\varepsilon)\Big)-\sum_{k=2}^{n}\frac{(k-2)!}{\varepsilon^{k-1}}\Big((g\varphi^o)^{(n-k)}(\varepsilon)+(-1)^{(k+1)}(g\varphi^o)^{(n-k)}(-\varepsilon)\Big)\right)\end{aligned}$$

Since we have the following different cases :

	m even and f odd		m odd and f even	
	j even	j odd	j even	j odd
g	odd	even	even	odd
n	even	odd	odd	even

and since $\varphi^o(r) = \varphi^o(-r)$, we always have

$$\frac{g(r)\varphi^o(r)+(-1)^{n+1}g(-r)\varphi^o(-r)}{r^n} = \varphi^o(r)\frac{g(r)+(-1)^{n+1}g(-r)}{r^n} = \frac{2\varphi^o(r)g(r)}{r^n} = 2\frac{\varphi^o(r)f(r)}{r^m}r^2.$$

So

$$\begin{aligned}&\left\langle T_{\frac{\operatorname{sgn}(r)}{r^n}}, g(r)\varphi^o\right\rangle\\ =&\frac{2}{\omega_2}\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^3\setminus B(0,\varepsilon)}\frac{f(\|x\|)}{\|x\|^m}\varphi^o(\|x\|)\,\mathrm{d}x\\ &+\frac{1}{(n-1)!}\left(\ln\varepsilon\Big((g\varphi^o)^{(n-1)}(\varepsilon)+(g\varphi^o)^{(n-1)}(-\varepsilon)\Big)-\sum_{k=2}^{n}\frac{(k-2)!}{\varepsilon^{k-1}}\Big((g\varphi^o)^{(n-k)}(\varepsilon)+(-1)^{(k+1)}(g\varphi^o)^{(n-k)}(-\varepsilon)\Big)\right)\\ =&\frac{2}{\omega_2}\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^3\setminus B(0,\varepsilon)}\frac{f(\|x\|)}{\|x\|^m}\varphi^o(\|x\|)\,\mathrm{d}x + F_\varepsilon(g\varphi^o)\\ =&\frac{2}{\omega_2}\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^3\setminus B(0,\varepsilon)}\frac{f(\|x\|)}{\|x\|^m}\varphi^o(\|x\|)\,\mathrm{d}x + F_\varepsilon(g\varphi^o)\end{aligned}$$

But if we define the tempered distribution u by

$$\langle u,\varphi\rangle := \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^3\setminus B(0,\varepsilon)}\frac{f(\|x\|)}{\|x\|^m}\varphi(x)\,\mathrm{d}x + \frac{\omega_2}{2}F_\varepsilon(g\varphi^o)$$

it is straightforward that u is radial distribution (clearly $u\circ A = u$ for any $A\in\mathcal{O}(n)$, and from Proposition 1.2,

$$\langle u,\varphi\rangle = \langle u,\varphi^{\mathrm{rad}}\rangle = \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^3\setminus B(0,\varepsilon)}\frac{f(\|x\|)}{\|x\|^m}\varphi^o(\|x\|)\,\mathrm{d}x + \frac{\omega_2}{2}F_\varepsilon(g\varphi^o) = \frac{\omega_2}{2}\left\langle T_{\frac{\operatorname{sgn}(r)}{r^n}}, g(r)\varphi^o\right\rangle. \tag{3.25}$$

Putting togheter (3.24) and (3.25), we have

$$\langle T,\varphi\rangle = \langle u,\varphi\rangle + \frac{\omega_2}{2}C(j+2,n)(-1)^{n-1}\langle\delta_0^{(n-1)},\varphi^o\rangle$$

so,

$$\langle PT, \varphi\rangle = \langle u, P\varphi\rangle + \frac{\omega_2}{2} C(j+2,n)(-1)^{n-1} \underbrace{\langle \delta_0^{(n-1)}, (P\varphi)^o\rangle}_{=((P\varphi)^o)^{(n-1)}(0)=0} = \langle u, P\varphi\rangle.$$

The fact that $((P\varphi)^o)^{(n-1)}(0) = 0$ is a straightforward consequence of (3.23) and the following Lemma 3.2 (with $g = 1$).

Now, since we have

$$\lim_{\varepsilon\to 0^+} F_\varepsilon(gP\varphi^o) = 0$$

which is a consequence from the easy following lemma

* * *

Lemma 3.2 *For all $\ell \in \mathbb{N}$ such that $l < d$, there exists a smooth function h_ℓ such that*

$$\big(g(P\varphi)^o\big)^{(\ell)}(\varepsilon) = h_\ell(\varepsilon)\varepsilon^{d-\ell}.$$

Proof:

By induction : for $l = 0$, since g is a smooth function (*cf.* Lemma 6.1), and

$$\begin{aligned}(P\varphi)^o &= \frac{1}{\omega_2}\int_{\mathbb{S}^2} P\varphi(\varepsilon\sigma)\,\mathrm{d}\sigma\\ &= \frac{1}{\omega_2}\int_0^{2\pi}\int_0^{\pi} P\varphi(\varepsilon\Theta(\theta,\psi))\sin(\varphi)\,\mathrm{d}\sigma\\ &= \varepsilon^d\frac{1}{\omega_2}\int_{\mathbb{S}^2} P(\sigma)\varphi(\varepsilon\sigma)\,\mathrm{d}\sigma\end{aligned}$$

we have (it is straightforward that $\varepsilon \mapsto \frac{1}{\omega_2}\int_{\mathbb{S}^2} P(\sigma)\varphi(\varepsilon\sigma)\,\mathrm{d}\sigma$ is a smooth function)

$$g(P\varphi)^o(\varepsilon) = h_0(\varepsilon)\varepsilon^d$$

with $h_0(\varepsilon) := g(\varepsilon)\frac{1}{\omega_2}\int_{\mathbb{S}^2} P(\sigma)\varphi(\varepsilon\sigma)\,\mathrm{d}\sigma$.

For $l + 1$:

$$\begin{aligned}\Big(\big(g(P\varphi)^o\big)^{\ell}\Big)' &= (h_\ell(\varepsilon)\varepsilon^{d-\ell})'\\ &= h'_\ell(\varepsilon)\varepsilon^{d-\ell} + h_\ell(\varepsilon)\varepsilon^{d-\ell-1}\\ &= h_{\ell+1}(\varepsilon)\varepsilon^{d-(\ell+1)}\end{aligned}$$

with $h_{\ell+1}(\varepsilon) := h'_\ell(\varepsilon)\varepsilon + h_\ell(\varepsilon)$.

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we have

$$\langle PT, \varphi\rangle = \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^3\setminus B(0,\varepsilon)} \frac{f(\|x\|)}{\|x\|^m} P\varphi(x)\,\mathrm{d}x = \int_{\mathbb{R}^3} \frac{f(\|x\|)}{\|x\|^m} P\varphi(x)\,\mathrm{d}x.$$

2) Similarly, we will show the result for $T_1 = f(r)T_{\frac{1}{r^m}}$ (f even or odd, according to m).

From (5.34), we have (with $g(r) := \frac{f(r)}{r^j}$)

$$\frac{\omega_2}{2}\langle f(r)T_{\frac{1}{r^m}}, r^2\varphi^o\rangle = \frac{\omega_2}{2}\langle g(r)T_{\frac{1}{r^n}}, \varphi^o\rangle \tag{3.26}$$

and (with (5.36)),

$$\left\langle T_{\frac{\operatorname{sgn}(r)}{r^n}}, g(r)\varphi^o \right\rangle = \lim_{\varepsilon\to 0^+} \int_{r>\varepsilon} \frac{g(r)\varphi^o(r) - (-1)^{n+1} g(-r)\varphi^o(-r)}{r^n}\,\mathrm{d}r$$
$$- \frac{1}{(n-1)!}\sum_{k=1}^{n-1}\frac{(k-1)!}{\varepsilon^k}\Big((g\varphi^o)^{(n-k-1)}(\varepsilon) - (-1)^k (g\varphi^o)^{(n-k-1)}(-\varepsilon)\Big)$$

Since we have the following different cases :

	m even and f even		m odd and f odd	
	j even	j odd	j even	j odd
g	even	odd	odd	even
n	even	odd	odd	even

and since $\varphi^o(r) = \varphi^o(-r)$, we always have

$$\frac{g(r)\varphi^o(r) - (-1)^{n+1} g(-r)\varphi^o(-r)}{r^n} = \varphi^o(r)\frac{g(r) - (-1)^{n+1} g(-r)}{r^n} = \frac{2\varphi^o(r) g(r)}{r^n} = 2\frac{\varphi^o(r) f(r)}{r^m} r^2.$$

So

$$\begin{aligned}\left\langle T_{\frac{1}{r^n}}, g(r)\varphi^o \right\rangle =& \frac{2}{\omega_2}\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^3\setminus B(0,\varepsilon)} \frac{f(\|x\|)}{\|x\|^m}\varphi^o(\|x\|)\,\mathrm{d}x \\ &- \frac{1}{(n-1)!}\sum_{k=1}^{n-1}\frac{(k-1)!}{\varepsilon^k}\Big((g\varphi^o)^{(n-k-1)}(\varepsilon) - (-1)^k (g\varphi^o)^{(n-k-1)}(-\varepsilon)\Big) \\ =& \frac{2}{\omega_2}\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^3\setminus B(0,\varepsilon)} \frac{f(\|x\|)}{\|x\|^m}\varphi^o(\|x\|)\,\mathrm{d}x + F'_\varepsilon(g\varphi^o) \\ =& \frac{2}{\omega_2}\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^3\setminus B(0,\varepsilon)} \frac{f(\|x\|)}{\|x\|^m}\varphi^o(\|x\|)\,\mathrm{d}x + F'_\varepsilon(g\varphi^o)\end{aligned}$$

But if we define the tempered distribution u by

$$\langle u, \varphi\rangle := \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^3\setminus B(0,\varepsilon)} \frac{f(\|x\|)}{\|x\|^m}\varphi(x)\,\mathrm{d}x + \frac{\omega_2}{2}F'_\varepsilon(g\varphi^o)$$

it is straightforward that u is radial distribution (clearly $u\circ A = u$ for any $A \in \mathcal{O}(n)$, and from Proposition 1.2,

$$\langle u, \varphi\rangle = \langle u, \varphi^{\mathrm{rad}}\rangle = \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^3\setminus B(0,\varepsilon)} \frac{f(\|x\|)}{\|x\|^m}\varphi^o(\|x\|)\,\mathrm{d}x + \frac{\omega_2}{2}F'_\varepsilon(g\varphi^o) = \frac{\omega_2}{2}\left\langle T_{\frac{1}{r^n}}, g(r)\varphi^o \right\rangle. \quad (3.27)$$

With (3.26) and (3.27), we have directly

$$\langle T, \varphi\rangle = \langle u, \varphi\rangle$$

so,

$$\langle PT, \varphi\rangle = \langle u, P\varphi\rangle.$$

Now, since we have (using Lemma 3.2),

$$\lim_{\varepsilon\to 0^+} F'_\varepsilon(gP\varphi^o) = 0,$$

then

$$\langle PT, \varphi\rangle = \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^3\setminus B(0,\varepsilon)} \frac{f(\|x\|)}{\|x\|^m}P\varphi(x)\,\mathrm{d}x = \int_{\mathbb{R}^3}\frac{f(\|x\|)}{\|x\|^m}P\varphi(x)\,\mathrm{d}x.$$

•

4 Appendix

4.1 Distributions associated to $\frac{1}{x^n}$ and $\frac{\mathrm{sgn}(x)}{x^n}$

$$T_{\frac{1}{x^n}} := \frac{(-1)^{n-1}}{(n-1)!}\big(\ln(|x|)\big)^{(n)} \tag{4.28}$$

$$T_{\frac{\mathrm{sgn}(x)}{x^n}} := \frac{(-1)^{n-1}}{(n-1)!}\big(\mathrm{sgn}(x)\ln(|x|)\big)^{(n)} \tag{4.29}$$

Remark 4.1 *1. We then have directly de followings inductive properties*

$$T_{\frac{1}{x^n}} = \frac{-1}{n-1}T'_{\frac{1}{x^{n-1}}} \tag{4.30}$$

$$T_{\frac{\mathrm{sgn}(x)}{x^n}} = \frac{-1}{n-1}T'_{\frac{\mathrm{sgn}(x)}{x^{n-1}}}. \tag{4.31}$$

2. $T_{\frac{1}{x^n}}$ and $T_{\frac{\mathrm{sgn}(x)}{x^n}}$ sont des distributions tempérés car elles sont, à constantes près, issues respectivement des dérivations successives (au sens des distributions) des fonctions $x \mapsto x\ln(|x|) - x$ et $x \mapsto |x|\ln(|x|) - x$ qui sont continues et à croissance lente, donc elles-mêmes des distributions tempérées.

Proposition 4.2 *1.*

$$x^n T_{\frac{1}{x^n}} = 1 \tag{4.32}$$

$$x^n T_{\frac{\mathrm{sgn}(x)}{x^n}} = \mathrm{sgn}(x) \tag{4.33}$$

2.

$$x^n T_{\frac{1}{x^{m+n}}} = T_{\frac{1}{x^m}} \tag{4.34}$$

$$x^n T_{\frac{\mathrm{sgn}(x)}{x^{m+n}}} = T_{\frac{\mathrm{sgn}(x)}{x^m}} - (-1)^{m-1}C(m,n)\delta_0^{(m-1)} \tag{4.35}$$

where δ_0 is the Dirac distribution in 0, with the convention $\delta_0^{(0)} := \delta_0$ and with[1]

$$C(m,n) = \frac{2}{(m-1)!}\sum_{k=0}^{n-1}\frac{1}{m+k} = \frac{2(H_{m+n-1} - H_{m-1})}{(m-1)!}.$$

Proof:

•

1. Where H_n is the n-th Harmonic number :

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

Proposition 4.3 *For all $\varphi \in \mathcal{S}(\mathbb{R})$, we have (with the conventions $\varphi^{(0)} = \varphi$ and $\sum_{k=a}^{b} f(k) = 0$ if $a > b$)*

1.

$$\left\langle T_{\frac{1}{x^n}}, \varphi \right\rangle = \lim_{\varepsilon \to 0^+} \int_{x>\varepsilon} \frac{\varphi(x) - (-1)^{n-1}\varphi(-x)}{x^n} \, \mathrm{d}x - \frac{1}{(n-1)!} \sum_{k=1}^{n-1} \frac{(k-1)!}{\varepsilon^k} \Big(\varphi^{(n-1-k)}(\varepsilon) - (-1)^k \varphi^{(n-1-k)}(-\varepsilon) \Big) \tag{4.36}$$

2.

$$\left\langle T_{\frac{\mathrm{sgn}(x)}{x^n}}, \varphi \right\rangle = \lim_{\varepsilon \to 0^+} \int_{x>\varepsilon} \frac{\varphi(x) - (-1)^{n}\varphi(-x)}{x^n} \, \mathrm{d}x + \frac{1}{(n-1)!} \left(\ln\varepsilon \Big(\varphi^{(n-1)}(\varepsilon) + \varphi^{(n-1)}(-\varepsilon) \Big) - \sum_{k=2}^{n} \frac{(k-2)!}{\varepsilon^{k-1}} \Big(\varphi^{(n-k)}(\varepsilon) + (-1)^{(k-1)} \varphi^{(n-k)}(-\varepsilon) \Big) \right) \tag{4.37}$$

Proof:

1. By induction (or direct calculation).
2. By induction.

•

5 Micellaneous

Lemma 5.1 *Let f be a smooth function and $j \in \mathbb{N}^*$ such that*

$$f(0) = f'(0) = \cdots = f^{(j-1)}(0) = 0$$

then the function

$$g : x \mapsto \frac{f(x)}{x^j}$$

is also smooth.

Proof:

The Taylor's Theorem with integral remainder gives

$$f(x) = \frac{x^j}{j!} \int_0^1 (1-t)^j f^{(j)}(tx) \, \mathrm{d}t$$

so

$$g(x) = \frac{1}{j!} \int_0^1 (1-t)^j f^{(j)}(tx) \, \mathrm{d}t$$

and the Leibniz integral rule, allows us to conclude the smoothness of g. •

Références

[1] Loukas Grafakos and Gerald Teschl. On fourier transforms of radial functions and distributions. *Journal of Fourier Analysis and Applications*, 19(1) :167–179, 2013.